A COMPREHENSIVE COURSE IN PURE MATHEMATICS

ADVANCED MATHEMATICS

A PURE COURSE

PERKINS & PERKINS

COALVILLE TECHNICAL COLLEGE
LIBRARY

T08886

510
PER

ADVANCED MATHEMATICS

A PURE COURSE

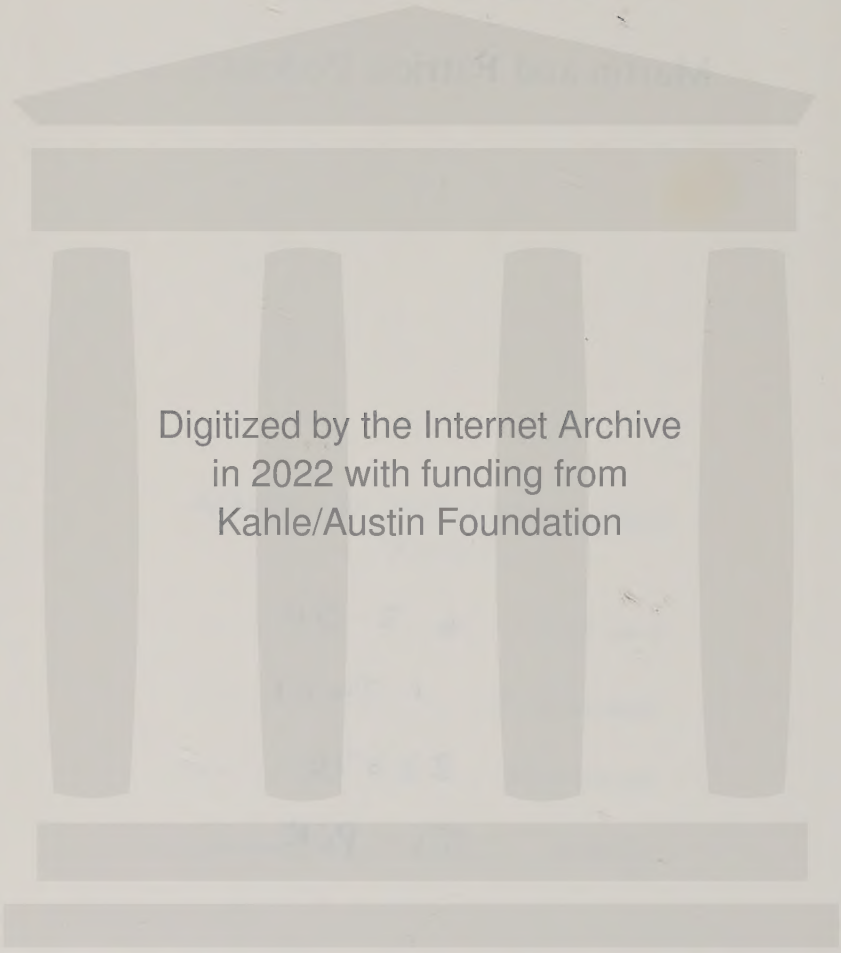
This book is to be returned on or before
the last date stamped below.

Dec 11-92

WITHDRAWN

COALVILLE TECHNICAL COLLEGE LIBRARY
BRIDGE ROAD COALVILLE, LEICESTER LE6 3QR

LIBREX



Digitized by the Internet Archive
in 2022 with funding from
Kahle/Austin Foundation

https://archive.org/details/advancedmathemat0000perk_d6h5

ADVANCED MATHEMATICS

A PURE COURSE

Martin and Patricia Perkins

Unwin Hyman

Published in 1987 by Unwin Hyman Ltd
15–17 Broadwick Street
London W1V 1FP

© Martin and Patricia Perkins 1987

All rights reserved. No part of this publication may be reproduced, stored in a retrieval system, or transmitted in any form or by any means, electronic, mechanical, photo-copying, recording or otherwise, without the prior permission of Unwin Hyman.

British Library Cataloguing in Publication Data

Perkins, Martin
Advanced mathematics: a pure course.
1. Mathematics——1961–
I. Title II. Perkins, Patricia
510 QA39.2

ISBN 0 7135 2821 4

Printed in Great Britain at
The Alden Press Ltd., Oxford.

Contents

Preface viii

List of symbols x

1 Real numbers and their properties 1
- 1.1 The language of sets 1
- 1.2 Rational and irrational numbers 3
- 1.3 Binary operations on real numbers 5
- 1.4 Introduction to inequalities 7
- 1.5 Surds and indices 9
- 1.6 Logarithms 12

2 Functions and graphs 17
- 2.1 Mappings and functions 17
- 2.2 The graph of a function 19
- 2.3 Points, lines and curves 21
- 2.4 Gradient of a straight line 25
- 2.5 The equation of a line 28
- 2.6 Linear inequalities 32

3 Polynomials and equations 36
- 3.1 The quadratic function 36
- 3.2 Quadratic equations 39
- 3.3 Sum and product of roots 42
- 3.4 Non-linear simultaneous equations 45
- 3.5 Products and factors 47
- 3.6 Polynomial equations 52

4 The calculator and numerical work 56
- 4.1 Accuracy and types of error 56
- 4.2 Some functions and their inverses 59

4.3 Simple flow diagrams 63
4.4 Efficient methods of calculation 68
4.5 Reduction of laws to straight line form 71

5 Limits and differentiation 80
5.1 The limit of a function 80
5.2 The gradient of a curve 83
5.3 Differentiation from first principles 86
5.4 Differentiation rules 89
5.5 Elementary curve sketching 93
5.6 The second derivative 100

6 Introduction to integration 106
6.1 The reverse of differentiation 106
6.2 The area under a curve 109
6.3 Properties of definite integrals 114
6.4 Volumes of revolution 117
6.5 Integration as the limit of a sum 120
6.6 Mean value 124

7 Elementary trigonometry 128
7.1 Trigonometric ratios for any angle 128
7.2 The inverse functions and simple equations 133
7.3 Properties of trigonometric ratios 137
7.4 Addition formulae 141
7.5 Double and half angle formulae 144
7.6 Sum and product formulae 147
7.7 The expression $a\cos\theta + b\sin\theta$ 150

8 Trigonometry of triangles 155
8.1 Review of elementary geometry 155
8.2 Sine and cosine rules 157
8.3 Problems in two dimensions 163
8.4 Angles, lines and planes 164
8.5 Problems in three dimensions 168

9 Vectors 175
9.1 Introduction to vectors 175
9.2 Elementary operations on vectors 178
9.3 Unit vectors and Cartesian components 182
9.4 Position vectors 185

10 Further differentiation 190
10.1 Composite functions 190
10.2 Functions defined implicitly 193
10.3 Products and quotients 196
10.4 Maxima and minima 201
10.5 Related rates of change 204
10.6 Small changes and errors 206

11 Trigonometric functions 211
11.1 Circular measure 211
11.2 General solutions of trigonometric equations 213
11.3 Useful limits and approximations 216
11.4 Derivatives of trigonometric functions 219
11.5 Integration of trigonometric functions 222
11.6 Inverse trigonometric functions 225

12 Permutations and combinations 231
12.1 Permutations 231
12.2 Combinations 234
12.3 Elementary theory of probability 237

13 Series and the binomial theorem 242
13.1 Sequences and series 242
13.2 Arithmetic progressions 244
13.3 Geometric progressions 247
13.4 Infinite geometric series 250
13.5 The Σ notation 252
13.6 The binomial theorem 255
13.7 Use of the binomial series 259

14 Graphs and parameters 265
14.1 Odd and even functions 265
14.2 Composite functions 267
14.3 Addition of functions 270
14.4 The function $(ax+b)/(cx+d)$ 272
14.5 Parametric equations 274
14.6 Differentiation and parameters 276

15 Coordinate geometry 282
15.1 The locus of a point 282
15.2 The circle 285
15.3 Tangents and normals 290
15.4 The parabola 294
15.5 Translations and change of origin 298

16 Inequalities 305
16.1 Basic inequalities 305
16.2 Graphical approach 307
16.3 Analytical methods 309
16.4 Modulus inequalities 312

17 Mathematical reasoning 315
17.1 Deductive and inductive processes 315
17.2 Implication and equivalence 318
17.3 Types of proof 321
17.4 Proof by induction 323

18 Some functions and their properties 331
18.1 Symmetry, continuity and differentiability 331
18.2 Some graphs with related equations 336
18.3 Inverse functions 340
18.4 Polynomials; remainders and factors 344
18.5 Graphs of rational functions 348
18.6 Partial fractions 353

19 Exponential and logarithmic functions 363
19.1 Exponential functions 363
19.2 Logarithmic functions 368
19.3 $\ln x$ defined in integral form 372
19.4 Integration using logarithms 375
19.5 Introduction to hyperbolic functions 379

20 Methods of integration 383
20.1 Standard forms 383
20.2 Use of partial fractions 387
20.3 Integration by substitution 391
20.4 Integration by parts 397
20.5 Use of inverse trigonometric functions 401
20.6 Properties of the definite integral 404

21 Vectors and scalar product 413
21.1 Properties of vectors 413
21.2 Equations of lines and planes 417
21.3 Scalar product 424
21.4 Further lines and planes 429

22 Series and expansions 439
22.1 Summation of finite series 439
22.2 Maclaurin's theorem 444
22.3 The binomial series 446
22.4 Exponential and logarithmic series 452
22.5 The expansions of $\sin x$ and $\cos x$ 457

23 Some numerical methods 461
23.1 Numerical integration 461
23.2 Approximate solutions to equations 469
23.3 The Newton-Raphson method 475
23.4 Other iterative processes 479

24 Differential equations 491
24.1 Differential equations and their solution curves 491
24.2 Elementary methods of solution 495
24.3 Equations with separable variables 501
24.4 Formulation of differential equations 506
24.5 The equations $d^2y/dx^2 \pm n^2y = \text{constant}$ 509

25 Further coordinate geometry 520
25.1 Angles, distances and areas 520
25.2 Further work on circles 525
25.3 Loci using parametric forms 529
25.4 Conic sections 535

26 Complex numbers 544
26.1 Algebraic form 544
26.2 Roots of polynomial equations 548
26.3 Geometrical representation 552
26.4 Modulus-argument form 556
26.5 Loci in the Argand diagram 562
26.6 De Moivre's theorem for an integral index 570

Formulae for reference 580

Answers 584

Index 649

Preface

This is one of two books designed to cover most Advanced Level single subject Mathematics syllabuses. The present volume provides a complete pure course. Both books should also be suitable for subsidiary level mathematics courses of various kinds. Students in schools and colleges embarking on advanced work come from a variety of backgrounds. We have therefore tried to introduce each new topic gradually to cater for those to whom it is unfamiliar.

We believe that the split between 'modern' and 'traditional' mathematics has been damaging and consider that the subject should be seen as a whole: we have attempted in this book to use what we see as the best in these two approaches to create a coherent forward-looking course. SI units are used throughout and we assume that students will have access to an adequate 'scientific' calculator.

Topics are presented in a logical sequence, but the text is arranged in short chapters to allow individual students or teachers to vary the order of study. The chapters are divided into numbered sections each followed by an exercise. The early questions in these exercises are intended to provide plenty of routine practice in newly acquired skills, whereas the aim of later questions is to develop a greater understanding of the ideas involved. There is a selection of further questions in the miscellaneous exercise at the end of each chapter.

We are most grateful for permission to use questions from past G.C.E. Advanced Level examinations. These are acknowledged as follows:

University of London University Entrance and Schools Examination Council (L)
The Associated Examining Board (AEB)
Joint Matriculation Board (JMB)
University of Cambridge Local Examinations Syndicate (C)
Oxford and Cambridge Schools Examination Board (O & C)
Oxford Delegacy of Local Examinations (O)
Southern Universities Joint Board (SU)
Welsh Joint Education Committee (W)

Our thanks are also due to the staff of Messrs. Bell and Hyman, especially Peter Kaner and Anne Forsyth for their helpful criticism, and to many friends and colleagues for their encouragement and patience.

M.L.P.
P.P.

List of symbols

s.f.	significant figures
d.p.	decimal places
$\approx$	is approximately equal to
$\equiv$	is identically equal to
$<$	is less than
$\leqslant$	is less than or equal to
$>$	is greater than
$\geqslant$	is greater than or equal to
$\Rightarrow$	implies
$\Leftrightarrow$	is equivalent to
$\in$	is an element of
:	such that
$\subseteq$	is a subset of
$\subset$	is a proper subset of
$\varnothing$	empty set
$\mathscr{E}$	universal set
A'	complement of set A
$n(A)$	number of elements in the set A
$A \times B$	Cartesian product of sets A and B
$\mathbb{N}$	set of natural numbers $\{0,1,2,3,\ldots\}$
$\mathbb{Z}$	set of integers
$\mathbb{Z}^+$	set of positive integers
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	set of real numbers
$\mathbb{R}^2$	set of ordered pairs of real numbers (x, y)
$\cup$	union
$\cap$	intersection
$f: x \to y$	the function f maps x to y
$f(x)$	image of x under f
f^{-1}	inverse of function f
gf	function f followed by function g

$[a, b]$	the closed interval $\{x \in \mathbb{R}: a \leqslant x \leqslant b\}$
(a, b)	the open interval $\{x \in \mathbb{R}: a < x < b\}$
$\lim_{x \to a} f(x)$	limit of $f(x)$ as x tends to a
∞	infinity
δ	small increment
$f'(x)$ or $\frac{dy}{dx}$	first derivative of $y = f(x)$
$f''(x)$ or $\frac{d^2y}{dx^2}$	second derivative of $y = f(x)$
$\int f(x)\,dx$	indefinite integral
$\int_a^b f(x)\,dx$	definite integral
$\Big[f(x)\Big]_a^b$	$f(b) - f(a)$
$\sqrt{x}$	non-negative square root of x
$\lvert x \rvert$	modulus (or magnitude) of x
$[x]$	integral part of x
e^x, $\exp x$	exponential function of x
$\log_a x$	logarithm to the base a of x
$\ln x$	the natural logarithm of x, $\log_e x$
$\lg x$	the common logarithm of x, $\log_{10} x$
$\sin^{-1}$	inverse function of sin with range $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$
$\cos^{-1}$	inverse function of cos with range $[0, \pi]$
$\tan^{-1}$	inverse function of tan with range $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$
$n!$	n factorial
${}_nP_r$	number of permutations of r objects chosen from n
${}_nC_r$	number of combinations of r objects chosen from n
u_n	nth term of a sequence or series
S_n	sum to n terms of a series
$\sum_{r=1}^{n} f(r)$	$f(1) + f(2) + \ldots + f(n)$
$\binom{n}{r}$	the binomial coefficient: ${}_nC_r = \frac{n!}{r!(n-r)!}$ for $n \in \mathbb{Z}^+$, $\frac{n(n-1)\ldots(n-r+1)}{r!}$ for $n \in \mathbb{Q}$
$\mathbf{PQ}$	vector represented by line segment $\overrightarrow{PQ}$
$\lvert\mathbf{PQ}\rvert$ or PQ	magnitude of vector $\mathbf{PQ}$

$\lvert\mathbf{a}\rvert$, a	magnitude of vector $\mathbf{a}$
$\hat{\mathbf{a}}$	unit vector in the direction of $\mathbf{a}$
$\mathbf{i}, \mathbf{j}, \mathbf{k}$	unit vectors in the directions of the x-, y- and z-axes
$\mathbf{a}.\mathbf{b}$	scalar product of $\mathbf{a}$ and $\mathbf{b}$
i	square root of -1
z	complex number, $z = x + iy = r(\cos\theta + i\sin\theta)$
$\mathrm{Re}(z)$	real part of z, $\mathrm{Re}(z) = x = r\cos\theta$
$\mathrm{Im}(z)$	imaginary part of z, $\mathrm{Im}(z) = y = r\sin\theta$
z^*	complex conjugate of z, $z^* = x - iy$
$\lvert z\rvert$	modulus of z, $\lvert z\rvert = \sqrt{(x^2 + y^2)} = r$
$\arg z$	principal argument of z, $\arg z = \theta$, where $\sin\theta = x/r$, $\cos\theta = y/r$ and $-\pi < \theta \leqslant \pi$
$\mathbb{C}$	set of complex numbers

The use of the symbols $=$ and $\approx$

In numerical work it is important to distinguish between exact values and approximate values. When giving exact answers we use the symbol $=$. For approximate results we use either the symbol $=$ (together with an appropriate phrase, such as 'by calculator', 'to 3 s.f.'), or the symbol $\approx$. The use of the symbol $\approx$ may show that a result has been obtained by an approximate method, but often indicates that a more accurate result obtained by calculator has been rounded to the given number of significant figures (or, in the case of angles, to the nearest tenth of a degree).

1 Real numbers and their properties

1.1 The language of sets

A *set* is a collection of objects. These objects are the *elements* of the set. A set may be defined either by listing the elements or by describing their properties, e.g. $A = \{2,4,7\}$, B is the set of all triangles, $C = \{\ldots, -5, -3, -1, 1, 3, \ldots\}$ or {odd integers}. Using the colon : to mean 'such that', we may also write $P = \{p : p \text{ is a prime number}\}$, which is read '$P$ is the set of all elements p such that p is a prime number'. Whatever form is chosen it must always be possible to decide from the definition whether or not an object belongs to the set.

The statement $x \in S$ means 'x is an element of S' or 'x belongs to S'. Similarly $x \notin S$ means 'x is not an element of the set S'. Thus, for the sets defined above, $2 \in A$ but $2 \notin C$ and $9 \in C$ but $9 \notin P$.

Two sets are *equal* if they contain exactly the same elements, e.g. $\{16, 25, 36, 49, 64, 81\} = \{36, 81, 16, 25, 64, 49\} = \{\text{two digit squares}\}$. The way in which the elements are listed or described is not important.

Sometimes all the elements of one set are contained in another. If every element of a set A also belongs to a set B, then A is a *subset* of B, written $A \subseteq B$. If A is a subset of B and there is at least one element of B which is not an element of A, then A is a *proper subset* of B and we write $A \subset B$. For instance, if $X = \{a,c\}$, $Y = \{a,c,e\}$ and $Z = \{a, b, c, d\}$, then $X \subset Y$ and $X \subset Z$ but $Y \not\subset Z$. We may also write $Y \supset X$ and $Z \supset X$ using the symbol $\supset$ to mean 'contains as a proper subset'.

Three important properties of the set inclusion relation follow from the definition.

(1) Any set A is a subset of itself, i.e. $A \subseteq A$.

(2) $A \subseteq B$ and $B \subseteq C \Rightarrow A \subseteq C$.

(3) $A \subseteq B$ and $B \subseteq A \Leftrightarrow A = B$.

[Note that if p and q represent statements, then $p \Rightarrow q$ means 'if p then q' or 'p implies q'. The symbol $\Leftarrow$ means 'is implied by'. We write $p \Leftrightarrow q$ meaning 'p implies and is implied by q' when the statements p and q are equivalent.]

All the objects under consideration in any mathematical discussion form a set called the *universal set* (or *universe of discourse*), often denoted by $\mathscr{E}$. For instance, in a geometrical problem the universal set may be a set of points, a set of lines or a set of plane figures. Once a universal set $\mathscr{E}$ has been defined any other set considered must be one of its subsets. Thus for any set A, $A \subseteq \mathscr{E}$.

The set with no elements is called the *null* or *empty set*, written $\varnothing$ or sometimes $\{\ \}$. The empty set $\varnothing$ is considered to be a subset of any set A, i.e. $\varnothing \subseteq A$.

The *complement* of a set A is the set containing all the elements of $\mathscr{E}$ which are not elements of A and is written A'. For instance, if $\mathscr{E} = \{1,2,3,4,5\}$ and $A = \{1,3,5\}$ then $A' = \{2,4\}$. It follows from the definition that for any set A, $(A')' = A$. We also find that $\varnothing' = \mathscr{E}$ and $\mathscr{E}' = \varnothing$.

Consider now two sets X and Y. If $x \in X$ and $y \in Y$, then (x, y) is called an *ordered pair*. Although the set $\{3,4\}$ is equal to the set $\{4,3\}$, the ordered pairs $(3,4)$ and $(4,3)$ are not the same. The set of all ordered pairs (x, y) where $x \in X$ and $y \in Y$ is called the *product set* or *Cartesian product* of X and Y, written $X \times Y$. For example, if $X = \{1,2,3\}$ and $Y = \{4,5\}$,

then $X \times Y = \{(1,4),(1,5),(2,4),(2,5),(3,4),(3,5)\}$
and $Y \times X = \{(4,1),(5,1),(4,2),(5,2),(4,3),(5,3)\}$.

This example demonstrates the fact that in general $X \times Y \neq Y \times X$.

Exercise 1.1

1. State whether the following statements are true or false:
(a) $4 \in \{\text{even integers}\}$ (b) $\{a\} \in \{a,b,c\}$
(c) $51 \notin \{\text{prime numbers}\}$ (d) $\varnothing \subset \{3,7,11,12\}$
(e) $\{x,y\} = \{\{x\},\{y\}\}$ (f) $0 \subset \{0,1,2\}$
(g) $\{p : p \text{ is prime and } p \neq 2\} \subset \{x : x \text{ is an odd integer}\}$
(h) $\{n : n \text{ is divisible by 2, 3 and 4}\} = \{k : k \text{ is a multiple of 24}\}$.

2. Use the symbols $=$ and $\subset$ to write down any relationships between the following:
$A = \{1,2,0\}$, $B = \{2,1\}$, $C = \{0,1,2\}$, $D = \{2,3\}$, $E = \{3\}$.

3. Use the symbols $=$, $\subset$ and $\in$ to write down any relationships between the following:
$\varnothing$, $X = \{0\}$, $Y = \{0,1\}$, 0, $P = \{\{0\}\}$, $Q = \{\varnothing\}$.

4. In each of the following cases state the connection between the statements p and q by writing $p \Leftrightarrow q$, $p \Rightarrow q$ or $p \not\Rightarrow q$.
(a) p: $x \in A$ and $A \subset B$, q: $x \in B$
(b) p: $1 \in X$ and $2 \in X$, q: $\{1,2\} \subseteq X$
(c) p: $a = x$ and $b = y$, q: $\{a,b\} = \{x,y\}$
(d) p: $x \notin A$ and $x \in B$, q: A is a proper subset of B.

5. Write down the connection between the sets X and Z if
(a) $W \subset Y$, $X \subset W$ and $Y \subset Z$, (b) $X \subseteq Y$, $Y \subseteq Z$ and $Z \subseteq X$.

6. Find the sets A, B, C and D which are distinct subsets of the set $\{1,2,3\}$ given that:
$2 \in C$, $3 \notin D$, $A \subset B$, $B \subset C$, $B \subset D$ and $D \not\subset C$.

7. List the elements of the following product sets, where $A = \{x\}$, $B = \{p,q\}$ and $C = \{f,g,h\}$
(a) $A \times C$, (b) $A \times B$, (c) $B \times A$, (d) $A \times A$.

8. If $A \times B = \{(1,1), (2,1), (1,2), (3,2), (3,1), (2,2)\}$, list the elements of the sets A and B.

9. If the universal set $\mathscr{E} = \{0,1,2\}$, write down the complements of the following sets
(a) $\{1\}$, (b) $\varnothing$, (c) $\{0,1\}$, (d) $\{0,1,2\}$.

10. Discuss the following statements
(a) $x \in A' \Leftrightarrow x \notin A$
(b) $A = \{\text{even integers}\} \Leftrightarrow A' = \{\text{odd integers}\}$
(c) $A \subset B \Leftrightarrow B' \subset A'$
(d) $A \subset B$ and $B' \subset C' \Rightarrow A \not\subset C$.

1.2 Rational and irrational numbers

Most people are familiar with the counting numbers 0,1,2,3,4,... The set $\{0,1,2,3,4,\ldots\}$ containing all such numbers is called the set of *natural numbers* $\mathbb{N}$. Any two natural numbers, a and b, may be added to produce a third, $a+b$. However, the solution of the equation $a+x=b$, where a, $b \in \mathbb{N}$, is a natural number only if $a \leqslant b$, 'a is less than or equal to b'.

To solve the equation $a+x=b$ for all elements a, $b \in \mathbb{N}$, it is necessary to extend the set of natural numbers to create a new set containing all numbers of the form $b-a$. For example, the set must include as elements $3-8$, i.e. -5, and $4-6$, i.e. -2. All such numbers form the set of *integers*, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. Clearly the set $\mathbb{N}$ is a subset of $\mathbb{Z}$ and we may write $\mathbb{N} \subset \mathbb{Z}$. The set of *positive integers* $\{1,2,3,4,\ldots\}$ is denoted by $\mathbb{Z}^+$. Since the set $\mathbb{N}$ contains all non-negative integers including zero, $\mathbb{Z}^+ \subset \mathbb{N}$, i.e. $\mathbb{Z}^+$ is a proper subset of $\mathbb{N}$.

The addition, subtraction or multiplication of two integers always produces another integer. However, to solve all equations of the type $bx = a$, where a, $b \in \mathbb{Z}$ and $b \neq 0$, a further extension of the number system is needed. The enlarged set must include elements such as $3/4$, $-7/11$, $10/3$. The set containing all numbers of the form a/b, where a is any integer and b is any non-zero integer, is the set of *rational numbers* $\mathbb{Q}$. Since any integer k can be expressed in the form a/b by writing $a = k$ and $b = 1$, we may write $\mathbb{Z} \subset \mathbb{Q}$.

The natural numbers, $\mathbb{N}$, the integers, $\mathbb{Z}$, and the rationals, $\mathbb{Q}$, are all subsets of the set of *real numbers* $\mathbb{R}$. The real numbers may be represented as the points on a straight line called the *real number line*. As shown in the diagram, the integers are

represented by points evenly spaced along the line. Every point on the line represents some real number.

Real numbers which are not rational are called *irrational numbers*. To prove that such numbers exist, we consider $\sqrt{2}$, the positive root of the equation $x^2 = 2$. If $\sqrt{2}$ is rational, then it can be expressed as a fraction in its lowest terms i.e. as a/b, where a and b are integers with no common factor greater than 1. $\sqrt{2} = \frac{a}{b} \Rightarrow 2 = \frac{a^2}{b^2} \Rightarrow a^2 = 2b^2 \Rightarrow a^2$ is an even number. Since no odd number has an even square, a itself must be an even number. If we let $a = 2c$, then $(2c)^2 = 2b^2$ so that $b^2 = 2c^2$. This implies that b is an even number. Thus a and b are both divisible by 2, which contradicts the assumption that a and b have no common factor greater than 1. Hence the assumption that $\sqrt{2}$ is rational must be false. We deduce that $\sqrt{2}$ is an irrational number.

All rational numbers and many irrational numbers, such as $\sqrt{2}$, which are roots of equations of the form $a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0$, where $a_0, a_1, \ldots, a_n \in \mathbb{Z}$, are known as *algebraic numbers*. Real numbers which are not algebraic are called *transcendental numbers*, the most familiar of these being π.

We now consider the expression of rational and irrational numbers as decimals. A proper fraction a/b is converted into decimal form by the process of long division. Each division by b leaves a remainder, which may be $0, 1, 2, \ldots, b-2$ or $b-1$. If at any stage this remainder is zero, then the process is complete and a/b has been expressed as a finite decimal. If at every stage a non-zero remainder is obtained, then there are $(b-1)$ different possible remainders. This means that in the first b divisions at least one remainder, R, must be repeated. After the repetition of R, the sequence of remainders following R must also be repeated. Thus a recurring decimal will be produced. Hence a rational number may be expressed either as a finite decimal or as a recurring decimal.

Conversely every finite decimal represents a rational number, e.g. $0{\cdot}45 = 45/100 = 9/20$. Any recurring decimal may be written in rational form by a method similar to the following:

let	$x = 0{\cdot}\dot{4}\dot{5} = 0{\cdot}454545\ldots$
then	$100x = 45{\cdot}454545\ldots$
subtracting	$99x = 45$
giving	$x = 45/99 = 5/11.$

These results together imply that in decimal form rational numbers are either finite or recurring and irrational numbers are infinite and non-recurring.

Exercise 1.2

1. If $X = \{2, 0, -2, 0{\cdot}\dot{7}, \sqrt[3]{2}, \pi, 7{\cdot}3\}$, write down all the elements of X which are (a) integers, (b) rational, (c) irrational, (d) algebraic, (e) transcendental.

2. Repeat question 1 for $X = \{7, -1, 4/3,\ \pi+2, 3{\cdot}142, \sqrt{\pi}, \sqrt{3}+2\}$.

3. List the elements of the set A when (i) $x \in \mathbb{N}$, (ii) $x \in \mathbb{Z}$, (iii) $x \in \mathbb{Q}$, (iv) $x \in \mathbb{R}$, given that

(a) $A = \{x : x^2 - 3x - 4 = 0\}$ (b) $A = \{x : x^2 - 5 = 0\}$

(c) $A = \{x : 2x^2 - 3x + 1 = 0\}$ (d) $A = \{x : x^2 + 5 = 0\}$

4. State which of the following rational numbers may be expressed as a finite decimal. (You need not write the numbers in decimal form.)

$$\frac{1}{2}, \frac{1}{3}, \frac{7}{16}, \frac{4}{9}, \frac{1}{25}, \frac{8}{11}, \frac{1}{625}, \frac{5}{64}, \frac{2}{7}, \frac{4}{45}, \frac{1}{150}, \frac{51}{75}.$$

5. Express the following recurring decimals in rational form:

(a) $0{\cdot}\dot{5}$, (b) $0{\cdot}4\dot{9}$, (c) $0{\cdot}\dot{2}\dot{7}$, (d) $0{\cdot}\dot{0}3\dot{7}$, (e) $0{\cdot}\dot{1}42857$.

1.3 Binary operations on real numbers

A *binary operation*, $*$, is a rule for combining two objects, a and b, to give a third object c, written $c = a * b$. In later chapters it will be shown that binary operations can be performed on objects such as vectors, functions and sets. However, we will consider here only binary operations in which two real numbers are combined to form another real number. The most important of these are addition, subtraction, multiplication and division.

When adding or multiplying two numbers the order in which the numbers are written is not important, e.g. $3+5 = 5+3$ and $4 \times 6 = 6 \times 4$. For this reason addition and multiplication are said to be *commutative*. However, $8-3 \neq 3-8$ and $15 \div 5 \neq 5 \div 15$. Thus subtraction and division are not commutative. More formally, the *commutative laws* for addition and multiplication state that

$$a+b = b+a, \quad ab = ba \quad \text{for all} \quad a, b \in \mathbb{R}.$$

We next examine the addition or multiplication of three or more real numbers. The order in which the operations are performed does not affect the result, e.g. $5+(4+3) = (5+4)+3$ and $5 \times (4 \times 3) = (5 \times 4) \times 3$. This property of addition and multiplication is called *associativity*. Subtraction and division are not associative, e.g. $5-(4-3) = 4$ but $(5-4)-3 = -2$; $24 \div (4 \div 2) = 12$ but $(24 \div 4) \div 2 = 3$. In general, the *associative laws* for addition and multiplication state that

$$a+(b+c) = (a+b)+c, \quad a(bc) = (ab)c \quad \text{for all} \quad a, b, c \in \mathbb{R}.$$

Another important property which involves both addition and multiplication is the *distributive law*. This is the rule used when 'removing brackets' or 'factorising' in elementary algebra. It states that

$$a(b+c) = ab+ac \quad \text{for all} \quad a, b, c \in \mathbb{R}.$$

Multiplication is said to be *distributive over addition.*

Within the set of real numbers there are two numbers with special properties, namely 0 and 1. We find that

$$a+0 = 0+a = a, \quad a \times 1 = 1 \times a = a \quad \text{for all} \quad a \in \mathbb{R}.$$

Both addition of 0 and multiplication by 1 leave a real number a unchanged (i.e. identical in value). Hence 0 and 1 are called *identity elements*. The number 0 is the identity for addition and 1 is the identity for multiplication.

This leads to the notion of an *inverse* under a binary operation. If an inverse exists then the combination of an element with its inverse gives the identity element. Thus, since $a+(-a) = (-a)+a = 0$, the inverse of a real number a with respect to addition is $(-a)$. Similarly, since $a \times \frac{1}{a} = \frac{1}{a} \times a = 1$ (assuming $a \neq 0$), the inverse of any non-zero real number a with respect to multiplication is $1/a$.

In further study of the structure of the real number system, it is often convenient to regard the subtraction of a number a as the addition of $(-a)$ and division by a, where $a \neq 0$, as multiplication by $1/a$.

We now consider the use made of inverses and the properties of binary operations in the solution of a simple equation.

$$\begin{aligned}
& 3(x+2) = 8 && \\
\Rightarrow \quad & 3x+6 = 8 && \text{distributive law} \\
\Rightarrow \quad & \{3x+6\}+(-6) = 8+(-6) && \text{using the inverse of 6 for } + \\
\Rightarrow \quad & 3x+\{6+(-6)\} = 2 && \text{associative law for } + \\
\Rightarrow \quad & 3x+0 = 2 && \\
\Rightarrow \quad & 3x = 2 && \text{property of 0} \\
\Rightarrow \quad & \tfrac{1}{3} \times (3x) = \tfrac{1}{3} \times 2 && \text{using the inverse of 3 for } \times \\
\Rightarrow \quad & (\tfrac{1}{3} \times 3)x = \tfrac{2}{3} && \text{associative law for } \times \\
\Rightarrow \quad & 1 \times x = \tfrac{2}{3} && \\
\Rightarrow \quad & x = \tfrac{2}{3} && \text{property of 1.}
\end{aligned}$$

In work with real numbers we usually take their properties for granted. However, the terms introduced here will be useful when comparing the behaviour of real numbers with that of other mathematical objects.

Exercise 1.3

1. Write down the inverses with respect to addition of the real numbers 5, $\frac{1}{4}$, -2, k and $-n$.

2. Write down the inverses with respect to multiplication of the real numbers 2, $-\frac{1}{3}$, -1, 4/5 and $1/x$.

3. Given that $x * y = x^y$, where x and y are positive integers, state whether the binary operation $*$ is (a) commutative, (b) associative.

4. Repeat question 3 given that $x * y$ is defined as the highest common factor of x and y.

5. Repeat question 3 given that $x * y = \frac{1}{2}(x+y)$.

6. Solve the following equations, stating which properties of real numbers are used.

(a) $3x = 12$ (b) $x-4 = 5$ (c) $2x+5 = 9$
(d) $2x = x-2$ (e) $3(x+2) = x$ (f) $x^2-2x = 0$.

1.4 Introduction to inequalities

An inequality is a statement involving one of the symbols $<$ 'is less than', $>$ 'is greater than', $\leqslant$ 'is less than or equal to', $\geqslant$ 'is greater than or equal to'. Inequalities such as $2x+3 < 7$, can be 'solved' by methods similar to those used to solve equations.

	$2x+3 < 7$
Subtract 3 from both sides	$2x < 4$
Divide both sides by 2	$x < 2$

The result can be represented as a set of points on the real number line. An open circle is used to indicate that $x = 2$ is not a solution of the inequality.

In general we find that when simplifying inequalities, any number may be added to or subtracted from both sides. However, when both sides are multiplied or divided by a number, the inequality remains valid only if the number is positive. If both sides of an inequality are multiplied or divided by a negative number, then the inequality sign must be reversed. For instance, $3 < 4$ but $-3 > -4$. Similarly, if $-x > 2$ then $x < -2$.

Example 1 Simplify the inequality $2x-5 \leqslant 5x+4$.

$$\begin{aligned} 2x-5 \leqslant 5x+4 &\Leftrightarrow \quad 2x \leqslant 5x+9 \\ &\Leftrightarrow -3x \leqslant 9 \\ &\Leftrightarrow \quad x \geqslant -3 \end{aligned}$$

This result is illustrated in the diagram below.

The set of numbers satisfying an equation or inequality is called the *solution set*. For instance, the solution set of the equation $2x+3 = 7$ is the set $\{2\}$ containing one element. The solution set of the inequality $2x+3 < 7$ is the set $\{x : x < 2\}$, which has infinitely many elements. [To avoid ambiguity we may describe this set more fully by writing $\{x \in \mathbb{R} : x < 2\}$.]

Example 2 Find, and represent on the real number line, the solution set of the inequalities $0 < 4x - 3 \leqslant 9$.

$$0 < 4x - 3 \leqslant 9 \Leftrightarrow 3 < 4x \leqslant 12$$
$$\Leftrightarrow \tfrac{3}{4} < x \leqslant 3$$

$\therefore$ the solution set is $\{x : \frac{3}{4} < x \leqslant 3\}$.

The *modulus* of a real number x, written $|x|$, is the magnitude of x. For instance, $|3| = 3$ and $|-5| = 5$. Using this notation inequalities such as $-2 < x < 2$ can be written in the form $|x| < 2$.

Example 3 Find the range of values of x for which $|3x| > 12$.

$$|3x| > 12 \Rightarrow 3x < -12 \quad \text{or} \quad 3x > 12$$
$$\Rightarrow x < -4 \quad \text{or} \quad x > 4.$$

Exercise 1.4

1. Represent the following sets on the real number line
(a) $\{x : x > -1\}$ (b) $\{x : x \leqslant 4\}$ (c) $\{x : 0 < x < 3\}$
(d) $\{x : -3 \leqslant x \leqslant 1\}$ (e) $\{x : |x| \leqslant 3\}$ (f) $\{x : |x| > 2\}$.

2. Find, and represent on the real number line, the solution sets of the following inequalities.
(a) $x + 2 < 9$ (b) $x - 4 \geqslant 1$ (c) $1 - x > 2$
(d) $2x - 5 \leqslant 3$ (e) $1 - 3x > 7$ (f) $5x + 3 < x$

3. Find the range of values of x satisfying the following inequalities.
(a) $2x - 3 < 4 - 5x$ (b) $2x + 9 \leqslant 4x - 7$
(c) $2(x + 1) < x - 1$ (d) $\frac{1}{2}(3x + 1) \geqslant \frac{1}{3}(4x + 5)$
(e) $|2x| \geqslant 10$ (f) $|\frac{1}{2}x| < 3$

4. Simplify the following inequalities.
(a) $-1 < 2x < 1$ (b) $0 \leqslant -3x \leqslant 6$
(c) $-2 < 5x - 7 < 3$ (d) $-3 \leqslant 1 - 4x < 3$
(e) $|x - 1| < 2$ (f) $|x + 3| \leqslant 7$

5. State whether the following statements are true or false.
(a) $x < k \Rightarrow 1 - x < 1 - k$ (b) $-x < k \Rightarrow -k < x$
(c) $x^2 > 1 \Rightarrow x > 1$ (d) $x^2 < 1 \Rightarrow x < 1$
(e) $x < y \Rightarrow x^2 < y^2$ (f) $x < y < 0 \Rightarrow x^2 > y^2$
(g) $x < y \Rightarrow \dfrac{1}{x} > \dfrac{1}{y}$ (h) $0 < x < y \Rightarrow \dfrac{1}{x} > \dfrac{1}{y}$
(i) $x < y \Rightarrow |x| < |y|$ (j) $x < y < 0 \Rightarrow |y| < |x|$
Check your answers by considering some numerical examples.

1.5 Surds and indices

The reader will be familiar with the basic laws of indices which state that, for positive integers m and n,

$$a^m \times a^n = a^{m+n} \quad \text{and} \quad (a^m)^n = a^{mn}.$$

For example, $a^2 \times a^3 = (a \times a) \times (a \times a \times a) = a^5$
and $(a^2)^3 = (a \times a)^3 = (a \times a) \times (a \times a) \times (a \times a) = a^6$.

The rule for division is that, for $m > n$, $a^m \div a^n = a^{m-n}$. By assuming that this rule also holds for $m \leqslant n$, we can give meanings to the expressions a^0 and a^{-n}, (provided that $a \neq 0$).

We write $a^0 = a^{n-n} = a^n \div a^n = 1$
and $a^{-n} = a^{0-n} = a^0 \div a^n = 1 \div a^n = 1/a^n$.

Thus it is consistent with the laws of indices to let $a^0 = 1$ and $a^{-n} = 1/a^n$.

We extend this process by assuming that the laws of indices also apply to fractional powers of positive real numbers. It then follows that $(a^{1/2})^2 = a^{1/2} \times a^{1/2} = a^1 = a$. Hence $a^{1/2}$ must be a square root of a. It is usual to take $a^{1/2}$ to be the positive square root $\sqrt{a}$. More generally, we can write

$$(a^{1/n})^n = a^{(1/n).n} = a^1 = a.$$

Hence, if n is a positive integer, $a^{1/n}$ is considered to be the positive nth root of a, written $\sqrt[n]{a}$. It follows that

$$a^{m/n} = (a^{1/n})^m = (\sqrt[n]{a})^m \quad \text{and} \quad a^{m/n} = (a^m)^{1/n} = \sqrt[n]{(a^m)},$$

where m and n are positive integers.

We can now summarise the laws of indices for rational powers of a positive real number a.

$a^m \times a^n = a^{m+n}$	$a^0 = 1, \quad a^{-n} = 1/a^n$
$a^m \div a^n = a^{m-n}$	
$(a^m)^n = a^{mn}$	$a^{m/n} = \sqrt[n]{(a^m)} = (\sqrt[n]{a})^m.$

Example 1 Evaluate (a) 2^{-2} (b) $16^{1/2}$ (c) $8^{2/3}$ (d) $\left(\frac{4}{9}\right)^{-1/2}$

(a) $2^{-2} = \frac{1}{2^2} = \frac{1}{4}$, (b) $16^{1/2} = \sqrt{16} = 4$, (c) $8^{2/3} = (\sqrt[3]{8})^2 = 2^2 = 4$,

(d) $\left(\frac{4}{9}\right)^{-1/2} = \left(\frac{9}{4}\right)^{1/2} = 9^{1/2}/4^{1/2} = \frac{\sqrt{9}}{\sqrt{4}} = \frac{3}{2} = 1\frac{1}{2}$.

[We have not considered rational powers of negative numbers at this stage because of the difficulties which arise. One problem is that no negative number has a real nth root when n is even. In the case of odd values of n, contradictory results can be produced. For instance, $(-8)^{1/3} = \sqrt[3]{(-8)} = -2$, but $(-8)^{1/3} = (-8)^{2/6} = \sqrt[6]{\{(-8)^2\}} = \sqrt[6]{64} = 2$.]

Equations involving indices can be solved in a variety of ways.

Example 2 Solve the equation $8^x = 32$.

Expressing both sides of the equation as powers of 2,
$8^x = 32 \Leftrightarrow (2^3)^x = 2^5 \Leftrightarrow 2^{3x} = 2^5$
$\therefore \quad 3x = 5$, i.e. $x = 1\frac{2}{3}$.

Example 3 Solve the equation $2^{2x} - 5.2^x + 4 = 0$.

Substituting $y = 2^x$, $\quad 2^{2x} - 5.2^x + 4 = 0$
$\Leftrightarrow \quad y^2 - 5y + 4 = 0$
$\Leftrightarrow \quad (y-1)(y-4) = 0$
$\Leftrightarrow \quad y = 0 \quad \text{or} \quad y = 4$
$\Leftrightarrow 2^x = 1 \quad \text{or} \quad 2^x = 4$
$\Leftrightarrow \quad x = 0 \quad \text{or} \quad x = 2.$

Example 4 Solve the equation $9^{x+1} - 3^{x+3} - 3^x + 3 = 0$.

Substituting $y = 3^x$, $\quad 9^{x+1} - 3^{x+3} - 3^x + 3 = 0$
$\Leftrightarrow 9.9^x - 3^3.3^x - 3^x + 3 = 0$
$\Leftrightarrow \quad 9y^2 - 27y - y + 3 = 0$
$\Leftrightarrow \quad (9y-1)(y-3) = 0$
$\Leftrightarrow \quad y = 1/9 \quad \text{or} \quad y = 3$
$\Leftrightarrow 3^x = 1/9 \quad \text{or} \quad 3^x = 3$
$\Leftrightarrow \quad x = -2 \quad \text{or} \quad x = 1.$

Numbers such as $\sqrt{3}$ and $\sqrt[3]{5}$ are called *surds*. It is often helpful to simplify expressions involving surds. For instance, since $(\sqrt{2}.\sqrt{3})^2 = \sqrt{2}.\sqrt{3}.\sqrt{2}.\sqrt{3} = (\sqrt{2})^2(\sqrt{3})^2 = 2.3 = 6$, we see that $\sqrt{2}.\sqrt{3} = \sqrt{6}$. This is an illustration of the useful result $(\sqrt{a})(\sqrt{b}) = \sqrt{(ab)}$.

Example 5 Simplify (a) $\sqrt{50}$, (b) $\sqrt{18}/\sqrt{2}$, (c) $\sqrt{12} - \sqrt{27} + \sqrt{3}$.

(a) $\sqrt{50} = \sqrt{(25.2)} = \sqrt{25}.\sqrt{2} = 5\sqrt{2}$
(b) $\sqrt{18}/\sqrt{2} = \sqrt{(18/2)} = \sqrt{9} = 3$
(c) $\sqrt{12} - \sqrt{27} + \sqrt{3} = \sqrt{4}.\sqrt{3} - \sqrt{9}.\sqrt{3} + \sqrt{3} = 2\sqrt{3} - 3\sqrt{3} + \sqrt{3} = 0.$

It is usual to *rationalise* fractions which involve surds, so that the denominators are integers. For instance, $1/\sqrt{2}$ may be written $\sqrt{2}/2$. To rationalise a denominator of the form $a + \sqrt{b}$, we use the fact that $(a+\sqrt{b})(a-\sqrt{b})$ reduces to the 'difference of squares' $a^2 - (\sqrt{b})^2$, i.e. $a^2 - b$.

Example 6 Simplify $\dfrac{1+\sqrt{3}}{2+\sqrt{3}}$.

$$\frac{1+\sqrt{3}}{2+\sqrt{3}} = \frac{(1+\sqrt{3})(2-\sqrt{3})}{(2+\sqrt{3})(2-\sqrt{3})} = \frac{2-\sqrt{3}+2\sqrt{3}-3}{4-2\sqrt{3}+2\sqrt{3}-3} = \frac{-1+\sqrt{3}}{1} = -1+\sqrt{3}.$$

The remaining examples concern surds and fractional indices in algebraic expressions.

Example 7 Simplify $x\sqrt{\left(\frac{1}{x^2}-1\right)}$.

$$x\sqrt{\left(\frac{1}{x^2}-1\right)} = x\sqrt{\left(\frac{1-x^2}{x^2}\right)} = x.\frac{\sqrt{(1-x^2)}}{\sqrt{(x^2)}} = \sqrt{(1-x^2)}.$$

Alternative method:

$$x\sqrt{\left(\frac{1}{x^2}-1\right)} = \sqrt{(x^2)}\sqrt{\left(\frac{1}{x^2}-1\right)} = \sqrt{x^2\left(\frac{1}{x^2}-1\right)} = \sqrt{(1-x^2)}.$$

Example 8 Simplify $(x+1)^{3/2}-(x+1)^{1/2}$.

$$\begin{aligned}(x+1)^{3/2}-(x+1)^{1/2} &= (x+1)^{1/2}(x+1)^1-(x+1)^{1/2}\\ &= (x+1)^{1/2}\{(x+1)-1\} = x(x+1)^{1/2}.\end{aligned}$$

Exercise 1.5

In questions 1 to 4 simplify the given expressions.

1. (a) 7^2, (b) 4^0, (c) 5^{-1}, (d) $36^{1/2}$,
   (e) $8^{1/3}$, (f) 2^{-3}, (g) $(12\frac{1}{4})^{1/2}$, (h) $27^{-1/3}$.

2. (a) $16^{3/4}$, (b) $(1\frac{1}{2})^{-1}$, (c) $64^{2/3}$, (d) $9^{-1/2}$,
   (e) $\left(\frac{3}{5}\right)^{-2}$, (f) $(3\frac{3}{8})^{1/3}$, (g) $\left(\frac{2}{5}\right)^{0}$, (h) $\left(\frac{1}{4}\right)^{-3/2}$

3. (a) $\sqrt{54}$, (b) $\sqrt{98}$, (c) $\sqrt{72}$, (d) $\sqrt{68}$,
   (e) $\sqrt{12}\times\sqrt{27}$, (f) $\sqrt{252}\div\sqrt{63}$, (g) $\sqrt{75}-\sqrt{48}$, (h) $\sqrt{32}-\sqrt{18}$.

4. (a) $\sqrt{32}\times\sqrt{15}\div\sqrt{24}$, (b) $\sqrt{80}-\sqrt{20}+\sqrt{45}$,
   (c) $\sqrt{84}\times\sqrt{140}\div\sqrt{120}$, (d) $\sqrt{112}-\sqrt{63}-\sqrt{28}$.

5. Express as a single square root
(a) $3\sqrt{2}$, (b) $5\sqrt{5}$, (c) $x\sqrt{y}$, (d) $2a^2\sqrt{b}$.

6. Rationalise the denominators of the following fractions, simplifying your answers

(a) $\frac{10}{\sqrt{5}}$ (b) $\frac{3+\sqrt{28}}{\sqrt{7}}$ (c) $\frac{3\sqrt{20}-\sqrt{60}}{2\sqrt{3}}$

(d) $\frac{\sqrt{5}+2}{\sqrt{5}-1}$ (e) $\frac{1+2\sqrt{2}}{5-3\sqrt{2}}$ (f) $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$.

In questions 7 to 9 solve the given equations.

7. (a) $9^x = 27$, (b) $4^x = 128$, (c) $5^{x+3} = 1$.

8. (a) $2^{x-3} = 4^{x+1}$ (b) $3^{2x}.3^{x-1} = 9$
   (c) $2^x.5^{x+1} = \frac{1}{2}$ (d) $3^x.2^{2x-3} = 18$.

9. (a) $2^{2x}-9.2^x+8=0$ (b) $3^{2x}-10.3^x+9=0$
(c) $4^x-3.2^{x+1}+8=0$ (d) $2^{2x+1}+4=2^{x+3}+2^x$
(e) $3^{2x-3}-4.3^{x-2}+1=0$ (f) $16^x-5.2^{2x-1}+1=0$.

10. Simplify the following expressions
(a) $x^2\sqrt{\left(1-\frac{1}{x^3}\right)}$ (b) $\frac{1}{x}\sqrt{(x^2+x^4)}$
(c) $(x-2)^{3/2}+2(x-2)^{1/2}$ (d) $(2x-1)^{-1/2}+(2x-1)^{1/2}$.

1.6 Logarithms

If a and x are positive real numbers and $x = a^p$, then p is the *logarithm* of x to the *base* a, written $\log_a x$.

Thus $$\boxed{x = a^p \Leftrightarrow \log_a x = p.}$$

Any statement in index form, such as $25 = 5^2$, has an equivalent logarithmic form, $\log_5 25 = 2$. To determine the logarithm to the base a of a number x, we must find the power of a which is equal to x.

Example 1 Find the logarithms to the base 4 of (a) 16, (b) 2, (c) $\frac{1}{4}$, (d) 4, (e) 1, (f) 8.

(a) $16 = 4^2$ $\therefore \log_4 16 = 2$ (b) $2 = 4^{1/2}$ $\therefore \log_4 2 = \frac{1}{2}$
(c) $\frac{1}{4} = 4^{-1}$ $\therefore \log_4 \frac{1}{4} = -1$ (d) $4 = 4^1$ $\therefore \log_4 4 = 1$
(e) $1 = 4^0$ $\therefore \log_4 1 = 0$ (f) $8 = 4^{3/2}$ $\therefore \log_4 8 = \frac{3}{2}$.

As illustrated in this example, for any base a,

$$\boxed{\log_a a = 1 \quad \text{and} \quad \log_a 1 = 0.}$$

Because logarithms are themselves indices, the laws of logarithms are closely related to the laws of indices.

Let $x = a^p, y = a^q$ then $\log_a x = p, \log_a y = q$.

Using the laws of indices:

$xy = a^p \times a^q = a^{p+q}$ $\therefore \log_a xy = p+q = \log_a x + \log_a y$
$\frac{x}{y} = a^p \div a^q = a^{p-q}$ $\therefore \log_a \frac{x}{y} = p-q = \log_a x - \log_a y$
$x^n = (a^p)^n = a^{pn}$ $\therefore \log_a x^n = pn = n \log_a x$.

Summarising:

$$\log_a xy = \log_a x + \log_a y$$
$$\log_a \frac{x}{y} = \log_a x - \log_a y$$
$$\log_a x^n = n \log_a x.$$

Example 2 Express $\log (x^3/\sqrt{y})$ in terms of $\log x$ and $\log y$.

$$\log (x^3/\sqrt{y}) = \log (x^3/y^{1/2}) = \log (x^3) - \log (y^{1/2}) = 3 \log x - \tfrac{1}{2} \log y.$$

Example 3 Express $\frac{1}{3} \log 8 - \log \frac{2}{5}$ as a single logarithm.

$$\frac{1}{3} \log 8 - \log \frac{2}{5} = \log 8^{1/3} + \log \left(\frac{2}{5}\right)^{-1} = \log 2 + \log \frac{5}{2} = \log \left(2 . \frac{5}{2}\right) = \log 5.$$

Example 4 Solve the equation $\log_6 x + \log_6 (x+5) = 2$.

$$\begin{aligned} \log_6 x + \log_6 (x+5) = 2 &\Rightarrow \log_6 x(x+5) = \log_6 36 \\ &\Rightarrow \quad x(x+5) = 36 \\ &\Rightarrow \quad x^2 + 5x - 36 = 0 \\ &\Rightarrow (x+9)(x-4) = 0. \end{aligned}$$

Hence either $x = -9$ or $x = 4$.

However, since $\log_6 x$ and $\log_6 (x+5)$ are undefined for $x = -9$, the only valid solution of the given equation is $x = 4$.

We now establish some important relationships between logarithms to different bases.

Let $\log_a x = p$ so that $x = a^p$.

Taking logarithms to the base x,

$$\log_x x = \log_x (a^p) \Leftrightarrow 1 = p \log_x a \Leftrightarrow \log_x a = 1/p$$

$$\therefore \quad \log_x a = 1/\log_a x.$$

Taking logarithms to the base b,

$$\log_b x = \log_b (a^p) \Leftrightarrow \log_b x = p \log_b a \Leftrightarrow p = \log_b x / \log_b a$$

$$\therefore \quad \log_a x = \log_b x / \log_b a.$$

In particular, when $b = 10$, $\log_a x = \log_{10} x / \log_{10} a$.

Example 5 Given that $\log_2 N = k$, express in terms of k

(a) $\log_2 N^2$, (b) $\log_2 2N$, (c) $\log_8 N$, (d) $\log_N 4$.

(a) $\log_2 N^2 = 2\log_2 N = 2k$, (b) $\log_2 2N = \log_2 2 + \log_2 N = 1 + k$,
(c) $\log_8 N = \log_2 N / \log_2 8 = \log_2 N / \log_2 2^3 = k/3$,
(d) $\log_N 4 = \log_2 4 / \log_2 N = \log_2 2^2 / \log_2 N = 2/k$.

Example 6 Solve the equation $2\log_4 x + 1 - \log_x 4 = 0$.

Substituting $\log_4 x = y$,

$$\begin{aligned} & 2\log_4 x + 1 - \log_x 4 = 0 \\ \Leftrightarrow \quad & 2y + 1 - 1/y = 0 \\ \Leftrightarrow \quad & 2y^2 + y - 1 = 0 \\ \Leftrightarrow \quad & (2y-1)(y+1) = 0 \\ \Leftrightarrow \quad & y = \tfrac{1}{2} \quad \text{or} \quad y = -1 \\ \Leftrightarrow \quad & \log_4 x = \tfrac{1}{2} \quad \text{or} \quad \log_4 x = -1 \\ \Leftrightarrow \quad & x = 4^{1/2} \quad \text{or} \quad x = 4^{-1} \end{aligned}$$

Hence $x = 2$ or $x = \frac{1}{4}$.

For some years the main use of logarithms in elementary work was in performing calculations involving multiplication and division. However, many people now use electronic calculators for such operations in preference to slide rules and sets of mathematical tables. Nevertheless, we now review briefly the use of tables of *common logarithms*, i.e. logarithms to the base 10. We will use the notation $\lg x$ instead of the more cumbersome $\log_{10} x$.

Logarithm tables usually give $\lg x$ for values of x between 1 and 10, e.g. $\lg 5{\cdot}3 = 0{\cdot}7243$. For other values of x, $\lg x$ is found by expressing x in *standard form*, i.e. in the form $k \times 10^n$, where $1 \leqslant k < 10$ and n is an integer. For example,

$$\begin{aligned} \lg 530 &= \lg(5{\cdot}3 \times 10^2) = \lg 5{\cdot}3 + \lg 10^2 = 0{\cdot}7243 + 2 = 2{\cdot}7243, \\ \lg 0{\cdot}53 &= \lg(5{\cdot}3 \times 10^{-1}) = \lg 5{\cdot}3 + \lg 10^{-1} = 0{\cdot}7243 - 1 = -0{\cdot}2757, \end{aligned}$$

which is often written as $\bar{1}{\cdot}7243$.

Anti-logarithm tables give values of 10^p for $0 \leqslant p < 1$. They are used to find a number given its logarithm, e.g. if $\lg x = 0{\cdot}78$, then $x = 10^{0{\cdot}78} = 6{\cdot}026$. For other values of p, 10^p is found by using the laws of indices.

If $\lg x = 1{\cdot}78$, then $x = 10^{1{\cdot}78} = 10^{0{\cdot}78} \times 10^1 = 60{\cdot}26$.
If $\lg x = \bar{1}{\cdot}78$, then $x = 10^{0{\cdot}78-1} = 10^{0{\cdot}78} \times 10^{-1} = 0{\cdot}6026$.
If $\lg x = -1{\cdot}22$, then $x = 10^{-1{\cdot}22} = 10^{0{\cdot}78} \times 10^{-2} = 0{\cdot}06026$.

[For further numerical work involving logarithms see §4.2.]

Exercise 1.6

1. Express the following statements in logarithmic form
(a) $16 = 2^4$ (b) $1/9 = 3^{-2}$ (c) $4 = 16^{1/2}$
2. Express the following statements in index form
(a) $\log_4 64 = 3$ (b) $\log_5 0{\cdot}2 = -1$ (c) $\log_9 27 = 1{\cdot}5$
3. Find the logarithms to the base 2 of
(a) 8 (b) $\frac{1}{4}$ (c) 1 (d) 2 (e) $\sqrt{2}$

4. Find the logarithms to the base 9 of
(a) 81 (b) 3 (c) 1/9 (d) 27 (e) 1
5. Find the following logarithms
(a) $\log_3 81$ (b) $\log_{27} 3$ (c) $\log_4 0{\cdot}5$ (d) $\log_{100} 10$
(e) $\log_8 0{\cdot}25$ (f) $\log_{0{\cdot}5} 8$ (g) $\log_6 6\sqrt{6}$ (h) $\log_5 0{\cdot}04$
6. Express in terms of $\log x$ and $\log y$
(a) $\log x^2 y$ (b) $\log \sqrt{(xy)}$ (c) $\log (x^4/y^3)$
7. Express in terms of $\log A$, $\log B$ and $\log C$
(a) $\log (AB^2/C^3)$ (b) $\log (A\sqrt{B}/C^{-2})$ (c) $\log \sqrt{(A^3B^2C)}$
8. Express as a single logarithm
(a) $\log 14 - \log 21 + \log 6$ (b) $4\log 2 + \frac{1}{2}\log 25$
(c) $\frac{3}{2}\log 9 - 2\log 6$ (d) $2\log(\frac{2}{3}) - \log(\frac{8}{9})$
9. Find the values of
(a) $\log_a 32/\log_a 2$ (b) $\log_x 125/\log_x 25$ (c) $\log_3 x/\log_9 x$
10. Given that $\log_3 2 = p$ and $\log_3 5 = q$, express in terms of p and q
(a) $\log_3 60$ (b) $\log_3 6{\cdot}4$ (c) $\log_{10} 2$.
11. Given that $\log_5 x = t$, express in terms of t
(a) $\log_5 5x^2$ (b) $\log_x 5$ (c) $\log_{25} x$ (d) $\log_x 0{\cdot}2$

In questions 12 to 14 solve the given equations.

12. (a) $\log_2 x^4 + \log_2 4x = 12$ (b) $\log_5 x + \log_5 (1/x^3) = 2$
13. (a) $\log_3 x + \log_3 (x+6) = 3$ (b) $\log_4 2x + \log_4 (x+1) = 1$
14. (a) $\log_3 x = 4\log_x 3$ (b) $2\log_4 x + 3\log_x 4 = 7$
(c) $3\log_8 x = 2\log_x 8 + 5$ (d) $\log_5 x + \log_x 25 = 3$
15. Solve the simultaneous equations: $\log_2 x^2 + \log_2 y^3 = 1$,
$\log_2 x - \log_2 y^2 = 4$.

Exercise 1.7 (*miscellaneous*)

In questions 1 to 4, assuming that $A = \{x \in \mathbb{R} : x^2 < 4\}$, $B = \{x \in \mathbb{Z} : 0 \leqslant x \leqslant 2\}$ and $C = \{-2, -1, 0, 1, 2\}$, decide whether the given statements are true or false.

1. (a) $A \subset C$, (b) $B \subset C$, (c) $B \subset A$, (d) $B \subset \mathbb{Z}$.

2. (a) $2 \in A$, (b) $\sqrt{2} \in A$, (c) $2 \in B$, (d) $\sqrt{2} \in B$.

3. (a) $(-1, 1) \in A \times B$, (b) $(1, -1) \in A \times B$,
(c) $(2, 2) \in A \times C$, (d) $(0, 0) \in A \times C$.

4. (a) $n \in A$ and $n \in B \Leftrightarrow n \in \{0, 1\}$,
(b) $n \notin B$ and $n \in C \Rightarrow n \in \{-2, -1, 0\}$,
(c) $n \notin A$ and $n \in C \Rightarrow n \in B$,
(d) $n \in A$ and $n \notin C \Leftrightarrow n \notin \mathbb{Z}$.

5. If the universal set $\mathscr{E}$ is the set of real numbers $\mathbb{R}$, find the complements of the following sets
(a) {rational numbers}, (b) {negative numbers},
(c) $\{x : x \leqslant 4\}$, (d) $\{x : x < 0 \text{ or } x > 0\}$,
(e) $\{x : x^2 < 0\}$, (f) $\{x : -1 < x < 1\}$.

6. Given that $x * y = \sqrt{(xy)}$, where x and y are real numbers, decide whether the binary operation $*$ is (a) commutative (b) associative.

7. Use the properties of real numbers to prove that
(a) $k+x = k+y \Rightarrow x = y$,
(b) $kx = ky$ and $k \neq 0 \Rightarrow x = y$.

8. Find the range of values of x satisfying the following inequalities.
(a) $3x+1 < x-5$, (b) $2(2x-3) \geqslant 5x+2$,
(c) $0 < 2x-1 \leqslant 3$, (d) $x(x+2) > 2x+9$,
(e) $|x+2| < 5$, (f) $|x-3| \geqslant 2$.

9. Arrange the following in order of magnitude:
$$7, \quad 2\sqrt{11}, \quad 4\sqrt{3}, \quad 5\sqrt{2}, \quad 3\sqrt{5}.$$

10. Simplify the following expressions
(a) $54^{1/3} \times 2^{-4/3}$, (b) $25^{1/6} \times 200^{1/3}$, (c) $54^{1/3} - 16^{1/3}$.

11. Find in the form $p+q\sqrt{2}$, where p and q are integers,
(a) the reciprocal of $3-2\sqrt{2}$, (b) the square of $3-2\sqrt{2}$,
(c) the square roots of $3-2\sqrt{2}$.

12. Solve the simultaneous equations
$$2^x + 3^y = 5, \quad 2^{x+3} - 3^{y+2} = 23.$$

13. Given that $\log_3 2 \approx 0{\cdot}63$, find the values of x for which
(a) $6^x . 2^{3-x} = 1$, (b) $3^{2x} = 3^x + 2$, (c) $2 . 9^x + 3^{x-1} + 4 = 2 . 3^{x+1}$.

14. Write down the value of x given that
(a) $\log_x 8 = \frac{3}{4}$, (b) $\log_x 0{\cdot}01 = \log_{0{\cdot}1} 100$,
(c) $\log_9 x = -\frac{1}{2}$, (d) $\log_3 x = \log_9 4$.

15. Simplify the following expressions
(a) $3\log_2\left(\frac{5}{3}\right) - 2\log_2\left(\frac{10}{9}\right) + \log_2\left(\frac{1}{30}\right)$,
(b) $\log_a \sqrt{(a^4+1)} - \frac{1}{2}\log_a\left(1 + \frac{1}{a^4}\right)$,
(c) $\log_4 \{(a^2+1)^2 - (a^2-1)^2\} - \log_2 2a$.

16. Solve the following equations
(a) $\log_2 4x = 8\log_x 2$, (b) $\log_9 x = \log_3 3x$.

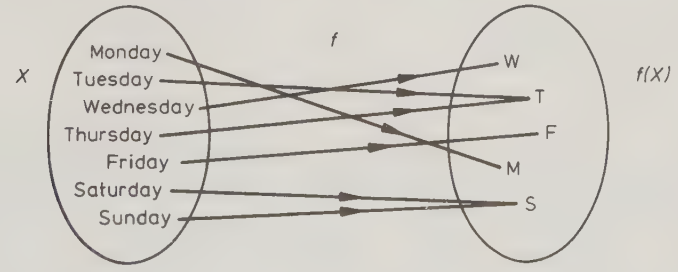
17. Prove that, if a, b and c are positive real numbers, then
$$(\log_a b)(\log_b c) = \log_a c.$$

2 Functions and graphs

2.1 Mappings and functions

A *mapping* or *function* f is a rule which assigns to an object x an *image* y. We write $f: x \to y$ or $x \xrightarrow{f} y$ and say 'f maps x to y'. The notation $y = f(x)$ is also used.

Consider a function f which maps each day of the week onto its initial letter. This function is represented in the diagram below. If X is the set containing the days of the week and Y is the set containing all letters of the alphabet, then f can be described as a mapping from X into Y, written $f: X \to Y$. The set X is called the *domain* of f and the set Y is called the *codomain*. By definition every element in the domain of a function has one and only one image in the codomain.



The set $\{W, T, F, M, S\}$ containing all the images under f is the *range* of the function, denoted by $f(X)$. Any element in the range is the image of one or more elements in the domain, e.g. F is the image of Friday, but T is the image of both Tuesday and Thursday. As every element of the range $f(X)$ belongs to the codomain, $f(X) \subset Y$. In this example, as there are letters of the alphabet, such as L and D, which are not images under f, the range $f(X)$ is a proper subset of the codomain Y.

We now examine some functions with special properties.

(1) Let X be the set of natural numbers $\{0,1,2,3,\ldots\}$ and let Y be the set of single digits $\{0, 1, 2, \ldots, 8, 9\}$. Let $f: X \to Y$ be the function which maps a natural number to its final digit.

As before X is the domain of f and Y is the codomain. However, as every element of Y is the final digit of at least one natural number, the set Y is also the range of f, i.e. $f(X) = Y$.

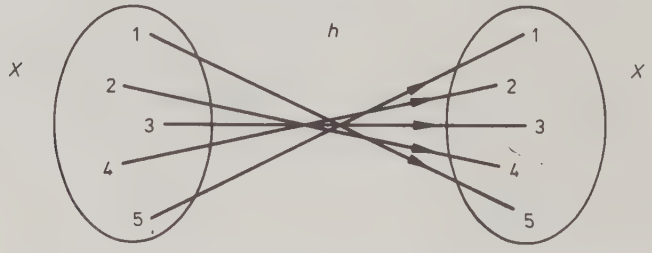
In general, if every element of the codomain Y of a function f is the image of at least one element of the domain X, then f is said to be a mapping of X *onto* Y. [A function which is 'onto' can also be called *surjective*.]

(2) Let g be the mapping of the set of natural numbers $\mathbb{N}$ into itself defined by $g: x \to x^2$.

In this case the set $\mathbb{N}$ is both the domain and the codomain of g, but the range of g is a proper subset of $\mathbb{N}$, $\{0,1,4,9,16,\ldots\}$. Hence g is a mapping 'into' rather than 'onto' $\mathbb{N}$. However, every element of the range is the image of only one element of the domain.

In general, if every element of the range $f(X)$ of a function f is the image of exactly one element of the domain X, then f is said to be *one-one*, sometimes written 1-1. A function which is not one-one may be described as *many-to-one*. Thus the function g defined above is one-one, but both the function f in (1) and the function defined earlier on the set of days of the week are many-to-one. [A one-one function can also be called *injective*.]

(3) Let $X = \{1,2,3,4,5\}$ and let h be a mapping from X to itself defined by $h: x \to 6-x$.



As shown in the diagram, the range of h is the set X itself. Moreover, every element of the range is the image of exactly one element of the domain. Hence h is a one-one mapping of X onto itself.

In general, any function which is a one-one mapping of a set X onto a set Y is said to form a *one-one correspondence* between X and Y. The symbol $\leftrightarrow$ is sometimes used to link object and image under a one-one correspondence. For example, under the mapping h, $1 \leftrightarrow 5$, $2 \leftrightarrow 4$, $3 \leftrightarrow 3$, $4 \leftrightarrow 2$, $5 \leftrightarrow 1$. [A function which is both one-one and 'onto' is said to be *bijective*.].

Exercise 2.1

1. Decide whether the following are valid definitions of a mapping f from the set $X = \{0,1,2,\ldots,10\}$ into the set of real numbers $\mathbb{R}$.

(a) f maps any element of X onto its cube,

(b) f maps any element of X onto its square root,

(c) f maps any element of X onto its reciprocal.

2. Decide whether the following are valid definitions of a mapping f from the set of all triangles into the set $\mathbb{R}$.
(a) f maps any triangle onto its area,
(b) f maps any triangle onto its height,
(c) f maps any triangle onto the number of vertices it has.

3. Let f be the function defined on the set $\mathbb{Z}^+$ of positive integers which maps each integer onto the sum of its digits.
(a) Find $f(4), f(19)$ and $f(301)$.
(b) Decide whether f is one-one or many-to-one.
(c) Are there any elements of $\mathbb{Z}^+$ which do not belong to the range of f? If so, give an example.
(d) Is f a mapping from $\mathbb{Z}^+$ onto itself?

4. Repeat question 3 for the function f from $\mathbb{Z}^+$ into itself denfined by $f: x \to 2x$.

5. Repeat question 3 for the function f from $\mathbb{Z}^+$ into itself defined by $f: x \to x^2 - x + 1$.

6. Repeat question 3 for the function f from $\mathbb{Z}^+$ into itself which maps a positive integer onto the sum of its factors, e.g. $f(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28$.

In questions 7 to 15 a mapping f from a set X into a set Y is defined. In each case find the range of the mapping, then decide whether (a) f maps X onto Y, (b) f is one-one, (c) f is a one-one correspondence between X and Y.

7. $X = Y = \mathbb{R}, f: x \to 2x$.

8. $X = Y = \mathbb{R}, f: x \to 2$.

9. $X = Y = \mathbb{R}, f: x \to \frac{1}{2}x$.

10. $X = Y = \mathbb{R}, f: x \to x^2$.

11. $X = Y = \mathbb{Z}, f: x \to x - 1$.

12. $X = Y = \mathbb{Z}, f: x \to 1 - x$.

13. $X = \{\text{all polygons}\}$, $Y = \{x \in \mathbb{R} : x > 0\}$, f maps a polygon onto the length of its perimeter.

14. $X = \{x \in \mathbb{R} : 0 < x < 1\}$, $Y = \mathbb{R}, f: x \to 1/x$.

15. $X = \{\text{all sets}\}$, $Y = \mathbb{N}$, f maps any set onto the number of subsets it has, e.g. $f: \{a, b, c\} \to 8$.

2.2 The graph of a function

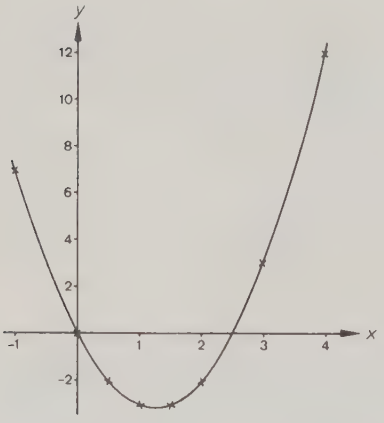
A function f from a set X to a set Y can be represented as the set of all ordered pairs (x, y) such that $f: x \to y$, i.e. as a subset of the product set $X \times Y$. By marking off the elements of X along a horizontal axis and the elements of Y along a

vertical axis, the function may be represented by a set of points with coordinates (x, y). This set of points is called the *graph* of the function.

Of particular interest are functions which map the set of real numbers $\mathbb{R}$ into itself. Such functions are represented as sets of ordered pairs (x, y) which are subsets of the product set $\mathbb{R} \times \mathbb{R}$, also written $\mathbb{R}^2$. Thus the graph of any function $f: \mathbb{R} \to \mathbb{R}$ is the set of all points (x, y) in the x, y plane such that $f: x \to y$. Points are plotted using horizontal and vertical axes called the x-axis and y-axis respectively. The coordinates (x, y) are often called the *Cartesian coordinates* of the point after René Descartes† who first described the system. The number x is called the x-coordinate or *abscissa* and y the y-coordinate or *ordinate*. Since every point (x, y) on the graph of a function f satisfies the relation $y = f(x)$, this is called the *Cartesian equation* of the graph. For instance, the graph of the function $f: x \to x+2$ has equation $y = x+2$.

It is clearly impossible to plot all the points on the graph of a function whose domain is the set of all real numbers. However, we can draw a graph of a function defined in a given interval of the form $\{x: a \leqslant x \leqslant b\}$.

Example 1 Draw the graph of the function defined by the equation $y = 2x^2 - 5x$, taking $-1 \leqslant x \leqslant 4$.



x	-1	0	1	2	3	4
y	7	0	-3	-2	3	12

To improve accuracy we consider two additional values of x.

x	$\frac{1}{2}$	$1\frac{1}{2}$
y	-2	-3

Although drawing a graph can be useful when considering a particular function, important features of the graph may lie beyond the points plotted. In later chapters we will see that there are ways of examining the general behaviour of a function. These can be used to produce a sketch indicating the form of the graph over the whole domain.

An equation $y = f(x)$ describes the relation between x and y as their values vary over the domain and range of the function f. Thus x and y are called *variables*. Since the value of x may be chosen freely from the domain of f, x is called the *independent variable*. However, the value of y depends on the value

† *Descartes, René* (1596–1650) French rationalist philosopher and mathematician. In mathematics his greatest achievements were advances in the theory of equations and the invention of coordinate geometry, as described in his essay *La Géométrie* (1637).

chosen for x, so y is called the *dependent variable*. The statement 'y is a function of x' is another way of expressing this relationship.

When defining a function on the set of real numbers, either in the form $f: x \to y$ or $y = f(x)$, the domain is usually taken to be the set of values of x for which a value of y can be determined. For instance, the domain of the function $f: x \to 1/(x-1)$ is the set containing all real numbers except 1, written $\mathbb{R}-\{1\}$. There is no value of x such that $f(x) = 0$. Hence the range of the function is the set $\mathbb{R}-\{0\}$. Since a negative number has no real square root, the domain of the function defined by $y = \sqrt{(1-x^2)}$ is the set $\{x \in \mathbb{R}: -1 \leqslant x \leqslant 1\}$. It follows that the range of the function is the set $\{y \in \mathbb{R}: 0 \leqslant y \leqslant 1\}$.

[In general an equation connecting variables x and y defines a *relation* on the set $\mathbb{R}$. Such an equation defines a function only if for every value of x in the domain there is exactly one value of y. For example, the equation $y^2 = x$ defines a relation, but not a function, because for any positive value of x there are two possible values of y.]

Exercise 2.2

In questions 1 to 6 write down the equation of the graph of the given function, then draw the graph taking values of x in the stated interval.

1. $f: x \to 2-x$, $-1 \leqslant x \leqslant 3$.
2. $f: x \to 2x+1$, $-2 \leqslant x \leqslant 2$.
3. $f: x \to 4-x^2$, $-3 \leqslant x \leqslant 1$.
4. $f: x \to x^3-2x^2$, $-2 \leqslant x \leqslant 3$.
5. $f: x \to 1/x^2$, $\frac{1}{2} \leqslant x \leqslant 4$.
6. $f: x \to 1/(1-x)$, $-5 \leqslant x \leqslant \frac{1}{2}$.

In questions 7 to 18 decide whether the given equation defines on some subset of $\mathbb{R}$ a function of the form $f: x \to y$. If so, determine the largest possible domain for f and the corresponding range, and state whether f is one-one or many-to-one.

7. $y = x+4$.
8. $x+y = 4$.
9. $y = 4x+1$.
10. $y = x^2+4$.
11. $y^2 = x+4$.
12. $y = 1-x^2$.
13. $y = \dfrac{1}{x^2}$.
14. $y = \dfrac{1}{x-1}$.
15. $y = \dfrac{1}{4-x}$.
16. $x^2+y^2 = 4$.
17. $y = \sqrt{x}$.
18. $y^3 = x^2$.

2.3 Points, lines and curves

Any equation connecting variables x and y may be considered as the Cartesian equation of a curve or graph in the x, y plane. The curve consists of the set of points whose coordinates (x, y) satisfy the equation. It is possible to decide

whether or not a point lies on a curve by substituting its coordinates into the equation.

Example 1 Which of the points $A(-1,1)$, $B(1,-2)$ and $C(-2,2)$ lie on the curve $x^2+y^2=5$?

At A: $x^2+y^2=(-1)^2+1^2=2$ i.e. $x^2+y^2\neq 5$
At B: $x^2+y^2=1^2+(-2)^2=5$ i.e. $x^2+y^2=5$
At C: $x^2+y^2=(-2)^2+2^2=8$ i.e. $x^2+y^2\neq 5$
$\therefore$ B lies on the curve $x^2+y^2=5$, but A and C do not.

Since the x-coordinate of any point on the y-axis is 0, points of intersection of a curve with the y-axis are found by substituting $x=0$ in the equation. Similarly points of intersection with the x-axis are found by substituting $y=0$. The point $(0,0)$, where the axes themselves intersect, is called the *origin*.

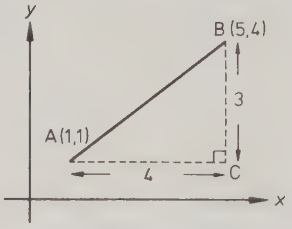
Example 2 Find the points of intersection of the curve $y=x^2-1$ with the coordinate axes.

$x=0\Rightarrow y=-1$
$\therefore$ the curve cuts the y-axis at the point $(0,-1)$.

$$y=0\Leftrightarrow x^2-1=0\Leftrightarrow(x+1)(x-1)=0$$
$$\Leftrightarrow x+1=0 \quad \text{or} \quad x-1=0$$
$$\Leftrightarrow x=-1 \quad \text{or} \quad x=1$$

$\therefore$ the curve cuts the x-axis at the points $(-1,0)$ and $(1,0)$.

We now consider pairs of points in the x,y plane and the straight lines joining them.

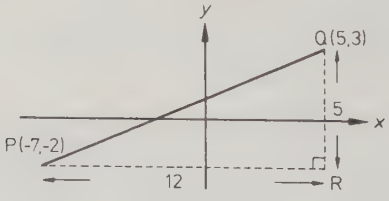


The diagram shows the points $A(1,1)$ and $B(5,4)$. In triangle ABC, AC and BC are parallel to the coordinate axes, so that $\angle ACB=90°$.

By Pythagoras' theorem,
$AB^2=AC^2+BC^2$

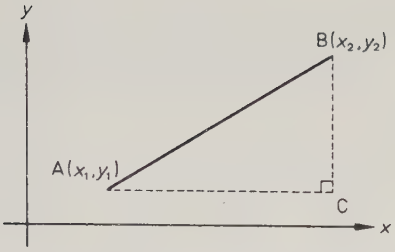
$$\therefore \quad AB^2=(5-1)^2+(4-1)^2=4^2+3^2=25$$

Hence the distance between the points A and B is 5.



Similarly, if P and Q are the points $(-7,-2)$ and $(5,3)$ respectively,
$PQ^2=PR^2+QR^2$

$$\therefore \quad PQ^2=(5+7)^2+(3+2)^2$$
$$=12^2+5^2=169$$
$$\therefore \quad PQ=13.$$

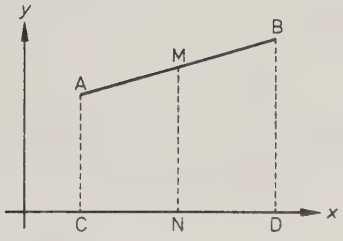


Thus, for a typical pair of points $A(x_1, y_1)$ and $B(x_2, y_2)$,
$AC = x_2 - x_1$, $BC = y_2 - y_1$

$$\therefore \quad \boxed{AB = \sqrt{\{(x_2-x_1)^2+(y_2-y_1)^2\}}.}$$

Example 3 Find the distance between the points $S(-1, 2)$ and $T(3, -1)$.

$$ST = \sqrt{\{(-1-3)^2+(2-(-1))^2\}} = \sqrt{\{4^2+3^2\}} = \sqrt{25} = 5.$$



Let M be the mid-point of the line joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$. If AC, MN and BD are drawn parallel to the y-axis then N is the mid-point of CD.

Since $CD = x_2 - x_1$,
$CN = \frac{1}{2}(x_2 - x_1)$

$$\therefore \quad ON = OC + CN = x_1 + \tfrac{1}{2}(x_2 - x_1) = \tfrac{1}{2}(x_1 + x_2),$$

i.e. the x-coordinate of M is $\frac{1}{2}(x_1 + x_2)$.

By a similar argument the y-coordinate of M is $\frac{1}{2}(y_1 + y_2)$

$$\therefore \quad \boxed{\text{the mid-point of } AB \text{ is } (\tfrac{1}{2}\{x_1+x_2\}, \tfrac{1}{2}\{y_1+y_2\}).}$$

Example 4 Find the mid-points of the lines joining the points (a) $(4, -1)$ and $(0, 3)$ (b) $(-5, 1)$ and $(-2, 5)$.

(a) The mid-point is $(\frac{1}{2}\{4+0\}, \frac{1}{2}\{-1+3\})$, i.e. $(2, 1)$.
(b) The mid-point is $(\frac{1}{2}\{-5-2\}, \frac{1}{2}\{1+5\})$, i.e. $(-3\frac{1}{2}, 3)$.

Example 5 Given that the points $A(5, -3)$, $B(3, 6)$, $C(-1, 7)$ are the vertices of a parallelogram $ABCD$, find the coordinates of D.

Since the diagonals of a parallelogram bisect each other, the diagonals AC and BD of the parallelogram $ABCD$ have the same mid-point.
The mid-point of AC is $(\frac{1}{2}\{5-1\}, \frac{1}{2}\{-3+7\})$, i.e. $(2, 2)$.
Letting D be the point (h, k), the mid-point of BD is $(\frac{1}{2}\{3+h\}, \frac{1}{2}\{6+k\})$.
$\therefore \quad \frac{1}{2}\{3+h\} = 2$ and $\frac{1}{2}\{6+k\} = 2$ i.e. $h = 1, k = -2$.
Hence the coordinates of D are $(1, -2)$.

Exercise 2.3

1. Find the points of intersection with the coordinate axes of the graphs with the following equations.

(a) $y = x-1$ (b) $y = 2x+5$
(c) $y = x^2+3x$ (d) $y = 2x^2+3x-2$
(e) $y = \dfrac{2}{x-2}$ (f) $y = \dfrac{x-3}{x-6}$

2. In each of the following cases determine which points lie on the given curve.

(a) $y = x^2+x$; $(1,3)$, $(-2,6)$, $(-3,6)$, $(0,2)$.
(b) $y = 1/(x-2)$; $(0,\frac{1}{2})$, $(3,1)$, $(-1,-1)$, $(6,\frac{1}{3})$.
(c) $y = 3x^2-7x+5$; $(2,3)$, $(-5,115)$, $(-2,29)$, $(1,4)$.
(d) $y = x^3-5x-9$; $(-2,-7)$, $(1,-11)$, $(4,35)$, $(-1,-5)$.
(e) $4x^2+y^2 = 20$; $(3,-4)$, $(-2,2)$, $(5,0)$, $(-1,-4)$.
(f) $x^2+y^2-8x+4y-5 = 0$; $(4,-2)$, $(7,-6)$, $(0,1)$, $(-1,-1)$.

3. Find the distances between the following pairs of points.

(a) $(0,7)$, $(12,-2)$ (b) $(-3,-5)$, $(-7,-8)$
(c) $(-2,14)$, $(3,2)$ (d) $(5,-1)$, $(11,7)$
(e) $(4,-10)$, $(-4,5)$ (f) $(-4,-4)$, $(3,20)$
(g) $(7,-4)$, $(5,-1)$ (h) $(-2,3)$, $(1,-3)$.

4. Find the mid-points of the lines joining the points given in question 3.

5. Given that B is the mid-point of the line segment AC, find the coordinates of C when

(a) $A(1,2)$, $B(3,1)$ (b) $A(6,2)$, $B(3,-2)$
(c) $A(-5,4)$, $B(-2,-1)$ (d) $A(4,-3)$, $B(3,-1)$.

6. Find the coordinates of the vertex D of the parallelogram $ABCD$ given

(a) $A(5,0)$, $B(-3,2)$, $C(1,-6)$ (b) $A(-2,7)$, $B(4,5)$, $C(6,-1)$
(c) $A(3,-4)$, $B(0,-7)$, $C(2,5)$ (d) $A(1,-5)$, $B(-3,7)$, $C(-2,8)$.

7. Determine whether the triangles with vertices given below are (i) isosceles, (ii) right-angled.

(a) $A(4,7)$, $B(-3,8)$, $C(9,2)$ (b) $P(3,-1)$, $Q(-4,2)$, $R(1,-6)$
(c) $S(-2,5)$, $T(4,3)$, $U(-5,-4)$ (d) $X(5,9)$, $Y(15,3)$, $Z(-1,-1)$.

8. Given points $P(5,5)$, $Q(-1,-3)$ and $R(-2,4)$, show that the mid-point M of the line PQ is the centre of the circle through P, Q and R.

9. Show that the point $P(4,5)$ lies on the perpendicular bisector of the line joining the points $A(5,-2)$ and $B(-3,4)$. Find the distance of P from the line AB.

10. Show that the triangle with vertices $D(2,5)$, $E(3,-7)$ and $F(-6,1)$ is isosceles and find its area.

11. Given that $A(0,-5)$, $B(-7,2)$, $C(2,11)$ are the vertices of a parallelogram $ABCD$, find the coordinates of D. Prove that $ABCD$ is a rectangle and find its area.

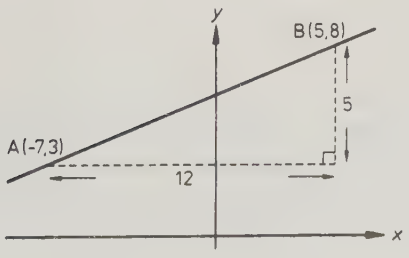
12. Given that $A(1,-7)$, $B(6,1)$ and $C(1,9)$ are the vertices of a parallelogram $ABCD$, find the coordinates of D. Prove that $ABCD$ is a rhombus and find its area.

13. The points $P(-1,-2)$, $Q(3,4)$ and $R(11,6)$ are the vertices of a parallelogram $PQRS$. A point T is taken on RS produced so that S is the mid-point of RT. Find the coordinates of S and T. Prove that $RP = RT$.

14. The coordinates of the points A, B, C are $(t,-1)$, $(-2,1)$ and $(4,3)$ respectively. Given that $AB = AC$, calculate the value of t. The line BC is produced to D so that $BC = CD$. Calculate the coordinates of D and the length of AD.

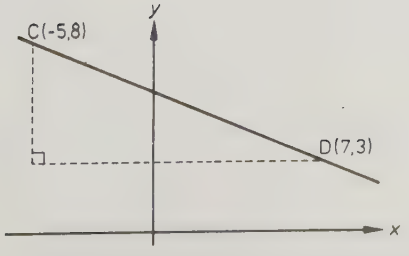
15. The vertices of a quadrilateral $ABCD$ are $A(-4,3)$, $B(2,1)$, $C(1,-2)$ and $D(-2,-3)$. Prove that $AB = AD$ and $BC = CD$. Find the area of the quadrilateral.

2.4 Gradient of a straight line



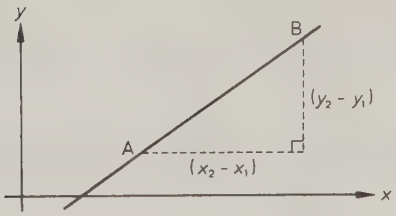
The *gradient* of a straight line is a quantity used for measuring the direction of a line in relation to the coordinate axes. Consider the straight line which passes through the points $A(-7,3)$ and $B(5,8)$. Moving from A to B results in increases of 12 in the x-coordinate and 5 in the y-coordinate. The gradient of AB is the ratio of these increases,

$$\text{i.e. gradient of } AB = \frac{\text{increase in } y}{\text{increase in } x} = \frac{5}{12}.$$



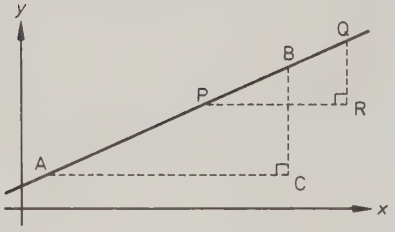
For the points $C(-5,8)$ and $D(7,3)$ the increase in x is 12, but there is a decrease in y of 5. This can be considered as an increase of -5

$$\therefore \text{ gradient of } CD = -\frac{5}{12}.$$



In general, for a line passing through the points $A(x_1, y_1)$ and $B(x_2, y_2)$, we have:

$$\text{gradient} = \frac{y_2 - y_1}{x_2 - x_1} \text{ or } \frac{y_1 - y_2}{x_1 - x_2}.$$



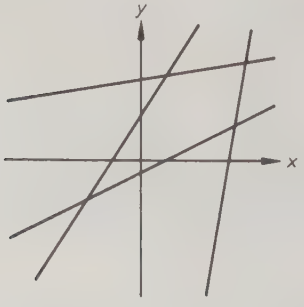
By considering similar triangles ABC and PQR, we see that $\frac{BC}{CA} = \frac{QR}{RP}$. Thus the value of the gradient of a line does not depend on the particular points used to calculate it, but only on the angle the line makes with the x-axis. As *parallel* lines make equal angles with the x-axis, they must have equal gradients.

Example 1 Find the gradients of the lines joining
(a) $A(4, -3)$ and $B(2, 1)$ (b) $C(4, 3)$ and $D(-2, 1)$.

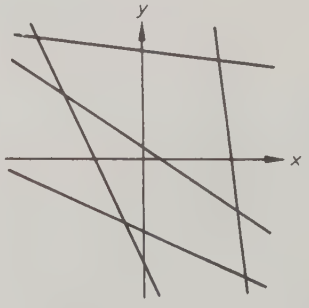
(a) Gradient of $AB = \dfrac{-3-1}{4-2} = \dfrac{-4}{2} = -2.$

(b) Gradient of $CD = \dfrac{3-1}{4-(-2)} = \dfrac{2}{6} = \dfrac{1}{3}.$

All the points on a line parallel to the x-axis have the same y-coordinate. Hence lines parallel to the x-axis have *zero gradient*. Since division by zero is undefined, we can give no finite numerical value to the gradient of lines joining points with the same x-coordinate. Thus lines parallel to the y-axis may be said to have *infinite gradient*. The diagrams below show the significance of the sign of the gradient of a line.

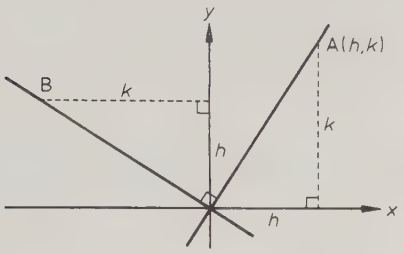


Positive gradients



Negative gradients

To find the relationship between the gradients of *perpendicular* lines, consider a pair of perpendicular lines through the origin with gradients m_1 and m_2.



Let $A(h,k)$ be a typical point on the first line and let B be a point on the second line such that $OB = OA$. If OA is rotated anti-clockwise through 90°, the point A moves to B. Hence the coordinates of B are $(-k,h)$. Thus,

gradient of $OA = m_1 = \frac{k}{h}$ and

gradient of $OB = m_2 = -\frac{h}{k}$

$$\therefore \quad m_1 m_2 = \frac{k}{h} \times \left(-\frac{h}{k}\right), \quad \text{i.e.} \quad m_1 m_2 = -1.$$

For any pair of perpendicular lines there is a corresponding pair through the origin, therefore the product of the gradients of perpendicular lines is -1.

Example 2 Write down the gradients of lines perpendicular to lines with gradients (a) 2, (b) $\frac{1}{2}$, (c) $-5/4$.

The gradients of the perpendicular lines are
(a) $-\frac{1}{2}$, (b) -2, (c) $4/5$.

Exercise 2.4

1. Find the gradients of the lines joining the points
(a) $(3,5)$ and $(-1,2)$ (b) $(4,6)$ and $(5,-1)$
(c) $(-2,6)$ and $(-1,4)$ (d) $(-2,-3)$ and $(-9,-6)$
(e) $(-4,0)$ and $(-3,5)$ (f) $(0,-7)$ and $(2,-5)$

2. Write down the gradients of lines perpendicular to those given in question 1.

3. By comparing the gradients of AB and BC, or otherwise, decide whether or not the points A, B and C are collinear.
(a) $A(1,-6)$, $B(3,-2)$, $C(-7,6)$ (b) $A(-2,6)$, $B(5,7)$, $C(-9,5)$
(c) $A(3,4)$, $B(-1,-8)$, $C(5,10)$ (d) $A(1,1)$, $B(6,-2)$, $C(-3,4)$.

4. Given the points $A(8,-1)$, $B(5,5)$, $C(-5,5)$ and $D(-3,1)$, show that (a) $ABCD$ is a trapezium, (b) AB is perpendicular to BD.

5. Given the points $X(-6,-2)$, $Y(-4,3)$ and $Z(6,-1)$, show that $\angle XYZ$ is a right angle. If M and N are the mid-points of XY and YZ, verify that MN is parallel to XZ.

6. The coordinates of the points P, Q and R are $(4,-3)$, $(-1,2)$ and $(2,t)$

respectively. Calculate the values of t given that (a) P, Q and R are collinear, (b) $\angle QPR = 90°$, (c) $\angle PRQ = 90°$.

7. The coordinates of the points A and B are $(1,6)$ and $(5,-2)$ respectively. Find the coordinates of the point C on the x-axis, given that (a) C also lies on AB, (b) $\angle ABC$ is a right angle, (c) C lies on the perpendicular bisector of AB.

8. A quadrilateral $ABCD$ has vertices $A(-1,-2)$, $B(5,0)$, $C(4,3)$ and $D(1,2)$. Show that $\angle ABC$ and $\angle BCD$ are right angles. Calculate the lengths of AB, BC and CD and hence find the area of the quadrilateral.

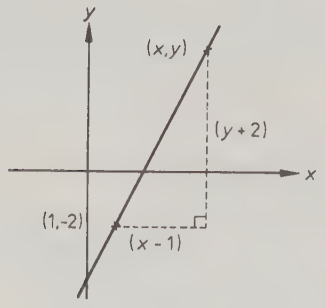
9. Given the points $P(7,-4)$, $Q(9,8)$, $R(-3,10)$, $S(-12,-7)$, show that (a) QS is perpendicular to PR, (b) QS passes through the mid-point of PR.

10. Given that the triangle with vertices $X(1,2)$, $Y(2,a)$ and $Z(-4,-3)$ is right-angled at Y, find the two possible values of a and the area of the triangle in each case.

11. A quadrilateral $ABCD$ has vertices $A(10,5)$, $B(5,-4)$, $C(-5,-7)$ and $D(-3,6)$. Show that AC is perpendicular to BD and calculate the area of the quadrilateral.

12. Show that the points $D(5,-3)$, $E(2,6)$, $F(-1,10)$ and $G(-4,9)$ are the vertices of a trapezium. Show also that DE is perpendicular to FG and that the area of the trapezium is 30 square units.

2.5 The equation of a line



The equation of a straight line is an equation connecting the variables x and y which is satisfied by any point (x,y) on the line. The gradient is used to find a suitable equation. Consider the line with gradient 2, which passes through the point $(1,-2)$. If the point (x,y) lies on the line, then its gradient is given by $\dfrac{y+2}{x-1}$.

Hence $\dfrac{y+2}{x-1} = 2$ for any point (x,y) on the line.

Rearranging, $y+2 = 2(x-1)$, i.e. $y+2 = 2x-2$

$\therefore$ the equation of the line is $y = 2x-4$.

The same method can be used to find the equation of a line with gradient m passing through the point (x_1, y_1).

The gradient of the line $= \dfrac{y-y_1}{x-x_1} = m$

$\therefore$ the equation of the line is $y-y_1 = m(x-x_1)$.

Example 1 Find the equation of the line with gradient $-\frac{1}{2}$ which passes through the point $(2, -3)$.

The equation of the line is $y+3 = -\frac{1}{2}(x-2)$

i.e. $2(y+3) = -(x-2)$

$2y+6 = -x+2$

$\therefore$ the required equation is $x+2y+4 = 0$.

Example 2 Find the equation of the line which passes through the points $(2, -2)$ and $(1, 3)$.

The gradient of the line $= \dfrac{-2-3}{2-1} = -5$

$\therefore$ the equation of the line is $y+2 = -5(x-2)$, i.e. $5x+y-8 = 0$.

The general form of the straight line equation emerges when we consider a line with gradient m passing through the point $(0, c)$ on the y-axis. Its equation is $y-c = m(x-0)$, i.e. $y = mx+c$. Hence the equation $y = mx+c$ represents a straight line with gradient m, where c is the *intercept* on the y-axis. In particular, $y = mx$ is a line through the origin $(0, 0)$ and $y = c$ is a line parallel to the x-axis. Lines parallel to the y-axis have equations of the form $x = k$.

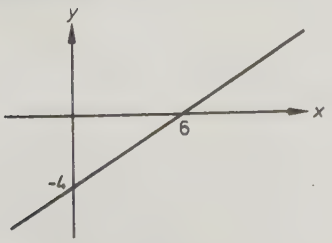
Example 3 Find the gradient of the line $3x+4y = 8$ and the intercept on the y-axis.

The equation may be written $4y = -3x+8$, i.e. $y = -\frac{3}{4}x+2$

$\therefore$ the gradient of the line is $-\frac{3}{4}$ and the intercept on the y-axis is 2, i.e. the line cuts the y-axis at the point $(0, 2)$.

A straight line is usually sketched by finding its points of intersection with both x- and y-axes.

Example 4 Sketch the line $2x-3y = 12$.



$x = 0 \Rightarrow -3y = 12 \Rightarrow y = -4$

$\therefore$ the line cuts the y-axis at $(0, -4)$.

$y = 0 \Rightarrow 2x = 12 \Rightarrow x = 6$

$\therefore$ the line cuts the x-axis at $(6, 0)$.

The general equation of a straight line can also be written $y-mx=c$. This form is sometimes used when obtaining the equation of a line with given gradient. The constant c is determined by substituting the coordinates of a point on the line.

Example 5 Find the equation of the line through $(-1,2)$ with gradient 3.

The equation of the line is $y-3x=2-3(-1)$, i.e. $y-3x=5$.

The properties of the gradients of parallel and perpendicular lines are used in the next example.

Example 6 Find the equations of the lines parallel and perpendicular to the line $3x-2y=1$, which pass through the point $(2,3)$.

Method 1 Rearranging the given equation, we obtain $y=\frac{3}{2}x-\frac{1}{2}$

$\therefore$ the gradient of the given line is 3/2.
Hence the gradient of any parallel line is 3/2.
The equation of the parallel line through $(2,3)$ is

$$y-3=\frac{3}{2}(x-2), \quad \text{i.e.} \quad 2(y-3)=3(x-2)$$
$$2y-6=3x-6$$

which may be written $3x-2y=0$.
The gradient of any perpendicular line is $-\frac{2}{3}$.
$\therefore$ the equation of the perpendicular line through $(2,3)$ is

$$y-3=-\tfrac{2}{3}(x-2), \quad \text{i.e.} \quad 3(y-3)=-2(x-2)$$
$$3y-9=-2x+4$$

which may be written $2x+3y=13$.

In time, the reader may prefer to adopt a quicker method based on the fact that any line parallel to $3x-2y=1$ has an equation of the form $3x-2y=c$, and any perpendicular line the form $2x+3y=k$.

Method 2 The equation of the required parallel line is

$$3x-2y=3.2-2.3, \quad \text{i.e.} \quad 3x-2y=0.$$

The equation of the required perpendicular line is

$$2x+3y=2.2+3.3, \quad \text{i.e.} \quad 2x+3y=13.$$

The point of intersection of two straight lines is the point which lies on both lines, therefore its coordinates must satisfy both equations. Hence these are found by solving the equations simultaneously.

Example 7 Find the point of intersection of the lines $3x+y=5$ and $3x-2y+1=0$.

At the point of intersection $3x+y=5$ (1)
and $3x-2y=-1$ (2)

Subtracting (2) from (1): $3y = 6 \quad \therefore y = 2$
Substituting in (1): $3x+2 = 5 \quad \therefore x = 1$
$\therefore$ the point of intersection is $(1,2)$.

[This method will break down if the two lines are parallel and thus have no point of intersection.]

Exercise 2.5

1. Find the equations of the straight lines through the given points with the given gradients.
(a) $(-2,3)$; 4 (b) $(2,-3)$; $-\frac{1}{2}$ (c) $(4,-1)$; -3
(d) $(-5,-7)$; $\frac{2}{3}$ (e) $(-12,5)$; $-5/12$ (f) $(t^2,2t)$; $1/t$.

2. Find the equations of the lines joining the following points.
(a) $(0,-7), (2,-5)$ (b) $(-2,6), (-1,4)$
(c) $(4,7), (-5,7)$ (d) $(4,3), (-4,-3)$
(e) $(-2,-3), (-9,-6)$ (f) $(6,2), (6,-3)$
(g) $(p^2,2p), (q^2,2q)$ (h) $(p,1/p), (q,1/q)$.

3. Find the gradients and intercepts on the y-axis of the following lines.
(a) $2y = 5x+9$, (b) $4y-7x+13 = 0$, (c) $2x+8y+19 = 0$,
(d) $3x-6y-17 = 0$, (e) $2y+5 = 0$, (f) $5x+9y-27 = 0$.

4. In each of the following cases sketch the given lines and find their points of intersection, if any.
(a) $4x+3y = 7, 3x-4y+1 = 0$, (b) $y = 4x-10, 4y = 11-x$,
(c) $2x-5y = 15, 3x-7y = 21$, (d) $5y = 17-4x, 8x+10y = 19$,
(e) $8x+7y = 23, 8y = 7x-6$, (f) $5y-2x = 3, 2y-5x = 4$.

5. Decide whether the pairs of lines given in question 4 are (i) perpendicular, (ii) parallel.

6. In each of the following cases find the equations of the lines parallel and perpendicular to the given line through the given point.
(a) $y = 2x-3, (-1,3)$ (b) $3x+2y = 7, (0,2)$
(c) $3x-5 = 0, (3,-4)$ (d) $3x-9y = 4, (-2,-1)$.

7. Find the coordinates of the foot of the perpendicular from the point $(-2,4)$ to the line $x-2y-5 = 0$.

8. Find the coordinates of the foot of the perpendicular from the point $(4,5)$ to the line $2x+5y = 4$.

9. Prove that the lines $3x-2y+7 = 0$, $5x+4y = 3$ and $2x+5y = 8$ are concurrent. Find the equation of the line through the point of intersection which has gradient 3.

10. Find the equation of the perpendicular bisector of the line joining the points $(-2, 3)$ and $(8, -7)$.

11. Given the points $A(9, -4)$, $B(2, 10)$ and $C(-4, 2)$, show that the perpendicular bisector of BC intersects AB on the x-axis.

12. The points $A(7, 3)$, $B(1, -4)$, $C(-5, -1)$ are three vertices of a trapezium $ABCD$. Given that $\angle BCD = 90°$, find (a) the equations of AD and CD, (b) the coordinates of D.

13. The points $P(-3, 1)$, $Q(-1, 5)$, $R(6, 4)$ are three vertices of a trapezium $PQRS$ in which PQ is parallel to SR. Given that the diagonals intersect at right angles, find the coordinates of S.

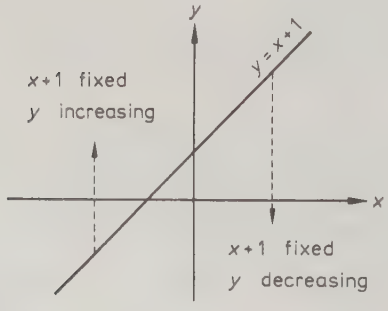
14. The point $A(7, 5)$ is one vertex of a parallelogram $ABCD$. The equations of the sides BC and CD are $3y = 2x - 9$ and $2y = 3x - 1$ respectively. Find the coordinates of B and D. Show that the diagonals cut at right angles.

15. The equations of two sides of a triangle PQR are $x + 2y + 4 = 0$ and $3x - 4y + 37 = 0$. Given that $\angle PQR = 90°$ and that P is the point $(6, -5)$, find the coordinates of Q.

16. Find the reflection of the point $P(3, 1)$ in the line l with equation $y = 2x$. Use your answer to find the reflection in l of the line $y = x - 2$.

2.6 Linear inequalities

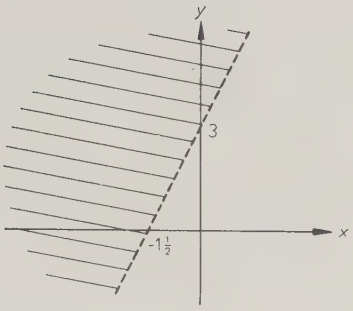
In the previous section it was shown that the equation $y = mx + c$ represents a straight line. We now consider inequalities, such as $y > mx + c$, and the sets of points they represent in the x, y plane.



Consider the line $y = x + 1$. At any point on the line, the value of y is equal to the value of $x + 1$. If we move away from the line in a direction parallel to the y-axis, then the value of $x + 1$ remains unchanged. However, the value of y either increases or decreases as shown in the diagram. Thus, for points above the line $y > x + 1$ and for points below the line $y < x + 1$.

By a similar argument, we find that the inequality $y > mx + c$ represents the set of points above the line $y = mx + c$, and $y < mx + c$ the set of points below the line $y = mx + c$.

Example 1 Sketch the region of the x, y plane represented by the inequality $y > 2x+3$.



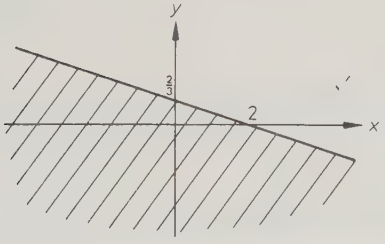
On the boundary of the region $y = 2x+3$.

$x = 0 \Rightarrow y = 3$
$y = 0 \Rightarrow 2x+3 = 0 \Rightarrow x = -1\frac{1}{2}$

$\therefore$ the boundary line cuts the axes at $(0, 3)$ and $(-1\frac{1}{2}, 0)$. Hence $y > 2x+3$ represents the shaded region shown in the diagram.
[The broken line is used to indicate that points on the boundary do not belong to the region.]

An alternative approach is used in the next example.

Example 2 Sketch the region of the x, y plane represented by the inequality $x+3y-2 \leqslant 0$.



On the boundary of the region $x+3y-2 = 0$.

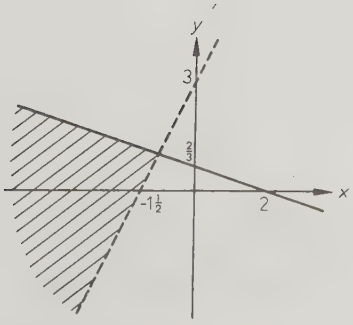
$x = 0 \Rightarrow 3y-2 = 0 \Rightarrow y = \frac{2}{3}$
$y = 0 \Rightarrow x-2 = 0 \Rightarrow x = 2$

$\therefore$ the boundary line cuts the axes at $(0, \frac{2}{3})$ and $(2, 0)$.

At the origin, $x+3y-2 = -2$

$\therefore$ the origin lies in the required region.

Hence $x+3y-2 \leqslant 0$ represents the set of points on and below the line $x+3y-2 = 0$.



Combining the results of Examples 1 and 2, we obtain the region in which the inequalities $y > 2x+3$ and $x+3y-2 \leqslant 0$ hold simultaneously.

Exercise 2.6

1. Sketch the region of the x, y plane represented by the following inequalities.
(a) $y > x+1$ (b) $y < 3x-6$ (c) $y \leqslant 4-x$ (d) $2y+x \geqslant 7$
(e) $2y < 5x$ (f) $3y-2x > 8$ (g) $2x-5 < 3$ (h) $y^2 \leqslant 4$.

2. Sketch the region of the x, y plane represented by the following simultaneous inequalities.
(a) $2y > x+5$, $x > -1$
(b) $x < y$, $x+y \leqslant 8$
(c) $-2 \leqslant x \leqslant 2$, $-3 \leqslant y \leqslant 3$
(d) $-x < y < x$, $2 < 2x+y < 4$
(e) $y+2 > 0$, $y < 2x+4$, $y+3x < 6$
(f) $x+3y \leqslant 9$, $y-3x \leqslant 9$, $3x-4y \leqslant 9$.

3. Given that x and y satisfy the inequalities $2x-y < 8$ and $3x+5y+1 > 0$, find the range of possible values for y.

4. Given that x and y satisfy the inequalities $4x+3y \leqslant 15$ and $3x-2y < 7$, find the range of possible values for x.

5. Given that $2x+y > 0$, $2y > x$ and $y < 3$, find the range of possible values of (a) x, (b) y, (c) $x+y$.

6. Repeat question 5 given that $x+4y \leqslant 17$, $y-3x \leqslant 14$, $x < 0$ and $y > 0$.

Exercise 2.7 (*miscellaneous*)

[Leave answers in surd form where appropriate.]

1. Find the range of the following functions and decide in each case whether the function is (i) one-one, (ii) 'onto'.
(a) f is the function which maps the set of real numbers $\mathbb{R}$ into itself such that $f: x \to x^2-4$.
(b) g is the function which maps the set of positive integers $\mathbb{Z}^+$ into the set of rational numbers $\mathbb{Q}$ defined by $g: x \to 1/x$.
(c) h is the function from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$ defined by $h: (r, s) \to r+s$.
2. Draw the graph of the function f defined by $f: x \to x(x-3)$ for the domain $-2 \leqslant x \leqslant 5$. Find the range of the function corresponding to this domain. Find also the largest possible domain for which the range is $0 \leqslant f(x) \leqslant 18$.
3. Given that the curve $y = (x+a)/(x+b)$ passes through the points $(2, 1\frac{1}{2})$ and $(-1, 3)$, find the coordinates of the points A and B at which the curve cuts the x- and y-axes and calculate the length of the chord AB.
4. Show that the point (t^2-5, t^3-5t) lies on the curve $y^2 = x^2(x+5)$ for all values of t. Show also that if P, Q and R are the points at which t takes the values -2, 1 and 3 respectively, then P, Q and R are collinear.
5. Given the points $A(-3, -2)$, $B(4, 2)$ and $C(-4, 1)$, show that $AB = BC$. If the point $P(h, k)$ is also equidistant from A and C, show that $3k = h+2$.
6. If the points $(2, -1)$, $(6, 7)$ and $(-2, 7)$ are three vertices of a parallelogram, find the three possible positions of the fourth vertex.

7. Find the equations of the following straight lines:
(a) the line through $(1,3)$ with gradient 2,
(b) the line through the points $(3,-5)$ and $(-2,4)$,
(c) the line through $(2,-1)$ parallel to the line $4x-3y=2$,
(d) the line through $(-3,6)$ perpendicular to the line $3x-5y=7$.

8. The sides AB, BC of the parallelogram $ABCD$ have equations $y+3x=1$ and $y=5x-7$ respectively. If the coordinates of D are $(5,10)$, find the coordinates of A, B and C.

9. A triangle has vertices $A(-3,0)$, $B(5,4)$, $C(7,-2)$. If D, E and F are the mid-points of BC, CA and AB respectively, find the point of intersection of the medians AD and BE. Show that this point also lies on the median CF.

10. Show that the points $(5,3)$, $(-1,-5)$ and $(-2,2)$ are three vertices of a square and find the coordinates of the fourth vertex. Find also the area of the square.

11. The points $(2,4)$, $(-1,2)$, $(2,-1)$ are the mid-points of the sides of a triangle. Find the coordinates of the vertices of the triangle.

12. The points A, B, C and D have coordinates $(3,-2)$, $(p,3)$, $(6,2)$ and (q,r) respectively. Given that quadrilateral $ABCD$ is a rhombus, find (a) the values of p, q and r, (b) the area of the rhombus.

13. The line $y=x+5$ is the perpendicular bisector of the line joining the points $P(3,10)$ and $Q(h,k)$. Find in terms of h and k (a) the coordinates of the mid-point of PQ and (b) the gradient of PQ. Deduce that $h+3=k$ and that $h+k=13$.

14. The equation of the side QR of a triangle PQR is $3x-4y=10$, the coordinates of P are $(3,6)$ and the gradient of PQ is 7. Given that $PQ=PR$, find (a) the coordinates of the foot of the perpendicular from P to QR, (b) the equation of the side PR.

15. Given that X, Y and Z are the feet of the perpendiculars from the origin O to the lines with equations $x=3$, $x+y=18$ and $8x-6y=25$, show that X, Y and Z lie on a straight line.

16. The line $3x+4y=30$ cuts the x-axis at A and the y-axis at B. Find (a) the perpendicular distance from the origin O to the line AB, (b) the equation of the perpendicular bisector of AB.

17. P is the point of intersection of the lines $3x-y=7$ and $x+4y+2=0$. Find (a) the equation of the line parallel to $2x+3y=40$ which passes through P, (b) the perpendicular distance of P from the line $2x+3y=40$.

18. The points $A(3,-1)$, $B(-2,4)$, $C(-1,7)$ are three vertices of the quadrilateral $ABCD$. Given that $\triangle ADC$ is the reflection in the line AC of $\triangle ABC$, find (a) the coordinates of D, (b) the area of quadrilateral $ABCD$.

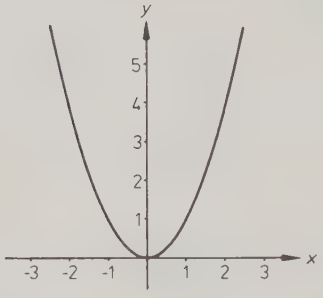
19. The points $(3,-2)$, $(4,5)$, $(-3,6)$ are the vertices of a triangle. Find the coordinates of the centre and the radius of the circumscribed circle.

20. Draw a diagram illustrating the region S of the x, y plane which is defined by the simultaneous inequalities $x+y \geqslant 7$, $2x+y \leqslant 13$, $2x+3y \leqslant 19$, and give the coordinates of the vertices of S. Prove that, if the line $y=kx$ intersects S, then $1/6 \leqslant k \leqslant 2\frac{1}{2}$.

The point P lies on $y=kx$ and is in the region S. Prove that, when $1/6 \leqslant k \leqslant 3/5$, the maximum value for the y-coordinate of P is $13k/(2+k)$, and find the corresponding expression when $3/5 \leqslant k \leqslant 2\frac{1}{2}$. (C)

3 Polynomials and equations

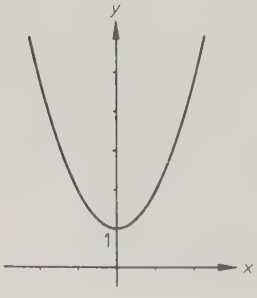
3.1 The quadratic function



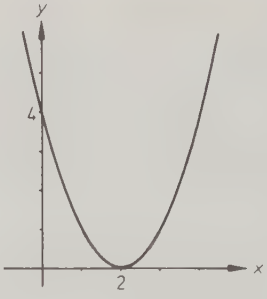
A *quadratic function* is a function of the form $f(x) = ax^2 + bx + c$, i.e. $f: x \to ax^2 + bx + c$, where $a \neq 0$. Thus the graph of a quadratic function has an equation of the form $y = ax^2 + bx + c$.

As shown in the diagram, the graph of $y = x^2$ is symmetrical about the y-axis. The lowest point of the graph, called the *minimum point*, occurs at the origin $(0, 0)$. Hence the *minimum value* of the function $f(x) = x^2$ is zero.

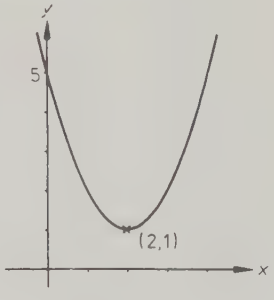
The graphs of some related functions are shown in the diagrams below.



$y = x^2 + 1$

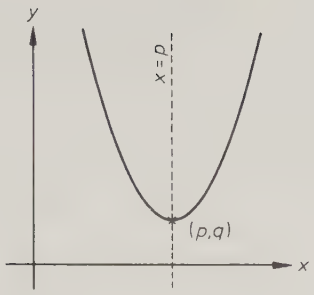


$y = (x-2)^2$



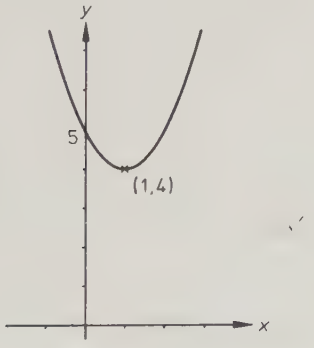
$y = (x-2)^2 + 1$

In each case a curve identical to the curve $y = x^2$ is produced. The term $+1$ in the equation $y = x^2 + 1$ moves the curve up 1 unit. In the graph of $y = (x-2)^2$ the original curve is moved 2 units to the right. The result of moving the curve 2 units to the right and 1 unit up is the graph with equation $y = (x-2)^2 + 1$, i.e. $y = x^2 - 4x + 5$.



Hence the equation $y = (x-p)^2+q$ represents a graph of the same shape as $y = x^2$, moved p units to the right and q units up. Thus it is symmetrical about the line $x = p$ and has a minimum point (p, q). This is confirmed by considering the range of the function $f(x) = (x-p)^2+q$. Since $(x-p)^2$ is never negative, the least possible value of $f(x)$ is given by $(x-p)^2 = 0$. Hence $f(x)$ takes its minimum value q when $x = p$, and its range is the set $\{y \in \mathbb{R} : y \geqslant q\}$.

Example 1 Find the minimum value of the function $f(x) = x^2-2x+5$ and sketch its graph.



$$f(x) = x^2-2x+5 = (x^2-2x+1)+4$$
$$= (x-1)^2+4$$

$\therefore$ $f(x)$ has a minimum value 4 given by $x = 1$.

Since $f(0) = 5$, the graph cuts the y-axis at the point $(0, 5)$.

[Note that x^2-2x+5 is expressed as $(x-1)^2+4$ by 'completing the square' of the form $x^2-2x+\ldots$. The -1 in the bracket is half the original coefficient of x.]

We now consider the graph of $y = ax^2+bx+c$ for values of a other than 1.

Example 2 Sketch the graph of $y = 5-4x-x^2$.

$y = 5-4x-x^2 = 5-(x^2+4x) = 9-(x^2+4x+4) = 9-(x+2)^2$

$\therefore$ y has a maximum value 9 given by $x = -2$.

$x = 0 \Rightarrow y = 5$ $\therefore$ the graph cuts the y-axis at $(0, 5)$.

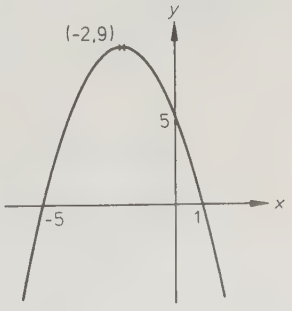
$y = 0 \Leftrightarrow 5-4x-x^2 = 0$

$\Leftrightarrow (5+x)(1-x) = 0$

$\Leftrightarrow 5+x = 0$ or $1-x = 0$

$\Leftrightarrow x = -5$ or $x = 1$

$\therefore$ the graph cuts the x-axis at $(-5, 0)$ and $(1, 0)$.

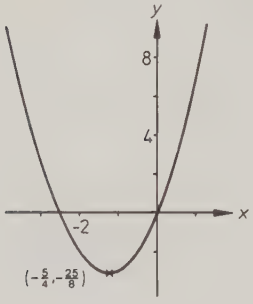


Example 3 Sketch the curve $y = 2x^2 + 5x$.

$$y = 2x^2 + 5x = 2\left\{x^2 + \frac{5}{2}x\right\}, \quad \text{but} \quad \left(x + \frac{5}{4}\right)^2 = x^2 + \frac{5}{2}x + \frac{25}{16}$$

$$\therefore \quad y = 2\left\{\left(x^2 + \frac{5}{2}x + \frac{25}{16}\right) - \frac{25}{16}\right\} = 2\left(x + \frac{5}{4}\right)^2 - \frac{25}{8}.$$

Hence y has a minimum value $-\frac{25}{8}$ given by $x = -\frac{5}{4}$.



$x = 0 \Rightarrow y = 0$
$y = 0 \Leftrightarrow 2x^2 + 5x = 0$
$\Leftrightarrow x(2x+5) = 0$
$\Leftrightarrow x = 0 \quad \text{or} \quad 2x + 5 = 0$
$\Leftrightarrow x = 0 \quad \text{or} \quad x = -\frac{5}{2}$

$\therefore$ the curve cuts the axes at the points $\left(-\frac{5}{2}, 0\right)$ and $(0, 0)$.

In general, the equation $y = ax^2 + bx + c$ represents a curve called a *parabola*. For $a > 0$, the curve has a minimum point and is symmetrical about a vertical axis through this point. However, as shown in Example 2, when $a < 0$ the curve is inverted and has a maximum point on the axis of symmetry.

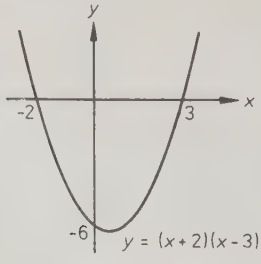
The last two examples in this section deal with quadratic inequalities.

Example 4 Prove that $2x^2 + 8x + 9 > 0$ for all real values of x.

$2x^2 + 8x + 9 = 2(x^2 + 4x) + 9 = 2(x^2 + 4x + 4) - 8 + 9 = 2(x+2)^2 + 1$.

For all real values of x, $(x+2)^2 \geqslant 0 \therefore 2x^2 + 8x + 9 > 0$.

Example 5 Find the values of x for which $x^2 - 1 < x + 5$.



$x^2 - 1 < x + 5$
$\Leftrightarrow \quad x^2 - x - 6 < 0$
$\Leftrightarrow (x+2)(x-3) < 0.$

The sketch shows that the curve $y = (x+2)(x-3)$ lies below the x-axis for $-2 < x < 3$.
Hence if $x^2 - 1 < x + 5$, then $-2 < x < 3$.

Exercise 3.1

1. Find the maximum or minimum value of each of the quadratic functions given below and sketch the graph of the function.
(a) x^2+4x-5 (b) $2x-x^2$ (c) x^2-3x+4 (d) x^2+4
(e) $4x^2-12$ (f) $4x^2-12x+9$ (g) $2+x-3x^2$ (h) $4x-6x^2-9$.

2. Find the range of each of the following functions.
(a) $f:x\to x^2-5$ (b) $f:x\to x^2-6x$ (c) $f:x\to 2x^2+2x+1$ (d) $f:x\to 6-5x-x^2$.

3. Express the function $f(x)=x^2-4x+8$ in the form $(x-p)^2+q$. Deduce that $f(x)$ is positive for all real values of x.

4. Prove that the expression $x-2-x^2$ is negative for all real values of x.

5. Prove that the following inequalities hold for all real values of x.
(a) $2x^2+5>4x$ (b) $2(2x-1)<3x^2$ (c) $9x^2-30x+25\geqslant 0$ (d) $2x<2x^2+1$.

6. Find the values of x for which
(a) $x^2-2x<0$ (b) $x^2+8\leqslant 6x+3$ (c) $x^2-4\geqslant 0$
(d) $4x+1<x^2+4$ (e) $3x^2+2x<1$ (f) $2x^2+7x+3\geqslant 0$.
[You may find sketch graphs helpful.]

7. Find the minimum value of the function $3x^2-12x+5$ and sketch the curve $y=3x^2-12x+5$. Find the range of values of c for which the function $3x^2-12x+c$ is positive for all values of x.

8. Find the range of values of k such that $x^2+3x+k\geqslant 0$ for all real values of x.

9. If $y=x^2+2px+q$, find the values of p and q in each of the following cases.
(a) $y=0$ when $x=2$ and when $x=4$,
(b) $y=1$ when $x=2$ and $y=7$ when $x=-4$,
(c) y has a minimum value of 2 when $x=-1$.

10. The maximum value of the function $f(x)=ax^2+bx+c$ is 10. Given that $f(3)=f(-1)=2$, find $f(2)$.

3.2 Quadratic equations

An equation of the form $ax^2+bx+c=0$ is called a *quadratic equation*. Many such equations are solved by factorising. Others may be solved by completing the square as follows:

$$\begin{aligned} x^2-6x+7=0 &\Leftrightarrow x^2-6x+9=2\\ &\Leftrightarrow (x-3)^2=2\\ &\Leftrightarrow x-3=\pm\sqrt{2}\\ &\Leftrightarrow x=3\pm\sqrt{2}. \end{aligned}$$

Hence if $x^2-6x+7=0$ then $x=1{\cdot}59$ or $4{\cdot}41$ (to 2 d.p.).

However, in some cases the method breaks down.

$$x^2-2x+5=0 \Leftrightarrow x^2-2x+1=-4$$
$$\Leftrightarrow \quad (x-1)^2=-4$$

Since $(x-1)^2$ can never be negative, there are no real values of x which satisfy this equation.

Let us now consider the general quadratic equation.

$$ax^2+bx+c=0 \Leftrightarrow \quad x^2+\frac{b}{a}x+\frac{c}{a}=0$$

$$\Leftrightarrow x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2}{4a^2}-\frac{c}{a}$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}.$$

Hence the nature of the roots of the equation $ax^2+bx+c=0$ depends on the sign of b^2-4ac.

(i) If $b^2-4ac>0$, then $x+\frac{b}{2a}=\pm\frac{\sqrt{(b^2-4ac)}}{2a}$

$\therefore$ the equation has two distinct real roots given by

$$x=\frac{-b\pm\sqrt{(b^2-4ac)}}{2a}.$$

(ii) If $b^2-4ac=0$, then $\left(x+\frac{b}{2a}\right)^2=0$

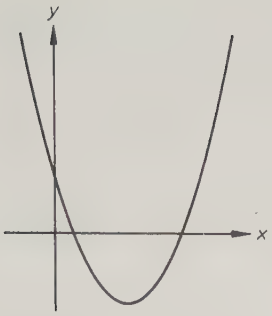
$\therefore$ the equation has two equal roots given by $x=-\frac{b}{2a}$.

(iii) If $b^2-4ac<0$, then $\left(x+\frac{b}{2a}\right)^2<0$, which is not possible for real values of x

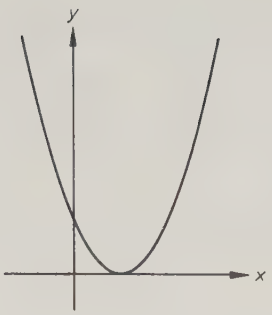
$\therefore$ the equation has no real roots.

The diagrams opposite show the curve $y=ax^2+bx+c$ in these three cases, for $a>0$. Each curve has a minimum point at $x=-\frac{b}{2a}$ and is symmetrical about the line $x=-\frac{b}{2a}$.

In case (iii) the graph shows that if $a>0$ and $b^2-4ac<0$ the function ax^2+bx+c is always positive. Similarly if $a<0$ and $b^2-4ac<0$ the function ax^2+bx+c is always negative.



(i) $b^2-4ac>0$



(ii) $b^2-4ac=0$

(iii) $b^2-4ac<0$

Example 1 Solve the equation $3x^2-4x-2=0$.

$$3x^2-4x-2=0 \Rightarrow x=\frac{-(-4)\pm\sqrt{\{(-4)^2-4.3.(-2)\}}}{2.3}$$

$$\Rightarrow x=\frac{4\pm\sqrt{(16+24)}}{6}$$

$$\Rightarrow x=\frac{4\pm\sqrt{40}}{6}$$

$\therefore$ the roots of the equation are $-0{\cdot}39$ and $1{\cdot}72$ (to 2 d.p.).

Example 2 Find the range of values of p for which the equation $x^2+px+p=0$ has real roots.

The equation $ax^2+bx+c=0$ has real roots if $b^2-4ac\geqslant 0$
$\therefore$ the equation $x^2+px+p=0$ has real roots if $p^2-4p\geqslant 0$
i.e. if $p(p-4)\geqslant 0$.
Since p and $(p-4)$ are both positive when $p>4$ and both negative when $p<0$, the equation has real roots when $p\leqslant 0$ and when $p\geqslant 4$.

Exercise 3.2

1. Find the real roots, if any, of the following quadratic equations, giving your answers in surd form.
(a) $x^2-3x+1=0$ (b) $2x^2+3x-1=0$ (c) $1-2x-x^2=0$
(d) $3x^2-4x+2=0$ (e) $4x^2-12x-27=0$ (f) $2x^2-6x+3=0$.

2. Find the values of k for which the following equations have equal roots.
(a) $x^2+2kx+k+6=0$, (b) $(x+1)(x+3)=k$.

3. Find the range of values of p for which each of the given equations has two distinct real roots.
(a) $x^2+2px-5p=0$ (b) $3x^2+3px+p^2=1$
(c) $x(x+3)=p(x-1)$ (d) $p(x^2-1)=3x+2$.

4. Given that the following expressions are positive for all real values of x, find in each case the set of possible values of a.
(a) x^2+a+a^2 (b) x^2+ax+a^2 (c) $x^2-2ax+3a$ (d) $ax^2+2ax+a^2$.

5. Find the values of p for which the expression $x^2+(p+3)x+2p+3$ is a perfect square.

6. Show that there is no real value of k for which the equation $x^2+(3-k)x+k^2+4=0$ has real roots.

7. Find any values of p for which the curve $y=3px^2+2px+1$ touches the x-axis.

8. Find the set of values of k for which the equation $kx^2+x+k-1=0$ has real roots, one positive and one negative.

9. Find the set of values of k for which the roots of the equation $x^2+kx-k+3=0$ are real and of the same sign.

10. Given that $y=px^2+2qx+r$, show that y will have the same sign for all real values of x if and only if $q^2<pr$.

3.3 Sum and product of roots

The quadratic equation with roots α and β may be written

$$(x-\alpha)(x-\beta)=0 \quad \text{i.e.} \quad x^2-(\alpha+\beta)x+\alpha\beta=0.$$

The equation $ax^2+bx+c=0$ may be written $x^2+\frac{b}{a}x+\frac{c}{a}=0$.

Hence if the equation $ax^2+bx+c=0$ has roots α and β,

$$\alpha+\beta=-\frac{b}{a}, \quad \alpha\beta=\frac{c}{a}.$$

Example 1 Write down the sum and product of the roots of the equation $3x^2-6x+2=0$.

The sum of the roots $=-\dfrac{(-6)}{3}=2$.

The product of the roots $=\dfrac{2}{3}$.

When the sum and product of the roots of a quadratic equation are known, the equation may be written down in the form

$$x^2-(\text{sum of roots})x+(\text{product of roots})=0.$$

Example 2 Write down the quadratic equation, the sum and product of whose roots are $\frac{3}{4}$ and -7 respectively.

The equation is $x^2-\frac{3}{4}x+(-7)=0$ i.e. $4x^2-3x-28=0$.

Example 3 Given that one root of the equation $3x^2+4x+k=0$ is three times the other, find k.

Let the roots of the given equation be α and 3α.

The sum of the roots $=\alpha+3\alpha=-\frac{4}{3}$ i.e. $4\alpha=-\frac{4}{3}$

The product of the roots $=\alpha.3\alpha=\frac{k}{3}$ i.e. $3\alpha^2=\frac{k}{3}$

Hence $\alpha=-\frac{1}{3}$ and $k=1$.

If α and β are the roots of a given quadratic equation, it is usually possible to find the value of a symmetrical function of α and β by expressing it in terms of $\alpha+\beta$ and $\alpha\beta$. [A symmetrical function of α and β is one which is unaltered if α and β are interchanged.]

Example 4 If α and β are the roots of the equation $3x^2-5x+1=0$, find the values of (i) $\frac{1}{\alpha}+\frac{1}{\beta}$ (ii) $\alpha^2+\beta^2$.

$$\alpha+\beta=\frac{5}{3}, \qquad \alpha\beta=\frac{1}{3}$$

(i) $\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\alpha+\beta}{\alpha\beta}=\frac{5}{3}\div\frac{1}{3}=\frac{5}{3}.\frac{3}{1}=5$

(ii) $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=\frac{25}{9}-2.\frac{1}{3}=\frac{19}{9}$.

Example 5 If α and β are the roots of the equation $x^2+7x+5=0$, find an equation whose roots are $\alpha+1$ and $\beta+1$.

Method 1: $\alpha+\beta=-7$, $\alpha\beta=5$
$(\alpha+1)+(\beta+1)=\alpha+\beta+2=-7+2=-5$
$(\alpha+1)(\beta+1)=\alpha\beta+\alpha+\beta+1=5-7+1=-1$
$\therefore$ the required equation is $x^2-(-5)x+(-1)=0$
i.e. $x^2+5x-1=0$.

Method 2: For the equation with roots $x=\alpha+1$ or $x=\beta+1$,
either $x-1=\alpha$ or $x-1=\beta$.

Since the equation with roots α and β is $x^2+7x+5=0$, the required equation must take the form

$$(x-1)^2+7(x-1)+5=0$$
$$x^2-2x+1+7x-7+5=0$$
$$\text{i.e.}\quad x^2+5x-1=0.$$

Exercise 3.3

1. Find the sums and products of the roots of the following equations.
(a) $2x^2-5x-3=0$ (b) $6+x-x^2=0$ (c) $3x^2-5=0$
(d) $x^2+px-p=0$ (e) $x(x-2)=5(x-1)$ (f) $4x-1/x=3$.

2. Find equations, the sums and products of whose roots are, respectively:
(a) $3, 2$ (b) $-\frac{1}{2}, \frac{3}{4}$ (c) $0, -4$
(d) $3/7, -1/14$ (e) $p-q, p+q$ (f) $a/b, 1/ab$.

3. Given that one root of the equation $2x^2-kx+k=0$, where $k\neq 0$, is twice the other, find k.

4. Given that the two roots of the equation $x^2+(7-p)x-p=0$ differ by 5, find the possible values of p.

5. If α and β are the roots of the equation $ax^2+bx+c=0$, prove that
(i) if $\beta=4\alpha$ then $4b^2=25ac$, (ii) if $\beta=\alpha+1$ then $a^2=b^2-4ac$.

6. If α and β are the roots of the equation $x^2-3x-2=0$, find the values of
(i) $\alpha+\beta$, (ii) $\alpha\beta$, (iii) $\alpha^2+\beta^2$, (iv) $\alpha^3+\beta^3$.

7. If α and β, where $\alpha>\beta$, are the roots of the equation $x^2+2x-5=0$, find the values of (i) $(\alpha+\beta)^2$, (ii) $(\alpha-\beta)^2$, (iii) $\alpha-\beta$, (iv) $\alpha^2-\beta^2$.

8. If α and β are the roots of the equation $2x^2-x-4=0$, find the values of
(a) $\alpha^2+\beta^2$ (b) $(\alpha-\beta)^2$ (c) $\alpha^3+\beta^3$
(d) $\dfrac{1}{\alpha}+\dfrac{1}{\beta}$ (e) $\dfrac{1}{\alpha^2}+\dfrac{1}{\beta^2}$ (f) $\dfrac{\alpha}{\beta^2}+\dfrac{\beta}{\alpha^2}$.

9. If $p+q=5$ and $p^2+q^2=19$, find the value of pq and write down an equation in x whose roots are p and q.

10. If $a-b=3$ and $a^2+b^2=65$, write down an equation in x whose roots are a and b.

11. If α and β are the roots of the equation $x^2-4x+2=0$, find equations whose roots are
(a) $3\alpha, 3\beta$, (b) $\alpha+3, \beta+3$, (c) $\alpha+3\beta, \beta+3\alpha$.

12. If α and β are the roots of the equation $x^2+x+k=0$, find equations whose roots are
(a) α^2, β^2, (b) $-\alpha, -\beta$, (c) $\alpha-1, \beta-1$.

13. If α and β are the roots of the equation $ax^2+bx+c=0$, find equations whose roots are
(a) $\frac{1}{\alpha}, \frac{1}{\beta}$, (b) $\alpha-\beta, \beta-\alpha$, (c) $\frac{\alpha}{\alpha+\beta}, \frac{\beta}{\alpha+\beta}$.

14. Given that the equations $x^2-2x+p=0$ and $2x^2-5x+q=0$ have a common root, by eliminating the terms in x^2, show that this common root is $q-2p$. Find expressions for the other roots of the equations.

15. Given that the equations $2x^2+3x+k=0$ and $3x^2+x-2k=0$ have a common root, find the possible values of k.

3.4 Non-linear simultaneous equations

In this section we deal with pairs of simultaneous equations in which at least one of the equations is non-linear. It is often necessary to solve such equations when finding the points of intersection of a line and a curve.

Example 1 Find the points of intersection of the line $y=x+1$ and the curve $2x^2+y^2=6$.

At the points of intersection $y = x+1$
and $2x^2+y^2=6$
$\therefore \quad 2x^2+(x+1)^2=6$
$2x^2+x^2+2x+1=6$
$3x^2+2x-5=0$
$(3x+5)(x-1)=0$

Hence either $3x+5=0$ or $x-1=0$
$x=-5/3$ $x=1$

$\therefore$ the points of intersection are $(-5/3, -2/3)$ and $(1,2)$.

Example 2 Show that the line $y=3x-1$ does not meet the curve $y=x^2+3x$.

At any points of intersection $y=3x-1$
and $y=x^2+3x$
$\therefore \quad x^2+3x=3x-1$, i.e. $x^2=-1$.

Since this equation has no real roots, there are no real points of intersection between the line and the curve.

Example 3 Find the points of intersection of the line $y=-x$ and the curve $y=x^3+6x^2+8x$.

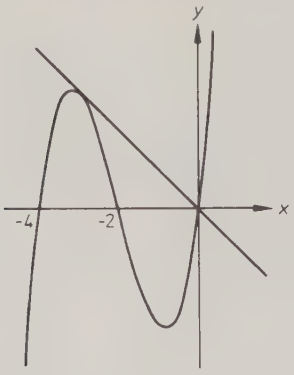
At the points of intersection $x^3+6x^2+8x = -x$

i.e. $x^3+6x^2+9x = 0$

$x(x+3)^2 = 0$

$\therefore$ either $x = 0, y = 0$ or $x = -3, y = 3$.

Hence the points of intersection are $(0,0)$ and $(-3,3)$.



The significance of the fact that $x = -3$ appears as a repeated root in the above example is shown in this sketch graph. The line $y = -x$ is a tangent to the curve at the point $(-3,3)$.

In general, when solving simultaneous equations, one linear and the other quadratic, a method similar to the following is used.

Example 4 Solve the equations $2x-3y = 4$ (1)

$x^2-3xy+3y^2 = 7$ (2)

Rearranging (1) $3y = 2x-4$

Equation (2) $\times 3$ $3x^2-9xy+9y^2 = 21$

Substituting for $3y$ $3x^2-3x(2x-4)+(2x-4)^2 = 21$

$3x^2-6x^2+12x+4x^2-16x+16 = 21$

$x^2-4x-5 = 0$

$(x-5)(x+1) = 0$

$\therefore$ either $x = 5$ or $x = -1$.

Substituting in (3) to find the corresponding values of y, we obtain the solutions $x = 5, y = 2$ and $x = -1, y = -2$.

In certain cases there is an alternative to the method of Example 4.

Example 5 Solve the equations $x+2y = 3$ (1)

$3x^2+7xy+2y^2 = 12$ (2)

Factorising (2) $(x+2y)(3x+y) = 12$

Dividing by (1) $3x+y = 4$

Multiplying (1) by 3 $3x+6y = 9$.

Subtracting $5y = 5$ $\therefore$ $y = 1$

Substituting in (1) $x+2 = 3$ $\therefore$ $x = 1$

Hence the equations have one soloution $x = 1, y = 1$.

[Note that when dealing with non-linear simultaneous equations it is always advisable to check that the solutions obtained satisfy the original equations.]

Exercise 3.4

1. Find any points of intersection between the given curves and straight lines.

(a) $y = x^2+1,\ y = 5x-3$ (b) $y = 3-x^2,\ y = 7-4x$

(c) $y = x-3,\ xy+2 = 0$ (d) $x^2+y^2+4x = 1,\ y = 2-x$

(e) $y = 2x-7,\ 2xy+3y+5x+11 = 0$ (f) $4x^2-9y^2 = 36,\ 2x+3y+2 = 0.$

In questions 2 to 14 solve the given pairs of simultaneous equations.

2. $x+y = 1$
$xy = -12.$

3. $3x = 2y$
$xy = 24.$

4. $x+y = 1$
$x^2-y^2 = 5.$

5. $x-5y = 8$
$x^2+9y^2+3x-xy = 30.$

6. $3x+y = 5$
$5x^2-2xy+y^2 = 5.$

7. $3x+2y = 10$
$3x^2-2xy+4y^2 = 25.$

8. $x-3y = 2$
$x^2-2xy-3y^2 = 12.$

9. $x+y = -1, \frac{1}{x}+\frac{1}{y} = \frac{1}{2}.$

10. $2x+y = 1, \frac{1}{x}-\frac{1}{y} = 2.$

11. $x+y = 1$
$x^3+y^3 = 91.$

12. $2x+3y = 3$
$8x^3+27y^3 = 117.$

13. $x^2-2xy+3y^2 = 3$
$x^2+2xy+3y^2 = 11.$

14. $4xy = -15$
$4x^3+4y^3 = 49.$

In questions 15 to 18 solve the given pairs of simultaneous equations by eliminating either the terms in x^2 or the terms in y^2.

15. $x^2+y^2+2x-3y = 3$
$2x^2-y^2+4x+18y = 54.$

16. $x^2+2y^2+4y = 4$
$x^2-y^2+2x-2y = 0.$

17. $x^2+xy+y^2 = 3$
$x^2-y^2 = 3.$

18. $x^2+y^2-3x = 9$
$x^2-y^2+x+4y = 3.$

3.5 Products and factors

A function of the form $P(x) = a_0x^n+a_1x^{n-1}+\ldots+a_n$ $(a_0 \neq 0)$, where n is a positive integer, is called a *polynomial* of degree n. The real numbers a_0, a_1,... are called the *coefficients* of x^n, x^{n-1},.... For instance, x^5-4x^2+3x is a polynomial of degree 5. A *linear* function, such as $ax+b$, is a polynomial of degree 1. A *quadratic* function is a polynomial of degree 2. A polynomial of degree 3 is often called a *cubic*.

The product of two polynomials can be found by either of the methods illustrated overleaf.

Example 1 Multiply $2x^3-x^2+3$ by $3x+1$.

$$\begin{aligned}(3x+1)(2x^3-x^2+3) &= 3x(2x^3-x^2+3)+1(2x^3-x^2+3)\\ &= 6x^4-3x^3+9x+2x^3-x^2+3\\ &= 6x^4-x^3-x^2+9x+3.\end{aligned}$$

OR

$$\begin{array}{r} 2x^3-x^2 \quad +3 \\ 3x+1 \\ \hline 2x^3-x^2 \quad +3 \\ 6x^4-3x^3 \quad +9x \\ \hline 6x^4- \ x^3-x^2+9x+3. \end{array}$$

When the complete product is not required, the coefficients of particular terms can be picked out.

Example 2 Find the coefficient of x^4 in the product of x^3-3x^2+x-2 and $2x^3+x^2-3x+1$.

The diagram shows the mental processes involved.

$$\underset{\textcircled{1}}{(x^3-3x^2+x-2)}\underset{\textcircled{2}}{(2x^3+x^2-3x+1)}$$

The coefficient of x^4 in the product

$$= 1.(-3)+(-3).1+1.2 = -3-3+2 = -4.$$

Sometimes it is possible to *factorise* a polynomial, i.e. express it as a product. This process is useful when simplifying fractions.

Example 3 Simplify $\dfrac{2x}{x^2-4}+\dfrac{7}{2x^2+x-6}$.

$$\begin{aligned}\frac{2x}{x^2-4}+\frac{7}{2x^2+x-6} &= \frac{2x}{(x-2)(x+2)}+\frac{7}{(2x-3)(x+2)}\\ &= \frac{2x(2x-3)}{(x-2)(x+2)(2x-3)}+\frac{7(x-2)}{(x-2)(x+2)(2x-3)}\\ &= \frac{4x^2-6x+7x-14}{(x-2)(x+2)(2x-3)}\\ &= \frac{4x^2+x-14}{(x-2)(x+2)(2x-3)}\\ &= \frac{(4x-7)(x+2)}{(x-2)(x+2)(2x-3)}\end{aligned}$$

$$\therefore \quad \frac{2x}{x^2-4}+\frac{7}{2x^2-x-6}=\frac{4x-7}{(x-2)(2x-3)}.$$

It is often difficult to find factors of a polynomial of degree 3 or more. Testing possible factors by long division is a cumbersome process as the next example shows.

Example 4 Divide x^4-3x+5 by $x-2$.

$$\begin{array}{r|l} & x^3+2x^2+4x+5 \\ \hline x-2 & x^4\qquad\quad -3x+5 \\ & x^4-2x^3 \\ & \hline & 2x^3 \\ & 2x^3-4x^2 \\ & \hline & 4x^2-3x \\ & 4x^2-8x \\ & \hline & 5x+5 \\ & 5x-10 \\ & \hline & 15 \end{array}$$

$\therefore$ when x^4-3x+5 is divided by $x-2$, the quotient is x^3+2x^2+4x+5 and the remainder is 15.

This result can be expressed in the form of an *identity*:

$$x^4-3x+5 \equiv (x-2)(x^3+2x^2+4x+5)+15.$$

[The symbol $\equiv$ means 'is identically equal to', i.e. equal for all values of x.]

More generally, we can consider the division of a polynomial $P(x)$ by $(x-a)$. If $Q(x)$ is the quotient and R the remainder, then we may write

$$P(x) \equiv (x-a)Q(x)+R.$$

Since this identity holds for all values of x, we substitute $x=a$ to obtain $P(a) = (a-a)Q(a)+R = R$. Thus, when a polynomial $P(x)$ is divided by $(x-a)$ the remainder is $P(a)$. This result is called the *remainder theorem.*

Example 5 Find the remainder when $P(x)=2x^3+7x^2-5x-4$ is divided by $x+3$.

$P(-3) = 2(-3)^3+7(-3)^2-5(-3)-4 = -54+63+15-4 = 20$

$\therefore$ the remainder when $P(x)$ is divided by $x+3$ is 20.

If for a polynomial $P(x)$ we find that $P(a)=0$, we deduce from the remainder

theorem that there is no remainder when $P(x)$ is divided by $(x-a)$. Thus we have the *factor theorem*, which states that

$$\boxed{\text{if } P(a)=0 \text{ then } (x-a) \text{ is a factor of } P(x).}$$

We use this result to test for linear factors of polynomials.

Example 6 Factorise $f(x)=6x^3+13x^2-4$.
Since the constant term is -4, we first try values of x which are factors of -4, i.e. ±1, ±2 or ±4.

$$f(1)=6.1^3+13.1^2-4=6+13-4\neq 0$$
$$f(-1)=6(-1)^3+13(-1)^2-4=-6+13-4\neq 0$$
$$f(2)=6.2^3+13.2^2-4=48+52-4\neq 0$$
$$f(-2)=6(-2)^3+13(-2)^2-4=-48+52-4=0.$$

Hence, by the factor theorem, $x+2$ is a factor of $f(x)$. By inspection or by long division,

$$f(x)=(x+2)(6x^2+x-2)$$
$$\therefore\quad f(x)=(x+2)(2x-1)(3x+2).$$

Note that in some cases a cubic polynomial is the product of a linear factor and an irreducible quadratic factor. For example,

$$x^3-a^3=(x-a)(x^2+ax+a^2)$$
$$x^3+a^3=(x+a)(x^2-ax+a^2).$$

Exercise 3.5

1. Find the product of
(a) $2x^2-x+7$ and $x+2$
(b) $3x^3-2x^2+5x-1$ and $2x-3$
(c) x^2+3x-2 and $4x^2-x+1$
(d) $5x^3-2x+3$ and x^2-1
(e) $3x^4-2x^3+6x-4$ and $3x+2$
(f) $x^5-x^4+x^3-x^2+x-1$ and $x+1$.

2. Find the coefficients of the given terms in the following products.
(a) $(2x-5)(x^3-x^2+2x-3)$; x, x^3.
(b) $(x^2-2x+3)(3x^2+x-2)$; x, x^2.
(c) $(x^3-x-1)(2x^2+3x-1)$; x^2, x^4.
(d) $(3x^3-x^2-2x+5)(x^3+2x^2-x-2)$; x^3, x^5.

3. Simplify the following expressions.

(a) $\dfrac{2}{x^2-1}-\dfrac{7}{2x^2-3x-5}$ (b) $\dfrac{1}{x+2}+\dfrac{6x}{x^3+8}$

(c) $\dfrac{x^4-x}{2x-3x^2}\times\dfrac{3x^2+x-2}{1-x^2}$ (d) $\dfrac{8x^3+1}{4x^2-8x+3}\div\left(2x-\dfrac{1}{1-2x}\right)$.

4. Find the remainder when
(a) $5x^2+3x-7$ is divided by $x-2$
(b) x^3-2x^2+8x-3 is divided by $x-5$
(c) $x^4-3x^3+4x^2-x+6$ is divided by $x+1$
(d) x^5+16 is divided by $x+2$
(e) $3x^2-2ax-4a^2$ is divided by $x+a$
(f) $a^3-3a^2b+2b^3$ is divided by $a-b$.

5. Find by long division the quotient and remainder when
(a) $3x^2-2x+5$ is divided by $x+1$
(b) $4x^3-4x^2+5x+1$ is divided by $2x-3$
(c) x^3-3 is divided by $x+3$
(d) $3x^5-x^4-6x^3+11x^2-1$ is divided by $3x-1$
(e) x^4-2x^3+6x-5 is divided by x^2-x-1
(f) $2x^4-x^3+3x^2-7$ is divided by x^2-2.

6. Factorise the following polynomials
(a) $2x^3-3x^2+1$
(b) $3x^3-2x^2-7x-2$
(c) x^4-x^2-72
(d) x^5+x^3+x
(e) $4x^3-13x+6$
(f) $4x^4-4x^3-9x^2+x+2$.

7. When the function x^2+ax+b is divided by $x-1$ and $x+2$, the remainders are 4 and 5 respectively. Find the values of a and b.

8. When x^3+px^2+qx+1 is divided by $x-2$ the remainder is 9; when divided by $x+3$ the remainder is 19. Find the values of p and q.

9. Given that the expression $2x^3+ax^2+b$ is divisible by $x+1$ and that there is a remainder of 16 when it is divided by $x-3$, find the values of a and b.

10. The expression ax^3-8x^2+bx+6 is exactly divisible by x^2-2x-3. Find the values of a and b.

11. The polynomials x^3+4x^2-2x+1 and x^3+3x^2-x+7 leave the same remainder when divided by $x-p$. Find the possible values of p.

12. Given that the expressions x^3-4x^2+x+6 and x^3-3x^2+2x+k have a common factor, find the possible values of k.

13. If $x-a$ is a factor of the expression $ax^3-3x^2-5ax-9$, find the possible values of a and factorise the expression for each of these values.

14. Prove that when a polynomial $P(x)$ is divided by $(ax-b)$ the remainder is $P(b/a)$. Hence find the remainder when (a) $12x^2-4x+9$ is divided by $2x-1$, (b) $9x^3-x+5$ is divided by $3x+2$.

15. Factorise (a) $8x^3-36x^2+46x-15$, (b) $12x^3+4x^2-5x-2$,
(c) $3x^3-x^2+4x+4$.

3.6 Polynomial equations

In practice it is often difficult to solve polynomial equations. However, sometimes solutions can be found by factorisation.

Example 1 Solve the equation $2x^4-5x^2-12=0$.

$$2x^4-5x^2-12=0 \Leftrightarrow (2x^2+3)(x^2-4)=0$$
$$\Leftrightarrow (2x^2+3)(x-2)(x+2)=0$$
$$\therefore \quad 2x^2+3=0 \quad \text{or} \quad x-2=0 \quad \text{or} \quad x+2=0.$$

Since $2x^2+3=0$ has no real roots, either $x=2$ or $x=-2$.

Example 2 Solve the equation $x^3+3x^2+x-1=0$.

Let $f(x)=x^3+3x^2+x-1$, then
$f(1)=1+3+1-1\neq 0$, $f(-1)=-1+3-1-1=0$
$\therefore$ $x+1$ is a factor of $f(x)$.
By division $f(x)=(x+1)(x^2+2x-1)$.

Hence if $x^3+3x^2+x-1=0$,
either $x+1=0$ or $x^2+2x-1=0$

$$x=-1 \qquad x=\frac{-2\pm\sqrt{(4+4)}}{2}$$
$$=-1\pm\sqrt{2}$$

$\therefore$ the roots of the given equation are -1, $-1-\sqrt{2}$ and $-1+\sqrt{2}$.

Equations involving surds can sometimes be expressed in polynomial form. However, since extra roots may be introduced in this process all solutions must be tested in the original equation.

Example 3 Solve the equation $\sqrt{(3-x)}-\sqrt{(7+x)}=2$.

Rearranging $\sqrt{(3-x)}=\sqrt{(7+x)}+2$.
Squaring both sides $3-x=\{\sqrt{(7+x)}+2\}^2$
i.e. $3-x=(7+x)+4\sqrt{(7+x)}+4$
$\therefore$ $-2x-8=4\sqrt{(7+x)}$
$\therefore$ $-(x+4)=2\sqrt{(7+x)}$.
Squaring both sides $x^2+8x+16=4(7+x)$
$\therefore$ $x^2+4x-12=0$
$(x+6)(x-2)=0$.
Hence either $x=-6$ or $x=2$.
If $x=-6$, $\sqrt{(3-x)}-\sqrt{(7+x)}=\sqrt{9}-\sqrt{1}=2$.
If $x=2$, $\sqrt{(3-x)}-\sqrt{(7+x)}=\sqrt{1}-\sqrt{9}=-2$.
Thus the only root of the given equation is $x=-6$.

In Example 1 a polynomial equation of degree 4 was found to have 2 real roots, whereas the cubic equation in Example 2 had 3 real roots. In general, if $x=a$ is a root of a polynomial equation $P(x)=0$ then $(x-a)$ is a factor of $P(x)$.

Thus, since a polynomial of degree n can have up to n factors of the form $(x-a)$, a polynomial equation of degree n can have up to n real roots.

In §3.2 it was shown that if the roots of the quadratic equation $ax^2+bx+c=0$ are α and β, then $\alpha+\beta=-b/a$ and $\alpha\beta=c/a$. Similar expressions can be obtained for equations of higher degree. For example, if the equation $ax^3+bx^2+cx+d=0$ has roots α, β and γ, then it may be written both in the form

$$x^3+\frac{b}{a}x^2+\frac{c}{a}x+\frac{d}{a}=0$$

and as

$$(x-\alpha)(x-\beta)(x-\gamma)=0$$

i.e.

$$x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\beta\gamma+\gamma\alpha)x-\alpha\beta\gamma=0.$$

Hence

$$\alpha+\beta+\gamma=-\frac{b}{a},\ \alpha\beta+\beta\gamma+\gamma\alpha=\frac{c}{a},\ \alpha\beta\gamma=-\frac{d}{a}.$$

Example 4 Given that 1 and -2 are two roots of the equation $3x^3-x^2+px+q=0$, find the third root and the values of p and q.

Since the sum of the roots is 1/3, the third root is

$$1/3-\{1+(-2)\},\quad \text{i.e.}\quad 4/3.$$

The product of the roots $=-\frac{q}{3}=1.(-2).\frac{4}{3}\quad\therefore\quad q=8.$

Since $x=1$ satisfies the equation, $3-1+p+q=0$

$$\therefore\quad p=-2-q=-10.$$

Hence the third root is 4/3 and the values of p and q are -10 and 8 respectively.

Exercise 3.6

In questions 1 to 14 solve the given equations.

1. $x^3-8x^2+5x+14=0$.
2. $6-x-4x^2-x^3=0$.
3. $4+7x+2x^2-x^3=0$.
4. $6x^3+31x^2+48x+20=0$.
5. $4x^4+23x^2-6=0$.
6. $x^4-11x^2+18=0$.
7. $x^3-6x-4=0$.
8. $x^3-6x^2+12x-9=0$.
9. $2x^3+x^2-6x-3=0$.
10. $x^4-5x^3+4x^2+6x-4=0$.
11. $\sqrt{(x+6)}-\sqrt{(x+1)}=1$.
12. $\sqrt{(12+x)}-\sqrt{(13-x)}=1$.
13. $\sqrt{(3x+1)}-\sqrt{(x-1)}=2$.
14. $\sqrt{(9-4x)}-\sqrt{(5-x)}=2$.
15. If $f(x)=x^3+px+q$ is exactly divisible by $(x+2)$ and $(x-3)$, find the values of p and q. With these values of p and q, find the roots of the equation $f(x)=0$.

16. Find the value of k if one root of the equation $12x^3+kx^2-17x+6=0$ is $\frac{1}{2}$. When k has this value, find the other roots of the equation.

17. Write down the sums and products of the roots of the following equations.
(a) $2x^3+3x^2-8x-12=0$ (b) $2x^3+5x^2-3x=0$
(c) $3x^3+4x^2-5x-2=0$ (d) $x^3-11x-6=0$.

18. Given that -1 and 4 are two roots of the equation $x^3+5x^2+ax+b=0$, find the third root and the values of a and b.

19. Given that 2 is a repeated root of the equation $2x^3+px^2+qx-4=0$, find the third root and the values of p and q.

20. Write down the sums and products of the roots of the following equations.
(a) $x^4-6x^3+9x^2-4=0$ (b) $2x^4-x^3-12x+3=0$
(c) $x^5-3x^3+2x=0$ (d) $2x^5-3x^4-2x^3+4x^2=1$.

21. Solve the equation $3x^4+4x^3-14x^2+4x+3=0$ by writing it in the form $a\left(x+\frac{1}{x}\right)^2+b\left(x+\frac{1}{x}\right)+c=0$.

22. Solve the equation $10x^4-37x^3+50x^2-37x+10=0$ by the method of the previous question.

Exercise 3.7 (*miscellaneous*)

1. Sketch the following curves, showing any maximum or minimum points.
(a) $y=x^2+2x$ (b) $y=2x^2+x-3$
(c) $y=3+4x-4x^2$ (d) $y=x^2-6x+10$.

2. Show that x^2-2x+2 is positive for all values of x. Hence find the range of values of x for which the expression x^3-x^2+2 is positive.

3. By means of a suitable substitution, solve the equation
$(x^2-3x)^2-9(x^2-3x)-10=0$.

4. Find the values of k for which the equation $x^2+2kx+3k=0$ has (a) equal roots, (b) real roots which differ by 4.

5. Given that the equation $x^2+px+q=0$ has roots α and β, find an equation whose roots are $\alpha(\alpha-1)$ and $\beta(\beta-1)$.

6. If α and β are the roots of the equation $x^2-x-3=0$, show that $\alpha^3+\beta^3=10$ and find a quadratic equation whose roots are α^2/β and β^2/α.

7. The function $f(x)=x^2+px+1$, where p is a constant, is zero when $x=\alpha$ and $x=\beta$; and the function $g(x)=x^2-9x+q$, where q is a constant, is zero when $x=\alpha+2\beta$ and $x=\beta+2\alpha$. Find p and q, and show that $f(3)=g(3)$. (JMB)

8. (a) The sum of the squares of the roots of the equation $x^2+px+q=0$ is 56 and the sum of the reciprocals of the roots is 2. Find the values of p and q. (b) Show that, for all real values of α, the equation $x^2+(3\alpha-2)x+\alpha(\alpha-1)=0$ has real roots for x. (C)

9. The roots of the equation $9x^2+6x+1=4kx$, where k is a real constant, are denoted by α and β. (a) Show that the equation whose roots are $1/\alpha$ and $1/\beta$ is $x^2+6x+9=4kx$. (b) Find the set of values of k for which α and β are real. (c) Find also the set of values of k for which α and β are real and positive. (L)

10. Given that a and b are non-zero constants, prove that if the equations $x^2+ax+b=0$ and $4x^2-ax+6b=0$ have a common root then $35a^2+4b=0$.

11. A straight line of gradient -2 passes through the point with x-coordinate -1 on the curve $y=2x^2-3x+8$. Find the coordinates of the point at which the line meets the curve again.

12. The line $y=2kx$ cuts the curve $y=x^2-4x+1$ at the points A and B. Without finding the coordinates of the points A and B, find the coordinates of the mid-point of the line AB in terms of k.

13. Solve the simultaneous equations $2x-y-1=x-2y=3x^2-8y^2$.

14. Solve the simultaneous equations $3x^2+y^2=7$, $4x^2-xy+y^2=6$.

15. If $P(x)=2x^5+3x^4-8x^3+px^2+q$ has factors $(x+1)$ and $(x-2)$, find the values of p and q. Obtain also the remaining factors.

16. If the roots of the equation $x^3+5x^2+hx+k=0$ are α, 2α and $\alpha+3$, find the values of α, h and k.

17. The quadratic polynomial $P(x)$ leaves a remainder of -6 on division by $(x+1)$, a remainder of -5 on division by $(x+2)$ and no remainder on division by $(x+3)$. Find $P(x)$ and solve the equation $P(x)=0$. (O&C)

18. Factorise the expression $n^4+2n^3-n^2-2n$. Given that n is an integer greater than 1, prove that the expression is divisible by 24.

19. The real numbers a, b, c are such that $b^2<4ac$ and $c>0$. Show that $ax^2+bx+c>0$ for all real numbers x and, conversely, that if $ax^2+bx+c>0$ for all real x, then $c>0$ and $b^2<4ac$. (AEB 1978)

20. For what range of values of c does the system of equations $x^2+xy+y^2=1$, $2x+y=c$ have real solutions? (O)

21. (a) Solve the equations $xy=4$, $x^2+x+y=6$. (b) Find the value of x^6+y^6 in terms of a and b where $x+y=a$, $x^2+y^2=b^2$. Show also that these equations give real values for x and y only if $a^2\leqslant 2b^2$. (W)

22. If $f(x)\equiv ax^3+(a+b)x^2+(a+2b)x+1$ is exactly divisible by $(x+1)$, express b in terms of a, and find the quotient when the division is carried out, expressing the coefficients in terms of a only. Prove that in this case the equation $f(x)=0$ has only one real root if $a^2-6a+1<0$. Show that this inequality implies that $3-2\sqrt{2}<a<3+2\sqrt{2}$.

4 The calculator and numerical work

4.1 Accuracy and types of error

In problems requiring a numerical answer it is sometimes possible to give an exact result, e.g. $1{\cdot}4 \times 2{\cdot}3 = 3{\cdot}22$. In practice, it is much more likely that an approximate solution will be obtained. Such a solution may be of little value unless its degree of accuracy can be estimated. This is done by examining sources of error in the original data and in the method of calculation. In this context possible inaccuracies due to mistakes or blunders are not taken into account. The use of the word *error* is confined to inaccuracies arising in other ways.

The error in calculations based on experimental data depends mainly on the accuracy of the instruments used. If, in an experiment, estimates are made of the errors in the readings, these can be used to find the total possible error in any calculated result.

Example 1 The length and breadth of a rectangular room are 9·7 m and 6·8 m respectively. If the possible error in each of these measurements is $\pm 0{\cdot}05$ m, find the two values between which the floor area of the room must lie. Find the area and give your answer to an appropriate degree of accuracy.

Minimum area of room $= (9{\cdot}65 \times 6{\cdot}75)\,\text{m}^2 = 65{\cdot}1375\,\text{m}^2$
Maximum area of room $= (9{\cdot}75 \times 6{\cdot}85)\,\text{m}^2 = 66{\cdot}7875\,\text{m}^2$
$\therefore$ the area lies between $65{\cdot}1375\,\text{m}^2$ and $66{\cdot}7875\,\text{m}^2$.
Area of room as measured $= (9{\cdot}7 \times 6{\cdot}8)\,\text{m}^2 = 65{\cdot}96\,\text{m}^2$.
Hence, taking possible errors into account the area of the room may be given as $66 \pm 1\,\text{m}^2$.

In some calculations the method itself leads to inaccuracies. For instance, to convert a temperature given in a weather forecast from degrees Celsius to degrees Fahrenheit one method is to 'double and add 30'. This formula is attractive because it is easier to use than the true formula, i.e. 'multiply by 9/5 and add 32', and because for temperatures from -5°C to 25°C the maximum error is only

$\pm 3°F$. Suppose, however, that this conversion formula were used to express an oven temperature of 200°C given in a recipe as 430°F. The error of 38°F in this result is clearly unacceptable. Thus we see that approximate methods must always be used with care, and that if such a process is used in the wrong situation the error in the result may be large.

Consider now a problem such as the division of 37 by 13. We can calculate the answer by long division, giving as many decimal places as necessary, but we cannot obtain an exact answer. For instance, working as far as the third decimal place, $37 \div 13 = 2{\cdot}846\ldots$ The answer 2·846 is said to be *truncated* (i.e. cut short) to 3 decimal places. Since the true value must lie between 2·846 and 2·847, the *truncation error* lies between 0 and 0·001. The result could also be given as $37 \div 13 = 2{\cdot}85$, *rounded* to 2 decimal places. This statement implies that $2{\cdot}845 \leqslant 37 \div 13 < 2{\cdot}855$. Hence the *rounding error* lies between $-0{\cdot}005$ and $+0{\cdot}005$.

To show how to estimate errors in a longer calculation, we consider the division of 37 by 13 performed using 4-figure logarithm tables.

Number	*Logarithm*	*Possible error*
37	1·5682	$\pm 5 \times 10^{-5}$
13	1·1139	$\pm 5 \times 10^{-5}$
2·846	0·4543	$\pm 1 \times 10^{-4}$

Thus the logarithm of $37 \div 13$ lies between 0·4542 and 0·4544. Allowing for rounding errors in the antilogarithm tables, both in the main table and the difference columns, we estimate that $37 \div 13 = 2{\cdot}846 \pm 0{\cdot}002$. Hence we may write $37 \div 13 = 2{\cdot}8$ correct to 1 decimal place, but we cannot be certain that the answer 2·85 is correct to 2 decimal places.

Truncation and rounding errors also arise when working with an electronic calculator. A typical calculator, which displays 8 digits or 6 digits and a 2-digit exponent, produced the following results.

$$37 \div 13 = 2{\cdot}8461538$$
$$3{\cdot}7 \div 13 = 0{\cdot}2846153$$
$$0{\cdot}037 \div 13 = 2{\cdot}84615 \times 10^{-3}.$$

We see that these values are truncated rather than rounded. Thus they are subject to a truncation error of between 0 and 1 in the final digit.

Using the same calculator the result

$$53 \div 13 = 4{\cdot}076923$$

is obtained. The calculator displays 7 digits rather than 8 only because the 8th digit would be a zero. The read-out should be interpreted as 4·0769230 truncated at the final zero.

To estimate error in longer calculations it is necessary to know how numbers are stored in the working register and memory of the calculator. This can be established by tests similar to the following:

Divide 1000 by 7: 142·85714
Subtract 140: 2·8571428.

These results show that a calculator which displays 8 digits may actually work with 10 digits.

In the case of scientific functions many electronic calculators produce rounded values. For example, using a calculator which rounds to 6 significant figures and suppresses unnecessary zeros,

$\lg 5{\cdot}3 = 0{\cdot}724276 \quad \lg 5{\cdot}2 = 0{\cdot}716003$
$\lg 53 = 1{\cdot}72428 \quad \lg 52 = 1{\cdot}716.$

The maximum rounding error in these values is $\pm 0{\cdot}5$ in the 6th significant figure.

Example 2 Allowing for rounding errors, find two values between which $(\lg 7 \div \lg 3)$ must lie. Calculate $\lg 7 \div \lg 3$ and give your answer to an appropriate number of significant figures.

Taking possible rounding errors into account

$$0{\cdot}8450975 < \lg 7 < 0{\cdot}8450985$$
$$0{\cdot}4771205 < \lg 3 < 0{\cdot}4771215$$

$$\therefore \quad \text{maximum value of } \frac{\lg 7}{\lg 3} = \frac{0{\cdot}8450985}{0{\cdot}4771205} = 1{\cdot}7712475$$

$$\text{minimum value of } \frac{\lg 7}{\lg 3} = \frac{0{\cdot}8450975}{0{\cdot}4771215} = 1{\cdot}7712417.$$

By calculator, $\lg 7 \div \lg 3 = 1{\cdot}7712446$
Hence $\lg 7 \div \lg 3 = 1{\cdot}7712$ to 5 s.f.

[In a longer calculation it would probably be unrealistic to give an answer to more than 4 significant figures.]

Thus although a calculator produces more accurate results than 4-figure tables, these results do contain unavoidable rounding and truncation errors. The approximate methods used within the calculator when determining the values of scientific functions may lead to further error, especially at the extremes of the calculator's range. Clearly it is not necessary to make detailed estimates of possible error after every calculation performed. However, it is important to know what degree of accuracy can be expected from the calculating aid one is using, be it mathematical tables, slide rule or calculator. Efforts can then be made to choose methods of working which minimise error.

Exercise 4.1

In questions 1 to 4 find two values between which z must lie, then calculate z giving your answer to an appropriate degree of accuracy.

1. $z = xy$ where $x = 8{\cdot}7 \pm 0{\cdot}1$, $y = 5{\cdot}3 \pm 0{\cdot}1$.

2. $z = x^2 + y^2$ where $x = 7{\cdot}3 \pm 0{\cdot}05$, $y = 6{\cdot}4 \pm 0{\cdot}05$.

3. $z = \frac{1}{x} + \frac{1}{y}$ where $x = 4{\cdot}16 \pm 0{\cdot}01$, $y = 6{\cdot}72 \pm 0{\cdot}01$.

4. $z = \sqrt{(x-y)}$ where $x = 62 \pm 0{\cdot}5$, $y = 45 \pm 0{\cdot}5$.

5. Find two values between which the product pq must lie given that $p = 4{\cdot}26$, $q = 2{\cdot}51$, (a) rounded to 2 decimal places, (b) truncated to 2 decimal places. In each case give the value of pq to an appropriate number of decimal places.

6. Repeat question 5 for the quotient p/q.

7. Use 4-figure logarithm tables to perform the following calculations. Estimate the possible errors and give your answers to an appropriate number of significant figures.

(a) $7{\cdot}34 \times 2{\cdot}69$ (b) $0{\cdot}581 \div 8{\cdot}07$
(c) $(3{\cdot}3)^2 \div 0{\cdot}76$ (d) $\sqrt{(4{\cdot}9 \times 32{\cdot}8)}$.

8. Answer the following questions about the calculator you are using. If possible, compare its performance with that of other calculators.

(a) How many digits are displayed?
(b) Are unnecessary zeros suppressed?
(c) How many digits are stored in the working register?
(d) In basic function operations, i.e. $+$, $-$, $\times$, $\div$, are the results truncated or rounded?
(e) Are the values of scientific functions, such as $\sqrt{x}$, $\lg x$, $\sin x$, truncated or rounded, and to how many figures?
(f) What degree of accuracy for the various calculator functions is claimed by the manufacturer?

9. Use an electronic calculator to perform the following calculations. By estimating the possible errors, give your answers to an appropriate number of significant figures.

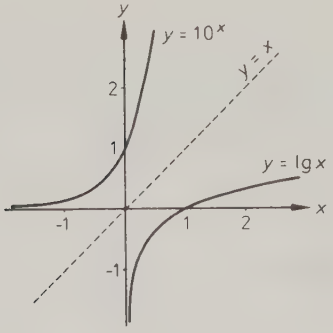
(a) $3{\cdot}6 \times 12{\cdot}7 \div 6{\cdot}84$ (b) $1/\sqrt{(5{\cdot}13)}$
(c) $\sin 40° \times \cos 20°$ (d) $\lg 3{\cdot}6 \div \lg 1{\cdot}8$.

[The answers to this question will vary according to the type of calculator used.]

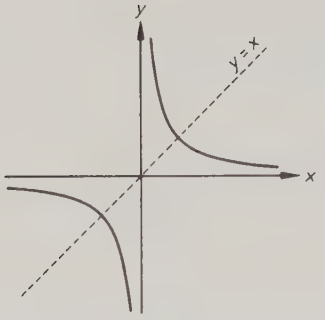
4.2 Some functions and their inverses

Consider a function f which is a one-one mapping of a set X onto a set Y. This means that every element of the range Y is the image of exactly one element of the domain X. Under these conditions it is possible to define an *inverse function* f^{-1} with domain Y and range X such that if $f(a) = b$ then $f^{-1}(b) = a$.

Any scientific calculator provides several examples of inverse functions. For instance, if $\lg a = b$ then $10^b = a$. Thus the function 10^x is the inverse of the function $\lg x$. It follows from this relationship that $10^{(\lg x)} = x$ and $\lg(10^x) = x$. [Discrepancies due to rounding errors may occur when testing such results. For instance, using certain calculators it is found that $10^{(\lg 9)} = 9{\cdot}00001$ and $\lg(10^{0{\cdot}29}) = 0{\cdot}289999$.]

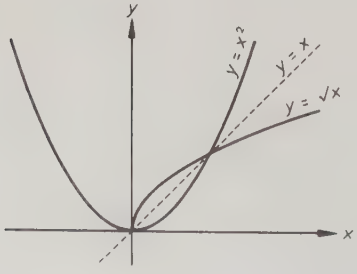


The sketch shows how the graphs $y = \lg x$ and $y = 10^x$ are related. If a point (a, b) lies on the graph $y = \lg x$, then the point (b, a) lies on the graph $y = 10^x$. Hence each graph is the reflection of the other in the line $y = x$.

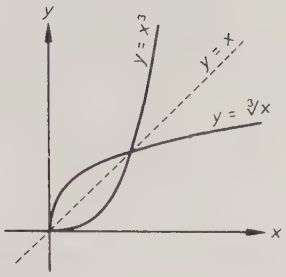


The function $1/x$ is its own inverse, since if $1/a = b$, then $1/b = a$. We find that in this case the graph of $y = 1/x$ is itself symmetrical about the line $y = x$.

The function $f: x \to x^2$ with domain the set of real numbers has no true inverse, since it is not one-one. For instance, the element 4 in the range is the image of both -2 and $+2$, so it is impossible to assign a single value to $f^{-1}(4)$. However, if the domain of the function $f: x \to x^2$ is the set of all real numbers greater than or equal to zero, every element of the range is the image of only one element of the domain. Thus we can define the inverse function as $f^{-1}: x \to \sqrt{x}$.



Comparing the graphs of $y = x^2$ and $y = \sqrt{x}$, we again see symmetry about the line $y = x$, but only for $x \geqslant 0$.



The functions x^y and $x^{1/y}$ are functions of two variables x and y, usually defined only for positive values of x. Using a constant value of y, functions such as x^3 and $\sqrt[3]{x}$ can be studied. From the symmetry of the graphs, we see that x^3 and $\sqrt[3]{x}$ are mutually inverse.

More generally, if $a^k = b$ and $a > 0$, then $b^{1/k} = \sqrt[k]{b} = a$. Hence the function $x^{1/k}$ is the inverse of the function x^k.

The following examples show some of the uses of the calculator functions discussed in this section.

Example 1 Evaluate $\left(2{\cdot}3 + \dfrac{1}{2{\cdot}3}\right)^2$.

By pressing the following keys: [2] [·] [3] [+] [1/x] [=] [x²]

$\left(2{\cdot}3 + \dfrac{1}{2{\cdot}3}\right)^2 = 7{\cdot}4790359$ (truncated to 8 digits).

Example 2 Evaluate $\sqrt[5]{(5{\cdot}6 \times 10^{-4})}$.

By pressing the following keys: [.5] [·] [6] [EXP] [4] [+/−] [x^1/y] [5] [=]

$\sqrt[5]{(5{\cdot}6 \times 10^{-4})} = 0{\cdot}223685$ (rounded to 6 s.f.).

Example 3 Find $\log_2 3$.

Using the formula $\log_a x = \log_{10} x / \log_{10} a$ (see §1.6)

$\log_2 3 = \lg 3 / \lg 2 \approx 1{\cdot}5849616$

Hence $\log_2 3 = 1{\cdot}5850$ (to 5 s.f.).

Alternative method:

$$\begin{aligned} \log_2 3 = x &\Leftrightarrow 2^x = 3 \\ &\Leftrightarrow \lg(2^x) = \lg 3 \\ &\Leftrightarrow x \lg 2 = \lg 3 \\ &\Leftrightarrow x = \lg 3 / \lg 2 \end{aligned}$$

$\therefore$ as before $\log_2 3 = 1{\cdot}5850$ (to 5 s.f.).

Example 4 Solve the equation $2^x . 3^{x-2} = 1$.

$$\begin{aligned} 2^x . 3^{x-2} = 1 &\Leftrightarrow 2^x . 3^x . 3^{-2} = 1 \\ &\Leftrightarrow (2.3)^x = 3^2 \\ &\Leftrightarrow 6^x = 9 \\ &\Leftrightarrow \lg(6^x) = \lg 9 \\ &\Leftrightarrow x \lg 6 = \lg 9 \\ &\Leftrightarrow x = \lg 9 / \lg 6 \approx 1{\cdot}2262954 \end{aligned}$$

$x = 1{\cdot}2263$ (to 5 s.f.)

Exercise 4.2

[Give numerical answers to 4 s.f.]

1. Write down the inverses of the following functions

(a) $f: x \to 3x$ (b) $f: x \to \frac{1}{2}x$ (c) $f: x \to x-1$
(d) $f: x \to 1-x$ (e) $f: x \to 2x+3$ (f) $f: x \to 4(x-1)$

2. Write down the equations of the reflections of the following graphs in the line $y = x$.

(a) $y = 2x$ (b) $y = 2/x$ (c) $y = 2-x$
(d) $y = x-2$ (e) $y = 2^x$ (f) $y = \lg(x-2)$.

3. Using the same pair of axes, draw accurate graphs of the functions x^y and $x^{1/y}$ for $y = 1{\cdot}5$, taking $0 \leqslant x \leqslant 4$.

4. Evaluate the following expressions

(a) $\sqrt{\left(1{\cdot}9 - \dfrac{1}{1{\cdot}9}\right)}$ (b) $\{(0{\cdot}34)^2 + (0{\cdot}27)^2\}^{-5/2}$
(c) $1/(7{\cdot}1 \times 10^{-3})^2$ (d) $\sqrt[3]{(0{\cdot}027 \times 10^4)}$.

5. Evaluate (a) $\log_4 7$, (b) $\log_3 10$, (c) $\log_5 2$

6. Solve the following equations

(a) $2^x = 9$, (b) $10^{3x} = 15$, (c) $8{\cdot}7^x = 3{\cdot}2$.

7. Using the same pair of axes, draw accurate graphs of $y = 3^x$ for $-2 \leqslant x \leqslant 2$ and $y = \log_3 x$ for $0{\cdot}1 \leqslant x \leqslant 10$.

8. Solve the following equations

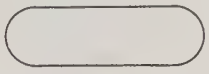
(a) $5^{2x} . 5^{1-x} = 40$ (b) $3^x . 4^{x+1} = 100$
(c) $2^{x-1} . 3^{x-2} = 1$ (d) $12^{x-1} = 3^{x+4}$.

9. By taking logarithms of both sides solve the equations
(a) $4^{x+2} = 7^{x-1}$, (b) $3^{2x} . 4^{1-x} = 13$.

10. Solve the equation $\log_3 x + \log_5 x = 1$.

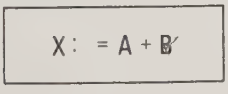
4.3 Simple flow diagrams

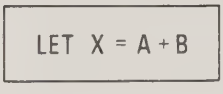
Any mathematical calculation consists of a sequence of steps performed one after the other. A rule or systematic process for carrying out a calculation is sometimes called an *algorithm*. The list of instructions describing how to perform the calculation or algorithm can be referred to as a *program*. In a computer program each step must be within the capabilities of the computer concerned. Programs are usually simpler to follow when they are given in the form of a *flow diagram* or *flow chart*. In a flow diagram the individual steps are written in 'boxes' of various shapes and the order in which the steps are to be performed is indicated by arrows. The shapes used in simple flow diagrams are:



for START and STOP

for other instructions.

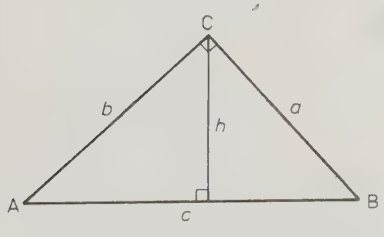
The instructions 

and 

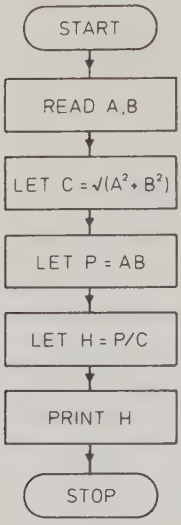
both mean: 'Calculate X, given that $X = A + B$'.

'Assign to X the value of x given in the data'

'Write down the value of X'



Example 1 In $\triangle ABC$, $\angle C = 90°$ and h is the height of the triangle taking AB as base. Carry out the procedure given in the flow diagram to find h given $a = 6$ and $b = 8$.



$A = 6, \quad B = 8$

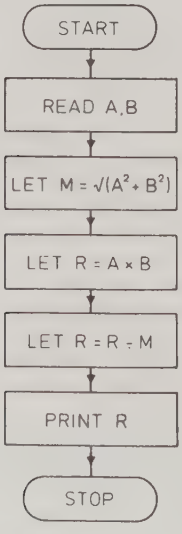
$C = \sqrt{(6^2 + 8^2)} = 10$

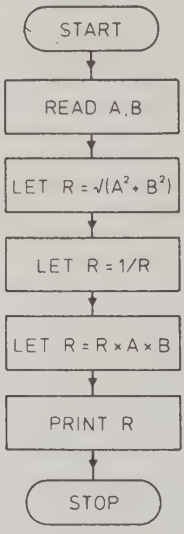
$P = 6 \times 8 = 48$

$H = 48/10 = 4{\cdot}8$

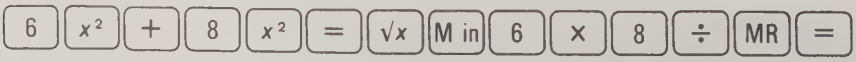
4·8

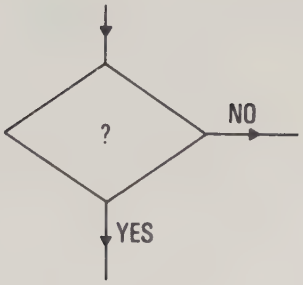
When writing a computer program, introducing a new variable involves using extra storage space in the computer memory. Thus in a flow diagram as few variables as possible are used. For instance, instead of writing 'LET $X = A + B$', we may write 'LET $A = A + B$' if we do not need to use the original value of A again. To illustrate this approach we give below two alternatives to the flow diagram in Example 1, both designed for use with a small calculator. The letters R and M are used to denote the contents of the working register and the memory.





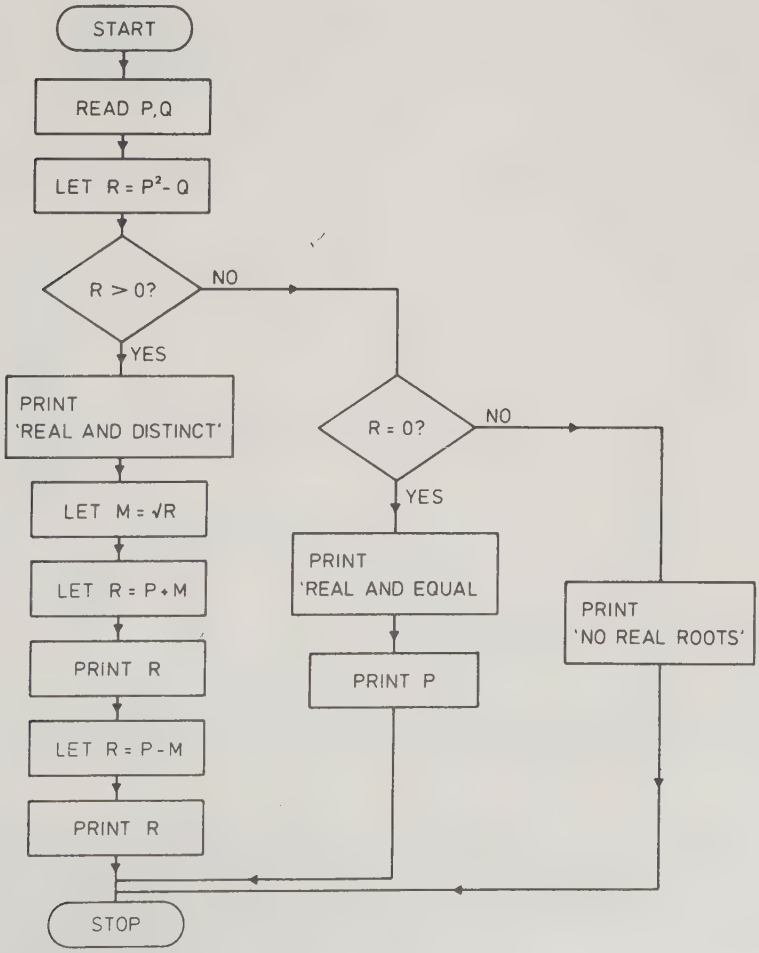
The reader is recommended to test these procedures for himself. In the first case a key sequence similar to the following would be used:





In some calculations the method used depends on the answer to various questions. In a flow diagram these questions are written in *decision boxes*, which have different exits for the answers YES and NO. The use of one of these exits will result in a *jump* to another part of the flow diagram.

Example 2 Construct a flow diagram for a procedure to determine the nature of the roots of the equation $x^2-2px+q=0$ and find their values if real.

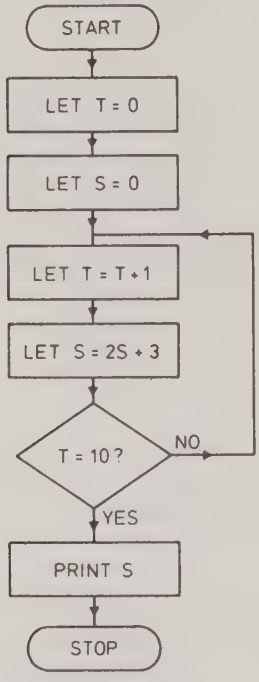


Testing the procedure for $p = 2$ and $q = 3$, we obtain the following results.

Working	*Print-out*
$P = 2, Q = 3$	
$R = 2^2 - 3 = 1$	
$R > 0$	REAL AND DISTINCT
$M = \sqrt{1} = 1$	
$R = 2 + 1 = 3$	3
$R = 2 - 1 = 1$	1

A flow diagram for a repetitive process may contain *loops* as shown in the next example.

Example 3 Construct a flow diagram for finding the sum to 10 terms, S_{10}, of the series $3 + 6 + 12 + 24 + \ldots$ using the relation $S_r = 2S_{r-1} + 3$.



The initial value of S is taken to be 0. Successive values of S and T can be tabulated as follows:

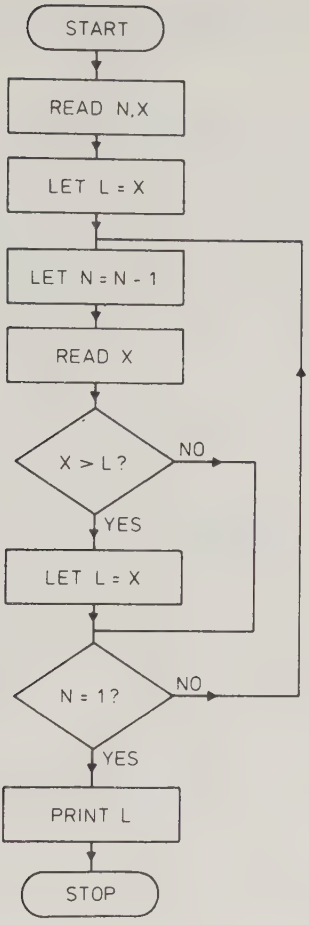
T	S
0	0
1	3
2	9
3	21
4	45
5	93
6	189
7	381
8	765
9	1533
10	3069

Thus the printed value of S is 3069.

In this example T was a *loop counter* used to indicate when the process had been carried out the correct number of times.

Some computer programs require the input of a list of numbers. In this context the instruction 'READ X' means 'assign to X the next number of the list'.

Example 4 Construct a flow diagram for a procedure to find the largest of a set of n real numbers.



If the given set of numbers is 5, −7, 8, 12, −3, 2, then the values of N, X and L are as follows:

N	X	L
6	5	5
5	−7	5
4	8	8
3	12	12
2	−3	12
1	2	12

Thus the print-out is the number 12, the largest element of the given set.

[Note that this flow diagram contains both a jump and a loop.]

Exercise 4.3

1. Carry out the procedure given in Example 1 for
(a) $a = 2, b = 1{\cdot}5$, (b) $a = 3{\cdot}5, b = 12$.

2. Carry out the procedure given in Example 2 for
(a) $p = 1, q = -15$ (b) $p = -0{\cdot}5, q = 0{\cdot}25$
(c) $p = 2{\cdot}3, q = 5{\cdot}3$ (d) $p = 3{\cdot}7, q = -4{\cdot}8$.

In questions 3 to 10 construct flow diagrams for carrying out the given processes.

3. Find and print out the total number of seconds in x hours, y minutes and z seconds.

4. Calculate x given that $a + b/x = c$, printing out suitable messages when $a = c$ or $b = 0$.

5. Print out the examination grade corresponding to a mark of $x\%$ given that the

lowest marks in each grade are: grade 1, 75%; grade 2, 60%; grade 3, 50%. Candidates with less than 50% fail.

6. Arrange three real numbers a, b, c in ascending order of magnitude.

7. Determine the number of real roots of the equation $x^3 - ax^2 - bx + ab = 0$ and print out their values.

8. Find the sum to 8 terms of the series $2 + 6 + 18 + 54 + \ldots$ using the relation $S_r = 3S_{r-1} + 2$.

9. Find the nth term of the Fibonacci series
$1 + 1 + 2 + 3 + 5 + 8 + 13 + \ldots$

10. Find the sum of the squares of a set of n numbers.

In each of the remaining questions construct a flow diagram for performing the given calculation using a pocket calculator with one memory.

11. Find the coordinates of the minimum point on the curve $y = x^2 + hx + k$.

12. Find the real roots, if any, of the equation $ax^2 + bx + c = 0$.

13. Find the coordinates of the point of intersection of the lines $ax + by = 1$ and $cx + dy = 1$, printing out suitable messages when the lines are parallel or coincident.

14. Find an approximate value for π using the result

$$\frac{\pi^2}{6} \approx \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots + \frac{1}{n^2}$$

for any given value of n.

4.4 Efficient methods of calculation

When performing a numerical calculation the main aims are to
(1) avoid mistakes
(2) use as few steps as possible
(3) minimise error
[In work with a calculator effective use of the memory may help to achieve all three aims.]

To avoid mistakes numerical answers should be checked whenever possible.

Example 1 The value 1·5 has been obtained for a root of the equation $2x^3 - x^2 + 9 = 0$. Check this result.

Without detailed calculation it is clear that

$$2(1{\cdot}5)^3-(1{\cdot}5)^2+9>0$$

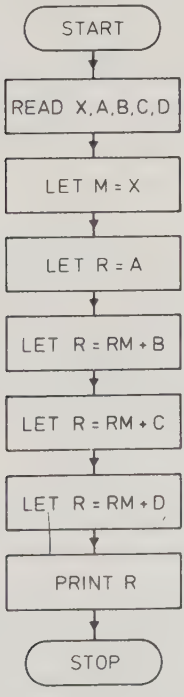
$\therefore$ 1·5 is not a root of the equation.
[In fact the correct value of the root is $-1{\cdot}5$.]

Example 2 A student using a calculator obtained the result $38{\cdot}76\div 0{\cdot}2497 = 6{\cdot}4422\times 10^{-3}$. Is this answer reasonable?

Rough check: $38{\cdot}76\div 0{\cdot}2497 \approx 40\div\frac{1}{4} = 160$
The answer $6{\cdot}4422\times 10^{-3}$ must be incorrect.

[On some calculators this wrong answer is obtained by inadvertently pressing the $\boxed{\div}$ key twice.]

The number of steps required in a calculation will depend on the type of calculator, table book or slide rule used. However, when evaluating a polynomial the method called *nested multiplication* is often the most efficient. For instance, rearranging the polynomial $3x^3-2x^2+4x-1$ as $\{(3x-2)x+4\}x-1$, we can quickly calculate that its value is 23 when $x = 2$.



This flow diagram gives the procedure for evaluating the polynomial ax^3+bx^2+cx+d using a calculator. In each step the same pattern of operations is repeated. Thus the method of nested multiplication can be summarised as follows:

(i) Take the coefficient of the highest power of x.
(ii) Multiply by x and add the next coefficient.
(iii) Repeat (ii) until constant term has been added.

When accuracy is an important consideration, it may be possible to reduce error by choosing a suitable method.

Example 3 Evaluate $5\sqrt{2}$ using 4-figure tables.

$\sqrt{2} = 1{\cdot}414 \pm 0{\cdot}0005 \quad \therefore \quad 5\sqrt{2} = 7{\cdot}07 \pm 0{\cdot}0025$
The possible rounding error is reduced by writing
$5\sqrt{2} = \sqrt{50} = 7{\cdot}071 \pm 0{\cdot}0005$.

Example 4 Evaluate $2 \div 7 \times 32$ using a calculator.

$2 \div 7 = 0{\cdot}2857142$ with error between 0 and 1×10^{-7}.
Assuming that the working register carries no additional digits, $2 \div 7 \times 32 = 9{\cdot}1428544$, subject to an error between 0 and 32×10^{-7}.
Hence the maximum number of significant figures that can be given in the answer is 5,

$$2 \div 7 \times 32 = 9{\cdot}1429 \quad \text{(to 5 s.f.)}$$

Accuracy may be improved by writing

$$2 \div 7 \times 32 = 2 \times 32 \div 7 = 9{\cdot}1428571.$$

Since the maximum possible truncation error is now 1×10^{-7}, we know that $2 \div 7 \times 32 = 9{\cdot}142857$ (to 7 s.f.)

Example 5 Find $\sqrt{(13{\cdot}2^2 - 12{\cdot}6^2)}$ using (a) 4-figure tables, (b) a calculator.

(a) $\sqrt{(13{\cdot}2^2 - 12{\cdot}6^2)} = \sqrt{(174{\cdot}2 - 158{\cdot}8)} = \sqrt{(15{\cdot}4)} = 3{\cdot}924$

Allowing for possible rounding errors of $\pm 0{\cdot}05$ in the squares of 13·2 and 12·6, the true value of the expression must lie between $\sqrt{(15{\cdot}3)} \approx 3{\cdot}912$ and $\sqrt{(15{\cdot}5)} \approx 3{\cdot}937$.
Hence $\sqrt{(13{\cdot}2^2 - 12{\cdot}6^2)} = 3{\cdot}9$ (to 2 s.f.)
A more accurate result is obtained by writing

$$\begin{aligned}\sqrt{(13{\cdot}2^2 - 12{\cdot}6^2)} &= \sqrt{\{(13{\cdot}2 + 12{\cdot}6)(13{\cdot}2 - 12{\cdot}6)\}} \\ &= \sqrt{\{25{\cdot}8 \times 0{\cdot}6\}} = \sqrt{(15{\cdot}48)} = 3{\cdot}934\end{aligned}$$

$\therefore \quad \sqrt{(13{\cdot}2^2 - 12{\cdot}6^2)} = 3{\cdot}93$ (to 3 s.f.)

(b) Since a calculator gives exact values for the squares of 13·2 and 12·6,

$\sqrt{(13{\cdot}2^2 - 12{\cdot}6^2)} = 3{\cdot}9344631$ (truncated to 8 digits)

Hence $\sqrt{(13{\cdot}2^2 - 12{\cdot}6^2)} = 3{\cdot}934463$ (to 7 s.f.)

Exercise 4.4

1. Draw flow diagrams for evaluating the polynomials (a) $2x^2 - 7x + 3$, (b) $x^3 + 3x^2 - 5x - 4$ and (c) $3x^4 - x^3 - 4x + 8$, by the method of nested multiplication. Test your procedures for $x = -1$ and $x = 2$.

2. Use a calculator to evaluate the following
(a) $3x^2-5x+1$ for $x = 2{\cdot}8$ and $x = -0{\cdot}6$
(b) $2x^3+x^2-8x-3$ for $x = -1{\cdot}3$ and $x = 2{\cdot}67$
(c) $x^5-3x^3+10x+5$ for $x = 1{\cdot}54$ and $x = -0{\cdot}09$
(d) $2x^4-x^3-6x^2+3x$ for $x = 3{\cdot}02$ and $x = -1{\cdot}4$.

In the remaining questions in the exercise, perform the given calculations by the most direct method, then decide whether there is a way of producing a more accurate result. Consider, if available, the use of 4-figure tables and various types of calculator. Illustrate your answers with flow diagrams where appropriate.

3. Evaluate (a) $3\sqrt{10}$, (b) $\frac{1}{2}\sqrt{6}$, (c) $4 \div 7{\cdot}36$.

4. Evaluate (a) $\dfrac{8{\cdot}71}{4{\cdot}03 \times 5{\cdot}64}$, (b) $\dfrac{1}{3{\cdot}61} \times \dfrac{1}{0{\cdot}52} \times 7{\cdot}3$.

5. Evaluate (a) $\sqrt{(25{\cdot}3^2-24{\cdot}7^2)}$, (b) $\sqrt{(25{\cdot}3^2+24{\cdot}7^2)}$.

6. Evaluate (a) $2 \lg 6$, (b) $(1{\cdot}76)^4$, (c) $10^{4 \lg 1{\cdot}76}$.

7. Find the sum of the first 8 terms of the series $1+x+x^2+x^3+\ldots$ given (a) $x = 2$, (b) $x = -3{\cdot}4$, (c) $x = 0{\cdot}8136$.

8. Find the sum of the series $1+2x+3x^2+\ldots$ as far as the first term which is less than 0·1, given (a) $x = 0{\cdot}5$, (b) $x = 0{\cdot}42$, (c) $x = 0{\cdot}61$.

[As a rough guide only, answers correct to 4 s.f. are provided for this exercise.]

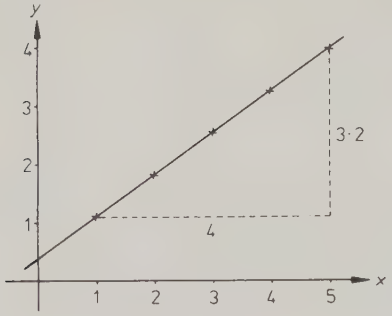
4.5 Reduction of laws to straight line form

In §2.5 we saw that an equation of the form $y = mx+c$ represents a straight line, where m is the gradient and c is the intercept on the y-axis. If a table of values of variables x and y leads to a straight line graph, we can use the gradient and the intercept on the y-axis to write down the law connecting x and y.

Example 1 Some values of the variables x and y are given in the table below:

x	1	2	3	4	5
y	1·2	2·0	2·8	3·6	4·4

Find a relationship between the variables, giving y in terms of x.



Plotting y against x, a straight line graph is obtained. Hence the relationship between x and y takes the form $y = mx + c$. The intercept on the y-axis is 0·4

$$\therefore \quad c = 0{\cdot}4.$$

The gradient can be found using the points $(1, 1{\cdot}2)$ and $(5, 4{\cdot}4)$

$$\therefore \quad m = \frac{4{\cdot}4 - 1{\cdot}2}{5 - 1} = \frac{3{\cdot}2}{4} = 0{\cdot}8.$$

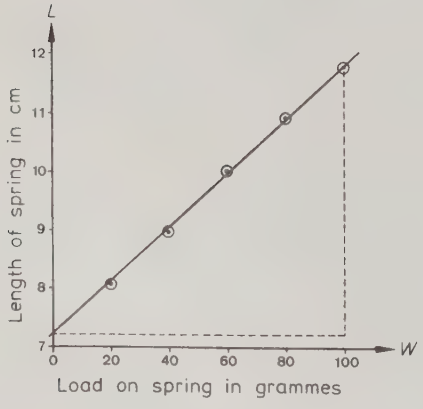
Hence the required relationship is $y = 0{\cdot}8x + 0{\cdot}4$.

If a graph is drawn from experimental data and the points on it all lie close to a straight line, we often assume that the relationship between the variables is approximately linear. We then draw the line that best fits the data and use points on it to determine the law connecting the variables. When finding the gradient of the line, errors are minimised by choosing two points near the ends of the line whose coordinates are easily read from the graph.

Example 2 The following observations were recorded when weights were suspended at the end of a coiled spring.

Load on spring in grammes, W	20	40	60	80	100
Length of spring in cm, L	8·1	9·0	10·0	10·9	11·8

Find an expression for L in terms of W, assuming that the relationship is linear.



The points representing the observations all lie close to a straight line, which cuts the vertical axis at $L = 7{\cdot}2$. Using the points $(0, 7{\cdot}2)$ and $(100, 11{\cdot}8)$ on the line, its gradient

$$= \frac{11{\cdot}8 - 7{\cdot}2}{100 - 0}$$

$$= \frac{4{\cdot}6}{100} = 0{\cdot}046.$$

Hence the relationship between L and W is approximately

$$L = 0{\cdot}046W + 7{\cdot}2.$$

[When estimating error in a result of this kind, it is necessary to consider the range of possible positions of the 'best' straight line graph, allowing for experimental errors.]

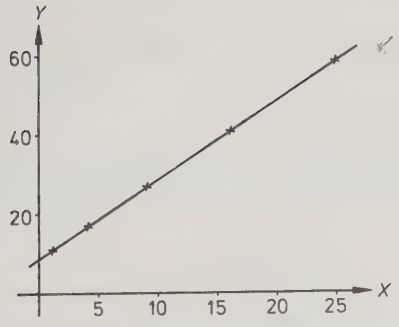
When variables are connected by a non-linear law, it is sometimes possible to find related variables which obey a linear law.

Example 3 The table below gives values of the variables x and y, which are related by an equation of the form $y^2 = ax^2 + b$. Plot y^2 against x^2 and use your graph to obtain the values of a and b.

x	1	2	3	4	5
y	3·32	4·12	5·20	6·40	7·68

Let $X = x^2$ and $Y = y^2$, then working to 3 s.f. the values of X and Y are:

X	1	4	9	16	25
Y	11·0	17·0	27·0	41·0	59·0



The intercept on the vertical axis is at $Y = 9$. Using the points $(1, 11)$ and $(25, 59)$ the gradient is

$$\frac{59-11}{25-1} = \frac{48}{24} = 2$$

$$\therefore \quad Y = 2X + 9.$$

Hence the variables x and y are related by the equation $y^2 = ax^2 + b$, where $a = 2$ and $b = 9$.

Logarithms are often used to express experimental laws in linear form.

Example 4 The following values of x and y are believed to obey a law of the form $y = ab^x$, where a and b are constants:

x	1	2	3	4	5
y	3·8	9·2	22·1	53·1	127·4

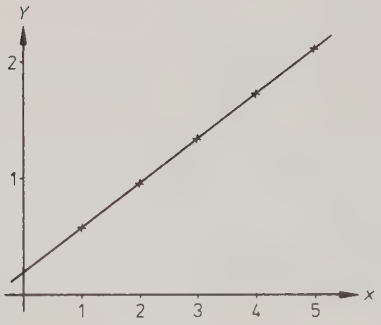
Show graphically that this is so and estimate the values of a and b correct to 1 decimal place.

Assuming that $y = ab^x$, then $\lg y = \lg(ab^x)$,

$$\text{i.e.} \quad \lg y = x \lg b + \lg a.$$

Hence plotting $\lg y$ against x should produce a straight line graph with gradient $\lg b$ and intercept $\lg a$.

x	1	2	3	4	5
$Y = \lg y$	0·58	0·96	1·34	1·73	2·11



The intercept on the vertical axis is at $Y = 0{\cdot}2$

$\therefore \quad \lg a \approx 0.2.$

Hence $a = 1{\cdot}6$ (to 1 d.p.)

Using the points $(1, 0{\cdot}58)$ and $(5, 2{\cdot}11)$ on the line, the gradient $= \dfrac{2{\cdot}11 - 0{\cdot}58}{5 - 1}$

$= 0{\cdot}3825$

$\therefore \quad \lg b \approx 0{\cdot}3825$ and hence $b = 2{\cdot}4$ (to 1 d.p.)

$\therefore$ x and y obey a law of the form $y = ab^x$, where $a = 1{\cdot}6$ and $b = 2{\cdot}4$ (to 1 d.p.).

Example 5 The following readings were obtained in an experiment:

x	20	40	60	80	100
y	55	150	280	430	600

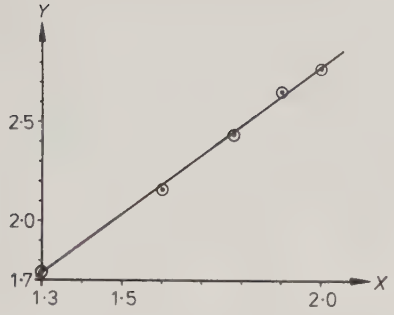
Show graphically that, allowing for small errors of observation, there is a relation between y and x of the form $y = ax^k$. Find approximate values for a and k.

Assuming that $y = ax^k$, then $\lg y = \lg(ax^k)$,

$$\text{i.e.} \quad \lg y = k \lg x + \lg a.$$

Hence plotting $\lg y$ against $\lg x$ should produce a straight line graph with gradient k and intercept $\lg a$.

$X = \lg x$	1·301	1·602	1·778	1·903	2·000
$Y = \lg y$	1·740	2·176	2·447	2·633	2·778



[In order to make full use of the graph paper the X-axis is marked from 1·3 to 2·0 and the Y-axis from 1·7 to 2·8. A suitable scale on both axes would then be 0·1 unit: 2 cm. This means that the true intercept on the Y-axis is not shown. Thus $\lg a$ cannot be read directly from the graph.]

The graph passes through the points (1·3, 1·73) and (2, 2·78)

$$\therefore \quad \text{the gradient} = \frac{2{\cdot}78 - 1{\cdot}73}{2 - 1{\cdot}3} = \frac{1{\cdot}05}{0{\cdot}7} = 1{\cdot}5.$$

Hence X and Y obey a law of the form $Y = 1{\cdot}5X + \lg a$.
Substituting $X = 2$, $Y = 2{\cdot}78$, $2{\cdot}78 = 3 + \lg a$

$$\therefore \quad \lg a = -0{\cdot}22 \quad \text{and} \quad a \approx 0{\cdot}60$$

Hence there is a relation between y and x of the form $y = ax^k$, where the approximate values of a and k are 0·60 and 1·5 respectively.

Exercise 4.5

[Give answers to 2 significant figures.]

1. By using the given table of values to draw a graph, find the relationship between the variables x and y in the form $y = mx + c$.

x	0	2	4	7	9
y	0·6	1·2	1·8	2·7	3·3

2. In an electrical experiment the following results were obtained.

V	86	80	70	60	50	40	30
R	282	267	252	231	208	176	154

Determine the law connecting R and V in the form $R = aV + b$.

3. In a test to determine the efficiency of a crane the following values were obtained for the effort E required to raise a load L. Find the law connecting E and L in the form $E = aL + b$.

L	10	20	30	40	50	60	70	80	90	100
E	1·0	1·7	2·1	2·6	3·2	3·8	4·2	4·9	5·5	6·0

4. In an experiment the following observations were recorded:

T	0	12	20	30	42	48	60
θ	7·6	3·9	1·1	−1·9	−5·7	−7·8	−11·8

Find the law connecting T and θ in the form $\theta = T_0 + kT$.

5. The table below gives experimental values of the variables u and v, which are connected by the equation $\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$, where f is a constant. By plotting the reciprocal of v against the reciprocal of u, estimate the value of f.

u	20	25	30	40	50
v	79	45	34	27	23·5

6. The following values for x and y have been found in an experiment:

x	1	2	3	4	5
y	8.29	11·2	8·67	0·84	−12·5

By plotting y/x against x, verify that x and y are connected by a law of the form $y = ax^2 + bx$ and find approximate values for a and b.

7. Two quantities, x and y, are known to be connected by a law of the form $x^n y = k$, where n and k are constants. Using the given table of values, plot $\lg y$ against $\lg x$ and hence find approximate values for n and k.

x	2	3	4	5	6
y	3·54	1·92	1·25	0·90	0·68

8. The given values of x and y are believed to obey a law of the form $y = ab^x$, where a and b are constants. Show graphically that this is so and estimate the values of a and b.

x	1	2	3	4	5
y	0·91	1·27	1·78	2·50	3·50

9. The table below gives experimental values of x and y, which are known to be related by a law of the form $a/x + b/y = 1$. By drawing a suitable graph find approximate values for a and b.

x	1	2	3	4	5
y	−3·25	−19·1	28·8	12·7	9·7

10. The variables x and y tabulated below are believed to satisfy a relationship of the form $y = a(x+2)^n$, where a and n are constants. Show graphically that this is so and obtain approximate values for a and n.

x	0	1	2	3	4	5
y	1·25	0·56	0·31	0·20	0·14	0·10

(AEB 1977)

Exercise 4.6 (*miscellaneous*)

1. The base and height of a triangle are measured as 16 cm and 23 cm respectively. If the possible error in these measurements is $\pm 0{\cdot}2$ cm, find two values between which the area of the triangle must lie. Find the area and give your answer to an appropriate degree of accuracy.

2. The radius of a circle is 5·95 cm rounded to 2 decimal places. Calculate the circumference by each of the methods given below. By estimating the possible error in each case give the answer to the appropriate number of significant figures.
(a) Use 4-figure tables taking $\lg \pi = 0{\cdot}4971$.
(b) Take $\pi = 22/7$ and use no calculating aids.
(c) Use a calculator and its value for π.

3. Find the inverses of the following functions.
(a) $f: x \to 1-2x$, (b) $g: x \to 5^x$, (c) $h: x \to -1/x$.
4. Find $\log_7 12$.

5. Solve the equation $3^{2x} - 5 \cdot 3^x + 4 = 0$.

6. Solve the equation $\log_3 x = \log_x 5$.

7. Suggest a relationship of the form $z = ap + 10b$, where a and b are integers, given the following experimental data.

p	145	160	170	185	195	200
z	65	170	225	300	360	410

8. Explain how a straight line graph of the form $y = mx + c$ may be drawn to represent the following relationships between variables u and v, where a and n are constants
(a) $u^2 = av^2 + n$ (b) $v = nu(a+u)$
(c) $u = av^n$ (d) $v = na^u$.

9. The following values of x and y are believed to obey a law of the form $y = kb^x$ where k and b are constants:

x	1	2	3	4	5
y	0·8	2·3	6·7	20·2	60·9

Show graphically that this is so and determine approximate values of k and b.
(AEB 1976)

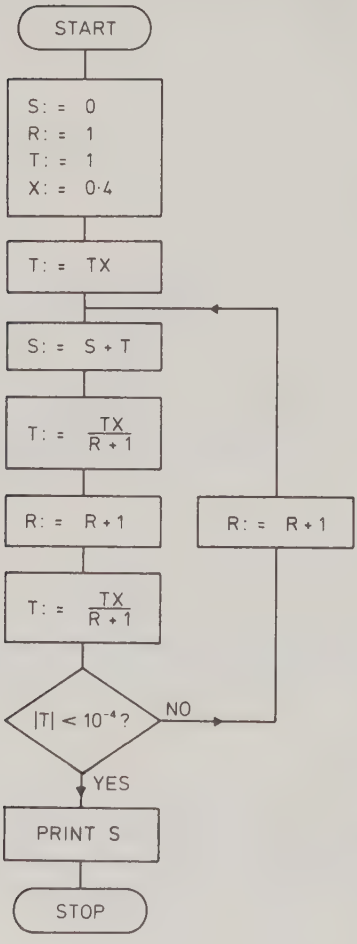
10. The table shows approximate values of a variable y corresponding to certain values of another variable x. By drawing a suitable linear graph, verify that these values of x and y satisfy approximately a relationship of the form $y = ax^k$. Use your graph to find approximate values of the constants a and k.

x	5	10	15	20	25	30
y	45	63	77	89	100	110

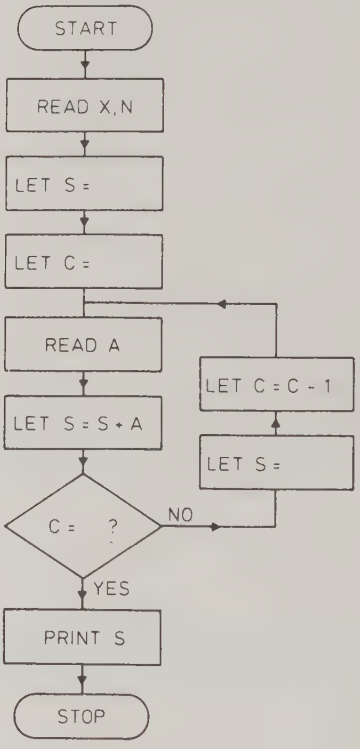
(L)

11. Carry out the procedure given in the flow diagram below, tabulating the successive values of S, T and R as you proceed. State the printed value of S to the number of decimal places that you consider appropriate. (JMB)

Question 11

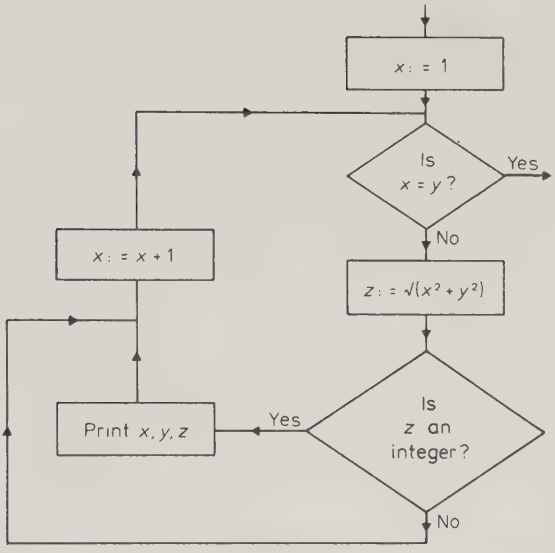


Question 12



12. Copy and complete the given flow diagram designed to evaluate the polynomial $a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$. Explain the function of the instruction 'READ A'. Demonstrate the use of the procedure to evaluate $2x^3 - 3x^2 + x - 7$ when $x = 2$, tabulating the values assigned to S, C and A throughout.

13. A *part* of a flow diagram is shown below. The purpose of the flow diagram is to find sets of positive integers x, y, z such that $x^2 + y^2 = z^2$, where $x < y$.



When the current value of y is 12, work through this section of the diagram, showing any values printed.

Complete the flow diagram so that y takes all integer values $1, 2, 3, \ldots, N$ in turn, where N is a positive integer to be read as data at the start. (C)

14. Draw a flow diagram for a procedure to determine whether a given integer N is prime. [You may assume that N is greater than 1.]

15. Draw a flow diagram for the process of reading 100 positive numbers into a computer, calculating the largest difference between any pair of these numbers and printing out this difference. (AEB 1975)

16. Construct a flow diagram to calculate the values of $3x^2 - 4x + 2$ using nested multiplication for $x = -1(0{\cdot}2)2$ (i.e. for $x = -1, -0{\cdot}8, -0{\cdot}6, \ldots, 1{\cdot}6, 1{\cdot}8, 2$) and to print out the least of these values.

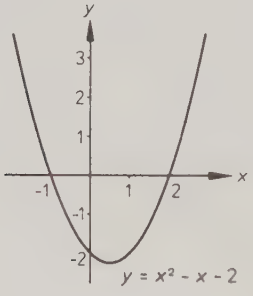
5 Limits and differentiation

5.1 The limit of a function

This chapter deals with the foundations of the study of calculus. The basis of this branch of mathematics is the idea of a limit.

Consider the function $f(x) = x^2 - x - 2$.

x	-1	$-0{\cdot}5$	$-0{\cdot}2$	$-0{\cdot}1$	$-0{\cdot}01$	$0{\cdot}01$	$0{\cdot}1$	$0{\cdot}2$	$0{\cdot}5$	1
$f(x)$	0	$-1{\cdot}25$	$-1{\cdot}76$	$-1{\cdot}89$	$-1{\cdot}9899$	$-2{\cdot}0099$	$-2{\cdot}09$	$-2{\cdot}16$	$-2{\cdot}25$	-2



The table shows that as the value of x approaches zero, from either above or below, then the value of $f(x)$ approaches -2. Thus -2 is called the limit of $f(x)$ as x approaches zero. Using the symbol $\rightarrow$, which is read 'tends to', we write:

$$\text{as } x \rightarrow 0,\ f(x) \rightarrow -2.$$

In general, if, as we consider values of x closer and closer to $x = a$, the value of a function $f(x)$ approaches a finite value l, then l is called the *limit* of $f(x)$ as x tends to a. This may be written:

$$\lim_{x \rightarrow a} f(x) = l \quad \text{or} \quad \text{as } x \rightarrow a, f(x) \rightarrow l.$$

Example 1 Find the limit as $h \rightarrow 0$ of $2 - 3h + h^2$.

As $h \rightarrow 0$, $2 - 3h + h^2 \rightarrow 2$.

Example 2 Find $\lim_{x \to 4} \left(\frac{x+3}{x-3} \right)$.

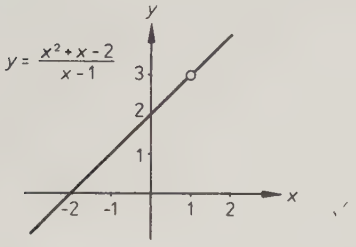
$$\lim_{x \to 4} \left(\frac{x+3}{x-3} \right) = \frac{4+3}{4-3} = 7.$$

As shown in these examples, we find that for many functions arising in elementary mathematics, $f(x) \to f(a)$ as $x \to a$. However, the notion of the limit of a function $f(x)$ as $x \to a$ is most useful when $f(x)$ is undefined at $x = a$.

Example 3 Find $\lim_{x \to 1} \left(\frac{x^2+x-2}{x-1} \right)$.

Since there is no definition of division by zero, the function $f(x) = \frac{x^2+x-2}{x-1}$ is not defined for $x = 1$.

However, when $x \neq 1$, $f(x) = \frac{(x+2)(x-1)}{x-1} = x+2$.



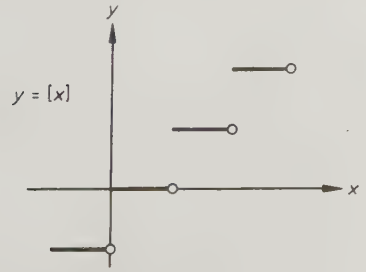
Hence we can obtain values of $f(x)$ as close as we like to 3, by taking values of x closer and closer to 1

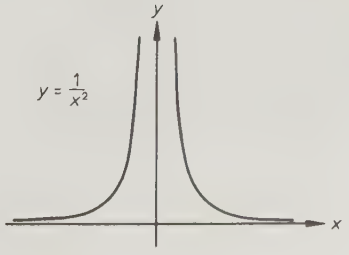
$$\therefore \lim_{x \to 1} \left(\frac{x^2+x-2}{x-1} \right) = \lim_{x \to 1} \left(x+2 \right) = 3.$$

If x is any real number, $[x]$ is defined as the greatest integer less than or equal to x, e.g. $[3\frac{3}{4}] = 3$, $[5] = 5$ and $[-1{\cdot}7] = -2$.

Example 4 Discuss the function $f(x) = [x]$ as $x \to 1$.

For values of x close to 1, but less than 1, $[x] = 0$.
For values of x close to 1, but greater then 1, $[x] = 1$
$\therefore$ $f(x) = [x]$ approaches no single value as $x \to 1$.
Hence, although $[x]$ is defined at $x = 1$, where it takes the value 1, the limit of $[x]$ as $x \to 1$ does not exist.





Example 5 Discuss the function $f(x) = 1/x^2$ as $x \to 0$.

As we examine values of x closer and closer to 0, $f(x)$ takes increasingly large values. Thus $f(x)$ has no limit as x tends to 0.

We may describe the behaviour of $f(x) = 1/x^2$ by writing:
as $x \to 0, f(x) \to \infty$, where $\to \infty$ is read 'tends to infinity'.

However, since a limit is a finite number, we may *not* state that the limit of $f(x)$ is infinity. It is sometimes convenient to say that $f(x)$ is infinite at $x = 0$, but it is *not* usually considered correct to write $f(0) = \infty$.

The concept of a limit may be extended to the value a function $f(x)$ approaches as x takes increasingly large values.

Example 6 Find the limits of $f(x) = 1/x$ as $x \to \infty$ and as $x \to -\infty$.

As x takes increasingly large positive values, $f(x)$ takes smaller and smaller positive values
$\therefore$ as $x \to \infty, f(x) \to 0$ from above.
Similarly, as x takes increasingly large negative values, $f(x)$ takes smaller and smaller negative values
$\therefore$ as $x \to -\infty, f(x) \to 0$ from below.

The following properties of limits are often useful when considering more complicated functions.

If, as $x \to a, f(x) \to l$ and $g(x) \to k$, then
(a) $f(x)+g(x) \to l+k$ (b) $f(x)-g(x) \to l-k$
(c) $f(x).g(x) \to lk$ (d) $f(x)/g(x) \to l/k$ $(k \neq 0)$.

Example 7 Find the limit of $f(x) = \dfrac{3x}{x+3}$ as $x \to \infty$.

Dividing numerator and denominator by x,

$$f(x) = \frac{3x}{x+3} = \frac{3}{1+3/x}.$$

As $x \to \infty, 3/x \to 0$ $\therefore$ as $x \to \infty, f(x) \to 3$.

Exercise 5.1

1. Find the limit as $x \to 3$ of
(a) $2x+3$, (b) $3-x$, (c) x^2+1, (d) $1-x^2+x^4$.

2. Find the limit as $x \to 2$ of

(a) $\dfrac{1}{x+1}$, (b) $1+\dfrac{1}{x}$, (c) $\dfrac{x-4}{x+2}$, (d) $\dfrac{x^2-4}{x-2}$.

3. Find the limit as $h \to 0$ of

(a) $3-h+h^2$, (b) $\dfrac{h-2h^2}{h}$, (c) $\dfrac{3h^3+2h^2}{h^2}$, (d) $\dfrac{h^3+1}{h+1}$.

4. Find the limit as $x \to 0$ of

(a) $\dfrac{(x+1)^2-(x-1)^2}{2x}$, (b) $\sqrt{\left(\dfrac{8x-5x^2+3x^3}{1-\{1-2x\}}\right)}$.

5. Discuss the function $f(x) = 1-[x]$ as $x \to 1$.

6. Discuss the function $f(x) = 1/x$ as $x \to 0$.

7. Discuss the values of the following functions as $x \to k$.

(a) $f(x) = (x-k)^2$, (b) $g(x) = \dfrac{1}{(x-k)^2}$, (c) $h(x) = \dfrac{1}{x-k}$.

8. Find the limit as $x \to \infty$ of

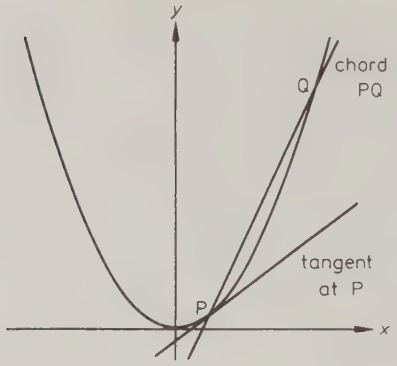
(a) $\dfrac{1}{x+1}$, (b) $\dfrac{1}{x^2}$, (c) $\dfrac{2x}{x-1}$, (d) $\dfrac{2x^2}{x^2-1}$.

9. Find (a) $\lim\limits_{x \to -1} \dfrac{3x^2+x-2}{2x^2+5x+3}$, (b) $\lim\limits_{x \to \infty} \dfrac{3x^2+x-2}{2x^2+5x+3}$,

(c) $\lim\limits_{x \to 1} \dfrac{x^3-1}{2x^3-3x+1}$, (d) $\lim\limits_{x \to -\infty} \dfrac{x^3-1}{2x^3-3x+1}$.

5.2 The gradient of a curve

In §2.4 gradient was defined as a quantity which indicates the direction of a straight line. It was found that the gradient of the straight line passing through the points (x_1, y_1) and (x_2, y_2) is $(y_2-y_1)/(x_2-x_1)$. However, since the direction of a curve is not constant, there is no similar formula for the gradient of a curve. Instead, the gradient of the curve $y = f(x)$ at the point $x = a$ is defined to be the gradient of the tangent to the curve at that point.



Consider the gradient of the curve $y = x^2$ at the point $P(1, 1)$. We cannot find the gradient of the tangent at $(1, 1)$ by the methods of Chapter 2, since we have the coordinates of only one point on it. However, we can obtain an approximate value for its gradient, by finding the gradient of a chord PQ, where Q is a point on the curve $y = x^2$ close to P. The table below gives successive approximations to the gradient of the tangent at P, obtained by taking Q closer and closer to P.

Value of x at Q	Value of y at Q	Gradient of PQ
2	4	3
1·5	2·25	2·5
1·1	1·21	2·1
1·01	1·0201	2·01
1·001	1·002001	2·001

This suggests that as Q approaches P, the values of the gradient of PQ approach 2.

To avoid the numerical work above, consider the point Q, where $x = 1+h$, $y = (1+h)^2$, then

$$\text{gradient of } PQ = \frac{(1+h)^2-1}{(1+h)-1} = \frac{2h+h^2}{h} = 2+h.$$

As $h \to 0$, $Q \to P$ and the gradient of PQ approaches that of the tangent at P.

$$\therefore \quad \text{gradient of tangent at } P = \lim_{h \to 0} (\text{gradient of } PQ) = \lim_{h \to 0} (2+h) = 2.$$

This limiting process also provides a method for finding the gradient of the curve $y = x^2$ at a general point $P(x, x^2)$.
Let Q be the point $(x+h, \{x+h\}^2)$.

The gradient of PQ is then $\dfrac{(x+h)^2-x^2}{(x+h)-x} = 2x+h$

$$\therefore \quad \text{gradient of tangent at } P = \lim_{h \to 0} (2x+h) = 2x.$$

Hence the gradient of the curve $y = x^2$ at any point is given by the function $2x$. This function is sometimes referred to as the *gradient function* for the curve.

In general, for the curve $y = f(x)$, if P is the point $(x, f(x))$ and Q is the point $(x+h, f(x+h))$, then the gradient of $PQ = \dfrac{f(x+h)-f(x)}{(x+h)-x} = \dfrac{f(x+h)-f(x)}{h}$.

$\therefore$ the gradient of the curve at any point is given by

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}, \quad \text{if this limit exists.}$$

One important application of this principle is to the motion of a particle in a straight line. The velocity of such a particle is the rate at which its displacement from a fixed point is changing.

Example 1 An object moves in a straight line so that its displacement from its starting point after t seconds is s metres, where $s = 20t - 3t^2$. If the object passes through the points P and Q when $t = a$ and $t = a+h$, respectively, find the average velocity of the object as it moves from P to Q. Deduce its velocity at the instant it passes through P.

The displacement PQ (in metres)

$= \{20(a+h) - 3(a+h)^2\} - \{20a - 3a^2\}$

$= 20a + 20h - 3a^2 - 6ah - 3h^2 - 20a + 3a^2$

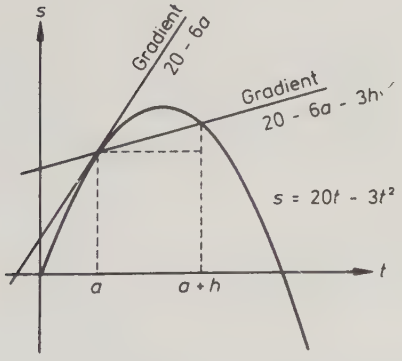
$= 20h - 6ah - 3h^2$.

The time taken to travel from P to $Q = h$ seconds

$\therefore$ the average velocity $= (20 - 6a - 3h)\,\text{m s}^{-1}$.

The velocity at P is the limiting value of this average velocity as $h \to 0$.

$\therefore$ the velocity of the object as it passes through P is $(20 - 6a)\,\text{m s}^{-1}$.



By comparing this example with the earlier work in this section, we see that the velocity of the particle at P is the same as the gradient of the graph $s = 20t - 3t^2$ at the point $t = a$.

Exercise 5.2

1. Write down a series of approximations to the gradient of the tangent at the point $P(0, 1)$ to the curve $y = 1 - x^2$ by finding the gradient of PQ, where Q is the point on the curve given by $x = 1, 0{\cdot}5, 0{\cdot}1, 0{\cdot}01, 0{\cdot}001$.

2. Repeat question 1 for the curve $y = 2x^2 - x + 1$.

3. If P and Q are the points on the curve $y = x^2 + 3$ with x-coordinates 1 and $1+h$ respectively, find the gradient of PQ. Deduce the gradient of the tangent to the curve at P.

4. Use the method of question 3 to find the gradients of the following curves at the given points

(a) $y = 3x^2$ at $(1, 3)$, (b) $y = x - x^2$ at $(2, -2)$,
(c) $y = 2x^2 + x + 1$ at $(0, 1)$, (d) $y = x^3$ at $(1, 1)$.

5. Find the gradient of the line joining the points $P(x, 3x^2 - 1)$ and $Q(x+h, 3\{x+h\}^2 - 1)$ on the curve $y = 3x^2 - 1$. By letting h tend to zero, find the gradient function for this curve.

6. Using the method of question 5 find the gradient functions for the following curves:

(a) $y = x^2 - x$, (b) $y = 1/x$, (c) $y = x^3$.

7. An object moves in a straight line so that its distance from its starting point after t seconds is s metres, where $s = 4t^2$. If the object passes through the points P and Q when $t = a$ and $t = a + h$ respectively, find the average velocity of the object as it moves from P to Q. Deduce its velocity at the instant it passes through P.

8. Repeat question 7 for (a) $s = 3t$, (b) $s = 12t - 5t^2$.

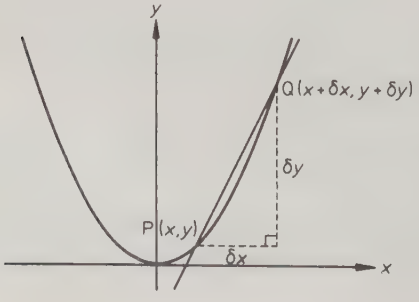
5.3 Differentiation from first principles

The process of finding the gradient function described in the previous section is called *differentiation* from first principles. For the curve $y = f(x)$ the gradient function is called the derived function or *derivative* of $f(x)$ and denoted by $f'(x)$, so that

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

For example, it has already been shown that if $f(x) = x^2$, then $f'(x) = 2x$.

An alternative expression for the derivative of the function $y = f(x)$ is obtained by using the Greek letter δ, 'delta', to mean 'a small increase in'.



Consider the points $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ on the curve $y = x^2$, where δx is a small increase in x and δy is the corresponding small increase in y.

At P $\quad y = x^2$

At Q $\quad y+\delta y = (x+\delta x)^2 = x^2+2x\delta x+(\delta x)^2$

$\therefore \quad \delta y = 2x\delta x+(\delta x)^2.$

The gradient of $PQ = \dfrac{\delta y}{\delta x} = 2x+\delta x.$

As $\delta x \to 0, \dfrac{\delta y}{\delta x} \to 2x$

$\therefore$ the gradient of the tangent at P is $2x$.

In this notation the derivative of y with respect to x is denoted by $\dfrac{dy}{dx}$, so that

$$\boxed{\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}.}$$

[Note that in this context d, dy, dx have no separate meaning.]

Both $f'(x)$ and dy/dx are widely used, so differentiation results may be written down using either notation.

Thus, if $f(x) = x^2$, then $f'(x) = 2x$

and, if $y = x^2$, then $\dfrac{dy}{dx} = 2x.$

We can also think of d/dx as a symbol indicating the operation of differentiation, writing $\dfrac{d}{dx}(x^2) = 2x.$

There is no need to refer to the graph of a function to carry out the process of differentiation.

Example 1 Differentiate from first principles the function $y = x^3$.

Let δx be a small increase in x and let δy be the corresponding small increase in y.

$$y = x^3$$
$$\Rightarrow y+\delta y = (x+\delta x)^3 = x^3+3x^2\delta x+3x(\delta x)^2+(\delta x)^3$$
$$\Rightarrow \quad \delta y = 3x^2\delta x+3x(\delta x)^2+(\delta x)^3$$
$$\Rightarrow \quad \frac{\delta y}{\delta x} = 3x^2+3x\delta x+(\delta x)^2$$
$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = 3x^2.$$

Alternative method

Let $f(x) = x^3$

then $f(x+h) = (x+h)^3 = x^3+3x^2h+3xh^2+h^3$

$$f(x+h)-f(x) = 3x^2h+3xh^2+h^3$$

$$\frac{f(x+h)-f(x)}{h} = 3x^2+3xh+h^2$$

$$\therefore \quad f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = 3x^2.$$

Example 2 Differentiate from first principles $y = 1/x$.

$$y = \frac{1}{x}$$

$$\Rightarrow y+\delta y = \frac{1}{x+\delta x}$$

$$\Rightarrow \quad \delta y = \frac{1}{x+\delta x}-\frac{1}{x} = \frac{x-(x+\delta x)}{x(x+\delta x)} = -\frac{\delta x}{x(x+\delta x)}$$

$$\Rightarrow \quad \frac{\delta y}{\delta x} = -\frac{1}{x(x+\delta x)}$$

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x\to 0}\frac{\delta y}{\delta x} = -\frac{1}{x^2}.$$

Exercise 5.3

1. If P is the point (x, y) and Q the point $(x+\delta x, y+\delta y)$ on the curve $y = x^2+2$, find the gradient of PQ and hence the gradient of the curve at P.

2. Repeat question 1 for the curves
(a) $y = 4-x^2$, (b) $y = x^2+2x$, (c) $y = \frac{2}{x}+3$.

In the remaining questions differentiate the given functions from first principles.

3. (a) $2x^2$ (b) $5x^2$ (c) $2x^2-1$.

4. (a) x (b) $4-x$ (c) $x+x^2$.

5. (a) $3x-1$ (b) $\frac{1}{2}x+5$ (c) 7.

6. (a) $-x^3$ (b) $2x^3$ (c) x^4.

7. (a) $x-\frac{1}{x}$ (b) $\frac{1}{x^2}$ (c) $\frac{1}{x^3}$.

5.4 Differentiation rules

From the results obtained in Exercise 5.3 a table of derivatives of powers of x can be made.

y	$\frac{1}{x^3}$	$\frac{1}{x^2}$	$\frac{1}{x}$	1	x	x^2	x^3	x^4	...
$\frac{dy}{dx}$	$-\frac{3}{x^4}$	$-\frac{2}{x^3}$	$-\frac{1}{x^2}$	0	1	$2x$	$3x^2$	$4x^3$	...

A pattern emerges when we rewrite the table:

y	x^{-3}	x^{-2}	x^{-1}	x^0	x^1	x^2	x^3	x^4
$\frac{dy}{dx}$	$-3x^{-4}$	$-2x^{-3}$	$-1x^{-2}$	$0x^{-1}$	$1x^0$	$2x^1$	$3x^2$	$4x^3$

It now seems reasonable to assume that for any integer n,

$$\boxed{\text{if } y = x^n, \quad \text{then } \frac{dy}{dx} = nx^{n-1}.}$$

[A formal proof of this statement when n is a positive integer is given in §13.6. Negative values of n are considered in §10.1.]

Some readers may prefer to think of this rule for differentiating powers of x in words:

'multiply by the index and subtract 1 from it'.

Example 1 Find $\frac{dy}{dx}$ if (a) $y = x^7$ (b) $y = \frac{1}{x^7}$.

(a) If $y = x^7$, then $\frac{dy}{dx} = 7x^6$.

[Multiply by the index 7, then subtract 1 to obtain the new index 6.]

(b) If $y = \frac{1}{x^7} = x^{-7}$, then $\frac{dy}{dx} = -7x^{-8} = -\frac{7}{x^8}$.

[In this case the index is -7 and subtracting 1 gives the new index -8.]

It was also found in Exercise 5.3 that, for instance, the derivative of $5x^2$ is $10x$, which is 5 times the derivative of x^2,

$$\text{i.e.} \quad \text{if } y = 5u \text{ where } u = x^2, \text{ then } \frac{dy}{dx} = 10x = 5\frac{du}{dx}.$$

Other examples show that the derivative of a sum is the same as the sum of the separate derivatives,

e.g. if $y = x^2 + x = u + v$ where $u = x^2$ and $v = x$,

$$\text{then } \frac{dy}{dx} = 2x + 1 = \frac{du}{dx} + \frac{dv}{dx}.$$

These are examples of two general rules which we now prove using the properties of limits.

(1) If $y = ku$, where k is a constant and u is a function of x, then $\dfrac{dy}{dx} = k\dfrac{du}{dx}$.

Proof Let δx be a small increase in x and let δu and δy be the corresponding increases in u and y.

$$\begin{aligned} & y = ku \\ \Rightarrow\ & y + \delta y = k(u + \delta u) = ku + k\delta u \\ \Rightarrow\ & \delta y = k\delta u \\ \Rightarrow\ & \frac{\delta y}{\delta x} = k\frac{\delta u}{\delta x} \end{aligned}$$

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} k\frac{\delta u}{\delta x} = k \lim_{\delta x \to 0} \frac{\delta u}{\delta x} = k\frac{du}{dx}.$$

(2) If $y = u + v$, where u and v are functions of x, then

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}.$$

Proof Let δx be a small increase in x and let δu, δv and δy be the corresponding increases in u, v and y.

$$\begin{aligned} & y = u + v \\ \Rightarrow\ & y + \delta y = (u + \delta u) + (v + \delta v) \\ \Rightarrow\ & \delta y = \delta u + \delta v \\ \Rightarrow\ & \frac{\delta y}{\delta x} = \frac{\delta u}{\delta x} + \frac{\delta v}{\delta x} \end{aligned}$$

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \left(\frac{\delta u}{\delta x} + \frac{\delta v}{\delta x} \right) = \lim_{\delta x \to 0} \frac{\delta u}{\delta x} + \lim_{\delta x \to 0} \frac{\delta v}{\delta x} = \frac{du}{dx} + \frac{dv}{dx}.$$

These results are used in the following examples.

Example 2 Find $\dfrac{dy}{dx}$ if (a) $y = x^3 - 4x^2$, (b) $y = 6 - \dfrac{7}{x^2}$.

(a) If $y = x^3 - 4x^2$, then $\dfrac{dy}{dx} = 3x^2 - 4.2x = 3x^2 - 8x$.

(b) If $y = 6 - \dfrac{7}{x^2} = 6 - 7x^{-2}$, then $\dfrac{dy}{dx} = 0 - 7(-2x^{-3}) = \dfrac{14}{x^3}$.

It is necessary to rearrange some functions in order to differentiate them.

Example 3 Find $f'(x)$ when (a) $f(x) = x(x-3)$ (b) $f(x) = \dfrac{x^2-4}{x}$.

(a) $f(x) = x(x-3) = x^2-3x$ $\quad\therefore\quad f'(x) = 2x-3$

(b) $f(x) = \dfrac{x^2+4}{x} = x + \dfrac{4}{x}$ $\quad\therefore\quad f'(x) = 1 - \dfrac{4}{x^2}$.

[Note that differentiating (x^2+4) and x separately then dividing would not give the derivative of the quotient.]

Example 4 Find the gradient of the curve $y = 2x^2+3x-1$ at the point $(1,4)$. Hence find the equation of the tangent to the curve at that point.

If $y = 2x^2+3x-1$, $\dfrac{dy}{dx} = 4x+3$ $\quad\therefore\quad$ when $x = 1$, $\dfrac{dy}{dx} = 7$.

Hence the gradient of the curve at the point $(1,4)$ is 7. Thus the tangent at this point is the straight line passing through $(1,4)$ with gradient 7

$\therefore$ its equation is $y-4 = 7(x-1)$, i.e. $y = 7x-3$.

In §5.2 we saw that the process of differentiation could be used to find the instantaneous velocity of a moving object.

Example 5 A particle moves along a straight line such that after t seconds its displacement from a fixed point is s metres, where $s = 10t^2-t^3$. Find (a) an expression for the velocity, $v\,\mathrm{m\,s^{-1}}$, after t seconds, (b) the velocity after 2 seconds and after 8 seconds.

(a) Since $s = 10t^2-t^3$, $v = \dfrac{ds}{dt} = 20t-3t^2$

$\therefore$ the velocity after t seconds is given by $v = 20t-3t^2$.

(b) When $t = 2$, $v = 20.2-3.4 = 28$

$\therefore$ the velocity after 2 seconds is $28\,\mathrm{m\,s^{-1}}$.

When $t = 8$, $v = 20.8-3.64 = 160-192 = -32$

$\therefore$ the velocity after 8 seconds is $-32\,\mathrm{m\,s^{-1}}$.

[The difference in sign shows that the direction of motion after 8 seconds is opposite to the direction after 2 seconds.]

It will be proved in §10.2 that the rule $\dfrac{d}{dx}(x^n) = nx^{n-1}$ can be applied to fractional powers of x.

Example 6 Differentiate (a) $x^{1/4}$, (b) $x^{-5/2}$, (c) $\sqrt{x}$, (d) $\dfrac{1}{\sqrt[3]{x}}$.

(a) If $y = x^{1/4}$, then $\dfrac{dy}{dx} = \frac{1}{4}x^{-3/4}$

(b) If $y = x^{-5/2}$, then $\dfrac{dy}{dx} = -\dfrac{5}{2}x^{-7/2}$

(c) If $y = \sqrt{x} = x^{1/2}$, then $\dfrac{dy}{dx} = \frac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}}$

(d) If $y = \dfrac{1}{\sqrt[3]{x}} = x^{-1/3}$, then $\dfrac{dy}{dx} = -\frac{1}{3}x^{-4/3} = -\dfrac{1}{3\sqrt[3]{(x^4)}}$

[For revision of surds and indices see §1.5.]

Exercise 5.4

Differentiate the following functions with respect to x:

1. (a) x^5, (b) x^{12}, (c) $3x^4$, (d) -5

2. (a) $2x+7$, (b) $3x^2-4x$, (c) $2x^3-7x^6$.

3. (a) $3+4x-x^2$, (b) $\frac{1}{2}x^3+\frac{1}{3}x^2$, (c) $5x-4x^4$.

4. (a) $x(2x+3)$, (b) $(3x-1)^2$, (c) $x(x+2)^2$.

5. (a) x^{-2}, (b) $1/x^4$, (c) $1/x^9$, (d) $2/x^3$.

6. (a) $x^2-\dfrac{1}{x}$, (b) $x^4+\dfrac{4}{x^2}$, (c) $\dfrac{1}{2x^2}-\dfrac{1}{3x^3}$.

7. (a) $\dfrac{1+x^2}{x}$, (b) $\dfrac{2x-5}{x^3}$, (c) $\dfrac{(x-1)(x+2)}{x}$.

8. Find $f'(x)$, $f'(0)$ and $f'(-2)$ given that
(a) $f(x) = \frac{1}{2}(x^3+1)^2$, (b) $f(x) = 3(x+1)^3$.

9. Find $f'(x)$, $f'(2)$ and $f'(-1)$ given that

(a) $f(x) = \dfrac{1}{x^2}(2x^3+1)$, (b) $f(x) = \left(\dfrac{x^2+1}{x}\right)^2$.

10. Find the gradients of the following curves for the given values of x

(a) $y = 3x^2-x+2$; $x = 1$, (b) $y = x^3-2x^2+5x$; $x = -2$,

(c) $y = \dfrac{x^4-1}{2x^3}$; $x = -1$, (d) $y = \dfrac{(x+2)(1-2x)}{x^2}$; $x = 2$.

11. Find the equations of the tangents to the following curves at the points with given x-coordinates

(a) $y = x^2-3x$; $x = 2$, (b) $y = 2+x-5x^2$; $x = -1$,

(c) $y = x^3 - 3x + 2$; $x = 0$,
(d) $y = (x^2 - 3)(2x^2 + 5)$; $x = -\frac{1}{2}$,
(e) $y = 1 - \frac{1}{x}$; $x = \frac{1}{2}$,
(f) $y = \frac{1 - x^3}{x^4}$; $x = 2$.

12. A particle moves along a straight line such that after t seconds its displacement from a fixed point is s metres, where $s = 5t^2 + 8$. Find (a) an expression for the velocity after t seconds, (b) the velocity after 3 seconds and after 5 seconds.

13. Repeat question 12 for $s = 27t - 3t^2$.

14. Given that $A = \pi r^2$, find $\frac{dA}{dr}$ when $r = 2$.

15. Given that $pv^3 = 54$, find $\frac{dp}{dv}$ when $v = 3$.

16. The gradient of the curve $y = x^2 + ax + b$ at the point $(2, 6)$ is 7. Find the values of a and b.

17. The gradient of the curve $y = ax^2 + b/x^2$ at the point $(\frac{1}{2}, 2\frac{1}{2})$ is -6. Find the values of a and b.

18. Find $\frac{dy}{dx}$ if

(a) $y = x^{1/3}$,
(b) $y = x^{3/2}$,
(c) $y = x^{-1/4}$,
(d) $y = \sqrt[5]{x}$,
(e) $y = 1/\sqrt{x}$,
(f) $y = \sqrt[3]{(8x)}$.

19. Differentiate with respect to x

(a) $x^{1/2} + x^{-3/2}$,
(b) $(2x - 1)\sqrt{x}$,
(c) $(1 + \sqrt{x})^2$,
(d) $\frac{x - 1}{\sqrt{x}}$,
(e) $\frac{x^3 + x^2}{\sqrt{x}}$,
(f) $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^3$.

20. Find the equation of the tangent to the curve $y = (\sqrt{x} + 1)\sqrt{x}$ at the point where $x = 4$.

5.5 Elementary curve sketching

In §3.1 it was shown that graphs of the form $y = ax^2 + bx + c$ all have the same general shape. Thus a quick sketch of a curve such as $y = x^2 - 5x + 4$ can be made by finding the points of intersection with the axes and checking the behaviour of the graph for large positive and large negative values of x.

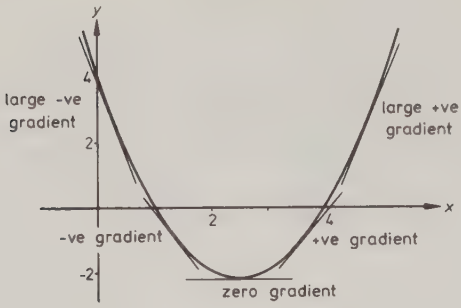
$x = 0 \Rightarrow y = 4$ $\quad\therefore$ the curve cuts the y-axis at $(0, 4)$

$y = 0 \Leftrightarrow x^2 - 5x + 4 = 0 \Leftrightarrow (x - 1)(x - 4) = 0$

$\Leftrightarrow x = 1$ or $x = 4$

$\therefore$ the curve cuts the x-axis at the points $(1, 0)$ and $(4, 0)$.

As $x \to \infty$, $y \to \infty$ and as $x \to -\infty$, $y \to \infty$.



It is clear from the sketch that as x increases, the gradient of the curve moves through negative values to zero and then through positive values.

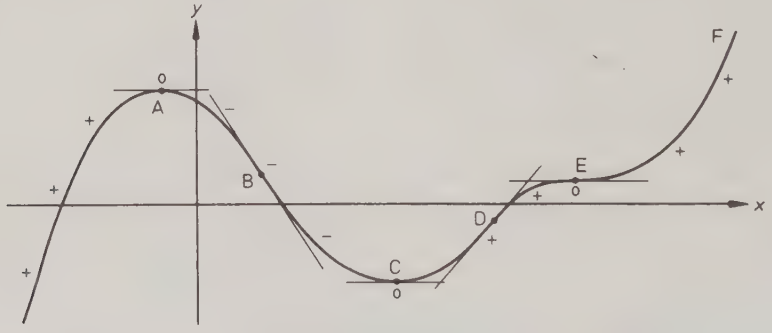
Differentiating $y = x^2 - 5x + 4$, $\dfrac{dy}{dx} = 2x - 5$.

Hence $2x - 5$ is the gradient function for the curve. As x increases, this does indeed behave as described above.

$$\frac{dy}{dx} = 0 \Leftrightarrow 2x - 5 = 0 \Leftrightarrow x = 2\tfrac{1}{2}$$

$\therefore$ the lowest point in the sketch occurs at $x = 2\frac{1}{2}$.

The graph of a more complicated function $y = f(x)$ is shown below.



The sign of the gradient as x increases is shown in the diagram. At A, C and E the tangents to the curve are parallel to the x-axis and the gradient is zero. At these points, which are called *stationary points*, y is said to have a *stationary value*. The point A is a *maximum point* and y takes a *maximum value* at A. Similarly, C is called a *minimum point* and y has a *minimum value* there. It is clear from the sketch that these are local maximum and minimum values, since, for instance, the value of y at F is greater than its value at A.

A and C are both called *turning points*, but although y has a stationary value at E, E is not a turning point. The gradient of the curve is zero at E, but does not change sign as the curve passes through E. Such points are called *points of inflexion*. In general, a point of inflexion is a point at which a curve stops bending

in one direction and starts bending the other way. In the diagram B, D and E are all points of inflexion and, as shown, the tangents at these points cross the curve.

Summarising, a point at which $dy/dx = 0$ is
(i) a maximum point, if the sign of dy/dx changes from + to −
(ii) a minimum point, if the sign of dy/dx changes from − to +
(iii) a point of inflexion, if the sign of dy/dx does not change.

This provides a new approach to the sketching of graphs.

Example 1 Sketch the graph of $y = 2+x-x^2$.

$x = 0 \Rightarrow y = 2$
$\therefore$ the curve cuts the y-axis at the point $(0,2)$.
$y = 0 \Leftrightarrow 2+x-x^2 = 0 \Leftrightarrow (2-x)(1+x) = 0$
$\Leftrightarrow x = -1$ or $x = 2$
$\therefore$ the curve cuts the x-axis at the points $(-1,0)$ and $(2,0)$.

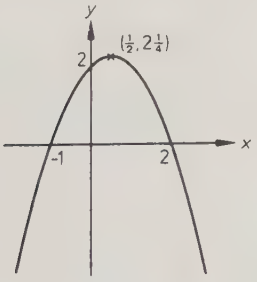
Differentiating $\dfrac{dy}{dx} = 1-2x$

$\therefore$ $\dfrac{dy}{dx} = 0 \Leftrightarrow 1-2x = 0 \Leftrightarrow x = \frac{1}{2}$

When $x < \frac{1}{2}$, $\dfrac{dy}{dx} > 0$ and when $x > \frac{1}{2}$, $\dfrac{dy}{dx} < 0$

$\therefore$ when $x = \frac{1}{2}$, y takes the maximum value of $2\frac{1}{4}$.

As $x \to \infty$, $y \to -\infty$ and as $x \to -\infty$, $y \to -\infty$.



We are now able to sketch the curve and label the maximum point $(\frac{1}{2}, 2\frac{1}{4})$.

Example 2 Sketch the curve $y = x^3-6x^2+9x$.

$x = 0 \Rightarrow y = 0$
$\therefore$ the curve cuts the y-axis at the origin $(0,0)$.
$y = 0 \Leftrightarrow x^3-6x^2+9x = 0 \Leftrightarrow x(x-3)^2 = 0$
$\Leftrightarrow x = 0$ or $x = 3$
$\therefore$ the curve cuts the x-axis at the points $(0,0)$ and $(3,0)$.

Differentiating $\dfrac{dy}{dx} = 3x^2 - 12x + 9 = 3(x-1)(x-3)$

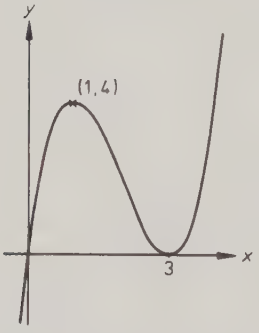
$\dfrac{dy}{dx} = 0 \Leftrightarrow (x-1)(x-3) = 0 \Leftrightarrow x = 1 \quad \text{or} \quad x = 3$

[We now determine the nature of the stationary points at $x = 1$ and $x = 3$, by drawing up a table showing the sign of dy/dx when x is less than 1, when x lies between 1 and 3 and when x is greater than 3.]

	$x < 1$	$1 < x < 3$	$x > 3$
$x-1$	$-$	$+$	$+$
$x-3$	$-$	$-$	$+$
dy/dx	$+$	$-$	$+$

$\therefore$ when $x = 1$, there is a maximum point $(1, 4)$
when $x = 3$, there is a minimum point $(3, 0)$

As $x \to \infty, y \to \infty$ and as $x \to -\infty, y \to -\infty$.



We can now sketch the curve $y = x^3 - 6x^2 + 9x = x(x-3)^2$ and we see that the repeated factor $(x-3)$ leads to a turning point at $x = 3$.

Example 3 Sketch the curve $y = (x+1)(x-3)^3$.

$x = 0 \Rightarrow y = -27$
$\therefore$ the curve cuts the y-axis at the point $(0, -27)$.

$y = 0 \Leftrightarrow (x+1)(x-3)^3 = 0 \Leftrightarrow x = -1 \quad \text{or} \quad x = 3$
$\therefore$ the curve cuts the x-axis at the points $(-1, 0)$ and $(3, 0)$.

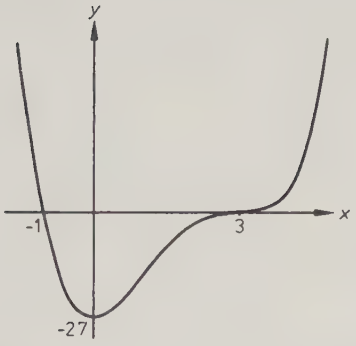
By multiplication $y = x^4 - 8x^3 + 18x^2 - 27$

Differentiating $\dfrac{dy}{dx} = 4x^3 - 24x^2 + 36x$

$= 4x(x^2 - 6x + 9) = 4x(x-3)^2.$

$\dfrac{dy}{dx} = 0 \Leftrightarrow 4x(x-3)^2 = 0 \Leftrightarrow x = 0 \quad \text{or} \quad x = 3$

	$x<0$	$0<x<3$	$x>3$
x	−	+	+
$(x-3)^2$	+	+	+
dy/dx	−	+	+



$\therefore$ when $x = 0$, there is a minimum point $(0, -27)$ and when $x = 3$, there is a point of inflexion $(3, 0)$. As $x \to \infty$, $y \to \infty$ and as $x \to -\infty$, $y \to \infty$.

Example 4 Sketch the graph $y = \dfrac{2x-5}{x}$.

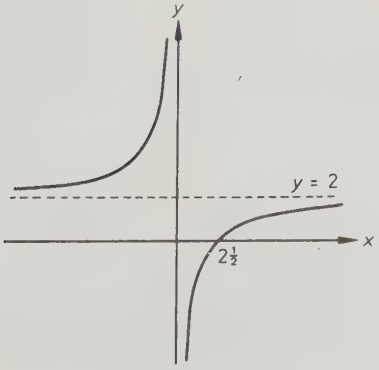
When $x = 0$ the value of y is undefined
$\therefore$ the curve does not cut the y-axis.

$y = 0 \Leftrightarrow 2x - 5 = 0 \Leftrightarrow x = 2\frac{1}{2}$
$\therefore$ the curve cuts the x-axis at the point $(2\frac{1}{2}, 0)$.

By division $\quad y = 2 - \dfrac{5}{x}$.

Differentiating $\quad \dfrac{dy}{dx} = \dfrac{5}{x^2}$.

Hence the gradient of the curve is always positive and there are no turning points.
As $x \to 0$ from above, $y \to -\infty$ and as $x \to 0$ from below, $y \to \infty$.
As $x \to \infty$, $y \to 2$ from below and as $x \to -\infty$, $y \to 2$ from above.



A straight line which a curve approaches more and more closely but never touches is called an *asymptote* to the curve. The curve $y = (2x-5)/x$ has two asymptotes, the y-axis and the line $y = 2$.

Example 5 Sketch the graph $y = x + \frac{1}{x}$.

When $x = 0$, the value of y is undefined.

$$y = 0 \Leftrightarrow x + \frac{1}{x} = 0 \Leftrightarrow x^2 + 1 = 0$$

$\therefore$ there is no real value of x for which $y = 0$.

Hence the curve cuts neither of the coordinate axes.

Differentiating $\frac{dy}{dx} = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2}$

$$\frac{dy}{dx} = 0 \Leftrightarrow x^2 - 1 = 0 \Leftrightarrow x = -1 \quad \text{or} \quad x = 1.$$

When $x = -1$, $y = -2$ and when $x = 1$, $y = 2$.

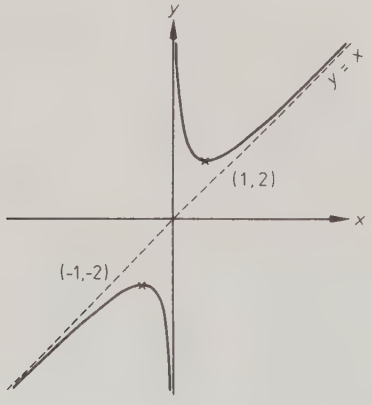
When x is just less than -1, $\frac{dy}{dx} > 0$ and when x is just greater than -1, $\frac{dy}{dx} < 0$ $\therefore$ the point $(-1, -2)$ is a maximum point.

When x is just less than 1, $\frac{dy}{dx} < 0$ and when x is just greater than 1, $\frac{dy}{dx} > 0$ $\therefore$ the point $(1, 2)$ is a minimum point.

As $x \to 0$ from above, $y \to \infty$ and as $x \to 0$ from below, $y \to -\infty$

$\therefore$ the y-axis is an asymptote to the curve.

As $x \to \infty$, $y \to \infty$ and as $x \to -\infty$, $y \to -\infty$.



Although we are now ready to sketch the curve, a more accurate sketch will be obtained by noting the fact that as $x \to \pm\infty$, the curve $y = x + \frac{1}{x}$ draws closer and closer to the line $y = x$. Hence the line $y = x$ is an asymptote to the curve.

To summarise, here is a list of the steps to take when sketching a curve.

(1) Find the points of intersection with the y-axis by putting $x = 0$.

(2) Find, if possible, the points of intersection with the x-axis by putting $y = 0$.

(3) Find the points at which $dy/dx = 0$.

(4) Determine the nature of these points by considering the sign of dy/dx.
(5) Note any values of x for which y is not defined and examine the behaviour of y as x approaches such values.
(6) Consider the values of y as $x \to \infty$ and as $x \to -\infty$.
(7) If further checks are needed, determine the sign of y for all values of x.

It is important to remember that if differentiating and solving $dy/dx = 0$ is difficult, it is often still possible to sketch a curve fairly quickly using the remaining techniques.

Exercise 5.5

Sketch the following curves showing clearly on each sketch the nature of any stationary points.

1. $y = x^2 - 2x - 3$.
2. $y = 2 - 3x - 2x^2$.
3. $y = (x-1)(x+2)$.
4. $y = 4x^2 - 4x + 3$.
5. $y = x^2(x-3)$.
6. $y = x^3 + 1$.
7. $y = 6x^2 - 12x - x^3$.
8. $y = (x+2)^2(2x-5)$.
9. $y = 2x + x^2 - 4x^3$.
10. $y = 8x^3 + 9x^2 - 15x - 2$.
11. $y = x^4(x+5)$.
12. $y = 4x^3 - 3x^4$.
13. $y = \dfrac{x+1}{x}$.
14. $y = \dfrac{2-3x}{x}$.
15. $y = 9x^3 - 45x^2 + 48x - 11$.
16. $y = x^4 - 2x^3 - 5x^2 + 6$.
17. $y = 4x + \dfrac{9}{x}$.
18. $y = \dfrac{x^3+4}{x^2}$.
19. $y = 2x^5 - 15x^4 + 30x^3 - 64$.
20. $y = x - \sqrt{x}$ $(x \geqslant 0)$.

21. The curve $y = x^3 + ax^2 + b$ has a minimum point at $(4, -11)$. Find the coordinates of the maximum point on the curve.

22. Given that the curve $y = x^3 + px^2 + qx + r$ passes through the point $(1, 1)$ and has turning points where $x = -1$ and $x = 3$, find the values of p, q and r.

23. Find the conditions that the curve $y = x^3 + ax^2 + bx + c$ should have (i) two distinct stationary points, (ii) one stationary point, (iii) no stationary points.

24. Write down the equations of the two asymptotes to the curve $y = (x^2 + k)/x$, where $k \neq 0$. Sketch the curve for (i) $k > 0$, (ii) $k < 0$.

5.6 The second derivative

For the function $y = f(x)$ the *first derivative* is $f'(x)$ or dy/dx. The *second derivative* is obtained by differentiating again and is written $f''(x)$ or $\dfrac{d^2y}{dx^2}$, which is shorthand for $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$.

Example 1 If $y = 3x + 2 + \dfrac{1}{x}$, find $\dfrac{d^2y}{dx^2}$.

Rearranging $\quad y = 3x + 2 + x^{-1}$

Differentiating $\quad \dfrac{dy}{dx} = 3 - x^{-2} = 3 - \dfrac{1}{x^2}$

and $\quad \dfrac{d^2y}{dx^2} = -(-2)x^{-3} = \dfrac{2}{x^3}$.

Example 2 If $f(x) = x^4 - 6x^2 + 7$, find $f(1)$, $f'(1)$ and $f''(1)$.

$f(x) = x^4 - 6x^2 + 7 \quad \therefore \quad f(1) = 1 - 6 + 7 = 2$
$f'(x) = 4x^3 - 12x \quad \therefore \quad f'(1) = 4 - 12 = -8$
$f''(x) = 12x^2 - 12 \quad \therefore \quad f''(1) = 12 - 12 = 0.$

If a particle moves in a straight line so that at time t its displacement from a fixed point is s, then the first derivative ds/dt gives the velocity v after t seconds. The second derivative d^2s/dt^2 is also the first derivative of v, dv/dt. Hence it is a measure of the rate at which the velocity is changing, i.e. it represents the acceleration of the particle and we write

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

Example 3 If a particle moves in a straight line, so that after t seconds its displacement from a fixed point is s metres, where $s = 8t^2 - t^4$, find its velocity and acceleration after 1 second.

Differentiating with respect to t: $\quad v = \dfrac{ds}{dt} = 16t - 4t^3$

$$a = \frac{dv}{dt} = 16 - 12t^2$$

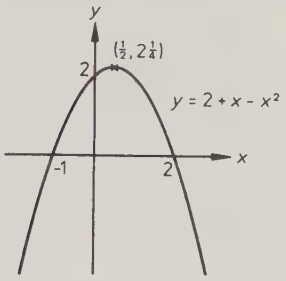
When $t = 1$, $v = 12$ and $a = 4$.
$\therefore$ after 1 second the velocity of the particle is $12\,\mathrm{m\,s^{-1}}$ and its acceleration is $4\,\mathrm{m\,s^{-2}}$.

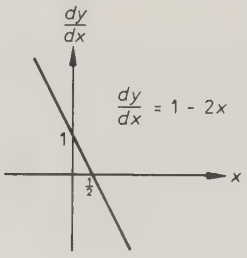
In general, the second derivative, d^2y/dx^2, can be used to examine how the gradient function, dy/dx, changes as x increases.

For the curve $y = 2 + x - x^2$ sketched in Example 1, §5.5,

$$\frac{dy}{dx} = 1 - 2x \quad \text{and} \quad \frac{d^2y}{dx^2} = -2.$$

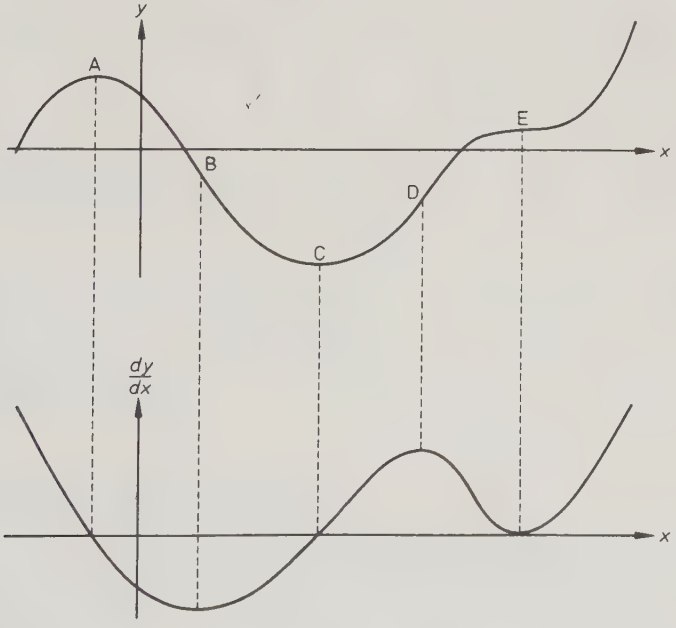
This means that the graph of this gradient function is a straight line, gradient -2.





The turning point on the curve corresponds to the point at which the graph of dy/dx cuts the x-axis.

The diagrams below show the graph of a function $y = f(x)$ and the corresponding graph of the gradient function dy/dx or $f'(x)$. The gradient function for this second graph is the second derivative d^2y/dx^2 or $f''(x)$.



At the maximum point A, the gradient of the curve is decreasing through zero and d^2y/dx^2 is negative. At the minimum point C, the gradient of the curve is increasing through zero and d^2y/dx^2 is positive. Points of inflexion, such as B, D and E, are points where dy/dx reaches either a maximum or a minimum value.

Thus at these points $d^2y/dx^2 = 0$. However, $d^2y/dx^2 = 0$ does not necessarily imply the presence of a point of inflexion.

Summarising, a point where $\frac{dy}{dx} = 0$ is a maximum point if $\frac{d^2y}{dx^2} < 0$, a minimum point if $\frac{d^2y}{dx^2} > 0$. When $\frac{d^2y}{dx^2} = 0$, further investigation is needed.

Example 4 Sketch the curve $y = 5x^4 - x^5$.

$x = 0 \Rightarrow y = 0$

$\therefore$ the curve cuts the y-axis at the origin $(0,0)$.

$y = 0 \Leftrightarrow 5x^4 - x^5 = 0 \Leftrightarrow x^4(5-x) = 0$

$\Leftrightarrow x = 0$ or $x = 5$

$\therefore$ the curve cuts the x-axis at the points $(0,0)$ and $(5,0)$.

$$\frac{dy}{dx} = 20x^3 - 5x^4 = 5x^3(4-x), \quad \frac{d^2y}{dx^2} = 60x^2 - 20x^3 = 20x^2(3-x)$$

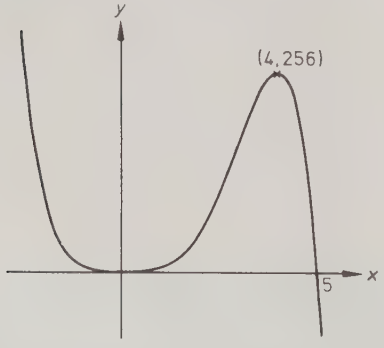
$$\therefore \quad \frac{dy}{dx} = 0 \Leftrightarrow 5x^3(4-x) = 0 \Leftrightarrow x = 0 \quad \text{or} \quad x = 4.$$

When $x = 4$, $y = 256$ and $\frac{d^2y}{dx^2} < 0$ $\therefore$ $(4, 256)$ is a maximum point.

When $x = 0$, $y = 0$ and $\frac{d^2y}{dx^2} = 0$ $\therefore$ to determine the nature of this point we must consider the sign of $\frac{dy}{dx}$.

When x is just less than 0, $\frac{dy}{dx} < 0$ and when x is just greater than 0, $\frac{dy}{dx} > 0$ $\therefore$ $(0,0)$ is a minimum point.

As $x \to \infty$, $y \to -\infty$ and as $x \to -\infty$, $y \to \infty$.



The sketch shows that there must be a point of inflexion between $x = 0$ and $x = 4$.

$$\frac{d^2y}{dx^2} = 0 \Rightarrow x = 0 \quad \text{or} \quad x = 3$$

and when $x = 3$, $y = 162$

$\therefore$ the coordinates of the point of inflexion are $(3, 162)$.

Exercise 5.6

In questions 1 to 6 find $\frac{d^2y}{dx^2}$.

1. $y = 3x^2 + 4x - 1$.
2. $y = 5x^3 - x^2 - x$.
3. $y = x^4 - 2x^3 + \frac{1}{x}$.
4. $y = \frac{(x^2-4)(x+1)}{x^2}$.
5. $y = \sqrt{x} + \frac{1}{\sqrt{x}}$.
6. $y = \frac{(x-2)(2x+3)}{\sqrt{x}}$.

Sketch the following curves using the second derivative, where possible, to determine the nature of any stationary points.

7. $y = x^3 - 3x$.
8. $y = x^4 - 2x^2 + 1$.
9. $y = x^4 - 2x^3$.
10. $y = 4x - 3 + \frac{1}{x}$.
11. $y = 2x^3 - 9x^2 + 12x - 3$.
12. $y = 3x^4 - 4x^3 - 12x^2 + 5$.

13. In each of the following cases find $f(-1)$, $f'(-1)$ and $f''(-1)$
(a) $f(x) = x^5 + 6x^4 - x^2$, (b) $f(x) = (x^3 - 2)/x$.

14. If $f(x) = x^4 - 4x^3 + 16x - 16$, show that $f(2) = f'(2) = f''(2) = 0$. Sketch the graph of the function showing any turning points and points of inflexion.

15. Show that the curve $y = x^3 - 3x^2 + 6x - 4$ has one point of inflexion and find the gradient of the curve at this point. Sketch the curve.

16. A particle moves along a straight line so that after t seconds its displacement from a fixed point A on the line is s metres. If $s = 3t^2(3-t)$, find (a) expressions for the velocity, v m s^{-1}, and the acceleration, a m s^{-2}, after t seconds, (b) the initial velocity and acceleration of the particle, (c) the velocity and acceleration after 3 seconds.

17. Repeat question 16 for $s = 8t^3 - 33t^2 + 27t$.

18. The curve $y = x^4 + ax^2 + bx + c$ passes through the point $(-1, 16)$ and at that point $\frac{d^2y}{dx^2} = -\frac{dy}{dx} = 16$. Find the values of a, b, c and sketch the curve.

19. Sketch the following curves, labelling any turning points and points of inflexion.
(a) $y = x^2 + \frac{1}{x}$, (b) $y = \frac{1}{x} - \frac{1}{x^2}$, (c) $y = \frac{1}{x} - \frac{1}{x^3}$.

20. Sketch the curve $y = x^4 + 4x$ and show that d^2y/dx^2 is zero at a point which is neither a turning point nor a point of inflexion.

Exercise 5.7 (*miscellaneous*)

1. Find the limit as $h \to 0$ of

(a) $2 - 3h + 5h^2$, (b) $\dfrac{h^2 - 4h}{h}$, (c) $\dfrac{1 - h^2}{1 - 2h}$.

2. Find the limit as $x \to \infty$ of

(a) $\dfrac{4}{2x - 3}$, (b) $\dfrac{2x^2 + 1}{x^2 - 7}$, (c) $\dfrac{x - 1}{x^2 - 3x + 5}$.

3. (a) Prove that, as x tends to zero, the limit of the expression

$\dfrac{(2 + 3x)^2 - 4(1 + x)^2}{6x}$ is $\dfrac{2}{3}$.

(b) Find the limit, as n tends to infinity, of

$\dfrac{[\sqrt{(4n + 3)} + 2\sqrt{(1 + n)}]}{\sqrt{n}}$. (AEB 1978)

4. Differentiate from first principles the functions
(a) $(x + 1)^3$, (b) $(x + 1)^{-1}$.

5. Prove that $\sqrt{(a + b)} - \sqrt{a} = \dfrac{b}{\sqrt{(a + b)} + \sqrt{a}}$. Use this result to differentiate the function $y = \sqrt{x}$ from first principles.

6. Differentiate with respect to x

(a) $(3x + 1)^2$, (b) $\dfrac{1}{x}(x + 2)(x + 3)$, (c) $(x + 2)\sqrt{x}$.

7. Find the equations of the tangents to the curve $y = x^3 - 2x^2 - 3x$ at the points where the curve crosses the x-axis.

8. Show that the tangent to the curve $y = (1 + x)^2(5 - x)$ at the point where $x = 1$ does not meet the curve again. Show also that the tangent at the point $(0, 5)$ cuts the curve at a turning point. Draw a rough sketch of the curve, showing these tangents.

9. Sketch the following curves indicating the approximate positions of any turning points

(a) $y = x^2 - 4$, (b) $x^3 - 4x$, (c) $y = x^4 - 4x^2$,

(d) $y = x - \frac{4}{x}$, (e) $y = 1 - \frac{4}{x^2}$, (f) $y = \frac{1}{x} - \frac{4}{x^3}$.

10. Find any maxima, minima or horizontal points of inflexion of the curve $y = \frac{x^3+3x-1}{x^2}$, stating, with reasons, the nature of each point. Sketch the curve, indicating clearly what happens as $x \to \pm\infty$. (SU)

11. A particle moves along a straight line such that after t seconds its displacement from a fixed point on the line is s metres, where $s = 7t+5t^2-2t^3$. Find (a) expressions for the velocity and acceleration of the particle after t seconds, (b) the velocity and acceleration of the particle initially and after 2 seconds.

12. A function $f(x)$ is defined as follows:
$f(x) = 2x^3-9x^2+12x$ when $0 \leqslant x \leqslant 2$
and $f(x) = 4(x-1)(3-x)$ when $2 < x \leqslant 3$.
Sketch the graph of the function.

13. If $f(x) = (x^2-1)(x-3)$ show that the minimum value of $f'(x)$ is -4. Sketch the graphs of the functions $f(x)$ and $f'(x)$.

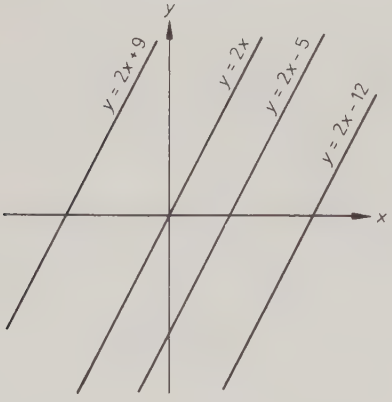
14. Show that the curve $y = ax^3+bx^2+cx+d$ $(a \neq 0)$ always has exactly one point of inflexion and write down its x-coordinate.

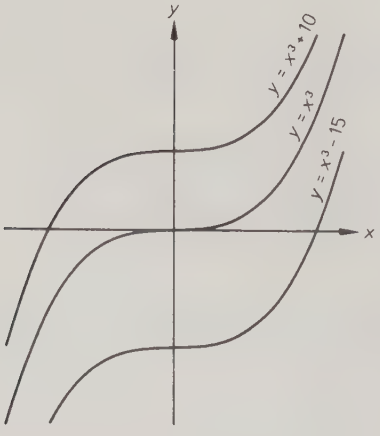
15. Show that there are no points of inflexion on the curve $y = x^4-4x^3+6x^2-4x$, then sketch the curve.

6 Introduction to integration

6.1 The reverse of differentiation

Any graph with gradient function given by $dy/dx = 2$ has gradient 2 at every point on it. Hence the graph must be a straight line with equation $y = 2x + c$, where c is some constant. Conversely, differentiating $y = 2x + c$, we obtain $dy/dx = 2$ for all values of c. Thus the *differential equation*, $dy/dx = 2$, is said to represent the family of all straight lines with gradient 2.





Consider now graphs with gradient function, $dy/dx = 3x^2$. Since $\frac{d}{dx}(x^3) = 3x^2$, the result $\frac{dy}{dx} = 3x^2$ may be derived from any equation of the form, $y = x^3 + c$, where c is a constant. Thus the equation $dy/dx = 3x^2$ represents the family of curves $y = x^3 + c$, some of which are shown in the diagram.

The operation of obtaining an expression for y in terms of x from the gradient function, dy/dx, is called *integration*. The function x^3+c is the *integral* of $3x^2$ with respect to x. Since the constant c may take any value, it is called an *arbitrary constant*.

The rule for integrating powers of x is found by reversing the rule for differentiation. We find that to integrate a power of x, we must add 1 to the index, then divide by the new index, i.e.

$$\text{if } \frac{dy}{dx} = x^n, \quad \text{then } y = \frac{x^{n+1}}{n+1} + c \quad \text{provided that } n+1 \neq 0.$$

This rule cannot be applied to the integration of x^{-1}, because there is no power of x whose derivative is x^{-1}.

Example 1 Find y if $\frac{dy}{dx}$ is (a) x^4, (b) $\frac{1}{x^3}$, (c) $6x$, (d) $x^{1/2}$.

(a) If $\frac{dy}{dx} = x^4$, then $y = \frac{1}{5}x^5 + c$.

(b) If $\frac{dy}{dx} = \frac{1}{x^3} = x^{-3}$, then $y = \frac{x^{-2}}{-2} + c = -\frac{1}{2x^2} + c$.

(c) If $\frac{dy}{dx} = 6x = 6.x^1$, then $y = 6.\frac{x^2}{2} + c = 3x^2 + c$.

(d) If $\frac{dy}{dx} = x^{1/2}$, then $y = \frac{x^{3/2}}{\frac{3}{2}} + c = \frac{2}{3}x^{3/2} + c$.

Example 2 Find $f(u)$, given that $f'(u) = (u-2)(2u+1)$.

If $f'(u) = (u-2)(2u+1) = 2u^2 - 3u - 2$

then $f(u) = \frac{2}{3}u^3 - \frac{3}{2}u^2 - 2u + c$.

It is sometimes possible to determine the value of the arbitrary constant in an integration problem, if additional information is given.

Example 3 Find V in terms of h, if $\frac{dV}{dh} = 4h^2 + h$ and $V = 2$ when $h = 1$.

$$\frac{dV}{dh} = 4h^2 + h \quad \Rightarrow \quad V = \frac{4}{3}h^3 + \frac{1}{2}h^2 + c$$

Since $V = 2$ when $h = 1$, $\quad 2 = \frac{4}{3} + \frac{1}{2} + c$

$$\therefore \quad c = 2 - \frac{4}{3} - \frac{1}{2} = \frac{1}{6}$$

Hence $V = \frac{4}{3}h^3 + \frac{1}{2}h^2 + \frac{1}{6} = \frac{1}{6}(8h^3 + 3h^2 + 1)$.

Example 4 The gradient of a curve is given by $\frac{dy}{dx} = 1 - \frac{1}{x^2}$. If the curve passes through the point $(1, 1)$, find its equation.

Integrating $y = x + \frac{1}{x} + c$

Since $y = 1$, when $x = 1$, $1 = 1 + 1 + c \quad \therefore c = -1$.

Hence the equation of the curve is $y = x + \frac{1}{x} - 1$.

Exercise 6.1

1. Sketch the families of curves with the following gradient functions: (a) 1, (b) $2x$, (c) $4x^3$.

2. Sketch the family of curves represented by each of the following differential equations

(a) $\frac{dy}{dx} = -\frac{1}{2}$, (b) $\frac{dy}{dx} = 1 - 2x$, (c) $\frac{dy}{dx} = 3x^2 - 6x$.

3. Find y if $\frac{dy}{dx}$ is (a) x^2, (b) $\frac{1}{x^2}$, (c) $8x^3$, (d) $\frac{6}{x^3}$.

In questions 4 to 12 integrate the given functions of x.

4. $3x^2 + 4x$. 5. $x^5 - 1$. 6. $(2x+1)^2$.

7. $6x - \frac{1}{6x^3}$. 8. $\left(x + \frac{1}{x}\right)^2$. 9. $\frac{x^2 - 1}{x^4}$.

10. $x^{1/3} + x^{-1/3}$. 11. $\frac{x-1}{\sqrt{x}}$. 12. $\sqrt{x^3} - \frac{1}{\sqrt{x^3}}$.

13. Find $f(t)$ given that $f'(t)$ is equal to

(a) $(t+6)(3t-4)$, (b) $\frac{(2t+1)^2}{t^4}$, (c) $\frac{(t-2)(2t-1)}{\sqrt{t}}$.

14. Find an expression for y in terms of x, if

$\frac{dy}{dx} = (x-1)(3x-5)$ and $y = 0$ when $x = 1$.

15. Given that $\frac{dy}{dx} = 2\left(x + \frac{1}{x^3}\right)$ and that $y = 3$ when $x = 1$, find the value of y when $x = \frac{1}{2}$.

16. The gradient of a curve is given by $\frac{dy}{dx} = 3x^2 - 4$. If the curve passes through the point $(2, -1)$, find its equation.

17. The gradient of a certain curve at a typical point (x, y) is $4(1-x)$. Given that the maximum value of y on the curve is 8, find its equation.

18. Find an expression for A in terms of h, given that

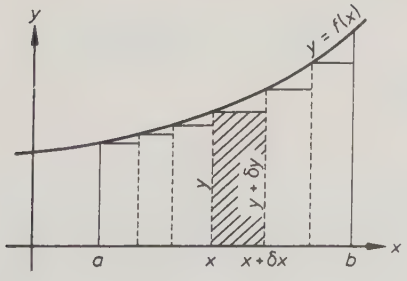
$\frac{dA}{dh} = 3\sqrt{h} - \frac{1}{2\sqrt{h}}$ and that $A = 10$ when $h = 4$.

19. Find an expression for y in terms of x, given that

$\frac{d^2y}{dx^2} = 6x - 10$ and that when $x = 1$, $y = 1$ and $\frac{dy}{dx} = 1$.

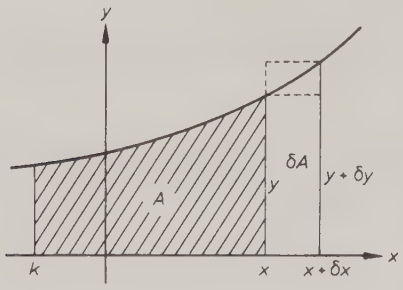
20. If y is a function of x and $\frac{d^2y}{dx^2} = 3 - \frac{5}{\sqrt[3]{x}}$, find the value of y when $x = 8$, given that $dy/dx = \frac{1}{2}$ and $y = 5$ when $x = 1$.

6.2 The area under a curve



The diagram shows part of a curve $y = f(x)$. The area bounded by the curve, the x-axis and the lines $x = a$ and $x = b$ can be determined by dividing it into strips. These strips are assumed to be approximately rectangular, so that a typical strip of width δx and height y has area $y\,\delta x$. The total area is then estimated by finding the sum of all such areas between $x = a$ and $x = b$. The smaller the width of each strip, the more accurate this estimate will be. Thus, as δx approaches zero, the sum of the areas of the strips approaches the required area under the curve. Since the area is derived from the limit of a sum of terms of the form $y\,\delta x$, as $\delta x \to 0$, the standard notation for this area is

$$\int_a^b y\,dx \quad \text{or} \quad \int_a^b f(x)\,dx, \quad \text{where} \quad \int \text{ is an elongated } S \text{ for sum.}$$



To find a way of evaluating this limit, consider the shaded area, A, between the curve $y = f(x)$ and the x-axis shown in the sketch. Since the value of A depends on the value of x, it may be possible to express A in terms of x.

Let δx be a small increase in x and let δy and δA be the corresponding increases in y and A. The area δA must lie between the areas of the two rectangles shown in the diagram, which have width δx and heights y and $y+\delta y$

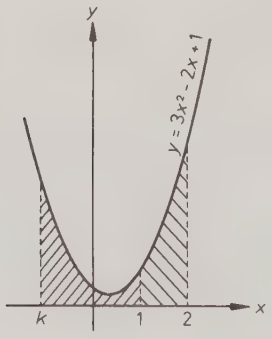
$$\text{i.e.} \quad y\,\delta x < \delta A < (y+\delta y)\,\delta x$$

$$\therefore \quad y < \frac{\delta A}{\delta x} < y+\delta y$$

As $\delta x \to 0$, $\delta y \to 0$ and $\dfrac{\delta A}{\delta x} \to \dfrac{dA}{dx}$

$$\therefore \quad \frac{dA}{dx} = y.$$

Hence an expression for A may be obtained by integrating $f(x)$.



For example, the area function A for the curve $y = 3x^2-2x+1$ is obtained by writing

$$\frac{dA}{dx} = 3x^2-2x+1$$

Thus

$$A = x^3-x+x+c,$$

where the value of the arbitrary constant c depends on the position of the boundary line $x = k$. However, the size of the area under the curve between $x = 1$ and $x = 2$ does not depend on the values of c and k.

At $x = 2$, $A = 8-4+2+c = 6+c$

At $x = 1$, $A = 1-1+1+c = 1+c$

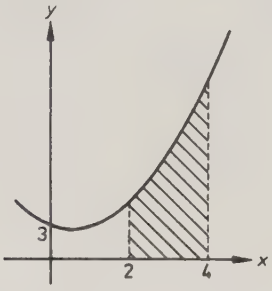
$\therefore$ the area bounded by the curve $y = 3x^2-2x+1$, the x-axis and the lines $x = 1$ and $x = 2$ is $(6+c)-(1+c)$, i.e. 5 square units.

More generally, if the area function for a curve $y = f(x)$ is $A = F(x)+c$, then the area under the curve between $x = a$ and $x = b$ is $F(b)-F(a)$.

Hence $$\int_a^b y\,dx = \int_a^b f(x)\,dx = F(b)-F(a).$$

The expression $F(b)-F(a)$ is often written $\Big[F(x)\Big]_a^b$.

Example 1 Find the area bounded by the curve $y = x^2-x+3$, the x-axis and the lines $x = 2$, $x = 4$.



$$\text{Required area} = \int_2^4 y\,dx$$
$$= \int_2^4 (x^2-x+3)\,dx$$
$$= \Big[\tfrac{1}{3}x^3-\tfrac{1}{2}x^2+3x\Big]_2^4$$
$$= (\tfrac{1}{3}.4^3-\tfrac{1}{2}.4^2+3.4)-(\tfrac{1}{3}.2^3-\tfrac{1}{2}.2^2+3.2)$$
$$= (64/3-8+12)-(8/3-2+6) = 18\tfrac{2}{3}.$$

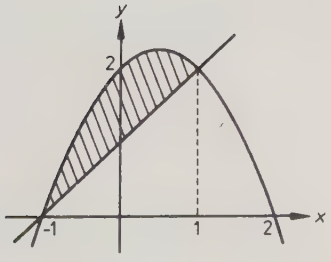
[We will assume that all areas obtained by integration are measured in square units.]

Example 2 Find the area of the region bounded by the curve $y = 2+x-x^2$ and the straight line $y = x+1$.

At the points of intersection of the line and the curve

$y = 2+x-x^2$ and $y = x+1$,
$2+x-x^2 = x+1$
$\Leftrightarrow$ $x^2 = 1$
$\Leftrightarrow$ $x = -1$ or $x = 1$

$\therefore$ the points of intersection are $(-1,0)$ and $(1,2)$.



The area under the curve from $x = -1$ to $x = 1$

$$= \int_{-1}^1 (2+x-x^2)\,dx$$
$$= \Big[2x+\tfrac{1}{2}x^2-\tfrac{1}{3}x^3\Big]_{-1}^1$$
$$= (2+\tfrac{1}{2}-\tfrac{1}{3})-(-2+\tfrac{1}{2}+\tfrac{1}{3}) = 3\tfrac{1}{3}$$

The area of the triangle under the line $= \tfrac{1}{2}.2.2 = 2$.

Hence the area bounded by the given curve and straight line is $3\tfrac{1}{3}-2$, i.e. $1\tfrac{1}{3}$.

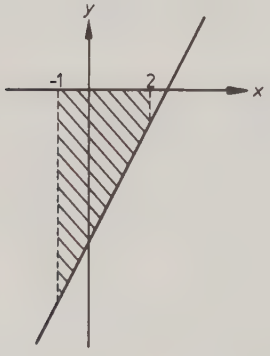
In §6.1 integration was described as the reverse of differentiation. However, we now see that it is also the process used to find areas under curves. Because of this relationship between integration and summation the integral of a function $f(x)$ is

denoted by $\int f(x)\,dx$ and referred to as an *indefinite integral.* The expression $\int_a^b f(x)\,dx$ is called a *definite integral.* The constants a and b are the *limits of integration,* a being the lower limit and b the upper.

Example 3 (a) Find $\int (2x-5)\,dx$, (b) Evaluate $\int_{-1}^{2} (2x-5)\,dx$.

(a) $\int (2x-5)\,dx = x^2-5x+c$

(b) $\int_{-1}^{2} (2x-5)\,dx = \Big[x^2-5x\Big]_{-1}^{2} = (2^2-5.2)-(\{-1\}^2-5.\{-1\}) = -6-6$
$= -12.$



The diagram shows the significance of the negative value of the above definite integral. The line $y = 2x-5$ is below the x-axis, i.e. y is negative, throughout the interval $x = -1$ to $x = 2$. This means that although the definite integral is negative, it is still numerically equal to the area bounded by the line $y = 2x-5$, the x-axis and the lines $x = -1, x = 2$.

As shown in the examples, for any given function $f(x)$, the definite integral $\int_a^b f(x)\,dx$ is not itself a function of x, but has a numerical value which depends solely on the limits a and b. If a different variable is used the value is unchanged.

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(\theta)\,d\theta.$$

Thus x (or t, or θ) may sometimes be referred to as a *dummy variable.*

Example 4 Evaluate $\int_1^4 \left(\sqrt{u} + \frac{1}{\sqrt{u}}\right) du.$

$$\int_1^4 \left(\sqrt{u} + \frac{1}{\sqrt{u}}\right) du = \int_1^4 (u^{1/2}+u^{-1/2})\,du$$
$$= \left[\tfrac{2}{3}u^{3/2}+2u^{1/2}\right]_1^4$$
$$= (\tfrac{2}{3}.4^{3/2}+2.4^{1/2})-(\tfrac{2}{3}.1^{3/2}+2.1^{1/2})$$
$$= \frac{16}{3}+4-\frac{2}{3}-2 = 6\tfrac{2}{3}.$$

Exercise 6.2

Find the following indefinite integrals

1. (a) $\int(6x+1)\,dx$. (b) $\int(3x-1)^2\,dx$. (c) $\int x(4x+1)\,dx$.

2. (a) $\int\left(3-\frac{1}{x^2}\right)dx$. (b) $\int\left(x-\frac{1}{x}\right)^2 dx$. (c) $\int\frac{3x+1}{x^3}\,dx$.

3. (a) $\int 4\sqrt{x}\,dx$. (b) $\int\frac{x-1}{\sqrt{x}}\,dx$. (c) $\int 4(x^{1/3}+x^{-1/3})\,dx$.

4. (a) $\int\left(3t^2-\frac{1}{t^2}\right)dt$. (b) $\int u(1-\sqrt{u})\,du$. (c) $\int\frac{s^3+1}{s^3}\,ds$.

Evaluate the following definite integrals

5. (a) $\int_{-1}^{1}(4x+3)\,dx$ (b) $\int_{1}^{3}(3x^2-x)\,dx$. (c) $\int_{1}^{25}x^{1/2}\,dx$.

6. (a) $\int_{1}^{2}\left(x+\frac{1}{x}\right)^2 dx$. (b) $\int_{-2}^{-1}\frac{2x^4+1}{x^4}\,dx$. (c) $\int_{1}^{4}\frac{5x^2-4}{\sqrt{x}}\,dx$.

7. (a) $\int_{-3}^{3}u(u-1)^2\,du$. (b) $\int_{1}^{27}y^{1/3}\,dy$. (c) $\int_{1}^{2}\frac{3t^4-1}{t^2}\,dt$.

8. Find the areas bounded by the x-axis and the following curves and straight lines.
(a) $y=4x^2, x=0, x=1$, (b) $y=x^3-x, x=1, x=2$,
(c) $y=18/x^3, x=1, x=3$, (d) $y=3\sqrt{x}, x=4, x=9$.

9. Find the areas enclosed by the x-axis and the following curves
(a) $y=(1-x)(x+2)$, (b) $y=3x^2-x^3$,
(c) $y=4x^2-4x-3$ (d) $y=x^4-8x$.

10. Find the areas of the regions bounded by the following curves and straight lines.
(a) $y=(5-x)(2+x), y=10$, (b) $y=(x-1)^2, y=9$,
(c) $y=x(3-x), y=x$, (d) $y=(x^2-1)(x-2), y=2-2x$,
(e) $y=2\sqrt{x}, 2x-3y+4=0$, (f) $y=9\left(x+\frac{1}{x^2}\right), y=7(x+1)$.

11. Find the area of the smaller closed region bounded by the curve $y=x^2$, the straight line $2x+y=8$ and the x-axis.

12. Sketch the curves $y = x^2$ and $y^2 = x$. Find the coordinates of the points of intersection of the curves. Calculate the area enclosed between the two curves.

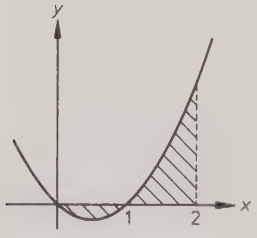
6.3 Properties of definite integrals

As shown in the previous section, the definite integral produces positive values for areas above the x-axis but negative values for areas below the x-axis. Thus difficulties may arise when a curve crosses the x-axis inside the range of integration.

Example 1 (a) Evaluate $\int_0^2 x(x-1)\,dx$. (b) Find the total area enclosed by the curve $y = x(x-1)$, the x-axis and the line $x = 2$.

(a) $\int_0^2 x(x-1)\,dx = \int_0^2 (x^2-x)\,dx = \left[\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2\right]_0^2 = (\tfrac{1}{3}.2^3 - \tfrac{1}{2}.2^2) - 0 = \tfrac{2}{3}$.

(b) $y = 0 \Leftrightarrow x(x-1) = 0$
$\Leftrightarrow x = 0$ or $x = 1$



$\therefore$ the curve cuts the x-axis at the points $(0,0)$ and $(1,0)$.

$$\int_0^1 (x^2-x)\,dx = \left[\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2\right]_0^1$$
$$= \tfrac{1}{3} - \tfrac{1}{2} = -\frac{1}{6}$$

$\therefore$ the shaded area below the x-axis is 1/6.

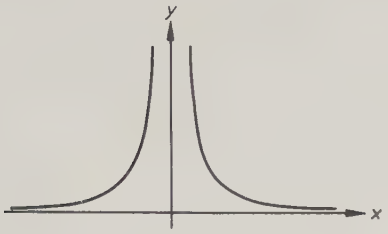
$$\int_1^2 (x^2-x)\,dx = \left[\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2\right]_1^2$$
$$= \frac{2}{3} - \left(-\frac{1}{6}\right) = \frac{5}{6}$$

$\therefore$ the shaded area above the x-axis is 5/6.

Hence the total area enclosed by the curve $y = x(x-1)$, the x-axis and the line $x = 2$ is 1.

[Note that the definite integral gave the difference between the area above the x-axis and the area below it, i.e. $\frac{5}{6} - \frac{1}{6} = \frac{2}{3}$, instead of the sum $\frac{5}{6} + \frac{1}{6} = 1$, which we required.]

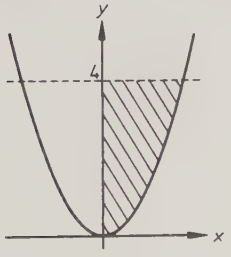
Another type of difficulty arises when we consider a curve such as $y = 1/x^2$.



The integral $\int_a^b \frac{1}{x^2}dx$ can be used to find areas under the curve as long as a and b are either both positive or both negative. However, there is a break or *discontinuity* in the curve at $x = 0$, as shown in the sketch. It is, therefore, quite meaningless to talk about the area under the curve between such values as $x = -1$ and $x = 2$. Although the integral $\int_{-1}^{2} \frac{1}{x^2}dx$ can be evaluated in the usual way, the result is not valid and does not represent an area under the curve $y = 1/x^2$.

The area bounded by a curve, the y-axis and lines $y = \alpha$ and $y = \beta$ is given by a definite integral of the form $\int_\alpha^\beta x\,dy$, in which the roles of x and y are interchanged.

Example 2 Find the area bounded by the curve $y = x^2$ and the line $y = 4$.



The required area is twice the shaded area shown in the sketch.

Shaded area

$$= \int_0^4 x\,dy = \int_0^4 y^{1/2}\,dy = \left[\tfrac{2}{3}y^{3/2}\right]_0^4$$

$$= 16/3 = 5\tfrac{1}{3}.$$

Hence the area bounded by the curve $y = x^2$ and the line $y = 4$ is $10\frac{2}{3}$.

Two general properties of the definite integral may be proved as follows:

Let $\int f(x)\,dx = F(x) + c$, then

$$\int_b^a f(x)\,dx = \Big[F(x)\Big]_b^a = F(a) - F(b) = -\{F(b) - F(a)\} = -\Big[F(x)\Big]_a^b$$

$$= -\int_a^b f(x)\,dx.$$

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \{F(b) - F(a)\} + \{F(c) - F(b)\} = F(c) - F(a)$$

$$= \int_a^c f(x)\,dx.$$

Summarising:

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx, \quad \int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx.$$

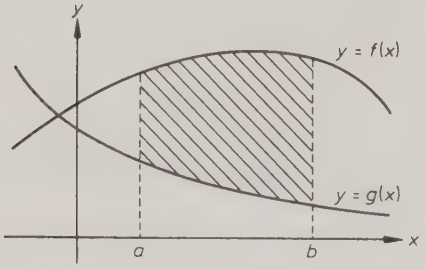
It may also be shown using the properties of limits that

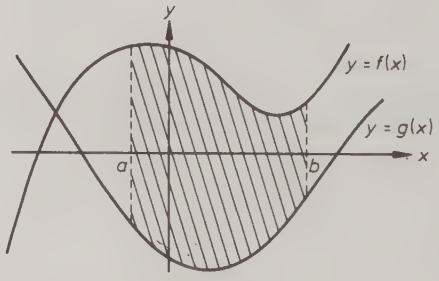
$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx \quad \text{where } k \text{ is a constant}$$

$$\int_a^b f(x)\,dx + \int_a^b g(x)\,dx = \int_a^b \{f(x)+g(x)\}\,dx.$$

These properties can be used when finding the area between two curves. For instance, in the cases illustrated below, the shaded area is given by

$$\int_a^b \{f(x)-g(x)\}\,dx.$$





Example 3 Find the area enclosed by the curves $y = 2+x-x^2$ and $y = 2-3x+x^2$.

At the points of intersection of the two curves

$$2+x-x^2 = 2-3x+x^2$$
$$\therefore \quad 4x-2x^2 = 0$$
$$\therefore \quad x(2-x) = 0$$

$\therefore$ either $x = 0, y = 2$ or $x = 2, y = 0$.

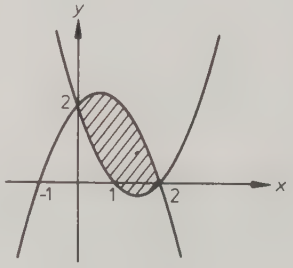
Hence the required area

$$= \int_0^2 (2+x-x^2)\,dx - \int_0^2 (2-3x+x^2)\,dx$$

$$= \int_0^2 (4x-2x^2)\,dx$$

$$= \left[2x^2 - \tfrac{2}{3}x^3\right]_0^2$$

$$= 8-16/3 = 2\tfrac{2}{3}.$$



Exercise 6.3

1. Find the total areas of the regions enclosed by the following curves and straight lines
(a) $y = (x+1)(x-3)$, $y = 0$ and $x = -3$,
(b) $y = x^3 - 3x^2$, $y = 0$ and $x = 4$,
(c) $y = x^3 + x^2 - 2x$ and the x-axis,
(d) $y = x^2 - 5 + \frac{4}{x^2}$ and the x-axis.

2. Find the area bounded by the curve $y = x^2$ and the line $y = 9$.

3. Find the area bounded by the curve $y^2 = 4 - x$ and the y-axis.

4. Find the area bounded by the curve $y = (x-1)^3$, the y-axis and the line $y = 8$.

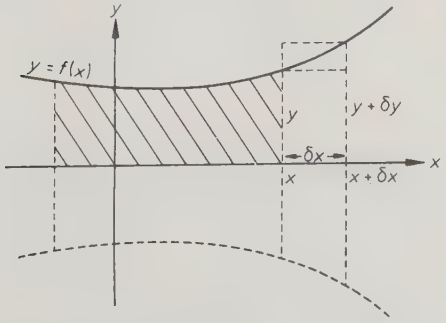
5. Find the area bounded by the curve $y = 1/(x+1)^2$, the y-axis and the line $y = 4$.

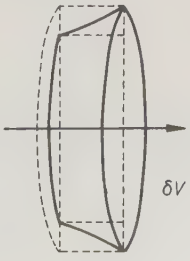
6. Find the areas of the finite regions bounded by the following curves and straight lines
(a) $y = x(4-3x)$ and $y = x$,
(b) $y = x^2 + 2x + 3$ and $y = 2(x+2)$,
(c) $y = 3 + 2x - x^2$ and $y = 3 - 2x$.

7. Sketch the following pairs of curves and find the area of the regions enclosed by them.
(a) $y = 2x^2$, $y = 3 - x^2$, (b) $y = x^2 - 3x$, $y = 2 - x^2$,
(c) $y = 3x^2 + 9x + 5$, $y = x^2 + x - 1$.

6.4 Volumes of revolution

Consider the volume, V, generated when the shaded area between the curve $y = f(x)$ and the x-axis is rotated completely about the x-axis.





Let δx be a small increase in x and let δy and δV be the corresponding increases in y and V. The volume δV must lie between the volumes of two cylinders or discs of thickness δx and radii y and $y+\delta y$,

$$\text{i.e.} \qquad \pi y^2\,\delta x < \delta V < \pi(y+\delta y)^2\,\delta x$$

$$\therefore \quad \pi y^2 < \frac{\delta V}{\delta x} < \pi(y+\delta y)^2$$

$$\text{As} \quad \delta x \to 0, \quad \delta y \to 0 \quad \text{and} \quad \frac{\delta V}{\delta x} \to \frac{dV}{dx}$$

$$\therefore \quad \frac{dV}{dx} = \pi y^2.$$

Hence the volume generated between $x = a$ and $x = b$ is given by

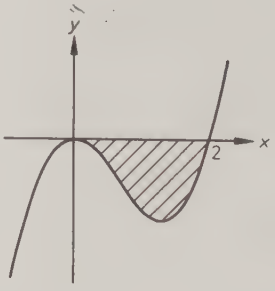
$$\boxed{\int_a^b \pi y^2\,dx.}$$

Example 1 Find the volume generated by rotating completely about the x-axis the area bounded by the curve $y = x^3 - 2x^2$ and the x-axis.

$y = 0 \Leftrightarrow x^3 - 2x^2 = 0 \Leftrightarrow x = 0$ or $x = 2$

$\therefore$ the curve cuts the x-axis at $(0,0)$ and $(2,0)$.

Hence the required volume $= \displaystyle\int_0^2 \pi y^2\,dx$



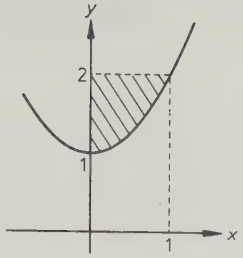
$$= \pi \int_0^2 (x^3 - 2x^2)^2\,dx$$

$$= \pi \int_0^2 (x^6 - 4x^5 + 4x^4)\,dx$$

$$= \pi \left[\frac{1}{7}x^7 - \frac{4}{6}x^6 + \frac{4}{5}x^5\right]_0^2$$

$$= \pi\left(\frac{1}{7}.2^7 - \frac{2}{3}.2^6 + \frac{4}{5}.2^5\right) = \pi.2^7\left(\frac{1}{7} - \frac{1}{3} + \frac{1}{5}\right) = \frac{128\pi}{105}.$$

Example 2 Find the volumes generated by rotating completely about (a) the y-axis, (b) the x-axis, the area bounded by that part of the curve $y = x^2 + 1$ for which x is positive, the y-axis and the line $y = 2$.



(a) The volume generated by rotation about the y-axis $= \int_1^2 \pi x^2\,dy$

$$= \int_1^2 \pi(y-1)\,dy$$

$$= \pi\left[\tfrac{1}{2}y^2 - y\right]_1^2$$

$$= \pi\{(\tfrac{1}{2}.2^2 - 2) - (\tfrac{1}{2} - 1)\} = \tfrac{1}{2}\pi.$$

(b) [Note that in this case the volume generated by rotation about the x-axis cannot be calculated directly.]

Let V_1 be the volume of the cylinder generated by rotating about the x-axis the rectangle bounded by the axes and the lines $x = 1$, $y = 2$.

Thus $V_1 = \pi.2^2.1 = 4\pi$.

Let V_2 be the volume generated by rotating about the x-axis the area bounded by the curve, the axes and the line $x = 1$.

Thus $V_2 = \int_0^1 \pi y^2\,dx = \pi\int_0^1 (x^2+1)^2\,dx$

$$= \pi\int_0^1 (x^4 + 2x^2 + 1)\,dx$$

$$= \pi\left[\frac{1}{5}x^5 + \frac{2}{3}x^3 + x\right]_0^1$$

$$= \pi\left(\frac{1}{5} + \frac{2}{3} + 1\right) = \frac{28\pi}{15}$$

Hence the required volume $= V_1 - V_2 = \left(4 - \frac{28}{15}\right)\pi = \frac{32\pi}{15}$.

Exercise 6.4

[Answers may be left in terms of π.]

1. Find the volumes of the solids formed by rotating completely about the x-axis the areas bounded by the x-axis and the given curves and lines.

(a) $y = 5x^2$, $x = -1$, $x = 3$,
(b) $y = 3x - x^2$, $x = 1$, $x = 2$,
(c) $y = 2x - 5$, $x = 1$, $x = 4$,
(d) $y = x^3 + 1$, $x = 2$,
(e) $y = 1 + \sqrt{x}$, $x = 4$, $x = 9$,
(f) $y = x + \frac{1}{x}$, $x = \frac{1}{2}$, $x = 2$.

2. Find the volumes of the solids formed by rotating completely about the x-axis the areas enclosed by each of the following curves and the x-axis (a) $y = x^2 - 4$, (b) $3y = x^2(3-x)$.

3. Find the volumes of the solids formed by rotating through 2 right angles about the x-axis the regions bounded by
(a) the curve $y^2 = x^3$ and the lines $x = 1, x = 2$,
(b) the curve $y^2 = 2(5-x)$ and the line $x = -1$.

4. Find the volumes of the solids formed by rotating completely about the y-axis the areas enclosed by the y-axis and the following curves and straight lines
(a) $y = \sqrt{x}, y = 2$,
(b) $y = x-1, y = 0, y = 1$,
(c) $y = 1/x, y = 1, y = 2$,
(d) $y = x^3, y = 8$.

5. Find the volumes of the solids formed when the region bounded by the curve $y = x^2+3$ and the straight lines $y = 0$, $x = 0$ and $x = 2$ is rotated completely about (a) the x-axis, (b) the y-axis.

6. Find the volumes of the solids formed when the smaller region enclosed by the curve $y^2 = 4-x$, the y-axis and the line $y = 1$ is rotated completely about (a) the x-axis, (b) the y-axis.

7. By considering a solid of revolution generated by the triangle with vertices $(0,0)$, (r,h) and $(0,h)$, find the volume of a cone height h, base radius r.

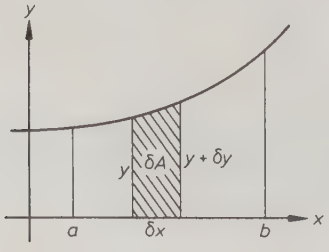
8. Given that the equation of the circle with radius a and centre the origin is $x^2+y^2 = a^2$, find by integration the volume of a sphere of radius a.

9. Find the volume generated by rotating completely about the x-axis the finite region bounded by the curve $y^2 = 4x$ and the straight line $y = x$.

10. Find the volume generated by rotating completely about the x-axis the finite region bounded by the curve $y = x^3$, the x-axis and the line $x+3y = 4$.

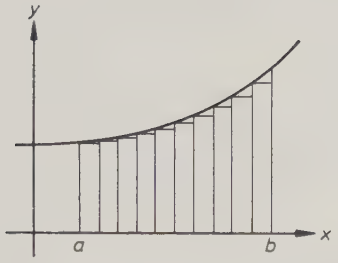
6.5 Integration as the limit of a sum

In §6.2 the definite integral $\int_a^b f(x)\,dx$ was introduced as the area bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$. We arrive at a formal definition of the definite integral by considering the area to be the sum of strips called *elements* of area.



The diagram shows a typical element of area δA. The value of $\int_a^b f(x)\,dx$ is the sum of the areas of all such elements. Using the Greek capital letter 'sigma' Σ to mean 'the sum of', this may be written

$$\int_a^b f(x)\,dx = \sum_{x=a}^{x=b} \delta A.$$



For small values of δx the area δA is approximately equal to the area of a rectangle with height y and width δx. Similarly the total area $\Sigma \delta A$ is approximately equal to the sum of the areas of such rectangles $\Sigma y\,\delta x$. If the number of strips is increased so that $\delta x \to 0$, then the difference between the sums $\Sigma \delta A$ and $\Sigma y\,\delta x$ also approaches zero. Hence the area under the curve $y = f(x)$ is given by the limit as δx tends to zero of $\Sigma y\,\delta x$. Thus the intuitive approach to the definite integral through areas is consistent with the following formal definition:

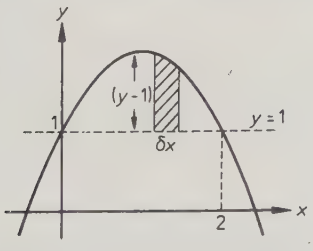
$$\int_a^b f(x)\,dx = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} f(x)\,\delta x.$$

The fact that the definite integral, considered as the limit of a sum, can be found using a process which is the reverse of differentiation is called the *fundamental theorem of calculus.*

These results indicate that it is often possible to use integration to find a quantity expressed as a sum of elements of the form $f(x)\,\delta x$. For example, the formula for a volume of revolution derived in §6.5 may also be obtained by considering a typical element of volume $\delta V \approx \pi y^2\,\delta x$. The volume generated by rotating about the x-axis the area bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$, is then given by

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} \pi y^2\,\delta x = \int_a^b \pi y^2\,dx.$$

Example 1 Find the volume generated by rotating completely about the line $y = 1$, the area bounded by the curve $y = 1 + 2x - x^2$ and the line $y = 1$.



At the points of intersection of the curve $y = 1 + 2x - x^2$ and the line $y = 1$,

$$1 + 2x - x^2 = 1$$
$$2x - x^2 = 0$$
$$x(2 - x) = 0$$

$\therefore$ either $x = 0$ or $x = 2$.

A typical element of the required volume is a disc, radius $(y-1)$ and thickness δx

$\therefore$ element of volume $\approx \pi(y-1)^2\,\delta x$

Hence the required volume $= \lim_{\delta x \to 0} \Sigma \pi (y-1)^2 \, \delta x$

$$= \int_0^2 \pi (y-1)^2 \, dx$$

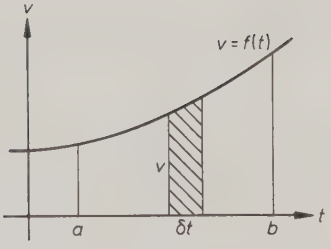
$$= \pi \int_0^2 (2x - x^2)^2 \, dx$$

$$= \pi \int_0^2 (4x^2 - 4x^3 + x^4) \, dx$$

$$= \pi \left[\frac{4}{3} x^3 - x^4 + \frac{1}{5} x^5 \right]_0^2$$

$$= \pi \left(\frac{4}{3} . 2^3 - 2^4 + \frac{1}{5} . 2^5 \right)$$

$$= \pi \left(\frac{32}{3} - 16 + \frac{32}{5} \right) = \frac{16\pi}{15}.$$

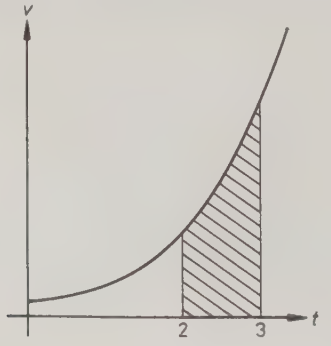


Consider now a particle moving along a straight line, so that its velocity at time t is given by $v = f(t)$. In a short interval of time δt, the displacement of the particle is approximately $v \, \delta t$. Hence the total displacement between $t = a$ and $t = b$ is given by $\lim_{\delta t \to 0} \sum_{t=a}^{t=b} v \, \delta t$.

Thus the total displacement of the particle in any given interval of time is represented by an area under the velocity-time graph and is found by evaluating a definite integral of the form $\int_a^b v \, dt$.

Example 2 A particle moves in a straight line so that after t seconds its velocity in $\mathrm{m\,s^{-1}}$ is given by $v = 2t^3 + 3t + 5$. Find the distance moved in the third second of the motion.

Since the velocity of the particle is positive throughout the motion, the distance moved in the third second



$$= \text{total displacement between } t = 2 \text{ and } t = 3$$

$$= \int_2^3 (2t^3 + 3t + 5) \, dt \text{ metres}$$

$$= \left[\frac{1}{2} t^4 + \frac{3}{2} t^2 + 5t \right]_2^3 \text{ metres}$$

$$= \left(\frac{81}{2} + \frac{27}{2} + 15 \right) - (8 + 6 + 10) \text{ metres}$$

$$= 45 \text{ metres.}$$

[Note that if a particle moving along a straight line changes direction in the interval from $t = a$ to $t = b$ then the total distance travelled in that interval is *not* equal to the total displacement of the particle between $t = a$ and $t = b$.]

Exercise 6.5

1. Find the volume of the solid formed by rotating completely about the line $y = 1$ the area bounded by the curve $y = x^2 + 2$, the y-axis and the lines $x = 1$ and $y = 1$.

2. Find the volume generated by rotating completely about the line $y = 3$, the area enclosed by the curve $y = 4x - x^2$ and the line $y = 3$.

3. A particle moves in a straight line so that after t seconds its velocity is v m s^{-1}. In each of the following cases sketch the velocity-time graph for the motion and find the total distance moved in the given interval of time.

(a) $v = 10 + 2t$, $t = 0$ to $t = 2$,
(b) $v = 16t - 3t^2$, the 3rd second,
(c) $v = \frac{1}{2}t^2 + 5$, $t = 1$ to $t = 3$,
(d) $v = 18t^2 - 4t^3$, the 4th second,
(e) $v = 5 + 4t - t^2$, $t = 1$ to $t = 6$,
(f) $v = 4t^3 - 16t$, $t = 0$ to $t = 3$.

4. Find the volume of the solid formed by rotating completely about the line $y = 2$ the region enclosed by the curve $y = x(2 - x)$ and the x-axis.

5. Find the volume of the solid formed by rotating through 2 right angles about the line $x = 2$ the area bounded by the curve $y = x^2 - 4x + 5$ and the line $y = 5$.

6. Use integration to find a formula for the volume of a pyramid of height h cm with a square base of side a cm.

7. The horizontal cross-section of a bowl at a height h cm above its base is a regular hexagon of side x cm. Given that $x^2 = 16(h + 1)$, find the volume of liquid in the bowl when it is filled to a depth of 8 cm.

8. By considering the area under the curve $y = \sqrt{x}$ between $x = 0$ and $x = 81$ as a sum of strips of width 1 unit, show that

(a) $\int_0^{81} \sqrt{x}\,dx < \sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{81}$,

(b) $\sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{80} < \int_0^{81} \sqrt{x}\,dx$.

Deduce that $477 < \sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{80} < 486$.

9. By considering the area under the curve $y = 1/x^2$ between $x = 1$ and $x = 100$, show that

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{100^2} < 2.$$

6.6 Mean value

The mean value of a function $f(x)$ between $x = a$ and $x = b$ is a type of average of the values of the function in that interval. For example, the mean value of a function $f(t)$ which represents the velocity of a moving object after time t is the same as the average velocity.

Example 1 An object is fired vertically upwards so that after time t its velocity is given by $v = 40 - 10t$. Find the average velocity between $t = 1$ and $t = 3$.

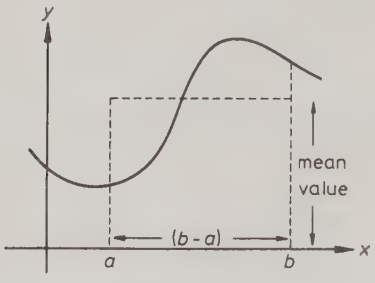
The total displacement between $t = 1$ and $t = 3$

$$= \int_1^3 (40 - 10t)\,dt = \Big[40t - 5t^2\Big]_1^3 = 75 - 35 = 40$$

$\therefore$ the average velocity between $t = 1$ and $t = 3$

$$= \frac{40}{3-1} = \frac{40}{2} = 20.$$

Similarly, the average or mean velocity of an object moving in a straight line, between $t = a$ and $t = b$, is found by evaluating $\dfrac{1}{(b-a)}\displaystyle\int_a^b v\,dt$. The general definition of mean value is framed along these lines.



The *mean value* of a function $f(x)$ between $x = a$ and $x = b$ is

$$\frac{1}{(b-a)}\int_a^b f(x)\,dx.$$

Geometrically, this is the height of the rectangle, with base $(b-a)$, whose area is equal to the area under the curve from $x = a$ to $x = b$.

Example 2 The volume, $V\,\text{cm}^3$, of water in a hemispherical bowl is given by $V = \frac{1}{3}\pi(30h^2 - h^3)$ where $h\,\text{cm}$ is the depth of the water. Find the mean volume of water in the bowl as the depth increases from 0 to 4 cm.

$$\text{Mean volume (in cm}^3) = \frac{1}{4-0}\int_0^4 V\,dh = \frac{1}{4}\int_0^4 \frac{1}{3}\pi(30h^2 - h^3)\,dh$$

$$= \frac{1}{12}\pi\left[10h^3 - \frac{1}{4}h^4\right]_0^4$$

$$= \frac{1}{12}\pi\left(10.4^3 - \frac{1}{4}.4^4\right)$$

$$= \frac{1}{12}\pi.9.4^3 = 48\pi.$$

Exercise 6.6

1. A particle is moving in a straight line such that after t seconds its velocity is $v\,\mathrm{m\,s^{-1}}$, where $v = 6t + 12t^2$. Find
(a) the mean velocity during the first 2 seconds of the motion,
(b) the mean acceleration between $t = 1$ and $t = 5$.

2. Repeat question 1 for $v = 2t + \dfrac{5}{t^2}$.

3. Find the mean values of the following functions in the given intervals.
(a) $2x^3 - x + 1$, $-1 \leqslant x \leqslant 1$, (b) $(x+3)(2x-5)$, $1 \leqslant x \leqslant 5$,
(c) $1 - \dfrac{1}{x^2}$, $0{\cdot}1 \leqslant x \leqslant 1$, (d) $(5x-3)\sqrt{x}$, $1 \leqslant x \leqslant 4$.

4. The volume, $V\,\mathrm{cm}^3$, of water in a conical vessel is given by $V = \pi h^3/12$, where $h\,\mathrm{cm}$ is the depth of the water. Find the mean volume of water in the bowl as the depth increases from 2 cm to 8 cm.

5. A mass of gas of volume v at pressure p expands according to the law $pv^{3/4} = 30$. Find the mean pressure as the gas expands from $v = 1$ to $v = 16$.

6. Given that $x = 4t - 1$, find the mean values in the interval from $t = 1$ to $t = 4$ of (a) x^2, (b) $1/(x+1)^2$, (c) $(x+1)^{3/2}$.

Exercise 6.7 (miscellaneous)

[Answers may be left in terms of π, where appropriate.]

1. Sketch the families of curves with the following gradient functions

(a) $\dfrac{dy}{dx} = 2(1-x)$, (b) $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}}$, (c) $\dfrac{dy}{dx} = 1 + \dfrac{1}{x^3}$.

2. Integrate with respect to x

(a) $4x^5 - x^3$, (b) $4 - \dfrac{3}{x^2}$, (c) $\dfrac{x^4+1}{x^3}$.

3. Find the following indefinite integrals

(a) $\displaystyle\int (2-3t)^2\,dt$, (b) $\displaystyle\int \left(5x^4 - \frac{1}{5x^4}\right)dx$, (c) $\displaystyle\int \frac{1}{\sqrt[3]{u}}\,du$.

4. Given that $\dfrac{dv}{dt} = (t+1)(3t-7)$ and that $v = 36$ when $t = 5$, show that $v = (t+1)^2(t-4)$.

5. The gradient of a curve is given by $\dfrac{dy}{dx} = ax + \dfrac{b}{x^2}$. If the curve passes through

the point $(-1, -4)$ and has a turning point at the point $(1,0)$, find its equation and sketch the curve.

6. Evaluate (a) $\int_1^2 \frac{(x+1)(x-2)}{x^4}\,dx$, (b) $\int_1^4 \left(\sqrt{x} - \frac{1}{x}\right)^2 dx$.

7. Find the area enclosed by the curve $y = x - \frac{1}{x^2}$, the x-axis and the line $x = 2$.

8. Find the area enclosed by the curve $y = 2x^2 + 1$ and the line $y = 3$.

9. Find the total area of the regions enclosed by the curve $y = x^4 - x^3$, the x-axis and the line $x = 2$.

10. Find the area enclosed by the curve $y^2 - 2y = x$ and the y-axis.

11. Find the equation of the tangent at the point $(1,0)$ to the curve $y = (x-1)(x^2+1)$. Find the area of the finite region enclosed by the curve and this tangent.

12. Sketch the region in the x-y plane within which the inequalities $y > (x-2)^2$ and $y < x$ are satisfied. Determine the area of this region.

13. Sketch the curves $y = x^2(4x-5)$ and $y = x(3x-4)$ on the same diagram, then find the area of the finite region enclosed between them.

14. When the area bounded by the curve $y = 15x(k-x)$ and the x-axis is rotated completely about the x-axis the volume of the solid generated is 240π. Find the value of k.

15. The region R is bounded by the curve $y = 4/x$, the x- and y-axes and the lines $x = 2$, $y = 4$. Find the volume of the solid formed when R is rotated completely about (a) the x-axis, (b) the y-axis.

16. The region R is bounded by the curve $4y = x^2$ and the lines $x = 4$, $y = 1$. Find the volume of the solid formed when R is rotated completely about (a) the x-axis, (b) the y-axis, (c) the line $y = 1$, (d) the line $x = 4$.

17. The region R is bounded by the curve $y^2 = 2x$ and the line $y = x$. Find the volume of the solid formed when R is rotated completely about (a) the x-axis, (b) the y-axis.

18. The curves $cy^2 = x^3$ and $y^2 = ax$ (where $a > 0$ and $c > 0$) intersect at the origin O and at a point P in the first quadrant. The areas of the regions enclosed by the arcs OP, the x-axis and the ordinate through P are A_1 and A_2 for the two curves; the volumes of the two solids formed by rotating these regions through four right angles about the x-axis are V_1 and V_2 respectively. Prove that $A_1/A_2 = 3/5$ and $V_1/V_2 = 1/2$.

19. Find dy/dx given that $y^2 = x(3-x)^2$. State the values of x for which dy/dx is (a) zero, (b) infinite. Sketch the curve $y^2 = x(3-x)^2$ and find the volume of the solid formed when the area enclosed by the loop of the curve is rotated through 180° about the x-axis.

20. Evaluate $\int_{-2}^{4} [x]\,dx$, where $[x]$ is defined as the greatest integer less than or equal to x. (See §5.1.)

21. Evaluate $\int_{-4}^{2} |x|\,dx$, where $|x|$ is defined as the magnitude of x. (See §1.4.)

22. A particle moves in a straight line such that after t seconds its velocity is $v\,\mathrm{m\,s^{-1}}$, where $v = 4-4t-15t^2$. Find the *total* distance moved by the particle in (a) the first two seconds, (b) the third second of its motion.

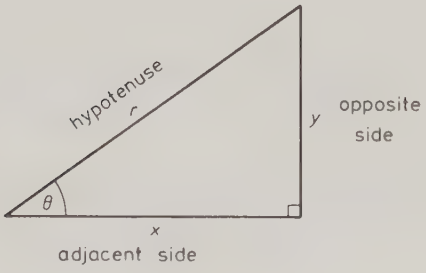
23. Show that over the range $0 \leqslant x \leqslant 6$, the mean value of the function $9x(6-x)$ is two-thirds of the maximum value of the function. (AEB 1975)

24. Find the mean area of vertical cross-sections of a sphere of radius a.

7 Elementary trigonometry

7.1 Trigonometric ratios for any angle

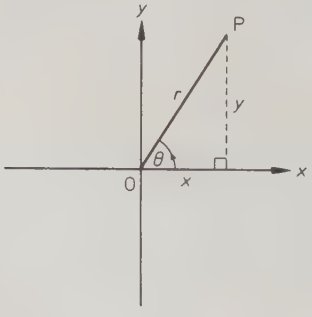
Elementary trigonometry is based on the fact that in any set of similar right-angled triangles the ratios of corresponding pairs of sides remain constant. Using the right-angled triangle in the diagram, the sine and cosine of angle θ are defined as follows:



$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{y}{r}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{x}{r}.$$

In more advanced work it is useful to extend these definitions so that they can be applied to angles of any magnitude.

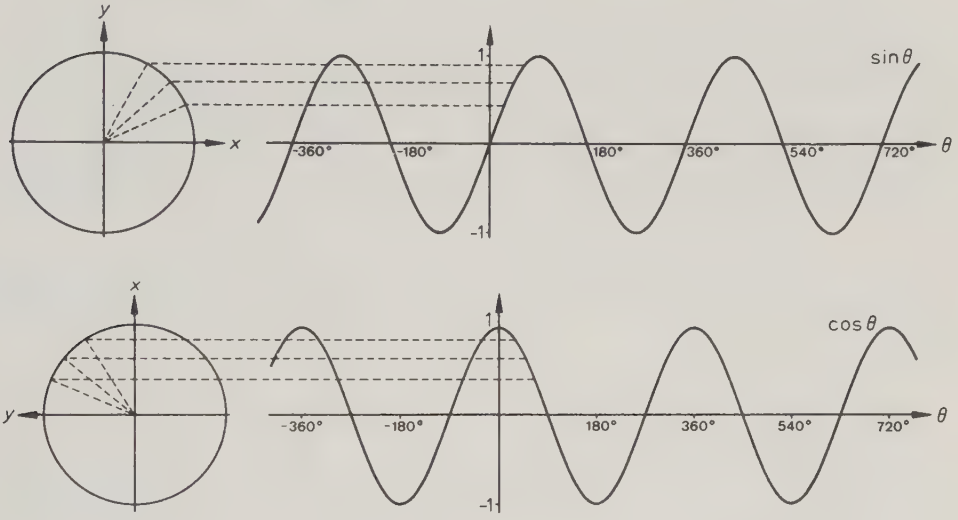


Consider a point P with coordinates (x, y). Let $OP = r$ and let θ be the angle OP makes with the positive direction of the x-axis. Positive angles are measured anti-clockwise from the x-axis and negative angles clockwise.

Sine and cosine may now be defined for any real value of θ by writing:

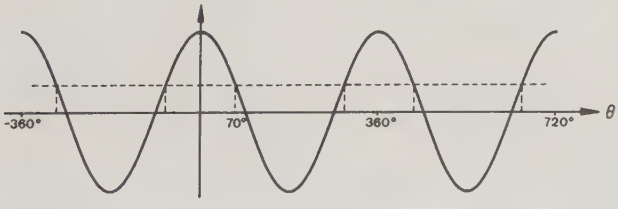
$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r}$$

The graphs of $\sin\theta$ and $\cos\theta$ may be plotted directly as shown below. Various points P on a circle of radius 1 unit are used, so that $\sin\theta = y$ and $\cos\theta = x$.



After OP has been rotated through 360°, $\sin\theta$ and $\cos\theta$ pass through the sets of values they have already taken. For this reason $\sin\theta$ and $\cos\theta$ are described as *periodic functions* with *period* 360°.

Example 1 Using a sketch graph if necessary, write down all the angles between $-360°$ and 720° with the same cosine as 70°.



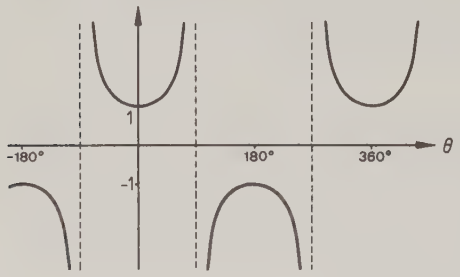
From the sketch the angles with the same cosine as 70° are: $-290°$, $-70°$, 70°, 290°, 430° and 650°.

The sine and cosine of an angle can be used to define the remaining trigonometric ratios, tangent, cotangent, secant, cosecant.

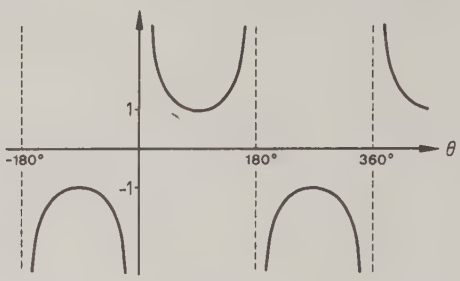
$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \operatorname{cosec}\theta = \frac{1}{\sin\theta}.$$

Because of their relationship with $\cos\theta$ and $\sin\theta$, $\sec\theta$ and $\operatorname{cosec}\theta$ must also be periodic functions with period 360°. Since the range of both $\cos\theta$ and $\sin\theta$ is the set of real numbers from -1 to $+1$, neither $\sec\theta$ nor $\operatorname{cosec}\theta$ can take values between -1 and $+1$.

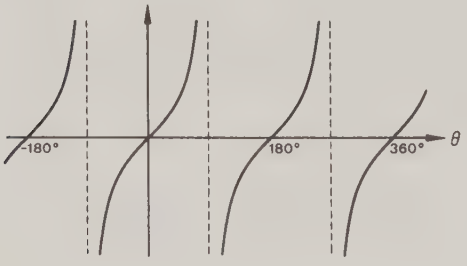


$\sec\theta$

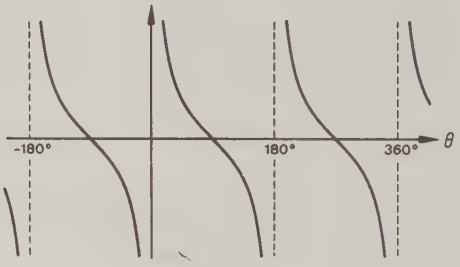


$\operatorname{cosec}\theta$

However, the functions $\tan\theta$ and $\cot\theta$ take all real values and have a period of 180°.



$\tan\theta$



$\cot\theta$

In terms of the coordinates (x, y) of the point P considered earlier, the six trigonometric ratios may now be defined as follows:

$$\cos\theta = \frac{x}{r} \qquad \sin\theta = \frac{y}{r} \qquad \tan\theta = \frac{y}{x}$$

$$\sec\theta = \frac{r}{x} \qquad \operatorname{cosec}\theta = \frac{r}{y} \qquad \cot\theta = \frac{x}{y}$$

The x- and y-axes divide the plane into four regions called *quadrants*.

Second quadrant	First quadrant
Third quadrant	Fourth quadrant

x −ve y +ve	x +ve y +ve
x −ve y −ve	x +ve y −ve

When P lies in the first quadrant, θ is acute and all the trigonometric ratios are positive. By examining the signs of x and y in the other quadrants, the signs of $\sin\theta$, $\cos\theta$ and $\tan\theta$ for any value of θ can be determined.

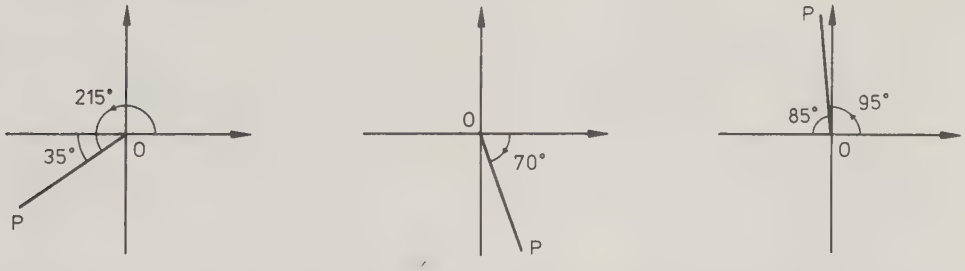
$\sin\theta$ +ve $\cos\theta$ −ve $\tan\theta$ −ve	$\sin\theta$ +ve $\cos\theta$ +ve $\tan\theta$ +ve
$\sin\theta$ −ve $\cos\theta$ −ve $\tan\theta$ +ve	$\sin\theta$ −ve $\cos\theta$ +ve $\tan\theta$ −ve

SIN +ve	ALL +ve
TAN +ve	COS +ve

The above diagrams show which ratios are positive in each quadrant. There are many mnemonics for the initial letters of All, Sin, Tan, Cos, e.g. All Stations To Crewe, All Steam Tugs Chug.

The trigonometric ratios for any angle θ are numerically equal to the corresponding ratios for the acute angle between OP and the x-axis. The quadrant in which OP lies determines the sign of the ratio, positive or negative.

Example 2 Express the following in terms of ratios of acute angles:
(a) $\sin 215°$, (b) $\cos(-70°)$, (c) $\tan 95°$.

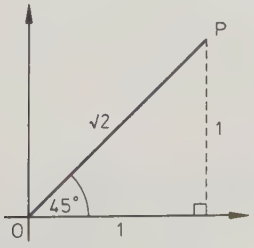


(a) [We note mentally that the acute angle between OP and the x-axis is 35° and that OP lies in the 3rd quadrant.] From the sketch, $\sin 215° = -\sin 35°$.
(b) $\cos(-70°) = \cos 70°$, (c) $\tan 95° = -\tan 85°$.
[These results can also be obtained using sketch graphs of $\sin\theta$, $\cos\theta$ and $\tan\theta$.]

In general calculators, slide rules or mathematical tables are used to write down values of trigonometric ratios. However, in some cases such aids are not needed.

$\sin 0° = 0$	$\cos 0° = 1$	$\tan 0° = 0$
$\sin 90° = 1$	$\cos 90° = 0$	$\tan 90°$ is undefined.

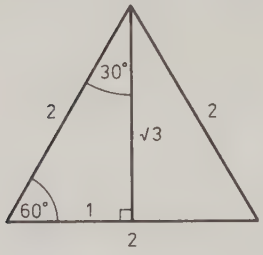
[Considering values of θ close to 90° we find that as $\theta \to 90°$ from below, $\tan\theta \to \infty$ and as $\theta \to 90°$ from above, $\tan\theta \to -\infty$.]



Let P be the point $(1, 1)$, so that $OP = \sqrt{2}$ and $\theta = 45°$.

Hence $\sin 45° = \cos 45° = \dfrac{1}{\sqrt{2}}$

and $\tan 45° = 1$.



Using an equilateral triangle with sides each 2 units in length and height $\sqrt{3}$ units, we have

$$\sin 30° = \frac{1}{2} \qquad \sin 60° = \frac{\sqrt{3}}{2}$$

$$\cos 30° = \frac{\sqrt{3}}{2} \qquad \cos 60° = \frac{1}{2}$$

$$\tan 30° = \frac{1}{\sqrt{3}} \qquad \tan 60° = \sqrt{3}.$$

Exercise 7.1

1. Express the following in terms of ratios of acute angles
(a) $\cos 165°$, (b) $\sin(-35°)$, (c) $\tan 310°$, (d) $\sin 112°$,
(e) $\tan(-95°)$, (f) $\cos 318°$, (g) $\sin 195°$, (h) $\cos(-237°)$.

2. Express the following in terms of ratios of acute angles
(a) $\tan 124°$, (b) $\sec(-23°)$, (c) $\cot 248°$, (d) $\sin 702°$,
(e) $\operatorname{cosec} 280°$, (f) $\cot(-15°)$, (g) $\cos 186°$ (h) $\sec 99°$.

3. Using sketch graphs if necessary, write down all the angles between $-360°$ and $720°$ with
(a) the same sine as $40°$, (b) the same cosine as $10°$,
(c) the same tangent as $75°$, (d) the same secant as $180°$.

4. Find all the values of θ between $0°$ and $360°$ such that
(a) $\cos\theta = \cos 25°$, (b) $\sin\theta = \sin 50°$,
(c) $\tan\theta = \tan 42{\cdot}3°$, (d) $\sin\theta = -\sin 63{\cdot}4°$.

5. Find, without using tables or a calculator, the values of the following
(a) $\sin 120°$, (b) $\cot 30°$, (c) $\cos(-210°)$, (d) $\tan(-45°)$,
(e) $\cos 135°$, (f) $\sin(-300°)$, (g) $\operatorname{cosec} 210°$, (h) $\sec 750°$.

6. Find, without using tables or a calculator, the values of the following
(a) $\cos 270°$, (b) $\tan 120°$, (c) $\sin(-180°)$, (d) $\sec 240°$,
(e) $\cot 450°$, (f) $\cos 540°$, (g) $\operatorname{cosec} 225°$, (h) $\sin(-90°)$.

7. Find the possible values of θ and ϕ between $0°$ and $360°$, given that
(a) $\cos^2\theta + \sin^2\phi = 0$, (b) $\tan^2\theta + \cot^2\phi = 0$,
(c) $2\sin\theta + \cos\phi = 3$, (d) $3\sin\theta - 2\cos\phi = 5$.

8. Find the range of possible values of an acute angle θ given that
(a) $\tan 2\theta > 0$, (b) $\cos 3\theta < 0$,
(c) $\sin 5\theta < 0$, (d) $\cot 6\theta > 0$.

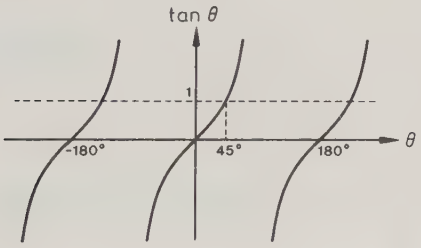
9. In each of the following cases sketch the given pair of curves on the same diagram

(a) $y = 2\sin\theta$, $y = \sin 2\theta$,
(b) $y = \frac{1}{2}\cos\theta$, $y = \cos\frac{1}{2}\theta$,
(c) $y = -\sin\theta$, $y = \sin(-\theta)$,
(d) $y = -\cos\theta$, $y = \cos(-\theta)$.

7.2 The inverse functions and simple equations

In the previous section the trigonometric ratios were defined for any given angle. We now consider the reverse process of determining an angle given the value of one of the trigonometric ratios. One approach to this problem is through sketch graphs.

Example 1 Find all the values of θ between $-180°$ and $+180°$ such that $\tan\theta = 1$.



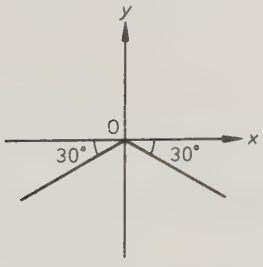
$\tan\theta = 1 = \tan 45°$
$\therefore$ from the sketch
$\theta = -135°$ or $45°$.

[Note that if there is no restriction on the value of θ, the equation $\tan\theta = 1$ has an infinite solution set

$\{\ldots, -315°, -135°, 45°, 225°, 405°, \ldots\}$.]

An alternative approach is to use the notion of the rotating line segment OP discussed in §7.1.

Example 2 Find all the values of θ between $0°$ and $360°$ such that $\sin\theta = -\frac{1}{2}$.



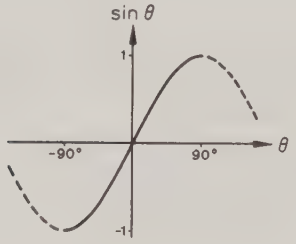
We note mentally that if $\sin\theta$ is $-$ve then OP is in the 3rd or 4th quadrant. If $\sin 30° = \frac{1}{2}$ then the acute angle between OP and the x-axis is $30°$. The sketch shows the two possible positions of OP.

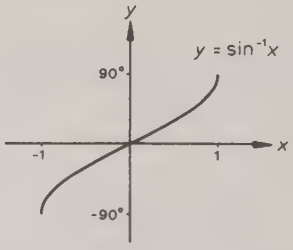
$\sin\theta = -\frac{1}{2} = -\sin 30°$
$\therefore \quad \theta = 210° \quad \text{or} \quad 330°$.

[More generally, by considering further rotations of OP, the solution set of the equation $\sin\theta = -\frac{1}{2}$ is found to be

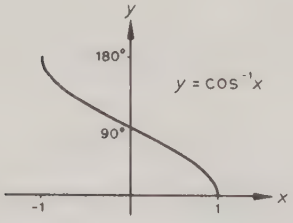
$\{\ldots, -150°, -30°, 210°, 330°, 570°, \ldots\}$.]

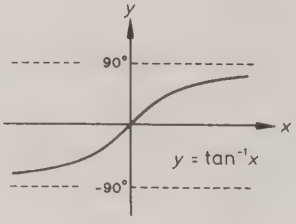
As illustrated in the above example, if $\sin a = b$ then for any given value of b there are many possible values of a. Thus, in the language of §4.2, the function $f: \theta \to \sin\theta$ has no true inverse. However, if the domain of the sine function is restricted so that $-90° \leqslant \theta \leqslant 90°$, then an inverse function can be defined. The *inverse sine* of x, written $\sin^{-1} x$ (or $\arcsin x$), is the angle between $-90°$ and $90°$ whose sine is x. Hence if $y = \sin^{-1} x$, then $\sin y = x$ and $-90° \leqslant y \leqslant 90°$. These values, $-90°$ to $+90°$, are sometimes called the *principal values* of the inverse sine. [To avoid confusion between $\sin^{-1} x$ and $(\sin x)^{-1}$, i.e. $1/\sin x$, the symbol $\sin^{-1}$ may be read as 'inverse sine', 'sine inverse' or 'sine-minus one', but *not* 'sine to the minus one'.] Similarly, the *inverse cosine* of x, written $\cos^{-1} x$ (or $\arccos x$), is the angle between $0°$ and $180°$ whose cosine is x. Hence if $y = \cos^{-1} x$, then $\cos y = x$ and $0° \leqslant y \leqslant 180°$.





The *inverse tangent* of x, written $\tan^{-1} x$ (or $\arctan x$), is the angle between $-90°$ and $90°$ whose tangent is x. Hence if $y = \tan^{-1} x$, then $\tan y = x$ and $-90° < y < 90°$.





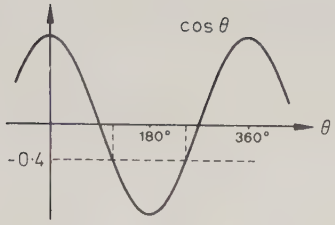
In the following example we demonstrate the use of both calculator inverse functions and mathematical tables.

Example 3 Solve the equation $\cos\theta = -0{\cdot}4$ for values of θ between $0°$ and $360°$, giving your answers to the nearest tenth of a degree.

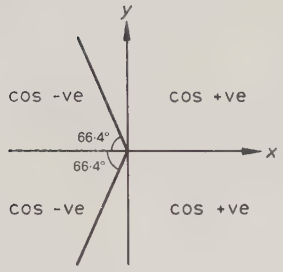
Method A (using calculator inverse cosine function)

We must first find the angle between $0°$ and $180°$ whose cosine is $-0{\cdot}4$.

As a first answer we write down the angle to three decimal places.



Thus $\cos\theta = -0{\cdot}4 = \cos 113{\cdot}578°$
$= 113{\cdot}6°$ (to nrst $0{\cdot}1°$)
$\therefore \quad \theta = 113{\cdot}6°$ or $360° - 113{\cdot}6°$
i.e. $\theta = 113{\cdot}6°$ or $246{\cdot}4°$



Method B (using table of cosines)
$\cos\theta = -0{\cdot}4 = -\cos 66{\cdot}422°$
$= -\cos 66{\cdot}4°$ (nrst $0{\cdot}1°$)
$\therefore \quad \theta = 180° - 66{\cdot}4°$ or
$180° + 66{\cdot}4°$
i.e. $\theta = 113{\cdot}6°$ or $246{\cdot}4°$

Example 4 Find the values of θ between $0°$ and $360°$ such that $\tan 2\theta = -2$, giving your answer to the nearest tenth of a degree. [Note that since we require values of θ between $0°$ and $360°$, we must look for values of 2θ between $0°$ and $720°$.]

$\tan 2\theta = -2 = -\tan 63{\cdot}435°$

$\therefore$ between $0°$ and $720°$, $2\theta = 116{\cdot}565°, 296{\cdot}565°, 476{\cdot}565°$ or $656{\cdot}565°$.

Hence, between $0°$ and $360°$, $\theta = 58{\cdot}3°, 148{\cdot}3°, 238{\cdot}3°$ or $328{\cdot}3°$.

The last example concerns a quadratic equation in $\cos\theta$. Since brackets would be rather cumbersome $(\cos\theta)^2$ is usually written $\cos^2\theta$.

Example 5 Solve the equation $2\cos^2\theta - 5\cos\theta + 2 = 0$ for values of θ from $0°$ to $360°$.

$$2\cos^2\theta - 5\cos\theta + 2 = 0 \Leftrightarrow (2\cos\theta - 1)(\cos\theta - 2) = 0$$
$$\Leftrightarrow \cos\theta = \tfrac{1}{2} \text{ or } \cos\theta = 2$$

Since there is no value of θ for which $\cos\theta = 2$, we must have $\cos\theta = \frac{1}{2} = \cos 60°$. Hence $\theta = 60°$ or $300°$.

Exercise 7.2

Give answers to the nearest tenth of a degree.

1. Find all the values of θ between $0°$ and $360°$ such that
(a) $\sin\theta = 0{\cdot}3$, (b) $\tan\theta = 1{\cdot}5$,
(c) $\cos\theta = -0{\cdot}73$, (d) $\sin\theta = -0{\cdot}62$.

2. Find all the values of θ between $-180°$ and $+180°$ such that
(a) $\cos\theta = 0{\cdot}6$, (b) $\sin\theta = 0{\cdot}8$,
(c) $\tan\theta = -0{\cdot}62$, (d) $\cos\theta = -0{\cdot}342$.

3. Find all the values of θ between 0° and 360° such that

(a) $\sin\theta = 1/\sqrt{2}$, (b) $\tan\theta = -1/\sqrt{3}$, (c) $\cot\theta = 1$,
(d) $\cos\theta = -\frac{1}{2}$, (e) $\sec\theta = -1$, (f) $\operatorname{cosec}\theta = 2$.

4. Solve the following equations for $0° \leqslant x \leqslant 360°$

(a) $4\sin x = 3\cos x$, (b) $\sin x + \cos x = 0$,
(c) $4\tan x = 3\sec x$, (d) $2\tan x \operatorname{cosec} x = 3$.

5. Solve the following equations for $0° \leqslant \theta \leqslant 360°$

(a) $\cos(\theta + 10°) = -0{\cdot}44$, (b) $\sin(\theta - 30°) = 0{\cdot}207$,
(c) $\sin(\theta + 50°) = 0{\cdot}541$, (d) $\tan(70° - \theta) = -1{\cdot}16$,
(e) $\tan 2\theta = 0{\cdot}7$, (f) $\cos\frac{1}{3}\theta = 0{\cdot}34$,
(g) $\sin\frac{1}{2}\theta = 0{\cdot}83$, (h) $\sec 2\theta = -1$.

6. Solve the following equations for $-90° \leqslant x \leqslant 90°$

(a) $\sin 3x = 1/\sqrt{2}$, (b) $\tan 4x = 0$,
(c) $\cot 5x + 1 = 0$, (d) $2\cos 3x + 1 = 0$.

7. Solve the following equations for $0° \leqslant x \leqslant 360°$

(a) $4\sin^2 x = 3$, (b) $2\cos^2 x = \cos x$,
(c) $5\sin x \cos x = \sin x$, (d) $\sin^2 x - \sin x - 2 = 0$,
(e) $2\tan^2 x = \sec x \tan x$, (f) $3\cos^2 x = 2\sin x \cos x$.

8. By rearranging as quadratic equations, solve the following for $-180° \leqslant \theta \leqslant 180°$

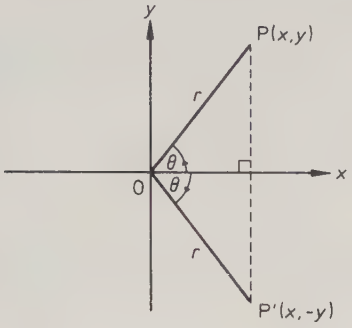
(a) $2\tan\theta - \cot\theta = 1$, (b) $2\operatorname{cosec}\theta + 7 = 4\sin\theta$,
(c) $2\cos\theta + \sec\theta = 3$, (d) $2\sin\theta\tan\theta = 3\sin\theta + 5\cos\theta$.

9. Solve the given pairs of simultaneous equations, where x and y can take any values from $-180°$ to $+180°$ inclusive.

(a) $x + y = 180°$, $\tan(x - y) = \sqrt{3}$ (b) $x - y = 120°$, $\sin(x + y) = 0{\cdot}5$.

10. Draw flow charts for procedures to solve the equations (i) $a\sin x = b$, (ii) $a\cos x = b$, (iii) $a\sin x = b\cos x$, using a pocket calculator. Print out all possible values of x from 0° to 360° inclusive or an appropriate message if the equation has no solution. Check that your procedures deal with $a = 0$, $b = 0$ and $a = b$.

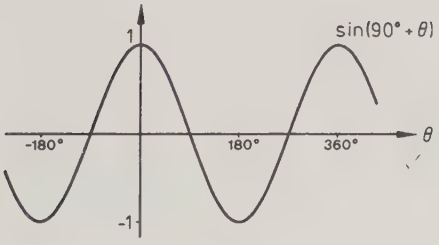
7.3 Properties of trigonometric ratios

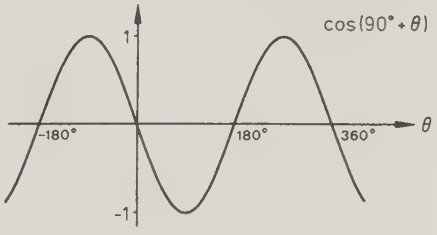


It follows from the definitions of $\sin\theta$, $\cos\theta$ and $\tan\theta$ that

$\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$,
$\tan(-\theta) = -\tan\theta$.

These results can also be obtained by considering the graphs of $\sin(-\theta)$, $\cos(-\theta)$ and $\tan(-\theta)$, which are the reflections in the y-axis of the graphs of $\sin\theta$, $\cos\theta$ and $\tan\theta$.





The diagrams above show that

$\sin(90°+\theta) = \cos\theta$ $\qquad$ $\cos(90°+\theta) = -\sin\theta$

By a similar process it is found that

$\sin(180°+\theta) = -\sin\theta$ $\qquad$ $\cos(180°+\theta) = -\cos\theta$

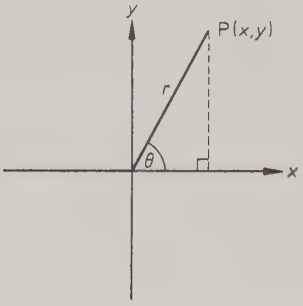
Substituting $-\theta$ for θ in these results

$\sin(90°-\theta) = \cos(-\theta)$ $\qquad$ $\cos(90°-\theta) = -\sin(-\theta)$

$\therefore\ \sin(90°-\theta) = \cos\theta$ $\qquad$ $\therefore\ \cos(90°-\theta) = \sin\theta$

Similarly $\sin(180°-\theta) = \sin\theta$, $\cos(180°-\theta) = -\cos\theta$.

[The graphs of $\sin(90°-\theta)$, $\cos(90°-\theta)$, $\sin(180°-\theta)$ and $\cos(180°-\theta)$ may be obtained by reflection then translation of the graphs of $\sin\theta$ and $\cos\theta$.]



Three basic properties of the trigonometric ratios are derived from Pythagoras'† theorem.

For all values of θ, $x^2+y^2=r^2$,

so that $\dfrac{x^2}{r^2}+\dfrac{y^2}{r^2}=1.$

Thus $\boxed{\cos^2\theta+\sin^2\theta=1.}$

Dividing by $\cos^2\theta$: $\dfrac{\cos^2\theta}{\cos^2\theta}+\dfrac{\sin^2\theta}{\cos^2\theta}=\dfrac{1}{\cos^2\theta}$

$\therefore\ \boxed{1+\tan^2\theta=\sec^2\theta.}$

Dividing by $\sin^2\theta$: $\dfrac{\cos^2\theta}{\sin^2\theta}+\dfrac{\sin^2\theta}{\sin^2\theta}=\dfrac{1}{\sin^2\theta}$

$\therefore\ \boxed{\cot^2\theta+1=\operatorname{cosec}^2\theta.}$

Such relationships between the trigonometric ratios are called *identities*, because they hold for all values of θ. The word 'equation' is used for relationships which hold for only certain values of θ.

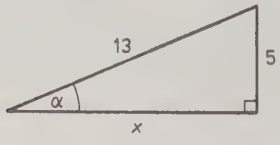
Example 1 If $\sin\theta=5/13$, where θ is an obtuse angle, find $\cos\theta$ and $\tan\theta$.

$$\cos^2\theta=1-\sin^2\theta=1-\left(\frac{5}{13}\right)^2=1-\frac{25}{169}=\frac{144}{169}$$

Since θ is obtuse, $\cos\theta=-12/13$

and $\tan\theta=\dfrac{\sin\theta}{\cos\theta}=\dfrac{5}{13}\Big/\left(-\dfrac{12}{13}\right)=-5/12.$

Alternative method



The diagram shows a right-angled triangle constructed so that $\sin\alpha=5/13$.
By Pythagoras' theorem, $x=12$.
Hence $\cos\alpha=12/13$, $\tan\alpha=5/12$.

† *Pythagoras* (6th cent. B.C.) Greek philosopher. He founded a religious brotherhood which devoted much of its attention to geometry and the theory of numbers. It is believed that Pythagoras or his early followers discovered the theorem that bears his name.

Since θ is an obtuse angle such that $\theta = 5/13$, $\cos\theta = -12/13$ and $\tan\theta = -5/12$.

Example 2 Simplify $\dfrac{\sin^2\theta}{1-\cos\theta}$.

$$\frac{\sin^2\theta}{1-\cos\theta} = \frac{1-\cos^2\theta}{1-\cos\theta} = \frac{(1-\cos\theta)(1+\cos\theta)}{1-\cos\theta} = 1+\cos\theta.$$

Example 3 Eliminate θ from the equations

$$x = 1+2\cos\theta,\ y = 3\sin\theta.$$

Rearranging $\dfrac{x-1}{2} = \cos\theta, \dfrac{y}{3} = \sin\theta$

Since $\cos^2\theta+\sin^2\theta = 1$, $\dfrac{(x-1)^2}{4} + \dfrac{y^2}{9} = 1.$

Example 4 Solve the equation $\sec^2\theta+\tan\theta-1 = 0$ for $0° \leqslant \theta \leqslant 360°$.

Substituting $\sec^2\theta = 1+\tan^2\theta$,

$\sec^2\theta+\tan\theta-1 = 0 \Leftrightarrow (1+\tan^2\theta)+\tan\theta-1 = 0$

$\Leftrightarrow \tan^2\theta+\tan\theta = 0$

$\Leftrightarrow \tan\theta(\tan\theta+1) = 0$

$\therefore$ either $\tan\theta = 0$ or $\tan\theta = -1$

$\theta = 0°, 180°, 360°$ $\theta = 135°, 315°.$

Hence the required solutions are 0°, 135°, 180°, 315° and 360°.

Exercise 7.3

1. Sketch the graphs of the following functions and use them to verify the results given in the text

(a) $\sin(-\theta)$, $\cos(-\theta)$, (b) $\sin(180°+\theta)$, $\cos(180°+\theta)$,

(c) $\sin(90°-\theta)$, $\cos(90°-\theta)$, (d) $\sin(180°-\theta)$, $\cos(180°-\theta)$.

2. By drawing sketch graphs, or otherwise, express the following in terms of $\sin\theta$ or $\cos\theta$

(a) $\sin(360°+\theta)$, (b) $\cos(360°-\theta)$, (c) $\cos(270°+\theta)$,

(d) $\sin(\theta-90°)$, (e) $\cos(\theta-180°)$, (f) $\sin(270°-\theta)$.

3. By drawing sketch graphs, or otherwise, express the following in terms of $\tan\theta$ or $\cot\theta$

(a) $\tan(90°+\theta)$, (b) $\tan(180°+\theta)$, (c) $\tan(90°-\theta)$,

(d) $\tan(180°-\theta)$, (e) $\tan(\theta-360°)$, (f) $\tan(\theta-90°)$.

In questions 4 to 7 do not use tables or calculators.

4. If $\cos\theta = 4/5$ and θ is an acute angle, find $\sin\theta$, $\tan\theta$ and $\sec\theta$.

5. If $\tan\theta = -8/15$ and θ is obtuse, find $\sin\theta$, $\sec\theta$ and $\cot\theta$.

6. If $\cot x = 7/24$ and $90° < x < 360°$, find $\sin x$, $\cos x$ and $\operatorname{cosec} x$.

7. If $\sin x = 12/13$ and $0° < x < 90°$, find $\cos x$, $\cos(x+90°)$ and $\tan(x+180°)$.

8. Simplify the following expressions, where $0° < \theta < 90°$:

(a) $\dfrac{\sqrt{(1-\cos^2\theta)}}{\tan\theta}$, (b) $\dfrac{\sin\theta}{\sqrt{(1-\sin^2\theta)}}$, (c) $\dfrac{\tan\theta}{\sqrt{(1+\tan^2\theta)}}$.

9. Simplify the following expressions

(a) $\sec^2\theta - \tan^2\theta$, (b) $(\sin^2\theta - 2)^2 - 4\cos^2\theta$,

(c) $\dfrac{\cos^4\theta - \sin^4\theta}{\cos\theta - \sin\theta}$, (d) $\dfrac{\sin\theta}{\operatorname{cosec}\theta - \cot\theta}$.

10. Prove the following identities

(a) $\sec\theta - \cos\theta = \sin\theta\tan\theta$, (b) $\sec^2\theta + \operatorname{cosec}^2\theta = \sec^2\theta\operatorname{cosec}^2\theta$,

(c) $\dfrac{1-\sin\theta}{1+\sin\theta} = (\sec\theta - \tan\theta)^2$, (d) $\dfrac{\cot^2\theta - 1}{\cot^2\theta + 1} = 1 - 2\sin^2\theta$.

11. Eliminate θ from the following pairs of equations

(a) $x = \cos\theta$, $y = \sin\theta$, (b) $x = 1 - 2\sin\theta$, $y = 1 + 3\cos\theta$,

(c) $x = \sec\theta$, $y = \tan\theta$, (d) $x = \sin\theta + \cos\theta$, $y = \sin\theta - \cos\theta$.

12. Solve for t and θ, where $0° \leqslant \theta \leqslant 360°$, the simultaneous equations $\sin\theta = 2t$, $2\cos\theta + 3t = 0$.

In questions 13 to 22 solve the given equations for $0° \leqslant \theta \leqslant 360°$.

13. $3\cos^2\theta + 5\sin\theta - 1 = 0$.

14. $8\sin^2\theta + 2\cos\theta - 5 = 0$.

15. $2\sin\theta = 3\cot\theta$.

16. $\sec\theta\tan\theta = 2$.

17. $\tan^2\theta = \sec\theta + 5$.

18. $7\sin^2\theta + \cos^2\theta = 5\sin\theta$.

19. $2\sec^2\theta + 3\tan\theta = 4$.

20. $4\cot^2\theta - 3\operatorname{cosec}^2\theta = 2\cot\theta$.

21. $\sec\theta = 3\cos\theta + \sin\theta$.

22. $4\sec^2\theta = 5\tan\theta + 3\tan^2\theta$

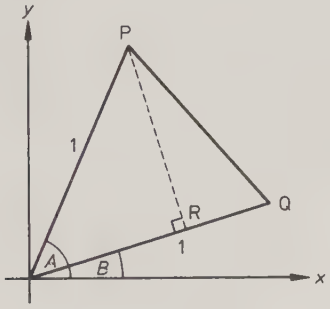
23. Express $2\sin x - \cos^2 x$ in the form $(\sin x + a)^2 + b$ and hence find the maximum and minimum values of the expression.

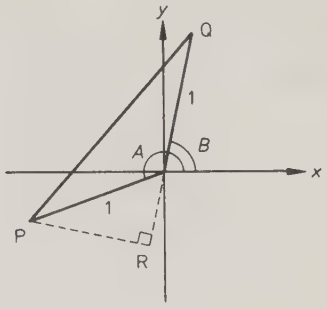
24. Find the range of the function $f : x \to \sec^2 x - 2\tan x + 1$.

7.4 Addition formulae

This section deals with expressions for $\sin(A \pm B)$, $\cos(A \pm B)$ in terms of sines and cosines of A and B.

In the diagrams below, OP and OQ make angles A and B respectively with the positive direction of the x-axis, so that $\angle POQ = A - B$. Letting $OP = OQ = 1$, the coordinates of P and Q are $P(\cos A, \sin A)$ and $Q(\cos B, \sin B)$.





The distance between the points P and Q is given by

$$\begin{aligned} PQ^2 &= (\cos A - \cos B)^2 + (\sin A - \sin B)^2 \\ &= \cos^2 A - 2\cos A \cos B + \cos^2 B + \sin^2 A - 2\sin A \sin B + \sin^2 B \\ &= (\cos^2 A + \sin^2 A) + (\cos^2 B + \sin^2 B) - 2(\cos A \cos B + \sin A \sin B) \\ &= 2 - 2(\cos A \cos B + \sin A \sin B). \end{aligned}$$

Applying Pythagoras' theorem to $\triangle PQR$, $PQ^2 = PR^2 + RQ^2$

For all values of A and B this leads to

$$\begin{aligned} PQ^2 &= \{\sin(A-B)\}^2 + \{1 - \cos(A-B)\}^2 \\ &= \sin^2(A-B) + \cos^2(A-B) + 1 - 2\cos(A-B) \\ &= 2 - 2\cos(A-B). \end{aligned}$$

Comparing these two expressions for PQ^2, we have

$$\cos(A-B) = \cos A \cos B + \sin A \sin B. \tag{1}$$

Substituting $-B$ for B in this expression

$$\begin{aligned} \cos(A+B) &= \cos A \cos(-B) + \sin A \sin(-B) \\ \therefore\ \cos(A+B) &= \cos A \cos B - \sin A \sin B. \end{aligned} \tag{2}$$

Substituting $90° - A$ for A in (1)

$$\begin{aligned} \cos(90° - A - B) &= \cos(90° - A)\cos B + \sin(90° - A)\sin B \\ \therefore\ \sin(A+B) &= \sin A \cos B + \cos A \sin B. \end{aligned} \tag{3}$$

Substituting $-B$ for B in (3)

$$\begin{aligned} \sin(A-B) &= \sin A \cos(-B) + \cos A \sin(-B) \\ \therefore\ \sin(A-B) &= \sin A \cos B - \cos A \sin B. \end{aligned} \tag{4}$$

Expressions for $\tan(A+B)$ and $\tan(A-B)$ can be deduced from identities (1) to (4).

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}.$$

Dividing numerator and denominator by $\cos A \cos B$

$$\tan(A+B) = \left(\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}\right) \Big/ \left(1 - \frac{\sin A}{\cos A} \cdot \frac{\sin B}{\cos B}\right)$$

$$\therefore \quad \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

Similarly, $$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}.$$

Summarising:
$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$
$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$
$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$
$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$
$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}, \quad \tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}.$$

Example 1 Find, without using tables, expressions in surd form for (a) $\sin 15°$, (b) $\tan 105°$.

(a) $\sin 15° = \sin(60° - 45°) = \sin 60° \cos 45° - \cos 60° \sin 45°$

$$= \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} - \frac{1}{2} \cdot \frac{1}{\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}} \quad \text{or} \quad \frac{1}{4}(\sqrt{6}-\sqrt{2})$$

(b) $\tan 105° = \tan(60° + 45°) = \dfrac{\tan 60° + \tan 45°}{1 - \tan 60° \tan 45°}$

$$= \frac{\sqrt{3}+1}{1-\sqrt{3}} = \frac{(\sqrt{3}+1)^2}{(1-\sqrt{3})(1+\sqrt{3})} = \frac{3+2\sqrt{3}+1}{1-3} = -2-\sqrt{3}.$$

Example 2 Solve the equation $\cos(\theta - 30°) = 2\sin\theta$ for values of θ between 0° and 360°.

$$\cos(\theta - 30°) = 2\sin\theta$$
$$\Rightarrow \quad \cos\theta\cos 30° + \sin\theta \sin 30° = 2\sin\theta$$
$$\Rightarrow \quad \frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta = 2\sin\theta$$
$$\Rightarrow \quad \sqrt{3}\cos\theta = 3\sin\theta$$

Hence $\tan\theta = \dfrac{1}{\sqrt{3}} = \tan 30°$, so that $\theta = 30°$ or $210°$.

Example 3 Find, without using tables, the values of

(a) $\dfrac{1}{\sqrt{2}}(\cos 75° - \sin 75°)$ (b) $\dfrac{1+\tan 15°}{1-\tan 15°}$

(a) $\frac{1}{\sqrt{2}}(\cos 75° - \sin 75°) = \cos 45° \cos 75° - \sin 45° \sin 75° = \cos(45° + 75°)$
$= \cos 120° = -\frac{1}{2}$.

(b) $\frac{1+\tan 15°}{1-\tan 15°} = \frac{\tan 45° + \tan 15°}{1 - \tan 45° \tan 15°} = \tan(45° + 15°) = \tan 60° = \sqrt{3}$.

Exercise 7.4

Answer questions 1 to 10 without using tables or calculators.

1. Find expressions in surd form for
(a) $\sin 75°$, (b) $\cos 105°$, (c) $\tan(-15°)$, (d) $\cot 75°$.

2. Find the values of
(a) $\sin 80° \cos 70° + \cos 80° \sin 70°$, (b) $\frac{1}{\sqrt{2}}\cos 15° - \frac{1}{\sqrt{2}}\sin 15°$,
(c) $\cos 105° \cos 15° + \sin 105° \sin 15°$, (d) $\frac{\sqrt{3}}{2}\sin 60° + \frac{1}{2}\cos 60°$.

3. Find the values of
(a) $\frac{\tan 40° + \tan 20°}{1 - \tan 40° \tan 20°}$, (b) $\frac{\tan 75° - 1}{\tan 75° + 1}$,
(c) $\cos 75° + \sin 75°$, (d) $\sqrt{3}\cos 15° - \sin 15°$.

4. If $\sin A = 12/13$ and $\sin B = 4/5$, where A and B are acute angles, find $\sin(A+B)$ and $\cos(A+B)$.

5. If $\cos A = 5/7$ and $\sin B = 1/5$, where A is acute and B is obtuse, find $\sin(A-B)$ and $\cos(A-B)$.

6. If $\tan A = 1/2$ and $\tan B = -1/3$, where $180° < A < 360°$ and $-90° < B < 90°$, find $A - B$.

7. If $\tan(A-B) = 2$ and $\tan B = \frac{1}{4}$, find $\tan A$.

8. Simplify the following expressions
(a) $\sin(A+B) + \sin(A-B)$, (b) $\cos(A+B) + \cos(A-B)$,
(c) $\cos A \cos(A-B) + \sin A \sin(A-B)$.

9. Solve the equation $\cos 3x \cos 2x - \sin 3x \sin 2x = 0{\cdot}5$ for $0° \leqslant x \leqslant 180°$.

10. Solve the equation $\sin 5x \cos 3x - \cos 5x \sin 3x = 1$ for $-180° \leqslant x \leqslant 180°$.

11. Solve the following equations for $0° \leqslant \theta \leqslant 360°$
(a) $2\tan\theta = 3\tan(45° - \theta)$, (b) $\sin\theta = 2\sin(60° - \theta)$,
(c) $\cos(\theta - 30°) = \cos(\theta + 30°)$, (d) $\sin(\theta + 45°) = 5\cos(\theta - 45°)$.

12. Express $\cot(A+B)$ in terms of $\cot A$ and $\cot B$.

13. By writing $A = \tan^{-1} x$ and $B = \tan^{-1} y$, show that under certain conditions $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$.

14. Use the result of question 13 to show that

(a) $\tan^{-1}\dfrac{1}{3} + \tan^{-1}\dfrac{1}{2} = \tan^{-1} 1$, (b) $\tan^{-1}\dfrac{1}{5} + \tan^{-1}\dfrac{1}{8} = \tan^{-1}\dfrac{1}{3}$.

7.5 Double and half angle formulae

Substituting $B = A$ in the expressions obtained in the previous section for $\sin(A+B)$, $\cos(A+B)$ and $\tan(A+B)$:

$$\sin 2A = \sin(A+A) = \sin A\cos A + \cos A\sin A = 2\sin A\cos A$$
$$\cos 2A = \cos(A+A) = \cos A\cos A - \sin A\sin A = \cos^2 A - \sin^2 A$$
$$\tan 2A = \tan(A+A) = \frac{\tan A + \tan A}{1-\tan A\tan A} = \frac{2\tan A}{1-\tan^2 A}.$$

Using the identity $\cos^2 A + \sin^2 A = 1$,

$$\cos 2A = \cos^2 A - (1-\cos^2 A) = 2\cos^2 A - 1$$
$$\cos 2A = (1-\sin^2 A) - \sin^2 A = 1 - 2\sin^2 A.$$

Thus we have several useful results:

$$\begin{aligned} \cos 2A &= \cos^2 A - \sin^2 A \\ &= 2\cos^2 A - 1 \\ &= 1 - 2\sin^2 A \end{aligned} \qquad \begin{aligned} \sin 2A &= 2\sin A\cos A \\ \tan 2A &= \frac{2\tan A}{1-\tan^2 A}. \end{aligned}$$

Example 1 Solve the equation $1 - 2\sin\theta - 4\cos 2\theta = 0$ for values of θ between $-180°$ and $+180°$.

Substituting $\cos 2\theta = 1 - 2\sin^2\theta$,

$$1 - 2\sin\theta - 4\cos 2\theta = 0 \Leftrightarrow 1 - 2\sin\theta - 4(1 - 2\sin^2\theta) = 0$$
$$\Leftrightarrow \quad 8\sin^2\theta - 2\sin\theta - 3 = 0$$
$$\Leftrightarrow \quad (4\sin\theta - 3)(2\sin\theta + 1) = 0$$
$$\therefore \quad \sin\theta = \tfrac{3}{4} = \sin 48{\cdot}6° \quad \text{or} \quad \sin\theta = -\tfrac{1}{2} = -\sin 30°$$
$$\theta = 48{\cdot}6°, 131{\cdot}4° \qquad \theta = -150°, -30°.$$

Hence the required solutions are $-150°$, $-30°$, $48{\cdot}6°$, $131{\cdot}4°$.

Example 2 Express $\sin 3A$ in terms of $\sin A$.

$$\sin 3A = \sin(2A + A) = \sin 2A\cos A + \cos 2A\sin A$$
$$= 2\sin A\cos A\cos A + (\cos^2 A - \sin^2 A)\sin A$$

$$= 3\sin A\cos^2 A - \sin^3 A$$
$$= 3\sin A(1-\sin^2 A) - \sin^3 A$$
$$\therefore \quad \sin 3A = 3\sin A - 4\sin^3 A.$$

Example 3 Simplify $\dfrac{\sin x}{1+\cos x}$.

$$\sin x = 2\sin \tfrac{1}{2}x\cos \tfrac{1}{2}x, \quad \cos x = 2\cos^2 \tfrac{1}{2}x - 1$$
$$\therefore \quad \frac{\sin x}{1+\cos x} = \frac{2\sin \frac{1}{2}x\cos \frac{1}{2}x}{2\cos^2 \frac{1}{2}x} = \frac{\sin \frac{1}{2}x}{\cos \frac{1}{2}x} = \tan \tfrac{1}{2}x.$$

Two further important identities are obtained by rearranging expressions for $\cos 2A$:

$$\boxed{\cos^2 A = \tfrac{1}{2}(1+\cos 2A), \quad \sin^2 A = \tfrac{1}{2}(1-\cos 2A).}$$

Example 4 Express $\cos^4 A$ in terms of cosines of multiples of A.

$$\cos^4 A = (\cos^2 A)^2 = \tfrac{1}{4}(1+\cos 2A)^2 = \tfrac{1}{4}(1+2\cos 2A+\cos^2 2A)$$
$$= \tfrac{1}{4}(1+2\cos 2A+\tfrac{1}{2}\{1+\cos 4A\}) = \tfrac{1}{8}(2+4\cos 2A+1+\cos 4A)$$
$$\therefore \quad \cos^4 A = \tfrac{1}{8}(3+4\cos 2A+\cos 4A).$$

If other methods fail, equations can often be solved and identities proved by expressing $\sin\theta$, $\cos\theta$ and $\tan\theta$ in terms of t, where $t = \tan \frac{1}{2}\theta$.

Using the formula for $\tan 2A$, $\tan\theta = \dfrac{2t}{1-t^2}$.

$$\cos\theta = \cos^2 \tfrac{1}{2}\theta - \sin^2 \tfrac{1}{2}\theta = \cos^2 \tfrac{1}{2}\theta(1-\tan^2 \tfrac{1}{2}\theta) = \frac{1-\tan^2 \frac{1}{2}\theta}{\sec^2 \frac{1}{2}\theta} = \frac{1-t^2}{1+t^2}.$$

We may now write $\sin\theta = \cos\theta\tan\theta$ or obtain the result independently,

$$\sin\theta = 2\sin \tfrac{1}{2}\theta\cos \tfrac{1}{2}\theta = 2\tan \tfrac{1}{2}\theta\cos^2 \tfrac{1}{2}\theta = \frac{2\tan \frac{1}{2}\theta}{\sec^2 \frac{1}{2}\theta} = \frac{2t}{1+t^2}.$$

Thus
$$\boxed{\sin\theta = \frac{2t}{1+t^2}, \quad \cos\theta = \frac{1-t^2}{1+t^2}, \quad \tan\theta = \frac{2t}{1-t^2}.}$$

The main disadvantage of using these 't formulae' to solve trigonometric equations is that polynomial equations involving large powers of t may be produced. It is not always possible to solve these by elementary methods.

Example 5 Solve the equation $5\tan\theta + \sec\theta + 5 = 0$ for values of θ between $0°$ and $360°$.

If $\tan\frac{1}{2}\theta = t$, then provided that $t^2 \neq 1$, the equation becomes

$$5\left(\frac{2t}{1-t^2}\right) + \left(\frac{1+t^2}{1-t^2}\right) + 5 = 0$$
$$10t + (1+t^2) + 5(1-t^2) = 0$$
$$-4t^2 + 10t + 6 = 0$$
$$2t^2 - 5t - 3 = 0$$
$$(t-3)(2t+1) = 0$$

$\therefore$ either $t = 3$ or $t = -\frac{1}{2}$

$\tan\frac{1}{2}\theta = \tan 71{\cdot}565°$ $\quad$ $\tan\frac{1}{2}\theta = -\tan 26{\cdot}565°$

Hence, between 0° and 180°, $\frac{1}{2}\theta = 71{\cdot}565°$ or $153{\cdot}435°$.
Thus the required values of θ are 143·1° or 306·9° (nrst 0·1°).

Exercise 7.5

Answer questions 1 to 7 without using tables or calculators.

1. Given that x is an acute angle such that $\cos x = 3/5$, find $\sin 2x$, $\cos 2x$ and $\sin 3x$.

2. Given that θ is an obtuse angle such that $\sin\theta = 2/3$, find $\cos 2\theta$, $\tan 2\theta$ and $\cos 4\theta$.

3. Given that A is an acute angle such that $\tan A = \frac{1}{2}$, find $\sin 2A$, $\tan 2A$ and $\tan 3A$.

4. Given that x is an obtuse angle such that $\cos 2x = 1/8$, find $\cos x$, $\tan x$ and $\sec\frac{1}{2}x$.

5. Given that $\sin A = -0{\cdot}8$ where $0° < A < 270°$, find $\cos\frac{1}{2}A$, $\sin\frac{1}{2}A$ and $\sin\frac{3}{2}A$.

6. Find the values of the following expressions
(a) $(\sin 67\frac{1}{2}° + \cos 67\frac{1}{2}°)^2$, (b) $\cos^2 15° - \sin^2 15°$,
(c) $\tan 75° \cos^2 75°$, (d) $1 - 2\sin^2 22\frac{1}{2}°$.

7. Solve the following equations for $0° \leqslant x \leqslant 180°$
(a) $\sin 2x = \sin x$, (b) $\cos 2x = \cos x$,
(c) $\tan 2x + \tan x = 0$, (d) $\sin 2x - \tan x = 0$.

In questions 8 to 15 prove the given identities.

8. $\cos^4 A - \sin^4 A = \cos 2A$.

9. $\cot\theta - \tan\theta = 2\cot 2\theta$.

10. $\dfrac{\sin 3A}{\sin A} + \dfrac{\cos 3A}{\cos A} = 4\cos 2A$.

11. $\dfrac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} = \dfrac{\cos 2\theta}{1+\sin 2\theta}$.

12. $\dfrac{\sec 2x - 1}{\sec 2x + 1} = \sec^2 x - 1$.

13. $\dfrac{\sin 2\theta - \cos 2\theta + 1}{\sin 2\theta + \cos 2\theta + 1} = \tan\theta$.

14. $\operatorname{cosec} x - \cot x = \tan \frac{1}{2}x$.

15. $\tan(\frac{1}{2}x + 45°) + \cot(\frac{1}{2}x + 45°) = 2 \sec x$.

16. Solve the following equations for $0° \leqslant x \leqslant 360°$

(a) $\cos 2x = 5 \cos x + 2$,
(b) $3 \cos 2x + 1 = 2 \sin x$,
(c) $4 \sin^2 x - \tan^2 x = 1$,
(d) $4 \sin x = 7 \tan 2x$.

17. By expressing $\cos 2\theta$ and $\sin 2\theta$ in terms of $\tan \theta$, solve the following equations for $0° \leqslant \theta \leqslant 360°$

(a) $\cos 2\theta - 2 \sin 2\theta = 2$,
(b) $5 \cos 2\theta - 2 \sin 2\theta = 2$.

18. Using the substitution $\tan \frac{1}{2}x = t$, solve the following equations for $-180° \leqslant x \leqslant 180°$

(a) $9 \cos x - 8 \sin x = 1$,
(b) $\sin x + 1 = 3 \cos x$,
(c) $2 \cos x - \sin x + 1 = 0$,
(d) $7 \sin x + 9 \cos x = 3$.

19. By squaring $\sin^2 \theta + \cos^2 \theta$, prove that $\sin^4 \theta + \cos^4 \theta = \frac{1}{4}(\cos 4\theta + 3)$. Hence solve the equation $\sin^4 \theta + \cos^4 \theta = \frac{1}{2}$ for $0° < \theta < 360°$.

20. Express $\cos 3\theta$ in terms of $\cos \theta$ and hence solve the equation $\cos 3\theta + 2 \cos 2\theta + 4 \cos \theta + 2 = 0$ for $0° < \theta < 360°$.

7.6 Sum and product formulae

In §7.4 it was shown that

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$
$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

Adding
$$\sin(A+B) + \sin(A-B) = 2 \sin A \cos B \quad (1)$$

Subtracting
$$\sin(A+B) - \sin(A-B) = 2 \cos A \sin B \quad (2)$$

Similarly
$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$
$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

Adding
$$\cos(A+B) + \cos(A-B) = 2 \cos A \cos B \quad (3)$$

Subtracting
$$\cos(A+B) - \cos(A-B) = -2 \sin A \sin B. \quad (4)$$

These results provide a way of expressing a product of sines and cosines as a sum or a difference. Results (1), (3) and (4) cover all possible products.

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$
$$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$
$$2 \sin A \sin B = -\{\cos(A+B) - \cos(A-B)\}.$$

Example 1 Express as a sum or a difference (a) $2\cos 4\theta\cos\theta$, (b) $\sin 2x\cos 3x$.

(a) $2\cos 4\theta\cos\theta = \cos(4\theta+\theta)+\cos(4\theta-\theta) = \cos 5\theta+\cos 3\theta.$

(b) $\sin 2x\cos 3x = \frac{1}{2}\{\sin(2x+3x)+\sin(2x-3x)\}$
$= \frac{1}{2}\{\sin 5x+\sin(-x)\} = \frac{1}{2}\{\sin 5x-\sin x\}.$

Example 2 Find, without using tables, the value of $2\sin 15^\circ\sin 45^\circ$.

$$\begin{aligned}2\sin 15^\circ\sin 45^\circ &= -\{\cos(15^\circ+45^\circ)-\cos(15^\circ-45^\circ)\}\\ &= -\cos 60^\circ+\cos(-30^\circ)\\ &= -\frac{1}{2}+\frac{\sqrt{3}}{2} = \frac{1}{2}(\sqrt{3}-1).\end{aligned}$$

Results (1), (2), (3) and (4) can also be used to express sums and differences of sines and cosines as products.

Letting $A+B = X$, $A-B = Y$ so that $A = \frac{1}{2}(X+Y)$, $B = \frac{1}{2}(X-Y)$,

$$\begin{aligned}\sin X+\sin Y &= 2\sin\tfrac{1}{2}(X+Y)\cos\tfrac{1}{2}(X-Y)\\ \sin X-\sin Y &= 2\cos\tfrac{1}{2}(X+Y)\sin\tfrac{1}{2}(X-Y)\\ \cos X+\cos Y &= 2\cos\tfrac{1}{2}(X+Y)\cos\tfrac{1}{2}(X-Y)\\ \cos X-\cos Y &= -2\sin\tfrac{1}{2}(X+Y)\sin\tfrac{1}{2}(X-Y).\end{aligned}$$

As these identities can be used to 'factorise' sums and differences, they are sometimes called the *factor formulae.*

Example 3 Simplify $\dfrac{\sin(A+B)-\sin A}{\cos(A+B)+\cos A}$.

$$\begin{aligned}\sin(A+B)-\sin A &= 2\cos\tfrac{1}{2}\{(A+B)+A\}\sin\tfrac{1}{2}\{(A+B)-A\}\\ &= 2\cos(A+\tfrac{1}{2}B)\sin\tfrac{1}{2}B.\end{aligned}$$

Similarly $\cos(A+B)+\cos A = 2\cos(A+\frac{1}{2}B)\cos\frac{1}{2}B$

$$\therefore\quad \frac{\sin(A+B)-\sin A}{\cos(A+B)+\cos A} = \frac{2\cos(A+\frac{1}{2}B)\sin\frac{1}{2}B}{2\cos(A+\frac{1}{2}B)\cos\frac{1}{2}B} = \tan\tfrac{1}{2}B.$$

Example 4 Solve the equation $\sin 5\theta+\sin 3\theta = 0$ for values of θ between 0° and 180°.

$$\begin{aligned}\sin 5\theta+\sin 3\theta = 0 &\Rightarrow 2\sin\tfrac{1}{2}(5\theta+3\theta)\cos\tfrac{1}{2}(5\theta-3\theta) = 0\\ &\Rightarrow \qquad \sin 4\theta\cos\theta = 0\\ &\therefore\quad \sin 4\theta = 0 \qquad \text{or} \quad \cos\theta = 0\\ &\qquad 4\theta = 0^\circ, 180^\circ, 360^\circ, 540^\circ, 720^\circ \qquad \theta = 90^\circ\\ &\qquad \theta = 0^\circ, 45^\circ, 90^\circ, 135^\circ, 180^\circ\end{aligned}$$

Hence the solutions are $\theta = 0^\circ, 45^\circ, 90^\circ, 135^\circ, 180^\circ$.

Example 5 Solve the equation $\cos 5\theta + \cos 3\theta + \cos\theta = 0$ for values of θ between 0° and 180°.

$$\begin{aligned} & \cos 5\theta + \cos 3\theta + \cos\theta = 0 \\ \Rightarrow\quad & (\cos 5\theta + \cos\theta) + \cos 3\theta = 0 \\ \Rightarrow\quad & 2\cos 3\theta\cos 2\theta + \cos 3\theta = 0 \\ \Rightarrow\quad & \cos 3\theta(2\cos 2\theta + 1) = 0 \end{aligned}$$

$\therefore$ $\cos 3\theta = 0$ or $\cos 2\theta = -\frac{1}{2}$

$3\theta = 90°, 270°, 450°$ $\quad$ $2\theta = 120°, 240°$

$\theta = 30°, 90°, 150°$ $\quad$ $\theta = 60°, 120°$.

Hence the required values of θ are 30°, 60°, 90°, 120°, 150°.

Exercise 7.6

1. Express as a sum or a difference

(a) $2\sin 2x\cos x$, (b) $2\cos 3\theta\cos 5\theta$,

(c) $6\sin 3A\sin 2A$, (d) $\sin 2t\cos 4t$,

(e) $2\cos(A+B)\cos(A-B)$, (f) $\sin(x+75°)\sin(x+15°)$.

2. Find, without using tables, the values of

(a) $4\sin 75°\cos 45°$, (b) $20\sin 22\frac{1}{2}°\sin 67\frac{1}{2}°$,

(c) $8\cos 52\frac{1}{2}°\cos 37\frac{1}{2}°\cos 15°$, (d) $\tan 82\frac{1}{2}°\cot 52\frac{1}{2}°$.

3. Solve the following equations for $0° \leqslant \theta \leqslant 360°$

(a) $2\sin 2\theta\cos\theta = \sin 3\theta$, (b) $2\cos(\theta+120°)\cos(\theta+60°) = 1$,

(c) $2\sin(\theta+30°) = \sec\theta$, (d) $\sin(\theta+30°)\cos(\theta-40°) = 0{\cdot}2$.

4. Express as a product

(a) $\sin x + \sin 7x$, (b) $\cos 3A + \cos 5A$,

(c) $\sin(\theta+\alpha) - \sin(\theta-\alpha)$, (d) $\cos 3x + \sin(x-90°)$.

5. Simplify the following expressions

(a) $\dfrac{\cos 3\theta + \cos 7\theta}{\sin 3\theta + \sin 7\theta}$, (b) $\dfrac{\cos 5x - \cos x}{\sin 5x - \sin x}$,

(c) $\dfrac{\cos A - \cos 2A + \cos 3A}{\sin A - \sin 2A + \sin 3A}$, (d) $\dfrac{\cos\theta - 2\cos 3\theta + \cos 5\theta}{\cos\theta + 2\cos 3\theta + \cos 5\theta}$.

6. Find the values of

(a) $\dfrac{\cos 50° - \cos 70°}{\sin 70° - \sin 50°}$, (b) $\dfrac{\sin\theta + \sin(\theta+120°)}{\cos(60°-\theta) + \cos\theta}$.

7. Solve the following equations for $0° \leqslant x \leqslant 180°$

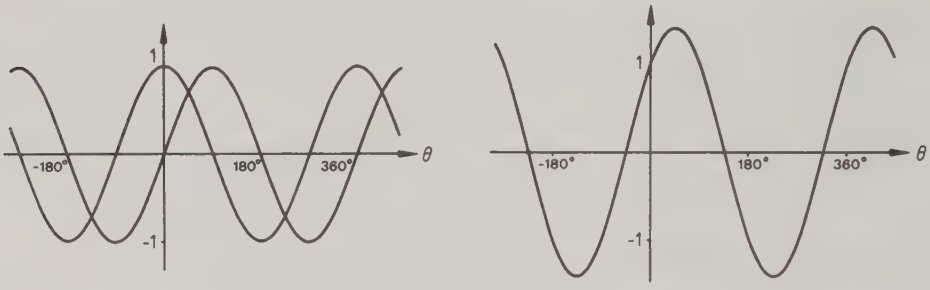
(a) $\cos 3x = \cos x$, (b) $\sin 5x = \sin x$,

(c) $\sin(3x+60°) = \sin x$, (d) $\sin 3x = \cos x$.

8. Solve the following equations for $0° \leqslant \theta \leqslant 180°$
(a) $\sin\theta + \sin 2\theta + \sin 3\theta = 0$,
(b) $\sin 5\theta - \sin\theta = \cos 3\theta$,
(c) $\cos 3\theta - \sin 3\theta = \cos 2\theta - \sin 2\theta$.

7.7 The expression $a\cos\theta + b\sin\theta$

A sketch of the graph of a function of the form $a\cos\theta + b\sin\theta$ can be made by combining the graphs of $a\cos\theta$ and $b\cos\theta$. The result for the function $\cos\theta + \sin\theta$ is shown below.



However, the exact form of the graph becomes clear when we apply one of the factor formulae derived in the previous section.

$$\cos\theta + \sin\theta = \cos\theta + \cos(90° - \theta) = 2\cos 45° \cos(\theta - 45°)$$
$$= 2 \cdot \frac{1}{\sqrt{2}} \cos(\theta - 45°) = \sqrt{2}\cos(\theta - 45°).$$

It seems reasonable to investigate whether the function $a\cos\theta + b\sin\theta$ can be expressed in the form $r\cos(\theta - \alpha)$, where r is positive.

Since $$r\cos(\theta - \alpha) = r(\cos\theta\cos\alpha + \sin\theta\sin\alpha)$$
$$= (r\cos\alpha)\cos\theta + (r\sin\alpha)\sin\theta$$

the expressions $a\cos\theta + b\sin\theta$ and $r\cos(\theta - \alpha)$ will be equal for all values of θ if

$$r\cos\alpha = a, \qquad r\sin\alpha = b$$

$\therefore$ $a^2 + b^2 = r^2(\cos^2\alpha + \sin^2\alpha) = r^2$ i.e. $r = \sqrt{(a^2 + b^2)}$

and $$\frac{b}{a} = \frac{r\sin\alpha}{r\cos\alpha} = \tan\alpha \qquad \text{i.e. } \tan\alpha = b/a.$$

Hence the function $a\cos\theta + b\sin\theta$ can be expressed in the form $r\cos(\theta - \alpha)$, where $r = \sqrt{(a^2 + b^2)}$, $\tan\alpha = b/a$.

[Note that since r is assumed positive, the value of α must be chosen so that $\cos\alpha$ has the same sign as a and $\sin\alpha$ the same sign as b.]

Example 1 Express $3\cos\theta + 4\sin\theta$ in the form $r\cos(\theta - \alpha)$. Hence find the maximum and minimum values of $3\cos\theta + 4\sin\theta$.

If $3\cos\theta+4\sin\theta = r\cos(\theta-\alpha) = r\cos\alpha\cos\theta+r\sin\alpha\sin\theta$
then $r\cos\alpha = 3$ and $r\sin\alpha = 4$.

Since both $r\cos\alpha$ and $r\sin\alpha$ are positive, we may take $r > 0$ and $0° < \alpha < 90°$.

$\therefore\quad r = \sqrt{(3^2 + 4^2)} = 5$ and $\tan\alpha = 4/3$, $\alpha = 53{\cdot}13°$

Hence $3\cos\theta + 4\sin\theta = 5\cos(\theta - 53{\cdot}1°)$.

Since the maximum and minimum values of $\cos(\theta - 53{\cdot}1°)$ are $+1$ and -1 respectively, the maximum and minimum values of $3\cos\theta + 4\sin\theta$ must be $+5$ and -5.

Example 2 Solve the equation $3\cos\theta-\sin\theta = 1$ for values of θ between 0° and 360°.

Let $3\cos\theta-\sin\theta = r\cos(\theta+\alpha) = r\cos\alpha\cos\theta-r\sin\alpha\sin\theta$
then $r\cos\alpha = 3$ and $r\sin\alpha = 1$

$\therefore\quad r = \sqrt{(3^2 + 1^2)} = \sqrt{10}$ and $\tan\alpha = 1/3$, $\alpha = 18{\cdot}435°$.

Hence the equation $3\cos\theta - \sin\theta = 1$
may be written $\sqrt{10}\cos(\theta + 18{\cdot}435°) = 1$
$\therefore\quad \cos(\theta + 18{\cdot}435°) = 1/\sqrt{10} = \cos 71{\cdot}565°$
$\therefore\quad \theta + 18{\cdot}435° = 71{\cdot}565°$ or $288{\cdot}435°$.

Hence the required solutions are 53·1° and 270°.

In the above example it was more convenient to use $r\cos(\theta+\alpha)$ instead of $r\cos(\theta-\alpha)$, because this led to a value of α between 0° and 90°. The expressions $r\sin(\theta+\alpha)$ and $r\sin(\theta-\alpha)$ may also be used when appropriate.

Exercise 7.7

1. Express each of the following functions in the form $r\cos(\theta\pm\alpha)$, where $r > 0$ and $0° < \alpha < 90°$. Hence find the maximum and minimum values of the function, giving the values of θ between $-360°$ and $+360°$ for which these occur. Sketch the graph of the function in this interval.
(a) $\cos\theta-\sin\theta$, (b) $\sqrt{3}\cos\theta-\sin\theta$,
(c) $5\cos\theta+12\sin\theta$, (d) $2\cos\theta+\sin\theta$.

2. Using an appropriate expression of the form $r\cos(\theta\pm\alpha)$, solve the following equations for $0° \leqslant \theta \leqslant 360°$.
(a) $4\cos\theta-3\sin\theta = 1$, (b) $2\cos\theta+5\sin\theta = 4$,
(c) $3\cos\theta+7\sin\theta+3 = 0$, (d) $15\cos\theta-8\sin\theta+10 = 0$.

3. Express $5\sin\theta-8\cos\theta$ in the form $r\sin(\theta-\alpha)$, where $r > 0$ and $0° < \alpha < 90°$. Hence solve the following equations for $0° \leqslant \theta \leqslant 360°$.
(a) $5\sin\theta-8\cos\theta = 6$, (b) $5\sin\theta-8\cos\theta = 5$.

4. Solve the following equations for $0° \leqslant x \leqslant 360°$,

(i) by expressing $\cos x$ and $\sin x$ in terms of $t = \tan \frac{1}{2}x$, (ii) using an expression of the form $r\cos(x-\alpha)$.

(a) $3\cos x - 5\sin x = 2$, (b) $4\cos x - 6\sin x = 5$.

5. Construct a flow chart for a procedure to solve the equation $a\cos\theta + b\sin\theta = c$, where $a > 0$, using a pocket calculator. The output of your procedure should be the values of θ, for $0° \leqslant \theta \leqslant 360°$, or a suitable message if there are no solutions. Consider whether accuracy or efficiency would be improved by incorporating in your chart branches or jumps for special cases such as $a = b$ or $b = c$.

Exercise 7.8 (miscellaneous)

1. Find all values of θ from 0° to 360° inclusive, such that

(a) $\sin\theta = \cos 20°$, (b) $\sin 2\theta = -\sin 50°$,

(c) $\cos\theta = -\cos(-35°)$, (d) $\tan 3\theta = \cot 15°$.

2. Given that $\cos x = -0{\cdot}8$, where $0° < x < 180°$, find the values of

(a) $\cos(x-180°)$, (b) $\cos(x+90°)$, (c) $\sin(90°-x)$,

(d) $\sin(x+180°)$, (e) $\tan(x+360°)$, (f) $\cot(x-90°)$.

3. Solve the following equations for $0° \leqslant x \leqslant 360°$.

(a) $2\sin x = 3\cos x$, (b) $3\cos x - \sec x = \frac{1}{2}$,

(c) $3\tan x = 5\sin x$, (d) $5\cos 2x + 1 = 11\sin x$.

4. Given that θ is an obtuse angle and that $\cos\theta = k$, find the following in terms of k.

(a) $\cos 2\theta$, (b) $\sin 3\theta$, (c) $\cos\frac{1}{2}\theta$, (d) $\tan\frac{1}{2}\theta$.

5. Eliminate θ from the following pairs of equations.

(a) $x = 1+\sin\theta$, $y = \frac{1}{2}\cos\theta$,

(b) $x = \tan\theta+\sec\theta$, $y = \sec\theta-\tan\theta$,

(c) $x = \cos\theta$, $y = \cos 2\theta+1$,

(d) $x = 1+\cos\theta$, $y = \sin 2\theta$.

6. Solve the following equations for $0° \leqslant x \leqslant 180°$.

(a) $\dfrac{\tan x}{1-\tan^2 x} = \dfrac{1}{5}$, (b) $\dfrac{\tan x}{1+\tan^2 x} = \dfrac{1}{5}$,

(c) $\sin 3x = \sin x$, (d) $\sin 4x + \sin 2x = 0$.

7. Sketch the curves $y = 2\sin x$ and $y = 3\cos x$ for $0° \leqslant x \leqslant 360°$. Find the range of values of x in this interval for which $2\sin x \geqslant 3\cos x$.

8. Given that $\tan A = \frac{2}{3}$ and $\tan B = \frac{1}{2}$, find the value of

$$\frac{\cos 2A + \cos 2B}{\sin 2A - \sin 2B}.$$

9. Sketch the graphs of the following functions for $-360° \leqslant \theta \leqslant 360°$, showing clearly the maximum and minimum values of each function.
(a) $2\cos\theta\sin\theta$, (b) $2\sin^2\theta$,
(c) $\cos\theta+\sqrt{3}\sin\theta$, (d) $(\cos\theta+\sqrt{3}\sin\theta)^2$.

10. Prove the following identities

(a) $\dfrac{1-\cos 2A}{1+\cos 2A} = \tan^2 A = \dfrac{\tan^2 A+1}{\cot^2 A+1}$,

(b) $\dfrac{\sin\theta}{1+\cos\theta} + \dfrac{1-\cos\theta}{\sin\theta} = 2\tan\tfrac{1}{2}\theta$,

(c) $\dfrac{\cos^4 A+\sin^4 A}{\cos^4 A-\sin^4 A} = \tfrac{1}{2}(\cos 2A+\sec 2A)$,

(d) $\dfrac{\sin(\theta+15°)+\sin(\theta-15°)}{\cos(\theta-15°)+\cos(\theta+15°)} = \tan\theta$.

11. (a) Prove that $15\cos 2\theta+20\sin 2\theta+7 \equiv 2(11\cos\theta-2\sin\theta)(\cos\theta+2\sin\theta)$. Hence, or otherwise, find all angles θ between 0° and 180° for which $15\cos 2\theta + 20\sin 2\theta + 7 = 0$, giving your answers to the nearest tenth of a degree.
(b) If $\sin(\theta+\alpha) = \lambda\sin(\theta-\alpha)$, where λ is a numerical constant ($\lambda \neq 1$), find an expression for $\tan\theta$ in terms of $\tan\alpha$ and λ. In the case when $\lambda = \frac{1}{2}$ and $\alpha = 30°$, find the values of θ which lie between 0° and 360°. (C)

12. (a) Solve the equation $\cos x+\cos 2x+\cos 3x = 0$, giving all values between 0° and 360° (inclusive).
(b) Express $3\sin x+4\cos x$ in the form $R\cos(x-\alpha)$, where R is a positive number. Hence, or otherwise, solve the equation $3\sin x+4\cos x = 2$, giving all values between 0° and 360°.
(c) If x and y are angles between 0° and 360° (inclusive), and $\sin^2 x+\cos^2 y = 2$, state what the possible values are of x and y. (SU)

13. Express $\tan 3\theta$ in terms of $\tan\theta$. Hence solve the equation $2\tan 3\theta = 9\tan\theta$ for $0° \leqslant \theta \leqslant 180°$.

14. Write down expressions for $\tan\theta$ and $\sec\theta$ in terms of t, where $t = \tan\dfrac{\theta}{2}$, and show that $\sec\theta+\tan\theta = \tan\left(45° + \dfrac{\theta}{2}\right)$. Find a solution in the interval $0° < \theta < 90°$ of the equation $\sec\theta+\tan\theta = \cot 2\theta$. (JMB)

15. (a) Express $7\sin x-24\cos x$ in the form $R\sin(x-\alpha)$, where R is positive and α is an acute angle. Hence or otherwise solve the equation $7\sin x-24\cos x = 15$, for $0° < x < 360°$.
(b) Solve the simultaneous equations $\cos x+\cos y = 1$, $\sec x+\sec y = 4$, for $0° < x < 180°, 0° < y < 180°$. (AEB)

16. By expressing $\sin\theta$ and $\cos\theta$ in terms of $\tan\frac{1}{2}\theta$, solve the equation $\sin\theta+2\cos\theta=1$, for $0°\leqslant\theta\leqslant 360°$. Explain why the method breaks down for the equation $2\sin\theta-\cos\theta=1$ and find the values of θ, between 0° and 360°, which satisfy this equation.

17. Prove that (a) $\cos A\cos 2A-\cos 2A\cos 5A=\sin 3A\sin 4A$
(b) $\sin A\cos 2A+\sin 2A\cos 5A=\sin 3A\cos 4A$.

18. (a) Prove the identities $\cos\theta+\cos 3\theta+\cos 5\theta+\cos 7\theta\equiv 4\cos\theta\cos 2\theta\cos 4\theta\equiv\sin 8\theta/2\sin\theta$.
(b) Solve for $0°<\theta<360°$ the equation $\sec\theta-3\tan\theta=2$. (W)

19. Find, to the nearest 0·1°, the acute angle α for which $4\cos\theta-3\sin\theta\equiv 5\cos(\theta+\alpha)$. Calculate the values of θ in the interval $-180°\leqslant\theta\leqslant 180°$ for which the function $f(\theta)=4\cos\theta-3\sin\theta-4$ attains its greatest value, its least value and the value zero. (JMB)

20. Solve the following equations for $0°\leqslant x\leqslant 180°$.
(a) $2\tan 3x\cot x+1=0$, (b) $3(\sec 2x-\tan 2x)=2\tan x$,
(c) $\cot 5x+\tan 2x=0$, (d) $4\sin x\cos 2x\sin 3x=1$.

21. Prove that, if the equation $a\sin x+b\cos x=c$ is to have a solution, then $a^2+b^2\geqslant c^2$. Find all the solutions of the equation $7\sin x+6\cos x=9$ between 0° and 360°, giving your answers to the nearest tenth of a degree. (O)

22. Write down the values of the following in degrees.
(a) $\sin^{-1}\frac{1}{2}$, (b) $\cos^{-1}(-1)$, (c) $\tan^{-1}(-1)$, (d) $\cot^{-1}\sqrt{3}$,
(e) $\sin^{-1}x+\cos^{-1}x$, (f) $\tan^{-1}x+\tan^{-1}\dfrac{1}{x}$,
(g) $\tan^{-1}\dfrac{2}{3}+\tan^{-1}\dfrac{1}{5}$, (h) $\tan^{-1}\dfrac{5}{2}-\tan^{-1}\dfrac{3}{7}$.

23. Prove that $\cos 3\theta=4\cos^3\theta-3\cos\theta$. Hence solve the equation
$$x^3-3x-\sqrt{2}=0$$
by using a suitable substitution of the form $x=k\cos\theta$, then finding possible values of θ between 0° and 180°. Give your answers in surd form.

24. (a) Find all solutions of the equation $\sin 3\theta=\sin^2\theta$ for which $0°\leqslant\theta<360°$.
(b) Express $\lambda\sin\theta+(1-\lambda)\cos\theta$ in the form $R\sin(\theta+\phi)$, where $R(R>0)$ and $\tan\phi$ are to be given in terms of λ. Write down an expression, in terms of λ, for the minimum value of $\lambda\sin\theta+(1-\lambda)\cos\theta$ as θ varies, and show that, for all λ, this minimum is less than or equal to $-\frac{1}{2}\sqrt{2}$. (C)

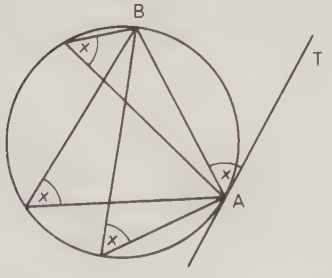
8 Trigonometry of triangles

8.1 Review of elementary geometry

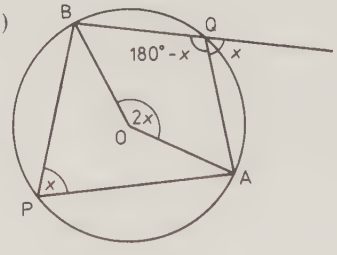
[This introductory section contains a review of some of the language and results of elementary geometry which may be required when solving trigonometrical problems.]

In elementary geometrical work the notions of points and distances, straight lines and angles are introduced. Development of these basic ideas leads to the study of plane figures such as triangles and parallelograms. One set of important results involving circles is illustrated here. In diagram (1) angles subtended by the

(1)



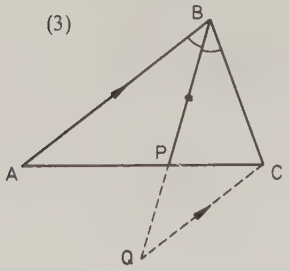
(2)



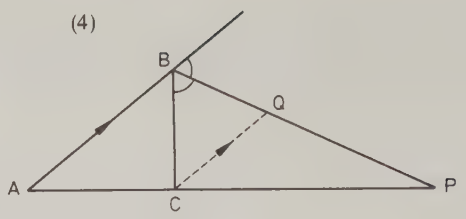
chord AB, lying in the same segment of the circle, are equal. The angle between the tangent AT and the chord AB is equal to any angle in the alternate segment. In diagram (2) the angle subtended by AB at the centre O is twice the angle subtended by AB at P on the circumference of the circle. Opposite angles of the cyclic quadrilateral $APBQ$ are supplementary and any exterior angle is equal to the opposite interior angle.

The ratio properties of similar figures have many applications. One result which may be proved using similar triangles is the angle bisector theorem. In the triangles ABC shown overleaf, BP bisects $\angle A$ internally in diagram (3) and externally in diagram (4).

(3)



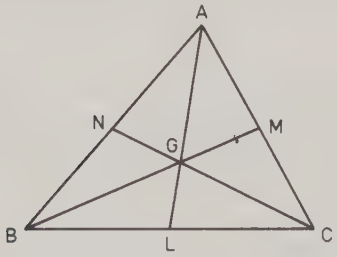
(4)



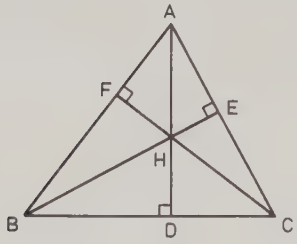
In both cases we have $\dfrac{AP}{PC} = \dfrac{AB}{BC}$.

[The proof uses similar triangles ABP and CQP.]

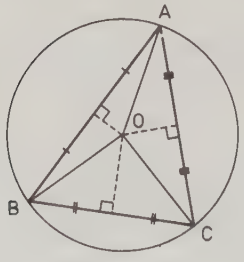
We now list some further properties of the triangle.



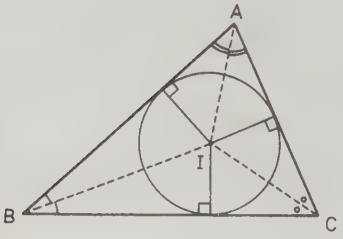
A *median* of a triangle is a line joining a vertex to the mid-point of the opposite side. The medians AL, BM and CN of triangle ABC are concurrent at a point G, called the *centroid* of the triangle. G divides each median in the ratio 2 : 1.



An *altitude* of a triangle is a perpendicular from a vertex to the opposite side. The altitudes AD, BE and CF of triangle ABC are concurrent at a point H called the *orthocentre* of the triangle.



The *circumcircle* or circumscribed circle of a triangle is the circle which passes through the vertices of the triangle. The centre of this circle is called the *circumcentre*. It is the point of intersection of the perpendicular bisectors of the sides of the triangle.



The *inscribed circle* or incircle of a triangle is the circle which touches the three sides of the triangle internally. The centre of this circle is called the *incentre*. It is the point of intersection of the bisectors of the angles of the triangle.

Exercise 8.1

1. The points A, B and C lie on a circle centre O. How are $\angle A$, $\angle B$ and $\angle C$ of the quadrilateral $OABC$ related?

2. The points P, Q and R lie on a circle and $\angle PRQ$ is obtuse. The tangents to the circle at P and Q intersect at T. If $\angle TPQ = x$, $\angle PRQ = y$ and $\angle PTQ = z$, state the relationships between (a) x and y, (b) y and z.

3. In triangle PQR the internal bisector of $\angle Q$ meets PR at X and the external bisector of $\angle Q$ meets PR produced at Y. Given that $PQ:QR = 3:2$, find the ratio $PR:XY$. Prove that $\angle XQY$ is a right-angle. Hence show that Q lies on the circle with diameter XY.

4. A point P moves in a plane containing two fixed points A and B, such that the ratio AP/PB is a constant not equal to 1. Show that the locus of P is a circle. [This locus is called *Apollonius*'† *circle*.]

5. By showing that the medians of a triangle divide the triangle into six smaller triangles of equal area, prove that the centroid of the triangle divides each median in the ratio 2:1.

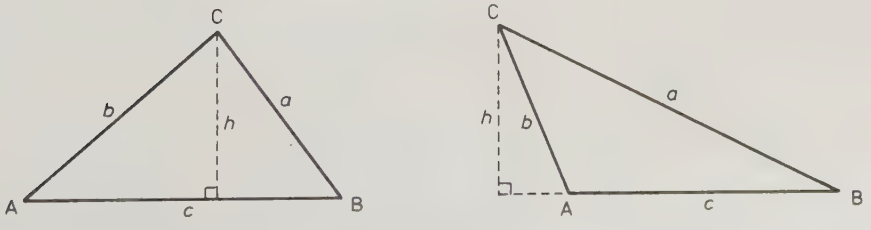
6. Given that AD, BE and CF are the altitudes of an acute angled triangle ABC, prove that the orthocentre of $\triangle ABC$ is the incentre of $\triangle DEF$. Find the position of the incentre of $\triangle DEF$ when $\angle A$ is obtuse. [$\triangle DEF$ is called the *pedal triangle* of $\triangle ABC$ because its vertices are the 'feet' of the altitudes.]

8.2 Sine and cosine rules

To establish the sine rule, consider a triangle ABC as shown below. Let h be the height of the perpendicular (or altitude) from C to AB. In both diagrams,

$$\sin A = \frac{h}{b}, \sin B = \frac{h}{a}$$

$$\therefore \quad \frac{a}{\sin A} = \frac{b}{\sin B} \left(= \frac{ab}{h} \right).$$

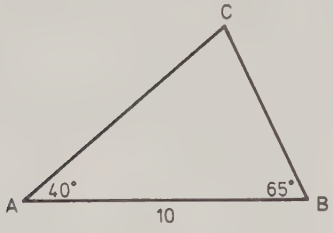


† *Apollonius of Perga* (3rd cent. B.C.) Greek mathematician known as 'the great geometer'. His treatise on conic sections forms the basis of much of the later work on the subject.

Similarly, it may be shown that $\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$, leading to the *sine rule*:

$$\boxed{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.}$$

Example 1 Solve the triangle ABC in which $A = 40°$, $B = 65°$ and $c = 10$.



$A+B+C = 180°$

$\therefore \quad C = 180° - 40° - 65° = 75°.$

By the sine rule,

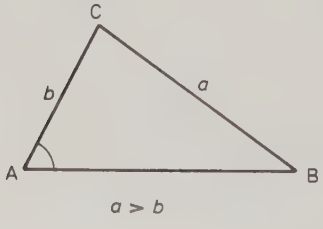
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\therefore \quad \frac{a}{\sin 40°} = \frac{b}{\sin 65°} = \frac{10}{\sin 75°}$$

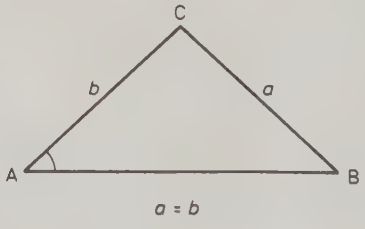
Hence $a = \dfrac{10 \sin 40°}{\sin 75°} \approx 6{\cdot}65, \quad b = \dfrac{10 \sin 65°}{\sin 75°} \approx 9{\cdot}38$

$\therefore$ the remaining sides and angles of the triangle are $C = 75°$, $a = 6{\cdot}65$ and $b = 9{\cdot}38$.

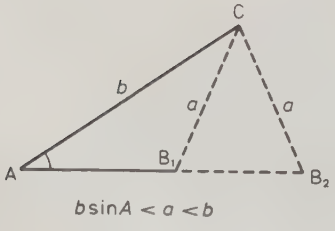
The sine rule can also be applied when two sides and one angle of a triangle are given. However, in this case it may sometimes be possible to construct two triangles satisfying the given conditions. For instance, let us assume that in $\triangle ABC$, the angle A and the sides a and b are given. These diagrams show that the ambiguous case arises when a is less than b, but greater than $b \sin A$.



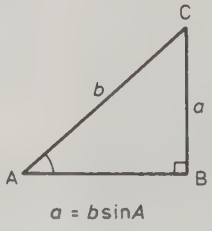
$a > b$



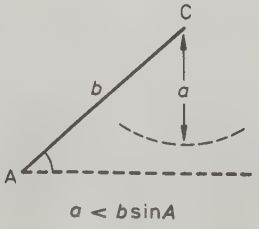
$a = b$



$b \sin A < a < b$



$a = b \sin A$



$a < b \sin A$

Example 2 If in triangle ABC, $a = 10$, $b = 16$ and $A = 30°$, find the two possible values of c.

$$\text{Using the sine rule} \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\therefore \quad \sin B = \frac{b \sin A}{a} = \frac{16 \sin 30°}{10} = \frac{16 . \frac{1}{2}}{10} = 0{\cdot}8.$$

Since $b > a$, $B > A$ and B may be either acute or obtuse

$$\therefore \quad \text{either} \quad \begin{matrix} B = 53{\cdot}13° \\ C = 96{\cdot}87° \end{matrix} \quad \text{or} \quad \begin{matrix} B = 180° - 53{\cdot}13° = 126{\cdot}87° \\ C = 23{\cdot}13°. \end{matrix}$$

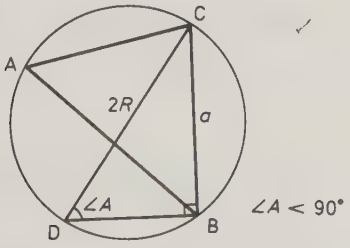
If B is acute, $c = \dfrac{a \sin C}{\sin A} = \dfrac{10 \sin 96{\cdot}87°}{\frac{1}{2}} = 20 \sin 96{\cdot}87° \approx 19{\cdot}9$

If B is obtuse, $c = \dfrac{a \sin C}{\sin A} = \dfrac{10 \sin 23{\cdot}13°}{\frac{1}{2}} = 20 \sin 23{\cdot}13° \approx 7{\cdot}86.$

Hence the two possible values of c are 7·86 and 19·9.

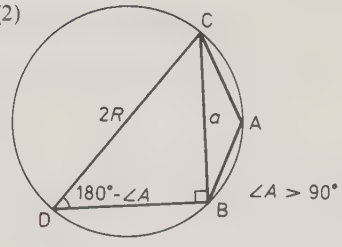
Another proof of the sine rule involves the construction of the circumcircle of $\triangle ABC$. In both diagrams, CD is a diameter of the circle, so that $\angle CBD = 90°$. Using the geometrical properties of a circle, in (1) $\angle D = \angle A$ and in (2) $\angle D = 180° - \angle A$.

(1)



$\angle A < 90°$

(2)



$\angle A > 90°$

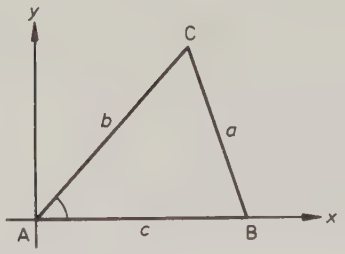
Let R be the radius of the circumcircle of $\triangle ABC$, then using $\triangle BCD$, in (1) $\sin A = a/2R$ and in (2) $\sin A = \sin(180° - A) = a/2R$

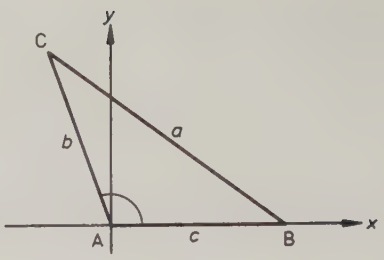
$$\therefore \quad \text{in both cases,} \quad \frac{a}{\sin A} = 2R.$$

By similar arguments it may be shown that

$$\boxed{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R,}$$

which is an extension of the basic sine rule.





To prove the cosine rule we use Cartesian coordinates and the distance between two points. In both diagrams, the coordinates of B and C are $(c, 0)$ and $(b\cos A, b\sin A)$ respectively,

$$\begin{aligned}\therefore \quad a^2 = BC^2 &= (b\cos A - c)^2 + (b\sin A - 0)^2 \\ &= b^2\cos^2 A - 2bc\cos A + c^2 + b^2\sin^2 A \\ &= b^2(\cos^2 A + \sin^2 A) + c^2 - 2bc\cos A.\end{aligned}$$

Since $\cos^2 A + \sin^2 A = 1$, we have established the *cosine rule*

$$\boxed{a^2 = b^2 + c^2 - 2bc\cos A.}$$

[Note that this proof is valid for an angle A of any magnitude.]

Example 3 Solve the triangle ABC in which $a = 5$, $b = 8$ and $C = 100°$.

Using the cosine rule in the form $c^2 = a^2 + b^2 - 2ab\cos C$,

$$c^2 = 5^2 + 8^2 - 2.5.8\cos 100° = 89 - 80\cos 100°$$

$$\therefore \quad c \approx 10{\cdot}1.$$

[The approximate value for c given here is the value $c = 10{\cdot}143561$, obtained by calculator, rounded to 3 significant figures. This more accurate value is stored in the calculator memory for use when finding angle A.]

Using the sine rule, $$\frac{\sin A}{a} = \frac{\sin C}{a}$$

$$\therefore \quad \sin A = \frac{a\sin C}{c} = \frac{5\sin 100°}{c} \quad \text{and} \quad A \approx 29{\cdot}04°.$$

Since $A + B + C = 180°$, $B = 50{\cdot}96°$.

Hence the remaining side and angles of the triangle are $c = 10{\cdot}1$, $A = 29{\cdot}0°$ and $B = 51{\cdot}0°$.

The cosine rule can be rearranged when finding unknown angles:

$$\boxed{\cos A = \frac{b^2 + c^2 - a^2}{2bc}.}$$

Example 4 Solve the triangle ABC in which $a = 8$, $b = 9$ and $c = 10$.

[Since any obtuse angle in the triangle will be opposite the longest side c it may be easier to find angles A and B first.]

Using the cosine rule, $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$

$$\therefore \quad \cos A = \frac{9^2+10^2-8^2}{2.9.10} = \frac{117}{180} = \frac{13}{20} = 0{\cdot}65.$$

$$\therefore \quad A \approx 49{\cdot}458°$$

Using the cosine rule, $\cos B = \dfrac{a^2+c^2-b^2}{2ac}$

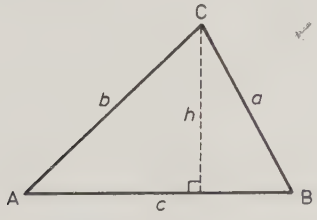
$$\therefore \quad \cos B = \frac{8^2+10^2-9^2}{2.8.10} = \frac{83}{160} = 0{\cdot}51875$$

$$\therefore \quad B \approx 58{\cdot}752°.$$

Since $A + B + C = 180°$, $C \approx 71{\cdot}790°$.

$\therefore$ the angles of the triangle are $A = 49{\cdot}5°$, $B = 58{\cdot}8°$ and $C = 71{\cdot}8°$.

[Some readers may prefer to find B using the sine rule. However, this does require the use of the approximate or even wrongly calculated value of A. As a check on accuracy, C can also be calculated independently.]



Let $\triangle$ represent the area of $\triangle ABC$ then

$$\triangle = \tfrac{1}{2}ch = \tfrac{1}{2}c.b\sin A$$

$\therefore$ the area of $\triangle ABC$ is $\tfrac{1}{2}bc\sin A$.

To obtain an expression for $\triangle$ in terms of a, b and c we write

$$\sin^2 A = 1-\cos^2 A = (1+\cos A)(1-\cos A).$$

Using the cosine rule it can then be shown that

$$4b^2c^2\sin^2 A = (\{b+c\}^2-a^2)(a^2-\{b-c\}^2).$$

Hence $$16\triangle^2 = (a+b+c)(b+c-a)(a+c-b)(a+b-c).$$

This result is expressed in a more convenient form using the *semi-perimeter* of the triangle, $s = \frac{1}{2}(a+b+c)$.

$$\triangle = \sqrt{\{s(s-a)(s-b)(s-c)\}}$$

This is *Hero's*† *formula* for the area of a triangle.

† *Hero (Heron) of Alexandria* (1st cent. A.D.) Greek mathematician. He invented many machines and demonstrated the law of reflection. The derivation of his formula for the area of a triangle appears in *Metrica*, his most important geometrical work.

Exercise 8.2

In questions 1 to 12 solve the triangle ABC.

1. $A = 60°, B = 50°, c = 5.$
2. $A = 47°, b = 8, C = 72°.$
3. $a = 5, b = 9, C = 51{\cdot}2°.$
4. $a = 2{\cdot}4, B = 45°, c = 5{\cdot}1.$
5. $a = 6, b = 8, c = 9.$
6. $a = 3{\cdot}6, b = 6, c = 4{\cdot}8.$
7. $A = 67{\cdot}3°, a = 10, b = 10.$
8. $A = 121°, b = 6{\cdot}9, c = 4{\cdot}7.$
9. $a = 6{\cdot}4, B = 23°, C = 54°.$
10. $B = 55°, b = 12, c = 9.$
11. $a = 15, b = 8, c = 10.$
12. $b = 6, c = 3, C = 30°.$

13. In triangle ABC, $a = 4$, $b = 5$ and $A = 48°$. Find the two possible values of c.

14. In triangle ABC, $b = 12$, $c = 9$ and $C = 34{\cdot}3°$. Find the two possible values of a.

15. In triangle XYZ, $\angle X = 108°$, $\angle Y = 25°$ and $XY = 10\,\text{cm}$. Calculate the area of the triangle.

16. A triangle has sides 13 cm, 7 cm and $5\sqrt{3}$ cm. Without using tables or calculators, find the smallest angle of the triangle and the diameter of the circumcircle.

17. The area of a triangle is $10\,\text{cm}^2$. Two of the sides are of lengths 5 cm and 7 cm. Calculate the possible lengths of the third side.

18. In quadrilateral $ABCD$, $AB = 7\,\text{cm}$, $BC = 8\,\text{cm}$, $CD = 5\,\text{cm}$ and $\angle ABC = 52°$. Given that $ABCD$ is a cyclic quadrilateral, find the radius of its circumscribing circle and the length of AD.

19. In triangle XYZ, $XY = p$, $XZ = q$ and M is the mid-point of YZ. If $\angle MXY = \theta$ and $\angle MXZ = \phi$, prove that $p\sin\theta = q\sin\phi$.

20. With the usual notation for a triangle ABC, prove that $a\cos B + b\cos A = c$.

21. Use the sine rule to prove, with the usual notation, that in triangle ABC,

$$\frac{b-c}{b+c} = \frac{\sin B - \sin C}{\sin B + \sin C}.$$

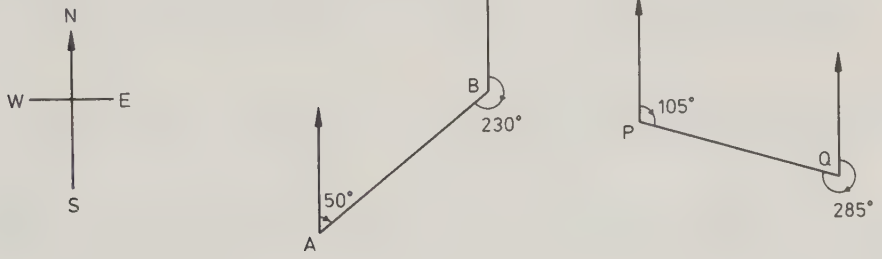
Deduce that $\tan\frac{1}{2}(B-C) = \dfrac{b-c}{b+c}\tan\frac{1}{2}(B+C) = \dfrac{b-c}{b+c}\cot\frac{1}{2}A$. Use this formula (called the *tangent rule*) to find B and C given (a) $A = 52°$, $b = 7$, $c = 5$, (b) $A = 40°$, $b = 4{\cdot}4$, $c = 2$.

22. The triangle ABC has incircle of radius r with centre at I. By considering triangles IBC, ICA and IAB, find an expression for the area of triangle ABC in terms of r and the semi-perimeter s. Calculate r given that $a = 5$, $b = 7$ and $c = 8$.

8.3 Problems in two dimensions

The next exercise provides further practice in the use of the sine and cosine rules. Some of the terms used in trigonometry problems are defined below.

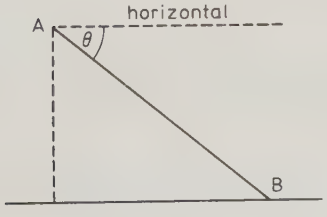
(1) *Bearings*



The bearing of B from A is 050° or N 50° E and that of A from B is 230° or S 50° W.

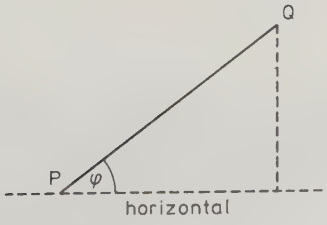
The bearing of Q from P is 105° or S 75° E and that of P from Q is 285° or N 75° W.

(2) *Angle of depression*



To an observer at A the angle of depression of an object at B is θ.

(3) *Angle of elevation*



To an observer at P the angle of elevation of an object at Q is ϕ.

The angle of elevation of the sun is sometimes called the *sun's altitude.*

Exercise 8.3

1. A ship A is 6·7 km from a lightship on a bearing of 308°. A second ship B is 8·3 km from the lightship on a bearing of 074°. Calculate the distance AB and the bearing of B from A to the nearest degree.

2. A, B and C are three towns. B is 10 km from A in the direction N 51° E. C is 15 km from B in the direction N 64° W. Calculate the distance and the bearing of A from C.

3. From a barge moving with constant speed along a straight canal the angle of elevation of a bridge is 10°. After 10 minutes the angle is 15°. How much longer will it be before the barge reaches the bridge, to the nearest second?

4. In a quadrilateral $PQRS$, $PQ = 10$ cm, $QR = 7$ cm, $RS = 6$ cm, $\angle PQR = 65°$ and $\angle PSR = 98°$. Find the length of PS.

5. A convex quadrilateral $ABCD$ has $AB = 15$, $BC = CD = 8$, $AD = 7$ and $AC = 13$. Show that the quadrilateral is cyclic and find BD.

6. A tower stands on a slope which is inclined at an angle of 17·2° to the horizontal. From a point further up the slope and 150 m from the base of the tower the angle of depression of the top of the tower is found to be 9·6°. Find the height of the tower.

7. AB is a level straight road 1500 m long. C and D are the bases of two church spires on opposite sides of the road. The angles BAC, ABC, BAD, ABD are 43°, 57°, 29° and 37° respectively. Find the distance between the spires.

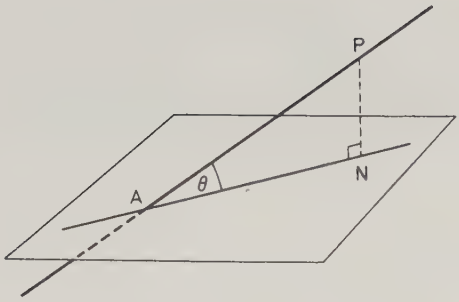
8. The line ABC is the tangent to a circle at the point B. X, Y and Z are points on the circle, such that $\angle ABX = 24°$, $\angle ABY = 63°$ and $\angle CBZ = 51°$. Given that the radius of the circle is 8 cm, find the area of $\triangle XYZ$.

9. In triangle ABC the median through A meets BC at M. Given that $AB = 8$ cm, $AM = 5$ cm and $AC = 6$ cm, use the cosine rule in triangles ABM and AMC to find BC. Find also $\angle ABC$.

10. In $\triangle ABC$, $AB = 9$ cm, $AC = 12$ cm, $\angle B = 2\theta$ and $\angle C = \theta$. Without using tables or calculators, find $\cos\theta$ and the length of BC.

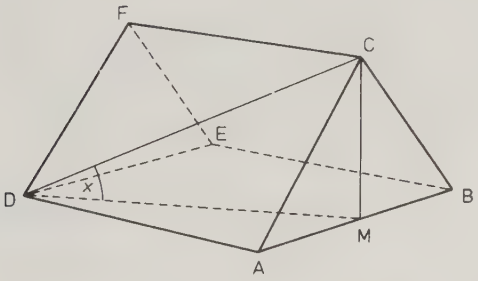
8.4 Angles, lines and planes

A line is said to lie in a plane if every point of the line is in the plane. A line is said to be parallel to a plane if the line never meets the plane. If a line neither lies in a plane nor is parallel to it, then the line intersects the plane in a single point. A line which is perpendicular to a plane is perpendicular to every line in the plane



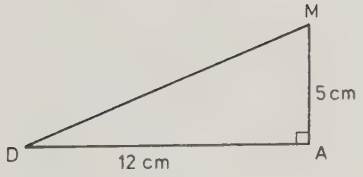
through this point of intersection. In general, a line and a plane intersect as shown in the diagram. If N is the foot of the perpendicular to the plane from a point P on the line, then AN is called the *projection* of AP on the plane. The angle between the line and the plane is defined as the angle θ between the line and its projection on the plane.

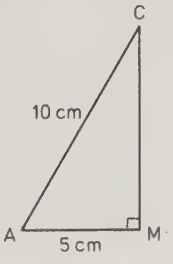
Example 1 The prism shown below has three identical rectangular faces. $AD = BE = CF = 12\,\text{cm}$ and the sides of triangles ABC, DEF are each 10 cm long. Find the angle x between the line CD and the plane $ABED$.

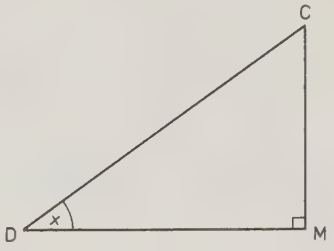


The perpendicular from C to the plane $ABED$ meets the plane at M, the midpoint of AB

$\therefore$ the angle between CD and the plane $ABED$ is $\angle CDM = x$.







Using Pythagoras' theorem in $\triangle ADM$,

$$DM = \sqrt{(AM^2 + AD^2)} = \sqrt{(5^2 + 12^2)}\,\text{cm} = \sqrt{169}\,\text{cm} = 13\,\text{cm}$$

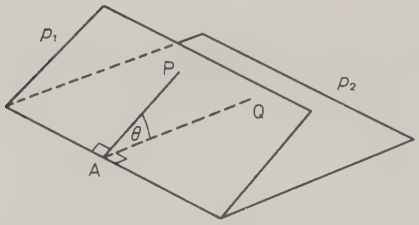
Using Pythagoras' theorem in $\triangle ACM$,

$$CM = \sqrt{(AC^2 - AM^2)} = \sqrt{(10^2 - 5^2)}\,\text{cm} = \sqrt{75}\,\text{cm} = 5\sqrt{3}\,\text{cm}$$

Hence, in $\triangle CMD$, $\tan x = CM/DM = 5\sqrt{3}/13 \quad \therefore \quad x \approx 33{\cdot}7°$.

The angle between CD and the plane $ABCD$ is 33·7°.

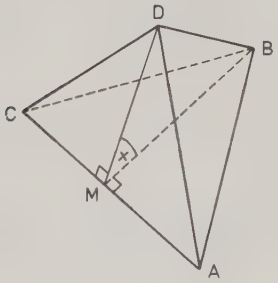
Two planes are said to be parallel if they do not intersect. Two planes which are not parallel have a line of intersection (or common line) as shown below.



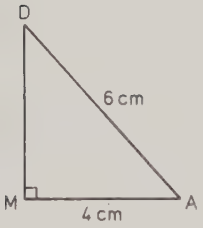
AP lies in the plane p_1 and AQ lies in a second plane p_2. If AP and AQ are *both* perpendicular to the line of intersection of the planes, then $\angle PAQ = \theta$ is the angle between the planes.

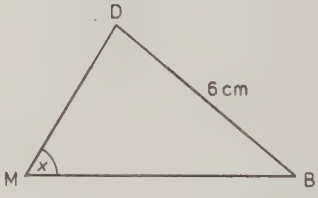
[θ is called the *dihedral angle* of the planes.]

Example 2 In a tetrahedron $ABCD$, $AD = BD = CD = 6\,\text{cm}$ and $AB = BC = CA = 8\,\text{cm}$. Find the angle between the plane ACD and the plane ABC.



AC is the line of intersection of the planes ACD and ABC. If M is the midpoint of AC, then since $AD = DC$ and $AB = BC$, both DM and BM are perpendicular to AC. Hence the angle between planes ACD and ABC is $\angle BMD = x$.





Using Pythagoras' theorem in $\triangle ADM$,

$$DM = \sqrt{(AD^2 - AM^2)} = \sqrt{(36-16)}\,\text{cm} = \sqrt{20}\,\text{cm} = 2\sqrt{5}\,\text{cm}.$$

Using Pythagoras' theorem in $\triangle ABM$,

$$BM = \sqrt{(AB^2 - AM^2)} = \sqrt{(64-16)}\,\text{cm} = \sqrt{48}\,\text{cm} = 4\sqrt{3}\,\text{cm}.$$

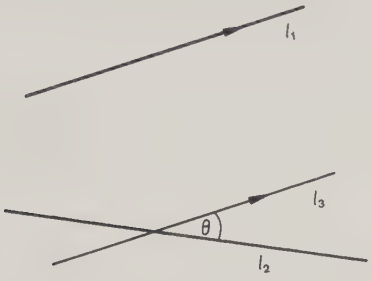
Using the cosine rule in $\triangle BDM$,

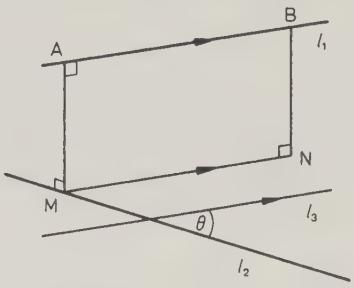
$$\cos x = \frac{BM^2 + DM^2 - BD^2}{2BM \,.\, DM} = \frac{48 + 20 - 36}{2.4\sqrt{3}.2\sqrt{5}} = \frac{32}{16\sqrt{15}} = \frac{2}{\sqrt{15}}$$

$$\therefore \quad x \approx 58{\cdot}9°.$$

Hence the angle between the planes ACD and ABC is 58·9°.

Two straight lines are said to be parallel if they are coplanar (i.e. lie in the same plane) but do not meet. Two lines which are not parallel and which do not meet are called *skew lines*. Skew lines l_1 and l_2 are shown below. The line l_3 is constructed so that l_3 intersects l_2 and is parallel to l_1. The angle between l_1 and l_2 is defined to be the angle θ between l_2 and l_3. BN is drawn perpendicular to the

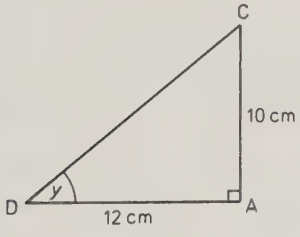




plane containing l_2 and l_3. If M is the point on l_2 such that MN is parallel to l_1, then $ABNM$ is a rectangle. This shows that it is always possible to construct a common perpendicular AM to the two skew lines l_1 and l_2. It can also be shown that AM is the shortest distance between l_1 and l_2.

Example 3 In the prism given in Example 1, find the angle between CD and BE.

Since AD is parallel to BE, the angle between CD and BE is equal to the angle between CD and AD, y.



In triangle ACD, $\tan y = \dfrac{10}{12} = \dfrac{5}{6}$,

$\therefore \quad y \approx 39{\cdot}8°$

Hence the angle between the skew lines CD and BE is 39·8°.

Exercise 8.4

1. Triangles ABC and XYZ are two faces of a right triangular prism. $AB = AC = XY = XZ = 7\,\text{cm}$ and $BC = YZ = 10\,\text{cm}$. The plane of each triangle is perpendicular to the edges AX, BY and CZ, which are of length 15 cm. Find the angles between the plane $BCZY$ and (a) the line AY, (b) the plane ABY and (c) the plane AYZ.

2. A triangle PQR lies in a horizontal plane and S is a point vertically above P. Given that $PQ = 5\,\text{cm}$, $PR = 10\,\text{cm}$, $RS = 14\,\text{cm}$ and $\angle RPQ = 60°$, find the angles between (a) RS and the plane PQR, (b) the planes QRS and PQR, (c) the lines PS and QR.

3. A triangle ABC lies in a horizontal plane. The points X, Y and Z are 8 cm vertically above A, B and C respectively. M is the mid-point of YZ. $AB = 5$ cm, $BC = 6$ cm and $\angle ABC = 90°$. Find the angles between (a) the planes AYZ and ABC, (b) the line AM and the plane ABC, (c) the lines AB and XC.

4. A square $ABCD$ of side 10 cm lies in a horizontal plane. A second square $ABXY$ lies in a plane inclined at an angle of 40° to the horizontal. Find the angles made with the horizontal by (a) AX and (b) the plane AXD.

5. $VABCD$ is a pyramid of height 15 cm standing symmetrically on a square base $ABCD$ of side 16 cm. Find the angles between (a) the edge VA and the base $ABCD$, (b) the face VAB and the base $ABCD$, (c) the edges VA and CD, (d) the edge VA and the line BD.

6. In the tetrahedron $PQRS$, $PQ = QR = RS = SP = 13$ cm and $PR = QS = 10$ cm. Find the angles between (a) the planes PQR and PSR, (b) the line QS and the plane PSR.

7. A pyramid $VABCD$ stands on a horizontal square base $ABCD$. The faces VAB, VBC, VCD,VDA are equilateral triangles of side $2a$. Find (a) the height of the vertex V above the base, (b) the angle between the faces VAB and VCD, (c) the angle between the faces VAB and VBC.

8. Each face of a tetrahedron $ABCD$ is an equilateral triangle of side 6 units. Without using tables or calculators, find (a) the cosine of the angle between any two faces, (b) the perpendicular distance of any vertex from the opposite face.

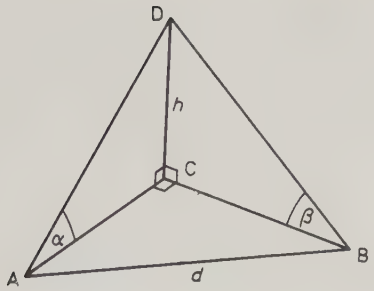
9. In a tetrahedron $ABCD$, $AB = AC = AD = 5$ cm and $BC = CD = DB = 8$ cm. Find the angles between (a) the edge AB and the face BCD, (b) the faces ABC and BCD, (c) the faces ABC and ACD, (d) the edge AD and the face ABC.

10. $ABCDPQRS$ is a cuboid on a horizontal base $ABCD$. The vertical edges PA, QB, RC and SD are of length 6 cm. $AB = 8$ cm and $BC = 9$ cm. Find the angles between the following pairs of lines (a) AR and BS, (b) BD and PR, (c) AS and DR, (d) AC and BS.

8.5 Problems in three dimensions

In some problems it is necessary to obtain a general expression connecting the lengths and angles involved.

Example 1 A man at A observes a tower CD due north of him, with height h and angle of elevation α. At a point B, due east of the tower and a distance d from A, the angle of elevation is β. If A, B and C lie in the same horizontal plane, find an expression connecting h, d, α and β.



$$\text{In } \triangle ACD, \cot\alpha = \frac{AC}{CD} = \frac{AC}{h}$$

$$\therefore \quad AC = h\cot\alpha.$$

$$\text{In } \triangle BCD, \cot\beta = \frac{BC}{CD} = \frac{BC}{h}$$

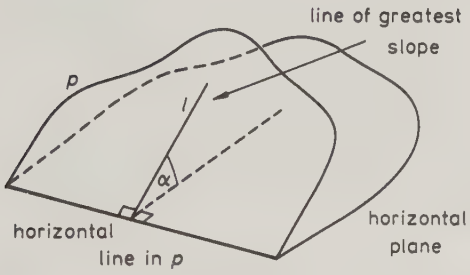
$$\therefore \quad BC = h\cot\beta.$$

Using Pythagoras' theorem in $\triangle ABC$,

$$AB^2 = AC^2 + BC^2$$

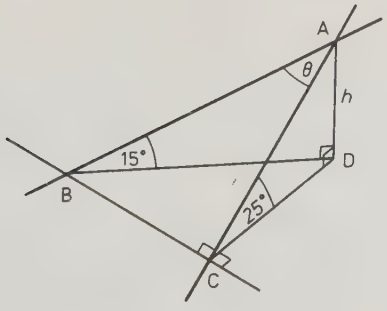
$$\therefore \quad d^2 = h^2\cot^2\alpha + h^2\cot^2\beta.$$

Hence the required expression is $d^2 = h^2(\cot^2\alpha + \cot^2\beta)$.



Consider a plane p inclined at an angle α to the horizontal. The diagram shows the plane p and its intersection with a horizontal plane. The inclination to the horizontal of any line l in p which is perpendicular to the line of intersection is α. Since no line in p can be inclined to the horizontal at an angle greater than α, l is called a *line of greatest slope.*

Example 2 On a plane hillside which slopes at an angle 25° to the horizontal, there are two straight roads. One lies along a line of greatest slope and the other makes an angle of 15° with the horizontal. Find the angle θ at which the roads intersect.



In the diagram AB and AC are segments of the two roads. Their point of intersection A is at a height h above the point D, which lies in the horizontal plane through BC.

$$\text{In } \triangle ACD, \quad \sin 25° = \frac{AD}{AC} = \frac{h}{AC} \qquad \therefore \quad AC = \frac{h}{\sin 25°}$$

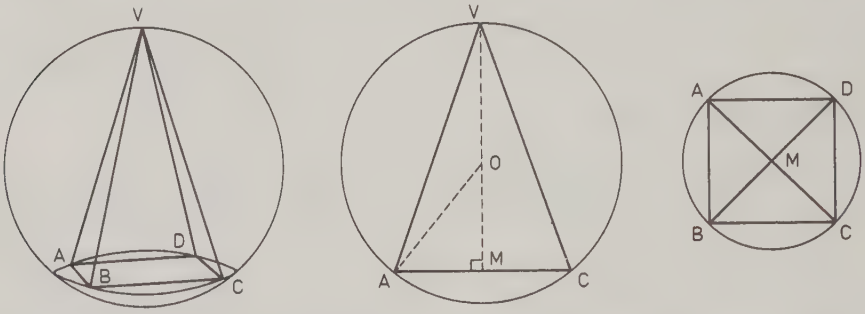
$$\text{In } \triangle ABD, \quad \sin 15° = \frac{AD}{AB} = \frac{h}{AB} \qquad \therefore \quad AB = \frac{h}{\sin 15°}$$

$$\text{In } \triangle ABC, \quad \cos\theta = \frac{AC}{AB} = \frac{h}{\sin 25°} \cdot \frac{\sin 15°}{h} = \frac{\sin 15°}{\sin 25°}$$

$\therefore$ the angle θ at which the roads intersect is 52·2°.

Problems involving spheres are often solved by drawing a number of circular sections through the sphere.

Example 3 A right pyramid with a square base is inscribed in a sphere of radius 9 cm. If the height of the pyramid is 16 cm, find the length of each side of the base.



In the diagrams above, O is the centre of the sphere and M is the mid-point of AC, so that VM is the height of the pyramid.

$\therefore$ $AO = VO = 9\,\text{cm}$, $VM = 16\,\text{cm}$, $OM = VM - VO = 7\,\text{cm}$.

Using Pythagoras' theorem in $\triangle AOM$,

$$AM^2 = AO^2 - OM^2 = 9^2 - 7^2 = 81 - 49 = 32.$$

Using Pythagoras' theorem in $\triangle ABM$,

$$AB^2 = AM^2 + MB^2 = 2AM^2 = 64.$$

Hence each side of the base of the pyramid is 8 cm long.

Exercise 8.5

1. Rectangles $ABED$ and $BCFE$ form two vertical faces of a box. Given that $AB = 20\,\text{cm}$, $BC = 36\,\text{cm}$, $AD = 15\,\text{cm}$ and $\angle ABC = 90°$, find $\angle DBF$ and the area of $\triangle DBF$.

2. Three points P, Q and R lie in the same horizontal plane. S, the top of a church spire, is h metres vertically above R. P is due west of the spire and Q is south-east of the spire. The angle of elevation of the spire from both P and Q is 30°. If the distance $PQ = 300\,\text{m}$, calculate h and find also the greatest angle of elevation of S from any point on PQ.

3. A clock rests on a stand so that its face makes an angle of 68° with the horizontal. Find the angle made with the horizontal by the minute hand at (a) 08.10, (b) 07.25.

4. A straight path up a hillside, which may be considered as a plane at 25° to the horizontal, makes an angle of 53° with a line of greatest slope. Find the angle the path makes with the horizontal. A second path making an angle of 19° with the horizontal cuts the first path. Find the angle between the two paths.

5. An observer on the ground notes that an aircraft on a bearing of 340° is at a height of 200 m and at an angle of elevation of 45°. The aircraft then travels 400 m due east rising to a height of 300 m. Find the new bearing and angle of elevation of the aircraft.

6. A small boat is due south of a lighthouse and to an observer at the top of the lighthouse, 40 m above sea level, its angle of depression is 30°. After travelling 200 m on a steady course and passing to the east of the lighthouse, the new angle of depression of the boat is 15°. Find the new bearing of the boat from the lighthouse. Find also the bearing of the boat from the lighthouse at its closest point of approach.

7. Two pylons 50 m high stand at points X and Y in the same horizontal plane as an observer at Z. The point Y is 100 m due east of X. The bearings of X and Y from Z are S 70° W and S 30° W respectively. Find the angles of elevation of the tops of the pylons from Z.

8. The triangle ABC lies in a horizontal plane and M is the mid-point of BC. $AM = 5\,\text{m}$, $AC = 4\,\text{m}$ and $\angle ACB = 90°$. A vertical pole CD of height h metres stands at C. The angle of elevation of D from A is α and the angle of elevation of D from B is β. Given that $\alpha + \beta = 45°$, find h, α and β.

9. A, B and C are three points in a horizontal plane. There are vertical posts AP and BQ at A and B, both of height h. The angles of elevation of P and Q from C are α and β respectively. Angles PCQ and ACB are θ and ϕ respectively. Show that
(a) if $\phi = 90°$, then $\cos\theta = \sin\alpha \sin\beta$,
(b) if $\theta = 90°$, then $\cos\phi = -\tan\alpha \tan\beta$.

10. A, B and C are three points on a plane which slopes at an angle of 30° to the horizontal, B and C being above the horizontal plane containing A. The line of greatest slope through A passes through the mid-point M of BC. $AM = 10\,\text{m}$, $BC = 16\,\text{m}$ and $\angle AMB = 60°$. Find the lengths of AB and AC. By adding to your diagram a horizontal line through A and lines of greatest slope through B and C, or otherwise, find the angles made by AB and AC with the horizontal plane through A.

11. A right pyramid with a square base is inscribed in a sphere. If the height of the pyramid is 10 cm and the length of each side of the base is 8 cm, find (without using tables or calculators) the radius of the sphere.

Exercise 8.6 (*miscellaneous*)

1. The points A, B, C and D lie on the circumference of a circle. $AB = 7\,\text{cm}$, $BC = 9\,\text{cm}$, $CA = 5\,\text{cm}$ and AD is a diameter of the circle. Find $\angle ABC$ and the length of CD.

2. Use the sine rule to prove that if P is a point on the side XZ of $\triangle XYZ$ such that $\angle XYP = \angle PYZ$, then $XP : PZ = XY : YZ$. By letting $\angle XYZ = 30°$, $\angle YZX = 60°$ and $YZ = 2$ units, show that $\cot 15° = 2+\sqrt{3}$.

3. The points X, Y and Z lie on a circle with centre O. If $XY = 8\,\text{cm}$, $YZ = 11\,\text{cm}$ and $ZX = 15\,\text{cm}$, find angles XOY, YOZ and ZOX. Find also the radius of the circle.

4. In the triangle ABC, X, Y and Z are the mid-points of BC, CA and AB, respectively. Using the cosine rule or otherwise, prove that $b^2+c^2 = 2AX^2+2BX^2$. Write down two other similar results, and hence show that $AX^2+BY^2+CZ^2 = \frac{3}{4}(a^2+b^2+c^2)$. (JMB)

5. In each of the following cases find (giving reasons for your answers) the number of triangles ABC satisfying the given conditions and, where possible, find by calculation the value(s) of c, leaving your answers in surd form, if you wish.
(i) $a = 5, b = 3, B = 30°$; (ii) $a = 5, b = 2, B = 30°$;
(iii) $a = 2, b = 3, B = 30°$; (iv) $a = 2, b = 3, B = 150°$. (C)

6. Construct a flow diagram for finding $\angle ABC$ in triangle ABC given a, b and $\angle BAC = x°$. Your diagram should produce an appropriate output for all cases.

7. The point P divides the side BC of $\triangle ABC$ internally in the ratio $m : n$. If $\angle B = \theta$, $\angle C = \phi$ and $\angle APC = \alpha$, show that $(m+n)\cot\alpha = n\cot\theta - m\cot\phi$.

8. Using the usual notation in $\triangle ABC$, prove that

$$\frac{a\cos A + b\cos B}{a\cos B + b\cos A} = \cos(A-B).$$

9. With the usual notation in $\triangle ABC$, write down an expression for $\cos A$ in terms of a, b and c. Using this result show that $\sin^2\frac{1}{2}A = \dfrac{(s-b)(s-c)}{bc}$ and $\cos^2\frac{1}{2}A = \dfrac{s(s-a)}{bc}$ where $s = \frac{1}{2}(a+b+c)$. Hence find expressions for $\tan^2\frac{1}{2}A$ and the area of the triangle in terms of a, b, c and s.

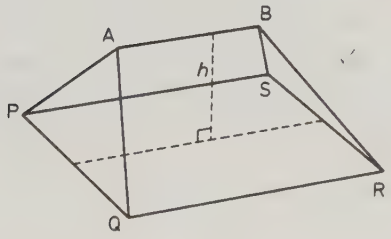
10. An isosceles triangle ABC, in which $AB = AC$ and $\angle A = 2\theta$, is inscribed in a circle of radius 5 cm. Prove that the two equal altitudes of the triangle have length $10\cos\theta\sin 2\theta\,\text{cm}$. If the sum of the lengths of the three altitudes is 10 cm, find the three angles of the triangle to the nearest degree. (O & C)

11. From a point A on level ground due S of a church the angle of elevation of the top T of its steeple is $\beta°$. From a point B on the ground whose distance from A is $2a$ on a bearing of N $\theta°$ E $(0 < \theta < 90)$ the angle of elevation of T is also $\beta°$. Show that the height of T above the ground is $a \sec\theta° \tan\beta°$ and find the bearing of the church from B. If D is the point on AB produced distant a from B show that the distance of D from the steeple is $a\sqrt{(3+\sec^2\theta°)}$ and that the bearing of the church from D is S$(\theta+\phi)°$ W, where $\sin\phi = \tan\theta/\sqrt{(3+\sec^2\theta)}$. (SU)

12. A, B, C are three landmarks. B is 15 km from A on a bearing of 140° and C is 15 km from B on a bearing of 216°. An observer on a ship finds that B is directly behind C and that A bears 015°. After sailing for 30 minutes directly towards A, the observer finds that C bears due east. Find the speed of the ship.

13. A and B are two points on level ground, and B is a metres due east of A; a tower, h metres high, is also on the same level ground. From A, the tower is in a direction N θ E and from B it is N ϕ W. From the top of the tower, the angle of depression of A is α and of B is β. Prove the following (in any order):
(i) $h\sin(\theta+\phi) = a\cos\phi\tan\alpha$,
(ii) $\cos\phi\tan\alpha = \cos\theta\tan\beta$,
(iii) $h^2(\cot^2\alpha - \cot^2\beta) - 2ha\cot\alpha\sin\theta + a^2 = 0$. (SU)

14.



The sketch represents a roof with a horizontal ridge AB at a height h above the (horizontal) ceiling level $PQRS$. $PQRS$ is a rectangle, and AB is parallel to PS and QR. The side of the roof $ABRQ$ makes an angle α with the horizontal, the end APQ makes an angle β with the horizontal, and θ is the angle between AQ and the horizontal. Prove that $\cot^2\alpha + \cot^2\beta = \cot^2\theta$. Prove also that $\cos^2\gamma + \cos^2\delta + \sin^2\theta = 1$, where γ and δ are the angles that AQ makes with PQ and QR respectively. (SU)

15. A rectangular lamina $ABCD$ has sides $AB = CD = 2a$ and $BC = AD = a$. The mid-point of CD is M and the mid-point of BM is N. The lamina is folded along BM so that the planes BMC and $BMDA$ are at right angles. For this folded figure, find, leaving your answers in surd form if desired, (i) the distance AN, (ii) the distance AC, (iii) the angle ACB, (iv) the tangent of the angle between the planes ADC and ADB. (O)

16. $ABCD$ is a regular tetrahedron (that is, a triangular pyramid with all its edges equal), and E, F are the midpoints of AB, CD respectively. Find the angles between (i) the planes $ABEF$ and $CDEF$, (ii) the plane $ABEF$ and the line AC, (iii) the lines AC and EF. (C)

17. A triangle ABC is in a horizontal plane and has $AB = 5\,\text{cm}$, $BC = 7\,\text{cm}$ and $AC = 8\,\text{cm}$. The points X, Y and Z are vertically above A, B and C respectively, such that $AX = 10\,\text{cm}$, $BY = 5\,\text{cm}$ and $CZ = 2\,\text{cm}$. Find $\angle BAC$ and $\angle YXZ$. If the lines AB and XY, AC and XZ intersect at the points P and Q respectively, show that triangles PAQ and PXQ are isosceles. Find the angle between the planes ABC and XYZ.

18. PTQ is the tangent to a circle at the point T. The points A and B on the circumference of the circle are such that TA and TB make acute angles α and β with TP and TQ respectively. If AB meets the diameter through T at N, prove that $TN = a\sin\alpha\sin\beta/\cos(\alpha-\beta)$, where a is the length of the diameter. If the points C and D on the circumference of the circle are such that TC and TD make acute angles γ and δ with TP and TQ respectively and CD meets the diameter through T at the same point N, prove that $\tan\alpha\tan\beta = \tan\gamma\tan\delta$. (O)

19. Prove that, for any triangle ABC, with the usual notation and for all values of the angle θ

$$c\sin\theta = a\sin(\theta-B)+b\sin(\theta+A).$$

A quadrilateral $ABCD$ is inscribed in a circle. By applying the above result to the triangle ABC and taking for θ the angle CAD, prove that
$AB.CD+AD.BC = AC.BD$. (JMB)

20. A road sign is surmounted by a plane equilateral triangle ABC of side 40 cm; its base BC is horizontal and the triangle lies in a vertical plane with BC pointing in the direction N 80° E. The sun is SW at an elevation of 30° and it casts a shadow $A'B'C'$ on a horizontal plane. If M' is the mid-point of $B'C'$, find the length of $A'M'$; hence, or otherwise, find the area of $A'B'C'$. (O & C)

21. A regular tetrahedron with edges of length $2a$ is inscribed in a sphere. Find the diameter of the sphere.

22. A pyramid has a horizontal square base $ABCD$ of side $2a$. Its vertex V is vertically above the centre O of the base, and the length of VO is $2a$. The line through V perpendicular to the face VAD meets the plane of the base at X. Find (i) the length of VX; (ii) the cosine of the angle between the faces VAB and VAD; (iii) the sine of the angle between the edge VC and the face VAB. (O)

23. In the region of three fixed buoys A, B and C at sea there is a plane stratum of oil-bearing rock. The depths of the rock below A, B and C are 900 m, 800 m and 1000 m respectively. B is 600 m due east of A and the bearings of C from A and B are 190° and 235° respectively. Calculate (i) the distance BC, (ii) the direction of the horizontal projection of the line of greatest slope of the plane, (iii) the angle this plane makes with the horizontal. (AEB 1978)

[It may be helpful to consider a horizontal plane at a depth of 900 m.]

9 Vectors

9.1 Introduction to vectors

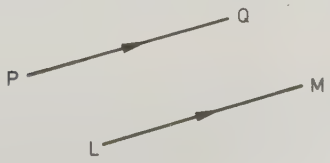
Some physical quantities, such as temperature, may be completely specified by a number referred to some unit of measurement, e.g. 25°C, 350°F. Such quantities, which have magnitude but which are not related to any definite direction in space, are called *scalar quantities*. Similarly, a number representing the magnitude of some physical quantity is called a *scalar*.

When measuring other quantities, such as wind velocity, it is necessary to give a direction as well as a number and a unit of measurement, e.g. 40 km/h from the north-east. These quantities, which have both magnitude and a definite direction in space, are called *vector quantities*. In general, a *vector* may be described as a number associated with a particular direction in space.

Any vector may be represented by a *directed line segment*, whose direction is that of the vector and whose length represents the magnitude of the vector.

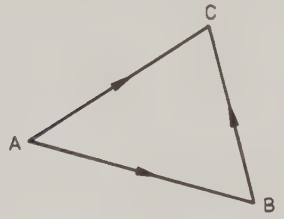
The vector represented here is denoted by **PQ** in bold type (or $\underset{\sim}{PQ}$ in manuscript). The *magnitude* or modulus of the vector is written $|\mathbf{PQ}|$ or PQ. The directed line segment joining P to Q is denoted by $\overrightarrow{PQ}$. However, since it is not usually necessary to distinguish between the line segment and the vector it represents, the notation $\overrightarrow{PQ}$ is in common use for both vector and line segment.

The magnitude of a vector is usually positive but may be zero, in which case the vector is called the *zero* or *null vector* and written **0** in bold type. The zero vector is the only vector with indeterminate direction.



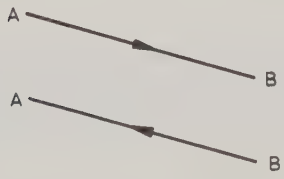
Two vectors are *equal* if they have the same magnitude and direction. Since PQ and LM are parallel and equal in length, $\mathbf{PQ} = \mathbf{LM}$.

A *displacement* is one of the simplest examples of a vector quantity. For instance, a car journey from a town A to a town B may be represented by a displacement vector **AB**. Its magnitude is the distance between A and B. Its direction is that of the straight line joining A to B.



Let us suppose that a man drives from A to B and then from B to C. On another occasion he drives directly from A to C. Since the result of these two journeys is the same, we may write $\mathbf{AB}+\mathbf{BC}=\mathbf{AC}$.

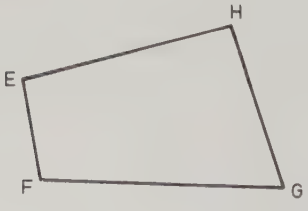
The displacement **AC** is the *sum* or *resultant* of the displacements **AB** and **BC**.



Consider now a journey from A to B, followed by the return journey from B to A. Since the result of these two journeys is a zero displacement, it is reasonable to write, $\mathbf{AB}+\mathbf{BA}=\mathbf{0}$ and $\mathbf{BA}=-\mathbf{AB}$.

All vectors represented by directed line segments may be manipulated in the same way as displacement vectors. For instance, given any three vectors **AB**, **BC** and **AC** represented by the sides of triangle ABC, we may write $\mathbf{AB}+\mathbf{BC}=\mathbf{AC}$. A vector which has the same magnitude as **AB**, but the opposite direction, is denoted by **BA** or $-\mathbf{AB}$.

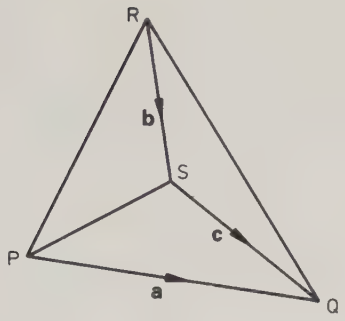
Example 1 Simplify $\mathbf{EF}+\mathbf{FG}-\mathbf{HG}$.



$$\begin{aligned}\mathbf{EF}+\mathbf{FG}-\mathbf{HG} &= \mathbf{EG}-\mathbf{HG}\\ &= \mathbf{EG}+\mathbf{GH}\\ &= \mathbf{EH}.\end{aligned}$$

It is often convenient to denote a vector by a single letter such as **a** in bold type (or $\underset{\sim}{a}$ in manuscript). Its magnitude is then written $|\mathbf{a}|$ or a.

Example 2 In the given diagram $\mathbf{PQ}=\mathbf{a}$, $\mathbf{RS}=\mathbf{b}$ and $\mathbf{SQ}=\mathbf{c}$. Find expressions for **RQ**, **PS** and **PR** in terms of **a**, **b** and **c**.



In $\triangle QRS$, $\mathbf{RQ} = \mathbf{RS} + \mathbf{SQ} = \mathbf{b} + \mathbf{c}$
In $\triangle PQS$, $\mathbf{PS} = \mathbf{PQ} + \mathbf{QS} = \mathbf{PQ} - \mathbf{SQ}$
$= \mathbf{a} - \mathbf{c}$
In $\triangle PQR$, $\mathbf{PR} = \mathbf{PQ} + \mathbf{QR} = \mathbf{PQ} - \mathbf{RQ}$
$= \mathbf{a} - (\mathbf{b} + \mathbf{c})$
$= \mathbf{a} - \mathbf{b} - \mathbf{c}$.

In general vectors do not have any definite position in space. However, certain vector quantities may be associated with a particular location. For instance, in mechanics, when considering the motion of an object, it may be necessary to take into account the line of action of a force as well as its magnitude and direction. Equal forces with different points of application may produce different effects. A vector which may be represented by any line segment of the appropriate magnitude and direction is sometimes called a *free vector*.

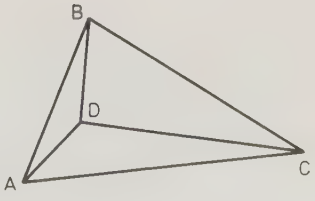
Exercise 9.1

1. Simplify (a) $\mathbf{PQ} + \mathbf{QR} + \mathbf{RS}$, (b) $\mathbf{AB} + \mathbf{DE} + \mathbf{CD} + \mathbf{BC}$,
(c) $\mathbf{AC} - \mathbf{BC}$, (d) $\mathbf{XY} + \mathbf{YX}$,
(e) $\mathbf{MN} - \mathbf{RP} - \mathbf{PN}$, (f) $\mathbf{QT} - \mathbf{QR} + \mathbf{PR} - \mathbf{ST}$.

2. Given that *ABCDEF* is a regular hexagon, decide whether the following statements are true or false.
(a) $\mathbf{BC} = \mathbf{EF}$, (b) $\mathbf{BF} = \mathbf{CE}$, (c) $|\mathbf{BE}| = |\mathbf{CF}|$, (d) $|\mathbf{AB}| = |\mathbf{DE}|$,
(e) **AC** and **DF** are in the same direction, (f) **AD** is parallel to **EF**.

3. Given that *ABCDPQRS* is a cuboid lettered so that *AP*, *BQ*, *CR* and *DS* are edges, decide whether the following statements are true or false.
(a) $\mathbf{PQ} = \mathbf{DC}$, (b) $\mathbf{BR} = \mathbf{PD}$, (c) $|\mathbf{RA}| = |\mathbf{BS}|$, (d) $|\mathbf{PA}| = |\mathbf{CR}|$,
(e) **DQ** and **AR** are in the same direction, (f) **AQ** is parallel to **CS**.

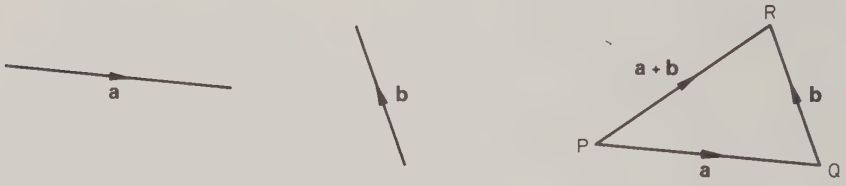
4. In the given diagram $\mathbf{AB} = \mathbf{x}$, $\mathbf{BC} = \mathbf{y}$ and $\mathbf{CD} = \mathbf{z}$. Find expressions for **AC**, **BD** and **AD** in terms of **x**, **y** and **z**.



5. In the pentagon *PQRST* $\mathbf{PQ} = \mathbf{a}$, $\mathbf{QR} = \mathbf{b}$, $\mathbf{ST} = \mathbf{c}$ and $\mathbf{QT} = \mathbf{d}$. Find expressions for **PT**, **RT**, **RS** and **PS** in terms of **a**, **b**, **c** and **d**.

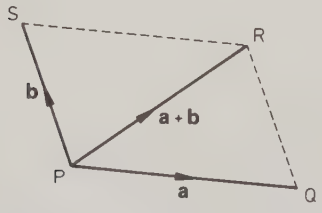
9.2 Elementary operations on vectors

In the previous section triangles were used to add displacements and other vectors represented by directed line segments. More formally, *vector addition* is a binary operation defined by the *triangle law*, as follows:



If two vectors **a** and **b** are represented by the sides $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ of a triangle, then $\mathbf{a}+\mathbf{b}$ is represented by the third side $\overrightarrow{PR}$.

In physics and mechanics it is sometimes more convenient to use the *parallelogram law* for vector addition, which states:



If two vectors **a** and **b** are represented by the sides $\overrightarrow{PQ}$ and $\overrightarrow{PS}$ of a parallelogram, then $\mathbf{a}+\mathbf{b}$ is represented by the diagonal $\overrightarrow{PR}$. [Since the line segments $\overrightarrow{PS}$ and $\overrightarrow{QR}$ are equal in magnitude and direction, the two addition laws are equivalent.]

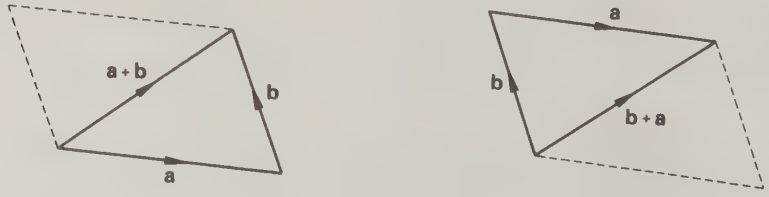
In $\triangle PQR$ shown above, $|\mathbf{a}|$, $|\mathbf{b}|$ and $|\mathbf{a}+\mathbf{b}|$ are represented by the lengths of the sides PQ, QR and PR respectively. For any points P, Q and R we have

$$PR \leqslant PQ + QR.$$

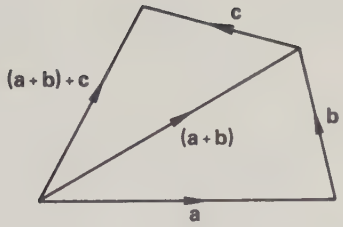
Hence

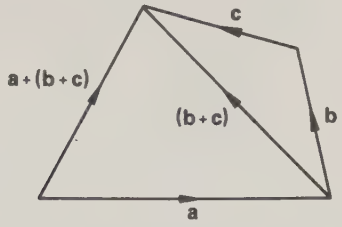
$$|\mathbf{a}+\mathbf{b}| \leqslant |\mathbf{a}| + |\mathbf{b}|.$$

[$|\mathbf{a}+\mathbf{b}| = |\mathbf{a}| + |\mathbf{b}|$ when P, Q and R are collinear.]



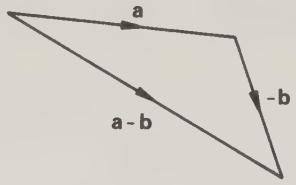
The two diagrams above show that $\mathbf{a}+\mathbf{b} = \mathbf{b}+\mathbf{a}$. Thus the *commutative law* holds for vector addition.

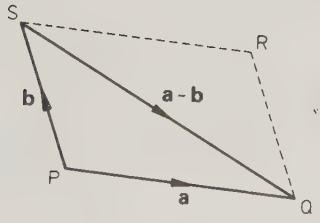




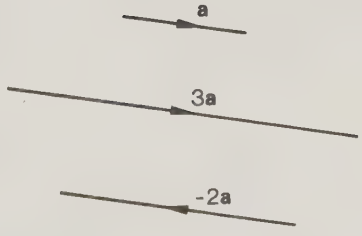
These diagrams show that $(\mathbf{a}+\mathbf{b})+\mathbf{c} = \mathbf{a}+(\mathbf{b}+\mathbf{c})$. Thus the *associative law* holds for vector addition. Hence the sum of a set of vectors is not affected by changes in the order or the grouping of the terms.

The *subtraction* of $\mathbf{b}$ from $\mathbf{a}$ may be defined as the address of $-\mathbf{b}$ to $\mathbf{a}$, i.e. $\mathbf{a}-\mathbf{b} = \mathbf{a}+(-\mathbf{b})$.





If the vectors $\mathbf{a}$ and $\mathbf{b}$ are represented by the sides $\overrightarrow{PQ}$ and $\overrightarrow{PS}$ of a parallelogram, then $\mathbf{a}-\mathbf{b}$ is represented by the diagonal $\overrightarrow{SQ}$.



Scalar multiples of a vector $\mathbf{a}$, such as $3\mathbf{a}$ and $-2\mathbf{a}$, can be defined using vector addition. For instance, $3\mathbf{a} = \mathbf{a}+\mathbf{a}+\mathbf{a}$ and $-2\mathbf{a} = -(\mathbf{a}+\mathbf{a})$. This leads to the following more general definition of *multiplication* of a vector by a scalar.

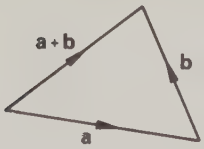
If λ is a scalar and $\mathbf{a}$ is a vector, then the magnitude of $\lambda\mathbf{a}$ is $|\lambda|$ times the magnitude of $\mathbf{a}$, i.e. $|\lambda\mathbf{a}| = |\lambda|a$. The direction of $\lambda\mathbf{a}$ is the same as that of $\mathbf{a}$ if λ is positive, but opposite to that of $\mathbf{a}$ if λ is negative.

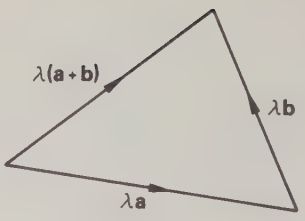
[*Division* of a vector by a scalar λ (not equal to zero) is defined as multiplication by $1/\lambda$.]

It follows from the associative and distributive properties of real numbers that

$$\lambda(\mu\mathbf{a}) = (\lambda\mu)\mathbf{a} \quad \text{and} \quad (\lambda+\mu)\mathbf{a} = \lambda\mathbf{a}+\mu\mathbf{a}.$$

We obtain a further distributive law by constructing a pair of similar triangles.



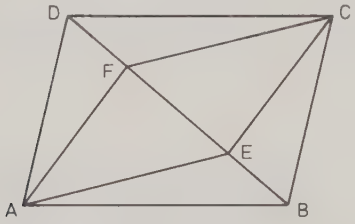


By applying the triangle law of addition to the second triangle, we find that

$$\lambda(\mathbf{a}+\mathbf{b}) = \lambda\mathbf{a}+\lambda\mathbf{b}.$$

We conclude this section with two examples showing vector methods applied to geometrical problems.

Example 1 $ABCD$ is a parallelogram. The points E and F lie on the diagonal BD and $BE = FD$. Prove that $AECF$ is a parallelogram.



Since $ABCD$ is a parallelogram,

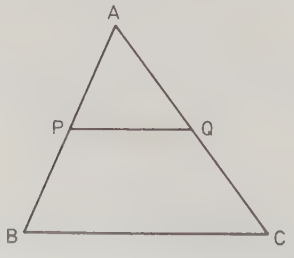
$$\mathbf{AB} = \mathbf{DC}.$$

Since B, E, F and D are collinear and $BE = FD$, $\mathbf{BE} = \mathbf{FD}$.

$$\therefore \quad \mathbf{AE} = \mathbf{AB}+\mathbf{BE} = \mathbf{DC}+\mathbf{FD} = \mathbf{FD}+\mathbf{DC} = \mathbf{FC}.$$

Hence, since AE and FC are parallel and equal in length, $AECF$ must be a parallelogram.

Example 2 Show that the line segment joining the mid-points of any two sides of a triangle is parallel to the third side and equal to half its length.



Consider the triangle ABC, where P and Q are the mid-points of the sides AB and AC respectively.

$$\begin{aligned}\mathbf{PQ} = \mathbf{PA}+\mathbf{AQ} &= \tfrac{1}{2}\mathbf{BA}+\tfrac{1}{2}\mathbf{AC}\\ &= \tfrac{1}{2}(\mathbf{BA}+\mathbf{AC})\\ &= \tfrac{1}{2}\mathbf{BC}.\end{aligned}$$

Hence PQ is parallel to BC and half its length.

Exercise 9.2

If **a**, **b** and **c** are displacements of 1 km due east, 2 km due north and 1 km in the direction S 30° W respectively, find the magnitudes and directions of the

displacements given in questions 1 to 3. (You may leave your answers in surd form.)

1. (a) $3\mathbf{a}$, (b) $5\mathbf{b}$, (c) $-\mathbf{c}$, (d) $-\frac{1}{2}\mathbf{b}$.

2. (a) $2\mathbf{a}+\mathbf{b}$, (b) $\mathbf{b}+2\mathbf{a}$, (c) $2\mathbf{a}-\mathbf{b}$, (d) $\mathbf{b}-2\mathbf{a}$.

3. (a) $\mathbf{a}+\mathbf{c}$, (b) $\mathbf{a}+2\mathbf{c}$, (c) $2\mathbf{a}+\mathbf{c}$, (d) $\mathbf{a}-\mathbf{c}$.

4. The diagonals of a parallelogram $ABCD$ intersect at M. If $\mathbf{AB}=\mathbf{p}$ and $\mathbf{AD}=\mathbf{q}$, express in terms of $\mathbf{p}$ and $\mathbf{q}$ the vectors **AC**, **BD**, **AM** and **MD**.

5. In a regular hexagon $ABCDEF$ $\mathbf{AB}=\mathbf{a}$ and $\mathbf{BC}=\mathbf{b}$. Find expressions in terms of $\mathbf{a}$ and $\mathbf{b}$ for **DE**, **DC**, **AD** and **BD**.

6. In the pentagon $PQRST$ $\mathbf{PQ}=\mathbf{a}$ and $\mathbf{RS}=\mathbf{b}$. If $\mathbf{PT}=2\mathbf{RS}$ and $\mathbf{TS}=2\mathbf{PQ}$, find expressions for **PR** and **QR** in terms of $\mathbf{a}$ and $\mathbf{b}$.

7. $PQRS$ is a parallelogram with $\mathbf{PQ}=2\mathbf{a}$ and $\mathbf{PS}=\mathbf{b}$. The point T is such that $\mathbf{PT}=2\mathbf{b}$. If the lines PR and QS intersect at X and the lines RS and QT intersect at Y, find expressions for the vectors **TR**, **PY**, **QY** and **XY** in terms of $\mathbf{a}$ and $\mathbf{b}$.

8. Simplify the following expressions, stating which vector property is being used at each stage.
(a) $\mathbf{x}+(\mathbf{y}+\mathbf{x})$, (b) $(\mathbf{u}-2\mathbf{v})+2(\mathbf{v}-\mathbf{w})$, (c) $3(\mathbf{p}+\mathbf{q})+4\mathbf{q}$.

9. Given that $\mathbf{HL}+\mathbf{KN}=\mathbf{KL}+\mathbf{HM}$, show that the points M and N are coincident.

10. Given that $\mathbf{OA}+\mathbf{OC}=\mathbf{OB}+\mathbf{OD}$, show that the quadrilateral $ABCD$ is a parallelogram.

11. If $ABCDEF$ is a hexagon in which $\mathbf{AB}=\mathbf{ED}$ and $\mathbf{BC}=\mathbf{FE}$, prove that $CDFA$ is a parallelogram.

12. If $PQRST$ is a pentagon in which $\mathbf{QR}+\mathbf{RS}=\mathbf{PT}$, prove that $\mathbf{PQ}=\mathbf{TS}$.

13. In a parallelogram $ABCD$, if M and N are the mid-points of AB and CD respectively, show that $AMCN$ is a parallelogram.

14. Prove that in any quadrilateral $PQRS$,
(a) $\mathbf{PQ}+\mathbf{RS}=\mathbf{PS}+\mathbf{RQ}$, (b) $\mathbf{PR}+\mathbf{QS}=\mathbf{PS}+\mathbf{QR}$.

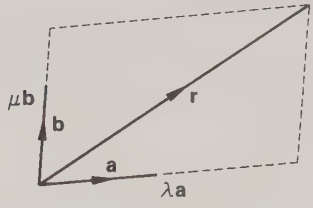
15. In quadrilateral $ABCD$ the points H, K, L and M are the mid-points of the sides AB, BC, CD and DA respectively. Prove that $HKLM$ is a parallelogram.

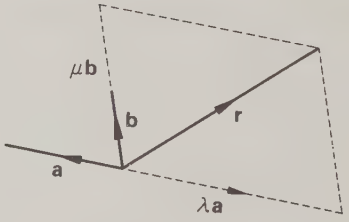
16. Prove that for any vectors $\mathbf{a}$ and $\mathbf{b}$ (i) $|\mathbf{a}-\mathbf{b}|\geqslant|\mathbf{a}|-|\mathbf{b}|$, (ii) $|\mathbf{a}-\mathbf{b}|\geqslant|\mathbf{b}|-|\mathbf{a}|$. Deduce that $|\mathbf{a}-\mathbf{b}|\geqslant||\mathbf{a}|-|\mathbf{b}||$. Under what conditions does $|\mathbf{a}-\mathbf{b}|=||\mathbf{a}|-|\mathbf{b}||$?

9.3 Unit vectors and Cartesian components

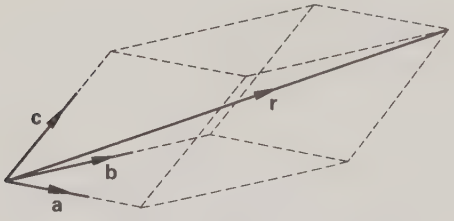
Resolving a vector into *component vectors* means expressing it as a sum of two or more non-parallel vectors.

Given two non-parallel vectors **a** and **b**, then for any vector **r** in the same plane it is possible to construct a parallelogram in which one diagonal represents **r** and the sides represent scalar multiples of **a** and **b**. For any particular vectors **a**, **b** and **r** there is only one way of constructing this parallelogram. This suggests that there is a unique expression for **r** of the form $\lambda\mathbf{a} + \mu\mathbf{b}$. Thus a vector **r** may be resolved into component vectors, $\lambda\mathbf{a}$ and $\mu\mathbf{b}$, in the directions of any two non-parallel vectors **a** and **b** in a plane containing **r**.



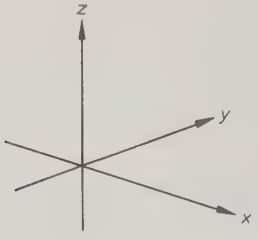


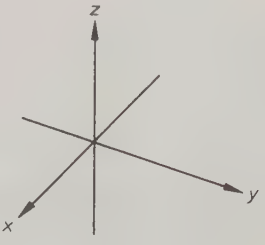
In three-dimensional problems it may be necessary to resolve a vector **r** into component vectors in three given directions. It may be demonstrated that this is possible, provided that the directions are not coplanar, by constructing a parallelepiped with **r** represented by a diagonal.



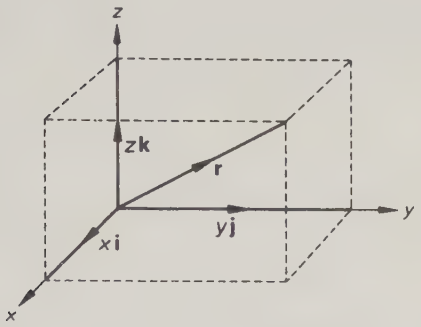
For any vectors **a**, **b** and **c** which are not coplanar, there is a unique expression for **r** of the form $\lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c}$.

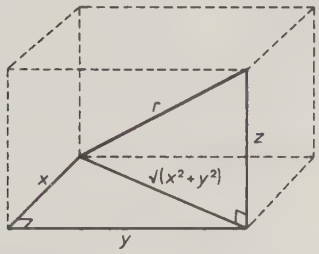
It is often convenient to resolve vectors into component vectors in the directions of the Cartesian coordinate axes. In three-dimensional work the third axis is the z-axis, which is constructed perpendicular to the x- and y-axes to form a *right-handed system*. This means that if we point the thumb of the right hand in the direction of the z-axis, then bend the fingers slightly, they will indicate the direction of a rotation from the x-axis to the y-axis. Two standard ways of drawing these axes are shown below.





Any vector with magnitude 1 is called a *unit vector*. The unit vectors in the directions of the x-, y- and z-axes are denoted by **i**, **j** and **k** respectively.





By constructing a cuboid with one diagonal representing **r** and sides parallel to the coordinate axes, we can demonstrate that there is a unique expression for any vector **r** of the form $x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$. The scalars x, y and z are called the *components* of **r** in the directions of **i**, **j** and **k**, or simply the *Cartesian components* of **r**.

Since **i**, **j** and **k** are unit vectors, x, y and z are the lengths of the sides of the cuboid constructed here. Applying Pythagoras' theorem in the two right-angled triangles illustrated, we find that the magnitude of **r** is given by

$$r = \sqrt{(x^2+y^2+z^2)}.$$

Example 1 If $\mathbf{a} = 3\mathbf{i}+2\mathbf{j}+7\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i}+4\mathbf{j}-5\mathbf{k}$, find $|\mathbf{a}|$ and $|\mathbf{a}-\mathbf{b}|$.

$|\mathbf{a}| = \sqrt{(3^2+2^2+7^2)} = \sqrt{(9+4+49)} = \sqrt{62}$
$\mathbf{a}-\mathbf{b} = (3\mathbf{i}+2\mathbf{j}+7\mathbf{k})-(2\mathbf{i}+4\mathbf{j}-5\mathbf{k}) = \mathbf{i}-2\mathbf{j}+12\mathbf{k}$
$\therefore\ |\mathbf{a}-\mathbf{b}| = \sqrt{(1^2+\{-2\}^2+12^2)} = \sqrt{(1+4+144)} = \sqrt{149}.$

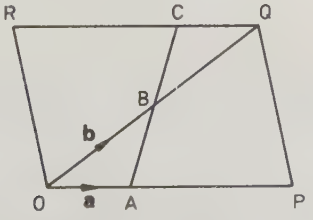
Since a represents the magnitude of a vector **a**, the vector $\mathbf{a}/a$ has magnitude 1 and the same direction as **a**. Thus the unit vector in the direction of a given vector **a** (sometimes denoted by $\hat{\mathbf{a}}$) may be expressed in the form $\mathbf{a}/a$.

Example 2 Find the unit vector in the direction of $\mathbf{a} = 2\mathbf{i}+2\mathbf{j}-\mathbf{k}$.

$a = \sqrt{(2^2+2^2+\{-1\}^2)} = \sqrt{(4+4+1)} = \sqrt{9} = 3$
$\therefore$ the unit vector in the direction of **a**

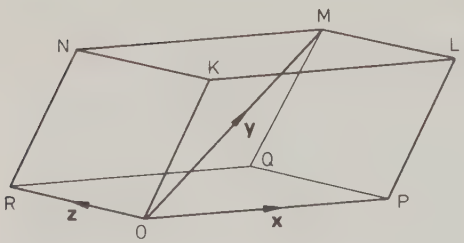
$$= \tfrac{1}{3}(2\mathbf{i}+2\mathbf{j}-\mathbf{k}) = \tfrac{2}{3}\mathbf{i}+\tfrac{2}{3}\mathbf{j}-\tfrac{1}{3}\mathbf{k}.$$

Exercise 9.3



1. In the given diagram $\mathbf{OA} = \mathbf{a}$ and $\mathbf{OB} = \mathbf{b}$. The points P, Q and R are constructed so that $\mathbf{AP} = 2\mathbf{OA}$, $\mathbf{BQ} = \mathbf{OB}$ and $OPQR$ is a parallelogram. Express the following vectors as sums of component vectors in the directions of **a** and **b**

(a) **OQ**, (b) **QR**, (c) **PR**, (d) **AB**, (e) **CQ**, (f) **OC**.



In the parallelepiped shown in the diagram $\mathbf{OP} = \mathbf{x}$, $\mathbf{OM} = \mathbf{y}$ and $\mathbf{OR} = \mathbf{z}$. In questions 2 and 3 resolve the given vectors into component vectors in the directions of of x, y and z.

2. (a) **PQ**. (b) **RQ**, (c) **MN**, (d) **LM**, (e) **PR**, (f) **RM**.

3. (a) **ON**, (b) **KM**, (c) **OK**, (d) **NP**, (e) **KQ**, (f) **LR**.

4. In quadrilateral $ABCD$, $\mathbf{AB} = 2\mathbf{i}-\mathbf{j}$, $\mathbf{BC} = 3\mathbf{i}+4\mathbf{j}$, $\mathbf{AD} = \mathbf{i}+5\mathbf{j}$ and M is the mid-point of CD. Express in terms of **i** and **j** the vectors
(a) **AC**, (b) **BD**, (c) **CD**, (d) **DM**, (e) **AM**, (f) **BM**.

5. In tetrahedron $PQRS$, $\mathbf{PQ} = 3\mathbf{i}-\mathbf{j}-\mathbf{k}$, $\mathbf{PR} = -\mathbf{i}+4\mathbf{j}+\mathbf{k}$ and $\mathbf{SQ} = 3\mathbf{i}-4\mathbf{j}-3\mathbf{k}$. The point T lies on RS produced such that $RS = ST$. Express in terms of **i**, **j** and **k** the vectors
(a) **QR**, (b) **PS**, (c) **RS**, (d) **PT**.

6. Find the magnitude of each of the following vectors
(a) $5\mathbf{i}-12\mathbf{j}$, (b) $-\mathbf{i}+2\mathbf{k}$, (c) $8\mathbf{j}+6\mathbf{k}$,
(d) $2\mathbf{i}+\mathbf{j}-2\mathbf{k}$, (e) $2\mathbf{i}-6\mathbf{j}+3\mathbf{k}$, (f) $-\mathbf{i}+7\mathbf{j}-5\mathbf{k}$.

7. Find unit vectors in the direction of
(a) $8\mathbf{i}-4\mathbf{j}-\mathbf{k}$, (b) $16\mathbf{i}-8\mathbf{j}-2\mathbf{k}$, (c) $-8\mathbf{i}+4\mathbf{j}+\mathbf{k}$.

In questions 8 to 10 take $\mathbf{a} = \mathbf{i}-2\mathbf{j}+\mathbf{k}$, $\mathbf{b} = 3\mathbf{i}-\mathbf{j}-\mathbf{k}$, $\mathbf{c} = \mathbf{i}+3\mathbf{j}-2\mathbf{k}$.

8. Find (a) $|\mathbf{a}+\mathbf{b}|$, (b) $|\mathbf{a}-\mathbf{b}|$, (c) $|2\mathbf{b}-2\mathbf{a}|$.

9. Find (a) $|\mathbf{b}+\mathbf{c}|$, (b) $|\mathbf{a}+\mathbf{b}+\mathbf{c}|$, (c) $|5\mathbf{a}-\mathbf{b}+3\mathbf{c}|$.

10. Find unit vectors in the directions of
(a) $2\mathbf{b}-\mathbf{c}$, (b) $5\mathbf{a}-5\mathbf{b}+3\mathbf{c}$, (c) $5\mathbf{a}-\mathbf{b}+3\mathbf{c}$.

11. If $\mathbf{a} = \lambda\mathbf{i}-3\mathbf{j}+3\lambda\mathbf{k}$ and $|\mathbf{a}| = 7$, find the possible values of λ.

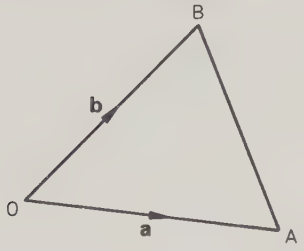
12. Given that $\mathbf{a} = 24\mathbf{i}-7\mathbf{j}$ and that $|\mathbf{b}| = 15$, find the range of possible values of $|\mathbf{a}+\mathbf{b}|$.

13. Given that **a** and **b** are two non-zero vectors which are not parallel, prove that if $p\mathbf{a} = q\mathbf{b}$, then $p = q = 0$. Deduce that if $\lambda\mathbf{a}+\mu\mathbf{b} = s\mathbf{a}+t\mathbf{b}$ then $\lambda = s$ and $\mu = t$.

14. Use the result of question 13 to find the values of k and l given that
(a) $k\mathbf{a}+(1-k)\mathbf{b} = l(\mathbf{a}+\mathbf{b})$, (b) $k(\mathbf{a}+\mathbf{b})+2\mathbf{b} = l(2\mathbf{a}+\mathbf{b})$.

9.4 Position vectors

Taking a fixed point O as origin, the position of any point P can be specified by giving the vector **OP**, which is then called the *position vector* of P.



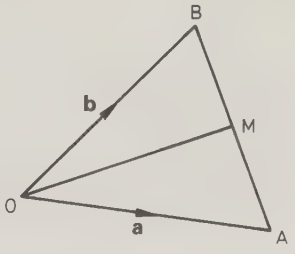
If A and B are two points with position vectors **a** and **b** respectively, then

$$\mathbf{AB} = \mathbf{AO} + \mathbf{OB} = -\mathbf{OA} + \mathbf{OB} = \mathbf{OB} - \mathbf{OA}$$
$$\therefore \quad \mathbf{AB} = \mathbf{b} - \mathbf{a}.$$

Hence the distance between the points A and B is given by $|\mathbf{b} - \mathbf{a}|$.

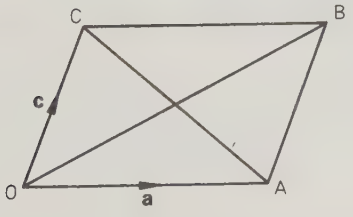
Consider now the mid-point M of AB.

$$\mathbf{AM} = \mathbf{MB}$$
$$\Rightarrow \mathbf{OM} - \mathbf{OA} = \mathbf{OB} - \mathbf{OM}$$
$$\Rightarrow \quad 2\mathbf{OM} = \mathbf{OA} + \mathbf{OB}$$
$$\Rightarrow \quad \mathbf{OM} = \tfrac{1}{2}(\mathbf{OA} + \mathbf{OB})$$



$\therefore$ the position vector of the mid-point of AB is $\frac{1}{2}(\mathbf{a} + \mathbf{b})$.

Example 1 Prove that the diagonals of a parallelogram bisect each other.

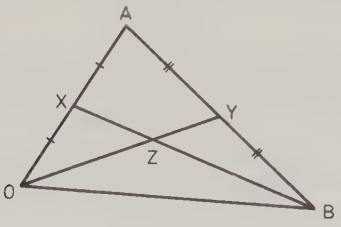


In parallelogram $OABC$, let **a** and **c** be the position vectors of A and C, then the position vector of the mid-point of AC is $\frac{1}{2}(\mathbf{a} + \mathbf{c})$.

By the parallelogram law, $\mathbf{OB} = \mathbf{a} + \mathbf{c}$.
$\therefore$ the position vector of the mid-point of OB is $\frac{1}{2}(\mathbf{a} + \mathbf{c})$.
Since the diagonals have the same mid-point, they must bisect each other.

If P is a point on the straight line through given points A and B, then there must exist some scalar t, such that $\mathbf{AP} = t\mathbf{AB}$. It is often helpful to assume a relationship of this form when dealing with collinear points.

Example 2 In the given diagram X and Y are the mid-points of OA and AB respectively. If $\mathbf{a}$ and $\mathbf{b}$ are the position vectors of the points A and B, find the position vector of the point Z.



Since X is the mid-point of OA,

$$OX = \tfrac{1}{2}\mathbf{a}.$$

Since Y is the mid-point of AB,

$$\mathbf{OY} = \tfrac{1}{2}(\mathbf{a}+\mathbf{b}).$$

$$\mathbf{XB} = \mathbf{OB}-\mathbf{OX} = \mathbf{b}-\tfrac{1}{2}\mathbf{a}.$$

Since Z lies on XB, for some scalar t, $\mathbf{XZ} = t\mathbf{XB} = t(\mathbf{b}-\frac{1}{2}\mathbf{a})$.

$\therefore$ $\mathbf{OZ} = \mathbf{OX}+\mathbf{XZ} = \frac{1}{2}\mathbf{a}+t(\mathbf{b}-\frac{1}{2}\mathbf{a}) = \frac{1}{2}(1-t)\mathbf{a}+t\mathbf{b}$.

However, since Z also lies on OY, its position vector $\mathbf{OZ}$ must also be a scalar multiple of $(\mathbf{a}+\mathbf{b})$

$$\therefore \quad \tfrac{1}{2}(1-t) = t \quad \text{and} \quad t = \tfrac{1}{3}.$$

Hence the position vector of Z is $\frac{1}{3}(\mathbf{a}+\mathbf{b})$.

When working in a Cartesian coordinate system, the point P with coordinates (x, y, z) has position vector $x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors in the directions of the coordinate axes.

Example 3 The position vectors of points A and B with respect to an origin O are $\mathbf{a} = \mathbf{i}-2\mathbf{j}+7\mathbf{k}$ and $\mathbf{b} = 5\mathbf{i}+\mathbf{j}-5\mathbf{k}$. Find (i) the distance between the origin and the point A, (ii) the distance between the points A and B, (iii) the position vector of the mid-point of AB.

(i) the distance between the origin and the point A
$= |\mathbf{a}| = \sqrt{(1^2+\{-2\}^2+7^2)} = \sqrt{(1+4+49)} = \sqrt{54}$.

(ii) $\mathbf{AB} = \mathbf{b}-\mathbf{a} = (5\mathbf{i}+\mathbf{j}-5\mathbf{k})-(\mathbf{i}-2\mathbf{j}+7\mathbf{k}) = 4\mathbf{i}+3\mathbf{j}-12\mathbf{k}$
$\therefore$ the distance between the points A and B
$= |\mathbf{b}-\mathbf{a}| = |4\mathbf{i}+3\mathbf{j}-12\mathbf{k}| = \sqrt{(16+9+144)} = 13$.

(iii) the position vector of the mid-point of AB
$= \frac{1}{2}(\mathbf{a}+\mathbf{b}) = \frac{1}{2}(1+5)\mathbf{i}+\frac{1}{2}(-2+1)\mathbf{j}+\frac{1}{2}(7-5)\mathbf{k} = 3\mathbf{i}-\frac{1}{2}\mathbf{j}+\mathbf{k}$.

Exercise 9.4

1. In a regular hexagon $OABCDE$ the position vectors of A and B relative to O are $\mathbf{a}$ and $\mathbf{b}$ respectively. Find expressions in terms of $\mathbf{a}$ and $\mathbf{b}$ for the vectors $\mathbf{AB}$ and $\mathbf{BC}$. Find also the position vectors of the points C, D and E.

2. In a quadrilateral $ABCD$ the point M is the mid-point of the diagonal BD. Given that the position vectors of A, B and C are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively and that

$ABCM$ is a parallelogram, find in terms of **a**, **b** and **c** the vectors **BA**, **BC** and **BM**. Hence find the position vector of the point D.

3. The points A, B, C, D have position vectors **a**, **b**, **c** and **d** respectively. Prove that the lines joining the mid-points of opposite edges of the tetrahedron $ABCD$ bisect each other and give the position vector of the point of intersection.

4. Write down, in terms of unit vectors **i** and **j** in the directions of the x- and y-axes, the position vectors of the points $A(2,3)$, $B(5,-1)$ and $C(4,4)$. If D is the fourth vertex of the parallelogram $ABCD$, find the vector **AD**. Hence find the position vector and the coordinates of the point D.

5. The points A and B have position vectors **a** and **b** respectively. In each of the following cases, find the distance between A and B, the unit vector in the direction of **AB** and the position vector of the mid-point M of AB.
(a) $\mathbf{a} = 3\mathbf{i}-5\mathbf{j}-\mathbf{k}$, $\mathbf{b} = \mathbf{i}-\mathbf{j}+3\mathbf{k}$,
(b) $\mathbf{a} = -7\mathbf{i}+13\mathbf{j}$, $\mathbf{b} = 5\mathbf{i}-5\mathbf{j}-4\mathbf{k}$,
(c) $\mathbf{a} = \mathbf{i}-\mathbf{j}+3\mathbf{k}$, $\mathbf{b} = \mathbf{i}+2\mathbf{j}+2\mathbf{k}$,
(d) $\mathbf{a} = 5\mathbf{i}+4\mathbf{j}+\mathbf{k}$, $\mathbf{b} = -\mathbf{i}+\mathbf{j}-2\mathbf{k}$.

6. Write down in terms of **i**, **j** and **k** the position vectors of the points $A(2,-1,5)$, $B(0,2,-1)$ and $C(-2,4,3)$. Hence find the length of AB and the coordinates of the mid-point of BC. Find also the coordinates of the point D, given that it is the fourth vertex of the parallelogram $ABCD$.

7. The points X, Y and Z have position vectors $-2\mathbf{i}+2\mathbf{j}-3\mathbf{k}$, $2\mathbf{i}+4\mathbf{j}-5\mathbf{k}$ and $-4\mathbf{i}+\mathbf{j}-2\mathbf{k}$ respectively. By finding **XY** and **XZ**, show that X, Y and Z are collinear.

8. Use vector methods to show that the points $X(3,-4,0)$, $Y(-1,8,-8)$ and $Z(6,-13,6)$ are collinear.

9. The points A, B and C have position vectors $\mathbf{a} = -\mathbf{i}+2\mathbf{j}+3\mathbf{k}$, $\mathbf{b} = 8\mathbf{i}+7\mathbf{j}-9\mathbf{k}$ and $\mathbf{c} = 2\mathbf{i}-3\mathbf{j}-\mathbf{k}$. Prove that the triangle ABC is right-angled and find its area.

10. The points A, B, C and D have position vectors **a**, **b**, 4**b** and $k(\mathbf{a}-\mathbf{b})$ respectively. Find the value of k given that
(a) **AD** is parallel to **b**,
(b) **BD** is parallel to **a**,
(c) **CD** is parallel to $\mathbf{a}+\mathbf{b}$,
(d) A, C and D are collinear.

11. The points P and Q have position vectors **p** and **q** with respect to an origin O. X is the mid-point of PQ and Y is the point on OX such that $OY = 2YX$. If PY meets OQ at Z, find expressions in terms of **p** and **q** for **OX**, **OY** and **PY**. If **PZ** $= k$**PY** find **OZ** in terms of **p**, **q** and k. Deduce that the position vector of Z is $\frac{1}{2}\mathbf{q}$.

12. The points A and B have position vectors **a** and **b** with respect to an origin O.

The points P, Q and R are defined such that $3\mathbf{OP} = \mathbf{OA}$, $3\mathbf{OQ} = 2\mathbf{OB}$, $2\mathbf{PR} = \mathbf{RQ}$ and S is the point of intersection of AB and OR produced. Find the position vector of R. If $\mathbf{AS} = k\mathbf{AB}$ and $\mathbf{OS} = l\mathbf{OR}$, find the values of k and l. Hence write down the position vector of S.

13. The points A, B, P and Q are defined as in question 12. X is the point of intersection of AQ and BP. Y is the point of intersection of AB and OX produced. Find the position vectors of X and Y.

Exercise 9.5 (*miscellaneous*)

1. Given that $\mathbf{SR}+\mathbf{QT} = \mathbf{SQ}+\mathbf{PT}$, show that the points P, Q and R are collinear.

2. If P is any point and D, E, F are the mid-points of the sides BC, CA and AB respectively of the triangle ABC, show that $\mathbf{PA}+\mathbf{PB}+\mathbf{PC} = \mathbf{PD}+\mathbf{PE}+\mathbf{PF}$.

3. In a tetrahedron $PQRS$, if H and K are the mid-points of the edges PQ and RS respectively, show that

$$\mathbf{PR}+\mathbf{QR}+\mathbf{PS}+\mathbf{QS} = 4\mathbf{HK}.$$

4. (i) In the quadrilateral $ABCD$, X and Y are the mid-points of the diagonals AC and BD respectively. Show that (a) $\overrightarrow{BA}+\overrightarrow{BC} = 2\overrightarrow{BX}$,
(b) $\overrightarrow{BA}+\overrightarrow{BC}+\overrightarrow{DA}+\overrightarrow{DC} = 4\overrightarrow{YX}$.
(ii) The point P lies on the circle through the vertices of a rectangle $QRST$. The point X on the diagonal QS is such that $\overrightarrow{QX} = 2\overrightarrow{XS}$. Express $\overrightarrow{PX}$, $\overrightarrow{QX}$ and $(\overrightarrow{RX}+\overrightarrow{TX})$ in terms of $\overrightarrow{PQ}$ and $\overrightarrow{PS}$. (L)

5. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are vectors representing the edges of a parallelepiped, find vectors representing the four diagonals.

6. In a tetrahedron $OABC$ $\mathbf{OA} = \mathbf{a}$, $\mathbf{OB} = \mathbf{b}$ and $\mathbf{OC} = \mathbf{c}$. The points P and Q are constructed such that $\mathbf{OA} = \mathbf{AP}$ and $2\mathbf{OB} = \mathbf{BQ}$. The point M is the mid-point of PQ. Find, in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, expressions for (a) $\mathbf{AB}$, (b) $\mathbf{PQ}$, (c) $\mathbf{CQ}$, (d) $\mathbf{QM}$, (e) $\mathbf{MB}$, (f) $\mathbf{OM}$.

7. If $\mathbf{a} = 2\mathbf{i}-4\mathbf{j}+2\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i}+4\mathbf{j}-5\mathbf{k}$, find (a) the magnitudes of $\mathbf{a}$, $\mathbf{b}$ and $5\mathbf{a}+2\mathbf{b}$, (b) the unit vector in the direction of $\mathbf{a}+2\mathbf{b}$.

8. If P and Q are the points with position vectors

$$\mathbf{p} = -\mathbf{i}+3\mathbf{j}+2\mathbf{k} \text{ and } \mathbf{q} = \mathbf{i}-7\mathbf{j}+4\mathbf{k}$$

respectively, find (a) the distance PQ, (b) the unit vector in the direction of PQ, (c) the position vector of the mid-point of PQ.

9. Given that $\mathbf{a} = 6\mathbf{i}+(p-10)\mathbf{j}+(3p-5)\mathbf{k}$ and that $|\mathbf{a}| = 11$, find the possible values of p.

10. Given that the points A, B and C have position vectors $\mathbf{a} = \mathbf{i}-2\mathbf{j}+2\mathbf{k}$, $\mathbf{b} = 3\mathbf{i}-\mathbf{k}$ and $\mathbf{c} = -\mathbf{i}+\mathbf{j}+4\mathbf{k}$, prove that the triangle ABC is isosceles.

11. In the parallelogram $OPQR$, the position vectors of P and R with respect to O are $6\mathbf{i}+\mathbf{j}$ and $3\mathbf{i}+4\mathbf{j}$ respectively. M and N are the mid-points of the sides PQ and QR respectively. Find (a) the unit vector in the direction of $\vec{MN}$, (b) the magnitude of the vector $\vec{OM}$. (L)

12. In triangle OAB, $\mathbf{OA} = \mathbf{a}$ and $\mathbf{OB} = \mathbf{b}$. Points X, Y and Z are constructed so that $2\mathbf{AX} = 3\mathbf{AB}$, $\mathbf{OB} = \mathbf{BY}$ and Z is the point of intersection of OX and AY. If $\mathbf{AZ} = k\mathbf{AY}$, express $\mathbf{OZ}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and k. Show that $\mathbf{OZ}$ is a scalar multiple of $3\mathbf{b}-\mathbf{a}$. Hence find the value of k and the position vector of Z.

13. In a parallelogram $OABC$, M is the mid-point of AB and P is the point of intersection of OM and AC. If the position vectors of A and B with respect to O as origin are $\mathbf{a}$ and $\mathbf{b}$, find the position vector of P.

14. The points $A(3,-2,0)$, $B(-2,5,-4)$ and $C(-1,0,4)$ are three vertices of a rhombus. Use vector methods to find the fourth vertex and the area of the rhombus.

15. Prove that three distinct points A, B, C with position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are collinear if and only if there exist non-zero scalars l, m, n such that $l\mathbf{a}+m\mathbf{b}+n\mathbf{c} = 0$ where $l+m+n = 0$. Find the condition that the point with position vector $s\mathbf{a}+t\mathbf{b}$ should lie on the straight line which passes through A and B.

16. In triangle ABC $\mathbf{AB} = \mathbf{p}$ and $\mathbf{AC} = \mathbf{q}$. If G is the point of intersection of the medians of the triangle, find an expression for $\mathbf{AG}$ in terms of $\mathbf{p}$ and $\mathbf{q}$. Use your result to find the position vector of G, given that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are the position vectors of A, B and C respectively.

10 Further differentiation

10.1 Composite functions

If f and g are functions defined on the set of real numbers, such that $f: x \to u$ and $g: u \to y$, then the *composite function* gf is defined by writing $gf: x \to y$. Thus $gf(x)$, the image of x under gf, is obtained by applying f followed by g.

For example, if $f: x \to x-2$ and $g: u \to 3u$, then $f(5) = 3$ and $g(3) = 9$
$\therefore \quad gf(5) = 9$.

In this case a general expression for $gf(x)$ can be obtained by writing $gf(x) = g(u)$, where $u = f(x) = x-2$. Hence $gf(x) = 3u = 3(x-2)$, i.e. $gf: x \to 3(x-2)$.

[Note that since $gf(x) = g(f(x))$, the function gf may also be described as a *function of a function.*]

Example 1 If $f(x) = x^2$ and $g(x) = x+1$, find $fg(x)$, $gf(x)$, $f^2(x)$ and $g^2(x)$.

$$fg(x) = f(u) \quad \text{where} \quad u = g(x)$$
i.e. $$fg(x) = u^2 \quad \text{where} \quad u = x+1$$
$\therefore$ $$fg(x) = (x+1)^2 = x^2+2x+1.$$

Similarly
$$gf(x) = g(x^2) = x^2+1$$
$$f^2(x) = ff(x) = f(x^2) = (x^2)^2 = x^4$$
$$g^2(x) = g(x+1) = (x+1)+1 = x+2.$$

It is sometimes convenient to reverse the process and consider a function as a composite function. If y is a given function of x, then this may be done by expressing y in terms of u, where u is a function of x. For instance, given $y = (2x+1)^5$, then $y = u^5$ where $u = 2x+1$.

In general, for a composite function of the form $y = fg(x)$, we may write $y = f(u)$ where $u = g(x)$.

To obtain a rule for differentiating functions of this type, let δx be a small increase in x and let δu and δy be the corresponding increases in u and y.

As δx, δu and δy tend to zero,

$$\frac{\delta y}{\delta x} \to \frac{dy}{dx}, \quad \frac{\delta y}{\delta u} \to \frac{dy}{du} \quad \text{and} \quad \frac{\delta u}{\delta x} \to \frac{du}{dx}$$

$$\therefore \quad \text{since} \quad \frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \cdot \frac{\delta u}{\delta x} \quad \text{we find that} \quad \boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}.$$

This formula is known as the *chain rule.*

Example 2 Find $\frac{dy}{dx}$ if $y = (2x+1)^5$.

Let $y = u^5$ where $u = 2x+1$

then $\frac{dy}{du} = 5u^4$ $\qquad$ $\frac{du}{dx} = 2$

$\therefore$ $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 5u^4 . 2 = 10(2x+1)^4.$

Example 3 Find $\frac{dy}{dx}$ if $y = \sqrt{\left(1 - \frac{1}{x^2}\right)^3}$

Let $y = \sqrt{(u^3)} = u^{3/2}$ where $u = 1 - \frac{1}{x^2} = 1 - x^{-2}$

then $\frac{dy}{du} = \frac{3}{2}u^{1/2} = \frac{3}{2}\sqrt{u}$ $\qquad$ $\frac{du}{dx} = 2x^{-3} = \frac{2}{x^3}$

$\therefore$ $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{3}{2}\sqrt{u} \cdot \frac{2}{x^3} = \frac{3}{x^3}\sqrt{\left(1 - \frac{1}{x^2}\right)}.$

A student who is confident that he understands this technique may write down solutions in one of the following ways.

Example 4 Differentiate $\frac{1}{1+x^3}$.

Either If $y = \frac{1}{1+x^3} = (1+x^3)^{-1}$

then $\frac{dy}{dx} = -1 . (1+x^3)^{-2} . 3x^2 = -\frac{3x^2}{(1+x^3)^2}$

or $\frac{d}{dx}\left(\frac{1}{1+x^3}\right) = \frac{d}{dx}\{(1+x^3)\}^{-1} = -1 . (1+x^3)^{-2} . \frac{d}{dx}(1+x^3)$

$$= -(1+x^3)^{-2} . 3x^2 = -\frac{3x^2}{(1+x^3)^2}.$$

This rule can be extended to 'chains' of three or more functions.

Example 5 Find $\dfrac{dy}{dx}$ if $y = \{1+(x^2-1)^3\}^{1/3}$.

Let $y = u^{1/3}, \qquad u = 1+v^3, \qquad v = x^2-1$

$$\frac{dy}{du} = \tfrac{1}{3}u^{-2/3} \qquad \frac{du}{dv} = 3v^2 \qquad \frac{dv}{dx} = 2x$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx} = \tfrac{1}{3}u^{-2/3}\,.\,3v^2\,.\,2x = 2xv^2/u^{2/3}.$$

Hence $$\frac{dy}{dx} = 2x(x^2-1)^2/\{1+(x^2-1)^3\}^{2/3}.$$

The differentiation rule $\dfrac{d}{dx}(x^n) = nx^{n-1}$ can now be proved for negative values of n, assuming that the rule holds for positive values of n and that $\dfrac{d}{dx}\left(\dfrac{1}{x}\right) = -\dfrac{1}{x^2}$.

Let $y = x^n = x^{-m}$ where m is positive,

then $y = u^m$ where $u = x^{-1}$

$$\frac{dy}{du} = mu^{m-1} \qquad \frac{du}{dx} = -x^{-2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = mu^{m-1}\,.\,(-x^{-2}) = -mx^{-m+1}\,.\,x^{-2}$$

Hence $$\frac{dy}{dx} = -mx^{-m-1} = nx^{n-1}.$$

Exercise 10.1

1. If $f(x) = x^2-1$ and $g(x) = 2x+1$, find
(a) $gf(x)$, (b) $fg(x)$, (c) $f^2(x)$, (d) $g^2(x)$.

2. If $f(x) = 2x$, $g(x) = x-1$ and $h(x) = x^2$, find
(a) $fgh(x)$, (b) $f^2g(x)$, (c) $hgf(x)$, (d) $gh^2(x)$.

3. If $f(x) = 2x-3$ and $fg(x) = 2x+1$, find $g(x)$.

4. If $f(x) = x-1$ and $gf(x) = 3+2x-x^2$, find $g(x)$.

5. Express y in terms of u, given that $u = 2x-1$ and
(a) $y = \sqrt{(2x-1)}$ (b) $y = (x-\frac{1}{2})^5$
(c) $y = 1/(1-2x)$ (d) $y = x(2x-1)^8$.

Differentiate with respect to x:

6. $(x+3)^6$. 7. $(2x-1)^4$. 8. $(5-3x)^7$.

9. $(3x-2)^{-2}$. 10. $(x^3-1)^5$. 11. $(4-x^2)^{-4}$.

12. $(1+3x)^{1/2}$. 13. $(6x+1)^{-1/3}$. 14. $(3x^2-1)^{5/2}$.

15. $\left(x-\frac{1}{x}\right)^{-3}$. 16. $\frac{1}{\sqrt{(1-x^2)}}$. 17. $\frac{1}{(1+\sqrt{x})^2}$.

18. Find the gradient of the curve $y=(2x^2-1)^3$ at the point where $x=-1$. Hence find the equation of the tangent to the curve at this point.

19. Find the equation of the tangent to the curve $y=\sqrt{\left(1+\frac{6}{x}\right)}$ at the point where $x=2$.

20. A particle moves along a straight line so that after t seconds its displacement, s metres, from a fixed point on the line is given by $s^3=3t-2$. Find the velocity and acceleration of the particle when $t=1$.

21. An object moves along a straight line such that after t seconds its velocity is $v\,\mathrm{m\,s^{-1}}$, where $v=t+4/(t+1)$. Find an expression for its acceleration after t seconds. Find also the minimum speed of the object during its motion.

22. Differentiate the following functions with respect to x by substituting $u=x+1$.

(a) $x(x+1)^5$, (b) $\frac{x-1}{x+1}$, (c) $\frac{x}{\sqrt{(x+1)}}$.

23. Differentiate with respect to x:

(a) $\sqrt{\{(2x-3)^4-1\}}$, (b) $\frac{1}{4+\sqrt{(4x^2+1)}}$.

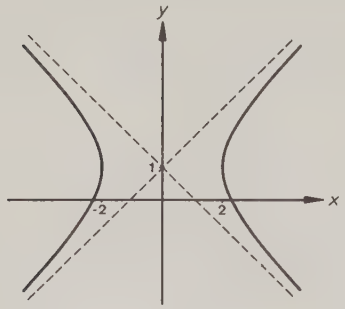
24. Find the turning points on the curve $y=x(5-x)^4$ and determine their nature. Sketch the curve.

25. Find any turning points on the curve $y=4/(x^2-2x+5)$ and show that the x-axis is an asymptote to the curve. Hence sketch the curve.

10.2 Functions defined implicitly

In previous sections we have considered functions defined explicitly by equations of the form $y=f(x)$. Sometimes a function $f: x \to y$ is defined *implicitly* using an equation such as $x^2+2y-y^2=5$.

[Note that since a function is a rule assigning one and only one value of y to every value of x, restrictions on the values of x and y are often required to complete such a definition.



The sketch of the curve $x^2+2y-y^2 = 5$ shows that in this case possible restrictions would be $|x| > 2$, $y > 1$. However, in general an expression for dy/dx can be obtained without considering such restrictions.]

Given $x^2+2y-y^2 = 5$, dy/dx is found by differentiating both sides of the equation with respect to x.

$$\frac{d}{dx}(x^2) = 2x, \frac{d}{dx}(2y) = 2\frac{dy}{dx}, \frac{d}{dx}(5) = 0.$$

The derivative of y^2 is obtained using the chain rule:

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2).\frac{dy}{dx} = 2y\frac{dy}{dx}.$$

Thus $x^2+2y-y^2 = 5 \Rightarrow 2x+2\dfrac{dy}{dx} - 2y\dfrac{dy}{dx} = 0$

$$\Rightarrow \qquad \frac{dy}{dx} - y\frac{dy}{dx} = -x$$

$\therefore$ provided that $y \neq 1$, $\dfrac{dy}{dx} = \dfrac{x}{y-1}$.

Example 1 Given that $(x+1)^2+(y-2)^2 = 1$, find $\dfrac{dy}{dx}$.

Differentiating with respect to x:

$$2(x+1)+2(y-2)\frac{dy}{dx} = 0$$

$$\therefore \quad (y-2)\frac{dy}{dx} = -(x+1).$$

Hence provided that $y \neq 2$ $\dfrac{dy}{dx} = -\dfrac{x+1}{y-2}$.

Example 2 Find the gradient of the curve $x^3+y^3 = 4y^2$ at the point $(2, 2)$.

Differentiating with respect to x: $3x^2+3y^2\dfrac{dy}{dx} = 8y\dfrac{dy}{dx}$

Substituting $x = 2, y = 2$: $\quad 12 + 12\dfrac{dy}{dx} = 16\dfrac{dy}{dx}$

$$\therefore \quad \frac{dy}{dx} = 3$$

Hence at the point $(2, 2)$ the gradient of the curve is 3.

Using this approach it may be proved that if $\dfrac{d}{dx}(x^n) = nx^{n-1}$ for integral values of n, then the result also holds for fractional values of the form $n = p/q$, where p and q are integers.

Let $\quad y = x^n = x^{p/q} \quad$ then $\quad y^q = x^p.$

Differentiating with respect to x, $\quad qy^{q-1}\dfrac{dy}{dx} = px^{p-1}$

$$\therefore \quad \frac{dy}{dx} = \frac{px^{p-1}}{qy^{q-1}} = \frac{px^p y}{qxy^q} = \frac{px^p x^n}{qx \,.\, x^p} = nx^{n-1}.$$

Given an equation connecting x and y, dy/dx is a measure of the rate of change of y as x increases. Similarly dx/dy is the rate of change of x as y increases. To establish the relationship between these derivatives, let δx and δy be corresponding small increases in x and y, then $\dfrac{\delta y}{\delta x} \cdot \dfrac{\delta x}{\delta y} = 1.$

As $\delta x \to 0, \delta y \to 0, \dfrac{\delta y}{\delta x} \to \dfrac{dy}{dx}$ and $\dfrac{\delta x}{\delta y} \to \dfrac{dx}{dy}$

$$\therefore \quad \frac{dy}{dx} \cdot \frac{dx}{dy} = 1 \quad \text{so that} \quad \frac{dx}{dy} = 1 \Big/ \frac{dy}{dx}.$$

Consider, for example, the relation $x^2 - y = 0$ for positive values of x and y.

$y = x^2$	and	$x = y^{1/2}$
$\dfrac{dy}{dx} = 2x$		$\dfrac{dx}{dy} = \frac{1}{2}y^{-1/2} = \dfrac{1}{2x}$
$\dfrac{d^2y}{dx^2} = 2$		$\dfrac{d^2x}{dy^2} = -\dfrac{1}{4}y^{-3/2} = -\dfrac{1}{4x^3}$

Hence $\dfrac{dx}{dy} = 1 \Big/ \dfrac{dy}{dx}$ but $\dfrac{d^2x}{dy^2} \neq 1 \Big/ \dfrac{d^2y}{dx^2}.$

Thus although it is tempting to think of derivatives such as dy/dx as fractions, higher derivatives clearly do *not* behave like fractions.

Exercise 10.2

Find $\dfrac{dy}{dx}$ in terms of x and y when

1. $y^2 = 6x.$ 2. $x^2 + y^2 = 5.$ 3. $4x^2 + y^2 = 4.$

4. $y^3 = x^2+1$. 5. $(y-1)^2 = 2x$. 6. $3y^2 = x(x^2-3)$.

7. $x^2+y^2-2x+y+1 = 0$ 8. $(x+y)^3 = 3x+5$.

9. Find the gradient of the curve $(x-3)^2+(y+4)^2 = 5$ at the point $(1, -3)$.

10. Find the gradient of the curve $y(y+2) = 4x-1$ at each of the points on it where $x = 4$.

11. Find the equation of the tangent to the curve $y^2 = 3x+1$ at the point $(1, -2)$.

12. Find the equation of the tangent to the curve $x^2-y^2 = 9$ at the point $(5, 4)$.

13. Find the x-coordinates of the stationary points on the curve $x^3+4x^2+3y^2+5x-2 = 0$.

14. Find the coordinates of the stationary points on the curve $x^2+y^2-6x-8y = 0$.

10.3 Products and quotients

Consider a function of x, y, which can be expressed as a product of two other functions of x, u and v. Let δx be a small increase in x and let δu, δv and δy be the corresponding small increases in u, v and y, then

$$\begin{aligned} & y = uv \\ \Rightarrow \quad & y+\delta y = (u+\delta u)(v+\delta v) = uv+u\delta v+v\delta u+\delta u\delta v \\ \Rightarrow \quad & \delta y = u\delta v+v\delta u+\delta u\delta v \\ \Rightarrow \quad & \frac{\delta y}{\delta x} = u\frac{\delta v}{\delta x}+v\frac{\delta u}{\delta x}+\delta u\frac{\delta v}{\delta x}. \end{aligned}$$

As $\delta x \to 0$, $\delta u \to 0$, $\dfrac{\delta y}{\delta x} \to \dfrac{dy}{dx}$, $\dfrac{\delta u}{\delta x} \to \dfrac{du}{dx}$, $\dfrac{\delta v}{\delta x} \to \dfrac{dv}{dx}$

$$\therefore \quad \boxed{\frac{dy}{dx} = u\frac{dv}{dx}+v\frac{du}{dx}.}$$

Example 1 Find $\dfrac{dy}{dx}$ if $y = (x^2+4)(x^5+7)$.

Let $y = uv$ where $u = x^2+4$ $v = x^5+7$

$$\frac{du}{dx} = 2x \qquad \frac{dv}{dx} = 5x^4$$

$$\therefore \quad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = (x^2+4).5x^4 + (x^5+7).2x = 5x^6 + 20x^4 + 2x^6 + 14x$$

$$\therefore \quad \frac{dy}{dx} = 7x^6 + 20x^4 + 14x.$$

Clearly, this result could have been obtained by writing $y = x^7 + 4x^5 + 7x^2 + 28$, then differentiating. The next two examples show how the product rule and the chain rule can be used together to differentiate more complicated functions.

Example 2 Find $\frac{dy}{dx}$ if $y = (x^2-1)^3(3x+1)^4$.

Let $u = (x^2-1)^3$ $\qquad$ $v = (3x+1)^4$

then $\frac{du}{dx} = 3(x^2-1)^2.2x$ $\qquad$ $\frac{dv}{dx} = 4(3x+1)^3.3$

$= 6x(x^2-1)^2$ $\qquad$ $= 12(3x+1)^3$.

$$\therefore \quad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = (x^2-1)^3.12(3x+1)^3 + (3x+1)^4.6x(x^2-1)^2$$
$$= 6(x^2-1)^2(3x+1)^3\{2(x^2-1) + x(3x+1)\}$$
$$= 6(x^2-1)^2(3x+1)^3(5x^2+x-2).$$

Example 3 Find $\frac{dy}{dx}$ if $y = (x+4)\sqrt{(x^2-1)}$.

Let $u = x+4$ $\qquad$ $v = \sqrt{(x^2-1)} = (x^2-1)^{1/2}$

$\frac{du}{dx} = 1$ $\qquad$ $\frac{dv}{dx} = \frac{1}{2}(x^2-1)^{-1/2}.2x = \frac{x}{\sqrt{(x^2-1)}}$

$$\therefore \quad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = (x+4).\frac{x}{\sqrt{(x^2-1)}} + \sqrt{(x^2-1)}.1$$
$$= \frac{x(x+4)}{\sqrt{(x^2-1)}} + \frac{x^2-1}{\sqrt{(x^2-1)}}$$
$$= (2x^2+4x-1)/\sqrt{(x^2-1)}.$$

Consider now a quotient, $y = u/v$, where u and v are functions of x. Let δx be a small increase in x and let δu, δv and δy be the corresponding small increases in u, v and y, then

$$y = \frac{u}{v}$$

$$\Rightarrow \quad y + \delta y = \frac{u+\delta u}{v+\delta v}$$

$$\Rightarrow \quad \delta y = \frac{u+\delta u}{v+\delta v} - \frac{u}{v} = \frac{v(u+\delta u) - u(v+\delta v)}{v(v+\delta v)} = \frac{v\delta u - u\delta v}{v(v+\delta v)}$$

$$\Rightarrow \quad \frac{\delta y}{\delta x} = \frac{v\dfrac{\delta u}{\delta x} - u\dfrac{\delta v}{\delta x}}{v(v+\delta v)}.$$

As $\delta x \to 0, \delta v \to 0, \dfrac{\delta y}{\delta x} \to \dfrac{dy}{dx}, \dfrac{\delta u}{\delta x} \to \dfrac{du}{dx}$ and $\dfrac{\delta v}{\delta x} \to \dfrac{dv}{dx}$

$$\therefore \quad \boxed{\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.}$$

Example 4 Find $\dfrac{dy}{dx}$ if $y = \dfrac{x}{x+1}$.

Method 1 using the quotient rule:

Let $y = \dfrac{u}{v}$ where $u = x \qquad v = x+1$

$$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = 1$$

$$\therefore \quad \frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} = \frac{(x+1).1 - x.1}{(x+1)^2} = \frac{x+1-x}{(x+1)^2} = \frac{1}{(x+1)^2}.$$

Method 2 using the product rule:

Let $y = uv$ where $u = x \qquad v = (x+1)^{-1}$

$$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = -1.(x+1)^{-2}.1$$

$$= -1/(x+1)^2$$

$$\therefore \quad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = x.\frac{-1}{(x+1)^2} + \frac{1}{(x+1)}.1$$

$$= \frac{-x+(x+1)}{(x+1)^2} = \frac{1}{(x+1)^2}$$

Example 5 Find $\dfrac{dy}{dx}$ if $y = \sqrt{\left(\dfrac{x+1}{x^2+1}\right)}$.

$$y = \sqrt{\left(\frac{x+1}{x^2+1}\right)} \qquad y^2 = \frac{x+1}{x^2+1}.$$

Differentiating: $\quad 2y\dfrac{dy}{dx} = \dfrac{(x^2+1).1 - (x+1).2x}{(x^2+1)^2}$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2y}.\frac{(x^2+1-2x^2-2x)}{(x^2+1)^2} = \left(\frac{x^2+1}{x+1}\right)^{1/2}.\frac{(1-2x-x^2)}{2(x^2+1)^2}.$$

Hence $\quad \dfrac{dy}{dx} = \dfrac{1-2x-x^2}{2(x+1)^{1/2}(x^2+1)^{3/2}}.$

[This function can also be differentiated by writing $y = \sqrt{u}$ where $u = (x+1)/(x^2+1)$ and using the chain rule.]

The next two examples show the use of the product formula when y is an implicit function of x.

Example 6 Find $\frac{dy}{dx}$ if $y^3 - xy^2 - x^3 = 1$.

Differentiating with respect to x:

$$\frac{d}{dx}(y^3) - \frac{d}{dx}(xy^2) - \frac{d}{dx}(x^3) = 0$$

$$3y^2\frac{dy}{dx} - \left(x.2y\frac{dy}{dx} + y^2.1\right) - 3x^2 = 0$$

$$(3y^2 - 2xy)\frac{dy}{dx} - y^2 - 3x^2 = 0$$

$$\therefore \quad \frac{dy}{dx} = \frac{3x^2 + y^2}{3y^2 - 2xy}.$$

Example 7 If $y^3 + x^3 = 3x + 7$ show that

$$y^2\frac{d^2y}{dx^2} + 2y\left(\frac{dy}{dx}\right)^2 + 2x = 0.$$

Differentiating with respect to x: $3y^2\frac{dy}{dx} + 3x^2 = 3$

$$\therefore \quad y^2\frac{dy}{dx} + x^2 = 1.$$

Differentiating again with respect to x:

$$y^2\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}.\frac{d}{dx}(y^2) + 2x = 0$$

$$y^2\frac{d^2y}{dx^2} + \frac{dy}{dx}.2y\frac{dy}{dx} + 2x = 0$$

$$\therefore \quad y^2\frac{d^2y}{dx^2} + 2y\left(\frac{dy}{dx}\right)^2 + 2x = 0.$$

Exercise 10.3

Differentiate the following with respect to x, simplifying your answers when possible.

1. $(x+3)(x^4-1)$.
2. $(x-1)(x^2+x-6)$.
3. $(3x^2-1)(2-5x^3)$.
4. $(x^2-2)(x^2+2+1/x^2)$.

5. $x^2(x+3)^3$.

6. $x(3x-1)^5$.

7. $(x+1)^2(x+2)^5$.

8. $(2x+3)^3(1-x)^4$.

9. $x^2\sqrt{(1+x^2)}$.

10. $(x+3)\sqrt{(1-x^3)}$.

11. $\dfrac{x-2}{x+2}$.

12. $\dfrac{x^2}{2x+1}$.

13. $\dfrac{x+3}{2-3x}$.

14. $\dfrac{x^2+1}{x^3+1}$.

15. $\dfrac{x^2-2}{x^2+1}$.

16. $\dfrac{x^2+x-1}{1-x^2}$.

17. $\dfrac{x^2-2}{(x+2)^2}$.

18. $\dfrac{x^2}{(x-1)^3}$.

19. $\dfrac{(2x+1)^2}{(4x+3)^3}$.

20. $\dfrac{x}{\sqrt{(1-x^2)}}$.

21. $\dfrac{x^3}{\sqrt{(1-2x^2)}}$.

22. $\dfrac{\sqrt{(x^2+4)}}{x+1}$.

23. $\left(\dfrac{x-1}{2-x}\right)^2$.

24. $\sqrt{\left(\dfrac{2x}{x-1}\right)}$.

25. $\sqrt{\left(\dfrac{1+x^2}{x}\right)}$.

26. Find $\dfrac{dy}{dx}$ in terms of x and y, given that (a) $x^2-xy+y^2=1$, (b) $x^2y+y^3=4$, (c) $x^2+y^2-6xy+3x-2y+5=0$.

27. Find the gradient of the curve $x^2+2xy-2y^2+x=2$ at the point $(-4,1)$.

28. Find the equation of the tangent to the curve $(x+1)y^2=x-2$ at the point $(-2,2)$.

29. Find $\dfrac{d^2y}{dx^2}$ given that

(a) $y=\dfrac{1}{x^2+1}$, (b) $y=\dfrac{x}{x-1}$, (c) $y=\sqrt{(x^2+1)}$.

30. Differentiate the following with respect to x

(a) $\dfrac{(x^2+1)(x-1)^2}{2x-1}$, (b) $\dfrac{x\sqrt{(x-1)}}{x+1}$, (c) $\sqrt{\left\{\dfrac{x^3+1}{(x+1)^3}\right\}}$.

31. If $x^2-y^2=1$, prove that $y\dfrac{d^2y}{dx^2}+\left(\dfrac{dy}{dx}\right)^2=1$.

32. Find the values of $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ at the point $(1,3)$ on the curve $3x^2+y^2=4y$.

33. If $y = \sqrt{\left(\frac{6x}{x+2}\right)}$, find the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when $x = 4$.

34. If $y^2 - 2xy = 2x$, prove that $(x-y)\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = \left(\frac{dy}{dx}\right)^2$.

10.4 Maxima and minima

In Chapter 5 it was shown that maximum and minimum points on a curve $y = f(x)$ can be found by examining points at which $dy/dx = 0$. Thus any maximum or minimum values of the function $f(x)$ occur when $f'(x) = 0$. The nature of such stationary values is determined either by studying the behaviour of $f'(x)$ or by finding the sign of the second derivative $f''(x)$.

Example 1 Find the maximum and minimum values of the function $f(x) = \frac{4x}{x^2+4}$ and sketch its graph.

Differentiating, $f'(x) = \frac{4\{(x^2+4).1 - x.2x\}}{(x^2+4)^2} = \frac{4(4-x^2)}{(x^2+4)^2}$

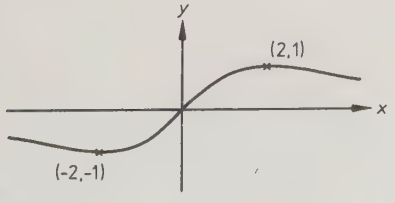
$$f'(x) = 0 \Leftrightarrow 4 - x^2 = 0 \Leftrightarrow x = -2 \quad \text{or} \quad x = 2.$$

When x is just less than -2, $f'(x) < 0$ and when x is just greater than -2, $f'(x) > 0$

$\therefore$ when $x = -2$, $f(x)$ takes the minimum value $f(-2) = -1$.

When x is just less than 2, $f'(x) > 0$ and when x is just greater than 2, $f'(x) < 0$

$\therefore$ when $x = 2$, $f(x)$ takes the maximum value $f(2) = 1$.

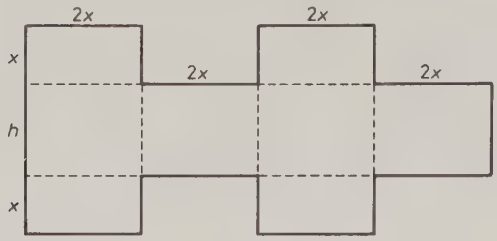


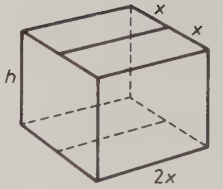
Hence the curve $y = \frac{4x}{x^2+4}$ has a minimum point $(-2, -1)$ and a maximum point $(2, 1)$. The curve cuts the axes only at the origin. As $x \to \infty$, $y \to 0$ from above and as $x \to -\infty$, $y \to 0$ from below,

$\therefore$ the x-axis is an asymptote to the curve.

These methods can also be applied to finding maximum and minimum values in practical problems.

Example 2 A carton of volume $V\,\text{m}^3$ is made from a piece of cardboard as shown below. If the area of cardboard used is $6\,\text{m}^3$, find expressions for h and V in terms of x and the value of x which produces a box of maximum volume.





The area of cardboard used $= (4.2x.h+4.2x.x)\,\text{m}^2$
$= 8x(h+x)\,\text{m}^2$

$\therefore \quad 8x(h+x) = 6 \quad \text{and} \quad h+x = 6/8x = 3/4x$

$\therefore$ in terms of x, $\quad h = \dfrac{3}{4x} - x.$

The volume of the box $= V\,\text{m}^3 = 4hx^2\,\text{m}^3$

$\therefore$ in terms of x, $V = 4x^2\left(\dfrac{3}{4x} - x\right) = 3x - 4x^3.$

Differentiating with respect to x,

$$\frac{dV}{dx} = 3-12x^2, \quad \frac{d^2V}{dx^2} = -24x$$

$$\frac{dV}{dx} = 0 \Leftrightarrow 3-12x^2 = 0 \Leftrightarrow x^2 = \tfrac{1}{4}$$

$\therefore$ the only positive value of x for which $\dfrac{dV}{dx} = 0$ is $x = \frac{1}{2}$.

When $x = \frac{1}{2}$, $\dfrac{d^2V}{dx^2} < 0$ and $V = 1$

$\therefore \quad x = \frac{1}{2}$ gives a maximum value for V of 1.

It is important to note the greatest value of a function in a given interval is not always a 'maximum value'.

Example 3 Find the greatest value of $y = x^3 - 5x^2 + 7x$ in the interval from $x = 0$ to $x = 4$.

Differentiating, $\quad \dfrac{dy}{dx} = 3x^2 - 10x + 7 = (3x-7)(x-1)$

$$\frac{d^2y}{dx^2} = 6x-10$$

$$\frac{dy}{dx} = 0 \Leftrightarrow (3x-7)(x-1) = 0 \Leftrightarrow x = 1 \text{ or } x = 2\tfrac{1}{3}.$$

When $x = 1$, $\dfrac{d^2y}{dx^2} < 0$ $\quad\therefore$ y takes a maximum value.

When $x = 2\frac{1}{3}$, $\dfrac{d^2y}{dx^2} > 0$ $\quad\therefore$ y takes a minimum value.

However, in the interval from $x = 2\frac{1}{3}$ to $x = 4$, dy/dx is positive, so that the value of y must be increasing.
When $x = 1$, $y = 1-5+7 = 3$.
When $x = 4$, $y = 64-80+28 = 12$.

Hence although the function has a 'local' maximum value of 3 when $x = 1$, its greatest value in the interval $x = 0$ to $x = 4$ is 12.

Exercise 10.4

In questions 1 to 6 find any maximum or minimum values of the given function and sketch its graph.

1. $(2x-5)^4$.
2. $x(x+4)^3$.
3. $x^2(x-3)^4$.
4. $\dfrac{2x-1}{x^2+2}$.
5. $\dfrac{3x}{x^2+x+1}$.
6. $\dfrac{x^2-1}{x^2+1}$.

7. Find the greatest and least values of $y = x^3+x^2-5x-4$ in the interval from $x = -3$ to $x = 3$.

8. Find the greatest and least values of $y = x^2+16/x$ in the interval from $x = \frac{2}{3}$ to $x = 3$.

9. Find the range of values taken by y on the curve $y = 12/(x^2+3)$. Find also the coordinates of the points of inflexion on the curve.

10. Repeat question 9 for the curve $y = 12x/(x^2+3)$.

11. The line $x = t$ meets the curves $y = x(x-3)$ and $y = 5x-x^2$ at the points A and B. Find the maximum length of AB as t varies between 0 and 4.

12. A square of side x cm is cut from each of the corners of a rectangular piece of cardboard 15 cm by 24 cm. The cardboard is then folded to form an open box of depth x cm. Show that the volume of the box is $(4x^3-78x^2+360x)\,\text{cm}^3$. Find the value of x for which the volume is a maximum.

13. A man wishes to fence in a rectangular enclosure of area 200 m^2. One side of the enclosure is formed by part of a wall already in position. What is the least possible length of fencing required for the other three sides?

14. A lump of modelling clay of volume 72 cm^3 is moulded into the shape of a cuboid with edges of lengths x cm, $2x$ cm and y cm. Find the minimum surface area of this cuboid.

15. A right circular cylinder of height $2h$ is contained in a sphere of radius R, the circular edges of the cylinder touching the sphere. The volumes of the cylinder and the sphere are denoted by V and W respectively. Express V in terms of R and h. By finding the maximum value of V, as h varies, show that $V/W \leqslant 1/\sqrt{3}$. (W)

16. A closed hollow right-circular cone has internal height a and the internal radius of its base is also a. A solid circular cyclinder of height h just fits inside the cone with the axis of the cylinder lying along the axis of the cone. Show that the volume of the cylinder is $V = \pi h(a-h)^2$. If a is fixed, but h may vary, find h in terms of a when V is maximum. (JMB)

17. $ABCDE$ is a pentagon of fixed perimeter P cm. Its shape is such that ABE is an equilateral triangle and $BCDE$ is a rectangle. If the length of AB is x cm, find the value of P/x for which the area of the pentagon is a maximum.

18. A right circular cylinder is of radius r cm and height pr cm. The total surface area of the cylinder is S cm^2 and its volume is V cm^3. Find an expression for V in terms of p and S. If the value of S is fixed, find the value of p for which V is a maximum.

10.5 Related rates of change

The chain rule can be used to solve many practical problems involving the rates of change of more than two variables.

Example 1 A piece of paper is burning round the circumference of a circular hole. After t seconds, the radius, r cm, of the hole is increasing at the rate of $0{\cdot}5\,\text{cm}\,\text{s}^{-1}$. Find the rate at which the area, $A\,\text{cm}^2$, of the hole is increasing when $r = 5$.

The rate of increase of the radius of the hole in $\text{cm}\,\text{s}^{-1}$ is given by

$$\frac{dr}{dt} = 0{\cdot}5$$

The area of the hole in cm^2 is $A = \pi r^2$

$$\therefore \quad \frac{dA}{dr} = 2\pi r = 10\pi \quad \text{when} \quad r = 5.$$

The rate of increase of the area of the hole in $\text{cm}^2\,\text{s}^{-1}$ is

$$\frac{dA}{dt} = \frac{dA}{dr}.\frac{dr}{dt} = 10\pi \times 0{\cdot}5 = 5\pi.$$

Hence the area of the hole is increasing at a rate of $5\pi\,\text{cm}^2\,\text{s}^{-1}$ when the radius is 5 cm.

Example 2 Water is being poured into a conical vessel at a rate of $10\,\text{cm}^3\,\text{s}^{-1}$. After t seconds, the volume, $V\,\text{cm}^3$, of water in the vessel is given by $V = \frac{1}{6}\pi x^3$, where $x\,\text{cm}$ is the depth of the water. Find, in terms of x, the rate at which the water is rising.

Since $V = \frac{1}{6}\pi x^3$ where x is a function of t, $\dfrac{dV}{dt} = \dfrac{dV}{dx}\cdot\dfrac{dx}{dt}$.

The rate at which the volume is increasing is the rate at which the water is being poured into the vessel

$$\therefore \quad \frac{dV}{dt} = 10.$$

By differentiation $\dfrac{dV}{dx} = \frac{1}{2}\pi x^2$.

Hence $10 = \frac{1}{2}\pi x^2 \cdot \dfrac{dx}{dt}$, i.e. $\dfrac{dx}{dt} = \dfrac{20}{\pi x^2}$

$\therefore$ the rate at which the water is rising is $20/\pi x^2\,\text{cm}\,\text{s}^{-1}$.

Example 3 A particle moves along a straight line so that its velocity, $v\,\text{m}\,\text{s}^{-1}$, when it is $s\,\text{m}$ from a fixed point, is given by $v = s^2 + 3$. Find an expression for its acceleration, $a\,\text{m}\,\text{s}^{-2}$, in terms of s.

$$a = \frac{dv}{dt} = \frac{dv}{ds}\cdot\frac{ds}{dt} = v\frac{dv}{ds}.$$

By differentiation, $\dfrac{dv}{ds} = 2s$

$$\therefore \quad a = v\frac{dv}{ds} = 2s(s^2 + 3).$$

Exercise 10.5

[Answers may be left in terms of π where appropriate.]

1. The side of a square is increasing at the rate of 3 cm/s. Find the rate of increase of the area when the length of the side is 10 cm.

2. The volume of a cube is increasing at the rate of $12\,\text{cm}^3/\text{s}$. Find the rate of increase of the length of an edge when the volume of the cube is $125\,\text{cm}^3$.

3. The area of a circle is increasing at the rate of $12\,\text{cm}^2/\text{s}$. Find the rate of increase of the circumference when the radius is 3 cm.

4. The area of a rectangle $x\,\text{cm}$ by $y\,\text{cm}$ is constant. If $dx/dt = 2$ when $3x = y$, find the corresponding value of dy/dt.

5. If $M = (2p+3)^4$, find $\dfrac{dM}{dt}$ when $p = 1$, given that $\dfrac{dp}{dt} = 2$.

6. If $r = \dfrac{1+\theta}{1+\theta^2}$, find $\dfrac{d\theta}{dt}$ when $\theta = 2$, given that $\dfrac{dr}{dt} = 14$.

7. A particle moves along a straight line so that its velocity, v m s^{-1}, when it is s m from a fixed point, is given by $v = (1+s^2)/s$. Find an expression for the acceleration of the particle in terms of s.

8. A particle moves along a straight line so that after t seconds its displacement s m from a fixed point satisfies the equation $s^2 + s = t$. Find an expression for its acceleration after t seconds in terms of s.

9. The volume of a spherical bubble is increasing at the rate of 6 cm^3/s. Find the rate at which the surface area of the bubble is increasing when the radius is 3 cm.

10. The radius of the base of a right circular cylinder is r cm and its height is $2r$ cm. Find (a) the rate at which its volume is increasing, when the radius is 2 cm and is increasing at 0·25 cm/s, (b) the rate at which the total surface area is increasing when the radius is 5 cm and the volume is increasing at 5π cm^3/s.

11. A horizontal trough is 4 m long and 1 m deep. Its cross-section is an isosceles triangle of base 1·5 m with its vertex downwards. Water runs into the trough at the rate of 0·03 m^3 s^{-1}. Find the rate at which the water level is rising after the water has been running for 25 seconds.

12. Water runs into a conical vessel fixed with its vertex downwards at the rate of 3π cm^3/s, filling the vessel to a depth of 15 cm in a time of one minute. Find the rate at which the depth of water in the vessel is increasing when the water has been running for 7·5 seconds.

10.6 Small changes and errors

If δx is a small change in x and δy is the corresponding small change in y,

$$\text{then} \quad \frac{\delta y}{\delta x} \approx \frac{dy}{dx}, \quad \text{i.e.} \quad \boxed{\delta y \approx \frac{dy}{dx}\delta x.}$$

Example 1 In an experiment, the diameter, x cm, of a sphere is measured and the volume, V cm^3, calculated using the formula $V = \dfrac{1}{6}\pi x^3$. If the diameter is found

to be 10 cm with a possible error of 0·1 cm, estimate the possible error in the volume calculated from this reading.

$$\frac{dV}{dx} = \tfrac{1}{2}\pi x^2 = 50\pi \quad \text{when} \quad x = 10$$

$$\therefore \quad \text{if} \quad \delta x = 0{\cdot}1 \quad \text{then} \quad \delta V \approx \frac{dV}{dx}\delta x \approx 50\pi \times 0{\cdot}1 \approx 5\pi.$$

Hence the possible error in the volume is $5\pi\,\text{cm}^3$.

Example 2 The time, T seconds, taken for one complete swing of a pendulum, length l m, is given by $T = 2\pi\sqrt{\frac{l}{g}}$, where g is constant. If a 1% error is made in measuring the length of a pendulum, estimate the percentage error in the value of T.

$$T = 2\pi\sqrt{\frac{l}{g}} = \frac{2\pi}{\sqrt{g}}.l^{1/2} \qquad \therefore \quad \frac{dT}{dl} = \frac{2\pi}{\sqrt{g}}.\tfrac{1}{2}l^{-1/2} = \frac{\pi}{\sqrt{(gl)}}$$

$$\therefore \quad \delta T \approx \frac{dT}{dl}\delta l = \frac{\pi}{\sqrt{(gl)}}.\delta l \quad \text{and} \quad \frac{\delta T}{T} \approx \frac{\pi}{\sqrt{(gl)}}.\frac{1}{2\pi}\sqrt{\frac{g}{l}}.\delta l = \frac{1}{2}.\frac{\delta l}{l}.$$

Since $\dfrac{\delta l}{l} = \dfrac{1}{100}$, $\qquad \dfrac{\delta T}{T} \approx \dfrac{1}{200}$

$\therefore$ the error in the value of T is approximately 0·5%.

In the next example we find an approximate value for $y + \delta y$ rather than just δy.

Example 3 Find an approximate value for $\sqrt[3]{(1003)}$.

Let $y = \sqrt[3]{x} = x^{1/3}$, then $\dfrac{dy}{dx} = \tfrac{1}{3}x^{-2/3}$

$$\therefore \quad \delta y \approx \frac{dy}{dx}\delta x = \tfrac{1}{3}x^{-2/3}\delta x.$$

Substituting $x = 1000$ and $\delta x = 3$, $\quad \delta y \approx \dfrac{1}{3} \times \dfrac{1}{100} \times 3 = 0{\cdot}01.$

Hence $\sqrt[3]{(1003)} \approx 10 + 0{\cdot}01 = 10{\cdot}01$.

Exercise 10.6

[Answers may be left in terms of π where appropriate.]

1. If $y = x^2(x-2)$, find the approximate increase in y when x increases from 3 to 3·02.

2. If $y = 2x^3 + 1$, find the approximate increase in x which causes y to increase from 3 to 3·06.

3. If $T = 2\pi\sqrt{\frac{l}{10}}$, find the approximate decrease in T if l is reduced from 2·5 to 2·4.

4. The radius of a circle is 6 cm. Find the approximate reduction in the area of the circle, if the radius is reduced by 0·1 cm.

5. A cylindrical hole is said to be 25 cm deep and 6 cm in diameter. Find the error in the calculated volume if there is an error of (a) 0·1 cm in the diameter, (b) 0·3 cm in the depth.

6. If $y = 3x^4$, find the approximate percentage increase in y when x in increased by 0·5%.

7. If a 2% error is made in measuring the diameter of a sphere, find approximately the resulting percentage errors in the volume and surface area of the sphere.

8. The length of a rectangle is twice its breadth. If the area of the rectangle increases by 5%, find the percentage increase in its perimeter.

9. The pressure P and volume V of a certain mass of gas are connected by the formula $PV = k$ where k is constant. If the pressure increases by 1%, what is the approximate change in the volume?

10. If $y = x^{3/2}$, find $\frac{dy}{dx}$ when $x = 4$. Hence find approximate values for (a) $(4{\cdot}01)^{3/2}$, (b) $(3{\cdot}98)^{3/2}$.

11. Find approximately the values of $7x^2 - 3x + \frac{4}{x}$ when (a) $x = 2{\cdot}01$, (b) $x = 1{\cdot}98$.

12. Find approximately the values of

(a) $\sqrt{(4{\cdot}004)}$, (b) $\frac{1}{5{\cdot}05}$, (c) $\sqrt[3]{(124{\cdot}7)}$, (d) $\frac{1}{\sqrt{(0{\cdot}96)}}$.

Exercise 10.7 (miscellaneous)

1. Differentiate with respect to x

(a) $\frac{1}{(5+2x-x^2)^3}$, (b) $\sqrt{(2x^3+5)}$.

2. If $y = \left(1 - \frac{1}{x}\right)^4$, prove that $(x^2 - x)\frac{dy}{dx} = 4y$.

3. Find the equation of the tangent to the curve $y = 1/(x-1)^2$ at the point where $x = 2$.

4. Find the gradient of the curve $x^2 + y^2 = 10y$ at each of the points where $x = 3$.

5. Find the equation of the tangent to the curve $(x+3)^2 - 4(y-2)^2 = 9$ at the point $(2,4)$.

6. Differentiate with respect to x:

(a) $(2x+3)(x^2+1)^4$, (b) $(x^2-2)\sqrt{(2x-1)}$,

(c) $\frac{4x^2+3x+5}{(x+1)^2}$, (d) $\sqrt{\left(\frac{x+5}{x+3}\right)}$.

7. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when $y = \sqrt{(1+4x^2)}$.

8. If $x^4 - x^2y^2 + y^4 = 5$, find $\frac{dy}{dx}$ in terms of x and y.

9. Find the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point $(1, -2)$ on the curve $3x^2 + 2xy + y^2 = 3$.

10. Prove that if $y = \sqrt{(3x^2+2)}$, then $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 3$.

11. Find any stationary values of the function $x^2(x-5)^3$ and sketch its graph.

12. Find any turning points and points of inflexion on the graph $y = x/(x^2+1)$. Hence sketch the curve.

13. In each of the following cases, express f', the derivative of f (with respect to x) in terms of g'. The number a is constant. (i) $f(x) = g(x+g(a))$. (ii) $f(x) = g(a+g(x))$. (iii) $f(x) = g(x^2)$. (AEB 1975)

14. Given that $(x+y) = (x-y)^2$, prove by differentiating implicitly, or otherwise, that $1 - \frac{dy}{dx} = \frac{2}{2x-2y+1}$. Hence, or otherwise, prove that $\frac{d^2y}{dx^2} = \left(1 - \frac{dy}{dx}\right)^3$ (JMB)

15. A piece of wire 80 cm in length is cut into three parts, two of which are bent into equal circles and the third into a square. Find the radius of the circles if the sum of the enclosed areas is a minimum. (AEB 1975)

16. The diagonals of a rhombus are of lengths $2x$ cm and $(10-x)$ cm. As x varies, find (a) the maximum area of the rhombus, (b) the minimum length of its perimeter.

17. The height h and the base radius r of a right circular cone vary in such a way that the volume remains constant. Find the rate of change of h with respect to r at the instant when h and r are equal.

18. A right circular cone is of height 4 cm. The radius of the base is increasing at the rate of 0·5 cm/s. When the radius is 3 cm, find the rate of increase of (a) the volume of the cone, (b) the curved surface area.

19. Find approximate values for

(a) $(8{\cdot}003)^{2/3}$, (b) $\dfrac{1}{4{\cdot}98}$, (c) $\sqrt[4]{(16{\cdot}032)}$.

20. Find approximately the values of $x^3+2x-\dfrac{16}{x^2}$ when

(a) $x = 4{\cdot}02$, (b) $x = 3{\cdot}96$.

21. Find the percentage increase in the circumference of a circle, which will result in an increase in the area of the circle of approximately 3%.

22. In an experiment the value of a quantity f is calculated using the formula $\dfrac{1}{f} = \dfrac{1}{u}+\dfrac{1}{v}$. When $u = 20$, the value of v is found to be 30 with a possible error of 0·5. Find the corresponding error in the value of f.

23. A right pyramid having a square base is inscribed in a sphere of radius R, all five vertices of the pyramid lying on the sphere. The height of the pyramid is x; show that the four vertices forming the base of the pyramid lie on a circle of radius r, where $r^2 = 2Rx-x^2$. Hence, or otherwise, show that the volume V, of the pyramid is given by the formula $V = \frac{2}{3}x^2(2R-x)$. If R is fixed but x may vary, find the greatest possible value of V. (C)

24. Water starts running into an empty basin at the rate of 6π cm^3/s. The basin is in the shape of the surface formed when the curve $4y = x^2$ is rotated completely about the y-axis. Show that when the depth of the water is y cm, the volume of water in the basin is $2\pi y^2$ cm^3. Find the rate at which the water level is rising when the water has been running for 3 seconds.

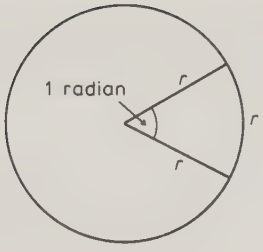
25. A hemispherical bowl of radius a cm is initially full of water. The water runs out of a small hole at the bottom of the bowl at a constant rate which is such that it would empty the bowl in 24 s. Given that, when the depth of the water is x cm, the volume of water is $\frac{1}{3}\pi x^2(3a-x)$ cm^3, prove that the depth is decreasing at a rate of $a^3/\{36x(2a-x)\}$ cm/s. Find after what time the depth of water is $\frac{1}{2}a$ cm, and the rate at which the water level is then decreasing. (O&C)

11 Trigonometric functions

11.1 Circular measure

In elementary work, angles are usually measured in degrees, but it is often more convenient to use another unit called a radian.

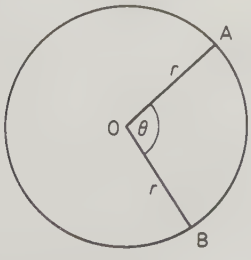
An angle of 1 *radian* at the centre of a circle is subtended by an arc equal in length to the radius of the circle. Since the circumference of the circle is an arc of length $2\pi r$, it subtends an angle of 2π radians at the centre



$\therefore$ one complete revolution $= 360° = 2\pi$ radians.

Hence $1 \text{ rad} = \dfrac{360°}{2\pi} \approx 57{\cdot}3°$ and $1° = \dfrac{2\pi}{360} \text{rad} \approx 0{\cdot}0175 \text{ rad}.$

Because of their relation to circular arcs, radians are referred to as *circular measure* and sometimes denoted by ^{c}, e.g. $2\pi^c = 360°$. This symbol and the abbreviation rad are used only when it is necessary to distinguish between radians and degrees. Usually it is assumed that an angle is measured in radians unless otherwise stated.



Consider now two points A and B on a circle, centre O and radius r. Let $\angle AOB = \theta$ (in radians). Since an arc of length r subtends an angle of 1 radian at O, the length of arc AB must be $r\theta$.

Since the area of the circle is πr^2, the area of sector AOB must be $\dfrac{\theta}{2\pi} \times \pi r^2$, i.e. $\frac{1}{2}r^2\theta$.

$$\therefore \quad \boxed{\text{length of arc } AB = r\theta, \quad \text{area of sector } AOB = \tfrac{1}{2}r^2\theta.}$$

Solutions of trigonometric equations are sometimes given in radians.

Example 1 Solve the equation $\cos 2\theta + \cos\theta + 1 = 0$ where $0 \leqslant \theta \leqslant 2\pi$.

$$\begin{aligned} \cos 2\theta + \cos\theta + 1 = 0 &\Leftrightarrow (2\cos^2\theta - 1) + \cos\theta + 1 = 0 \\ &\Leftrightarrow \quad 2\cos^2\theta + \cos\theta = 0 \\ &\Leftrightarrow \quad \cos\theta(2\cos\theta + 1) = 0 \end{aligned}$$

$$\therefore \quad \cos\theta = 0 \quad \text{or} \quad \cos\theta = -\tfrac{1}{2} = -\cos\tfrac{1}{3}\pi$$
$$\theta = \frac{1}{2}\pi, \frac{3}{2}\pi \qquad \theta = \frac{2}{3}\pi, \frac{4}{3}\pi$$

Hence the required solutions are $\dfrac{1}{2}\pi, \dfrac{2}{3}\pi, \dfrac{4}{3}\pi$ and $\dfrac{3}{2}\pi$.

[The equivalent solutions in degrees are 90°, 120°, 240° and 270°.]

Exercise 11.1

1. Express the following angles in degrees:
(a) π rad, (b) $\pi/4$ rad, (c) $\pi/6$ rad, (d) $4\pi/3$ rad.

2. Express the following angles in radians (as multiples of π):
(a) 90°, (b) 60°, (c) 150°, (d) 315°.

3. Find, without using tables
(a) $\sin\dfrac{3\pi}{2}$, (b) $\cos\dfrac{2\pi}{3}$, (c) $\tan\dfrac{5\pi}{4}$, (d) $\sec\left(-\dfrac{\pi}{6}\right)$.

4. Give the values in radians of
(a) $\cos^{-1}\dfrac{1}{\sqrt{2}}$, (b) $\tan^{-1}(-\sqrt{3})$, (c) $\sin^{-1}\dfrac{1}{2}$, (d) $\cos^{-1}(-1)$.

5. An arc PQ subtends an angle of 1·5 radians at the centre O of a circle with radius 10 cm. Find the length of the arc PQ and the area of the sector POQ.

6. P and Q are points on the circumference of a circle centre O and radius 4 cm. The tangent to the circle at P meets OQ produced at T. If $\angle POQ = \frac{1}{4}\pi$ radians, find the area of the region bounded by PT, TQ and the arc PQ.

7. AB is a diameter of a circle radius r and C is a point on the circumference

such that $\angle ABC = \theta$ radians. Find expressions for the area and the perimeter of the region bounded by AB, BC and the arc AC.

8. A and B are points on the circumference of a circle centre O, radius 5 cm. If $\angle AOB = \theta$ radians, find the values of θ for which (a) the area of sector OAB is $15\,\text{cm}^2$, (b) the length of the perimeter of sector OAB is 12 cm, (c) the chord AB is of length $5\sqrt{3}$ cm, (d) the area of triangle OAB is $6{\cdot}25\,\text{cm}^2$.

9. AB is an arc of a circle with centre O and radius 12 cm. A second circle with centre O and radius $(12+x)$ cm cuts OA and OB produced at the points C and D. If $\angle AOB = 0{\cdot}8$ radians and the difference between the areas of the sectors AOB and COD is $72\,\text{cm}^2$, find the value of x.

10. Solve the following equations in radians for $0 \leqslant \theta \leqslant 2\pi$
(a) $2\sin\theta = 1$, (b) $4\cos^2\theta = 1$, (c) $\sec^2\theta = 2\tan\theta$.

11. Solve the following equations in radians for $0 \leqslant \theta \leqslant \pi$
(a) $\sin 2\theta = \cos 4\theta$, (b) $\sin 6\theta = \cos 3\theta$.

12. Sketch on the same diagram the graphs $y = \sqrt{2}\cos x$ and $y = \sin 2x$, for $-\pi \leqslant x \leqslant 2\pi$. Find the ranges of values of x in that interval for which $\sin 2x > \sqrt{2}\cos x$.

In questions 13 to 15 you may use a calculator, giving your answers to 3 significant figures. Check that the calculator is adjusted for work in radians.

13. A chord AB subtends an angle of 0·75 rad at the centre of a circle of radius 20 cm. Find the area and the length of the perimeter of the minor segment of the circle cut off by AB.

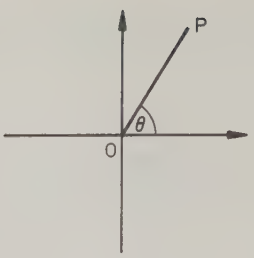
14. AB is the diameter of a circle radius 12 cm and P is a point on the circumference such that $\angle PAB = 0{\cdot}6$ rad. The tangent to the circle at P cuts AB produced at T. Find the area of the region bounded by BT, TP and arc PB.

15. A chord AB of a circle of radius 5 cm subtends an angle θ at the centre O. A second chord CD is parallel to AB and on the same side of O as AB. If $AB = 2$ cm and $\angle COD = \theta + \frac{1}{2}\pi$, find θ and the area of the part of the circle lying between AB and CD.

11.2 General solutions of trigonometric equations

The trigonometric equation $\sin\theta = 0$ has an infinite solution set $\{\ldots, -2\pi, -\pi, 0, \pi, \ldots\}$. Since all the solutions are of the form $n\pi$ where n is an integer, $\theta = n\pi$ is said to be the *general solution* of the equation. We now consider three more general cases.

(1) $\cos\theta = \cos\alpha$



Geometrically, the roots of this equation are represented by the possible positions of a line OP, which rotates about the origin as shown in the diagram. In every complete revolution of OP, θ passes through two values which satisfy $\cos\theta = \cos\alpha$. These values correspond with the positions of OP given by $\theta = \alpha$ and $\theta = -\alpha$. Hence every root takes the form $\alpha + 2n\pi$ or $-\alpha + 2n\pi$ where n is an integer, $\therefore$ the general solution of the equation $\cos\theta = \cos\alpha$ is

$$\theta = 2n\pi \pm \alpha, \quad \text{where } n \text{ is an integer.}$$

(2) $\sin\theta = \sin\alpha$

In this case, since $\sin(\pi - \alpha) = \sin\alpha$, the two possible positions of OP in every complete revolution are given by $\theta = \alpha$ and $\theta = \pi - \alpha$, $\therefore$ every root of the equation takes either the form $\alpha + 2n\pi$ or $(\pi - \alpha) + 2n\pi$, where n is an integer. Hence the general solution of the equation $\sin\theta = \sin\alpha$ is

$$\theta = 2n\pi + \alpha \quad \text{or} \quad 2n\pi + (\pi - \alpha) \quad \text{where } n \text{ is an integer.}$$

This means that θ is either an even multiple of π plus α or an odd multiple of π minus α. Hence the general solution of the equation $\sin\theta = \sin\alpha$ can also be written in the form

$$\theta = n\pi + (-1)^n\alpha, \quad \text{where } n \text{ is an integer.}$$

(3) $\tan\theta = \tan\alpha$

The function $\tan\theta$ takes every real value exactly once in every rotation of OP through π radians. Since one solution of the equation is $\theta = \alpha$, the general solution of $\tan\theta = \tan\alpha$ is

$$\theta = n\pi + \alpha \quad \text{where } n \text{ is an integer.}$$

Summarising:

$$\begin{aligned} &\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha \\ &\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n\alpha \\ &\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha \end{aligned}$$

where in every case n is an integer.

There are a few special cases in which these general forms can be simplified.

$\cos\theta = 0 \quad \Rightarrow \theta = (2n+1)\dfrac{\pi}{2}$ $\qquad$ $\sin\theta = 0 \quad \Rightarrow \theta = n\pi$

$\cos\theta = 1 \quad \Rightarrow \theta = 2n\pi$ $\qquad$ $\sin\theta = 1 \quad \Rightarrow \theta = (4n+1)\dfrac{\pi}{2}$

$\cos\theta = -1 \Rightarrow \theta = (2n+1)\pi$ $\qquad$ $\sin\theta = -1 \Rightarrow \theta = (4n-1)\dfrac{\pi}{2}$

Example 1 Find the general solution of the equation $\sin 2\theta - \sqrt{3}\cos\theta = 0$.

$$\sin 2\theta - \sqrt{3}\cos\theta = 0 \Leftrightarrow 2\sin\theta\cos\theta - \sqrt{3}\cos\theta = 0$$
$$\Leftrightarrow \quad \cos\theta(2\sin\theta - \sqrt{3}) = 0$$

$\therefore$ either $\cos\theta = 0$ or $\sin\theta = \dfrac{\sqrt{3}}{2} = \sin\dfrac{\pi}{3}$

$$\theta = n\pi + \frac{\pi}{2} \qquad \theta = n\pi + (-1)^n\frac{\pi}{3}$$

Hence the general solution of the equation is

$$\theta = (2n+1)\frac{\pi}{2} \quad \text{or} \quad n\pi + (-1)^n\frac{\pi}{3}, \quad \text{where } n \text{ is an integer.}$$

Example 2 Find the general solution of the equation $\sin 3\theta = \sin\theta$.

$$\sin 3\theta = \sin\theta$$

$\therefore$ either $3\theta = 2n\pi + \theta$ or $3\theta = 2n\pi + \pi - \theta$

$2\theta = 2n\pi$ $\qquad$ $4\theta = (2n+1)\pi$

$\theta = n\pi$ $\qquad$ $\theta = (2n+1)\dfrac{\pi}{4}$

Hence the general solution is $\theta = n\pi$ or $\theta = (2n+1)\dfrac{\pi}{4}$ where n is an integer.

Example 3 Find the general solution of the equation $\cos 4\theta = \sin 5\theta$ giving your answer in degrees.

$$\cos 4\theta = \sin 5\theta \Leftrightarrow \cos 4\theta = \cos(90° - 5\theta)$$

$\therefore$ either $4\theta = 360n° + 90° - 5\theta$ or $4\theta = 360n° - (90° - 5\theta)$

$9\theta = 360n° + 90°$ $\qquad$ $-\theta = 360n° - 90°$

$\theta = 40n° + 10°$ $\qquad$ $\theta = 90° - 360n°$.

Since every solution of the form $90° - 360n°$ can also be expressed in the form $40n° + 10°$, the required general solution is simply $\theta = 40n° + 10°$, where n is an integer.

Example 4 Find the general solution of the equation $3\cos\theta + 4\sin\theta = 2$, giving your answer in degrees and minutes.

Let $3\cos\theta + 4\sin\theta \equiv r\cos(\theta - \alpha) \equiv r\cos\alpha\cos\theta + r\sin\alpha\sin\theta$

then $r\cos\alpha = 3$ and $r\sin\alpha = 4$

$\therefore$ $r = \sqrt{(3^2 + 4^2)} = 5$ and $\tan\alpha = 4/3$, $\alpha = 53{\cdot}13°$.

Hence $3\cos\theta + 4\sin\theta = 2 \Leftrightarrow 5\cos(\theta - 53{\cdot}13°) = 2$

$$\therefore \quad \cos(\theta - 53{\cdot}13°) = 0{\cdot}4 = \cos 66{\cdot}42°$$

$\therefore$ either $\theta - 53{\cdot}13^\circ = 360n^\circ + 66{\cdot}42^\circ$
$\theta = 360n^\circ + 119{\cdot}6^\circ$
or $\theta - 53{\cdot}13^\circ = 360n^\circ - 66{\cdot}42^\circ$
$\theta = 360n^\circ - 13{\cdot}3^\circ$.

Thus the general solution is $\theta = 360n^\circ + 119{\cdot}6^\circ$ or $\theta = 360n^\circ - 13{\cdot}3^\circ$ where n is an integer.

Exercise 11.2

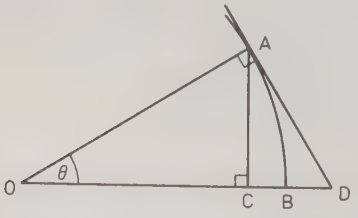
Find, in radians, the general solutions of the following equations.

1. $\cos\theta = \cos\frac{2}{5}\pi$.
2. $\tan\theta = \tan\frac{1}{3}\pi$.
3. $\sin\theta = \sin\frac{\pi}{8}$.
4. $\cos\theta = \cos\pi$.
5. $\tan\theta = -1$.
6. $2\sin\theta = 1$.
7. $\sin^2\theta = 1$.
8. $2\cos^2\theta = 1$.
9. $\sin 2x = \sin x$.
10. $\cos 2x = \cos x$.
11. $2\cos 2x = 1 - 4\cos x$.
12. $\sec^2 x = 2\tan x$.
13. $\tan\theta = \tan 4\theta$.
14. $\tan 2\theta = \tan(\frac{1}{2}\pi - 3\theta)$.
15. $\tan 3\theta = \cot\theta$.
16. $\cos 3\theta = \sin 5\theta$.

Find, to the nearest $0{\cdot}1^\circ$, the general solutions of the following equations.

17. $\cos\theta + 3\sin\theta = 2$.
18. $16\cos\theta + 30\sin\theta = 17$.

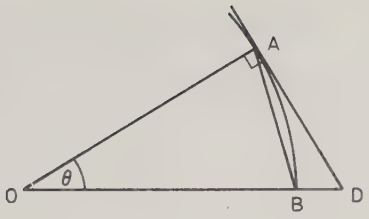
11.3 Useful limits and approximations



If $OA = OB = 1$ and θ is measured in radians, then

$OC = \cos\theta$. arc $AB = \theta$
$AC = \sin\theta$, $AD = \tan\theta$

When θ is small, $OC \approx OB$ and $AC \approx AB \approx AD$.
Hence $\cos\theta \approx 1$ and $\sin\theta \approx \theta \approx \tan\theta$.



In a more formal approach to small values of θ, areas are considered.

$$\begin{array}{rccccc} & \triangle AOB & < & \text{sector } AOB & < & \triangle AOD \\ \Rightarrow & \frac{1}{2}r^2\sin\theta & < & \frac{1}{2}r^2\theta & < & \frac{1}{2}r^2\tan\theta \\ \Rightarrow & \sin\theta & < & \theta & < & \tan\theta \\ \Rightarrow & \dfrac{\sin\theta}{\sin\theta} & < & \dfrac{\theta}{\sin\theta} & < & \dfrac{\tan\theta}{\sin\theta} \\ \Rightarrow & 1 & < & \dfrac{\theta}{\sin\theta} & < & \dfrac{1}{\cos\theta} \end{array}$$

As $\theta \to 0$, $\cos\theta \to 1$ $\quad\therefore\quad \dfrac{1}{\cos\theta} \to 1$

Hence, as $\theta \to 0$, $\quad \dfrac{\theta}{\sin\theta} \to 1.$

Thus

$$\boxed{\lim_{\theta\to 0}\frac{\theta}{\sin\theta} = 1 \quad \text{and} \quad \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1.}$$

As before, we see that for small values of θ, $\sin\theta \approx \theta$.
Similarly $\sin\frac{1}{2}\theta \approx \frac{1}{2}\theta$ $\quad\therefore \cos\theta = 1 - 2\sin^2\frac{1}{2}\theta \approx 1 - 2(\frac{1}{2}\theta)^2 \approx 1 - \frac{1}{2}\theta^2$.

$\therefore$ for small values of θ in radians, we write

$$\sin\theta \approx \theta, \quad \tan\theta \approx \theta \quad \text{and} \quad \cos\theta \approx 1 - \tfrac{1}{2}\theta^2$$

[or $\cos\theta \approx 1$ when less accuracy is required].

These limits and approximations have many applications especially in calculus. Some of their uses are simply illustrated in the following examples.

Example 1 Find an approximate value for $\sin 2°$ given $\pi = 3{\cdot}1416$.

$$2° = \frac{2}{360} \times 2\pi = \frac{\pi}{90} \approx \frac{3{\cdot}1416}{90} \approx 0{\cdot}0349$$

$\therefore$ an approximate value for $\sin 2°$ is 0·0349.

[The true value of $\sin 2°$ correct to 6 significant figures is 0·0348995.]

Example 2 Find an approximation for the expression $\dfrac{\sin 2\theta \tan 2\theta}{1 - \cos\theta}$ when θ is small.

$$\sin 2\theta \approx 2\theta, \quad \tan 2\theta \approx 2\theta, \quad \cos\theta \approx 1-\tfrac{1}{2}\theta^2$$

$$\therefore \quad \frac{\sin 2\theta \tan 2\theta}{1-\cos\theta} \approx \frac{2\theta . 2\theta}{1-(1-\frac{1}{2}\theta^2)} = \frac{4\theta^2}{\frac{1}{2}\theta^2} = 8.$$

Hence when θ is small $\dfrac{\sin 2\theta \tan 2\theta}{1-\cos\theta} \approx 8.$

Example 3 Evaluate $\lim\limits_{\theta\to 0} \dfrac{\sin 2\theta + \sin 4\theta}{\theta}$

$$\frac{\sin 2\theta + \sin 4\theta}{\theta} = \frac{\sin 2\theta}{\theta} + \frac{\sin 4\theta}{\theta} = 2 . \frac{\sin 2\theta}{2\theta} + 4 . \frac{\sin 4\theta}{4\theta}$$

$$\therefore \quad \lim_{\theta\to 0} \frac{\sin 2\theta + \sin 4\theta}{\theta} = 2\left(\lim_{\theta\to 0} \frac{\sin 2\theta}{2\theta}\right) + 4\left(\lim_{\theta\to 0} \frac{\sin 4\theta}{4\theta}\right)$$
$$= 2 . 1 + 4 . 1 = 6.$$

Exercise 11.3

1. Find approximate values for the following expressions without using tables or calculators, given that $\pi = 3{\cdot}1416$.
(a) $\sin 1{\cdot}2°$ (b) $\tan 0{\cdot}7°$ (c) $\sin 0{\cdot}015°$.

2. Find approximate values of the following expressions when θ is small,
(a) $\dfrac{\theta \sin\theta}{1-\cos\theta}$, (b) $\dfrac{\tan 3\theta}{2\theta}$, (c) $\dfrac{\sin 2\theta \tan\theta}{1-\cos 3\theta}$.

3. Find an expression of the form $a+b\theta+c\theta^2$ which is approximately equal to the following when θ is small,
(a) $\sin(\theta+\frac{1}{3}\pi)$, (b) $\cos(\theta-\frac{1}{4}\pi)$, (c) $\cos\theta\cos 2\theta$.

4. Find the limit as $\theta \to 0$ of
(a) $\dfrac{\sin\theta}{2\theta}$, (b) $\dfrac{\sin 3\theta}{\theta}$, (c) $\dfrac{\sin 5\theta}{\sin 4\theta}$.

5. Find the limit as $x \to 0$ of
(a) $\dfrac{1-\cos 2x}{x^2}$, (b) $\dfrac{\cos 3x - \cos x}{\cos 4x - \cos 2x}$.

6. Use areas to prove that as $\theta \to 0$, $\dfrac{\tan\theta}{\theta} \to 1$.

11.4 Derivatives of trigonometric functions

The function $\sin x$ is differentiated from first principles as follows:

$$y = \sin x \quad \text{(where } x \text{ is in radians)}$$

$$\Rightarrow y+\delta y = \sin(x+\delta x)$$

$$\Rightarrow \quad \delta y = \sin(x+\delta x) - \sin x = 2\cos(x+\tfrac{1}{2}\delta x)\sin\tfrac{1}{2}\delta x$$

$$\Rightarrow \quad \frac{\delta y}{\delta x} = \frac{2\cos(x+\frac{1}{2}\delta x)\sin\frac{1}{2}\delta x}{\delta x} = \cos(x+\tfrac{1}{2}\delta x).\frac{\sin\frac{1}{2}\delta x}{\frac{1}{2}\delta x}.$$

As $\delta x \to 0$, $\cos(x+\frac{1}{2}\delta x) \to \cos x$ and $\dfrac{\sin\frac{1}{2}\delta x}{\frac{1}{2}\delta x} \to 1$

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \cos x . 1.$$

Hence $$\frac{d}{dx}(\sin x) = \cos x.$$

Similarly it can be shown that $\dfrac{d}{dx}(\cos x) = -\sin x.$

Example 1 Differentiate (a) $\sin^3 x$, (b) $\cos(4-3x)$, (c) $\sin x°$.

(a) Let $y = \sin^3 x = u^3$ where $u = \sin x$

$$\frac{dy}{du} = 3u^2 \qquad \frac{du}{dx} = \cos x$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = 3u^2.\cos x = 3\sin^2 x \cos x.$$

Hence $$\frac{d}{dx}(\sin^3 x) = 3\sin^2 x \cos x.$$

(b) Let $y = \cos(4-3x) = \cos u$ where $u = 4-3x$

$$\frac{dy}{du} = -\sin u \qquad \frac{du}{dx} = -3$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = -\sin u.(-3) = 3\sin u.$$

Hence $$\frac{d}{dx}(\cos(4-3x)) = 3\sin(4-3x).$$

(c) Let $y = \sin x°$, then expressing $x°$ in radians $y = \sin\dfrac{\pi}{180}x$

$$\therefore \quad \frac{dy}{dx} = \cos\frac{\pi}{180}x.\frac{\pi}{180} = \frac{\pi}{180}\cos x°.$$

Hence $$\frac{d}{dx}(\sin x°) = \frac{\pi}{180}\cos x°.$$

Example 2 Find the maximum and minimum values of the function $f(x) = 2\sin x + \cos 2x$ for $0 < x < \pi$.

Differentiating, $f'(x) = 2\cos x - 2\sin 2x = 2\cos x - 4\sin x\cos x$
$= 2\cos x(1 - 2\sin x)$

$f'(x) = 0 \Leftrightarrow \cos x(1 - 2\sin x) = 0$
$\Leftrightarrow \cos x = 0 \quad \text{or} \quad \sin x = \frac{1}{2}$

$\therefore$ for $0 < x < \pi, f'(x) = 0$ when $x = \frac{1}{6}\pi, \frac{1}{2}\pi$ or $\frac{5}{6}\pi$.

	$0 < x < \frac{1}{6}\pi$	$\frac{1}{6}\pi < x < \frac{1}{2}\pi$	$\frac{1}{2}\pi < x < \frac{5}{6}\pi$	$\frac{5}{6}\pi < x < \pi$
$\cos x$	+	+	−	−
$1 - 2\sin x$	+	−	−	+
$f'(x)$	+	−	+	−

$\therefore$ at $x = \frac{1}{6}\pi, f(x)$ takes the maximum value $f\left(\frac{1}{6}\pi\right) = 1\frac{1}{2}$,

at $x = \frac{1}{2}\pi, f(x)$ takes the minimum value $f\left(\frac{1}{2}\pi\right) = 1$.

at $x = \frac{5}{6}\pi, f(x)$ takes the maximum value $f\left(\frac{5}{6}\pi\right) = 1\frac{1}{2}$.

The function $\tan x$ is differentiated by expressing it as the quotient $\dfrac{\sin x}{\cos x}$ and using the formula $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \left(v\dfrac{du}{dx} - u\dfrac{dv}{dx}\right)\Big/v^2$ derived in §10.3.

If $y = \tan x = \dfrac{\sin x}{\cos x} = \dfrac{u}{v}$ where $u = \sin x, \quad v = \cos x$

then $\dfrac{du}{dx} = \cos x, \quad \dfrac{dv}{dx} = -\sin x$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{\cos x \,.\, \cos x - \sin x \,.\, (-\sin x)}{\cos^2 x} = \dfrac{\cos^2 x + \sin^2 x}{\cos^2 x} = \dfrac{1}{\cos^2 x} = \sec^2 x$

$$\therefore \quad \frac{d}{dx}(\tan x) = \sec^2 x.$$

Similarly it can be shown that $\dfrac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$.

To differentiate $\sec x$ we write:

$$y = \sec x = (\cos x)^{-1} = u^{-1} \quad \text{where} \quad u = \cos x$$

$\therefore \quad \dfrac{dy}{du} = -\dfrac{1}{u^2}$ and $\dfrac{du}{dx} = -\sin x.$

Hence $\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dx} = -\dfrac{1}{u^2}(-\sin x) = \dfrac{\sin x}{\cos^2 x} = \dfrac{1}{\cos x}\times\dfrac{\sin x}{\cos x}$

$$\therefore \quad \frac{d}{dx}(\sec x) = \sec x \tan x.$$

Similarly it can be shown that $\dfrac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x.$

Example 3 If $y^2 = \tan 2x + \sec 2x$, show that when $y \neq 0$, $\dfrac{dy}{dx} = y \sec 2x$.

Differentiating with respect to x,

$$y^2 = \tan 2x + \sec 2x \Leftrightarrow 2y\frac{dy}{dx} = 2\sec^2 2x + 2\sec 2x \tan 2x$$

$$\therefore \quad y\frac{dy}{dx} = \sec 2x(\sec 2x + \tan 2x) = y^2 \sec 2x.$$

Hence, when $y \neq 0$, $\dfrac{dy}{dx} = y \sec 2x$.

The table below shows the results obtained in this section.

y	$\sin x$	$\cos x$	$\tan x$	$\cot x$	$\sec x$	$\operatorname{cosec} x$
$\dfrac{dy}{dx}$	$\cos x$	$-\sin x$	$\sec^2 x$	$-\operatorname{cosec}^2 x$	$\sec x \tan x$	$-\operatorname{cosec} x \cot x$

Exercise 11.4

Differentiate the following with respect to x, simplifying your answers where possible.

1. (a) $\sin 2x$, (b) $3\cos 4x$, (c) $4\tan \frac{1}{2}x$.
2. (a) $\cos^4 x$, (b) $\sqrt{(\sin x)}$, (c) $\sin^2 5x$.
3. (a) $(x + \sin x)^3$, (b) $\sin (x^2)$, (c) $\tan (6x^\circ)$.
4. (a) $x^4 \cos x$, (b) $(4x^2 + 1)\tan x$, (c) $x^2 \sin 3x$.
5. (a) $\operatorname{cosec} (x + 1)$, (b) $2\cot (1 - 2x)$, (c) $\sec (3x - 4)$.
6. (a) $\tan (\cos x)$, (b) $\sec (1 + \sqrt{x})$, (c) $\cot (1/x)$.
7. (a) $\sin x \cos 2x$, (b) $\cos^3 x \sin 3x$, (c) $\cos x \tan x$.

8. (a) $\dfrac{\sin x}{x^2}$, (b) $\dfrac{1}{\cos x - \sin x}$, (c) $\dfrac{\cos 3x}{1+\sin 3x}$.

In questions 9 to 12 find the coordinates of any stationary points on the given curve for $0 \leqslant x \leqslant 2\pi$. Hence sketch the curve in that interval.

9. $y = 4\sin x - \cos 2x$.

10. $y = x - 2\sin x$.

11. $y = \tan x(\tan x + 2)$.

12. $y = \sin^3 x \cos x$.

13. A particle P moves in a straight line such that after t seconds its displacement from a fixed point O is s metres, where $s = 10\sin 3t$. Find (a) the time at which P first returns to O, (b) the maximum distance from O reached by P, (c) the maximum speed attained by the particle and its position each time this speed is reached.

14. A right circular cone has semi-vertical angle θ and height 15 cm. If θ is increasing at the rate of k rad/s, where k is constant, find the rate of change of the volume of the cone when $\theta = \frac{1}{4}\pi$.

15. Points A and B lie on the circumference of a circle with centre O and radius 10 cm. If $\angle AOB = \theta$ and θ is increasing at the rate of 0·1 rad/s, find the rate of increase when $\theta = \frac{1}{3}\pi$ of (a) the area of $\triangle AOB$, (b) the perimeter of the region enclosed by the chord AB and the arc AB.

16. Given that $\sqrt{3} \approx 1{\cdot}7321$ and $0{\cdot}1^\circ \approx 0{\cdot}00175$ rad, find approximate values for (a) $\cos 30{\cdot}7^\circ$, (b) $\tan 45{\cdot}3^\circ$, (c) $\sin 60{\cdot}5^\circ$.

17. In a trapezium $ABCD$ the sides AB, DC are parallel and $\angle ABC$ is a right angle. If $AD = DB = 10$ cm and $\angle DAB = \theta$, find the value of $\tan\theta$ for which the perimeter, p cm, is a maximum. Hence find the range of values taken by p for $0 < \theta < \frac{1}{2}\pi$.

18. A particle moves in a straight line such that after t seconds its displacement from a fixed point is s metres, where $s = 4\cos t - \cos 2t$. If the particle first comes to rest after T seconds, where $T > 0$, find (a) its acceleration at time T, (b) the maximum speed it attains for $0 < t < T$.

11.5 Integration of trigonometric functions

From the differentiation results obtained so far:

$$\int \sin x\,dx = -\cos x + c \qquad \int \cos x\,dx = \sin x + c$$

$$\int \sec^2 x\,dx = \tan x + c \qquad \int \operatorname{cosec}^2 x\,dx = -\cot x + c$$

$$\int \sec x \tan x \, dx = \sec x + c \qquad \int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + c.$$

These results can be extended as shown in the following examples.

Example 1 Integrate (a) $\cos 2x$, (b) $\sin^4 x \cos x$.

(a) Since $\frac{d}{dx}(\sin 2x) = 2\cos 2x$, $\int \cos 2x \, dx = \frac{1}{2}\sin 2x + c$.

(b) Since $\frac{d}{dx}(\sin^5 x) = 5\sin^4 x \cos x$, $\int \sin^4 x \cos x \, dx = \frac{1}{5}\sin^5 x + c$.

In some cases trigonometric identities are used to change the form of the integrand (i.e. the function to be integrated).

Example 2 Integrate (a) $\tan^2 x$, (b) $\sin^2 x$.

(a) $\int \tan^2 x \, dx = \int (\sec^2 x - 1)\, dx = \tan x - x + c$.

(b) $\int \sin^2 x \, dx = \int \frac{1}{2}(1 - \cos 2x)\, dx = \frac{1}{2}(x - \frac{1}{2}\sin 2x) + c = \frac{1}{2}x - \frac{1}{4}\sin 2x + c$.

Example 3 Find $\int \sin 5x \cos x \, dx$.

$$2\sin 5x \cos x = \sin 6x + \sin 4x$$

$$\therefore \int \sin 5x \cos x \, dx = \int (\tfrac{1}{2}\sin 6x + \tfrac{1}{2}\sin 4x)\, dx$$

$$= \frac{1}{2}\cdot\frac{1}{6}(-\cos 6x) + \frac{1}{2}\cdot\frac{1}{4}(-\cos 4x) + c$$

$$= -\frac{1}{12}\cos 6x - \frac{1}{8}\cos 4x + c.$$

Example 4 Evaluate $\int_0^{\pi/6} \cos^5 x \, dx$.

$$\cos^5 x = (\cos^2 x)^2 \cos x = (1 - \sin^2 x)^2 \cos x = (1 - 2\sin^2 x + \sin^4 x)\cos x,$$

$$\therefore \int_0^{\pi/6} \cos^5 x \, dx = \int_0^{\pi/6} (\cos x - 2\sin^2 x \cos x + \sin^4 x \cos x)\, dx$$

$$= \left[\sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x\right]_0^{\pi/6}$$

$$= \left(\frac{1}{2} - \frac{2}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{5}\left(\frac{1}{2}\right)^5\right) - 0 = \frac{1}{2} - \frac{1}{12} + \frac{1}{160} = \frac{203}{480}.$$

Exercise 11.5

Find the following indefinite integrals

1. (a) $\int \sin 3x\,dx$, (b) $\int \cos \tfrac{1}{2}x\,dx$, (c) $\int 4\sec^2 2x\,dx$.

2. (a) $\int \cos^2 x\,dx$, (b) $\int \cot^2 x\,dx$, (c) $\int \sin^2 x \cos x\,dx$.

3. (a) $\int \sin x \cos^3 x\,dx$, (b) $\int \sin x \cos 3x\,dx$, (c) $\int \cos 2x \cos x\,dx$.

Evaluate the following definite integrals

4. (a) $\int_0^{\pi/4} \cos 2x\,dx$, (b) $\int_0^{\pi/12} \sin^2 3x\,dx$, (c) $\int_0^{\pi/2} \sin x \cos^2 x\,dx$.

5. (a) $\int_0^{\pi/3} \tan^2 x\,dx$, (b) $\int_0^{\pi/3} \sin 3x \cos 2x\,dx$, (c) $\int_0^{\pi/4} \sin 3x \sin x\,dx$.

6. (a) $\int_0^{\pi/3} \dfrac{\sin x}{\cos^2 x}\,dx$, (b) $\int_{\pi/4}^{\pi/2} \dfrac{\cos 2x}{\sin^2 x}\,dx$, (c) $\int_0^{\pi/2} \cos^3 x\,dx$.

7. Differentiate $\tan^3 x$ with respect to x and use your result to find $\int \tan^4 x\,dx$.

8. Show that (a) $\int 4 \sin x \cos x\,dx = 2\sin^2 x + \text{constant}$,

(b) $\int 4 \sin x \cos x\,dx = -\cos 2x + \text{constant}$.

Write down a third expression for the same integral and explain how all three expressions can be correct.

9. Express $\cos^4 x$ in terms of $\cos 2x$ and $\cos 4x$. Hence evaluate $\int_0^{\pi/4} \cos^4 x\,dx$.

10. Find the area of the region bounded by the curve $y = 2/(1+\cos 2x)$, the x-axis, the y-axis and the line $x = \frac{1}{4}\pi$.

11. Find the volume of the solid formed when the region bounded by the curve $y = 3 - 2\cos x$, the x-axis, the y-axis and the line $x = \pi$ is rotated completely about the x-axis.

12. Find the mean value of $\sin^3 x$ over the interval $0 \leqslant x \leqslant \pi$.

11.6 Inverse trigonometric functions

The inverse trigonometric functions $\sin^{-1} x$, $\cos^{-1} x$ and $\tan^{-1} x$ were introduced in §7.2. At that stage it was convenient to use angles measured in degrees. However, in more advanced work, these functions are usually evaluated in radians.

To differentiate the function $\sin^{-1} x$,

let $y = \sin^{-1} x$, then $\sin y = x$

Differentiating $\cos y \dfrac{dy}{dx} = 1$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\cos y}$$

But $\cos^2 y + \sin^2 y = 1 \quad \therefore \quad \cos^2 y = 1 - \sin^2 y$

Since $\cos y$ is positive for $-\frac{1}{2}\pi < y < \frac{1}{2}\pi$,

$$\cos y = \sqrt{(1 - \sin^2 y)} = \sqrt{(1 - x^2)}.$$

Hence $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{(1-x^2)}}$, i.e. $\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{(1-x^2)}}$.

By a similar method, it can be shown that

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{(1-x^2)}}$$

[This result can also be obtained by using the fact that $\sin^{-1} x + \cos^{-1} x = \frac{1}{2}\pi$.]

To differentiate the function $\tan^{-1} x$,

let $y = \tan^{-1} x$, then $\tan y = x$

Differentiating $\sec^2 y \dfrac{dy}{dx} = 1$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sec^2 y}.$$

But $\sec^2 y = 1 + \tan^2 y = 1 + x^2$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{1+x^2}, \quad \text{i.e.} \quad \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}.$$

Summarising:

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{(1-x^2)}} \quad \text{and} \quad \int \frac{dx}{\sqrt{(1-x^2)}} = \sin^{-1} x + c$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \quad \text{and} \quad \int \frac{dx}{1+x^2} = \tan^{-1} x + c.$$

Example 1 Find $\dfrac{dy}{dx}$ if $y = \sin^{-1}(3x-1)$.

Let $y = \sin^{-1} u$ where $u = 3x-1$

then $\dfrac{dy}{du} = \dfrac{1}{\sqrt{(1-u^2)}}$ $\dfrac{du}{dx} = 3$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{3}{\sqrt{(1-u^2)}} = \frac{3}{\sqrt{(1-\{3x-1\}^2)}} = \frac{3}{\sqrt{(6x-9x^2)}}.$$

To obtain more general integration results, we consider the functions $\sin^{-1}\dfrac{x}{a}$ and $\tan^{-1}\dfrac{x}{a}$.

Let $y = \sin^{-1} u$ where $u = \dfrac{x}{a}$

$$\frac{dy}{du} = \frac{1}{\sqrt{(1-u^2)}} \qquad \frac{du}{dx} = \frac{1}{a}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{\sqrt{(1-u^2)}}\cdot\frac{1}{a} = \frac{1}{a\sqrt{\left(1-\dfrac{x^2}{a^2}\right)}} = \frac{1}{\sqrt{(a^2-x^2)}}.$$

Hence $$\frac{d}{dx}\left(\sin^{-1}\frac{x}{a}\right) = \frac{1}{\sqrt{(a^2-x^2)}}.$$

Similarly, $$\frac{d}{dx}\left(\tan^{-1}\frac{x}{a}\right) = \frac{1}{\left(1+\dfrac{x^2}{a^2}\right)}\cdot\frac{1}{a} = \frac{a}{a^2\left(1+\dfrac{x^2}{a^2}\right)} = \frac{a}{a^2+x^2}.$$

Thus $$\boxed{\int\frac{dx}{\sqrt{(a^2-x^2)}} = \sin^{-1}\frac{x}{a} + c, \quad \int\frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c.}$$

Example 2 Evaluate $\displaystyle\int_0^3 \frac{dx}{\sqrt{(9-x^2)}}$.

$$\int_0^3 \frac{dx}{\sqrt{(9-x^2)}} = \left[\sin^{-1}\frac{x}{3}\right]_0^3 = \sin^{-1}1 - \sin^{-1}0 = \tfrac{1}{2}\pi.$$

Example 3 Find (a) $\displaystyle\int\frac{dx}{1+4x^2}$, (b) $\displaystyle\int\frac{dx}{\sqrt{(4-9x^2)}}$.

(a) [The fact that $4x^2 = (2x)^2$ suggests that the answer may involve $\tan^{-1} 2x$.]

$$\frac{d}{dx}(\tan^{-1} 2x) = \frac{1}{1+(2x)^2} \cdot 2 = \frac{2}{1+4x^2}$$

$$\therefore \quad \int \frac{dx}{1+4x^2} = \tfrac{1}{2}\tan^{-1} 2x + c.$$

(b) [The fact that $4-9x^2 = 2^2-(3x)^2$ suggests that the answer may involve $\sin^{-1}\frac{3x}{2}$.]

$$\frac{d}{dx}\left(\sin^{-1}\frac{3x}{2}\right) = \frac{1}{\sqrt{\left(1-\frac{9x^2}{4}\right)}} \cdot \frac{3}{2} = \frac{3}{\sqrt{\left\{4\left(1-\frac{9x^2}{4}\right)\right\}}} = \frac{3}{\sqrt{(4-9x^2)}}$$

$$\therefore \quad \int \frac{dx}{\sqrt{(4-9x^2)}} = \frac{1}{3}\sin^{-1}\frac{3x}{2} + c.$$

Exercise 11.6

Differentiate with respect to x:

1. $\sin^{-1}\frac{1}{2}x$.
2. $\tan^{-1} 3x$.
3. $\cos^{-1}(2x+1)$
4. $x\tan^{-1} x$.
5. $\sin^{-1}(\cos x)$.
6. $(1-x^2)^{1/2}\sin^{-1} x$.
7. $\sin^{-1}\left(\frac{1}{x}\right)$.
8. $\frac{\tan^{-1} 2x}{1+4x^2}$.
9. $\tan^{-1}\left(\frac{1+x}{1-x}\right)$.

Integrate the following functions:

10. $\frac{1}{\sqrt{(25-x^2)}}$.
11. $\frac{1}{9+x^2}$.
12. $\frac{1}{\sqrt{(1-4x^2)}}$.
13. $\frac{1}{1+9x^2}$.
14. $\frac{1}{\sqrt{(16-9x^2)}}$.
15. $\frac{1}{9+4x^2}$.

Evaluate the following definite integrals:

16. $\int_0^1 \frac{dx}{\sqrt{(4-x^2)}}$.
17. $\int_0^2 \frac{dx}{4+x^2}$.
18. $\int_{1/6}^{1/3} \frac{dx}{\sqrt{(1-9x^2)}}$.
19. $\int_0^{\sqrt{3}/5} \frac{dx}{1+25x^2}$.
20. $\int_0^{\sqrt{3}/4} \frac{dx}{16x^2+9}$.
21. $\int_{-3/4}^{0} \frac{dx}{\sqrt{(9-16x^2)}}$.

Exercise 11.7 (miscellaneous)

1. Two circles with centres A and B intersect at points P and Q, such that $\angle APB$ is a right angle. If $AB = x$ cm and $\angle PAQ = \frac{1}{3}\pi$ radians, find in terms of x the length of the perimeter and the area of the region common to the two circles.

2. The points A, B and C lie on a circle and $\angle ACB = \frac{1}{3}\pi$ radians. Show that AB divides the circle into major and minor segments whose areas are in the ratio $(8\pi + 3\sqrt{3}) : (4\pi - 3\sqrt{3})$.

3. Two parallel chords of lengths x cm and y cm lie on the same side of the centre of a circle with radius r cm. Construct a flow chart for a procedure to determine the perimeter and the area of the part of the circle between the two chords.

4. Solve, for $0 \leqslant \theta \leqslant 2\pi$, the equations
(a) $\cos 3\theta = \cos \theta$, (b) $\sin \theta + \sin 2\theta = \sin 3\theta$.

5. Find, in radians, the general solutions of the equations
(a) $\cos 2x = \sin x$, (b) $\cos x + \cos 3x + \cos 5x = 0$.

6. Find, in degrees, the general solution of the equation $6\cos\theta - 4\sin\theta = 5$.

7. Find the limit as $x \to 0$ of $\dfrac{\sin(x+\alpha) - \sin\alpha}{\sin 2x}$.

8. Differentiate the function $\cos x$ from first principles.

9. Write down the derivatives of $\sin x$ and $\cos x$. Hence find the derivatives of $\cot x$ and $\operatorname{cosec} x$.

10. Differentiate with respect to x
(a) $\sin^2(2x-5)$, (b) $x^4 \tan 4x$,
(c) $\dfrac{\sec x + \tan x}{\sec x - \tan x}$, (d) $\dfrac{\sin x}{\sqrt{(\cos 2x)}}$.

11. Sketch the following curves for $0 \leqslant x \leqslant 2\pi$, showing clearly the positions of any stationary points
(a) $y = 2\cos x + \sin 2x$, (b) $y = 3\sin x - \sin 3x$.

12. Find the following indefinite integrals
(a) $\int \cos(2x-1)\,dx$, (b) $\int \sin^2 \frac{1}{2}x\,dx$,
(c) $\int \tan x \sec^2 x\,dx$, (d) $\int \sin 4x \cos x\,dx$.

13. Evaluate (a) $\int_{\pi/6}^{\pi/2} \sin^5 x \cos x \, dx$, (b) $\int_0^{\pi/4} \cos x \cos 2x \cos 3x \, dx$.

14. Differentiate $x \sin x$ with respect to x and use your result to find $\int x \cos x \, dx$. By a similar method find $\int x^2 \sin x \, dx$.

15. A particle moves in a straight line such that after t seconds its acceleration is $a \, \mathrm{m\,s^{-2}}$, where $a = 8 \sin 2t$. If the velocity of the particle after $\pi/3$ seconds is $5 \, \mathrm{m\,s^{-1}}$, find its displacement from its initial position at that instant.

16. Given that R is the region in the first quadrant enclosed by the curves $y = 2\cos x$, $y = \sin 2x$ and the y-axis, find the volume of the solid formed by rotating R completely about the x-axis.

17. Use the identity $1 - \sin^2 x = \cos^2 x$ to find the integral $\int \frac{dx}{1 - \sin x}$. Hence find the mean value of the function $1/(1 - \sin x)$ over the interval $-\pi/3 \leqslant x \leqslant \pi/6$.

18. Differentiate with respect to x
(a) $x/\sin^{-1} x$, (b) $\tan^{-1}(\sec x)$, (c) $\cos(\sin^{-1} x)$.

19. Find (a) $\int \frac{dx}{\sqrt{(9 - x^2)}}$, (b) $\int \frac{dx}{9x^2 + 25}$.

20. Evaluate (a) $\int_0^1 \frac{dx}{1 + 3x^2}$, (b) $\int_{\sqrt{2}}^2 \frac{dx}{\sqrt{(8 - x^2)}}$.

21. Given that p and q are solutions of the equation $x \tan x = 1$,
(i) show that $\int_0^1 \cos px \cos qx \, dx = 0$ when $p \neq q$,
(ii) find an expression for $\int_0^1 \cos^2 px \, dx$ entirely in terms of p, not involving any trigonometric functions. (AEB 1976)

22. The equal sides AB and AC of an isosceles triangle have length a, and the angle at A is denoted by 2θ. Show that the radius of the inscribed circle of the triangle is given by $r = \frac{a \sin\theta \cos\theta}{1 + \sin\theta}$. Show that, if a is constant and θ varies between $0°$ and $90°$, the maximum value of r occurs when $\sin\theta = (\sqrt{5} - 1)/2$. [You need not verify that this value of θ gives a maximum rather than a minimum.] (W)

23. The points A and B are on the same horizontal level, and at a distance b apart. A particle P falls vertically from rest at B, so that, at time t, its depth below

B is kt^2, where k is constant. At this time, the angle of depression of P from A is θ. Prove that $\frac{d\theta}{dt} = \frac{2bkt}{b^2+k^2t^4}$. Show that $\frac{d\theta}{dt}$ is greatest (and not least) when $\theta = \frac{1}{6}\pi$. (C)

24. Differentiate $\tan^{-1}(x+a)$ with respect to x and use your result to find

(a) $\int \frac{dx}{x^2+4x+5}$, (b) $\int \frac{dx}{x^2+2x+5}$.

25. A rectangular sheet of paper $ABCD$ is folded about the line joining points P on AB and Q on AD so that the new position of A is on CD. If $AB = a$ and $AD = b$, where $a \geqslant 2b/\sqrt{3}$, show that the least possible area of the triangle APQ is obtained when the angle AQP is equal to $\pi/3$. What is the significance of the condition $a \geqslant 2b/\sqrt{3}$? (O)

12 Permutations and combinations

12.1 Permutations

A *permutation* is an arrangement of objects chosen from a given set. In this section we discuss ways of finding the total number of permutations of a set of objects under various conditions. For example, consider the number of ways in which we could arrange four guests in a row of four numbered chairs at a concert.

Any one of the 4 people may sit in the first chair. When the first person is seated, there are 3 people to be considered for the second place. This means that for each of the 4 ways of choosing the first person, there are 3 ways of choosing the second person to be seated. Similarly, in each of these $4 \times 3 = 12$ cases, we are left with 2 ways of choosing the occupant of the third chair. When 3 people are seated, the fourth chair must be filled by the 1 remaining person. Hence the total number of ways of seating the 4 people is $4.3.2.1$, i.e. 24.

Using the letters A, B, C, D to represent the 4 people, these 24 arrangements are as follows:

A B C D	B A C D	C A B D	D A B C
A B D C	B A D C	C A D B	D A C B
A C B D	B C A D	C B A D	D B A C
A C D B	B C D A	C B D A	D B C A
A D B C	B D A C	C D A B	D C A B
A D C B	B D C A	C D B A	D C B A

Since products of the type $4.3.2.1$ frequently arise in this type of work, the *factorial* notation is used. We write $4.3.2.1 = 4!$ which is read 'four factorial' or 'factorial four'. Thus for any positive integer n, $n! = n(n-1)(n-2)\ldots 3.2.1$.

The result of the above seating problem can now be generalised.

> The number of permutations or arrangements of n different objects is $n!$.

Example 1 In how many different ways can 4 people chosen from a set of 6 be seated in a row of four chairs?

There are 6 possible choices for the first chair, then 5 choices for the second, 4 for the third and 3 for the fourth. Hence the total number of ways of seating 4 people out of a set of six is 6.5.4.3, i.e. 360.
[In factorial notation this number is 6!/2!]

In general, the number of permutations or arrangements of r different objects chosen from a set of n objects, written ${}_nP_r$, is given by ${}_nP_r = \dfrac{n!}{(n-r)!}$.

When $r = n$ this expression becomes ${}_nP_n = n!/0!$, but as shown above ${}_nP_n = n!$. Hence, it is convenient to define 0! to be 1. This is equivalent to assuming that there is exactly one way of arranging a set containing no objects.

These methods can be extended to other types of problem.

Example 2 How many different three-digit numbers can be formed using the digits 0, 1, 2, 3, 4 (excluding numbers which begin with 0) if (a) no digit may be repeated, (b) repetitions are allowed?

(a) Since the first digit must not be 0, the number of ways of choosing the first digit is 4. As no digit may be repeated, there are 4 digits to choose from for the second position, then 3 for the third. Hence the total number of ways of forming the three-digit number is 4.4.3, i.e. 48.
(b) If repetitions are allowed, any of the 5 digits may be used in the second and third positions. Hence the total number of three-digit numbers is 4.5.5, i.e. 100.

Difficulties arise when some of the objects to be arranged are identical. For instance, let us consider the number of ways of arranging the letters of the word TOTTER.

If we label the Ts with suffixes, then for the word $T_1OT_2T_3ER$ there are 6!, i.e. 120 different arrangements. However, many of these would be indistinguishable with the suffixes removed, e.g. the set

$$RT_1ET_2T_3O, \quad RT_2ET_1T_3O, \quad RT_3ET_1T_2O,$$
$$RT_1ET_3T_2O, \quad RT_2ET_3T_1O, \quad RT_3ET_2T_1O.$$

Since T_1, T_2, T_3 can be arranged in 3!, i.e. 6 different ways, all 120 arrangements can be grouped into sets of 6, like the one already listed. Hence, with no suffixes, the number of different arrangements will be $6!/3! = 120/6 = 20$.

In general, the number of permutations of n objects, r of which are identical, is $n!/r!$.

Example 3 In how many ways can the letters of the word NECESSITIES be arranged?

The total number of letters in the word is 11.
The letters E and S each occur 3 times. The letter I occurs twice.

Hence the total number of arrangements $= \dfrac{11!}{3!3!2!} = 554\,400.$

Exercise 12.1

1. Without using a calculator, find the values of

(a) 7!, (b) $6!-5!$, (c) $\dfrac{8!}{4!}$, (d) $\dfrac{10!}{3!7!}$, (e) ${}_4P_1$, (f) ${}_8P_5$.

2. Express in factorial notation
(a) 5.4.3.2, (b) 10.9.8.7, (c) $n(n-1)(n-2)$.

3. Find the number of ways of arranging 6 different books on a shelf.

4. Find the number of arrangements of 8 items on a shopping list.

5. A lady has 8 house plants. In how many ways can she arrange 6 of them in a line on a window sill?

6. In a competition 6 household products chosen from 10 are to be listed in order of preference. In how many ways can this be done?

7. There are 25 entrants in a gymnastics competition. In how many different ways can the gold, silver and bronze medals be awarded?

8. Find the number of arrangements of 4 different letters chosen from the word PROBLEM which (a) begin with a vowel, (b) end with a consonant.

9. How many different four-digit numbers greater than 6000 can be formed using the digits 1, 2, 4, 5, 6, 8, if (a) no digit can be repeated, (b) repetitions are allowed?

10. In how many ways can 3 books be distributed among 10 people if (a) each person can receive any number of books, (b) nobody can be given more than 1 book, (c) nobody can be given more than 2 books?

11. Find the number of different ways in which the letters of the following words can be arranged
(a) NUMBER, (b) POSSIBLE, (c) PEPPER, (d) STATISTICS.

12. How many different six-digit numbers can be formed using the digits 2, 3, 3, 3, 4, 4? How many of these are even?

13. In how many different ways can 8 books be arranged on a shelf if 2 particular books must be placed next to each other?

14. Three girls and 4 boys are to sit in a row of 7 chairs. If the girls wish to sit in adjacent chairs, how many different arrangements are possible?

15. In how many ways can 6 children form a circle to play a game? If Joan refuses to stand next to Tom, how many possible arrangements are there?

16. Find the number of permutations of the word PARABOLA. In how many of these permutations are (a) all three As together, (b) no two As together?

17. How many numbers less than 3500 can be formed using one or more of the digits 1, 3, 5, 7, if (a) no digit can be repeated, (b) repetitions are allowed?

18. Find the total number of possible arrangements of 3 letters chosen from the word CALCULUS.

19. A set of 10 flags, 5 red, 3 blue and 2 yellow, are to be arranged in a line along a balcony. If flags of the same colour are indistinguishable, find the number of arrangements in which (a) the three blue flags are together, (b) the yellow flags are not together, (c) the red flags occupy alternate positions in the line. If there is room for only 9 of the flags, find the total number of possible arrangements.

20. On a shelf there are 4 saucers of different colours and 4 matching cups. In how many ways can the cups be arranged on the saucers, so that no cup is on a matching saucer?

12.2 Combinations

In some problems the order in which objects are arranged is not important. The number of *combinations* of r different objects out of a set of n is the number of different selections, irrespective of order, and is denoted by ${}_nC_r$, or sometimes $\binom{n}{r}$.

Let us suppose that a set of 3 cards is to be dealt from a pack of 52 playing cards. The number of permutations of 3 cards chosen from $52 = 52!/49!$ $= 52.51.50$. However, since any set of 3 cards can be arranged in 3! different ways, each set of 3 cards will appear 6 times in the list of all possible arrangements. For instance, the set containing cards X, Y and Z will appear as XYZ, XZY, YXZ, YZX, ZXY and ZYX. Hence the total number of ways of selecting 3 cards $= \dfrac{52!}{3!49!} = \dfrac{52.51.50}{3.2.1} = 22\,100.$

In general, the number of ways of selecting r objects from n unlike objects $= {}_nC_r = \dfrac{n!}{r!(n-r)!}$.

Example 1 A committee of 5 is to be formed from 12 men and 8 women. In how many ways can the committee be chosen so that there are 3 men and 2 women on it?

Number of ways of choosing 3 men from 12

$$= {}_{12}C_3 = \frac{12!}{3!9!} = \frac{12.11.10}{3.2.1} = 220.$$

Number of ways of choosing 2 women from 8

$$= {}_8C_2 = \frac{8!}{2!6!} = \frac{8.7}{2.1} = 28.$$

$\therefore$ the total number of ways of forming the committee

$$= {}_{12}C_3 \times {}_8C_2 = 220 \times 28 = 6160.$$

There is no general formula for dealing with selections made from sets containing objects which are not all different, as there was for permutations of such sets.

Example 2 How many ways are there of selecting 4 letters from the letters in the word TOTTER?

Number of selections containing one T and three other letters = 1.
Number of selections containing two Ts and two letters from the remaining three $= {}_3C_2 = 3$.
Number of selections containing three Ts and one letter from the remaining three $= {}_3C_1 = 3$.
Hence the total number of ways of selecting 4 letters from TOTTER is 7.

When we select r objects from n, we are dividing the set containing n objects into 2 sets containing r and $n-r$ objects respectively. It is not, therefore, surprising to find that the number of ways of selecting $n-r$ objects from n unlike objects is the same as the number of ways of selecting r objects.

$$ {}_nC_{n-r} = \frac{n!}{(n-r)!(n-\{n-r\})!} = \frac{n!}{(n-r)!r!} = {}_nC_r.$$

Another identity involving the quantity ${}_nC_r$ is proved in the next example.

Example 3 Prove that ${}_nC_r + {}_nC_{r-1} = {}_{n+1}C_r$.

$$ {}_nC_r + {}_nC_{r-1} = \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-\{r-1\})!}$$

$$= \frac{n!}{r!(n-r+1)!}\{(n-r+1)+r\}$$

$$= \frac{(n+1)!}{r!(n+1-r)!} = {}_{n+1}C_r.$$

Exercise 12.2

1. Evaluate (a) ${}_7C_4$, (b) ${}_5C_4$, (c) ${}_6C_1$, (d) ${}_8C_0$.

2. Verify that ${}_nC_{n-r} = {}_nC_r$ for the cases
(a) $n = 8, r = 5$; (b) $n = 10, r = 2$.

3. In how many ways can
(a) 4 photographs be chosen from 10 proofs,
(b) 3 representatives be chosen from 20 students,
(c) a hand of 5 cards be dealt from a set of 13,
(d) 11 players be selected from 12 cricketers?

4. Nine people are to go on a journey in cars which can take 2, 3 and 4 passengers respectively. In how many different ways can the party travel, assuming that the seating arrangements inside the cars are not important?

5. In how many ways can a set of 12 unlike objects be divided into (a) 2 sets of 6, (b) 3 sets of 4, (c) 6 sets of 2?

6. A team of 5 students, including a captain and a reserve, is to be selected for a general knowledge contest. In how many ways can the team be chosen from a short list of 12?

7. Find the number of different selections of two letters which can be made from the letters of the word PROBABILITY. How many of these selections do not contain a vowel?

8. A chess team of 5 players is to be selected from 15 boys. In how many ways can the team be chosen if (a) no more than one of the three best players is to be included, (b) at least one of the four youngest players is to be included?

9. Find the number of ways in which 8 books can be distributed to 2 boys, if each boy is to receive at least 2 books.

10. Find the number of different selections of three letters which can be made from the letters of the word PARALLELOGRAM. How many of these selections contain the letter P?

11. A tennis team of 6 players consists of a 1st pair, a 2nd pair and a 3rd pair. In how many ways can the team be selected, if there are (a) only 6 players available, (b) 9 players available?

12. A committee of 6 is to be formed from 13 men and 7 women. In how many ways can the committee be selected given that (a) it must consist of 4 men and 2 women, (b) it must have at least one member of each sex?

12.3 Elementary theory of probability

In mathematics probability is the numerical value assigned to the likelihood that a particular event will take place. For instance, if we throw an unbiased die, we have equal chances of scoring any of the numbers 1, 2, 3, 4, 5 and 6. Since there is one chance in six of throwing a 3 the probability of the event occurring is said to be 1/6. Similarly when tossing a coin the probability that it lands heads is considered to be 1/2. This does not, of course, mean that in any two tosses we expect the coin to fall heads once, only that in a long series of tosses the number of heads will be approximately half the total number of throws.

To arrive at a formal definition, consider the set S of all possible *outcomes* of an experiment or *trial*. A set of this kind is sometimes called a *possibility space* or *sample space*. Any *event* A can be represented by the subset A of S, which contains all the outcomes in which the event occurs. If S contains a finite number of *equally likely* outcomes, then the probability $P(A)$ that the event A will occur is given by

$$P(A) = \frac{n(A)}{n(S)}, \quad \text{i.e.} \quad \frac{\text{no. of favourable outcomes}}{\text{no. of possible outcomes}}.$$

If the event A is impossible, then $A = \varnothing$ and $P(A) = 0$. However, if the event A is certain to occur, then $A = S$ and $P(A) = 1$. Otherwise $P(A)$ will take some value between 0 and 1.

[When probabilities are calculated from experimental data, a definition such as the following may be used.

$$\text{Estimated probability} = \frac{\text{no. of successes}}{\text{no. of trials}}.$$

However, this value for the probability could be unreliable unless large numbers of observations are used.]

Example 1 What is the probability of drawing an ace at random from a pack of cards?

Since there are 4 aces in a pack of 52 cards, the probability of drawing an ace is 4/52, i.e. 1/13.

Example 2 If a letter is chosen at random from the word FACETIOUS, what is the probability that it is a vowel?

Since there are 5 vowels out of a total of 9 letters, the probability of choosing a vowel is 5/9.

Returning to the problem of throwing an unbiased die, the probability of throwing a 3 is 1/6, but the probability of not throwing a 3 is 5/6. We notice that the sum of these probabilities is 1. More generally, in a set S of equally likely

possible outcomes, if A' denotes the subset of outcomes in which the event A does not occur, then A and A' are called *complementary* events and

$$P(A') = \frac{n(A')}{n(S)} = \frac{n(S) - n(A)}{n(S)} = 1 - \frac{n(A)}{n(S)}$$

$$\therefore \quad \boxed{P(A') = 1 - P(A).}$$

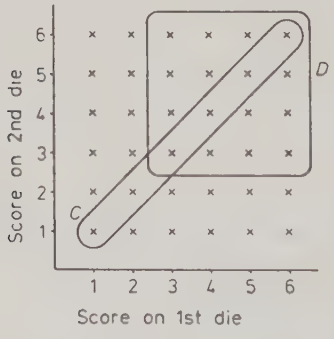
Hence, it seems reasonable to assume that in any sample space, if p is the probability that an event occurs and q is the probability that the event does not occur, then $p + q = 1$.

For experiments in which a trial is a throw of two or more dice, we can construct the sample space from the set of outcomes for a single die, $S = \{1, 2, 3, 4, 5, 6\}$. For instance, in the two-dice case, the possible outcomes can be expressed as ordered pairs such as $(1, 3)$, $(4, 6)$ and $(6, 4)$ formed from the elements of S.

A *sample space diagram* can be used to find probabilities as shown in the next example.

Example 3 If two dice are thrown together, what is the probability of the following events?

A: scoring a total of 2
B: scoring a total of 3
C: the same score on both dice
D: 3 or more on each die.



The diagram shows that when two dice are thrown, there are 36 possible outcomes. A score of 2 is obtained only when the score on each die is 1

$$\therefore \quad P(A) = \frac{1}{36}.$$

There are two outcomes $(1, 2)$ and $(2, 1)$ which produce a score of 3

$$\therefore \quad P(B) = \frac{2}{36} = \frac{1}{18}.$$

The set of outcomes in which both dice show the same score has 6 elements, as indicated in the diagram

$$\therefore \quad P(C) = \frac{6}{36} = \frac{1}{6}.$$

The diagram also shows that there are 16 outcomes in which both dice show 3 or more

$$\therefore \quad P(D) = \frac{16}{36} = \frac{4}{9}.$$

Example 4 If 4 cards are selected at random from a pack of 52, what is the probability that exactly 3 of them are diamonds?

The number of ways of selecting 3 diamonds from 13 $= {}_{13}C_3$.
The number of ways of selecting 1 card from the 39 which are not diamonds $= 39$.
$\therefore$ the number of ways of selecting 4 cards, 3 of which are diamonds $= 39 \times {}_{13}C_3$.
The number of ways of selecting any 4 cards from 52 $= {}_{52}C_4$.
Hence the probability that exactly 3 cards out of 4 are diamonds

$$= 39 \times {}_{13}C_3 \div {}_{52}C_4 = 39 \times \frac{13!}{3!10!} \times \frac{4!48!}{52!} \approx 0{\cdot}0412.$$

For a trial in which the possible outcomes are not equally likely, it is more difficult to arrive at a precise definition of the probability of an event. However, the subsets of a sample space S are regarded as events whatever the exact nature of the set S. In all cases $P(A)$, the probability of an event A, is a number or a weighting associated with the subset A. We can consider P to be a *probability mapping* which maps any subset A onto a real number p, where $0 \leqslant p \leqslant 1$. In any experiment the set of events (i.e. the set of subsets of the sample space) to be considered together with the appropriate probability mapping may be referred to as the *probability space*.

Exercise 12.3

In questions 1 to 4 find the probability of events A, B and C.

1. A die is thrown. A: scoring a four, B: scoring an odd number, C: scoring four or less.

2. A card is drawn from a pack. A: drawing a red card, B: drawing a seven, C: drawing a king, a queen or a jack.

3. Three coins are tossed. A: three heads, B: at least two tails, C: at least one of each.

4. It rained on exactly two days last week. A: it rained on Monday and Tuesday, B: it rained on two consecutive days, C: it rained on neither Monday nor Tuesday.

5. Two unbiased dice are thrown. Find the probability that the product of the scores is (a) odd, (b) a multiple of 3, (c) a multiple of 12.

6. A domino is drawn from a standard set of 28. Find the probability that the sum of its spots is (a) 2, (b) 8, (c) odd, (d) even.

7. An integer is chosen at random from the first 200 positive integers. Find the probability that it is (a) not divisible by 5, (b) a perfect square, (c) divisible by both 5 and 2, (d) divisible by neither 2 nor 7.

8. In a cafeteria 80% of the customers order chips and 60% order peas. If 20% of those ordering peas do not want chips, find the probability that a customer chosen at random orders chips but not peas.

9. The points A, B, C, D and E are the vertices of a regular pentagon. All possible lines joining pairs of these points are drawn. If two of these lines are chosen at random, that is the probability that their point of intersection is (a) inside the pentagon, (b) one of the points A, B, C, D, E.

10. A box contains 4 discs numbered 1, 2, 2, 3. A disc is drawn from the box then replaced and a second disc is drawn. With the aid of a sample space diagram or otherwise, find the probability that (a) the total score is 6, (b) the total score is 4, (c) the numbers drawn are different, (d) the difference between the two numbers drawn is less than two.

11. Two cards are selected at random without replacement from a set of five numbered 2, 3, 4, 4, 5. With the aid of a diagram or otherwise, find the probability that the numbers on the cards (a) are both even, (b) have a difference of 2, (c) have a sum of 7, (d) have a sum of 8.

12. Write down in factorial form the number of different arrangements of the letters of the word EQUILIBRIUM. One of these arrangements is chosen at random. Find the probability that (a) the first two letters of the arrangement are consonants, (b) all the vowels are together.

13. A hand of three cards is dealt from a well shuffled pack of 52. Find the probability that the hand contains (a) exactly one ace, (b) three cards of the same suit, (c) no two cards of the same suit.

14. On a plate of 12 assorted cakes, 3 are doughnuts. If 3 cakes are selected at random, find the probabilities that 0, 1, 2, 3 doughnuts are chosen.

15. A bag contains 25 clothes pegs, 15 plastic pegs and 10 wooden pegs. If 4 pegs are taken from the bag at random, find the probability that (a) all 4 are plastic, (b) 2 are plastic and 2 are wooden.

16. Four books are selected at random from a shelf containing 3 cookery books, 5 novels and 2 biographies. Find the probability that the four books (a) are all novels, (b) are 2 novels and 2 cookery books, (c) include at least one biography, (d) include at least one of each type.

Exercise 12.4 (miscellaneous)

1. How many even five-digit numbers can be formed using the digits 0, 1, 2, 3, 4, 5, 6 (excluding numbers which start with 0) if (a) no digit may be repeated, (b) repetition is allowed?

2. Find the number of different ways in which the letters of the word ISOSCELES can be arranged. How many of these arrangements (a) begin and end with E, (b) begin with S, (c) end with a vowel?

3. A man has found 6 records that he likes in a shop, but cannot decide which of them to buy. Assuming that he buys at least one, find the number of different selections he could make.

4. Six lines are drawn, no two of which are parallel. If no more than two of the lines pass through any one point, find the number of triangles formed.

5. In how many ways can a group of 9 people attending a conference be split into three sets of 3 people for discussion? Later the same people form new sets of 3. In how many ways can this be done if no two people remain together?

6. Find the number of ways in which 5 books can be distributed between three people A, B and C if the books are (a) all different, (b) indistinguishable.

7. A drawer contains 4 different pairs of socks. Find the probability that (a) if 2 socks are selected at random they will form a pair, (b) if 4 socks are selected at random they will form two pairs.

8. (a) Two dice are thrown together, and the scores added. What is the probability that (i) the total score exceeds 8, (ii) the total score is 9, or the individual scores differ by 1, or both?
(b) A bag contains 3 red balls and 4 black ones. 3 balls are picked out, one at a time and not replaced. What is the probability that there will be 2 red and 1 black in the sample?
(c) A committee of 4 is to be chosen from 6 men and 5 women. One particular man and one particular woman refuse to serve if the other person is on the committee. How many different committees may be formed? (SU)

9. Two cards are drawn simultaneously from a pack of 52. What is the probability that both are spades if one card is known to be (a) black, (b) red, (c) a spade, (d) a king?

10. A pack of 52 cards contains 4 suits each of 13 cards. If 13 cards are taken at random from the pack what is the probability that exactly 10 of them are spades? [You may take $\binom{52}{13} = 6{\cdot}35 \times 10^{11}$.] (O&C)

13 Series and the binomial theorem

13.1 Sequences and series

Below are some sets of numbers given in a definite order. In each case the numbers are produced according to some simple rule.

(a) 2, 4, 6, 8, 10, ...
(b) 1, 4, 9, 16, 25, ...
(c) 1, −1, 1, −1, 1, ...
(d) 8, −4, 2, −1, $\frac{1}{2}$, ...
(e) 10, 11, 9, 12, 8, 13, ...
(f) 1, 1, 2, 3, 5, 8, ...

These sets of numbers are called *sequences*. Each member of a sequence is a *term*. For instance, in (a) 6 is the third term of the sequence.

One way of stating the rule by which a sequence is obtained is to write down a formula for the *nth term*, often denoted by u_n. For sequences (a), (b) and (c) given above we may write,

(a) $u_n = 2n$ and $u_6 = 12, u_7 = 14$,
(b) $u_n = n^2$ and $u_6 = 36, u_7 = 49$,
(c) $u_n = (-1)^{n+1}$ and $u_6 = -1, u_7 = 1$.

In some cases the formula for u_n is not obvious. In (d) to obtain each successive term we divide by 2, then change the sign. Thus, at each stage we are multiplying by $-\frac{1}{2}$, which means that

$$u_n = 8 . (-\tfrac{1}{2})^{n-1}, \text{ so that } u_6 = 8 . (-\tfrac{1}{2})^5 = -\frac{8}{2^5} = -\tfrac{1}{4}.$$

In (e) a general formula for u_n would be difficult to find, but we can give separate formulae for odd and even terms of the sequence.

$u_{2n+1} = 10 - n$ and for $n = 3, u_7 = 7$,
$u_{2n} \quad = 10 + n$ and for $n = 4, u_8 = 14$.

In (f) each term is the sum of the two previous terms. This can be expressed formally as a relation between u_n and u_{n-1}, u_{n-2}:

$$u_n = u_{n-1} + u_{n-2} \qquad (n \geqslant 3).$$

In this case more advanced techniques are needed to find a formula for u_n in terms of n.

If a sequence ends after a certain number of terms it is said to be *finite*. A sequence which continues indefinitely is said to be *infinite*. The sequence $1, 3, 5, \ldots, 2n-1$ is finite with n terms, whereas $1, 3, 5, \ldots, 2n-1, \ldots$ denotes an infinite sequence.

A sum of a sequence of numbers is called a *series*. $1+3+5+\ldots+97+99$ is a finite series and $1+3+5+\ldots$ indicates an infinite series. The nth term of both these series is $u_n = 2n-1$, where $1 \leqslant n \leqslant 50$ in the finite case.

S_n is used to denote the sum of the first n terms of a series, so that $S_n = u_1 + u_2 + \ldots + u_n$.

Example 1 Find S_1, S_5 and S_6 for the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$.

S_1 is simply the first term $\quad \therefore \quad S_1 = 1$

$S_5 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \quad \therefore \quad S_5 = 1\frac{15}{16}$

$S_6 = S_5 + \frac{1}{32} = 1\frac{15}{16} + \frac{1}{32} \quad \therefore \quad S_6 = 1\frac{31}{32}$.

Example 2 The sum of the first n terms of a series is given by $S_n = n^2 + n$. Write down the first three terms of the series and find an expression for the nth term, u_n.

$S_1 = 1^2 + 1 = 2, \quad S_2 = 2^2 + 2 = 6, \quad S_3 = 3^2 + 3 = 12,$

$\therefore$ the first three terms of the series must be $2+4+6+\ldots$.

$S_n = n^2 + n, \quad S_{n-1} = (n-1)^2 + (n-1) = n^2 - n$

$\therefore \quad u_n = S_n - S_{n-1} = n^2 + n - (n^2 - n) = 2n.$

Hence the nth term of the series is $2n$.

Exercise 13.1

In questions 1 to 10 write down the next two terms of the given sequence and an expression for the nth term.

1. $5, 10, 15, 20, \ldots$
2. $4, 7, 10, 13, \ldots$
3. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$
4. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots$
5. $1, 2, 4, 8, \ldots$
6. $0, 3, 8, 15, 24, \ldots$
7. $\frac{1}{12}, \frac{1}{6}, \frac{1}{3}, \frac{2}{3}, \cdots$
8. $\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \cdots$

9. $1, -2, 3, -4, \ldots$

10. $11, 8, 13, 6, 15, \ldots$

In questions 11 to 16 find the first three terms and the nth term of the series with the given sum to n terms.

11. $S_n = 4+7n^2$.

12. $S_n = 3n-7n^3$.

13. $S_n = 2^n$.

14. $S_n = \frac{1}{n}$.

15. $S_n = \frac{1}{4}n^2(n+1)^2$.

16. $S_n = \frac{1}{6}n(n+1)(2n+1)$.

17. Given that $1^3+2^3+3^3+\ldots+n^3 = \frac{1}{4}n^2(n+1)^2$, find the sum of the first 20 terms of the series $2+16+54+128+250+\ldots$.

18. Given that $1^2+2^2+3^2+\ldots+n^2 = \frac{1}{6}n(n+1)(2n+1)$, find the sum of the series $2^2+4^2+6^2+\ldots+50^2$. Use your result to find the sum of the series $1^2+3^2+5^2+\ldots+49^2$.

13.2 Arithmetic progressions

An *arithmetic progression* is a series in which one term is obtained from the previous term by adding a fixed number.

For example: (a) $1+2+3+4+\ldots+98+99$,
(b) $6+10+14+18+\ldots+46+50$,
(c) $10+7+4+1+\ldots-47-50$.

This fixed number is called the *common difference*. In the above examples the common differences are 1, 4 and -3 respectively.

An arithmetic progression is completely defined when the first term a and the common difference d are given:

$$a+(a+d)+(a+2d)+\ldots.$$

Example 1 Write down the first three terms, the 10th term and the nth term of the A.P. (arithmetic progression) with first term -20 and common difference 3.

The first three terms are -20, -17 and -14.
The 10th term, $u_{10} = -20+(3\times 9) = 7$.
The nth term, $u_n = -20+3(n-1) = 3n-23$.

More generally, the nth term of the A.P. with first term a and common difference d is $\boxed{u_n = a+(n-1)d.}$

To illustrate the general approach to the summation of arithmetic progressions, we consider the following:

$$\begin{array}{r} 1+\ 2+\ 3+\ldots+\ 98+\ 99 \\ 99+\ 98+\ 97+\ldots+\ 2+\ 1 \\ \hline 100+100+100+\ldots+100+100 \end{array}$$

Since there are 99 columns,

$$2(1+2+3+\ldots+99) = 99\times 100.$$

Hence $1+2+3+\ldots+99 = 4950.$

Applying this method to an A.P. with n terms, first term a, last term l and common difference d:

$$S_n = a+(a+d)+\ldots+(l-d)+l$$
$$S_n = l+(l-d)+\ldots+(a+d)+a$$
$$\therefore\quad 2S_n = (a+l)+(a+l)+\ldots+(a+l)+(a+l) = n(a+l)$$

$\therefore$ the sum of the first n terms, $\boxed{S_n = \frac{1}{2}n(a+l).}$

As l is the nth term of the series, $l = a+(n-1)d$

$$\therefore\quad S_n = \tfrac{1}{2}n\{a+a+(n-1)d\} = \tfrac{1}{2}n\{2a+(n-1)d\}.$$

Hence the sum of the first n terms of an arithmetic progression with first term a and common difference d is $\frac{1}{2}n\{2a+(n-1)d\}$.

Example 2 Find the sums of the series

(a) $6+10+14+\ldots+50,$ (b) $10+7+4+\ldots-50.$

(a) The series has first term 6, common difference 4

$\therefore$ the number of terms $= \dfrac{50-6}{4}+1 = 12.$

Hence the sum of the series $= \frac{1}{2}.12(6+50) = 336.$

(b) The series has first term 10, common difference -3

$\therefore$ the number of terms $= \dfrac{-50-10}{-3}+1 = 21.$

Hence the sum of the series $= \frac{1}{2}.21(10-50) = -420.$

Example 3 Find the sum of the first 20 terms of the A.P. with first term 3 and common difference $\frac{1}{2}$.

$$S_{20} = \tfrac{1}{2}.20(2.3+19.\tfrac{1}{2}) = 10\left(6+\frac{19}{2}\right) = 60+95 = 155$$

$\therefore$ the sum of the first twenty terms is 155.

Example 4 In an A.P. the sum of the first 10 terms is 520 and the 7th term is double the 3rd term. Find the first term a and the common difference d.

$S_{10} = \frac{1}{2}.10(2a+9d) = 520 \quad \therefore \quad 2a+9d = 104.$ (1)

The 7th term, $u_7 = a+6d$, and the 3rd term, $u_3 = a+2d$

$\therefore \quad a+6d = 2(a+2d)$, i.e. $2d = a$ (2)

Substituting in (1) $13d = 104 \quad \therefore \quad d = 8$
Substituting in (2) $a = 16.$

Hence the first term of the A.P. is 16 and the common difference is 8.

If three numbers a, b and c are in arithmetic progression, then b is called the *arithmetic mean* of a and c.

The common difference $= b-a = c-b, \quad \therefore \quad b = \frac{1}{2}(a+c)$. Thus the arithmetic mean of any two numbers, p and q, is the 'average', $\frac{1}{2}(p+q)$.

Exercise 13.2

1. Write down the stated term and the nth term of the following A.P.s
(a) $7+11+15+\ldots$ (7th), (b) $18+11+4+\ldots$ (6th),
(c) $-7-5-3-\ldots$ (23rd), (d) $3+3\frac{1}{2}+4+\ldots$ (16th).

2. Find the sums of the following series
(a) $5+9+13+\ldots+101.$ (b) $83+80+77+\ldots+5,$
(c) $-17-12-7-\ldots+33,$ (d) $1+1\frac{1}{4}+1\frac{1}{2}+\ldots+9\frac{3}{4}.$

3. Find the sums of the following A.P.s
(a) $4+11+\ldots$ to 16 terms, (b) $3+8\frac{1}{2}+\ldots$ to 20 terms,
(c) $19+13+\ldots$ to 10 terms, (d) $-9-1+\ldots$ to 8 terms.

4. Find the sum of the A.P. $-7-3+1+\ldots$ from the seventh to the thirtieth term inclusive.

5. Find the sum of all odd numbers between 0 and 500 which are divisible by 7.

6. Show that the sum $1+3+5+\ldots+(2n-1)$ is always a perfect square.

7. The first and last terms of an A.P. with 25 terms are 29 and 179. Find the sum of the series and its common difference.

8. The rth term of a series is $10-3r$. Find the first three terms of the series and the sum of the first 18 terms.

9. Given that the first and third terms of an A.P. are 13 and 25 respectively, find the 100th term and the sum of the first 15 terms.

10. A piece of string of length 5 m is cut into n pieces in such a way that the lengths of the pieces are in arithmetic progression. If the lengths of the longest and the shortest pieces are 1 m and 25 cm respectively, calculate n.

11. The second and seventh terms of an A.P. are -5 and 10 respectively. Find the fifth term and the least number of terms that must be taken for their sum to exceed 200.

12. The tenth term of an A.P. is 10 and the sum of the first 10 terms is -35. Find the first term and the common difference of the progression.

13. The sum of the first four terms of an A.P. is twice the fifth term. Show that the common difference is equal to the first term.

14. In an A.P. the sum of the first 15 terms is 615 and the 13th term is six times the 2nd term. Find the first three terms.

15. Find the arithmetic mean of

(a) 3 and 27, (b) 3 and -27, (c) $\frac{1}{3}$ and $\frac{1}{27}$, (d) $\lg 3$ and $\lg 27$.

16. Three numbers in A.P. have sum 33 and product 1232. Find the numbers.

17. The sum of three numbers in A.P. is 30 and the sum of their squares is 398. Find the numbers.

18. The sum of the first n terms of a certain series is $3n^2+n$. Show that the series is an A.P. and find the first term and the common difference.

19. Show that the sum to 20 terms of the series

$$\log a+\log(ab)+\log(ab^2)+\log(ab^3)+\ldots$$

can be written in the form $\log(a^x b^y)$ and find the values of x and y.

20. In an A.P. the sum of the first $2n$ terms is equal to the sum of the next n terms. If the first term is 12 and the common difference is 3, find the non-zero value of n.

13.3 Geometric progressions

A *geometric progression* (G.P.) is a series in which any term is obtained from the previous term by multiplying by a fixed number.

For example: (a) $1+2+4+8+\ldots+128+256$,

(b) $27-9+3-1+\ldots+\frac{1}{27}-\frac{1}{81}$.

This fixed number is called the *common ratio*. In the above examples the common ratios are 2 and $-\frac{1}{3}$ respectively.

A geometric progression is completely defined when the first term a and the common ratio r are given:

$$a+ar+ar^2+ar^3+\ldots.$$

The nth term of this G.P. is $\boxed{u_n = ar^{n-1}.}$

Example 1 Write down the formula for the nth term and find the number of terms in series (a) and (b) above.

(a) The nth term $= 1\times 2^{n-1} = 2^{n-1}$.

If $2^{n-1} = 256 = 2^8$, then $n = 9$

$\therefore$ the series has 9 terms.

(b) The nth term $= 27\times(-\frac{1}{3})^{n-1}$

If $27\times(-\frac{1}{3})^{n-1} = -\dfrac{1}{81}$, then $(-\frac{1}{3})^{n-1} = -\dfrac{1}{3^7} = (-\frac{1}{3})^7$

$\therefore$ $n = 8$ and the series must have 8 terms.

We illustrate the method by which G.P.s are summed by considering the sum S of series (a).

$$\begin{aligned} S &= 1+2+4+\ldots+256 \\ 2S &= \quad 2+4+8+\ldots\quad +512. \end{aligned}$$

Subtracting:
$$\begin{aligned} 2S-S &= 512-1 \\ \therefore\ S &= 511. \end{aligned}$$

Applying this method to a G.P. with n terms, first term a and common ratio r, we have

$$\begin{aligned} S_n &= a+ar+ar^2+\ldots+ar^{n-1} \\ rS_n &= \quad ar+ar^2+\ldots+ar^{n-1}+ar^n. \end{aligned}$$

Subtracting:
$$\begin{aligned} S_n-rS_n &= a-ar^n \\ S_n(1-r) &= a(1-r^n) \end{aligned}$$

$$\therefore\ S_n = \frac{a(1-r^n)}{1-r} \qquad (r \neq 1)$$

$\therefore$ the sum of the first n terms of a geometric progression with first term a and common ratio r is given by

$$\boxed{S_n = \frac{a(1-r^n)}{1-r} \quad \text{or} \quad S_n = \frac{a(r^n-1)}{r-1}.}$$

[The second expression is more convenient if $r > 1$.]

Example 2 Find the sum of the first 6 terms of the series
(a) $2-6+18-\ldots$ (b) $14+7+3\frac{1}{2}+\ldots$

(a) The series is a G.P. with first term 2 and common ratio -3

$$\therefore \quad \text{the sum} = \frac{2\{1-(-3)^6\}}{1-(-3)} = \frac{2(1-729)}{4} = -364.$$

(b) The series is a G.P. with first term 14 and common ratio $\frac{1}{2}$

$$\therefore \quad \text{the sum} = \frac{14\{1-(\frac{1}{2})^6\}}{1-\frac{1}{2}} = \frac{14(1-1/64)}{\frac{1}{2}} = 14.2.\frac{63}{64} = \frac{441}{16}.$$

Example 3 A G.P. has first term 10 and common ratio 1·5. How many terms of the series are needed to reach a sum greater than 200?

$$\text{The sum of } n \text{ terms} = \frac{10\{(1{\cdot}5)^n-1\}}{1{\cdot}5-1} = 20\{(1{\cdot}5)^n-1\}$$

$$\begin{aligned} 20\{(1{\cdot}5)^n-1\} > 200 &\Rightarrow (1{\cdot}5)^n-1 > 10 \\ &\Rightarrow \quad (1{\cdot}5)^n > 11 \\ &\Rightarrow \lg(1{\cdot}5)^n > \lg 11 \\ &\Rightarrow \quad n\lg 1{\cdot}5 > \lg 11 \\ &\Rightarrow \qquad n > \frac{\lg 11}{\lg 1{\cdot}5} \approx 5{\cdot}9. \end{aligned}$$

Hence 6 terms are needed to reach a sum greater than 200.

If three positive numbers a, b and c are in geometric progression, then b is called the *geometric mean* of a and c.

$$\text{The common ratio} = \frac{b}{a} = \frac{c}{b} \qquad \therefore \quad b^2 = ac.$$

Hence the geometric mean of two numbers, p and q, is $\sqrt{(pq)}$.

Exercise 13.3

1. Write down the stated term and the nth term of the following G.P.s
(a) $\frac{1}{2}+1+2+\ldots$ (8th), (b) $162+54+18+\ldots$ (6th),
(c) $200-50+12\frac{1}{2}-\ldots$ (5th), (d) $-\frac{4}{9}-\frac{2}{3}-1-\ldots$ (7th).

2. Find the number of terms in each of the following G.P.s and the sum of the series.
(a) $\frac{1}{4}+\frac{1}{2}+\ldots+64$, (b) $\frac{1}{4}-\frac{1}{2}+\ldots+64$,
(c) $1000+200+\ldots+0{\cdot}32$, (d) $2-3+\ldots+22\frac{25}{32}$.

3. Find the sums of the following G.P.s
(a) $100+10+\ldots$ to 7 terms, (b) $1-\frac{1}{3}+\ldots$ to 6 terms,
(c) $3-6+\ldots$ to n terms, (d) $a^p+a^{p+3}+a^{p+6}+\ldots$ to k terms.

4. The first term of a G.P. with positive terms is 80. If the sum of the first three terms is 185, find the common ratio.

5. Find two distinct numbers p and q such that p, q, 10 are in arithmetic progression and q, p, 10 are in geometric progression.

6. Find the geometric mean of

(a) 3 and 27, (b) $\frac{1}{3}$ and $\frac{1}{27}$, (c) 10^3 and 10^{27}.

7. Given that the geometric mean of the numbers $4x-3$ and $9x+4$ is $6x-1$, find the value of x.

8. The second and fifth terms in a G.P. are 405 and -120 respectively. Find the seventh term and the sum of the first seven terms.

9. In a G.P. the second term exceeds the first by 20 and the fourth term exceeds the second by 15. Find the two possible values of the first term.

10. If the sum of the first two terms of a G.P. is 162 and the sum of its first four terms is 180, find the sum of the first six terms. Find also the two possible values of the sixth term.

11. The sum of the first six terms of a G.P. is nine times the sum of the first three terms. Find the common ratio.

12. The sum of $(n+12)$ terms of the G.P. $2+4+8+\ldots$ is twice the sum of n terms of the G.P. $3+12+48+\ldots$. Calculate the value of n.

13. The sum of the first seven terms of a G.P. is 7 and the sum of the next seven terms is 896. Find the common ratio of the progression. If the kth term is the first term of the G.P. which is greater than 1, find k.

14. Find the sum of the first n terms of the G.P. $\frac{1}{12}+\frac{1}{4}+\frac{3}{4}+\ldots$. How many terms of the series are needed to reach a sum greater than 100?

15. A G.P. has first term 16 and common ratio $\frac{3}{4}$. If the sum of the first n terms is greater than 60, find the least possible value of n.

13.4 Infinite geometric series

Consider the infinite geometric progression (or geometric series)

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\left(\frac{1}{2}\right)^{n-1}+\ldots.$$

The sum of the first n terms is

$$S_n = \frac{1\{1-(\frac{1}{2})^n\}}{1-\frac{1}{2}} = 2\{1-(\tfrac{1}{2})^n\} = 2 - \frac{1}{2^{n-1}}.$$

As n increases, $\frac{1}{2^{n-1}}$ approaches 0 and S_n takes values closer and closer to 2.

Thus as $n \to \infty$, $1/2^{n-1} \to 0$ and $S_n \to 2$.

Since S_n approaches a finite limit, as n increases, the infinite series is said to be *convergent* with sum 2.

More generally, for a geometric series with first term a and common ratio r,

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - r^n\left(\frac{a}{1-r}\right).$$

The value of S_n, as n increases, depends on the value of r^n. If

$|r| < 1$, i.e. if $-1 < r < 1$, then as $n \to \infty$, $r^n \to 0$ and $S_n \to a/(1-r)$.

In this case, we say that the series $a+ar+ar^2+\ldots$ *converges* and its *sum to infinity*, denoted by S or S_∞, is $\frac{a}{1-r}$.

[If $r > 1$, then as $n \to \infty$, $r^n \to \infty$. If $r < -1$, then as $n \to \infty$, r^n oscillates between large positive and large negative values. If $r = 1$, then $S_n = na$. If $r = -1$, then either $S_n = a$ or $S_n = 0$. In none of these cases does S_n approach a finite limit and the series is said to be *divergent*.]

Hence, provided that $|r| < 1$, the sum to infinity of the geometric series $a+ar+ar^2+\ldots$ is $\frac{a}{1-r}$.

Example 1 Find the sum to infinity of the series $18-6+2-\ldots$.

This is a geometric series with first term 18 and common ratio $-\frac{1}{3}$

$\therefore$ the sum to infinity $= \frac{18}{1-(-\frac{1}{3})} = 18\Big/\frac{4}{3} = 18.\frac{3}{4} = 13\frac{1}{2}$.

Example 2 Express as a fraction in its lowest terms the recurring decimal $0\dot{3}7\dot{0}$.

$0\dot{3}7\dot{0} = 0{\cdot}370370370\ldots = 0{\cdot}37+0{\cdot}00037+0{\cdot}00000037+\ldots$.

This is a geometric series with first term 0·37 and common ratio 1/1000, i.e. 0·001

$\therefore \quad 0\dot{3}7\dot{0} = \frac{0{\cdot}37}{1-0{\cdot}001} = \frac{0{\cdot}37}{0{\cdot}999} = \frac{370}{999} = \frac{10}{27}$.

Example 3 Find the sum of the infinite series $1+2x+4x^2+8x^3+\ldots$, stating for which values of x your result is valid.

This is a geometric series with first term 1 and common ratio $2x$

$\therefore$ its sum is $1/(1-2x)$.

The result is valid for $|2x|<1$, i.e. for $|x|<\frac{1}{2}$.

Exercise 13.4

1. Find the sums to infinity of the following geometric series
(a) $6+2+\frac{2}{3}+\ldots$, (b) $1-\frac{1}{2}+\frac{1}{4}-\ldots$,
(c) $10+1+0{\cdot}1+\ldots$, (d) $45-30+20-\ldots$.

2. Express as fractions in their lowest terms
(a) $0{\cdot}\dot{5}\dot{4}$, (b) $0{\cdot}\dot{0}7\dot{2}$, (c) $0{\cdot}5\dot{7}4\dot{0}$.

3. A geometric series with common ratio 0·8 converges to the sum 250. Find the fourth term of the series.

4. The first and fourth terms of a geometric series are 135 and -40 respectively. Find the common ratio of the series and its sum to infinity.

5. The sum of the first n terms of a geometric series is $8-2^{3-2n}$. Find the first term of the series, its common ratio and its sum to infinity.

6. Find the sums of the following infinite series, stating the values of x for which your results are valid.
(a) $1+3x+9x^2+\ldots$, (b) $1-\frac{1}{2}x+\frac{1}{4}x^2-\ldots$,
(c) $2-4x+8x^2-\ldots$, (d) $x+\frac{1}{3}x^2+\frac{1}{9}x^3+\ldots$.

7. A geometric series has first term 35 and common ratio 2^x. State the set of values of x for which the series is convergent. Find the value of x for which the sum to infinity of the series is 40.

8. Find the sum to n terms of the geometric series $108+60+33\frac{1}{3}+\ldots$. If k is the least number which exceeds this sum for all values of n, find k. Find also the least value of n for which the sum exceeds 99% of k.

13.5 The Σ notation

As writing out a series can often be cumbersome, the Greek capital letter Σ (pronounced 'sigma') is used to mean 'the sum of'.

$\sum_{r=1}^{n} u_r \left(\text{or} \sum_{1}^{n} u_r \right)$ means 'the sum of all the terms u_r from $r = 1$ to $r = n$', i.e.

$$S_n = \sum_{r=1}^{n} u_r = u_1 + u_2 + u_3 + \ldots + u_n.$$

Example 1 Find $\sum_{r=3}^{6} r(r-1)$.

$$\sum_{r=3}^{6} r(r-1) = 3(3-1) + 4(4-1) + 5(5-1) + 6(6-1)$$
$$= 6 + 12 + 20 + 30 = 68.$$

Example 2 Write in Σ notation the series

(a) $2 + 5 + 10 + 17 + 26 + \ldots + 401$

(b) $3 + 2 + \frac{11}{7} + \frac{12}{9} + \frac{13}{11} + \ldots + \frac{20}{25}$.

(a) The rth term of the series, $u_r = r^2 + 1$.
Since 401 is given by $r = 20$, the series has 20 terms.
Hence in Σ notation the series is $\sum_{r=1}^{20} (r^2 + 1)$.

(b) The rth term, $u_r = \frac{r+8}{2r+1}$.
The final term, $\frac{20}{25}$, is given by $r = 12$.
Hence in Σ notation the series is $\sum_{r=1}^{12} \frac{r+8}{2r+1}$.

Some basic rules for manipulating expressions involving Σ can be established as follows:

$$\sum_{1}^{n} (ku_r) = ku_1 + ku_2 + \ldots + ku_n$$
$$= k(u_1 + u_2 + \ldots + u_n) = k \sum_{1}^{n} u_r.$$

In particular $\sum_{1}^{n} k = k + k + \ldots + k = kn.$

$$\sum_{1}^{n} (u_r + v_r) = (u_1 + v_1) + (u_2 + v_2) + \ldots + (u_n + v_n)$$
$$= (u_1 + u_2 + \ldots + u_n) + (v_1 + v_2 + \ldots + v_n)$$
$$= \sum_{1}^{n} u_r + \sum_{1}^{n} v_r.$$

Similarly $\sum_1^n (u_r - v_r) = \sum_1^n u_r - \sum_1^n v_r$.

[Note that, in general, $\sum_1^n u_r v_r = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$ is *not* equal to $\left(\sum_1^n u_r\right)\left(\sum_1^n v_r\right)$.]

Exercise 13.5

1. Write in full and hence evaluate:

(a) $\sum_{r=1}^{4} (r^3 + 3r)$, (b) $\sum_{r=10}^{12} (150 - r^2)$, (c) $\sum_{r=2}^{7} (4r+3)$,

(d) $\sum_{r=1}^{6} \frac{120}{r}$, (e) $\sum_{r=1}^{6} \sin\frac{r\pi}{3}$, (f) $\sum_{r=0}^{5} (-1)^r(1+2^{r+1})$.

2. Find expressions for the following series in the form $\sum_1^n f(r)$.

(a) $5+7+9+\ldots+27$, (b) $1+8+27+\ldots+4096$

(c) $-2+3-4+5-\ldots+41$, (d) $360-180+90-\ldots+5\frac{5}{8}$,

(e) $\frac{1}{2}+\frac{3}{4}+\frac{5}{6}+\ldots+\frac{19}{20}$, (f) $\frac{1}{4}+\frac{4}{7}+\frac{9}{10}+\ldots+\frac{144}{37}$.

3. Evaluate the following without writing down the series in full.

(a) $\sum_1^{20} (4r+5)$, (b) $\sum_1^{16} (-1)^r(2r-1)$, (c) $\sum_0^{50} (25-2r)$

(d) $\sum_1^{9} 5(-2)^{r-1}$, (e) $\sum_0^{\infty} (\frac{1}{2})^r$, (f) $\sum_2^{\infty} (0{\cdot}1)^r$.

4. By expressing r in terms of s, or otherwise, show that

(a) $\sum_{r=0}^{n-1} r(r+1) = \sum_{s=1}^{n} s(s-1)$, (b) $\sum_{r=1}^{n} (3r+2) = \sum_{s=3}^{n+2} (3s-4)$.

5. Show that (a) $\sum_1^{100} (2-r)(2+r) = 400 - \sum_1^{100} r^2$

(b) $\sum_{21}^{40} (r-5) = 300 + \sum_1^{20} r$.

6. Show that $1^3+3^3+5^3+\ldots+25^3 = \sum_1^{25} r^3 - 8\sum_1^{12} r^3$.

7. Prove that $\sum_1^n (au_r + bv_r) = a\sum_1^n u_r + b\sum_1^n v_r$.

13.6 The binomial theorem

This section concerns powers of binomial (or 'two term') expressions such as $(a+b)$. By multiplication it is found that

$$\begin{aligned}
(a+b)^0 &= 1\\
(a+b)^1 &= a+b\\
(a+b)^2 &= a^2+2ab+b^2\\
(a+b)^3 &= a^3+3a^2b+3ab^2+b^3\\
(a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4\\
(a+b)^5 &= a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5
\end{aligned}$$

These coefficients form an array known as Pascal's† triangle:

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
\multicolumn{11}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots}
\end{array}$$

Each entry in the array is the sum of the two entries on either side of it in the previous line. This means that the triangle can easily be extended to provide the coefficients in the expansions of higher powers of $(a+b)$. To see why coefficients in successive lines are related in this way, we consider

$$(a+b)^6 = (a+b)(a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5)$$

In the product there are two terms in a^4b^2, $b.5a^4b = 5a^4b^2$ and $a.10a^3b^2 = 10a^4b^2$. Hence the coefficient of $a^4b^2 = 5+10 = 15$.

Although Pascal's triangle can always be used to obtain expansions of $(a+b)^n$, for large values of n the method becomes rather long. To find another approach, we consider the product

$$\begin{aligned}
&(a_1+b_1)(a_2+b_2)(a_3+b_3)\\
&\quad = (a_1+b_1)(a_2a_3+a_2b_3+b_2a_3+b_2b_3)\\
&\quad = a_1a_2a_3+a_1a_2b_3+a_1b_2a_3+a_1b_2b_3+b_1a_2a_3+b_1a_2b_3+b_1b_2a_3+b_1b_2b_3.
\end{aligned}$$

Writing $a_1 = a_2 = a_3 = a$ and $b_1 = b_2 = b_3 = b$, we see that the terms a^2b and ab^2 each appear three times, giving

$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3.$$

† *Pascal, Blaise* (1623–1662) French theologian, mathematician and inventor. His *Traité du Triangle Arithmétique* is one of several essays on mathematical subjects. He also anticipated the invention of differential calculus and contributed to the foundation of the theory of probability.

Extending this method to the product

$$(a+b)^6 = (a+b)(a+b)(a+b)(a+b)(a+b)(a+b)$$

we find that each individual term in the expansion is the product of an 'a' or a 'b' from each of the six brackets. A term in a^4b^2 arises when 4 a's and 2 b's are selected. Since there are ${}_6C_2$ ways of selecting b's from 2 of the 6 brackets and a's from the rest, the term a^4b^2 must be produced ${}_6C_2$ times. (See §12.2.) Hence the coefficient of a^4b^2 is ${}_6C_2 = \dfrac{6!}{2!4!} = 15$, as shown above. This approach leads to the general result:

$$\boxed{(a+b)^n = a^n + {}_nC_1 a^{n-1}b + {}_nC_2 a^{n-2}b^2 + \ldots + {}_nC_{n-1}ab^{n-1} + b^n}$$

i.e. $(a+b)^n = a^n + na^{n-1}b + \dfrac{n(n-1)}{2!}a^{n-2}b^2 + \ldots + nab^{n-1} + b^n$. This is called the *binomial theorem.*

A general term in the expansion, such as the term in $a^{n-r}b^r$, takes the form:

$${}_nC_r a^{n-r}b^r = \frac{n!}{r!(n-r)!}a^{n-r}b^r = \frac{n(n-1)\ldots(n-r+1)}{r!}a^{n-r}b^r.$$

Example 1 Use the binomial theorem to expand $(1+x)^7$.

$$(1+x)^7 = 1 + {}_7C_1x + {}_7C_2x^2 + {}_7C_3x^3 + {}_7C_4x^4 + {}_7C_5x^5 + {}_7C_6x^6 + x^7$$

$$= 1 + \frac{7!}{1!6!}x + \frac{7!}{2!5!}x^2 + \frac{7!}{3!4!}x^3 + \frac{7!}{4!3!}x^4 + \frac{7!}{5!2!}x^5 + \frac{7!}{6!1!}x^6 + x^7$$

$$= 1 + 7x + \frac{7.6}{1.2}x^2 + \frac{7.6.5}{1.2.3}x^3 + \frac{7.6.5}{1.2.3}x^4 + \frac{7.6}{1.2}x^5 + 7x^6 + x^7$$

$$= 1 + 7x + 21x^2 + 35x^3 + 35x^4 + 21x^5 + 7x^6 + x^7.$$

Example 2 Use the binomial theorem to expand $(3p+2q)^4$.

$$(3p+2q)^4 = (3p)^4 + {}_4C_1(3p)^3 2q + {}_4C_2(3p)^2(2q)^2 + {}_4C_3 3p(2q)^3 + (2q)^4$$

$$= 3^4p^4 + 4.3^3.2p^3q + 6.3^2.2^2p^2q^2 + 4.3.2^3pq^3 + 2^4q^4$$

$$\therefore \quad (3p+2q)^4 = 81p^4 + 216p^3q + 216p^2q^2 + 96pq^3 + 16q^4.$$

Example 3 Find the coefficient of x^4 in the expansion of $(2x-1)^{15}$.

By the binomial theorem the term in $x^4 = {}_{15}C_{11}(2x)^4(-1)^{11}$.

Hence the coefficient of $x^4 = -{}_{15}C_{11}.2^4 = -\dfrac{15.14.13.12}{4.3.2.1}.2^4$

$$= -21\,840.$$

Example 4 Write down the first three terms in the expansion of $(1-x)^{10}$. Hence find an approximate value for $(0{\cdot}99)^{10}$.

Using the binomial theorem

$$(1-x)^{10} = 1^{10} + {}_{10}C_1 1^9(-x) + {}_{10}C_2 1^8(-x)^2 + \dots$$

$$= 1 - 10x + \frac{10.9}{2.1}x^2 - \dots$$

$$\therefore \quad (1-x)^{10} = 1 - 10x + 45x^2 - \dots$$

Substituting $x = 0{\cdot}01$, we find that

$$(0{\cdot}99)^{10} \approx 1 - 10(0{\cdot}01) + 45(0{\cdot}01)^2 = 1 - 0{\cdot}1 + 0{\cdot}0045$$

$$\therefore \quad (0{\cdot}99)^{10} \approx 0{\cdot}9045.$$

Example 5 Expand $(1+x+2x^2)^6$ as far as the term in x^3.

$$(1+x+2x^2)^6 = 1 + 6(x+2x^2) + \frac{6.5}{2.1}(x+2x^2)^2$$

$$+ \frac{6.5.4}{3.2.1}(x+2x^2)^3 + \dots .$$

Neglecting terms in x^4 and higher powers:

$$(1+x+2x^2)^6 = 1 + 6(x+2x^2) + 15(x^2+4x^3) + 20x^3 + \dots$$

$$= 1 + 6x + 27x^2 + 80x^3 + \dots .$$

One important application of the binomial theorem is to the proof of the rule for differentiating powers of x introduced in §5.4, namely

$$\frac{d}{dx}(x^n) = nx^{n-1}, \quad \text{where } n \text{ is a positive integer.}$$

Let δx be a small increase in x and let δy be the corresponding small increase in y, then

$$y = x^n$$

$$\Rightarrow y + \delta y = (x+\delta x)^n$$

$$= x^n + {}_nC_1 x^{n-1}\,\delta x + {}_nC_2 x^{n-2}(\delta x)^2 + \dots + (\delta x)^n$$

$$\Rightarrow \quad \delta y = nx^{n-1}\,\delta x + \frac{n(n-1)}{2}x^{n-2}(\delta x)^2 + \dots + (\delta x)^n$$

$$\Rightarrow \quad \frac{\delta y}{\delta x} = nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}\,\delta x + \dots + (\delta x)^{n-1}$$

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0}\frac{\delta y}{\delta x} = nx^{n-1}.$$

Exercise 13.6

1. Use the binomial theorem to expand:
(a) $(x+y)^4$, (b) $(a-b)^7$, (c) $(2+p^2)^6$,
(d) $(2h-k)^5$, (e) $\left(x+\frac{1}{x}\right)^3$, (f) $\left(z-\frac{1}{2z}\right)^8$.

2. Find the given terms in the following expansions:
(a) $(1+x)^{10}$, 5th term, (b) $(2-3x)^8$, term in x^2,
(c) $(2a+b)^{12}$, 10th term, (d) $(p-3q^2)^7$, term in p^4q^6,
(e) $\left(x-\frac{1}{x}\right)^6$, constant term, (f) $\left(x^2+\frac{1}{x}\right)^9$, term in $\frac{1}{x^3}$.

3. Expand and simplify $\left(2x+\frac{1}{x^2}\right)^5+\left(2x-\frac{1}{x^2}\right)^5$.

4. The coefficient of x^3 in the expansion of $(1+x)^n$ is four times the coefficient of x^2. Find the value of n.

5. In the binomial expansion of $(1+\frac{1}{3})^n$ the fourth and fifth terms are equal. Find the value of n.

6. The coefficient of x^5 in the expansion of $(1+5x)^8$ is equal to the coefficient of x^4 in the expansion of $(a+5x)^7$. Find the value of a.

7. If the first three terms of the expansion of $(1+ax)^n$ in ascending powers of x are $1-4x+7x^2$, find n and a.

8. Use the expansion of $(a+b)^4$ to evaluate $(1{\cdot}03)^4$ correct to 2 decimal places.

9. Use the expansion of $(2-x)^5$ to evaluate $(1{\cdot}98)^5$ correct to 5 decimal places.

10. Obtain the expansion in ascending powers of x of $(1+2x)^{15}$ as far as the term in x^3. Hence evaluate $(1{\cdot}002)^{15}$ correct to 5 decimal places.

11. Find the first four terms in the expansions, in ascending powers of x, of (a) $(1+x)^7$, (b) $(1+x-x^2)^7$.

12. Expand $(1+2x+3x^2)^8$ in ascending powers of x as far as the term in x^3.

13. Find the first three terms in the expansion in ascending powers of x of $(1-3x)(1+2x)^6$.

14. Find the coefficient of the given power of x in the expansion of
(a) $(1+x^2)(2-3x)^7$, x^3, (b) $(1-3x-2x^2)(1+x^2)^{10}$, x^{20},
(c) $x\left(x-\frac{2}{x^2}\right)^{12}$, x^4, (d) $\left(x+\frac{1}{x}\right)^2(1-x)^5$, x^2.

13.7 Use of the binomial series

By the binomial theorem, if n is a positive integer

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\ldots+x^n.$$

This result can be extended to other values of n by writing

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\ldots$$
$$\ldots+\frac{n(n-1)\ldots(n-r+1)}{r!}x^r+\ldots.$$

When n is a positive integer this expansion is the same as that obtained using the binomial theorem. Otherwise the expansion produces an infinite series called the *binomial series*. It can be shown that this series is a valid expansion of $(1+x)^n$ when $|x|<1$, i.e. when $-1<x<1$.

Example 1 Expand $1/(1-x)^2$ in ascending powers of x, giving the first four terms and the term in x^r.

$$\frac{1}{(1-x)^2} = (1-x)^{-2} = 1+(-2)(-x)+\frac{(-2)(-3)}{2!}(-x)^2$$
$$+\frac{(-2)(-3)(-4)}{3!}(-x)^3+\ldots$$
$$+\frac{(-2)(-3)\ldots(-2-r+1)}{r!}(-x)^r+\ldots$$

$$\therefore\quad \frac{1}{(1-x)^2} = 1+2x+3x^2+4x^3+\ldots+(r+1)x^r+\ldots$$

[Note: The coefficients of the terms in the expansions of $(1-x)^{-1}$, $(1-x)^{-2}$, $(1-x)^{-3}$, . . . form the diagonal lines in Pascal's triangle given in §13.6.]

Example 2 Find the expansion of $(1+2x)^{3/2}$ in ascending powers of x as far as the term in x^3. State the range of values of x for which the expansion is valid.

$$(1+2x)^{3/2} = 1+\frac{3}{2}\cdot 2x+\frac{\frac{3}{2}\cdot\frac{1}{2}}{2!}(2x)^2+\frac{\frac{3}{2}\cdot\frac{1}{2}\cdot(-\frac{1}{2})}{3!}(2x)^3+\ldots$$
$$= 1+3x+\frac{3}{2}x^2-\frac{1}{2}x^3+\ldots$$

The expansion is valid for $|2x|<1$, i.e. for $|x|<\frac{1}{2}$.

Example 3 Find the first three terms in the expansion of $(4+x)^{-1/2}$ in ascending powers of x. Deduce an approximate value of $1/\sqrt{(4{\cdot}16)}$.

[Note that the binomial series cannot be used to expand $(4+x)^{-1/2}$ directly. We must rearrange to create an expression of the form $(1+\ldots)^{-1/2}$.]

$$(4+x)^{-1/2} = \{4(1+\tfrac{1}{4}x)\}^{-1/2} = 4^{-1/2}(1+\tfrac{1}{4}x)^{-1/2} = \tfrac{1}{2}(1+\tfrac{1}{4}x)^{-1/2}$$

$$\therefore \quad (4+x)^{-1/2} = \tfrac{1}{2}\left\{1+(-\tfrac{1}{2})(\tfrac{1}{4}x)+\frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}(\tfrac{1}{4}x)^2+\ldots\right\}$$

$$= \frac{1}{2} - \frac{1}{16}x + \frac{3}{256}x^2 - \ldots$$

Substituting $x = 0{\cdot}16$, we have

$$(4{\cdot}16)^{-1/2} = \frac{1}{2} - \frac{1}{16}(0{\cdot}16) + \frac{3}{256}(0{\cdot}16)^2 - \ldots = 0{\cdot}5 - 0{\cdot}01 + 0{\cdot}0003 - \ldots .$$

Hence $1/\sqrt{(4{\cdot}16)} \approx 0{\cdot}4903$.

Exercise 13.7

In questions 1 to 3 expand the given function in a series of ascending powers of x giving the first three terms and the term in x^r.

1. $(1+x)^{-2}$, 2. $(1-x)^{-1}$, 3. $1/(1+2x)^3$.

In questions 4 to 9 expand the given function in a series of ascending powers of x giving the first three terms. State the values of x for which each expansion is valid.

4. $(1+x)^{1/2}$. 5. $\sqrt[3]{(1-3x)}$. 6. $1/\sqrt{(1+\frac{1}{2}x)}$.

7. $1/(3+x)$. 8. $\sqrt{(4-x)}$. 9. $(9-4x)^{3/2}$.

10. Find the first four non-zero terms in the expansion of $(1+2x^2)^{-1/2}$ in ascending powers of x.

11. Find the first four terms in the expansion of $(1-x)^{1/2}$ in ascending powers of x. Deduce the value of $\sqrt{(0{\cdot}9)}$ correct to 4 decimal places.

12. Expand $(1+3x)^{-1/3} + (1-4x)^{-1/4}$ as a series in ascending powers of x, giving the first three non-zero terms.

13. Expand $(2-x)^{-2}$ as a series of ascending powers of x as far as the term in x^4. Deduce the value of $1/(1{\cdot}8)^2$ correct to 3 significant figures.

14. Expand $(1+2x)^{1/2}$ in ascending powers of x as far as the term in x^3. By substituting $x = 1/25$, find an approximate value for $\sqrt{3}$ giving your answer to 3 decimal places.

15. Expand $(1-3x)/(1+4x)$ in ascending powers of x as far as the term in x^3, stating the values of x for which the expansion is valid.

16. Expand the following in ascending powers of x as far as the term in x^3
(a) $1/(1+x-x^2)$, (b) $\sqrt{(1+x+2x^2)}$.

Exercise 13.8 (miscellaneous)

1. Write down an expression for the nth term of the sequence 0, 2, 6, 12, 20, ... Find which term of the sequence is equal to 210.

2. Find the first three terms and the nth term of the series whose sums are given by

(a) $S_n = 4n - \frac{1}{n}$, (b) $S_n = (n+1)!$

3. Evaluate the following:

(a) $\sum_{r=1}^{20} (r+2)$, (b) $\sum_{r=1}^{8} 2^r$, (c) $\sum_{r=1}^{n} 2r$.

4. Given that the arithmetic mean of $1/(a+b)$ and $1/(b+c)$ is $1/(a+c)$, find in terms of b the arithmetic mean of a^2 and c^2.

5. Find the least value of n for which the nth term of the series $12+10{\cdot}7+9{\cdot}4+8{\cdot}1+\ldots$ is negative. Find also the least value of n for which the sum to n terms is negative.

6. The sum of the first twenty terms of an arithmetic progression is 45, and the sum of the first forty terms is 290. Find the first term and the common difference. Find the number of terms in the progression which are less than 100. (JMB)

7. The second, fourth and ninth terms of an arithmetic progression are in geometric progression. Find the common ratio of the geometric progression.

8. A rod one metre in length is divided into ten pieces whose lengths are in geometrical progression. The length of the longest piece is eight times the length of the shortest piece. Find, to the nearest millimetre, the length of the shortest piece. (JMB)

9. (i) The sum to infinity of a geometric progression is 3. When the terms of this geometric progression are squared a new geometric progression is obtained whose sum to infinity is 1·8. Find the first term and the common ratio of each series.
(ii) Express the recurring decimal $0{\cdot}3\dot{2}\dot{1}$ in the form p/q, where p and q are integers with no common factor. (AEB 1978)

10. (i) Write down the first four terms of a geometric series with sum to infinity 4/3 and first term 1.
(ii) Find the values of r for which the series

$$r^2 + \frac{r^2}{1+r^2} + \frac{r^2}{(1+r^2)^2} + \ldots \text{ is convergent, and find its sum.} \qquad \text{(AEB 1975)}$$

11. (i) Given that the sum of the first and second terms of an arithmetical progression is x and that the sum of the $(n-1)$th and nth terms is y, prove that the sum of the first n terms is $\frac{1}{4}n(x+y)$.
(ii) The sum of the first four terms of a geometric series of positive terms is 15 and the sum to infinity of the series is 16. Show that the sum of the first eight terms of the series differs from the sum to infinity by 1/16. (L)

12. (a) Prove that the sum of all the integers between m and n inclusive $(m, n \in \mathbb{Z}_+, n > m)$ is $\frac{1}{2}(m+n)(n-m+1)$. Find the sum of all the integers between 1000 and 2000 which are not divisible by 5.
(b) A geometric series has first term 2 and common ratio 0·95. The sum of the first n terms of the series is denoted by S_n and the sum to infinity is denoted by S. Calculate the least value of n for which $S - S_n < 1$. (C)

13. The first term of a geometric series is 2 and the second term is x. State the set of values of x for which the series is convergent. Show that when convergent the series converges to a sum greater than 1. If $x = \frac{1}{2}$, find the smallest positive integer n such that the sum of the first n terms differs from the sum to infinity by less than 2^{-10}. (L)

14. An arithmetic series and a geometric series have r as the common difference and the common ratio respectively. The first term of the arithmetic series is 1 and the first term of the geometric series is 2. If the fourth term of the arithmetic series is equal to the sum of the third and fourth terms of the geometric series, find the three possible values of r. When $|r| < 1$ find, in the form $p + q\sqrt{2}$, (i) the sum to infinity of the geometric series, (ii) the sum of the first ten terms of the arithmetic series. (AEB 1978)

15. If S is the sum of the series $1 + 3x + 5x^2 + \ldots + (2n+1)x^n$, prove, by considering $(1-x)S$, or otherwise, that if $x \neq 1$,

$$S = \frac{1 + x - (2n+3)x^{n+1} + (2n+1)x^{n+2}}{(1-x)^2}. \qquad \text{(O)}$$

16. (i) Find the set of values of x for which the series $\sum_{n=0}^{\infty} \left(\frac{2x}{x+1}\right)^n$ is convergent. If the sum to infinity is 3, find x.

(ii) Without using tables, find the value of

$$\left(2+\frac{1}{\sqrt{2}}\right)^5+\left(2-\frac{1}{\sqrt{2}}\right)^5. \qquad \text{(L)}$$

17. Obtain the first three terms in the expansion in ascending powers of x of
(a) $(1+2x-x^2)^6$, (b) $(1-2x)^5(1+x)^7$.

18. Find the term independent of x in the expansion of

(a) $\left(x^2-\frac{1}{3x}\right)^9$, (b) $\left(x-\frac{1}{x}\right)^8\left(x+\frac{1}{x^2}\right)^4$.

19. The coefficients of x^4 in the expansions of $(1+x)^{2n}$ and $(1+15x^2)^n$ are equal. Given that n is a positive integer, find its value.

20. By substituting $a=b=1$ in the binomial expansion of $(a+b)^8$, find the sum of ${}_8C_1+{}_8C_2+{}_8C_3+\ldots+{}_8C_8$. Using a similar method, evaluate ${}_8C_1-{}_8C_2+{}_8C_3-\ldots-{}_8C_8$.

21. The first three terms in the expansion of $(1+ax)^n$ are $1+12x+81x^2$. Find the values of a and n. Find also the coefficient of x^3 in the expansion.

22. Show that $\frac{1}{1-2x}-\frac{2}{2+x}=\frac{5x}{(1-2x)(2+x)}$. Hence expand $\frac{5x}{(1-2x)(2+x)}$ in ascending powers of x as far as the term in x^3.

23. Write down the first five terms of the expansion of $(1+3x)^{1/3}$ in ascending powers of x and state the range of values of x for which the infinite expansion of this form is valid. Hence find an approximation to the cube root of 0·97, correct to five places of decimals. (AEB 1976)

24. Write down the first four terms in the binomial expansion of $(1+2x)^{-3}$ in ascending powers of x. State the range of values of x for which it is valid. By putting $x=-0{\cdot}01$, find a value for 7^{-6} correct to four significant figures. (AEB 1978)

25. Write down the expansion in ascending powers of x up to the term in x^2 of (i) $(1+x)^{1/2}$, (ii) $(1-x)^{-1/2}$ and simplify the coefficients. Hence, or otherwise, expand $\sqrt{\left(\frac{1+x}{1-x}\right)}$ in ascending powers of x up to the term in x^2. By using $x=1/10$ obtain an estimate, to three decimal places, for $\sqrt{11}$. (JMB)

26. (a) If $x-\frac{1}{x}=u$, express $x^3-\frac{1}{x^3}$ and $x^5-\frac{1}{x^5}$ in terms of u.
(b) Assuming that $(1-2kx+x^2)^{-1/2}$ may be expanded in a series of ascending powers of x, obtain the expansion as far as the term in x^3, simplifying the coefficients. (C)

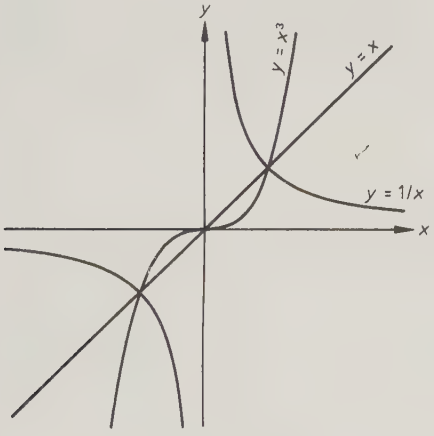
27. For each of the following series construct a flow diagram for a procedure to list the first 10 terms of the series and determine their sum.
(a) An arithmetic progression with the first term a and common difference d given as data.
(b) A geometric progression with the first term a and common ratio r given as data.
(c) The series $1+2x+3x^2+\ldots$ with the value of x given as data.

28. Construct a flow diagram to output the terms of the binomial expansion of $(\frac{1}{4}+\frac{3}{4})^n$, where n is a given positive integer. Test your procedure using a calculator and the value $n = 3$.

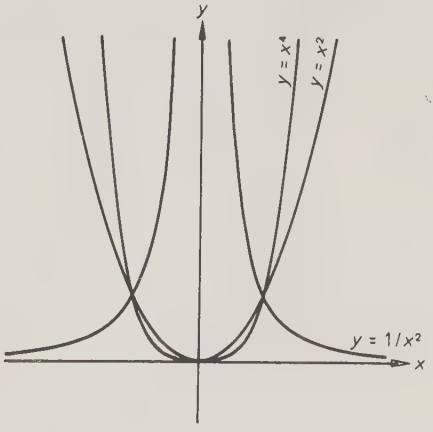
14 Graphs and parameters

14.1 Odd and even functions

The diagrams below show the graphs of $y = x^n$ for various integral values of n: in (1) odd values of n and in (2) even values.



Odd powers of x



Even powers of x

In diagram (1) each of the curves is symmetrical about the origin, i.e. each curve remains unchanged when rotated through 180° about the origin. In (2) the graphs of even powers of x are symmetrical about the y-axis, i.e. the curves remain unchanged when reflected in the y-axis. These symmetries arise because for odd powers $(-x)^n = -x^n$ and for even powers $(-x)^n = x^n$.

The words 'odd' and 'even' are used to describe functions with the same symmetries as the odd and even powers of x. Thus $f(x)$ is called an *odd function* if $f(-x) = -f(x)$. The graph is symmetrical about the origin, so that if (a, b) lies on the graph so does $(-a, -b)$. Similarly $f(x)$ is called an *even function* if $f(-x) = f(x)$. In this case the graph is symmetrical about the y-axis, so that if (a, b) lies on the graph so does $(-a, b)$.

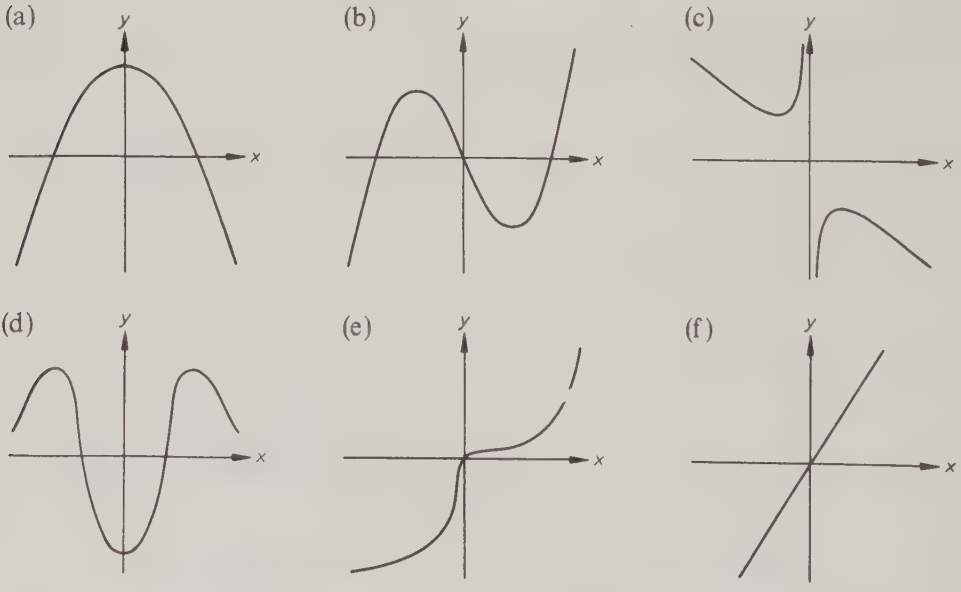
Example 1 Decide whether the following functions are odd, even or neither.
(a) $f(x) = x^3 + x$, (b) $g(x) = x^3 + 1$, (c) $h(x) = \cos 3x$.

(a) $f(-x) = (-x)^3 + (-x) = -x^3 - x = -(x^3 + x) = -f(x)$
$\therefore$ $f(x)$ is an odd function.
(b) $g(-x) = (-x)^3 + 1 = -x^3 + 1$
$\therefore$ $g(-x) \neq g(x)$ and $g(-x) \neq -g(x)$.
Hence $g(x)$ is neither odd nor even.
(c) $h(-x) = \cos 3(-x) = \cos(-3x) = \cos 3x = h(x)$
$\therefore$ $h(x)$ is an even function.

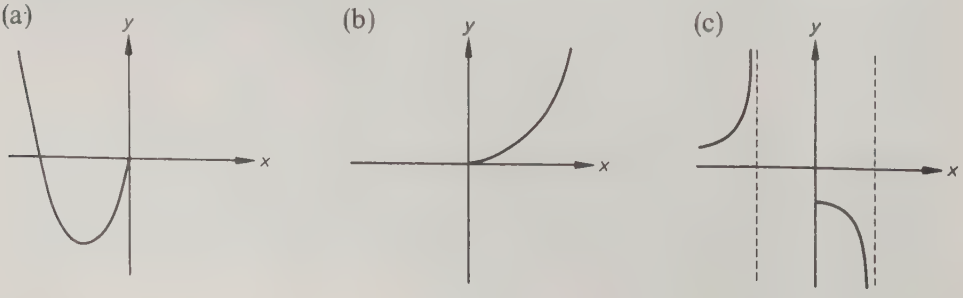
When sketching the graphs of odd and even functions, it is only necessary to consider positive values of x. The sketch is then completed using the symmetry of the function.

Exercise 14.1

1. State which of the following graphs represent odd or even functions.



2. Copy and complete each of the following sketches so that it represents (i) an odd function, (ii) an even function.



3. Decide whether the following functions are odd, even or neither.

(a) $3x^2+5$, (b) $2x-x^3$, (c) $(x-2)(x+4)$,

(d) $\dfrac{x^3+1}{x}$, (e) $\dfrac{x^3+x}{x^2}$, (f) $\dfrac{x}{x^3-x}$.

4. Decide whether the following functions are odd, even or neither.

(a) $\sin 2x$, (b) $\cos 5x$, (c) $\tan \pi x$,

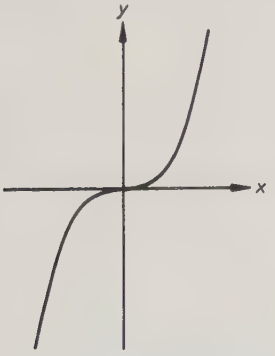
(d) $\sin^2 x$, (e) $\cos(x+\frac{1}{4}\pi)$, (f) $1+\cos x$.

14.2 Composite functions

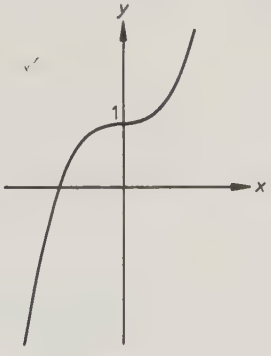
When investigating the properties of a function it is sometimes helpful to think of it as a composite function. This approach was used in §10.1 when differentiating various types of function. It can also be used when sketching graphs.

To illustrate the method let us suppose that $y = f(x)$ is the equation of a graph already familiar to us or relatively easy to sketch. We will now consider how our knowledge can be used to sketch composite functions of the form $f(x)+k$, $f(x-k)$, $1/f(x)$ and $|f(x)|$.

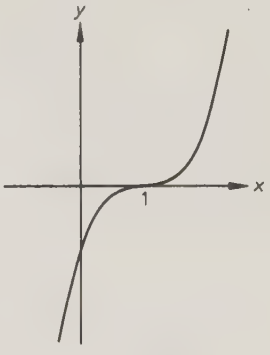
The graph of $f(x)+k$ is produced from the graph of $f(x)$ by performing a translation of k units in the direction of the y-axis. Similarly a translation of k units in the direction of the x-axis gives the graph of $f(x-k)$. This is readily verified by considering some examples.



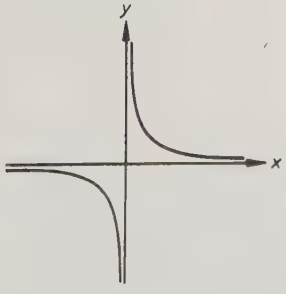
$y = x^3$



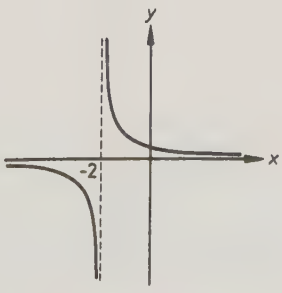
$y = x^3+1$



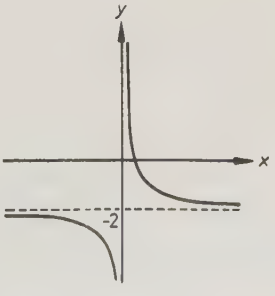
$y = (x-1)^3$



$y = \dfrac{1}{x}$



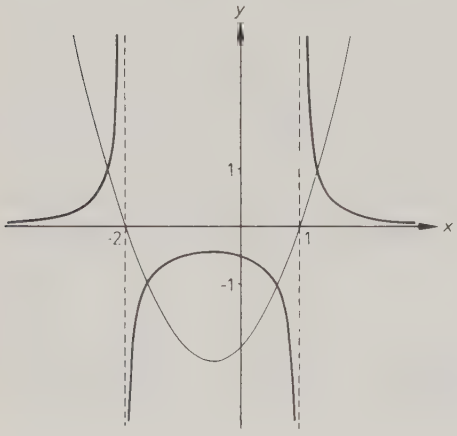
$y = \dfrac{1}{x+2}$



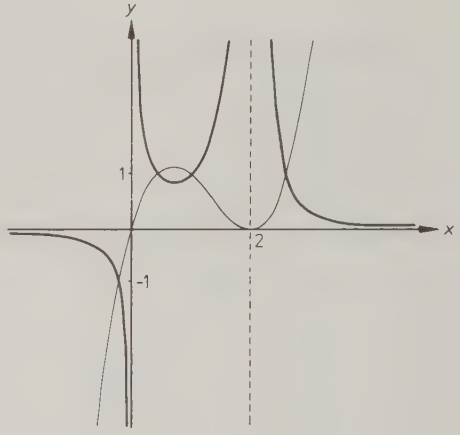
$y = \dfrac{1}{x} - 2$

When comparing the graphs of $f(x)$ and $1/f(x)$ it is useful to note the values of x for which $f(x) = 0$. If $f(a) = 0$ then the curve $y = f(x)$ cuts the x-axis at $(a, 0)$. However as $x \to a$, $1/f(x) \to \pm\infty$, which means that the line $x = a$ is an asymptote to the curve $y = 1/f(x)$.

Since as $f(x)$ increases, $1/f(x)$ must decrease, a non-zero maximum value of $f(x)$ corresponds to a minimum value of $1/f(x)$. Similarly when there is a minimum point on the curve $y = f(x)$, the corresponding point on the curve $y = 1/f(x)$, if defined, will be a maximum point.

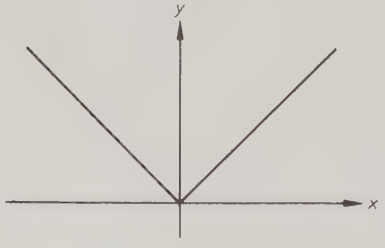


$y = (x-1)(x+2)$ and $y = \dfrac{1}{(x-1)(x+2)}$



$y = x(x-2)^2$ and $y = \dfrac{1}{x(x-2)^2}$

The sketches show the graphs of $y = f(x)$ and $y = 1/f(x)$ for $f(x) = (x-1)(x+2)$ and $f(x) = x(x-2)^2$. Since $f(x)$ and $1/f(x)$ always have the same sign, their graphs always lie on the same side of the x-axis. There are points of intersection where $f(x) = \pm 1$.

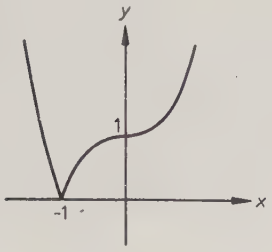


$y = |x|$

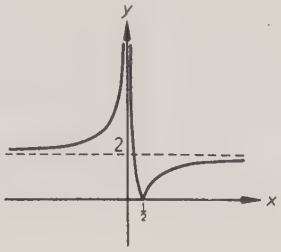
The modulus of a real number x, written $|x|$, was defined in §1.4 as the magnitude of x. Thus if $y = |x|$ then when $x \geqslant 0$, $y = x$ and when $x < 0$, $y = -x$.

Similarly the graph of $|f(x)|$ is the same as the graph of $f(x)$ when $f(x)$ is positive or zero. However, when $f(x)$ is negative, $|f(x)|$ takes the corresponding positive value. Hence when sketching the graph of $y = |f(x)|$ we must replace the parts of the curve $y = f(x)$ which lie below the x-axis by their reflection in the x-axis. For instance, using the graphs of $y = x^3 + 1$, $y = \dfrac{1}{x} - 2$ and

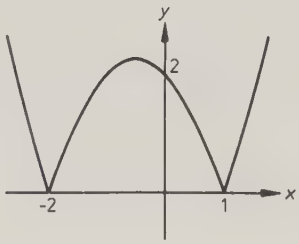
$y = (x-1)(x+2)$ which appear earlier in this section, the graphs of $y = |x^3+1|$, $y = \left|\frac{1}{x} - 2\right|$ and $y = |(x-1)(x+2)|$ can be produced.



$y = |x^3+1|$



$y = \left|\frac{1}{x} - 2\right|$



$y = |(x+1)(x+2)|$

Finally we note that when the turning points on the graph of $y = f(x)$ have been found, the coordinates of turning points on the graph of a composite function, such as $y = 1/f(x)$ or $y = |f(x)|$, can usually be deduced. Further differentiation should not be necessary.

Exercise 14.2

In questions 1 to 4 sketch the graphs with given equations

1. (a) $y = x^2$, (b) $y = x^2+3$, (c) $y = (x-3)^2$.

2. (a) $y = \frac{1}{x^2}$, (b) $y = \frac{1}{x^2} + 4$, (c) $y = \frac{1}{(x+4)^2}$.

3. (a) $y = \sin x$, (b) $y = 1+\sin x$, (c) $y = \sin(x-\frac{1}{3}\pi)$.

4. (a) $y = |x|$, (b) $y = |x|-1$, (c) $y = |x-1|$.

In questions 5 to 12 sketch the graphs of $f(x)$ and $1/f(x)$, giving the coordinates of any stationary points.

5. $f(x) = x^2-1$.

6. $f(x) = x^2+1$.

7. $f(x) = x(x-2)$.

8. $f(x) = (x+2)^2$.

9. $f(x) = x^2-4x+5$.

10. $f(x) = 2+x-x^2$.

11. $f(x) = x^2(x+3)$.

12. $f(x) = x^3-12x$.

In questions 13 to 18 sketch the graphs of $f(x)$ and $|f(x)|$, giving the coordinates of any stationary points.

13. $f(x) = x+2$.

14. $f(x) = 5-2x$.

15. $f(x) = x^2 - 2x - 3$.

16. $f(x) = 6x - x^2 - 9$.

17. $f(x) = 3 - \frac{1}{x}$.

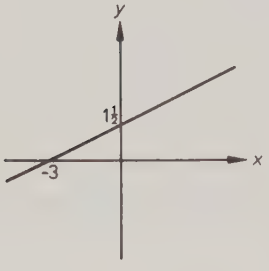
18. $f(x) = \frac{1}{x^2} - 1$.

19. Discuss the graphs of composite functions of the forms $[f(x)]$ and $f([x])$, such as $[x^2]$ and $([x])^2$, where $[x]$ is the integral part of x as defined in §5.1.

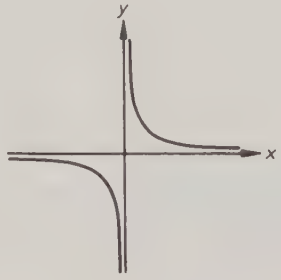
20. Discuss the graphs of functions of the form $\sin\{f(x)\}$ or $\cos\{f(x)\}$, such as $\sin(x^2)$.

14.3 Addition of functions

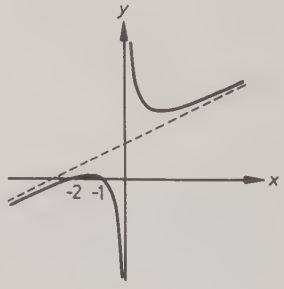
When a function $f(x)$ can be expressed as the sum of two simpler functions $g(x)$ and $h(x)$, it is sometimes possible to sketch the graph of $f(x)$ by combining the graphs of $g(x)$ and $h(x)$. This approach can be particularly useful when dealing with a function such as $y = (x^2+3x+2)/2x$. Rearranging we find that (i) $y = \frac{(x+1)(x+2)}{2x}$ and (ii) $y = \frac{1}{2}x + \frac{3}{2} + \frac{1}{x}$. The equation in form (i) tells us that the curve cuts the x-axis at the points $(-1,0)$ and $(-2,0)$ and that the y-axis is an asymptote to the curve. Form (ii) shows that the graph can be sketched by combining the graphs $y = \frac{1}{2}x + \frac{3}{2}$ and $y = \frac{1}{x}$.



$y = \frac{1}{2}x + \frac{3}{2}$



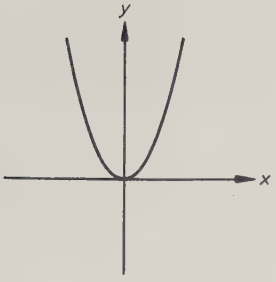
$y = \frac{1}{x}$



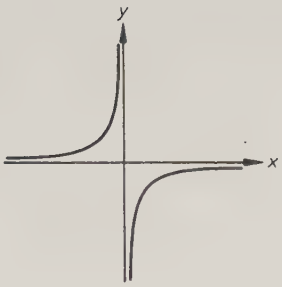
$y = \frac{1}{2}x + \frac{3}{2} + \frac{1}{x}$

The sketches show that the line $y = \frac{1}{2}x + \frac{3}{2}$ is an asymptote to the curve. The coordinates of the two turning points can be obtained using dy/dx in the usual way.

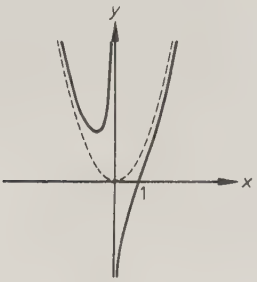
The next set of sketches show the same method applied to the curve $y = (x^3-1)/x$, i.e. $y = x^2 - \frac{1}{x}$.



$y = x^2$

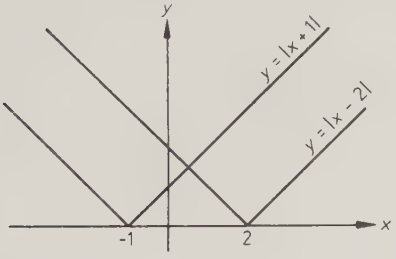


$y = -\frac{1}{x}$



$y = x^2 - \frac{1}{x}$

The sums of modulus functions need a different approach. For instance, consider the function $f(x) = |x+1| + |x-2|$.



The sketch of the graphs of $|x+1|$ and $|x-2|$ shows that the behaviour of $f(x)$ is likely to change when $x = -1$ and again when $x = 2$.

When $x < -1$, $x+1 < 0$ and $x-2 < 0$

$\therefore\ |x+1| = -(x+1)$ and $|x-2| = -(x-2)$

$\therefore\ f(x) = -(x+1)-(x-2) = -2x+1.$

When $-1 < x < 2$, $x+1 > 0$ and $x-2 < 0$

$\therefore\ |x+1| = x+1$ and $|x-2| = -(x-2)$

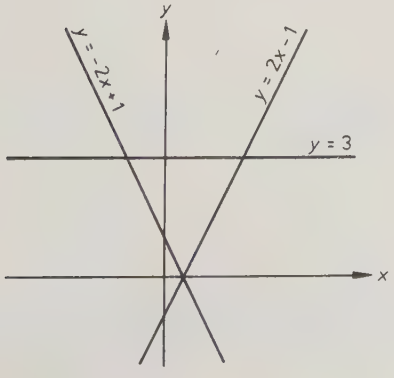
$\therefore\ f(x) = (x+1)-(x-2) = 3.$

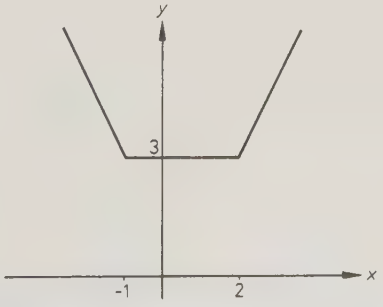
When $x > 2$, $x+1 > 0$ and $x-2 > 0$

$\therefore\ |x+1| = x+1$ and $|x-2| = x-2$

$\therefore\ f(x) = (x+1)+(x-2) = 2x-1.$

Thus the graph of $f(x) = |x+1|+|x-2|$ can now be sketched.





$y = |x+1| + |x-2|$

[Note that it is not generally advisable to apply the methods of this section to the sums of trigonometric functions. These methods would not, for instance, suggest that the function $a\cos x + b\sin x$ is of the form $r\cos(x \pm \alpha)$ as shown in §7.7.]

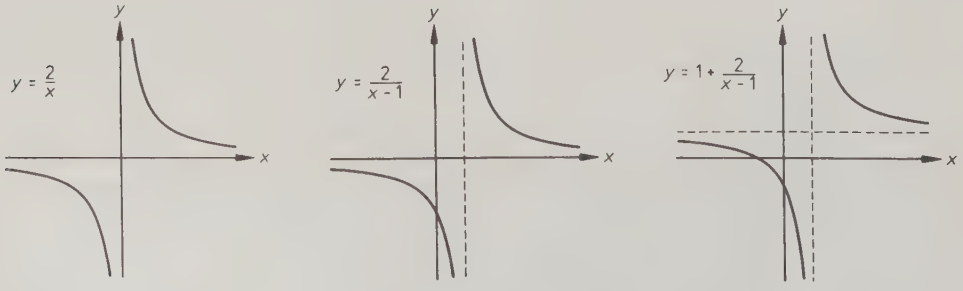
Exercise 14.3

Sketch the graphs with the following equations.

1. $y = 1 + \frac{1}{x}$.
2. $y = x - \frac{1}{x}$.
3. $y = x^2 + \frac{1}{x}$.
4. $y = 1 - \frac{1}{x^2}$.
5. $y = x + \frac{1}{x^2}$.
6. $y = x^2 + \frac{1}{x^2}$.
7. $y = \frac{x^2 + x + 1}{x}$.
8. $y = \frac{2x^2 - x - 1}{x}$.
9. $y = \frac{(x^2-1)^2}{x^2}$.
10. $y = \frac{(x+1)^2(2x-1)}{x^2}$.
11. $y = x + \frac{1}{x-1}$.
12. $y = |x| + \frac{1}{x}$.
13. $y = |x| + |x-4|$.
14. $y = |x-1| + |2x-1|$.

14.4 The function $(ax+b)/(cx+d)$

We consider first a typical function of this type, namely $y = \frac{x+1}{x-1}$, i.e. $y = 1 + \frac{2}{x-1}$.



As shown in the sketches the graph of this function can be obtained from the graph $y = 2/x$ by means of translations parallel to the x- and y-axes. In general

the function $y = (ax+b)/(cx+d)$ has a graph related to the graph $y = 1/x$ in a similar way. Thus to sketch such a graph it is usually sufficient to determine the points of intersection with the coordinate axes and the equations of the asymptotes.

Example 1 Sketch the curve $y = \dfrac{2x-9}{x+3}$.

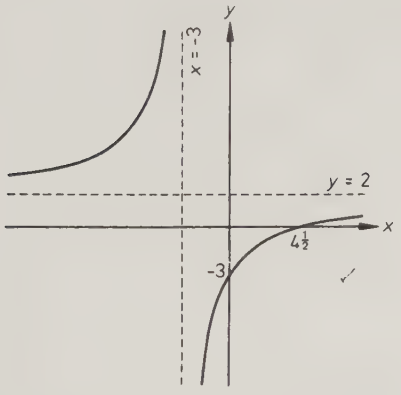
$x = 0 \Rightarrow y = -\dfrac{9}{3} = -3$

$\therefore$ the curve cuts the y-axis at the point $(0, -3)$

$y = 0 \Rightarrow 2x-9 = 0 \Rightarrow x = 4\frac{1}{2}$

$\therefore$ the curve cuts the x-axis at the point $(4\frac{1}{2}, 0)$.

As $x \to -3$, $y \to \pm\infty$ $\therefore$ the line $x = -3$ is an asymptote.



$$y = \frac{2x-9}{x+3} = \frac{2-9/x}{1+3/x}$$

$\therefore$ as $x \to \pm\infty$, $y \to 2$.
Hence the line $y = 2$ is an asymptote.

Exercise 14.4

In questions 1 to 9 sketch the given curve, stating the equations of the asymptotes.

1. $y = \dfrac{1}{x+2}$.
2. $y = \dfrac{3}{x-4}$.
3. $y = \dfrac{2}{2x-3}$.
4. $y = \dfrac{x}{x+2}$.
5. $y = \dfrac{x-2}{x+2}$.
6. $y = \dfrac{2x+3}{2x-3}$.
7. $y = \dfrac{2x-5}{x+2}$.
8. $y = \dfrac{1-3x}{3-x}$.
9. $y = \dfrac{5+2x}{2-5x}$.
10. Sketch the curve $y = x/(x-1)$ and write down the equation of its reflection in the y-axis.

11. Sketch the curve $y = (x-1)/(x+1)$ and write down the equation of its reflection in the x-axis.

12. If a, b, c and d are all positive, sketch the curve $y = \dfrac{ax+b}{cx+d}$ when (i) $ad-bc < 0$, (ii) $ad-bc = 0$, (iii) $ad-bc > 0$.

14.5 Parametric equations

In this and earlier chapters we have considered lines and curves defined by equations connecting the coordinates (x, y) of a typical point. It is sometimes more convenient to express x and y in terms of a third variable called a *parameter*. For instance, the equations $x = 1-t$, $y = t^2-4$ are the *parametric equations* of a curve. The same curve may also be referred to as the *locus* or path of the point $(1-t, t^2-4)$ as the parameter t varies.

The procedure used when making a sketch of a curve defined by parametric equations is similar to that used when the Cartesian equation is known.

(1) Find the points of intersection with the x- and y-axes by letting $y = 0$ and then $x = 0$.
(2) Note any restrictions on the values that x and y can take.
(3) Decide whether the curve is symmetrical in any way.
(4) If necessary plot a few points on the curve.

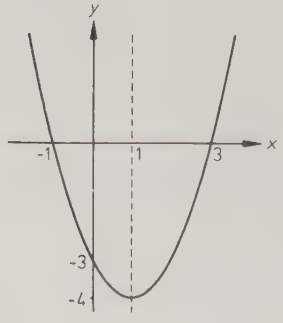
Example 1 Sketch the curve $x = 1-t$, $y = t^2-4$.

$x = 0 \Rightarrow 1-t = 0 \Rightarrow t = 1 \Rightarrow y = -3$

$\therefore$ the curve cuts the y-axis at the point $(0, -3)$.

$y = 0 \Rightarrow t^2-4 = 0 \Rightarrow$ either $t = -2$, $x = 3$ or $t = 2$, $x = -1$

$\therefore$ the curve cuts the x-axis at $(-1, 0)$ and $(3, 0)$.



As t varies x may take any value, but since t^2 is never negative the minimum value of y is -4. For any given value of y greater than -4, there are two possible values of x of the form $1 \pm k$. Hence the curve is symmetrical about the line $x = 1$.

It is sometimes easier to work with the Cartesian equation of a curve. This is obtained from the parametric equations by eliminating the parameter.

Example 2 Find the Cartesian equation of the curve $x = 1-t$, $y = t^2-4$.

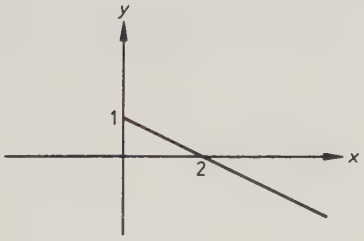
$x = 1-t \Leftrightarrow t = 1-x$
$\therefore \quad y = t^2-4 = (1-x)^2-4 = 1-2x+x^2-4.$
Hence the Cartesian equation is $y = x^2-2x-3$.

Example 3 Find the Cartesian equation of the locus of the point $(2t^2, 1-t^2)$ as t varies. Sketch the locus.

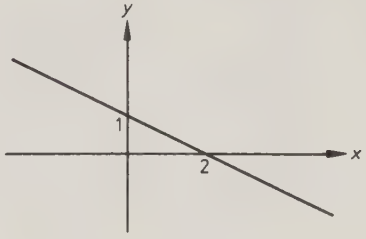
The parametric equations of the locus are

$$x = 2t^2 \tag{1}$$
$$y = 1-t^2 \tag{2}$$

Adding (1) to $2 \times$ (2) $\quad x+2y = 2$.
However, since t^2 can never be negative, $x \geqslant 0$.
Hence the Cartesian equation of the locus is $x+2y = 2$, where $x \geqslant 0$.



$x = 2t^2, y = 1-t^2$



$x+2y = 2$

Example 4 Find the Cartesian equation of the curve $x = 2\cos\theta+\sin\theta$, $y = \cos\theta-2\sin\theta$.

Eliminating $\sin\theta$, $\quad 2x+y = 5\cos\theta$
Eliminating $\cos\theta$, $\quad x-2y = 5\sin\theta$.
Squaring and adding these equations

$$(2x+y)^2+(x-2y)^2 = 25\cos^2\theta+25\sin^2\theta$$
$$4x^2+4xy+y^2+x^2-4xy+4y^2 = 25(\cos^2\theta+\sin^2\theta)$$
$$\therefore \quad 5x^2+5y^2 = 25.$$

Hence the required Cartesian equation is $x^2+y^2 = 5$.

The parametric equations of a curve may be used to find points of intersection with other lines or curves.

Example 5 Find the points of intersection of the line $3x-2y = 2$ with the curve $x = t-1$, $y = 1/t$.

Substituting for x and y in the equation of the line,

$$\begin{aligned} 3x-2y=2 &\Rightarrow 3(t-1)-2.\frac{1}{t}=2 \\ &\Rightarrow 3t(t-1)-2 \quad =2t \\ &\Rightarrow 3t^2-5t-2 \quad =0 \\ &\Rightarrow (3t+1)(t-2)=0 \\ &\Rightarrow t=-\tfrac{1}{3} \text{ or } t \quad =2 \end{aligned}$$

Hence the points of intersection are $(-1\frac{1}{3}, -3)$ and $(1, \frac{1}{2})$.

Exercise 14.5

1. Sketch the following graphs

(a) $x=4t, y=3-t$,
(b) $x=t-1, y=t^2+1$,
(c) $x=2t^2, y=4(t-1)$,
(d) $x=t^2-1, y=t^2+1$,
(e) $x=t+2, y=1/t$,
(f) $x=t^2-1, y=t^4+1$.

2. Find the Cartesian equations of the graphs in question 1.

3. Find the Cartesian equations of the following:

(a) $x=3\sin\theta$, $y=2\cos\theta$,
(b) $x=\sec\theta$, $y=5\tan\theta$,
(c) $x=1+\cos\theta$, $y=1-2\sin\theta$,
(d) $x=\cos\theta+\sin\theta$, $y=2\cos\theta+\sin\theta$,
(e) $x=\cos(\theta+\frac{1}{4}\pi)$, $y=\sqrt{2}\sin\theta$.

4. Find the points of intersection of the following:

(a) $x=t^2, y=t^3; y=3x$,
(b) $x=3t, y=3/t; y=2x+3$,
(c) $x=4t^2, y=8t; 3y+16=4x$,
(d) $x=2t+1, y=t^2; 2x-3y+13=0$,
(e) $x=\dfrac{1}{1+t^2}, y=\dfrac{t}{1+t^2}; x+y=1$,
(f) $x=\dfrac{t}{t+1}, y=\dfrac{1}{t+1}; x^2+y^2=25$.

5. Draw sketches of the following curves on graph paper. Plot sufficient points to give a good indication of the general shape of the curve.

(a) The semi-cubical parabola $x=t^2, y=t^3$.
(b) The ellipse $x=3\cos\theta, y=2\sin\theta$.
(c) The hyperbola $x=t-\dfrac{1}{t}, y=2\left(t+\dfrac{1}{t}\right)$.
(d) The astroid $x=\cos^3 t, y=\sin^3 t$.
(e) The cycloid $x=\theta-\sin\theta, y=1-\cos\theta$.

14.6 Differentiation and parameters

When a curve is defined by parametric equations of the form $x=f(t)$, $y=g(t)$, $\dfrac{dy}{dx}$ can be obtained as a function of t by writing

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{dy}{dt}\cdot\left(1\Big/\frac{dx}{dt}\right) = \frac{dy}{dt}\Big/\frac{dx}{dt}$$

Example 1 Find $\dfrac{dy}{dx}$ in terms of t if $x = 3t-2$, $y = t^3+1$.

$$\frac{dx}{dt} = 3 \qquad \frac{dy}{dt} = 3t^2$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{3t^2}{3} = t^2.$$

When expressed as a function of t, dy/dx represents the gradient of a curve at the point with parameter t.

Example 2 Find the equation of the tangent to the curve $y^2 = x-5$ at the point $(4t^2+5, 2t)$.

This curve can be represented by the parametric equations

$$x = 4t^2+5 \qquad y = 2t$$

$$\frac{dx}{dt} = 8t \qquad \frac{dy}{dt} = 2$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{2}{8t} = \frac{1}{4t}.$$

Hence the gradient of the tangent to the curve at the point $(4t^2+5, 2t)$ is $1/4t$.

The equation of the tangent is $y-2t = \dfrac{1}{4t}\left(x-\left\{4t^2+5\right\}\right)$,

$$\text{i.e. } 4ty-8t^2 = x-4t^2-5.$$

Thus the required equation is $x-4ty+4t^2-5 = 0$.

Remembering that second derivatives do *not* behave like fractions, we now obtain d^2y/dx^2 for the equations of Example 1, $x = 3t-2$, $y = t^3+1$. We look again at the method used to find dy/dx, then use a similar method to find d^2y/dx^2.

$$\frac{dy}{dx} = \frac{d}{dx}\left(t^3+1\right) = \frac{d}{dt}\left(t^3+1\right)\cdot\frac{dt}{dx} = \frac{d}{dt}\left(t^3+1\right)\Big/\frac{dx}{dt} = \frac{3t^2}{3} = t^2.$$

Similarly $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dx}\left(t^2\right) = \dfrac{d}{dt}\left(t^2\right)\cdot\dfrac{dt}{dx} = \dfrac{d}{dt}\left(t^2\right)\Big/\dfrac{dx}{dt} = \dfrac{2t}{3}.$

Example 3 If $x = 1+\dfrac{1}{t}$ and $y = t+\dfrac{1}{t}$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in terms of t.

$$\frac{dx}{dt} = -\frac{1}{t^2} \qquad \frac{dy}{dt} = 1-\frac{1}{t^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \left(1-\frac{1}{t^2}\right)\Big/\left(-\frac{1}{t^2}\right) = 1-t^2.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(1-t^2\right) = \frac{d}{dt}\left(1-t^2\right)\Big/\frac{dx}{dt} = -2t\Big/\left(-\frac{1}{t^2}\right) = 2t^3.$$

Exercise 14.6

Find $\frac{dy}{dx}$ in terms of t if:

1. $x = 2t+1, y = t^2-1.$
2. $x = t^3, y = 3t^2+2.$
3. $x = 4t, y = 1-\frac{1}{t}.$
4. $x = (1-2t)^3, y = t^2-t.$
5. $x = 4\cos t, y = 3\sin t.$
6. $x = 2\cos^3 t, y = 2\sin^3 t.$

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of t if:

7. $x = \frac{1}{t^2}, y = 1+t.$
8. $x = 6t^2, y = 12t-3t^4.$
9. $x = t^3, y = t^2+t.$
10. $x = (t+1)^2, y = t^2-1.$
11. $x = 2\sin t, y = \cos 2t.$
12. $x = \cos^2 t, y = \sin 2t.$
13. Given that $x = t^3-2t, y = 5t^2+\frac{1}{t}$, find the value of $\frac{dy}{dx}$ when $t = 1$.
14. The parametric equations of a curve are $x = t(t^2+1)$, $y = t^2+1$. Find, in its simplest form, the equation of the tangent to the curve at the point with parameter t.
15. Find the equation of the tangent to the curve $x^2-y^2 = 1$ at the point $(\sec\theta, \tan\theta)$.
16. The parametric equations of a curve are $x = (5-3t)^2$, $y = 6t-t^2$. Find dy/dx in terms of t and the coordinates of the stationary point on the curve.
17. Find the coordinates of the stationary points and the point of inflexion on the curve with parametric equations $x = t^2+1$, $y = t(t-3)^2$.
18. Given that $x = \theta-\sin\theta$, $y = 1-\cos\theta$, show that $y^2\frac{d^2y}{dx^2}+1 = 0$.

Exercise 14.7 (miscellaneous)

1. Decide whether the following functions are odd, even or neither

(a) $2x^5-3x^3+1$, (b) $\sin 2x-\sin x$, (c) $|x|$,

(d) $\dfrac{x}{2x^5-3x^3}$, (e) $x^3\tan x$, (f) $[x+\frac{1}{2}]$.

2. Given that $f(x)=ax^3+bx^2+cx+d$, state what can be deduced about the values of a, b, c, d in each of the following cases

(a) $f(x)$ is an even function,
(b) $f(x)$ is an odd function,
(c) $f(x)=|f(x)|$ for all values of x,
(d) $f(x)=-|f(x)|$ for all values of x.

3. Let $f(x)=\dfrac{ax+b}{x+c}$ where x, a, b, c are real and $x\neq\pm c$. Show that if f is an even function then $ac=b$. Deduce that if f is an even function then $f(x)$ must reduce to the form $f(x)=k$, where k is constant. Find all odd functions of the form $\dfrac{ax+b}{x+c}$. (JMB)

4. Sketch on the same diagram the graphs of $f(x)$ and $1/f(x)$ where (a) $f(x)=4x-x^2$, (b) $f(x)=(x-1)^2$, (c) $f(x)=x^2+2x+3$.

5. Sketch the graphs with the following equations

(a) $y=|2x-3|$, (b) $y=|x^2+3x|$,

(c) $y=\left|x+3+\dfrac{2}{x}\right|$, (d) $y=\left|\dfrac{x-2}{x+2}\right|$.

6. Find the coordinates of the point of intersection of the curve $y=\dfrac{x^3+1}{x}$ and the x-axis. Find also the gradients when $x=1$ and $x=-1$, the value of x at the stationary point, and the nature of this stationary point. Sketch this curve and also the curve $y=\dfrac{x}{x^3+1}$. (L)

7. Find the local maxima and minima, and the points of inflexion, if any, of the functions f and g defined by $f(x)=\dfrac{x^4}{4(x^2-1)}$, $g(x)=\dfrac{4(x^2-1)}{x^4}$, and sketch the graphs of these two functions on one diagram. (W)

8. Sketch the following curves

(a) $y=\dfrac{2x+1}{x-1}$, (b) $y=\dfrac{x-1}{2x+1}$,

(c) $y=\left|\dfrac{2x+1}{x-1}\right|$, (d) $y=\dfrac{2|x|+1}{|x|-1}$.

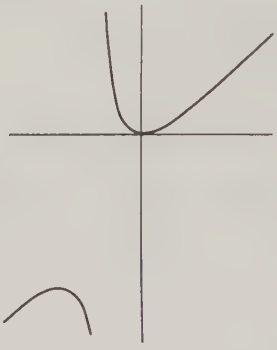
9. Below are listed five functions, numbered (1)–(5), and five graphs, lettered (A)–(E). *Four* of the graphs correspond to *four* of the functions.

$$(1)\ y=\frac{(x+1)^2}{x} \qquad (2)\ y=\frac{x}{(x+1)^2} \qquad (3)\ y=\frac{x}{x+1}$$

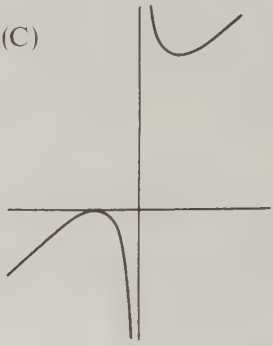
$$(4)\ y=\left|\frac{x}{x+1}\right| \qquad (5)\ y=\frac{x^2}{x+1}.$$

(A)

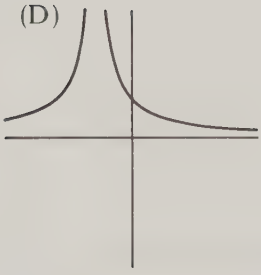
(B)



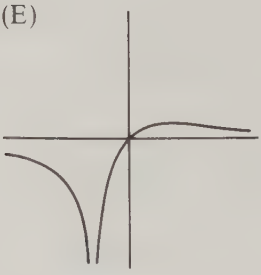
(C)



(D)



(E)



(a) Pair off the four corresponding functions and graphs. (Give your answers as four ordered pairs (n, X), n being a number and X a letter.)
(b) Sketch the graph of the fifth remaining function.
(c) Suggest a suitable function which could fit the fifth remaining graph. (O&C)

10. Sketch the following graphs
(a) $y=|x|+|2x-3|$, (b) $y=|x+3|-|x-2|$.

11. Show that the equation $|x+2|+|x|+|x-1|+|x-3|=6$ has an infinite number of real solutions. (C)

12. The function f is defined by $f: x \to |x-2|+a/x$ where $x \in \mathbb{R}$, $x \neq 0$, and $a \in \mathbb{R}$, $a \neq \pm 4$ or 0. Determine the number of turning points of f for different sets of values of a. (A turning point exists for $x=c$ if $f'(c)=0$ and $f'(x)$ changes sign as x passes through the value c.)

Sketch a graph of f (i) for $a = 3$ and (ii) for $a = -1$, showing carefully the nature of the curve near the point at which $x = 2$, and also for large values of $|x|$. (JMB)

13. Find the Cartesian equations of the following curves

(a) $x = 2\left(t + \frac{1}{t}\right), y = 3\left(t - \frac{1}{t}\right)$,

(b) $x = \frac{1+t^2}{1-t^2}, y = \frac{2t}{1-t^2}$,

(c) $x = \cos\theta - 2\sin\theta$
$y = 3\cos\theta + \sin\theta$,

(d) $x = 3\sin(\theta + \frac{1}{3}\pi)$
$y = 2\sin(\theta + \frac{1}{6}\pi)$.

14. The parametric equations of a curve are $x = 4 - t^2$, $y = 4t - t^3$. Find dy/dx in terms of t and hence find the coordinates of any turning points on the curve. Sketch the curve.

15. A curve is given parametrically by the equations $x = (1+t)^2$, $y = (1-t)^2$. Find dy/dx in terms of t, and hence find the point on the curve at which the gradient is zero. Find also the equation of the tangent to the curve at the point where $x = y$. (C)

16. A locus is defined parametrically by the equations $x = \frac{3}{t+1}$, $y = \frac{1-5t}{t+1}$. Find the points of intersection between this locus and the curve $xy = 3$. Find dy/dx for the locus and hence its Cartesian equation.

17. Given that $x + y = \sin(\theta + 75°)$, $x - y = \sin(\theta + 15°)$, express x and y each in terms of the sine or cosine of $\theta + 45°$. If x and y are the coordinates of a point P and θ varies, obtain in a simplified form the Cartesian equation of the locus of P. Sketch this locus. (JMB)

18. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of t, given that

(a) $x = 3t(3 - 4t^2)$, $y = (3-2t)(1-2t)^2$,

(b) $x = 2\cos t + \cos 2t$, $y = 2\sin t + \sin 2t$.

19. A curve joining the points $(0, 1)$ and $(0, -1)$ is represented parametrically by the equations

$$x = \sin\theta, y = (1 + \sin\theta)\cos\theta, \text{ where } 0 \leqslant \theta \leqslant \pi.$$

Find dy/dx in terms of θ, and determine the x, y coordinates of the points on the curve at which the tangents are parallel to the x-axis and of the point at which the tangent is perpendicular to the x-axis. Sketch the curve.

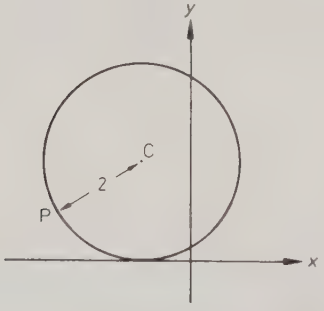
The region in the quadrant $x \geqslant 0$, $y \geqslant 0$ bounded by the curve and the coordinate axes is rotated about the x-axis through an angle of 2π. Show that the volume swept out is given by $V = \pi \int_0^1 (1+x)^2(1-x^2)dx$. Evaluate V, leaving your result in terms of π. (JMB)

15 Coordinate geometry

15.1 The locus of a point

A *locus* is the set of all points satisfying some condition. The Cartesian equation of a locus is obtained by expressing this condition as a relationship between the x- and y-coordinates of a typical point.

Example 1 Find the equation of the locus of points at a distance of 2 units from the point $C(-1, 2)$.



Let $P(x, y)$ be a point on the locus then

$$PC^2 = 2^2$$

$$\therefore \quad (x+1)^2+(y-2)^2 = 2^2$$

$$x^2+2x+1+y^2-4y+4 = 4$$

$\therefore$ the equation of the locus is

$$x^2+y^2+2x-4y+1 = 0.$$

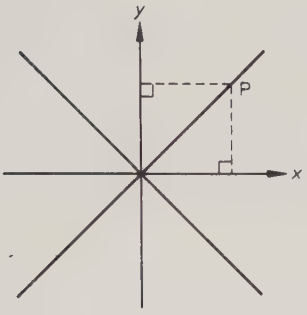
Clearly this equation must represent a circle with centre $(-1, 2)$ and radius 2.

Example 2 Find the equation of the locus of points 5 units above the x-axis.

For any point $P(x, y)$ on the locus, $y = 5$ and there is no restriction on the value of x. Hence the equation of the locus is simply $y = 5$.

Example 3 Find the equation of the locus of points equidistant from the x- and y-axes.

The distance of a point $P(x, y)$ from the x-axis is $|y|$. Similarly the distance of P from the y-axis is $|x|$. Since these distances are equal,



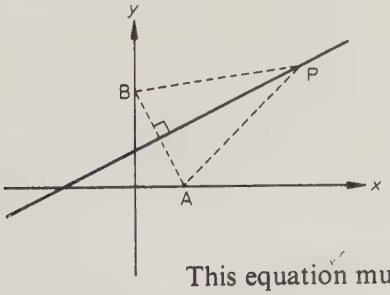
$$|x| = |y|.$$

Squaring to avoid the use of moduli, the equation of the locus may be written

$$x^2 = y^2 \quad \text{or} \quad x^2 - y^2 = 0.$$

This equation represents the pair of lines which bisect the angles between the x- and y-axes.

Example 4 Find the equation of the locus of a point which moves so that it is equidistant from the points $A(1, 0)$ and $B(0, 2)$.



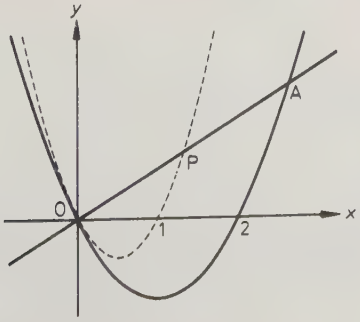
If $P(x, y)$ is any point on the locus then

$$PA^2 = PB^2$$
$$\therefore \quad (x-1)^2 + y^2 = x^2 + (y-2)^2$$
$$x^2 - 2x + 1 + y^2 = x^2 + y^2 - 4y + 4$$
$$\therefore \quad \text{the equation of the locus is}$$
$$2x - 4y + 3 = 0.$$

This equation must represent the perpendicular bisector of AB.

In some problems it is necessary to express the coordinates of a general point on a locus in terms of a parameter. The Cartesian equation is then obtained by eliminating the parameter.

Example 5 The line $y = mx$ and the curve $y = x^2 - 2x$ intersect at the origin O and meet again at a point A. If P is the mid-point of OA, find the equation of the locus of P as m varies.



At the points of intersection

$$x^2 - 2x = mx$$
$$x(x - m - 2) = 0$$
$$\therefore \quad x = 0 \quad \text{or} \quad x = m + 2$$
$$\therefore \quad \text{the coordinates of } A \text{ are}$$

$(m+2, m\{m+2\})$ and the coordinates of P are $(\frac{1}{2}\{m+2\}, \frac{1}{2}m\{m+2\})$.

Hence in parametric form the equations of the locus of P are

$$x = \tfrac{1}{2}(m+2) \quad (1)$$

$$y = \tfrac{1}{2}m(m+2) \quad (2)$$

Rearranging (1) $\quad m = 2x-2$

Substituting in (2) $\quad y = (x-1)2x.$

Hence the equation of the locus of P is $y = 2x(x-1)$.

Exercise 15.1

In questions 1 to 10 find the equation of the locus of a point P as it moves subject to the stated condition. In each case draw a sketch to illustrate your answer.

1. The distance of P from the point $(-3, 4)$ is 5 units.

2. P is equidistant from the points $(1, 0)$ and $(4, 3)$.

3. The perpendicular distance of P from the y-axis is 2 units.

4. A and B are the points $(-3, 1)$ and $(5, -1)$ respectively and $\angle PAB$ is a right angle.

5. P is equidistant from the lines $x = 1$ and $y = 1$.

6. The distance of P from the point $(0, 2)$ is equal to its distance from the x-axis.

7. The area of the rectangle with sides parallel to the axes and with diagonal OP is 9 square units.

8. A and B are the points $(-1, 2)$ and $(3, 2)$ respectively and $\angle APB$ is a right angle.

9. The distance of P from point $(3, 0)$ is equal to its perpendicular distance from the line $x+3 = 0$.

10. The distance of P from the point $(-2, 1)$ is twice its distance from the point $(4, 1)$.

11. Find the equation of the circle centre $(-1, 1)$ and radius $\sqrt{2}$. Show that this circle passes through the origin.

12. Find the equation of the perpendicular bisector of the line joining the points $A(2, 1)$ and $B(-4, -1)$. Hence find the point equidistant from A and B whose x-coordinate is -3.

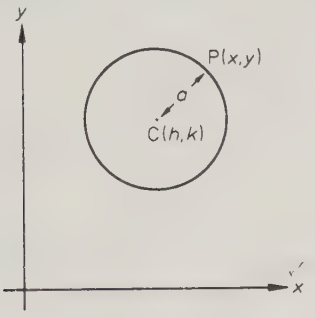
13. The line $y = mx$ meets the curve $y^2 = 4x$ at the origin O and at a point A. Find the equation of the locus of the mid-point of OA as m varies.

14. If P is the point $(t, t^2 - 4)$ and Q is the point $(2, 0)$, find the coordinates of the mid-point R of PQ. Deduce the equation of the locus of R as t varies.

15. A and B are the points $(a, 0)$ and $(0, b)$ respectively and P is the mid-point of AB. Find the equation of the locus of P as a and b vary given that (a) $a + b = c$, where c is constant, (b) triangle OAB is of constant area k, (c) AB is of constant length l.

16. The line $y = mx$ intersects the curve $y = x^2 - 1$ at the points A and B. Find the equation of the locus of the mid-point of AB as m varies.

15.2 The circle



Let $P(x, y)$ be a point on the circle, centre (h, k) and radius a, then $PC^2 = a^2$

$$\therefore \quad (x-h)^2 + (y-k)^2 = a^2.$$

Thus the equation of the circle, centre (h, k), radius a, is

$$\boxed{(x-h)^2 + (y-k)^2 = a^2.}$$

In particular, the equation of the circle with centre the origin and radius a is

$$\boxed{x^2 + y^2 = a^2.}$$

Example 1 Find the equation of the circle with centre $(1, -2)$ and radius 3.

The equation is $(x-1)^2 + (y+2)^2 = 3^2$

$$x^2 - 2x + 1 + y^2 + 4y + 4 = 9$$

i.e. $$x^2 + y^2 - 2x + 4y - 4 = 0.$$

Given the equation of a circle, it can be expressed in the form $(x-h)^2 + (y-k)^2 = a^2$ in order to write down its centre and radius.

Example 2 Find the centre and radius of the circle $x^2 + y^2 + 6x - 2y - 6 = 0$.

[The aim is to rearrange the equation by completing squares of the form $(x-h)^2$ and $(y-k)^2$.]

The equation of the circle may be rearranged as

$$x^2+6x+y^2-2y=6$$
$$x^2+6x+9+y^2-2y+1=6+9+1$$
i.e. $$(x+3)^2+(y-1)^2=4^2.$$

Hence the circle has centre $(-3, 1)$ and radius 4.

We see that the equation $x^2+y^2+2gx+2fy+c=0$ represents a circle for suitable values of g, f and c. Rearranging

$$x^2+2gx+y^2+2fy=-c$$
$$x^2+2gx+g^2+y^2+2fy+f^2=f^2+g^2-c$$
i.e. $$(x+g)^2+(y+f)^2=f^2+g^2-c.$$

Thus the equation $x^2+y^2+2gx+2fy+c=0$ represents a circle with centre $(-g, -f)$ and radius $\sqrt{(f^2+g^2-c)}$ provided that this is real.

Many problems in coordinate geometry concerning circles can be solved using their elementary properties.

Example 3 Find the equation of the tangent to the circle $x^2+y^2-4x+2y+3=0$ at the point $(3, -2)$.

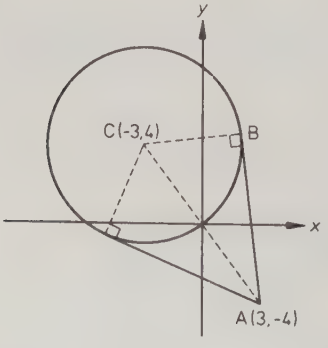
The centre of the circle is the point $(2, -1)$

$\therefore$ the gradient of the radius through $(3, -2) = \dfrac{-2-(-1)}{3-2} = -1.$

Since any tangent to a circle is perpendicular to the radius through its point of contact, the gradient of the tangent through $(3, -2)$ is 1.

Hence its equation is $y+2=1(x-3)$, i.e. $y=x-5$.

Example 4 Find the length of a tangent drawn from the point $A(3, -4)$ to the circle $x^2+y^2+6x-8y=0$.



Rearranging the equation of the circle

$$x^2+6x+9+y^2-8y+16=9+16$$
i.e. $$(x+3)^2+(y-4)^2=5^2$$

$\therefore$ the circle has centre $(-3, 4)$ and radius 5.

If C is the centre of the circle and B is the point of contact of a tangent through A, then

$AC^2=(3-\{-3\})^2+(-4-4)^2=6^2+8^2=100$ and BC, a radius of the circle, is of length 5.

Hence, using Pythagoras' theorem,

$AB^2 = AC^2 - BC^2 = 100 - 25 = 75 \quad \therefore \quad AB = 5\sqrt{3}.$

Thus the length of the tangents from A to the given circle is $5\sqrt{3}$.

Example 5 Find the equation of the circle whose diameter is the line joining the points $A(1,3)$ and $B(-2,5)$.

[One way of doing this is to write down the centre and the radius of the circle, then use these to obtain the equation. We give here an alternative method.]

If $P(x, y)$ is any point on the circle, then since AB is a diameter, the angle APB is a right angle.

Gradient of $AP = \dfrac{y-3}{x-1}$, gradient of $BP = \dfrac{y-5}{x+2}$.

Since AP is perpendicular to BP, $\dfrac{y-3}{x-1} \times \dfrac{y-5}{x+2} = -1$

$$\therefore \quad (x-1)(x+2)+(y-3)(y-5) = 0$$
$$x^2+x-2+y^2-8y+15 = 0.$$

Hence the equation of the circle is $x^2+y^2+x-8y+13 = 0$.

Example 6 Find the equation of the circle which passes through the points $(-1,0)$, $(1,2)$ and $(-5,4)$.

Let the equation of the circle be $x^2+y^2+2gx+2fy+c = 0$, then since the coordinates $(-1,0)$, $(1,2)$ and $(-5,4)$ must satisfy this equation

$1+0-2g+c = 0 \quad \therefore \quad -2g+c = -1 \quad (1)$
$1+4+2g+4f+c = 0 \quad \therefore \quad 2g+4f+c = -5 \quad (2)$
$25+16-10g+8f+c = 0 \quad \therefore \quad -10g+8f+c = -41 \quad (3)$

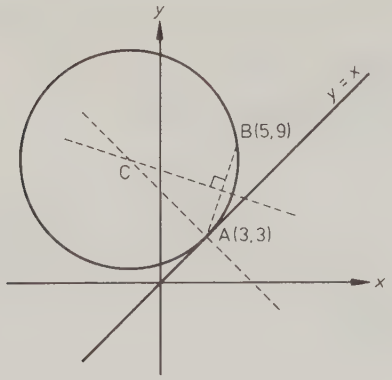
$2\times(2)-(3) \quad 14g+c = 31$
Subtracting (1) $\quad 16g = 32 \quad \therefore \quad g = 2$

Using (1) and (2) $\quad c = 3$ and $f = -3$.

Hence the required equation is $x^2+y^2+4x-6y+3 = 0$.

Another way of finding the equation of a circle through three given points is to use the fact that if points A and B lie on a circle then the centre of the circle lies on the perpendicular bisector of AB.

Example 7 Find the equation of the circle that touches the line $y = x$ at the point $A(3,3)$ and passes through the point $B(5,9)$.



If C is the centre of the circle then C must lie on the perpendicular bisector of AB. Since the line $y = x$ is a tangent to the circle, C must also lie on the line through A perpendicular to $y = x$.

The gradient of $AB = \dfrac{9-3}{5-3} = 3$.

The coordinates of the mid-point of AB are $(4, 6)$

$\therefore$ the perpendicular bisector passes through $(4, 6)$ and has gradient $-\frac{1}{3}$.

Hence its equation is $y-6 = -\frac{1}{3}(x-4)$ i.e. $x+3y = 22$ (1)

The line through A perpendicular to $y = x$ has gradient -1

$\therefore$ its equation is $y-3 = -1(x-3)$, i.e. $x+y = 6$ (2)

Subtracting (2) from (1) $\quad 2y = 16 \quad \therefore \quad y = 8, x = -2.$

Hence the centre of the circle is the point $(-2, 8)$.

The radius of the circle $= AC = \sqrt{\{(-2-3)^2+(8-3)^2\}} = \sqrt{50}$

Thus the equation of the circle is $(x+2)^2+(y-8)^2 = 50$

i.e. $x^2+y^2+4x-16y+18 = 0.$

Exercise 15.2

1. Find the equations of the circles with the following centres and radii

(a) $(3, 2)$, 4, (b) $(-1, -2)$, 1, (c) $(0, 0)$, 5,
(d) $(\frac{1}{2}, 0)$, 3/2, (e) $(4, -1)$, $\sqrt{3}$, (f) $(-3, 5)$, $2\sqrt{5}$.

2. Find the centres and radii of the following circles

(a) $x^2+y^2-2x-6y+1 = 0$, (b) $x^2+y^2 = 4$,
(c) $x^2+y^2+6x+8y = 0$, (d) $x^2+y^2-4x+2y+4 = 0$,
(e) $4x^2+4y^2-5 = 0$, (f) $2x^2+2y^2-6x+10y+7 = 0$.

3. Find the range of values of k for which each of the following equations represents a circle with non-zero radius.

(a) $x^2+y^2 = k$ (b) $x^2+ky^2-2x-8 = 0$,
(c) $kx^2+y^2+4y+9 = 0$, (d) $2x^2+2y^2+kxy-9 = 0$,
(e) $x^2+y^2+2x-6y+k = 0$, (f) $x^2+y^2+kx-2y+5 = 0$.

4. Find the equations of the tangents to the following circles at the given points
(a) $x^2+y^2=5$, $(-2,1)$, (b) $x^2+y^2-4x+2y=3$, $(0,-3)$,
(c) $x^2+y^2+6y-1=0$, $(3,-4)$, (d) $2x^2+2y^2+9x-4y+4=0$, $(-2,3)$.

5. Find the lengths of the tangents drawn from the following points to the given circles
(a) $(6,-1)$, $x^2+y^2=12$,
(b) $(-1,3)$, $x^2+y^2-8x+4y+19=0$,
(c) $(4,-2)$, $x^2+y^2-10y-4=0$,
(d) $(3,-4)$, $x^2+y^2+x-3y=0$.

6. Find the equations of the circles which pass through the following sets of points
(a) $(0,0)$, $(2,0)$, $(0,6)$, (b) $(5,6)$, $(5,-2)$, $(1,6)$,
(c) $(5,0)$, $(3,2)$, $(-3,-4)$, (d) $(2,3)$, $(-2,1)$, $(4,7)$.

7. Find the greatest and the least distance of a point P from the origin as it moves round the circle
(a) $x^2+y^2-24x-10y+48=0$,
(b) $x^2+y^2+6x-8y-24=0$.

8. Find the equation of the circle which has the points $(-7,3)$ and $(1,9)$ at the ends of a diameter. Find also the equations of the tangents to this circle which are parallel (a) to the x-axis, (b) to the y-axis.

9. Find the equation of the locus of a point P which moves so that its distance from the point $A(1,3)$ is twice its distance from the point $B(4,6)$. Show that the locus is a circle giving its centre and radius.

10. Find the equation of the circle which passes through the point $A(6,1)$ and touches the x-axis at the point $B(3,0)$. If this circle cuts the y-axis at the points P and Q, find the area of the quadrilateral $ABPQ$.

11. Find the equation of the circle which touches the line $3x-4y=3$ at the point $(5,3)$ and passes through the point $(-2,4)$.

12. Show that the circle $x^2+y^2-6x-4y+9=0$ touches the x-axis and that the circle $x^2+y^2-2x-6y+9=0$ touches the y-axis. Find the coordinates of their points of intersection, showing that these lie on a straight line through the origin.

13. Find the equation of the circle C_1 which has as the ends of a diameter the points $P(1,1)$ and $Q(6,2)$. Show that the point $R(7,-3)$ lies outside this circle and find the equation of the circle C_2 which passes through P, Q and R. Hence show that the centre of C_2 lies on C_1.

15.3 Tangents and normals

As we have seen in previous chapters, the gradient of the tangent at any point on a curve in the x, y plane is given by dy/dx. The *normal* to a curve at any point is the line perpendicular to the tangent. Thus if the gradients of the tangent and the normal at a particular point are m_1 and m_2, then $m_1 m_2 = -1$.

Example 1 Find the equations of the tangent and the normal to the curve $x^2 - 3xy + y^2 = 5$ at the point $(1, 4)$.

Differentiating the equation with respect to x

$$2x - 3x\frac{dy}{dx} - 3y + 2y\frac{dy}{dx} = 0$$

$\therefore$ at the point $(1, 4)$ $\quad 2 - 3\dfrac{dy}{dx} - 12 + 8\dfrac{dy}{dx} = 0$

i.e. $5\dfrac{dy}{dx} - 10 = 0$ so that $\dfrac{dy}{dx} = 2.$

Hence the gradients of the tangent and normal at $(1, 4)$ are 2 and $-\frac{1}{2}$ respectively.

The equation of the tangent is $y - 4 = 2(x - 1)$

i.e. $y = 2x + 2.$

The equation of the normal is $y - 4 = -\frac{1}{2}(x - 1)$

i.e. $x + 2y = 9.$

Example 2 Find the equation of the normal to the curve with parametric equations $x = 3t + 5$, $y = 1 - t^2$, which is parallel to the line $3x + 4y = 7$.

Differentiating $\dfrac{dx}{dt} = 3$ $\qquad \dfrac{dy}{dt} = -2t$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = -\frac{2t}{3}.$$

Hence the gradient of the normal at the point with parameter t is $3/2t$.

The gradient of the line $3x + 4y = 7$ is $-\frac{3}{4}$

$\therefore$ the normal to the curve is parallel to this line

when $\dfrac{3}{2t} = -\dfrac{3}{4}$, i.e. when $t = -2.$

Thus the required normal passes through the point $(-1, -3)$ on the curve.

$\therefore$ its equation is $3x+4y = 3(-1)+4(-3)$

i.e. $3x+4y+15 = 0.$

The next example shows a different way of obtaining the equation of a tangent to a curve.

Example 3 $P(2p, 2/p)$ and $Q(2q, 2/q)$ are two points on the curve $xy = 4$. Find the equation of the chord PQ. Deduce the equation of the tangent to the curve at P.

The gradient of $PQ = \dfrac{2/q-2/p}{2q-2p} = \dfrac{p-q}{pq(q-p)} = -\dfrac{1}{pq}$, assuming $p \neq q$.

$\therefore$ the equation of the chord PQ is

$$y - \frac{2}{p} = -\frac{1}{pq}(x-2p)$$

$$pqy - 2q = -x+2p$$

i.e. $$x+pqy = 2(p+q).$$

As we consider points Q closer and closer to P, the chord PQ approaches the tangent at P

$\therefore$ letting $q \to p$, we obtain the equation of the tangent at P,

$$x+p^2y = 4p.$$

We now consider the equations of tangents to a curve through a given point not on the curve.

Example 4 Using the result of Example 3, find the equations of the tangents to the curve $xy = 4$ which pass through the point $(3, 1)$ and give the coordinates of their points of contact.

The tangent at the point $(2p, 2/p)$ has equation $x+p^2y = 4p$. If this tangent passes through $(3, 1)$ then $3+p^2 = 4p$

i.e. $$p^2-4p+3 = 0$$

$$(p-1)(p-3) = 0$$

$\therefore$ either $p = 1$ or $p = 3$.

Hence the required tangents have equations $x+y = 4$ and $x+9y = 12$, their points of contact being $(2, 2)$ and $(6, \frac{2}{3})$ respectively.

Example 5 Find the equations of the tangents to the circle $x^2+y^2 = 5$ which pass through the point $(5, 0)$. Find also the equation of the chord of contact.

The equation of the line through $(5, 0)$ with gradient m is $y = m(x-5)$.

Substituting in the equation $x^2+y^2=5$ to find the points of intersection

$$\begin{aligned} x^2+m^2(x-5)^2 &= 5 \\ x^2+m^2x^2-10m^2x+25m^2 &= 5 \\ (1+m^2)x^2-10m^2x+25m^2-5 &= 0 \qquad (1) \end{aligned}$$

The line will be a tangent to the circle if this quadratic equation in x has equal roots

i.e. if $(-10m^2)^2-4(1+m^2)(25m^2-5)=0 \qquad [\,b^2-4ac=0\,]$

$\therefore \quad 100m^4-100m^4-80m^2+20=0$

$\therefore \quad 1-4m^2=0$

$\therefore$ either $m=\frac{1}{2}$ or $m=-\frac{1}{2}$.

Hence the equations of the tangents through $(5, 0)$ are

$y=\frac{1}{2}(x-5)$ and $y=-\frac{1}{2}(x-5)$

i.e. $x-2y=5$ and $x+2y=5$.

When equation (1) has equal roots

$$x=\frac{10m^2}{2(1+m^2)}=\frac{10.\frac{1}{4}}{2(1+\frac{1}{4})}=1 \qquad \left[x=\frac{-b}{2a}\right]$$

$\therefore$ the x-coordinate of both points of contact is 1.

Hence the equation of the chord of contact is $x=1$.

Example 6 Show that the line $y=3x-4$ is a tangent to the curve $y=x^2-x$. Find also the condition that the line $y=mx+c$ should be a tangent to the curve.

At the points of intersection between the curve $y=x^2-x$ and the line $y=3x-4$,

$$\begin{aligned} x^2-x &= 3x-4 \\ x^2-4x+4 &= 0 \\ (x-2)^2 &= 0. \end{aligned}$$

Since this equation has equal roots the line is a tangent to the curve.

At any points of intersection between the curve $y=x^2-x$ and the line $y=mx+c$,

$$x^2-x=mx+c$$

$$\therefore \quad x^2-(m+1)x-c=0$$

The condition that this equation has equal roots is

$$(m+1)^2-4.1.(-c)=0, \quad \text{i.e.} \quad (m+1)^2+4c=0.$$

Hence the line $y=mx+c$ will be a tangent to the curve $y=x^2-x$ if $(m+1)^2+4c=0$.

Exercise 15.3

In questions 1 to 10 find the equations of the tangent and normal to the given curve at the stated point.

1. $y = 3+5x-x^2$; $(4, 7)$.
2. $y = x\sqrt{(x-1)}$; $(2, 2)$.
3. $y = x^2 - \frac{1}{x}$; $(-1, 2)$.
4. $y = \frac{2x-1}{x+2}$; $(-3, 7)$.
5. $3x^2+y^2 = 39$; $(1, -6)$.
6. $x^2+5xy+2y^2 = 8$; $(1, 1)$.
7. $y^2 = x^3-2$; $(3, 5)$.
8. $\frac{1}{x} - \frac{1}{y} = \frac{1}{6}$; $(2, 3)$.
9. $x = (2t+1)^2$, $y = t^2-t$; $t = -1$.
10. $x = t(t^2+4)$, $y = t^2+4$; $t = 2$.

11. Show that the following lines are tangents to the given curves and find their points of contact
(a) $x+y = 4$; $y = 3x-x^2$,
(b) $2x+y = 5$; $x^2+y^2-2x+4y = 0$,
(c) $5x-3y = 8$; $x = t+\frac{1}{t}$, $y = t-\frac{1}{t}$.

12. Find the equation of the normal to the curve $x = 3t-2t^2$, $y = 2+t^2$ which is parallel to the line $5x-4y = 0$.

13. Find the equations of the tangents to the curve $y = x^3-10x+5$ which are parallel to the line $2x-y = 5$.

14. Find the values of m for which the line $y = m(2x-1)$ touches the curve $y = x^2+4x$.

15. Find the equations of the tangents to the curve $y = x^2+2x+4$ which pass through the point $(2, 3)$ and the coordinates of the points of contact.

16. Find the equations of the tangents from the origin to the circle $x^2+y^2-4x-2y+4 = 0$. Find also the length of the chord of contact.

17. Find the values of c for which the line $y = x+c$ is a tangent to the circle $x^2+y^2-4x+2y-3 = 0$.

18. Find the values of c for which the line $2x-3y = c$ is a tangent to the curve $x^2+2y^2 = 2$ and find the equation of the line joining the points of contact.

19. Find the equations of the tangents to the circle $x^2+y^2 = 10$ which are parallel to the line $y = 3x$.

20. Find the equations of the tangents to the circle $x^2+y^2 = 9$ which pass

through the point (0, 5). Find also the acute angle between these tangents, giving your answer in degrees.

21. Show that the equation of the tangent to the curve $x = 4\cos\theta$, $y = 2\sin\theta$ at the point with parameter θ is $x\cos\theta + 2y\sin\theta = 4$. Hence find the equations of the tangents which pass through the point $(5, 0)$ and the coordinates of their points of contact.

22. Find the equation of the normal to the curve $x = 2t$, $y = t^2$ at the point with parameter t. If this normal meets the x- and y-axes at the points A and B respectively, find the equation of the locus of the mid-point of AB.

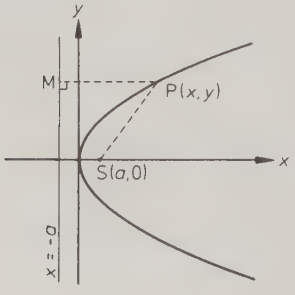
23. $P(p-1, p^2)$ and $Q(q-1, q^2)$ are two points on the curve $y = (x+1)^2$. Find the equation of the chord PQ and hence the equation of the tangent at P. If the tangent at P meets the line $x = -1$ at the point R, find the equation of the locus of the mid-point of PR.

24. $P(p^2, p^3)$ and $Q(q^2, q^3)$ are two points on the curve $y^2 = x^3$. Find the equation of the chord PQ and deduce the equation of the tangent at P. Given that the tangent at P passes through Q and is normal to the curve at Q, find the values of p and q.

15.4 The parabola

The locus of a point equidistant from a fixed point and a fixed line is called a *parabola*. The fixed point is the *focus* of the parabola and the fixed line is called the *directrix*.

The standard form of the equation of a parabola is obtained by letting the focus be the point $S(a, 0)$ and the directrix the line $x = -a$.



If $P(x, y)$ is any point on the parabola and M is the foot of the perpendicular from P to the directrix, then

$$SP^2 = PM^2$$

$$\therefore \quad (x-a)^2 + y^2 = (x+a)^2$$

$$\therefore \quad y^2 = 4ax$$

Thus the equation $y^2 = 4ax$ represents a parabola with focus $(a, 0)$, directrix $x = -a$.

The line of symmetry of the curve, in this case the x-axis, is called the *axis* of the parabola. The point where the curve cuts the axis, the origin $(0, 0)$, is referred to as the *vertex*.

Example 1 Find the equation of the tangent to the parabola $y^2 = 8x$ at the point $(2, 4)$.

$$y^2 = 8x$$

Differentiating with respect to x: $\quad 2y\dfrac{dy}{dx} = 8$

$$\therefore \quad \frac{dy}{dx} = \frac{8}{2y} = \frac{4}{y}.$$

$\therefore$ the gradient of the tangent at the point $(2, 4)$ is 1.

Hence its equation is $y-4 = 1(x-2)$, i.e. $y = x+2$.

By the same method it can be shown that the tangent at (x_1, y_1) to the curve $y^2 = 4ax$ has equation $yy_1 = 2a(x+x_1)$.

It is usually more convenient to express the equation $y^2 = 4ax$ in the parametric form $x = at^2$, $y = 2at$. Substituting in the original equation, we find that the point $(at^2, 2at)$ lies on the parabola for all values of the parameter t.

Example 2 Find the equations of the tangent and the normal to the curve $y^2 = 4ax$ at the point $(at^2, 2at)$.

Differentiating the parametric equations

$$x = at^2 \qquad y = 2at$$

$$\frac{dx}{dt} = 2at \qquad \frac{dy}{dt} = 2a$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{2a}{2at} = \frac{1}{t}$$

$\therefore$ the gradient of the tangent at the point $(at^2, 2at)$ is $1/t$.

Hence its equation is $y-2at = \dfrac{1}{t}(x-at^2)$, i.e.

$$ty-x = at^2.$$

The gradient of the normal at $(at^2, 2at)$ is $-t$.

Hence the equation of the normal is $y-2at = -t(x-at^2)$,

i.e. $y+tx = 2at+at^3$.

A *chord* of a parabola is a straight line which joins any two points on the curve. A chord which passes through the focus is a *focal chord*. The focal chord parallel to the directrix is called the *latus rectum*.

Example 3 Find the equation of the chord PQ of the parabola $x = at^2$, $y = 2at$, where P and Q are the points with parameters p and q respectively. Given that the chord PQ passes through the focus of the parabola, find the equation of the locus of the mid-point M of PQ as p and q vary.

The coordinates of P and Q are $(ap^2, 2ap)$ and $(aq^2, 2aq)$

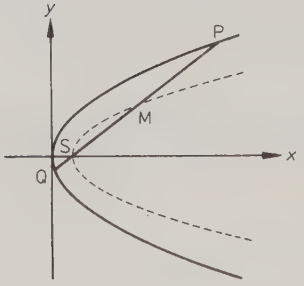
$\therefore$ the gradient of $PQ = \dfrac{2ap-2aq}{ap^2-aq^2} = \dfrac{2}{p+q}$.

Hence the equation of PQ is $y-2ap = \dfrac{2}{p+q}(x-ap^2)$

$$(p+q)(y-2ap) = 2(x-ap^2)$$
$$(p+q)y-2ap^2-2apq = 2x-2ap^2$$
i.e. $$(p+q)y-2x = 2apq.$$

Given that the focus $(a, 0)$ lies on this line

$$-2a = 2apq \qquad \therefore \quad pq = -1$$



The coordinates of M are

$$(\tfrac{1}{2}\{ap^2+aq^2\}, \tfrac{1}{2}\{2ap+2aq\})$$

$\therefore$ the equations of the locus of M are

$$x = \tfrac{1}{2}a(p^2+q^2), \quad y = a(p+q)$$
where $pq = -1$.

[To obtain the Cartesian equation of the locus, p and q must be eliminated from these three equations.]

Expressing x in terms of $p+q$ and pq,

$$x = \tfrac{1}{2}a\{(p+q)^2-2pq\}$$
$$\therefore \quad x = \tfrac{1}{2}a\left\{\frac{y^2}{a^2} - 2(-1)\right\} = \frac{y^2}{2a} + a.$$

Hence the equation of the locus of M is $y^2 = 2a(x-a)$.

Exercise 15.4

1. Sketch the following parabolas showing their foci and directrices

(a) $y^2 = 4x$, (b) $y^2 = 12x$, (c) $y^2 = -8x$,
(d) $y^2 = 4x+1$, (e) $y^2 = x$, (f) $y = x^2$,
(g) $x = 2t^2, y = 4t$, (h) $x = 3t^2, y = -6t$.

2. Write down the equations of the parabolas with the following foci and directrices

(a) $(2, 0); x = -2$, (b) $(5, 0); x = -5$,
(c) $(0, 1); y = -1$, (d) $(-3, 0); x = 3$,
(e) $(1, 1); x = -1$, (f) $(0, 2); y = 0$.

3. Derive the equations of the parabolas with the following foci and directrices

(a) $(3, 0)$; $x = 1$, (b) $(3, 1)$; $x = -3$,
(c) $(-1, 1)$; $x = -3$, (d) $(4, 2)$; $y = 3$.

4. Write down the parametric coordinates of a point on each of the following curves
(a) $y^2 = 8x$, (b) $y^2 = 24x$, (c) $y^2 = -16x$.

5. Find the equations of the tangents and normals to the following curves at the given points
(a) $y^2 = 16x$; $(1, 4)$, (b) $y^2 = 6x$; $(6, -6)$,
(c) $x = t^2$, $y = 2t$; $t = 2$, (d) $x = 3t^2$, $y = 6t$; $t = -1$.

6. The normal to the parabola $y^2 = 4ax$ at the point $P(at^2, 2at)$ meets the x-axis at A. Find the equation of the locus of the mid-point of AP as t varies.

7. The tangent to the parabola $y^2 = 4ax$ at the point $P(at^2, 2at)$ meets the x-axis at A and the y-axis at B. Find the equation of the locus of the mid-point of AB as t varies.

8. Show that, if the chord joining the points $P(ap^2, 2ap)$, $Q(aq^2, 2aq)$ on the parabola $y^2 = 4ax$ passes through $(a, 0)$, then $pq = -1$. Further, the tangent at P meets the line through Q parallel to the axis of the parabola at R. Prove that the line $x + a = 0$ bisects PR. (O & C)

9. Find the condition that the line $y = mx + c$ should be a tangent to the parabola $y^2 = 4ax$. Use this result to find the equations of the tangents to the curve $y^2 = 4x$ which pass through the point $(-2, 1)$.

10. The tangents at $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$ to the parabola $y^2 = 4ax$ meet at a point R. Find the coordinates of R. If R lies on the line $2x + a = 0$, find the equation of the locus of the mid-point of PQ.

11. The points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$ lie on the parabola $y^2 = 4ax$. Prove that if PQ is a focal chord then the tangents to the curve at P and Q intersect at right angles at a point on the directrix.

12. Find the gradient of the normal to the parabola $y^2 = 4ax$ at the point $P(ap^2, 2ap)$. Find the slope of the chord joining the point $P(ap^2, 2ap)$ to another point $Q(aq^2, 2aq)$. The normal at a point $P(ap^2, 2ap)$ meets the parabola again at a point $Q(aq^2, 2aq)$. By treating this line both as a normal and a chord, or otherwise, prove that $p^2 + pq + 2 = 0$.

The normal at a point $R(4a, 4a)$ meets the parabola again at a point S. The normal at S meets the parabola again at T. What are the coordinates of T? Find the length of RS, giving your answer in simplified surd form. (SU)

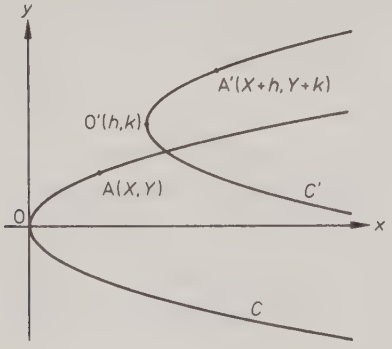
13. The tangents at the points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$ on the parabola $y^2 = 4ax$ intersect at the point R. Given that the tangent at P is perpendicular to the chord OQ, where O is the origin, find the equation of the locus of R as p varies.

14. Find the equation of the tangent to the parabola $y^2 = 4ax$ at the point $P(at^2, 2at)$. The line through O, parallel to this tangent, meets the parabola again at Q. Show that the line through P, parallel to the axis of the parabola, passes through the midpoint of OQ. Show also that, if the tangent and normal at P meet the x-axis at T and N respectively, the area of the triangle TPN is $2a^2t(1+t^2)$. (L)

15. The tangent to the parabola $y^2 = 4ax$ at the point $P(at^2, 2at)$ meets the x-axis at T. The straight line through P parallel to the axis of the parabola meets the directrix at Q. If S is the focus of the parabola, show that $PQTS$ is a rhombus. If M is the mid-point of PT and N is the mid-point of PM, find the equation of the locus of (i) M, (ii) N. (AEB 1978)

15.5 Translations and change of origin

In the previous section the parabola $y^2 = 4ax$ was considered in some detail. We now look at ways of using our knowledge of this 'standard' parabola to determine the properties of a parabola whose vertex is not at the origin.



The sketch shows the curve C with equation $y^2 = 4ax$ and its image C' under a translation with column vector $\begin{pmatrix} h \\ k \end{pmatrix}$. The image of the origin O under this translation is the point $O'(h, k)$. The image of a typical point $A(X, Y)$ on the curve C is the point $A'(X+h, Y+k)$. Thus at a typical point A' on C'

$$x = X+h, \quad y = Y+k \quad \text{where} \quad Y^2 = 4aX.$$

Eliminating X and Y we find that the Cartesian equation of C' is

$$(y-k)^2 = 4a(x-h).$$

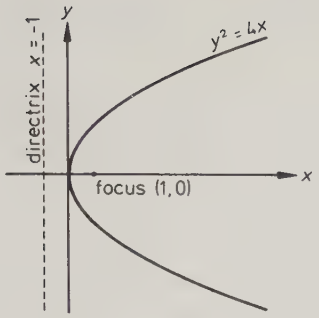
Hence the equation $(y-k)^2 = 4a(x-h)$ represents the image of the parabola $y^2 = 4ax$ under a translation in which the point $(0, 0)$ is mapped to the point (h, k).

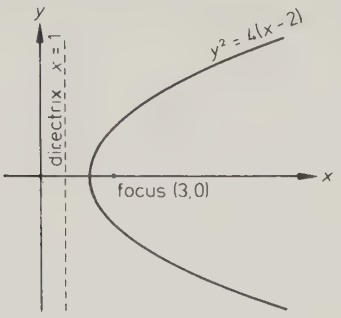
Example 1 Show that the equation $y^2 = 4x-8$ represents a parabola. Find its focus and directrix.

The equation may be written in the form $y^2 = 4(x-2)$.

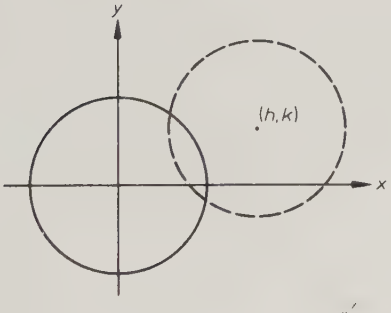
Hence the equation represents the image of the curve $y^2 = 4x$ under a translation which maps the point $(0, 0)$ to the point $(2, 0)$. The curve $y^2 = 4x$ is a parabola with focus $(1, 0)$ and directrix $x = -1$

$\therefore$ the curve $y^2 = 4x-8$ is a parabola with focus $(3, 0)$ and directrix $x = 1$.

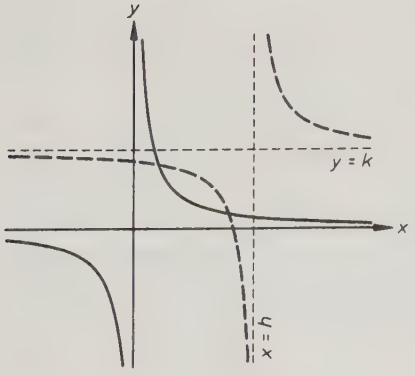




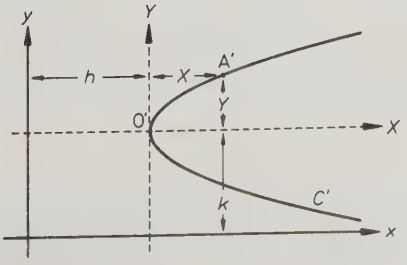
The result obtained for the parabola can be generalised. If the equation $f(x, y) = 0$ represents a curve in the x, y plane, then the equation $f(x-h, y-k) = 0$ represents the image of that curve under a translation with column vector $\begin{pmatrix} h \\ k \end{pmatrix}$.



For instance, the curve $x^2+y^2 = a^2$ is a circle, centre $(0, 0)$, radius a. The curve $(x-h)^2+(y-k)^2 = a^2$ is a circle, centre (h, k), radius a.



Similarly, the equation $xy = c^2$ represents a curve called a rectangular hyperbola with the lines $x = 0$, $y = 0$ as asymptotes. The curve $(x-h)(y-k) = c^2$ is a rectangular hyperbola with asymptotes $x = h$, $y = k$.



We now look briefly at a different approach to curves with equations of the form $f(x-h, y-k) = 0$. Consider again a typical point $A'(x, y)$ on the curve C' as defined earlier.

Relative to X- and Y-axes through a new origin $O'(h, k)$ the coordinates of A' are (X, Y), where $X = x-h$, $Y = y-k$. As before, since $Y^2 = 4aX$, the equation of C' must be $(y-k)^2 = 4a(x-h)$.

In general, if the point (x, y) has coordinates (X, Y) relative to a new origin at (h, k) then the equation $f(x-h, y-k) = 0$ may be written in the form $f(X, Y) = 0$.

Example 1 (alternative method) Let the point (x, y) have coordinates (X, Y) with respect to a new origin at $(2, 0)$, then $X = x-2$ and $Y = y$. The equation $y^2 = 4(x-2)$ then becomes $Y^2 = 4X$. Relative to the new X- and Y-axes the equation $Y^2 = 4X$ represents a parabola with focus $(1, 0)$ and directrix $X = -1$. Hence, relative to the original x- and y-axes, the equation $y^2 = 4x-8$ represents a parabola with focus $(3, 0)$ and directrix $x = 1$.

Exercise 15.5

1. Find the images of the points $(1, 3)$, $(0, 4)$ and $(-2, 5)$ under a translation in which the point $(0, 0)$ is mapped to the point
(a) $(1, 0)$, (b) $(0, -2)$, (c) $(-1, 2)$.

2. Find the coordinates of the image of the point $(0, 0)$ under a translation which maps the given curves as follows:
(a) $x^2+y^2 = 4 \rightarrow (x-1)^2+(y-2)^2 = 4$,
(b) $y^2 = 4x \rightarrow (y+1)^2 = 4(x-3)$,
(c) $x^2+y^2 = 1 \rightarrow x^2+y^2-2y = 0$,
(d) $xy = 4 \rightarrow xy+3x+2y+6 = 4$.

3. Show that each of the following equations represents a parabola and find its vertex, focus and directrix
(a) $y^2 = 4(x-3)$, (b) $(y-1)^2 = 8x$, (c) $y^2 = 4x+20$,
(d) $y^2+4y = x$, (e) $4y = x^2+4$, (f) $y = x^2-2x$.

4. Show that each of the following pairs of parametric equations represents a parabola and find its vertex, focus and directrix.
(a) $x = t^2+1$, $y = 2t+3$, (b) $x = 3t^2-2$, $y = 6t+5$,
(c) $x = 1-t^2$, $y = 2t$, (d) $x = 4t-3$, $y = 2t^2-1$.

5. Sketch the following pairs of related curves
(a) $y = x^3$, $y = x^3-3x^2+3x+1$,
(b) $xy = 1$, $xy = 2x+y-1$,
(c) $y^2 = -12x$, $y^2 = 12(y-x)$.

Exercise 15.6 (miscellaneous)

1. Find those points P on the curve with equation $y = x^2-2$ such that the normal to the curve at P passes through the point $(0, 0)$. (AEB 1977)

2. Find the condition that the line $y = mx + c$ should be a tangent to the circle $x^2 + y^2 = a^2$.

3. A curve is defined by the parametric equations

$$x = \theta - \sin\theta, \quad y = 1 - \cos\theta, \quad 0 < \theta < 2\pi.$$

Show that $dy/dx = \cot\theta/2$, and find the equation of the tangent and of the normal to the curve at the point where $\theta = \pi/2$. (JMB)

4. Find the equations of the tangent and the normal to the curve $y^2(y-1) = x^2(x+3)$ at the point $(1,2)$.

5. Find the centres and radii of the circles C_1 and C_2 whose equations are $x^2 + y^2 - 2x = 0$ and $x^2 + y^2 - 2y = 0$ respectively. Draw a figure to illustrate the circles. The line $y = mx$ through the origin O meets C_1 and C_2 again at P and Q. Find the coordinates of P and Q. Let A denote the second point of intersection of the two circles. Show that AP and AQ are perpendicular. (W)

6. Find the equation of the circle which passes through the points $A(2,0)$, $B(10,4)$ and $C(5,9)$ and show that it touches the y-axis. If the tangents at A and B intersect at D find the coordinates of D, the length BD and the angle ADB correct to the nearest degree. (SU)

7. The circle S has the equation $x^2 + y^2 - 6x - 8y = 0$. (i) Find the coordinates of the centre, and the radius, of S. (ii) Prove, by calculation, that the point A, with coordinates $(7,2)$, lies inside the circle S. (iii) Show that the chord of S which is bisected at A has equation $y - 2x + 12 = 0$. (iv) Find the equations of the two tangents to S which are parallel to the line $y = 2x$. (C)

8. Prove that the point $B(1,0)$ is the mirror-image of the point $A(5,6)$ in the line $2x + 3y = 15$. Find the equation of (a) the circle on AB as diameter, (b) the circle which passes through A and B and touches the x-axis. (L)

9. Find the equation of the tangent to the circle $x^2 + y^2 = a^2$ at the point $T(a\cos\theta, a\sin\theta)$. This tangent meets the line $x + a = 0$ at R. If RT is produced to P so that $RT = TP$, find the coordinates of P in terms of θ and find the coordinates of the points in which the locus of P meets the y-axis. (L)

10. A circle passes through the points A, B and C which have coordinates $(0,3)$, $(\sqrt{3},0)$ and $(-\sqrt{3},0)$ respectively. Find: (i) the equation of the circle, (ii) the length of the minor arc BC, (iii) the equation of the circle on AB as diameter. A line $y = mx - 3$ of variable gradient m cuts the circle ABC in two points L and M. Find in cartesian form the equation of the locus of the mid-point of LM.
(AEB 1976)

11. Find the equations of the two circles each of which touches both coordinate axes and passes through the point $(9,2)$. Find (i) the coordinates of the second

point of intersection of these circles, (ii) the equation of the common chord of the two circles. (C)

12. The point $A(4, 1)$ lies on the line whose equation is $3x-4y-8=0$. A circle touches this line at A and passes through the point $B(5, 3)$. Find the equation of the circle, and show that it touches the y-axis. Find also the equation of the line parallel to AB on which the circle cuts off a chord equal in length to AB. (C)

13. Sketch for $0 \leqslant t \leqslant 2\pi$ the curve given parametrically by $x = a\cos^3 t$, $y = a\sin^3 t$. Show that $dy/dx = -\tan t$. Find and simplify the equation of the tangent at the point where $t = \alpha$. If this tangent meets the axes at A and B, show that the length of AB is independent of x. (L)

14. A curve is given parametrically by the equations $x = a(2+t^2)$, $y = 2at$. Find the values of the parameter t at the points P and Q in which this curve is cut by the circle with centre $(3a, 0)$ and radius $5a$. Show that the tangents to the curve at P and Q meet on the circle, and that the normals to the curve at P and Q also meet on the circle. (L)

15. The parametric equations of a curve are $x = \cos 2t$, $y = 4\sin t$. Sketch the curve for $0 \leqslant t \leqslant \frac{1}{2}\pi$. Show that $dy/dx = -\operatorname{cosec} t$ and find the equation of the tangent to the curve at the point $A(\cos 2T, 4\sin T)$. The tangent at A crosses the x-axis at the point M and the normal at A crosses the x-axis at the point N. If the area of the triangle AMN is $12\sin T$, find the value of T between 0 and $\frac{1}{2}\pi$.
(AEB 1976)

16. A curve is defined by the parametric form $x = a(\cos 3\theta - 3\cos\theta)$, $y = a(\sin 3\theta + 3\sin\theta)$. Prove that $dy/dx = -\cot\theta$, and hence show that the normal to the curve at the point with parameter θ is given by $x\sin\theta - y\cos\theta + 4a\sin 2\theta = 0$. Prove that the distance between the points where the normal meets the coordinate axes is independent of θ. (O)

17. Find the equation of the tangent to the curve $y = \frac{5}{12}x^3 - \frac{13}{9}x$ at the point P at which $x = x_0$. Show that the x-coordinate of the point Q where this tangent meets the curve again is $-2x_0$, and find the values of x_0 for which the tangent at P is the normal at Q. (O)

18. Find the equations of the tangent and the normal to the parabola $y^2 = 4ax$ at the point P with parameter p. (i) Show that, if the tangent at P meets the directrix at L, then $PL = a(p^2+1)^{3/2}/p$. (ii) Show that, if the tangent at P is parallel to the normal at a point Q, then PQ passes through the focus of the parabola. (SU)

19. Find the equation of the normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ and the coordinates of the point at which this normal cuts the x-axis. Show that the equation of the circle which touches this parabola at the points $(at^2, 2at)$ and $(at^2, -2at)$ is $(x-2a-at^2)^2+y^2 = 4a^2(1+t^2)$. Find the values of t for which this circle passes through the point $(9a, 0)$. (L)

20. Prove that the chord joining the points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$ on the parabola $y^2 = 4ax$ has the equation $(p+q)y = 2x + 2apq$. A variable chord PQ of the parabola is such that the lines OP and OQ are perpendicular, where O is the origin. (i) Prove that the chord PQ cuts the axis of x at a fixed point, and give the x-coordinate of this point. (ii) Find the equation of the locus of the mid-point of PQ. (C)

21. Prove that the tangent at the point $(at^2, 2at)$ on the parabola $y^2 = 4ax$ has the equation $ty = x + at^2$. Find, in their simplest form, the coordinates of T, the point of intersection of the tangents at the points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$ on the parabola $y^2 = 4ax$. If PQ is of constant length d, show that T lies on the curve whose equation is

$$(y^2 - 4ax)(y^2 + 4a^2) = a^2d^2. \qquad \text{(C)}$$

22. Show that the equation of the normal to the parabola $y^2 = 4ax$ at the point $P(ap^2, 2ap)$ is $y + px = 2ap + ap^3$. Find the coordinates of R, the point of intersection of the normal at P and the normal at $Q(aq^2, 2aq)$. Given that the chord PQ passes through $S(a, 0)$, show that $pq = -1$ and find the equation of the locus of R. (AEB 1978)

23. Show that the equation of the normal to the parabola $y^2 = 4ax$ at the point $P(ap^2, 2ap)$ is $px + y = 2ap + ap^3$. The tangent at P meets the x-axis at A and the y-axis at B. The normal at P meets the x-axis at C and S is the point $(a, 0)$. Show that the areas of the triangles APS and SPC are equal. Show also that the locus of the mid-point of PS is a parabola through the mid-point of OS. If BS and OP meet at Q, show that the equation of the locus of Q is $2x^2 + y^2 = 2ax$. (AEB 1977)

24. Prove that the mid-points of chords of the parabola $y^2 = 4ax$ that are parallel to the line $y = mx$ lie on the line $y = 2a/m$. Hence, or otherwise, find the equation of the tangent to the parabola that is parallel to the line $y = mx$. (O)

25. Find the equation of the tangent to the parabola $y^2 = 4ax$ at the point $P(ap^2, 2ap)$. Show that the equation of the normal at P is $y = p(2a - x) + ap^3$. If the tangents at P and $Q(aq^2, 2aq)$ meet at T show that T is the point $(apq, ap + aq)$. The point N is the intersection of the normals at P and Q. Given that T lies on the line $x + 2a = 0$ show that N lies on the parabola with equation $y^2 = 4a(x - 4a)$. (L)

26. Prove that the line $y = mx + 15/4m$ is a tangent to the parabola $y^2 = 15x$ for all non-zero values of m. Using this result or otherwise find the equations of the common tangents to this parabola and the circle $x^2 + y^2 = 16$. (L)

27. Find the equation of the normal at the point $P(at^2, 2at)$ to the parabola $y^2 = 4ax$. The focus of the given parabola is the point S, $(a, 0)$. If PN is the normal at the point P and SN is parallel to the tangent at P, find the coordinates of the point N. Deduce that the locus of N for variable P is a parabola and find the coordinates of its vertex. (O & C)

28. Prove that the normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ has equation $y + tx = 2at + at^3$. The normals at the points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$ intersect at the point R. Find the coordinates of R in terms of $(p+q)$ and pq. If O is the vertex of the parabola and P and Q are variable points such that $P\hat{O}Q$ is a right angle, find the locus of R; verify that it is a parabola, and find the coordinates of its vertex. (O & C)

29. A fixed point $P(ap^2, 2ap)$ is taken on the parabola $y^2 = 4ax$. Two points Q and R are chosen on the parabola so that the lines PQ and PR are perpendicular. Prove that the line QR passes through a fixed point F, independent of Q and R, and that PF is normal to the parabola at P. (O)

30. A circle with centre at the point $P(h,k)$ touches the y-axis, and passes through the point $S(2,0)$. Show that P lies on the curve $y^2 = 4x - 4$, and sketch this curve. Show that the straight line joining $P(h,k)$ to the point $Q(2+h, 0)$ cuts the curve $y^2 = 4x - 4$ at right angles at P. (L)

31. Find the equation of the normal to the parabola $y^2 = 4x$ at the point $P(p^2, 2p)$. If the normals at P and $Q(q^2, 2q)$ meet at $R(\alpha, \beta)$ show that $\alpha = 2 + p^2 + pq + q^2$, $\beta = -pq(p+q)$. The point of intersection of PQ with the x-axis divides PQ internally in the ratio $1:\lambda$. Prove that $q = -\lambda p$. Given that $\lambda = 2$, find the Cartesian equation of the locus of R as p varies. Determine the coordinates of the point on this locus which is nearest the origin. (JMB)

32. Find the equations of the tangents to the curve $27y^2 = 4x^3$ at the points $P(3p^2, 2p^3)$ and $Q(3q^2, 2q^3)$. Show that these tangents intersect at the point $R(\alpha, \beta)$ where $\alpha = p^2 + pq + q^2$, $\beta = pq(p+q)$. The points P and Q move along the curve in such a way that the tangents at P and Q are always perpendicular. Prove that R moves on the parabola $y^2 = x - 1$. Verify that this parabola touches the curve $27y^2 = 4x^3$ at the points $(3/2, \pm 1/\sqrt{2})$. (JMB)

33. The straight line through the point $A(-a, 0)$ at an angle θ to the positive direction of the x-axis meets the circle $x^2 + y^2 = a^2$ at P, distinct from A. The circle on AP as diameter is denoted by C. (i) By finding the equation of C, or otherwise, show that if C touches the y-axis, then $\cos 2\theta = 2 - \sqrt{5}$. (ii) C meets the x-axis at M, distinct from A, and the tangents to C at A and M meet at Q. Find the coordinates of Q in terms of θ and show that, as θ varies, Q always lies on the curve $y^2 x + (x+a)^3 = 0$. (JMB)

16 Inequalities

16.1 Basic inequalities

The most important rules for manipulating inequalities are as follows:

(1) Any number may be added to or subtracted from both sides of an inequality, e.g.

$$x < y \Rightarrow x+3 < y+3.$$

(2) Both sides of an inequality may be multiplied or divided by the same *positive* number, e.g.

$$x < y \Rightarrow 3x < 3y.$$

(3) If both sides of an inequality are multiplied or divided by the same *negative* number, the inequality is reversed, e.g.

$$-2x < 6y \Rightarrow x > -3y.$$

[For further examples of the use of these rules see §1.4.]

In this chapter we will be considering two main types of inequality. This section deals briefly with basic inequalities which hold for all values of the variables involved. Later sections are concerned with finding the solution sets of inequalities which hold for only certain values of the variables. The distinction between these two types of inequality broadly corresponds to that made in earlier work between identities and equations.

Most basic inequalities are established using the fact that the square of a real number is never negative.

Example 1 Prove that $a^2+b^2 \geqslant 2ab$ for all real values of a and b.

As any square is positive or zero, $(a-b)^2 \geqslant 0$

$$\therefore \quad a^2-2ab+b^2 \geqslant 0$$

Hence $$a^2+b^2 \geqslant 2ab.$$

The result of Example 1 can be used to derive further inequalities.

Example 2 Prove that $a^2+b^2+c^2 \geqslant ab+bc+ca$ for all real values of a, b and c.

$$a^2+b^2 \geqslant 2ab, \quad b^2+c^2 \geqslant 2bc, \quad c^2+a^2 \geqslant 2ca.$$

Adding $$2(a^2+b^2+c^2) \geqslant 2(ab+bc+ca).$$

Hence $$a^2+b^2+c^2 \geqslant ab+bc+ca.$$

Example 3 Prove that $\dfrac{1}{a}+\dfrac{1}{b} \geqslant \dfrac{4}{a+b}$ for all positive values of a and b.

$$\frac{1}{a}+\frac{1}{b}-\frac{4}{a+b} = \frac{b(a+b)+a(a+b)-4ab}{ab(a+b)}$$

$$= \frac{a^2-2ab+b^2}{ab(a+b)} = \frac{(a-b)^2}{ab(a+b)}$$

$\therefore$ provided that a and b are positive $\dfrac{1}{a}+\dfrac{1}{b}-\dfrac{4}{a+b} \geqslant 0$

$$\text{i.e.} \quad \frac{1}{a}+\frac{1}{b} \geqslant \frac{4}{a+b}.$$

Alternative method

$$a^2+b^2 \geqslant 2ab \Rightarrow (a+b)^2 \geqslant 4ab.$$

Dividing both sides by $ab(a+b)$, assumed positive

$$\frac{a+b}{ab} \geqslant \frac{4}{a+b}$$

$$\therefore \quad \frac{1}{a}+\frac{1}{b} \geqslant \frac{4}{a+b}.$$

Exercise 16.1

1. State whether each of the following statements is true or false. If you decide that a statement is false, show, by a numerical example, that this is so.

(a) $x^2 < 4 \Rightarrow x < 2$,
(b) $x^2 > 4 \Rightarrow x > 2$,
(c) $x < 4 \Rightarrow \dfrac{1}{x} > \dfrac{1}{4}$,
(d) $x > 4 \Rightarrow \dfrac{1}{x} < \dfrac{1}{4}$.

2. Given that a and b are positive, state the range of values of x for which the following statements are true.

(a) $ax \leqslant bx \Rightarrow a \leqslant b$,
(b) $x-a < x-b \Rightarrow a > b$,
(c) $\dfrac{x}{a} \geqslant \dfrac{x}{b} \Rightarrow a \geqslant b$,
(d) $a > b \Rightarrow \dfrac{1}{x+a} < \dfrac{1}{x+b}$.

3. Prove that for any real numbers p and q

(a) $p^2+q^2 \geqslant 2pq$,
(b) $p^2+4q^2 \geqslant 4pq$,
(c) $(p+q)^2 \geqslant 4pq$,
(d) $(p+q)^2 \leqslant 2(p^2+q^2)$.

4. Prove that if a and b are positive, then

(a) $\frac{a}{b}+\frac{b}{a} \geqslant 2$, (b) $a^3+b^3 \geqslant ab(a+b)$.

5. Prove that for any real numbers p, q, r, s

(a) $p^4+q^4 \geqslant 2p^2q^2$, (b) $p^4+q^4+r^4+s^4 \geqslant 4pqrs$.

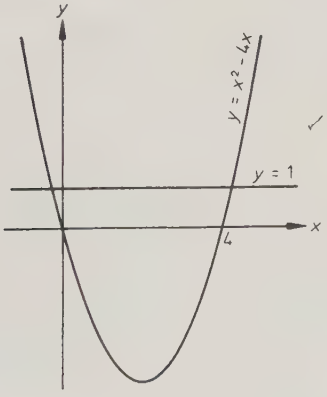
6. Prove that for any real numbers x, y, z

$3(xy+yz+zx) \leqslant (x+y+z)^2 \leqslant 3(x^2+y^2+z^2)$.

16.2 Graphical approach

We now consider the graphical approach to finding the set of values of x which satisfy an inequality of the form $f(x) < g(x)$. This method is particularly useful when the graphs of the functions $f(x)$ and $g(x)$ are fairly easy to sketch.

Example 1 Find the values of x for which $x^2-4x < 1$.



Consider the graphs of $y = x^2-4x$ and $y = 1$.

At their points of intersection

$$x^2-4x = 1$$

$$x^2-4x-1 = 0$$

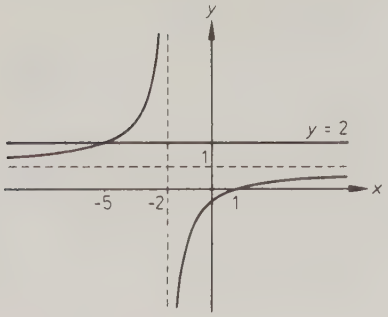
$$\therefore \quad x = \frac{4 \pm \sqrt{(4^2-4.1\{-1\})}}{2}$$

$$= \frac{4 \pm \sqrt{20}}{2} = 2 \pm \sqrt{5}.$$

The inequality $x^2-4x < 1$ is satisfied when the curve $y = x^2-4x$ is below the line $y = 1$

i.e. when $2-\sqrt{5} < x < 2+\sqrt{5}$.

Example 2 Find the values of x for which $0 \leqslant \frac{x-1}{x+2} \leqslant 2$.



Consider the curve $y = \dfrac{x-1}{x+2}$.

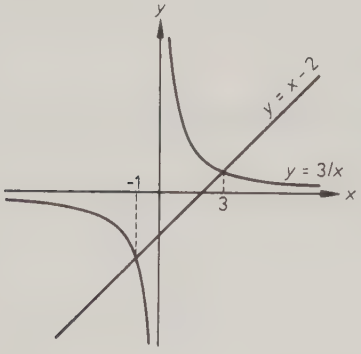
It cuts the axes at the points $(1,0)$ and $(0, -\frac{1}{2})$. The lines $x = -2$ and $y = 1$ are asymptotes to the curve. [See §20.4.]

$$\frac{x-1}{x+2} = 2 \Rightarrow x-1 = 2x+4$$
$$\Rightarrow x = -5$$

$\therefore$ the curve cuts the line $y = 2$ when $x = -5$.

Hence, from the sketch, $0 \leqslant \dfrac{x-1}{x+2} \leqslant 2$ when $x \leqslant -5$ and when $x \geqslant 1$.

Example 3 Find the values of x for which $x-2 > \dfrac{3}{x}$.



Consider the graphs $y = x-2$ and $y = \dfrac{3}{x}$.

At their points of intersection

$$x-2 = \frac{3}{x}$$

i.e. $x^2-2x-3 = 0$

$$(x+1)(x-3) = 0$$

$\therefore$ either $x = -1$ or $x = 3$.

The inequality $x-2 > \dfrac{3}{x}$ is satisfied when the line $y = x-2$ is above the curve $y = \dfrac{3}{2}$, i.e. when

$-1 < x < 0$ and when $x > 3$.

Exercise 16.2

With the aid of sketch graphs find the values of x which satisfy the following inequalities.

1. $x^2-x-6 < 0$.

2. $2x^2+7x+3 \geqslant 0$.

3. $6x-x^2 \geqslant 5$.

4. $x^2-2x > 5$.

5. $4x < 3-x^2$.

6. $x^3-3x^2 \leqslant 10x$.

7. $\dfrac{1}{x-3} > 1$.

8. $\dfrac{x}{x+1} < 2$.

9. $0 < \dfrac{2x-4}{x-1} < 1$.

10. $2 < \dfrac{x-7}{x-2} < 3$.

11. $3-x > \dfrac{2}{x}$.

12. $2x+5 < \dfrac{3}{x}$.

13. $\dfrac{3}{x+2} < x$.

14. $x-2 > \dfrac{2}{x-1}$.

15. $\dfrac{12}{x-3} < x+1$.

16. $\dfrac{1}{x-1} < \dfrac{1}{x+1}$.

17. $\dfrac{x+1}{2x-3} < \dfrac{1}{x-3}$.

18. $\dfrac{1}{x^2} < \dfrac{1}{x+2}$.

19. $\pi \sin x > 2x$.

20. $2\pi \cos x < 3x$.

16.3 Analytical methods

If $(x-a)$ is a factor of $f(x)$ then $f(x)$ may change sign as x passes through the value a. This statement forms the basis of the analytical approach to inequalities.

Example 1 Find the values of x for which $x(2x-5) > 3$.

$$x(2x-5) > 3 \Leftrightarrow \quad 2x^2-5x-3 > 0$$
$$\Leftrightarrow (2x+1)(x-3) > 0.$$

[We now construct a table of signs for the function $(2x+1)(x-3)$, noting that sign changes may occur at $x = -\frac{1}{2}$ and $x = 3$.]

	$x < -\frac{1}{2}$	$-\frac{1}{2} < x < 3$	$x > 3$
$2x+1$	$-$	$+$	$+$
$x-3$	$-$	$-$	$+$
$(2x+1)(x+3)$	$+$	$-$	$+$

Hence $x(2x-5) > 3$ when $x < -\frac{1}{2}$ or $x > 3$.

Example 2 Find the values of x for which $x(x^2-2) < x^2$.

$$x(x^2-2) < x^2 \Leftrightarrow \quad x^3-x^2-2x < 0$$
$$\Leftrightarrow x(x+1)(x-2) < 0.$$

The function $f(x) = x(x+1)(x-2)$ changes sign at $x = -1$, $x = 0$ and $x = 2$.

	$x < -1$	$-1 < x < 0$	$0 < x < 2$	$x > 2$
$x+1$	$-$	$+$	$+$	$+$
x	$-$	$-$	$+$	$+$
$x-2$	$-$	$-$	$-$	$+$
$f(x)$	$-$	$+$	$-$	$+$

Hence the inequality holds for $x < -1$ and $0 < x < 2$.

Great care must be taken when dealing with inequalities involving fractions. In general, it is inadvisable to 'multiply through' by a variable denominator which may be positive or negative. The most reliable method is to collect all the terms on one side of the inequality.

Example 3 Find the values of x for which $\dfrac{x^2-12}{x} > -1$.

$$\frac{x^2-12}{x} > -1 \Leftrightarrow \quad \frac{x^2-12}{x} + 1 > 0$$
$$\Leftrightarrow \quad \frac{x^2+x-12}{x} > 0$$
$$\Leftrightarrow \frac{(x+4)(x-3)}{x} > 0.$$

The function $f(x) = (x+4)(x-3)/x$ changes sign as x passes through the values -4, 0 and 3.

	$x < -4$	$-4 < x < 0$	$0 < x < 3$	$x > 3$
$x+4$	$-$	$+$	$+$	$+$
x	$-$	$-$	$+$	$+$
$x-3$	$-$	$-$	$-$	$+$
$f(x)$	$-$	$+$	$-$	$+$

Hence the inequality holds for $-4 < x < 0$ and $x > 3$.

Example 4 For what values of x is $\dfrac{x}{x+8} \leqslant \dfrac{1}{x-1}$?

$$\frac{x}{x+8} \leqslant \frac{1}{x-1} \Leftrightarrow \frac{x}{x+8} - \frac{1}{x-1} \leqslant 0$$

$$\Leftrightarrow \frac{x(x-1)-(x+8)}{(x+8)(x-1)} \leqslant 0$$

$$\Leftrightarrow \frac{x^2-2x-8}{(x+8)(x-1)} \leqslant 0$$

$$\Leftrightarrow \frac{(x+2)(x-4)}{(x+8)(x-1)} \leqslant 0.$$

The function $f(x) = \dfrac{(x+2)(x-4)}{(x+8)(x-1)}$ passes through the value zero when $x = -2$ and when $x = 4$. The function is undefined at $x = -8$ and $x = 1$ and has different signs on either side of these values.

	$x < -8$	$-8 < x < -2$	$-2 < x < 1$	$1 < x < 4$	$x > 4$
$x+8$	−	+	+	+	+
$x+2$	−	−	+	+	+
$x-1$	−	−	−	+	+
$x-4$	−	−	−	−	+
$f(x)$	+	−	+	−	+

Hence $\dfrac{x}{x+8} \leqslant \dfrac{1}{x-1}$ when $-8 < x \leqslant -2$ and $1 < x \leqslant 4$.

Exercise 16.3

Find the values of x which satisfy the following inequalities.

1. $x^2+2x < 15$.
2. $3x^2+2 > 7x$.
3. $x^2(x-5) > 6x$.
4. $x(x^2+4) < 5x^2$.
5. $\dfrac{x^2+12}{x} > 7$.
6. $\dfrac{x^2+6}{x} > 5$.
7. $\dfrac{(x-1)(x+3)}{(x-2)} < 0$.
8. $\dfrac{(2x-3)}{(x+2)(x-5)} > 0$.
9. $\dfrac{6}{x-1} > 1$.
10. $\dfrac{2x-4}{x-1} < 1$.

11. $\dfrac{8}{x+2} > x.$

12. $\dfrac{6}{x-4} < x+1.$

13. $\dfrac{5-x}{x^2-3x+2} < 1.$

14. $\dfrac{x+6}{x(x+1)} < 6.$

15. $\dfrac{1}{x+1} < \dfrac{1}{x+4}.$

16. $\dfrac{1}{x+2} > \dfrac{1}{2x-3}.$

17. $\dfrac{x}{x-2} > \dfrac{1}{x}.$

18. $\dfrac{x+1}{2x-3} < \dfrac{1}{x-3}.$

16.4 Modulus inequalities

The modulus notation was introduced in §1.4. From the definition it follows that if $|f(x)| < a$, where a is a positive constant, then $-a < f(x) < a$.

Example 1 Find the values of x for which $|2x+1| < 3$.

$$\begin{aligned} |2x+1| < 3 &\Leftrightarrow -3 < 2x+1 < 3 \\ &\Leftrightarrow -4 < 2x < 2 \\ &\Leftrightarrow -2 < x < 1. \end{aligned}$$

Hence the inequality is satisfied when $-2 < x < 1$.

Some inequalities can be simplified by squaring both sides. However, it is important to remember that this method is valid only when both sides of the inequality are positive or zero for all values of x.

Example 2 Find the values of x for which $2|x-1| < |x+3|$.

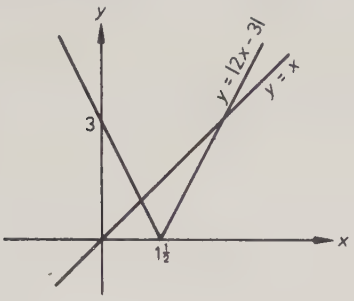
Since both sides of the inequality are positive or zero for all values of x

$$\begin{aligned} 2|x-1| < |x+3| &\Leftrightarrow 4(x-1)^2 < (x+3)^2 \\ &\Leftrightarrow 4x^2-8x+4 < x^2+6x+9 \\ &\Leftrightarrow 3x^2-14x-5 < 0 \\ &\Leftrightarrow (3x+1)(x-5) < 0. \end{aligned}$$

Hence the inequality holds if $-\frac{1}{3} < x < 5$.

In harder examples it may be possible to adapt the methods of Examples 1 and 2 or to use sketch graphs. [For work on graphs with equations of the form $y = |f(x)|$ see §§14.2, 14.3.]

Example 3 Find the values of x for which $|2x-3| > x$.



The graphs of $y = |2x-3|$ and $y = x$ have two points of intersection, where $2x-3 = x$, i.e. $x = 3$ and where $-(2x-3) = x$, i.e. $x = 1$. Hence, from the sketch, $|2x-3| > x$ when $x < 1$ and when $x > 3$.

Alternative method

Since $|2x-3|$ is never negative the inequality must hold when $x < 0$.

When $x \geqslant 0$, $|2x-3| > x \Leftrightarrow (2x-3)^2 > x^2$

$\Leftrightarrow 4x^2-12x+9 > x^2$

$\Leftrightarrow 3x^2-12x+9 > 0$

$\Leftrightarrow x^2-4x+3 > 0$

$\Leftrightarrow (x-1)(x-3) > 0$

$\Leftrightarrow x < 1$ or $x > 3$.

Thus, considering all values of x, the inequality holds for $x < 1$ and $x > 3$.

Exercise 16.4

Find the values of x which satisfy the following inequalities.

1. $|x-2| > 1$.
2. $|x+3| \leqslant 5$.
3. $|3x+5| < 4$.
4. $|2x-1| \geqslant 11$.
5. $|x| \leqslant |x-1|$.
6. $2|x+2| > |x+3|$.
7. $3|x-2| \geqslant |x+6|$.
8. $5|2x-3| < 4|x-5|$.
9. $2|x-2| > x$.
10. $|3x+4| \leqslant x+2$.
11. $|2x+1| < 3x+2$.
12. $|x+1| > x-3$.
13. $|x^2-3x-2| < 2$.
14. $|x(x-5)| > 6$.
15. $\left|\dfrac{x}{x+4}\right| < 2$.
16. $\left|\dfrac{x^2-4}{x}\right| \leqslant 3$.

Exercise 16.5 (miscellaneous)

In questions 1 to 18 find the set of values of x which satisfy the given inequalities.

1. $x(x+6) \geqslant 7$.
2. $x(x-2) < 1$.

3. $(x^2-1)(2x+1) < 0$.

4. $x(x^2+10) < 7x^2$.

5. $x^4-3x^2-4 > 0$.

6. $x^4-5x^2+6 > 0$.

7. $0 < \dfrac{x}{2x-3} < 1$.

8. $2 < \dfrac{4x^2-1}{x^2} < 3$.

9. $\dfrac{5}{x+1} < x-3$.

10. $\dfrac{x}{x-2} > \dfrac{1}{x+1}$.

11. $\dfrac{4x-1}{x^2-2x-3} < 3$.

12. $\dfrac{5x^2+2x-11}{x^2+1} > 4$.

13. $|3-2x| \leqslant |x+4|$.

14. $|x^2+1| < |x^2-9|$.

15. $|3x-2| < x$.

16. $|5x-6| > x^2$.

17. $\left|\dfrac{x+1}{x-1}\right| < 1$.

18. $\left|\dfrac{x}{x-2}\right| < 2$.

19. Find the values of x between $-\pi/2$ and $+5\pi/2$ for which
(a) $\cos x > \frac{1}{2}$, (b) $|\tan x| \leqslant 1$, (c) $4\sin^2 x < 1$.

20. By considering $(a^2+b^2)^2$ or otherwise, prove that $a^4+b^4 \geqslant a^3b+ab^3$ for all real values of a and b.

21. Prove that the geometric mean of two positive real numbers p and q is less than or equal to their arithmetic mean.

22. Find the ranges of values of x between 0 and 2π for which $\sin 2x > \cos x$. (O)

23. For each of the following expressions, state, with reasons, for what set of values of x it is greater than -1:

(i) $\dfrac{1}{x}$, (ii) $-(x-1)^2$, (iii) $\dfrac{2x-1}{x-2}$. (O&C)

24. Determine the range of values of x for which

(i) $\dfrac{6}{x+1} < x$, (ii) $\dfrac{6}{|x|+1} < |x|$. (O&C)

17 Mathematical reasoning

17.1 Deductive and inductive processes

Pure mathematics is the study of pattern and structure in systems involving sets of objects such as numbers or points in space. The theory of the subject is developed by logical argument based on precise definitions of the terms used. In general, the type of reasoning used in scientific work is different from that used in mathematics. A scientist observes some natural phenomenon, then uses the data he has collected to formulate laws governing the behaviour of the objects concerned. If results obtained from later experiments also satisfy these laws, they may be accepted by other scientists and used as the basis of further work. This process, by which particular items of data lead to a general theory, is called *inductive* reasoning. Scientific laws obtained in this way cannot be regarded as absolute truths. New evidence may be found which shows that the theory is untenable or that it needs modification.

A mathematician may use inductive processes to produce *conjectures* which he then attempts to prove. One such conjecture is that there are no positive integers a, b and c such that $a^n + b^n = c^n$, where n is any positive integer greater than 2. It is believed that the mathematician Fermat† may have proved this, but he left no record of his method of proof. The conjecture has since been proved for certain values of n, but the general statement has yet to be proved or disproved. Another famous mathematical conjecture, known as the "four colour problem", has only recently been proved. This is the statement that any geographical map (or division of a plane into regions) can be coloured using only four different colours, so that no two countries (or regions) with a common boundary have the same colour.

To *prove* a mathematical statement, we must establish its truth by logical reasoning. A mathematical *proof* is a chain of reasoning in which a statement or theorem is shown to follow from previously proved statements or from initial assumptions called premises, axioms or postulates. This process, in which one

†*Fermat, Pierre de* (1601–1665) French mathematician. He made important contributions to probability theory and the theory of numbers. He wrote his famous conjecture in the margin of his copy of the *Arithmetica* written by the Greek mathematician Diophantus in about A.D. 275.

statement follows logically from another, is called *deductive* reasoning or *deduction*. We now give a simple example to illustrate the use of the process.

Example 1 What can be deduced from the following three statements?

(a) John always enjoys a boiled egg for breakfast.
(b) There was no bacon left for breakfast today.
(c) If there is no bacon, John boils an egg for breakfast.

If we form these statements into a chain, taking (b) then (c) then (a), we may deduce that John enjoyed his breakfast today.

The next example shows that an apparently logical argument may be invalid.

Example 2 Decide whether the conclusion drawn from the three given statements is valid.

(a) There is a robin in Mrs. Brown's garden.
(b) No other bird in Mrs. Brown's garden has a red breast.
(c) Mrs. Brown can see a bird with a red breast.
Conclusion: Mrs. Brown can see a robin.

The argument is invalid for various reasons. Robins are not the only birds with red breasts. Mrs. Brown may not be watching the birds in her own garden. She may, for instance, be looking at a bullfinch in a neighbour's garden.

We now demonstrate an invalid mathematical argument.

Consider the equation:	$x + \frac{1}{x} = 1$	(1)
Multiply (1) by x,	$x^2 + 1 = x$	(2)
Multiply (2) by x,	$x^3 + x = x^2$	(3)
Rearranging (3),	$x^3 = x^2 - x$	
Rearranging (2),	$x^2 - x = -1$	
$\therefore$	$x^3 = -1$	(4)
Hence	$x = -1$	

Clearly $x = -1$ does not satisfy the original equation. The flaw in the argument becomes apparent when we consider the steps in more detail.

Rearranging (2),	$x^2 - x + 1 = 0$
Rearranging (3),	$x(x^2 - x + 1) = 0$
Adding,	$(x + 1)(x^2 - x + 1) = 0$
i.e.	$x^3 + 1 = 0$

We can now see that combining equations (2) and (3) to produce equation (4) results in the introduction of the factor $(x + 1)$. It is this extra factor which leads to the invalid solution $x = -1$.

[It can be shown that the equation $x + \frac{1}{x} = 1$ has no real roots.]

The way in which deductive reasoning is used in various types of mathematical proof is discussed later in the chapter. However, although deduction forms the basis of mathematical reasoning, in such fields as problem solving and original research less reliable processes such as intuition or "trial and error" may be involved.

In applied mathematics both inductive and deductive processes are used. An applied mathematician states in mathematical form laws obtained from experimental data. He may be interested in such things as the performance of an aircraft, the growth of a population of bacteria or the current in an electrical circuit. Using the laws governing the system under consideration, he sets up a mathematical model, often in the form of a set of equations. Predictions about the behaviour of the system can now be made using this model. In general, setting up a mathematical model is an inductive process, but using the model to make predictions involves mainly deductive reasoning.

Exercise 17.1

1. Decide what conclusion can be deduced from the following sets of statements.

(a) Most people disapprove of immorality.
Stealing is immoral.
The Knave of Hearts stole some tarts.

(b) N is an even number greater than 2.
A prime number has only itself and 1 as factors.
Any even number is divisible by 2.

(c) Boycott was the vice-captain of the England cricket team.
It is impossible to play cricket with a broken arm.
If the captain cannot play, the vice-captain takes over.
The England cricket captain broke his arm.

2. Decide whether the conclusions drawn from the following sets of statements are valid. Give reasons for your answer in each case.

(a) Swallow-tail butterflies feed on Milk Parsley.
Milk Parsley grows in the Norfolk fens.
Conclusion: There are swallow-tail butterflies in Norfolk.

(b) Mathematicians are highly intelligent.
People who are not absent-minded never wear odd socks.
Highly intelligent people are thought to be absent-minded.
Conclusion: Mathematicians are thought to wear odd socks.

(c) Opposite angles of a parallelogram are equal.
A rhombus is a parallelogram whose diagonals cut at right angles.
$ABCD$ has two pairs of equal angles.
The diagonals of $ABCD$ cut at right angles.
Conclusion: $ABCD$ is a rhombus.

3. Find the flaw in the following argument:

Let	$x = y$
then	$x^2 = xy$
and	$x^2 - y^2 = xy - y^2$
Factorising,	$(x + y)(x - y) = y(x - y)$
$\therefore$	$x + y = y$
Since $x = y$,	$2x = x$
Hence	$2 = 1$

4. Read the short story given below, then consider the statements which follow it. In each case decide whether the statement is (a) true, (b) false or (c) could be either true or false.

A pedestrian had just stepped off the pavement when a vehicle, travelling at high speed, came round the corner. The car driver quickly applied his brakes and the man on the zebra crossing narrowly escaped serious injury. The police were called to the scene of the incident and later the guilty party was prosecuted for dangerous driving.

(1) At the beginning of the story a man stepped off the pavement.
(2) The car was travelling at high speed.
(3) The man on the zebra crossing was not injured.
(4) After the incident the police were called to the scene.
(5) A driver was found guilty of dangerous driving.
(6) A man escaped serious injury because the car driver braked quickly.
(7) A vehicle came round the corner just after a pedestrian had stepped off the pavement.
(8) A man was crossing the road at the time of the incident.
(9) The car stopped on or near the zebra crossing.
(10) The car stopped.
(11) The car driver was prosecuted for dangerous driving.
(12) The incident involved at least one driver.
(13) The zebra crossing was near a bend in the road.
(14) The events described in the story could not have taken place on a dual carriageway.
(15) None of the people referred to in the story could have been riding a bicycle.
(16) It is possible that the incident involved two vehicles.
(17) The story states that a vehicle being driven at high speed knocked down a man on a zebra crossing, injuring him only slightly.
(18) Three people are mentioned in the story, apart from the police.

17.2 Implication and equivalence

It is sometimes convenient to use letters and symbols to represent statements and relations between them.

For two statements denoted by p and q,

$p \Rightarrow q$ means "p implies q" or "if p, then q",

$p \Leftarrow q$ means "p is implied by q" or "if q, then p",

$p \Leftrightarrow q$ means "p implies q and q implies p" i.e. "the statements p and q are equivalent".

Example 1 Use the symbols $\Rightarrow$ and $\Leftrightarrow$ to connect the following statements:

a: $\triangle ABC$ is isosceles b: In $\triangle ABC$, $\angle B = \angle C$
c: $\triangle ABC$ is equilateral d: In $\triangle ABC$, $AB = AC$

We see that $b \Rightarrow a, b \Rightarrow d; c \Rightarrow a, c \Rightarrow b, c \Rightarrow d; d \Rightarrow a, d \Rightarrow b$.
Since $b \Rightarrow d$ and $d \Rightarrow b$, we have $b \Leftrightarrow d$
i.e. the statements b and d are equivalent.

The symbols $\not\Rightarrow$ "does not imply" and $\not\Leftrightarrow$ "is not equivalent to" may also be used.

Example 2 Use the symbols $\not\Rightarrow$ and $\not\Leftrightarrow$ to connect statements a, b, c and d.

By considering an isosceles triangle ABC in which $AB = BC$, we see that $a \not\Rightarrow b$ and $a \not\Rightarrow d$. We also find that $a \not\Rightarrow c, b \not\Rightarrow c$ and $d \not\Rightarrow c$. Hence $a \not\Leftrightarrow b$, $a \not\Leftrightarrow c, a \not\Leftrightarrow d, b \not\Leftrightarrow c, c \not\Leftrightarrow d$.

The *negation* of a statement p is denoted by $\sim p$ (or sometimes p') and read "not p". The negations of a, b, c and d in *Example 1* are as follows:

$\sim a$: $\triangle ABC$ is not isosceles $\sim b$: In $\triangle ABC$, $\angle B, \neq \angle C$
$\sim c$: $\triangle ABC$ is not equilateral $\sim d$: In $\triangle ABC$, $AB \neq AC$

Here are some of the connections between these new statements:

$$\sim a \Rightarrow \sim c, \quad \sim b \Leftrightarrow \sim d, \quad \sim d \Rightarrow \sim c, \quad \sim d \not\Rightarrow \sim a.$$

These examples illustrate some basic properties of negation and implication.

(1) If $p \Rightarrow q$, then $\sim q \Rightarrow \sim p$.
(2) If $p \Leftrightarrow q$, then $\sim p \Leftrightarrow \sim q$.

It is also important to notice that writing $p \not\Rightarrow q$ is *not* equivalent to writing $p \Rightarrow \sim q$. For instance in *Example 2*, the statement $a \not\Rightarrow d$ means: "if $\triangle ABC$ is isosceles, then it does not necessarily follow that $AB = AC$". However, writing $a \Rightarrow \sim d$ would mean: "if $\triangle ABC$ is isosceles, then $AB \neq AC$", which is clearly false.

Other phrases which are used in mathematical statements, such as "if and only if" or "necessary and sufficient condition", can be replaced by implication symbols. The table below shows some sets of equivalent statements.

$p \Rightarrow q$	p only if q	q is a necessary condition that p
$p \Leftarrow q$	p if q	q is a sufficient condition that p
$p \Leftrightarrow q$	p if and only if q	q is a necessary and sufficient condition that p

Consider now the following statements concerning a positive integer n.

a: n is the square of an even number
b: n is a multiple of 4
c: n is an even number

We see that $a \Rightarrow b \Rightarrow c$ and that, using the phrases in the given table:

the statement b is true if a is true, but b is true only if c is true;
the truth of c is a necessary condition for b to be true, but a is a sufficient condition for b to be true.

Clearly implication signs provide the most economical way of expressing connections between statements. The process of deduction is based on a *law of implication* which we can now state in the form:

$$\text{if} \quad a \Rightarrow b \quad \text{and} \quad b \Rightarrow c, \quad \text{then} \quad a \Rightarrow c.$$

A deductive proof can always be reduced to a series of these steps or closely related ones such as, "if $a \Rightarrow b$ and $a \Rightarrow c$, where b and c together imply d, then $a \Rightarrow d$."

Exercise 17.2

1. Use the symbols $\Rightarrow$ and $\Leftrightarrow$ to connect the statements p, q and r.

(a) p: $ABCD$ is a square
q: $ABCD$ has four equal sides
r: $ABCD$ is a parallelogram

(b) p: Ivor is Welsh
q: Ivor lives in Cardiff
r: Ivor is British

(c) p: $KLMN$ is a rectangle 3 cm by 4 cm
q: The area of rectangle $KLMN$ is $12\,\text{cm}^2$
r: $KLMN$ is a rectangle with diagonals 5 cm long

(d) p: The integer n is an even multiple of 5
q: The integer n has final digit 0
r: The integer n has a pair of prime factors which differ by 3.

2. Use the symbol $\not\Rightarrow$ to connect the statements given in question 1.

3. Use the symbols $\Rightarrow$ and $\Leftrightarrow$ to connect the negations $\sim p$, $\sim q$ and $\sim r$ of the statements given in question 1.

4. Write down all possible connections between the statements p, q, $\sim p$ and $\sim q$ using the symbols $\Rightarrow$ and $\Leftrightarrow$.

(a) p: In quadrilateral $ABCD$, $AC = BD$
q: $ABCD$ is a rectangle

(b) p: In $\triangle ABC$, $\angle A$ is obtuse
q: In $\triangle ABC$, $\angle B = 90°$

(c) p: For integers x and y, $x - y$ is odd
q: For integers x and y, $x^2 - y^2$ is divisible by 4

(d) p: C lies on the circle with AB as diameter
q: In $\triangle ABC$, $\angle C$ is not a right angle.

5. Using only the terms and symbols $p, q, \sim p, \sim q, \Rightarrow, \Leftrightarrow, \not\Rightarrow$ and $\not\Leftrightarrow$, write down a statement equivalent to each of the following:

(a) $\sim p \Rightarrow \sim q$, (b) $p \Leftrightarrow \sim q$, (c) $\sim p \not\Rightarrow q$,
(d) $p \not\Leftrightarrow q$, (e) $p \Rightarrow \sim q$, (f) $\sim p \Leftrightarrow q$,

6. Use the symbol $\Rightarrow$ to state the connection between the statements p and q in each of the following cases.
(a) q is true only if p is true,
(b) p is a necessary condition that q,
(c) q is a sufficient condition that p,
(d) p is true if q is true,
(e) p is a sufficient condition that q,
(f) p is true only if q is true.

7. Use the terms "if", "only if", "necessary condition" and "sufficient condition" to express the connection between the statements, "$\triangle ABC$ is equilateral" and "$AB = AC$" in four different ways.

17.3 Types of proof

We first give an example of an elementary deductive proof.

Example 1 Prove that $a \times 0 = 0$ for all real values of a.
[You may assume the following properties of real numbers:
(i) $a + (b + c) = (a + b) + c$ associative law for addition
(ii) $a(b + c) = ab + ac$ distributive law
(iii) $a + 0 = a = 0 + a$ for all real a
(iv) any real number a has an additive inverse $(-a)$, such that $a + (-a) = 0 = (-a) + a$.]

	$b + 0 = b$	property (iii)
$\Rightarrow$	$a(b + 0) = ab$	both sides multiplied by a
$\Rightarrow$	$ab + a \times 0 = ab$	distributive law
$\Rightarrow$	$-ab + (ab + a \times 0) = -ab + ab$	$-ab$ added to both sides
$\Rightarrow$	$(-ab + ab) + a \times 0 = -ab + ab$	associative law for $+$
$\Rightarrow$	$0 + a \times 0 = 0$	property (iv)
$\Rightarrow$	$a \times 0 = 0$	property (iii)

Another important type of proof is called *proof by contradiction* (or "reductio ad absurdum".) To prove a theorem by this method we assume that the theorem does not hold, then show that this assumption leads to a contradiction. In its simplest form this process is equivalent to proving a statement p true by showing $\sim p$ to be

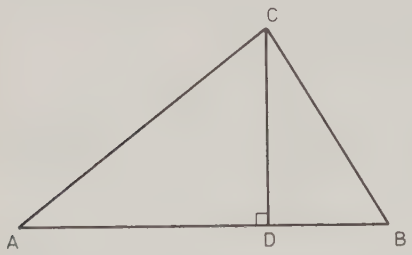
false. For instance, the standard proof of the statement "$\sqrt{2}$ is irrational" begins with the assumption that $\sqrt{2}$ is rational. In the example given here a statement of the form $p \Rightarrow q$ is proved by showing that $\sim q \Rightarrow \sim p$.

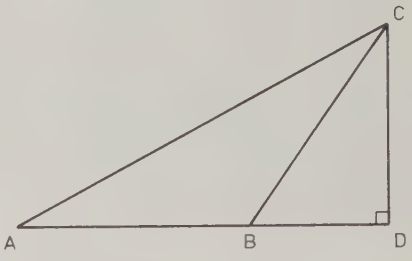
Example 2 Prove that in $\triangle ABC$, if $AB^2 + BC^2 = AC^2$, then $\angle ABC = 90°$.

[You may assume Pythagoras' theorem.]

Let us assume that in $\triangle ABC$, $\angle ABC \neq 90°$.

(1) $\angle ABC < 90°$ (2) $\angle ABC > 90°$





Case (1): By Pythagoras' theorem in $\triangle ADC$, $AC^2 = AD^2 + DC^2$

But $AD < AB$ and $DC < BC$, $\therefore$ $AC^2 < AB^2 + BC^2$.

Case (2): By Pythagoras' theorem in $\triangle ADC$, $AC^2 = AD^2 + DC^2$

$\therefore$ $AC^2 = (AB + BD)^2 + DC^2 = AB^2 + 2AB.BD + BD^2 + DC^2$

By Pythagoras' theorem in $\triangle BDC$, $BD^2 + DC^2 = BC^2$

$\therefore$ $AC^2 = AB^2 + 2AB.BD + BC^2 > AB^2 + BC^2$.

Thus, in both cases, if $AB^2 + BC^2 = AC^2$, the assumption that $\angle ABC \neq 90°$ leads to a contradiction.
Hence, if $AB^2 + BC^2 = AC^2$, then $\angle ABC = 90°$.

This result is, of course, the converse of Pythagoras' theorem. In general, the *converse* of the statement $p \Rightarrow q$ is the statement $q \Rightarrow p$.

As in the case of Fermat's famous conjecture, mentioned in §17.1, it is sometimes difficult to establish the truth of a general statement about such objects as real numbers or sets. However, a statement can be proved false by producing just one *counter-example*.

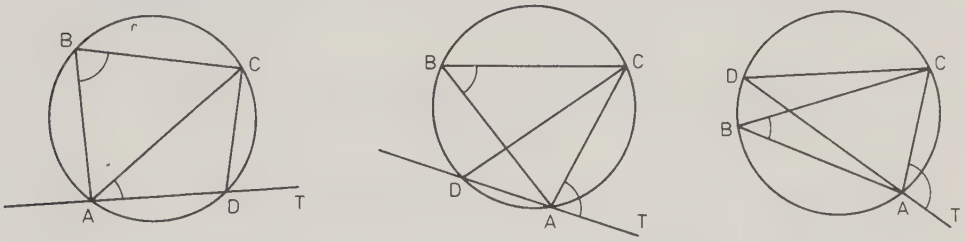
Example 3 Discuss the statement that if n is a positive integer, then $n^2 - n + 17$ is prime.

[To prove that this statement is true it would be necessary to show that it holds for all positive integral values of n. However, the statement can be proved false by finding one value of n, such that $n^2 - n + 17$ is not prime.]

If we test the statement by writing $n = 1, 2, 3, 4, 5 \ldots$, we obtain the prime numbers $17, 19, 23, 29, 37, \ldots$. This suggests that the statement could be true. However, if $n = 17$, $n^2 - n + 17 = 17^2$, which is not prime. Hence the statement is false.

Exercise 17.3

1. Using the given diagrams, show that if $\angle ABC = \angle CAT$, the assumption that AT cuts the circle again at a point D leads to a contradiction. What conclusion can be drawn from this?



2. Show that if $p_1, p_2, p_3, \ldots, p_n$ are prime numbers, then the number $p_1p_2p_3 \ldots p_n + 1$ is either a prime not already listed or a multiple of such a prime. Hence prove by contradiction that there are infinitely many prime numbers.

3. Write down the converses of the following statements. In each case decide whether (i) the statement, (ii) its converse are true or false.
(a) If a diameter of a circle bisects a chord, it is perpendicular to that chord.
(b) If $2x - 3 = 0$, then $2x^2 - x - 3 = 0$.
(c) If a pentagon is regular, it must be equilateral.
(d) If x is greater than y, then x^2 is greater than y^2.
(e) If a positive integer N is divisible by 9, then the sum of its digits is divisible by 9.
(f) If any set of six socks taken from a drawer contains a pair, then there must be five pairs of socks in the drawer.

In questions 4, 5 and 6 decide whether the given statements are true or false. If a statement is true, prove it; if it is false, give a counter-example.

4. (a) The square of any odd number is of the form $4k + 1$, where k is a non-negative integer.
(b) Any number of the form $4k + 1$ is the square of an odd number.

5. (a) If $0 < p < q$, then $\log p < \log q$.
(b) If $0 < p < q$, then $\int_0^p f(x)dx < \int_0^q f(x)dx$.

6. (a) If $n^3 - n$ is divisible by 4, then n must be divisible by 4.
(b) If n is a positive integer, then $n^3 - n$ is divisible by 3.

17.4 Proof by induction

The elementary deductive methods discussed so far are sometimes found to be inadequate when attempting to prove that a proposition, such as "the number $4^n + 5$ is divisible by 3", holds for all positive integral values of n. The difficulty is

that this general statements is equivalent to the infinite sequence of propositions "3 is a factor of $4^1 + 5, 4^2 + 5, 4^3 + 5, 4^4 + 5, \ldots$" However, propositions of this kind can often be proved using *mathematical induction.* Although the method is related to the type of inductive reasoning used in science, it is nevertheless a rigorous procedure in which conclusions are reached by sound deductive reasoning.

The principle of a proof by induction can be stated formally as follows:

> Given a proposition $P(n)$ involving a positive integer n, if (i) $P(k) \Rightarrow P(k+1)$ and (ii) $P(1)$ is true, then $P(n)$ is true for all positive integral values of n.

A useful mental picture of a proof by induction is created by considering the proposition that a man can climb a certain uniform staircase with n steps. The statement will be proved if we can show

(i) that the man is capable of climbing from any step to the one above and
(ii) that the man can reach the foot of the staircase.

Thus it may be helpful to think of a general proposition $P(n)$ as a staircase to be climbed. The steps of the staircase are the propositions $P(1), P(2), P(3), \ldots$. It is possible to proceed from one "step" to the next if we can prove that for any k, $P(k+1)$ may be deduced from $P(k)$. If, in addition, it is shown that $P(1)$ is true (i.e. that it is possible to set foot on the first step of the staircase), then it follows that $P(2)$ is true, then $P(3)$ is true and so on for all positive integral values of n.

Example 1 Prove that the number, $a_n = 4^n + 5$, is divisible by 3 for all positive integral values of n.

Consider the numbers $a_k = 4^k + 5$ and $a_{k+1} = 4^{k+1} + 5$.

$$a_{k+1} - a_k = (4^{k+1} + 5) - (4^k + 5)$$
$$= 4^{k+1} - 4^k = 4^k(4 - 1) = 4^k \times 3$$
$$\therefore \quad a_{k+1} = a_k + 4^k \times 3$$

Hence, if a_k is divisible by 3, so is a_{k+1}.
$\therefore$ if the proposition is true for $n = k$, it is also true for $n = k + 1$.
$a_1 = 4^1 + 5 = 9$, which is divisible by 3,
$\therefore$ the proposition is true for $n = 1$.
Hence, by induction, the number $4^n + 5$ must be divisible by 3 for all positive integral values of n.

When dealing with a finite series, it is sometimes possible to suggest a formula for the sum to n terms by examining the first few sums, $S_1, S_2, S_3, \ldots$. Such a result can often be proved by the method of induction.

Example 2 If S_n is the sum of the first n terms of the series

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \ldots + \frac{1}{n(n+1)} + \ldots,$$

find S_1, S_2 and S_3, then suggest a formula for S_n. Prove that the result is correct by the method of induction.

$$S_1 = \frac{1}{1 \times 2} = \frac{1}{2}, \qquad S_2 = S_1 + \frac{1}{2 \times 3} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3},$$

$$S_3 = S_2 + \frac{1}{3 \times 4} = \frac{2}{3} + \frac{1}{12} = \frac{3}{4}.$$

These results suggests that $S_n = \dfrac{n}{n+1}$

Assuming that this result is true for $n = k$, $S_k = \dfrac{k}{k+1}$

$$\therefore \quad S_{k+1} = S_k + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)}$$

$$= \frac{k^2 + 2k + 1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

$\therefore$ if the result holds for $n = k$, it also holds for $n = k + 1$.

When $n = 1$, $S_n = S_1 = \dfrac{1}{2}$ and $\dfrac{n}{n+1} = \dfrac{1}{1+1} = \dfrac{1}{2}$

$\therefore$ the result holds for $n = 1$.

Hence, by induction, it is true that $S_n = n/(n+1)$ for all positive integral values of n.

The method of induction can also be used to prove the binomial theorem:

$$(a+b)^n = a^n + {}_nC_1 a^{n-1}b + {}_nC_2 a^{n-2}b^2 + \ldots + {}_nC_r a^{n-r}b^r + \ldots + {}_nC_{n-1} ab^{n-1} + b^n.$$

Assuming that the theorem is true for $n = k$,

$$(a+b)^k = a^k + {}_kC_1 a^{k-1}b + {}_kC_2 a^{k-2}b^2 + \ldots + {}_kC_{r-1} a^{k-r+1}b^{r-1} + {}_kC_r a^{k-r}b^r + \ldots + b^k,$$

then, multiplying by $(a+b)$,

$$(a+b)^{k+1} = a^{k+1} + {}_kC_1 a^k b + {}_kC_2 a^{k-1}b^2 + \ldots + {}_kC_r a^{k-r+1}b^r + \ldots + a^k b + {}_kC_1 a^{k-1}b^2 + \ldots + {}_kC_{r-1} a^{k-r+1}b^r + \ldots + b^{k+1}.$$

But ${}_kC_1 + 1 = k + 1 = {}_{k+1}C_1$, and

$$
\begin{aligned}
{}_kC_r + {}_kC_{r-1} &= \frac{k!}{r!(k-r)!} + \frac{k!}{(r-1)!(k-\{r-1\})!} \\
&= \frac{k!}{r!(k-r+1)!}\{(k-r+1)+r\} \\
&= \frac{(k+1)!}{r!(k+1-r)!} = {}_{k+1}C_r
\end{aligned}
$$

$$
\therefore \quad (a+b)^{k+1} = a^{k+1} + {}_{k+1}C_1 a^k b + {}_{k+1}C_2 a^{k-1}b^2 + \ldots
\ldots + {}_{k+1}C_r a^{k+1-r}b^r + \ldots + b^{k+1}.
$$

$\therefore$ if the theorem is true for $n = k$ it is also true for $n = k + 1$.
Since $(a + b)^1 = a^1 + b^1$, the theorem is true for $n = 1$. Hence, by induction, the binomial theorem is true for all positive integral values of n.

[Note that the basic principle of induction can be extended in various ways. Suppose, for instance, that for some proposition $P(n)$, where n is a positive integer, it can be shown that $P(k) \Rightarrow P(k + 1)$, but that the proposition $P(1)$ is false or meaningless. Clearly $P(n)$ cannot be proved true for all positive integral values of n. However, provided that there is some positive integer m such that $P(m)$ is true, then by induction, $P(n)$ holds for all integral values of n greater than or equal to m.]

Exercise 17.4

In questions 1 to 8 prove, by induction, that the given statements are true for all positive integral values of n.

1. The sum of the first n terms of the series
$(1 \times 3) + (2 \times 4) + (3 \times 5) + \ldots + r(r + 2) + \ldots$ is $\frac{1}{6}n(n + 1)(2n + 7)$.

2. The sum of the first n terms of the series
$$\frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \frac{1}{5 \times 7} + \ldots + \frac{1}{(2r-1)(2r+1)} + \ldots \text{ is } \frac{n}{2n+1}.$$

3. $\sum_{r=1}^{n} r(r + 1) = \frac{1}{3}n(n + 1)(n + 2)$.

4. $\sum_{r=1}^{n} \frac{r}{2^r} = 2 - \frac{n+2}{2^n}$.

5. $n^3 + 3n^2 - 10n$ is divisible by 3.

6. $3^{2n} - 1$ is a multiple of 8.

7. $7^n + 4^n + 1^n$ is divisible by 6.

8. If $\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$, then $\mathbf{M}^n = \begin{pmatrix} 1 & 0 \\ 3n & 1 \end{pmatrix}$.

9. Prove that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$ for all positive integers n. Hence find the following sums:

(a) $1^2 + 2^2 + 3^2 + \ldots + 20^2$,
(b) $21^2 + 22^2 + 23^2 + \ldots + 40^2$,
(c) $2^2 + 4^2 + 6^2 + \ldots + 40^2$,
(d) $1^2 + 3^2 + 5^2 + \ldots + 39^2$.

10. Prove that the sum to n terms of the series

$$\frac{1}{1 \times 2 \times 3} + \frac{1}{2 \times 3 \times 4} + \frac{1}{3 \times 4 \times 5} + \ldots \text{ is } \frac{1}{4} - \frac{1}{2(n+1)(n+2)}.$$

Hence find the sum to infinity of the series.

11. If $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}$, prove that $\mathbf{A}^n = \begin{pmatrix} 2n+1 & -n \\ 4n & 1-2n \end{pmatrix}$, where n is a positive integer. Given that the transformation represented by $\mathbf{A}$ is applied to the points of the x, y plane n times in succession, find the image of

(a) the point $(0, -1)$,
(b) the line $y = x$.

12. Given that $f(x) = x/(x+1)$, find $f^2(x)$ i.e. $f(f(x))$. Find also $f^3(x)$ and suggest a possible form for $f^n(x)$. Prove that your result is correct by mathematical induction.

Exercise 17.5 (miscellaneous)

In questions 1 to 6 explain why the given chain of reasoning is unsound.

1.
$$\begin{aligned} & -6x + 9 = 2x + 1 \\ \Rightarrow \quad & x^2 - 6x + 9 = x^2 + 2x + 1 \\ \Rightarrow \quad & (x-3)^2 = (x+1)^2 \\ \Rightarrow \quad & x - 3 = x + 1 \\ \Rightarrow \quad & -3 = 1 \end{aligned}$$

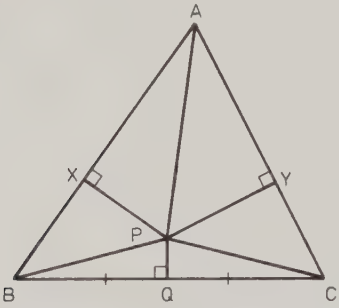
2.
$$\begin{aligned} & 2 > 1 \\ \Rightarrow \quad & 2\lg(0{\cdot}2) > \lg(0{\cdot}2) \\ \Rightarrow \quad & \lg(0{\cdot}2)^2 > \lg(0{\cdot}2) \\ \Rightarrow \quad & \lg(0{\cdot}04) > \lg(0{\cdot}2) \\ \Rightarrow \quad & 0{\cdot}04 > 0{\cdot}2 \end{aligned}$$

3.
$$\begin{aligned} & (2x+3)(3x+1) = (6x+5)(x+1) \\ \Rightarrow \quad & 6x^2 + 11x + 3 = 6x^2 + 11x + 5 \\ \Rightarrow \quad & 3 = 5 \end{aligned}$$

4.
$$\begin{aligned} & a + b > 0 \\ \Rightarrow \quad & a^2 + ab > 0 \\ \Rightarrow \quad & a^2 + 2ab + b^2 > ab + b^2 \\ \Rightarrow \quad & (a + b)^2 > (a + b)b \\ \Rightarrow \quad & a + b > b \end{aligned}$$

Hence if $a + b > 0$, then $a + b > b$.

5.



In $\triangle ABC$ the point P is constructed such that AP bisects $\angle A$ and PQ is the perpendicular bisector of BC. PX and PY are drawn perpendicular to AB and AC respectively.

In $\triangle$s APX and APY, $\angle AXP = \angle AYP$, $\angle PAX = \angle PAY$ and AP is common,
$\therefore$ $\triangle$s APX and APY are congruent.
Hence $PX = PY$ and $AX = AY$.
In $\triangle$s PXB and PYC, $\angle BXP = \angle CYP = 90°$, $PX = PY$ and $PB = PC$.
$\therefore$ $\triangle$s PXB and PYC are congruent and hence $XB = YC$.
Thus $AX + XB = AY + YC$ i.e. $AB = AC$.
Hence any triangle is isosceles.

6. To prove that $a_n = 7^n - 11$ is divisible by 6, where n is an integer greater than 1.

Proof: $a_{k+1} - a_k = (7^{k+1} - 11) - (7^k - 11)$
$= 7^{k+1} - 7^k = 7^k(7 - 1) = 7^k \times 6,$

$\therefore$ if a_k is divisible by 6, so is a_{k+1}.
Hence, by induction, a_n is divisible by 6 for all integral values of n greater than 1.

7. In proving a statement of the form $p \Rightarrow q$, it has been shown that p implies either r or s. What further steps are required to complete the proof?

8. In proving that the proposition $P(n)$ is true for all positive integral values of n, it has been shown that $P(k) \Rightarrow P(k + 2)$. What further steps are required to complete the proof?

9. Prove that if n is any positive integer,
$1 \times 2 \times 3 + 2 \times 3 \times 4 + \ldots + n(n + 1)(n + 2) = \frac{1}{4}n(n + 1)(n + 2)(n + 3)$.

10. Prove that $\dfrac{d}{dx}(x^n) = nx^{n-1}$ for all positive integral values of n.

11. Prove that $\displaystyle\sum_{r=2}^{n} \frac{1}{r^2 - 1} = \frac{3}{4} - \frac{2n + 1}{2n(n + 1)}$.

12. Given that $\mathbf{A} = \begin{pmatrix} 1+\cos\theta & \sin\theta \\ \sin\theta & 1-\cos\theta \end{pmatrix}$, prove that $\mathbf{A}^{n+1} = 2^n \times \mathbf{A}$, where n is a positive integer.

13. If $u_{n+1} = 2u_n + 1$ where n is a positive integer and $u_1 = 5$, prove by induction that $u_n = 3 \times 2^n - 1$.

14. If $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$, find $\mathbf{A}^2$, $\mathbf{A}^3$ and conjecture a form for $\mathbf{A}^n$ where n is any positive integer. Prove the truth of your conjecture by mathematical induction. (O & C)

15. Prove by induction that the sum of the first n terms of the series $\frac{1}{1 \times 2 \times 3} + \frac{4}{2 \times 3 \times 4} + \frac{7}{3 \times 4 \times 5} + \ldots$ is $\frac{n^2}{(n+1)(n+2)}$.
Find the limit of this sum as n tends to infinity.

16. If $S_n = 1.n + 2(n-1) + 3(n-2) + \ldots + (n-1).2 + n.1$, where n is a positive integer, prove that $S_{n+1} - S_n = \frac{1}{2}(n+1)(n+2)$. By induction, or otherwise, prove that $S_n = \frac{1}{6}n(n+1)(n+2)$. (O)

17. Prove that, if n is a positive integer greater than one, $n^5 - n$ is a multiple of 5. Hence, or otherwise, prove that all square numbers are of the form $5r$ or $5r \pm 1$ where r is an integer. (JMB)

18. Let A, B and C be subsets of some universal set $\mathscr{E}$. For the three statements below, prove those that are true and, by taking $\mathscr{E}$ to be the set of all positive integers, provide counter-examples to demonstrate the falsity of the others:
(i) $(A \cap B = A \cap C) \Rightarrow (B = C)$, (ii) $(A \cap B = A \cup B) \Rightarrow (A = B)$,
(iii) $(A \cap B) \cup C = (A \cup C) \cap B$.

[In this question Venn diagrams may be used to examine the plausibility of a statement but will not be accepted as proofs of truth or falsity.] (C)

19. The real function f, defined for all $x \in \mathbb{R}$, is said to be multiplicative if, for all $x \in \mathbb{R}$, $y \in \mathbb{R}$, $f(xy) = f(x)f(y)$. [$\mathbb{R}$ denotes the set of real numbers.] Prove that if f is a multiplicative function then
(i) either $f(0) = 0$ or $f(x) \equiv 1$,
(ii) either $f(1) = 1$ or $f(x) \equiv 0$,
(iii) $f(x^n) = \{f(x)\}^n$ for all positive integers n.
Give an example of a non-constant multiplicative function. (C)

20. The matrix $\mathbf{X} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ has real non-zero elements; the matrix $\mathbf{P}$ is of the form $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ where $a \neq b$, and $\mathbf{M} = \begin{pmatrix} 4 & 2 \\ -3 & -1 \end{pmatrix}$. Given that $\mathbf{XM} = \mathbf{PX}$, show

that $\frac{q}{p} = \frac{4-a}{3} = \frac{2}{1+a}$ and find a possible matrix $\mathbf{P}$ and a possible corresponding matrix $\mathbf{X}$. Show that, if n is a positive integer, $\mathbf{M}^n = \mathbf{X}^{-1}\mathbf{P}^n\mathbf{X}$. Deduce the elements of $\mathbf{M}^n$ in terms of n. (JMB)

21. Prove that if n is a positive integer, $10^n - 1$ is divisible by 9. Hence prove that a necessary and sufficient condition for a positive integer to be divisible by 9 is that the sum of its digits is divisible by 9.

18 Some functions and their properties

18.1 Symmetry, continuity and differentiability

A *function* f with *domain* X and *codomain* Y is a rule which assigns to each element $x \in X$ exactly one *image* $y \in Y$. The set containing the images under f of all the elements of the domain is called the *range* (or image set). In this chapter we will be considering functions with domain the set of real numbers $\mathbb{R}$ (or some subset of $\mathbb{R}$), and codomain $\mathbb{R}$. The notation used to define such functions will take one of the following forms: $f(x) = x^2 + 1$, $y = x^2 + 1$ or $f: x \to x^2 + 1$. Unless otherwise stated, the domain of a function f is assumed to be the set of all real numbers for which an image is defined. For instance, the domain of the function $f(x) = x/(x^2 - 1)$ is taken to be the set $\mathbb{R} - \{-1, 1\}$ i.e. all real numbers except -1 and $+1$. We now examine some of the properties possessed by various functions defined in $\mathbb{R}$.

A function $f(x)$ is said to be *odd* if $f(-x) = -f(x)$. The graph of an odd function is symmetrical about the origin, i.e. if (a, b) lies on the graph so does $(-a, -b)$. The function $f(x)$ is said to be *even* if $f(-x) = f(x)$. The graph of an even function is symmetrical about the y-axis i.e. if (a, b) lies on the graph so does $(-a, b)$.

Example 1 Decide whether each of the functions f, g and h is odd, even or neither, given that

$f(x) = x + 1/x,\ x \in \mathbb{R},\ x \neq 0,$
$g(x) = 2\cos x - x,\ x \in \mathbb{R},$
$h(x) = \sqrt{(4 - x^2)},\ x \in \mathbb{R},\ |x| \leqslant 2.$

$f(-x) = (-x) + 1/(-x) = -x - 1/x = -f(x)$

$\therefore$ f is an odd function.

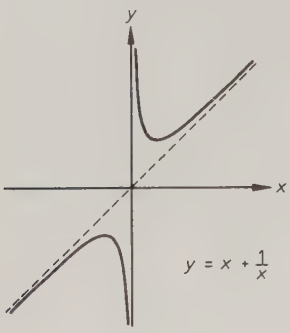
$g(-x) = 2\cos(-x) - (-x) = 2\cos x + x$

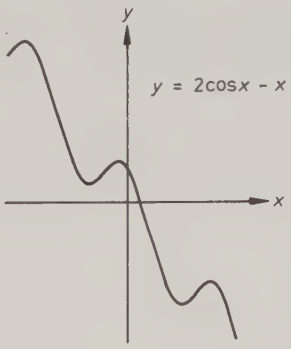
$\therefore$ g is neither odd nor even.

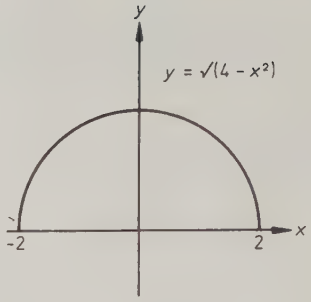
$h(-x) = \sqrt{(4 - \{-x\}^2)} = \sqrt{(4 - x^2)} = h(x)$

$\therefore$ h is an even function.

The sketches below show the graphs of f, g and h.



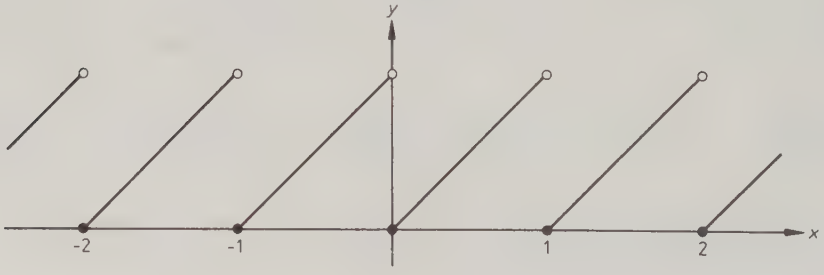




A function $f(x)$ is said to be *periodic* if there is a constant k such that $f(x + k) = f(x)$ for all values of x. The graph of the function forms a repeating pattern such that any translation parallel to the x-axis through k units leaves the pattern unchanged. The *period* of the function is the least positive constant a such that $f(x + a) = f(x)$ for all values of x. The trigonometric functions $\sin x$ and $\cos x$ are periodic and have period 2π. The function $\tan x$ is also periodic, but has period π.

Example 2 Sketch the graph of the function $f(x) = x - [x]$ and decide whether it is periodic. [Note that $[x]$ denotes the greatest integer less than or equal to x.]

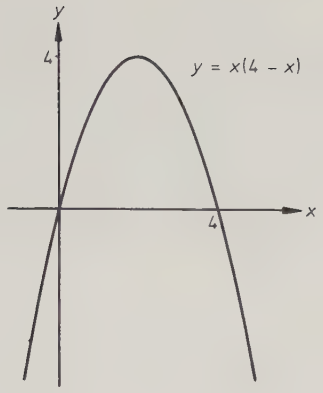
When x is an integer, $[x] = x$, $\therefore \quad f(x) = x - x = 0$
When $0 \leqslant x < 1$, $[x] = 0$, $\therefore \quad f(x) = x$
For all values of x, $f(x + 1) = (x + 1) - [x + 1]$
$= x + 1 - ([x] + 1)$
$= x - [x] = f(x)$

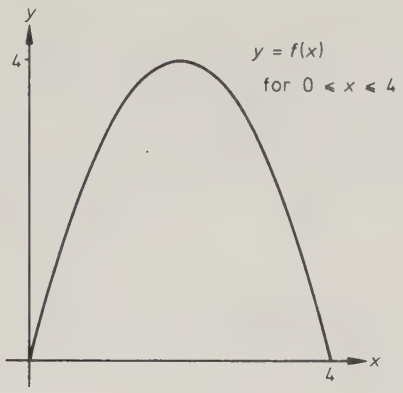


The sketch shows that $f(x)$ is periodic with period 1.
[Open and closed circles are used to show that, for instance, the value of $f(1)$ is 0 rather than 1.]

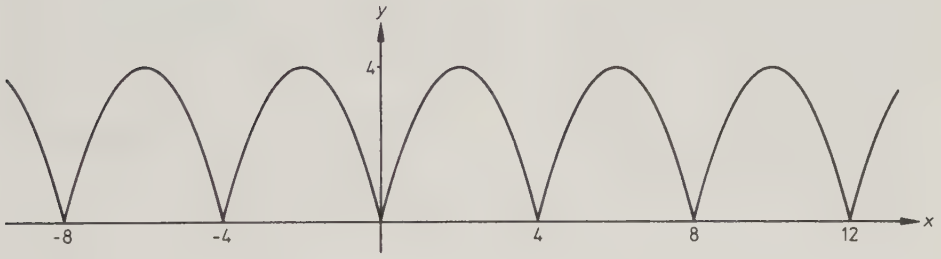
Example 3 Given that f is a periodic function such that for $0 \leqslant x \leqslant 4$, $f(x) = x(4 - x)$, sketch the graph $y = f(x)$ in each of the following cases,

(a) the period of f is 4,
(b) f is an odd function and has period 8.

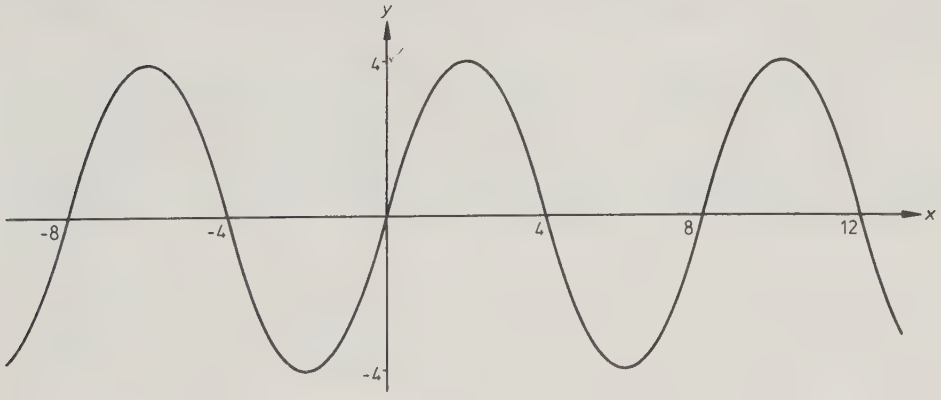




(a)



(b)

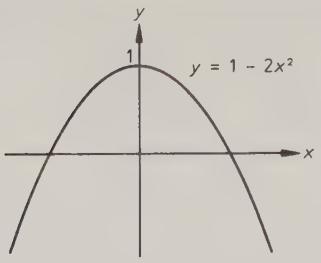


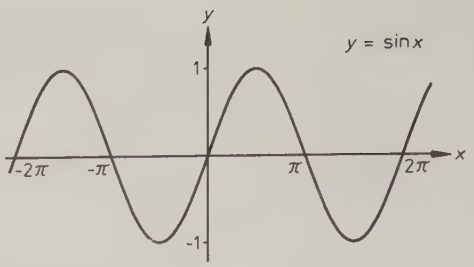
If f and g are functions defined on $\mathbb{R}$, then the *composite function* fg (also written $f \circ g$) is defined by $fg(x) = f(g(x))$.

Example 4 The functions f and g are defined by $f: x \to 1 - 2x^2$ and $g: x \to \sin x$. Sketch the graphs of the functions f, g and fg. State whether each of these functions is

(a) odd, even or neither,

(b) periodic, and if so give the period.

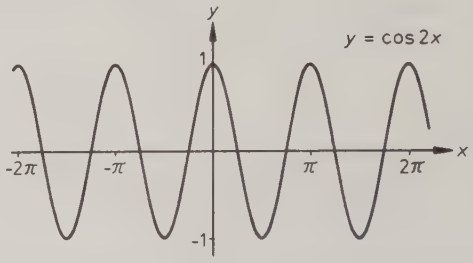




f is an even function but not periodic.
g is an odd function and periodic with period 2π.

$fg(x) = 1 - 2\sin^2 x = \cos 2x$

$\therefore$ fg is an even function and periodic with period π.

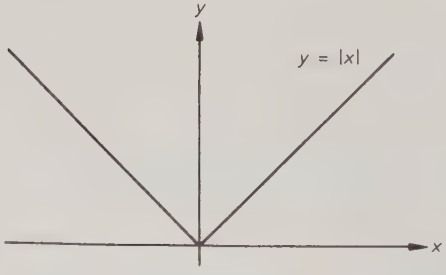


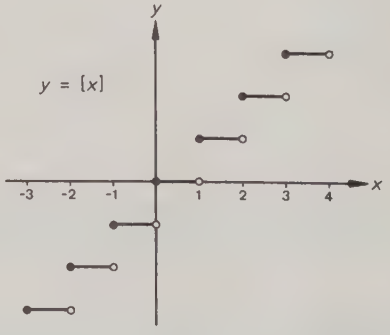
A function $f(x)$ is said to be *continuous* at $x = a$ if, when $x \to a$ from above and when $x \to a$ from below, $f(x) \to f(a)$

i.e. if $\lim_{x \to a} f(x) = f(a)$.

Less formally, $f(x)$ is continuous at $x = a$, if there is no break or *discontinuity* in its graph at that point. If the function $f(x)$ is continuous for all real values of x, it is called a *continuous function*. The graph of a continuous function is a single unbroken line or curve.

Many of the functions met in elementary mathematics are continuous e.g. $x^3 - x + 3, \sin x, |x|$. A function such as $\sqrt{(4 - x^2)}$, which is defined for $-2 \leqslant x \leqslant 2$, is continuous in that interval. However, other familiar functions have one or more discontinuities e.g. $1/x$ at $x = 0$; $\tan x$ at $x = \pm\frac{1}{2}\pi, \pm\frac{3}{2}\pi, \pm\frac{5}{2}\pi, \ldots$; $[x]$ at all integral values of x.





The *derivative* of a function $f(x)$ at $x = a$ is defined as

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

If this limit exists $f(x)$ is said to be *differentiable* at $x = a$. Given a function $f(x)$ the *derived function* $f'(x)$ can often be found using the standard rules for differentiation. However, any general formula for $f'(x)$ will be invalid where $f(x)$ has a discontinuity and possibly at certain other points. For instance, if $f(x) = 1/x$, the result $f'(x) = -1/x^2$ is meaningless when $x = 0$. The function $f(x) = [x]$ is not differentiable when x is an integer, but for non-integral values of x, $f'(x) = 0$. If we consider the function $f(x) = |x|$ near $x = 0$, we find that

for $h > 0$, $$\frac{f(0+h) - f(0)}{h} = \frac{h - 0}{h} = 1,$$

but for $h < 0$, $$\frac{f(0+h) - f(0)}{h} = \frac{-h - 0}{h} = -1.$$

Hence the expression $\dfrac{f(0+h) - f(0)}{h}$ approaches no single value as $h \to 0$. Thus although $|x|$ is a continuous function it is not differentiable at $x = 0$.

Exercise 18.1

In questions 1 to 4 decide whether the given functions are odd, even or neither.

1. (a) $x^4 - 6$, (b) $x^3 + 5$, (c) $x^3 + 5x$.

2. (a) $1 + \dfrac{1}{x^2}$, (b) $\dfrac{1}{1+x^2}$, (c) $\dfrac{x}{1+x^2}$.

3. (a) $1 - \cos x$, (b) $x^2 \sin x$, (c) $\sec x + \tan x$.

4. (a) $[x]$, (b) $[x] + \frac{1}{2}$, (c) $[x^2]$.

In questions 5 to 8 decide whether the given functions are periodic and if so state the period.

5. (a) $\sin 3x$, (b) $1 + \tan 2x$, (c) $\cos \frac{1}{4}(x + \pi)$.

6. (a) $x + \cos x$, (b) $\sin x + \cos x$, (c) $\sin x \cos x$.

7. (a) $\sin 2x - \sin 3x$, (b) $x \sin 3x$, (c) $\sin 3x \cos 2x$.

8. (a) $|\cos x|$, (b) $\sin |x|$, (c) $[\tan x]$.

In questions 9 to 12 sketch the graphs of the functions f, g, fg and gf. In each case state whether the function is odd, even or neither, give its range and, if periodic, its period.

9. $f(x) = x^2 - 1$, $g(x) = 2x + 1$.

10. $f(x) = \frac{1}{2}(\pi - x)$, $g(x) = \sin x$.

11. $f(x) = \cos x$, $g(x) = x^2$.

12. $f(x) = |x|$, $g(x) = x(x^2 - 3)$.

In questions 13 to 16 sketch the graphs of the given periodic functions.

13. $f(x) = x^2$ for $-2 < x \leqslant 2$, f has period 4.

14. $f(x) = |x|$ for $-1 < x \leqslant 1$, f has period 2.

15. $f(x) = x^3$ for $0 \leqslant x \leqslant 1$,
$f(x) = -x$ for $-1 < x < 0$, f has period 2.

16. $f(x) = x$ for $0 \leqslant x \leqslant k$,
$f(x) = k$ for $k < x < 2k$, f has period $2k$.

In questions 17 to 20 sketch the graphs of the given functions. State any values of x at which
(a) there is a discontinuity,
(b) the function is not differentiable.

17. $1/(x^2 - 1)$. 18. $|x^2 - 1|$. 19. $x + [x]$.

20. $f(x) = x$ for $|x| \leqslant 1$, $f(x) = x^2 - 2$ for $|x| > 1$.

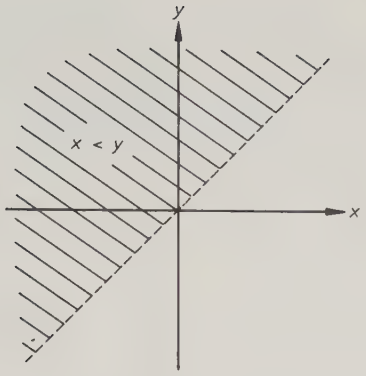
21. Given that f is a periodic function such that for $0 \leqslant x \leqslant 1$, $f(x) = x^2 - 2x$, sketch the graph of the function in each of the following cases
(a) f has period 1,
(b) f is an even function and has period 2,
(c) f is an odd function and has period 2.

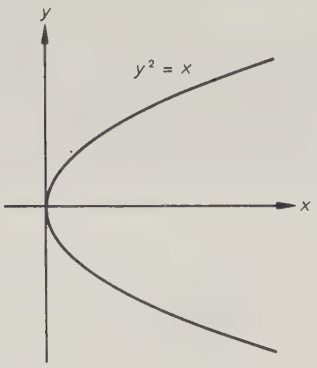
22. A function f is defined by $f(x) = x^2 + 3x$ for $x < 0$, $f(x) = kx$ for $x \geqslant 0$. Show that $f(x)$ is differentiable at $x = 0$ for only one value of k and state this value.

18.2 Some graphs with related equations

In this section we consider some graphs with equations of the form $y = f(x)$ together with graphs which have related equations such as $y = |f(x)|$, $y = 1/f(x)$, $y = \{f(x)\}^2$, $y^2 = f(x)$.

However, we first digress a little to note that not all graphs represent functions. There are many ways in which two real numbers x and y may be related, e.g. "$x < y$", "$y^2 = x$" or perhaps "$x - y$ is an integer". Each of these statements is said to define a *relation* on the set of real numbers. The graph of the relation is the set of points (x, y) such that x and y are related in the stated manner.



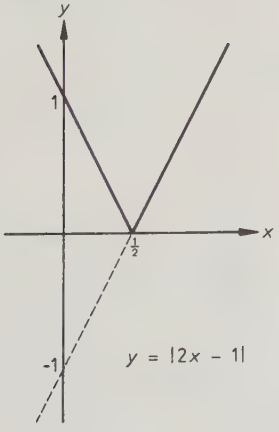


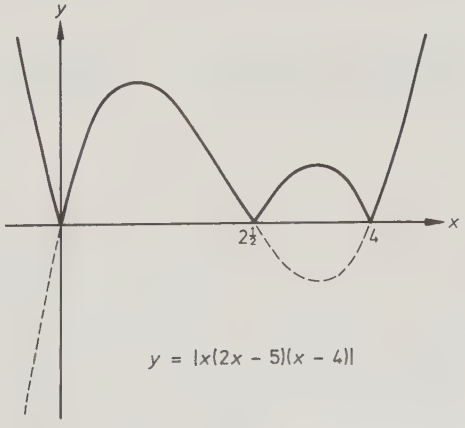
Thus, in general, an equation (or inequality) connecting x and y defines a relation on $\mathbb{R}$. Such an equation defines a function of x with domain $\mathbb{R}$ (or a subset of $\mathbb{R}$) only if there is at most one value of y corresponding to any given value of x. For instance the equation $y^2 = x$ specifies a relation, but not a function of the form $f: x \to y$, whereas the equation $y = \sqrt{x}$ defines a function with domain $\{x \in \mathbb{R}: x \geqslant 0\}$.

The equation $y = |f(x)|$

When $f(x) \geqslant 0$, the graph of $y = |f(x)|$ coincides with the graph of $y = f(x)$. When $f(x) < 0$, the graph of $y = |f(x)|$ is the reflection in the x-axis of the graph of $y = f(x)$.

[In the following sketches broken lines are used for the graph of $y = f(x)$ and unbroken lines for the graph of $y = |f(x)|$.]



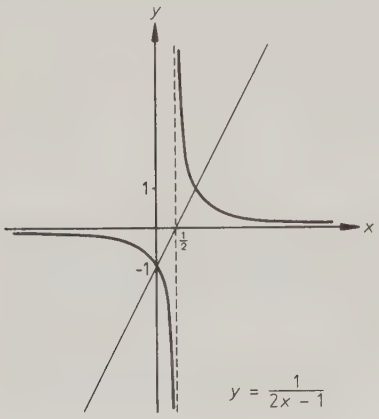


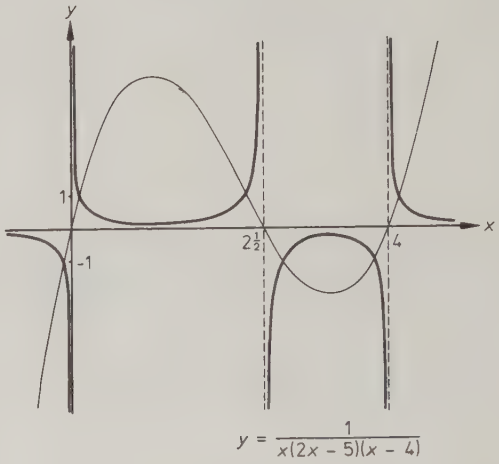
The equation $y = 1/f(x)$

If the graph of $y = f(x)$ meets the x-axis at $x = a$, then the line $x = a$ is an asymptote to the graph of $y = 1/f(x)$ and vice versa.

If there is a maximum (minimum) point on the graph of $y = f(x)$ when $x = k$, then provided that $f(k) \neq 0$, there is a minimum (maximum) point on the graph of $y = 1/f(x)$ when $x = k$.

The graphs of $y = f(x)$ and $y = 1/f(x)$ always lie on the same side of the x-axis and intersect when $f(x) = \pm 1$.





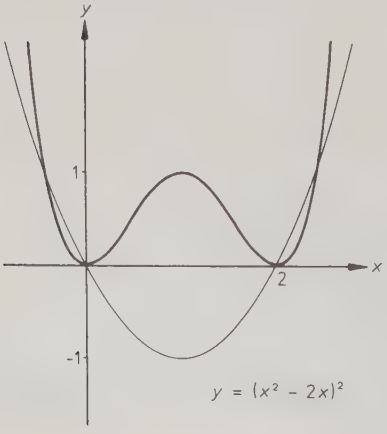
The equation $y = \{f(x)\}^2$

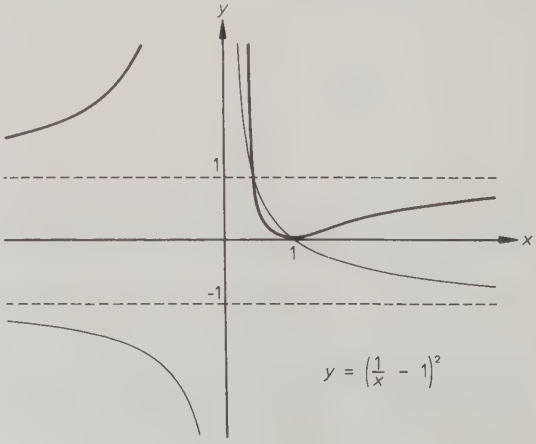
The graph of $y = \{f(x)\}^2$ lies entirely on or above the x-axis and cuts the graph of $y = f(x)$ when $f(x) = 0$ or 1.

$$y = \{f(x)\}^2 \Rightarrow \frac{dy}{dx} = 2f(x)f'(x)$$

Thus, since $dy/dx = 0$ when $f'(x) = 0$, there is a stationary point on the graph of $y = \{f(x)\}^2$ at any value of x for which there is a stationary point on the graph of $y = f(x)$.

Since $dy/dx = 0$ when $f(x) = 0$, any point at which the graph of $y = f(x)$ cuts the x-axis is a stationary point on the graph of $y = \{f(x)\}^2$.





The equation $y^2 = f(x)$

At values of x such that $f(x) < 0$, there are no points on the graph of $y^2 = f(x)$.

At values of x such that $f(x) > 0$, there are two possible values of y, namely $\pm\sqrt{\{f(x)\}}$. Hence the graph of $y^2 = f(x)$ is symmetrical about the x-axis.

It is also useful to note that if $0 < f(x) < 1$, then $\sqrt{\{f(x)\}} > f(x)$ and if $f(x) > 1$, then $\sqrt{\{f(x)\}} < f(x)$.

$$y^2 = f(x) \Rightarrow 2y\frac{dy}{dx} = f'(x)$$

$$\therefore \text{ if } f'(x) = 0, \text{ and } f(x) > 0, \text{ then } \frac{dy}{dx} = 0.$$

Thus, any stationary point above the x-axis on the curve $y = f(x)$ gives rise to a pair of stationary points on the curve $y^2 = f(x)$.

To determine how the graph of $y^2 = f(x)$ behaves near the x-axis, the expression for dy/dx must be considered further. For instance, suppose that $f(x) = x(x-3)^2$.

If $$y^2 = x(x-3)^2 = x^3 - 6x^2 + 9x$$

then differentiating with respect to x,

$$2y\frac{dy}{dx} = 3x^2 - 12x + 9 = 3(x-1)(x-3)$$

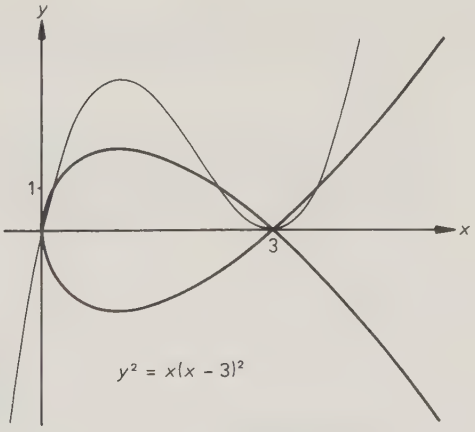
But $y = \pm(x-3)\sqrt{x}$

$\therefore$ provided that $x \neq 3$,

$$\frac{dy}{dx} = \pm\frac{3(x-1)}{2\sqrt{x}}$$

As $x \to 0$, $\frac{dy}{dx} \to \pm\infty$.

As $x \to 3$, $\frac{dy}{dx} \to \pm\sqrt{3}$.



Exercise 18.2

In questions 1 to 6 sketch the graphs with given equations

1. (a) $y = x^2$, (b) $y = \frac{1}{2}x^2$, (c) $y = x^2 - 4$, (d) $y = (x+2)^2$.

2. (a) $y = \sin x$, (b) $y = 2\sin x$, (c) $y = \sin 2x$, (d) $y = \sin(x + \frac{1}{2}\pi)$.

3. (a) $y = \frac{1}{x}$, (b) $y = \frac{4}{x}$, (c) $y = \frac{1}{x+1}$, (d) $y = \frac{x+2}{x+1}$.

4. (a) $y = |3 - 2x|$, (b) $y = |x^2 - 4|$, (c) $y = |x^3 - 3x^2|$.

5. (a) $y = |\cos x|$, (b) $y = |\cos x| - 1$, (c) $y = |2\cos x + 1|$.

6. (a) $y = \dfrac{1}{x-2}$, (b) $y = \dfrac{1}{2-x}$, (c) $y = \dfrac{1}{|x-2|}$.

In questions 7 to 16 sketch the given pair of curves in the same diagram. Label clearly any stationary points and asymptotes.

7. $y = x^2 + 2x$, $y = 1/(x^2 + 2x)$.

8. $y = x^2 + 2x + 1$, $y = 1/(x^2 + 2x + 1)$.

9. $y = x^2 + 2x + 2$, $y = 1/(x^2 + 2x + 2)$.

10. $y = x^3 + 2x^2 - 15x$, $y = 1/(x^3 - 2x^2 - 15x)$.

11. $y = (2x - 1)/(x + 2)$, $y = (x + 2)/(2x - 1)$.

12. $y = 2x - 3$, $y = (2x - 3)^2$.

13. $y = 4 - x^2$, $y = (4 - x^2)^2$.

14. $y = \sin x$, $y = \sin^2 x$.

15. $y = 2x/(1 - x)$, $y = 4x^2/(1 - x)^2$.

16. $y = 3x^2 - x^3$, $y^2 = 3x^2 - x^3$.

In questions 17 to 20 sketch the curves $y = f(x)$ and $y^2 = f(x)$ in the same diagram, showing clearly the behaviour of the curves near the x-axis.

17. $f(x) = x^3$.

18. $f(x) = x(x + 3)^2$.

19. $f(x) = (x + 1)(x - 2)^2$.

20. $f(x) = x^2(4 - x^2)$.

18.3 Inverse functions

Consider a function f with domain X and range Y. If every element of Y is the image of one and only one element of X, then f is said to be *one-one*. For any one-one function $f : X \to Y$ we define an *inverse function* $f^{-1} : Y \to X$ such that if $y = f(x)$ then $f^{-1}(y) = x$.

It follows from the definition that

$$f^{-1}f(x) = f^{-1}(y) = x \quad \text{and} \quad ff^{-1}(y) = f(x) = y$$

i.e. the composite functions $f^{-1}f$ and ff^{-1} are the identity mappings on the sets X and Y respectively.

Example 1 One-one functions f and g are defined on $\mathbb{R}$ by $f: x \to 3x - 1$ and $g: x \to \lg(1 - x)$, $x < 1$. Find expressions for f^{-1} and g^{-1}.

Let $y = f^{-1}(x)$ then $f(y) = x$

$\therefore \quad 3y - 1 = x$

$y = \frac{1}{3}(x + 1)$

Hence $f^{-1}: x \to \frac{1}{3}(x + 1)$.

Let $y = g^{-1}(x)$ then $g(y) = x$

$\therefore \quad \lg(1 - y) = x$

$1 - y = 10^x$

$y = 1 - 10^x$

Hence $g^{-1}: x \to 1 - 10^x$.

Example 2 A one-one function f is defined by $f: x \to 1/\sqrt{(x - 1)}$ where $x \in \mathbb{R}$, $x > 1$.
Find an expression for f^{-1}, stating its domain. Verify that $f^{-1}f(x) = x$ where $x \in \mathbb{R}$, $x > 1$.

$$\begin{aligned} y = f^{-1}(x) &\Rightarrow f(y) = x \\ &\Rightarrow 1/\sqrt{(y - 1)} = x \\ &\Rightarrow 1/(y - 1) = x^2 \\ &\Rightarrow 1 = x^2(y - 1) \\ &\Rightarrow y - 1 = 1/x^2 \\ &\Rightarrow y = 1 + 1/x^2 \end{aligned}$$

$\therefore \quad f^{-1}: x \to 1 + 1/x^2$.

The domain of f^{-1} is the same as the range of f
i.e. $\mathbb{R}^+$, the set of positive real numbers.

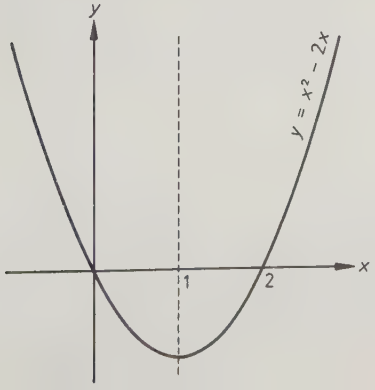
$f^{-1}f(x) = f^{-1}(u)$, where $u = f(x) = 1/\sqrt{(x - 1)}$

$\therefore \quad f^{-1}f(x) = 1 + 1/u^2 = 1 + (x - 1) = x$.

Example 3 A function f is defined on $\mathbb{R}$ by $f: x \to x^2 - 2x$. Explain why f^{-1} does not exist. Suggest a restricted domain for which f^{-1} does exist.

$f(0) = f(2) = 0$ i.e. 0 is the image under f of both 0 and 2,

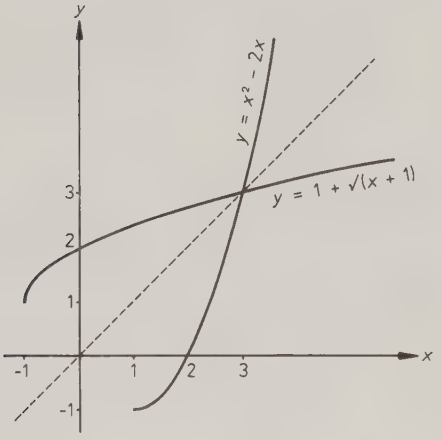
$\therefore$ the function f defined over $\mathbb{R}$ is not one-one and has no inverse.



The sketch of the graph $y = x^2 - 2x$ shows that for $x \geqslant 1$ there is only one value of x corresponding to each value of y, $\therefore$ over the domain $\{x \in \mathbb{R}: x \geqslant 1\}$ the function f is one-one and f^{-1} does exist. [Note that f^{-1} also exists for the restricted domain $\{x \in \mathbb{R}: x \leqslant 1\}$.]

In general, a function f has an inverse if and only if any line drawn parallel to the x-axis cuts the graph $y = f(x)$ exactly once. Clearly a function whose graph has one or more turning points does not have an inverse.

If a point (a, b) lies on the graph $y = f(x)$ then the point (b, a) lies on the graph $y = f^{-1}(x)$. Hence each graph is the reflection of the other in the line $y = x$.



The diagram illustrates this relationship for the function

$$f: x \to x^2 - 2x, \quad x \geqslant 1.$$

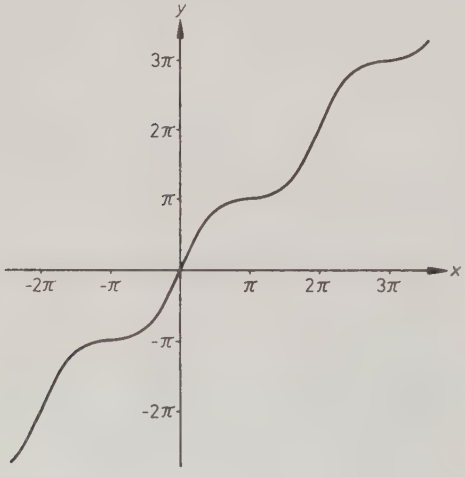
The equation of the graph of f^{-1} is $y^2 - 2y = x, y \geqslant 1$, which may be rearranged as

$$(y-1)^2 = x + 1, \ y \geqslant 1$$

i.e. $$y - 1 = \sqrt{(x+1)}$$

i.e. $$y = 1 + \sqrt{(x+1)}.$$

In some cases it is possible to show that a function f has an inverse, but difficult to find an expression for $f^{-1}(x)$. Consider, for example, the function $f: x \to x + \sin x$.



$$y = x + \sin x$$

$$\Rightarrow \quad \frac{dy}{dx} = 1 + \cos x$$

$$\Rightarrow \quad \frac{dy}{dx} \geqslant 0 \quad \text{for all} \quad x \in \mathbb{R}.$$

A sketch of the graph $y = x + \sin x$ shows that the function f must be one-one.

Hence the inverse f^{-1} exists and its graph has equation $y + \sin y = x$. However, it is impossible to rearrange this equation in the form $y = f^{-1}(x)$. Thus although f^{-1} exists, in practice it is difficult to find $f^{-1}(x)$ for many values of x.

Exercise 18.3

In questions 1 to 14 state whether or not the function f is one-one. If the function f is one-one, find an expression for f^{-1}, stating its domain. For any function that is not one-one, give two distinct values of x which have the same image.

1. $f: x \to 2x + 1$.
2. $f: x \to 5 - x$.
3. $f: x \to x(x + 2)$.
4. $f: x \to x^2 - 2,\ x \geqslant 0$.
5. $f: x \to 1 + \frac{1}{x},\ x \neq 0$.
6. $f: x \to \frac{1}{9 - x},\ x \neq 9$.
7. $f: x \to \sqrt{(3 - x)},\ x \leqslant 3$.
8. $f: x \to \sqrt{(1 - x^2)},\ -1 \leqslant x \leqslant 1$.
9. $f: x \to 3^x$.
10. $f: x \to \lg(x + 2), x > -2$.
11. $f: x \to \frac{1}{x^2 + 4}$.
12. $f: x \to \frac{2x}{x - 1},\ x \neq 1$.
13. $f: x \to \cos 3x$.
14. $f: x \to \sin^{-1} x,\ -1 \leqslant x \leqslant 1$.

15. One-one functions f and g are defined on $\mathbb{R}$ by $f: x \to 5 - 4x$ and $g: x \to 1 - x^2,\ x \geqslant 0$. Given that h is the composite function fg, find expressions for $h(x)$ and $h^{-1}(x)$. Verify that $h^{-1} = g^{-1}f^{-1}$.

16. Functions f, g and h are defined on $\mathbb{R}$ by $f: x \to 3 - x$, $g: x \to 3x/(x - 3)$, $x \neq 3$ and $h: x \to \sqrt{(4 - x^2)},\ 0 \leqslant x \leqslant 2$. Show that each of the functions is its own inverse. Find the composite function gf and verify that its inverse is fg.

17. Find the inverse of each of the following functions defined on $\mathbb{R}$. Verify that, over the appropriate domains, $f^{-1}f(x) = x$ and $ff^{-1}(x) = x$.

(a) $f: x \to (x - 2)^2,\ x \geqslant 2$,
(b) $f: x \to 1 + \lg(x - 1), x > 1$,
(c) $f: x \to \frac{1 - x}{1 + x},\ x \neq -1$,
(d) $f: x \to \frac{1}{\sqrt{(x^2 - 1)}},\ x > 1$.

In questions 18 to 23 the function f has domain $\mathbb{R}^+$, the set of positive real numbers. In each case sketch the graph of f and explain why no inverse function exists. Find the maximum domain of the form $\{x \in \mathbb{R} : x \geqslant k\}$ for which f would have an inverse f^{-1} and state the domain of the inverse.

18. $f: x \to x^2 - 4x + 8$.
19. $f: x \to 6x - x^2$.
20. $f: x \to |x - 4|$.
21. $f: x \to 2x^3 - 3x^2$.
22. $f: x \to x + \frac{1}{x}$.
23. $f: x \to x - \sqrt{x}$.

24. Decide whether the following functions have inverses, giving brief reasons for your answers.

(a) $f: x \to x^3 + 3x^2, x \in \mathbb{R}$,
(b) $f: x \to x^3 + 3x, x \in \mathbb{R}$,
(c) $f: x \to 2x + \cos x, x \in \mathbb{R}$,
(d) $f: x \to x + 2\sin x, x \in \mathbb{R}$,
(e) $f: x \to |\lg x|, x \in \mathbb{R}^+$,
(f) $f: x \to |x| + 2x, x \in \mathbb{R}$.

25. Differentiable functions f and g are defined on $\mathbb{R}$ such that f is the inverse of g. Prove that, if $f(a) = b$, then $f'(a)g'(b) = 1$.

18.4 Polynomials; remainders and factors

If, when a polynomial $P(x)$ is divided by $(x - a)$, the quotient is $Q(x)$ and the remainder is R, then

$$P(x) \equiv (x - a)Q(x) + R.$$

This identity leads to two familiar results.

The remainder theorem: When $P(x)$ is divided by $(x - a)$ the remainder is $P(a)$.

The factor theorem: If $P(a) = 0$, then $(x - a)$ is a factor of $P(x)$.

This approach can be extended to the division of a polynomial $f(x)$ by a polynomial $g(x)$ of degree less than or equal to the degree of $f(x)$. If the division gives quotient $Q(x)$ and remainder $R(x)$ then

$$f(x) \equiv g(x)Q(x) + R(x)$$

where $R(x)$ is of lower degree than $g(x)$.

[In particular, if $g(x)$ is a quadratic function then $R(x)$ is of the form $Ax + B$.]

Example 1 A polynomial $P(x)$ is a multiple of $x - 3$ and the remainder when $P(x)$ is divided by $x + 3$ is 12. Find the remainder when $P(x)$ is divided by $x^2 - 9$.

When $P(x)$ is divided by $x^2 - 9$, let the quotient be $Q(x)$ and the remainder $Ax + B$, then

$$P(x) \equiv (x^2 - 9)Q(x) + Ax + B.$$

Since $x - 3$ is a factor of $P(x)$, by the factor theorem,

$$P(3) = 0 \qquad \therefore \quad 3A + B = 0 \tag{1}$$

Since when $P(x)$ is divided by $x + 3$ the remainder is 12, by the remainder theorem,

$$P(-3) = 12 \qquad \therefore \quad -3A + B = 12 \tag{2}$$

Using (1) and (2), $A = -2$, $B = 6$.

Hence the remainder when $P(x)$ is divided by $x^2 - 9$ is $-2x + 6$.

Suppose now that a polynomial $f(x)$ has a repeated factor $(x - a)$,

so that $f(x) \equiv (x - a)^2 g(x)$

Differentiating, $f'(x) = (x - a)^2 g'(x) + 2(x - a)g(x)$

$= (x - a)\{(x - a)g'(x) + 2g(x)\}$

Hence, if $f(x)$ has a repeated factor $(x - a)$, then $(x - a)$ is also a factor of $f'(x)$.

It can further be shown, by the method of question 21, Exercise 18.4, that $(x - a)^2$ is a factor of a polynomial $f(x)$ if and only if $f(a) = f'(a) = 0$.

Example 2 Given that the polynomial $f(x) = x^3 + 3x^2 - 9x + k$ has a repeated linear factor, find the possible values of k.

Differentiating, $f'(x) = 3x^2 + 6x - 9$

$= 3(x^2 + 2x - 3) = 3(x - 1)(x + 3)$

$\therefore$ the repeated factor of $f(x)$ is $(x - 1)$ or $(x + 3)$.

If $(x - 1)$ is a factor of $f(x)$, then $f(1) = 0$

i.e. $1 + 3 - 9 + k = 0$ giving $k = 5$.

If $(x + 3)$ is a factor of $f(x)$, then $f(-3) = 0$

i.e. $-27 + 27 + 27 + k = 0$ giving $k = -27$.

Thus the possible values of k are 5 and -27.

When dealing with polynomial equations it may be helpful to use the relationships between roots and coefficients. In the case of the cubic equation $ax^3 + bx^2 + cx + d = 0$ with roots α, β and γ,

$$\alpha + \beta + \gamma = -\frac{b}{a}, \quad \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}, \quad \alpha\beta\gamma = -\frac{d}{a}.$$

Similar results can be obtained for equations of higher degree.

Example 3 Find the roots of the equation $4x^3 + 12x^2 - 15x + 4 = 0$ given that it has a repeated root.

Let $f(x) = 4x^3 + 12x^2 - 15x + 4$

then $f'(x) = 12x^2 + 24x - 15$

$= 3(4x^2 + 8x - 5) = 3(2x - 1)(2x + 5)$

Any repeated root of the equation $f(x) = 0$ is also a root of the equation $f'(x) = 0$

$\therefore$ the repeated root must be either $-5/2$ or $1/2$.

[It is possible to decide which of these values is the required root by evaluating $f(-5/2)$ and $f(1/2)$. However, the method given below may be quicker.]

The product of the roots of $f(x) = 0$ is $-4/4$ i.e. -1.

Thus, if the repeated root is $-5/2$, the remaining root is $-1 \div (-5/2)^2$ i.e. $-4/25$.

If the repeated root is $1/2$, the remaining root is $-1 \div (1/2)^2$ i.e. -4.

Since the sum of the roots of $f(x) = 0$ is $-12/4$ i.e. -3, the roots must be $1/2$, $1/2$, -4.

Let us consider two polynomials $P(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n$ and

$$Q(x) = b_0x^n + b_1x^{n-1} + \ldots + b_n.$$

It can be shown that if $P(x) \equiv Q(x)$ i.e. if $P(x)$ and $Q(x)$ are equal for all values of x, then $a_0 = b_0, a_1 = b_1, \ldots, a_n = b_n$.

One way of proving this result uses the fact that if $P(x) \equiv Q(x)$, then $P'(x) \equiv Q'(x)$, $P''(x) \equiv Q''(x), \ldots, P^{(n)}(x) \equiv Q^{(n)}(x)$.

Substituting $x = 0$, we find that $P(0) = Q(0) \Rightarrow a_n = b_n$,
$P'(0) = Q'(0) \Rightarrow a_{n-1} = b_{n-1}, \ldots, P^{(n)}(0) = Q^{(n)}(0) \Rightarrow a_0 = b_0$.

We deduce that if two polynomials are identically equal, then corresponding coefficients are equal. Problems involving such identities are solved by *comparing coefficients*, by substituting suitable values of x, or by a combination of these methods.

Example 4 Given that $x^2 - 3x + 7 \equiv a(x-1)^2 + b(x-1) + c$, find the values of a, b and c.

Comparing coefficients of x^2: $1 = a$ $\therefore$ $a = 1$
Putting $x = 1$: $1 - 3 + 7 = 0 + 0 + c$ $\therefore$ $c = 5$
Putting $x = 0$: $7 = a - b + c$ $\therefore$ $b = -1$
Hence $x^2 - 3x + 7 \equiv (x-1)^2 - (x-1) + 5$.

[Note that putting $x = 0$ is equivalent to comparing constant terms.]

Example 5 If $x^2 + 1$ is a factor of $3x^4 + x^3 - 4x^2 + px + q$, find the values of p and q.

Let $3x^4 + x^3 - 4x^2 + px + q \equiv (x^2 + 1)(ax^2 + bx + c)$.

Comparing coefficients of x^4: $3 = a$
x^3: $1 = b$
x^2: $-4 = a + c$
x: $p = b$
Comparing constant terms: $q = c$
Hence $p = 1$ and $q = -7$.

Exercise 18.4

1. Find the remainder when
(a) $x^3 - 4x^2 + 3x - 7$ is divided by $x - 4$,
(b) $x^4 + 2x^3 + 5x - 8$ is divided by $x + 3$,
(c) $2x^3 - 5x^2 + 3x + 1$ is divided by $2x + 1$,
(d) $x^3 - 2x^2 - 5$ is divided by $(x + 1)(x - 2)$,
(e) $x^4 + 3x^3 - 12x - 16$ is divided by $x^2 - 4$,
(f) $2x^4 - 3x^3 - 9x^2 - 7$ is divided by $x^2 - x - 6$.

2. When a polynomial $f(x)$ is divided by $(x - 2)$ the remainder is -2, and when it is divided by $(x - 3)$ the remainder is 3. Given that the remainder when $f(x)$ is divided by $(x - 2)(x - 3)$ is $px + q$, find the values of p and q.

3. When a polynomial is divided by $(x + 1)$ the remainder is 7, and when it is divided by $(x - 4)$ the remainder is -8. Find the remainder when the polynomial is divided by $(x + 1)(x - 4)$.

4. When the polynomial $x^3 + px^2 + qx + 7$ is divided by $x^2 - 3x + 2$, the remainder is $5x - 3$. Find the values of p and q.

5. When $P(x) = x^3 + ax^2 + bx + c$ is divided by $x^2 - 4$ the remainder is $2x + 11$. Given that $x + 1$ is a factor of $P(x)$, find the values of a, b and c.

6. Show, by putting $a = -b$, that $a + b$ is a factor of $(a + b + c)^3 - (a^3 + b^3 + c^3)$. Factorise the expression completely.

7. Show, by putting $x = y$, that $x - y$ is a factor of $(x - y)^3 + (y - z)^3 + (z - x)^3$. Factorise the expression completely.

8. Given that the polynomial $f(x) = 2x^3 - 3x^2 - 12x + k$ has a repeated linear factor, find the possible values of k and factorise $f(x)$ in each case.

9. Factorise the polynomial $3x^4 + 8x^3 + 16$ given that it has a repeated linear factor.

10. Find the values of a and b and factorise the polynomial

$$f(x) = x^3 + 4ax^2 + bx + 3a$$

given that it is divisible by $(x - 1)^2$.

11. Find the roots of the equation $x^3 - 6x^2 - 63x - 108 = 0$ given that it has a repeated root.

12. Given that $P(x) = 8x^3 - 12x^2 - 18x + k$, find the values of k such that the equation $P(x) = 0$ has a repeated root. Give the roots of the equation in each case.

13. The roots of the equation $2x^3 + 3x^2 + kx - 6 = 0$ are in arithmetic progression. By letting the roots be $\alpha - \lambda, \alpha, \alpha + \lambda$, or otherwise, find the value of k and solve the equation.

14. Given that the roots of the equation $3x^3 - 7x^2 + px + 24 = 0$ are in geometric progression, find the value of p and solve the equation.

15. If $ax^3 + 2x^2 + bx + 2b - c \equiv x^3 + 2x^2 + (c + 2)x + b + c$, find the values of a, b and c.

16. If $x^3 - 2x + 7 \equiv a(x-1)^3 + b(x-1)^2 + c(x-1) + d$, find the values of a, b, c and d.

17. Find constants a, b, c and d such that

$$n^3 \equiv an(n+1)(n+2) + bn(n+1) + cn + d.$$

18. Show that no constants a and b can be found such that

$$n^2 \equiv a(n+1)^2 + b(n+1).$$

19. Given that $x^4 - 6x^3 + 10x^2 + ax + b$ is a perfect square, find the values of a and b.

20. Given that $x^2 - x + 1$ is a factor of $2x^4 - 3x^3 + px^2 + qx - 3$, find the values of p and q.

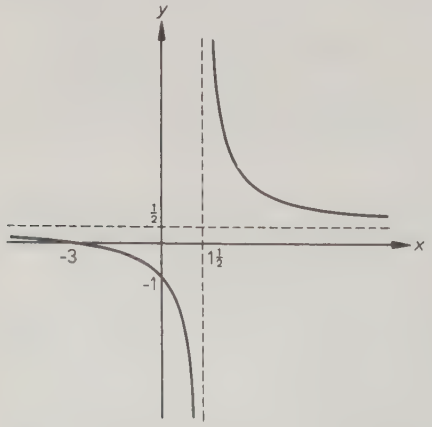
21. If $f(x)$ and $g(x)$ are polynomials and $f(x) \equiv (x-a)^2 g(x) + Ax + B$, find $f'(x)$ and hence find A and B in terms of $f(a)$ and $f'(a)$. Deduce that $(x-a)$ is a repeated factor of $f(x)$ if and only if $f(a) = f'(a) = 0$.

Find all real values of k for which the equation $x^3 - 3kx^2 + 2k + 2 = 0$ has repeated roots and, for each such k, solve the equation completely. (O & C)

18.5 Graphs of rational functions

A rational function is a function of the form $P(x)/Q(x)$, where $P(x)$ and $Q(x)$ are polynomials. The graph of this function cuts the x-axis where $P(x) = 0$ and has asymptotes parallel to the y-axis where $Q(x) = 0$.

One of the simplest types of rational function takes the form $(ax + b)/(cx + d)$. In general, the graph of this function is a curve called a *rectangular hyperbola* with asymptotes parallel to the x- and y-axes.



The diagram shows the graph of $y = (x+3)/(2x-3)$.
The curve cuts the axes at $(-3, 0)$ and $(0, -1)$.
As $x \to 1\frac{1}{2}$, $y \to \pm\infty$.

$$y = \frac{x+3}{2x-3} = \frac{1 + 3/x}{2 - 3/x}$$

$\therefore$ as $x \to \pm\infty$, $y \to \frac{1}{2}$.

Hence the lines $x = 1\frac{1}{2}$ and $y = \frac{1}{2}$ are asymptotes to the curve.

[If necessary, we can verify that such curves have no turning points by finding dy/dx.

In this case

$$\frac{dy}{dx} = \frac{(2x-3)1-(x+3)2}{(2x-3)^2} = -\frac{9}{(2x-3)^2}.$$

Thus the gradient of the curve is always negative.]

Another important set of rational functions are defined by expressions of the form $y = (ax+b)/(px^2+qx+r)$. When sketching the graphs of these functions it is helpful to find the range of values taken by y and also to note where changes in the sign of y occur.

We illustrate this approach by considering the graph of $y = 4x/(x^2+1)$.
The graph cuts the axes only at the origin $(0,0)$.
Since (x^2+1) is always positive, there can be no asymptotes parallel to the y-axis and the sign of y will always be the same as the sign of x.
As $x \to \infty, y \to 0$ from above and as $x \to -\infty, y \to 0$ from below,
$\therefore$ the x-axis is an asymptote to the curve.
To find the range of the function $y = 4x/(x^2+1)$, we rearrange the equation as a quadratic equation in x:

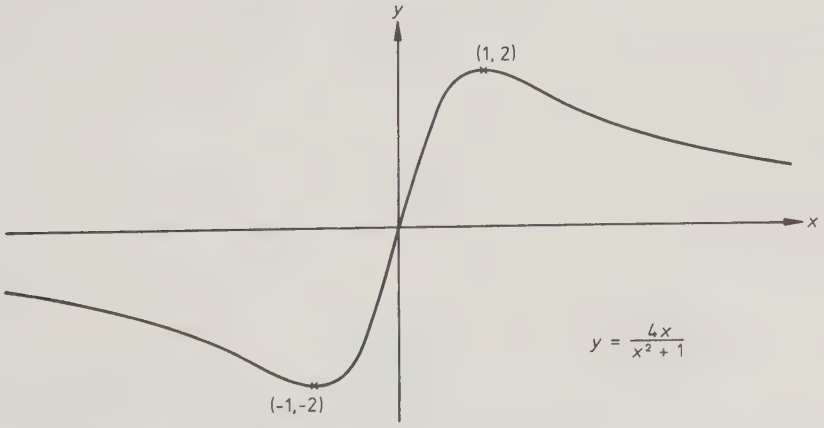
$$y(x^2+1) = 4x \quad \text{i.e.} \quad yx^2 - 4x + y = 0.$$

Since we are concerned only with values of y given by real values of x, the range of values taken by y is the set of values for which this quadratic equation has real roots.
The equation $ax^2+bx+c=0$ has real roots if $b^2-4ac \geqslant 0$,
$\therefore$ for real values of x, $(-4)^2 - 4y.y \geqslant 0$

$$16 - 4y^2 \geqslant 0$$
$$y^2 \leqslant 4$$

i.e. for real values of x, $-2 \leqslant y \leqslant 2$.

Thus all points of the graph lie on or between the lines $y = \pm 2$.



When $y = -2$ and when $y = 2$ the quadratic equation in x has equal roots, $x = -1$ and $x = 1$ respectively. Hence the line $y = -2$ is a tangent to the curve at the point $(-1,-2)$ and the line $y = 2$ is a tangent to the curve at the point $(1,2)$. It follows that $(-1,-2)$ must be a minimum point on the curve and $(1,2)$ a maximum point.

For every non-zero value of y in the interval $-2 < y < 2$ the quadratic equation in x has two distinct real roots. Hence there are no other turning points on the curve. [Note that a graph of similar form will be obtained for any function $(ax + b)/(px^2 + qx + r)$ if the denominator $(px^2 + qx + r)$ is never zero.]

Example 1 Sketch the curve $y = \dfrac{2x - 3}{x^2 + 2x - 3}$.

$x = 0 \Rightarrow y = 1$ $\quad\therefore$ the curve cuts the y-axis at the point $(0, 1)$.

$y = 0 \Leftrightarrow 2x - 3 = 0 \Leftrightarrow x = 1\frac{1}{2}$

$\therefore$ the curve cuts the x-axis at the point $(1\frac{1}{2}, 0)$.

$x^2 + 2x - 3 = 0 \Leftrightarrow (x + 3)(x - 1) = 0 \Leftrightarrow x = -3$ or $x = 1$

$\therefore$ the lines $x = -3$ and $x = 1$ are asymptotes to the curve.

As $x \to \pm\infty, y \to 0$ $\quad\therefore$ the x-axis is an asymptote to the curve.

The following table shows the changes in the sign of y.

	$x < -3$	$-3 < x < 1$	$1 < x < 1\frac{1}{2}$	$x > 1\frac{1}{2}$
$x + 3$	$-$	$+$	$+$	$+$
$x - 1$	$-$	$-$	$+$	$+$
$2x - 3$	$-$	$-$	$-$	$+$
y	$-$	$+$	$-$	$+$

Rearranging the equation as a quadratic in x:

$$y(x^2 + 2x - 3) = 2x - 3 \quad \text{i.e.} \quad yx^2 + (2y - 2)x - 3y + 3 = 0$$

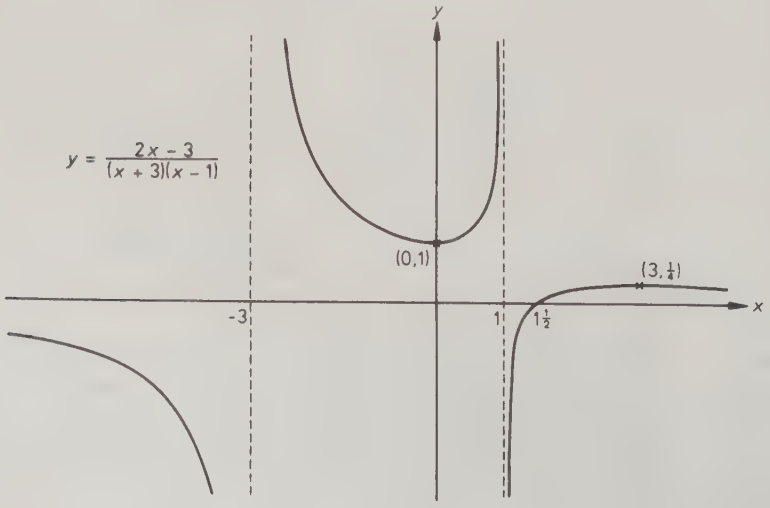
$\therefore$ for real values of x, $(2y - 2)^2 - 4y(-3y + 3) \geqslant 0$

$$4(y - 1)^2 + 12y(y - 1) \geqslant 0$$

$$4(y - 1)\{(y - 1) + 3y\} \geqslant 0$$

$$(y - 1)(4y - 1) \geqslant 0$$

$\therefore$ for real values of x, $y \leqslant \frac{1}{4}$ or $y \geqslant 1$.



Hence there are no points on the curve between the lines $y = \frac{1}{4}$ and $y = 1$.
There is a stationary point on the curve whenever the quadratic equation in x has repeated roots, i.e. when $y = \frac{1}{4}, x = 3$ and when $y = 1, x = 0$. Since y cannot take values between $\frac{1}{4}$ and 1, $(3, \frac{1}{4})$ is a maximum point and $(0, 1)$ a minimum point.

There is one other form that the graph of $y = (ax + b)/(px^2 + qx + r)$ can take. Consider, for example, the curve $y = x/(x + 2)(x - 1)$.
This curve cuts the axes only at the origin $(0, 0)$.
The asymptotes to the curve are the lines $x = -2$, $x = 1$ and the x-axis.
Rearranging the equation as a quadratic in x:

$$y(x^2 + x - 2) = x \quad \text{i.e.} \quad yx^2 + (y - 1)x - 2y = 0.$$

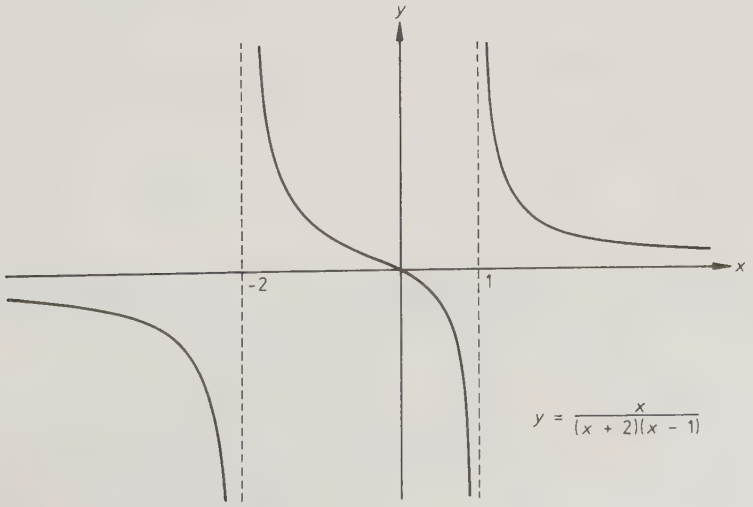
The nature of the roots of this quadratic equation depends on the sign of the expression

$$(y - 1)^2 - 4y(-2y) \quad \text{i.e.} \quad (y - 1)^2 + 8y^2.$$

Since this expression is always positive, the equation must have two real distinct roots for all non-zero values of y. Hence there are no stationary points on the curve and no restrictions on the values of y.

Again it is useful to determine the sign of y throughout the domain of the function.

	$x < -2$	$-2 < x < 0$	$0 < x < 1$	$x > 1$
$x + 2$	$-$	$+$	$+$	$+$
x	$-$	$-$	$+$	$+$
$x - 1$	$-$	$-$	$-$	$+$
y	$-$	$+$	$-$	$+$



[These methods can be extended to functions of the form $P(x)/Q(x)$ where both $P(x)$ and $Q(x)$ are quadratic functions.]

Exercise 18.5

In questions 1 to 6 sketch the given curve.

1. $y = \dfrac{1}{x-1}$ 2. $y = \dfrac{x}{x-1}$ 3. $y = \dfrac{x-1}{x}$

4. $y = \dfrac{3-x}{2+x}$ 5. $y = \dfrac{2x+1}{x-2}$ 6. $y = \dfrac{x}{3x+4}$

In questions 7 to 16 find the range of possible values of y. Sketch each curve showing clearly any asymptotes and turning points.

7. $y = \dfrac{5x-25}{x^2+3x-4}$ 8. $y = \dfrac{2x+1}{x^2+2}$

9. $y = \dfrac{x+1}{x(x+3)}$ 10. $y = \dfrac{x-1}{x(x+3)}$

11. $y = \dfrac{2-3x}{x^2-3x+2}$ 12. $y = \dfrac{15-10x}{4+x^2}$

13. $y = \dfrac{1-x}{1-x+x^2}$ 14. $y = \dfrac{x+6}{x^2+7x+10}$

15. $y = \dfrac{1}{x-1} + \dfrac{1}{4-x}$ 16. $y = \dfrac{1}{x-1} + \dfrac{1}{x-4}$

17. If a, b and c are all positive, sketch the curve $y = \dfrac{x-a}{(x-b)(x-c)}$ when

(i) $a < b < c$,
(ii) $b < a < c$,
(iii) $b < c < a$.

18. By considering the limiting value of y as $x \to \pm\infty$, show that the line $y = 1$ is an asymptote to the curve $y = (x^2+1)/(x^2-1)$. Sketch the curve.

19. Sketch the curves

(a) $y = \dfrac{3x^2}{x^2+x-2}$, (b) $y = \dfrac{3x^2}{x^2+x+2}$.

20. Sketch the curve $y = \dfrac{x^2 - 3x + 4}{x - 3}$ showing clearly any asymptotes and turning points.

18.6 Partial fractions

A sum of two fractions can be expressed as a single fraction by a method such as the following:

$$\frac{3}{3x-2} - \frac{x}{x^2+1} \equiv \frac{3(x^2+1) - x(3x-2)}{(3x-2)(x^2+1)} \equiv \frac{2x+3}{(3x-2)(x^2+1)}.$$

In this section we consider the reverse process by which a rational function is expressed as a sum of two or more simpler fractions called *partial fractions.*

We will deal first with *proper fractions* i.e. fractions in which the degree of the numerator is less than the degree of the denominator. The methods used in the examples which follow are based on the assumption that a proper fraction can be expressed as a sum of partial fractions which are also proper, such as

$$\frac{A}{x-a}, \quad \frac{Ax+B}{ax^2+bx+c}.$$

[A rigorous proof of this statement is beyond the scope of this course.]

Example 1 Express $\dfrac{2x-5}{(x+2)(x-1)}$ in partial fractions.

Let $$\frac{2x-5}{(x+2)(x-1)} \equiv \frac{A}{x+2} + \frac{B}{x-1}$$

then, multiplying both sides of the identity by $(x+2)(x-1)$,

$$2x - 5 \equiv A(x-1) + B(x+2)$$

Putting $x = 1$: $\quad 2 - 5 = 3B \quad \therefore \quad B = -1$

Putting $x = -2$: $\quad -4 - 5 = -3A \quad \therefore \quad A = 3$

$$\therefore \quad \frac{2x-5}{(x+2)(x-1)} \equiv \frac{3}{x+2} - \frac{1}{x-1}.$$

Example 2 Express $\dfrac{x+1}{2x^3 - 5x^2 + 2x}$ in partial fractions.

$$2x^3 - 5x^2 + 2x \equiv x(2x^2 - 5x + 2) \equiv x(x-2)(2x-1)$$

$$\therefore \text{ let } \quad \frac{x+1}{2x^3-5x^2+2x} \equiv \frac{A}{x} + \frac{B}{x-2} + \frac{C}{2x-1}$$

then $$x + 1 \equiv A(x-2)(2x-1) + Bx(2x-1) + Cx(x-2)$$

Putting $x = 0$: $1 = 2A$ $\quad\therefore\quad A = \frac{1}{2}$
Putting $x = 2$: $3 = 6B$ $\quad\therefore\quad B = \frac{1}{2}$
Putting $x = \frac{1}{2}$: $\frac{3}{2} = C \times \frac{1}{2} \times (-\frac{3}{2})$ $\quad\therefore\quad C = -2$

$$\therefore \quad \frac{x+1}{2x^3 - 5x^2 + 2x} \equiv \frac{1}{2x} + \frac{1}{2(x-2)} - \frac{2}{2x-1}.$$

So far we have determined unknown constants by substituting suitable values of x. In examples involving quadratic factors some constants can be obtained more quickly by comparing coefficients of powers of x.

Example 3 Express $\dfrac{4x^2}{(x-3)(x^2+3)}$ in partial fractions.

Let $$\frac{4x^2}{(x-3)(x^2+3)} \equiv \frac{A}{x-3} + \frac{Bx+C}{x^2+3}$$

then $$4x^2 \equiv A(x^2+3) + (Bx+C)(x-3)$$

Putting $x = 3$: $36 = 12A$ $\quad\therefore\quad A = 3$
Putting $x = 0$: $0 = 3A - 3C$ $\quad\therefore\quad C = 3$

Comparing coefficients of x^2:

$$4 = A + B \qquad \therefore \quad B = 1$$

$$\therefore \quad \frac{4x^2}{(x-3)(x^2+3)} \equiv \frac{3}{x-3} + \frac{x+3}{x^2+3}$$

In the next example we consider a fraction with the repeated factor $(x+1)^2$ in the denominator. Using the "proper fraction" rule, there will be a corresponding partial fraction of the form $(Ax+K)/(x+1)^2$. However, we obtained simpler partial fractions by writing $K = A + B$:

$$\frac{Ax+K}{(x+1)^2} \equiv \frac{Ax+A+B}{(x+1)^2} \equiv \frac{A(x+1)+B}{(x+1)^2} \equiv \frac{A}{x+1} + \frac{B}{(x+1)^2}.$$

Example 4 Express $\dfrac{x-5}{(x+1)^2(x-1)}$ in partial fractions.

Let $$\frac{x-5}{(x+1)^2(x-1)} \equiv \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1}$$

then $$x - 5 \equiv A(x+1)(x-1) + B(x-1) + C(x+1)^2$$

Putting $x = 1$: $-4 = 4C$ $\quad\therefore\quad C = -1$
Putting $x = -1$: $-6 = -2B$ $\quad\therefore\quad B = 3$

Comparing coefficients of x^2:

$$0 = A + C \qquad \therefore \quad A = 1$$

$$\therefore \quad \frac{x-5}{(x+1)^2(x-1)} \equiv \frac{1}{x+1} + \frac{3}{(x+1)^2} - \frac{1}{x-1}$$

A fraction in which the degrees of the numerator and denominator are equal can be expressed as the sum of a constant and a proper fraction,

e.g. $$\frac{x^2+1}{x^2-1} \equiv 1 + \frac{2}{x^2-1}.$$

We use this fact in the next example.

Example 5 Express $\dfrac{x^2+2}{(x-1)(2x+1)}$ in partial fractions.

Let $$\frac{x^2+2}{(x-1)(2x+1)} \equiv A + \frac{B}{x-1} + \frac{C}{2x+1}$$

then $$x^2 + 2 \equiv A(x-1)(2x+1) + B(2x+1) + C(x-1)$$

Putting $x = 1$: $\quad 3 = 3B \qquad \therefore \quad B = 1$

Putting $x = -\frac{1}{2}$: $\quad \frac{9}{4} = -\frac{3}{2}C \qquad \therefore \quad C = -\frac{3}{2}$

Comparing coefficients of x^2:

$$1 = 2A \qquad \therefore \quad A = \tfrac{1}{2}$$

$$\therefore \quad \frac{x^2+2}{(x-1)(2x+1)} \equiv \frac{1}{2} + \frac{1}{x-1} - \frac{3}{2(2x+1)}.$$

When dealing with fractions in which the degree of the numerator is greater than the degree of the denominator, we may either use long division to express the fraction as the sum of a polynomial and a proper fraction or extend the method of Example 5. If the degree of the numerator exceeds that of the denominator by n, then the expression in partial fractions must include a polynomial of degree n.

Example 6 Express $\dfrac{x^3-3x^2+1}{x^2-x-2}$ in partial fractions.

Method 1

$$\begin{array}{r} x-2 \\ x^2-x-2\overline{)\,x^3-3x^2\quad\quad+1} \\ x^3-x^2-2x \\ \hline -2x^2+2x+1 \\ -2x^2+2x+4 \\ \hline -3 \end{array}$$

$$\therefore \quad \frac{x^3-3x^2+1}{x^2-x-2} \equiv x - 2 - \frac{3}{x^2-x-2}$$

Let $$\frac{-3}{x^2-x-2} \equiv \frac{A}{x+1} + \frac{B}{x-2}$$

then $-3 \equiv A(x-2) + B(x+1)$

Putting $x = -1$: $-3 = -3A$ $\therefore \quad A = 1$

Putting $x = 2$: $-3 = 3B$ $\therefore \quad B = -1$

$$\therefore \quad \frac{x^3 - 3x^2 + 1}{x^2 - x - 2} \equiv x - 2 + \frac{1}{x+1} - \frac{1}{x-2}$$

Method 2

Let $$\frac{x^3 - 3x^2 + 1}{x^2 - x - 2} \equiv Ax + B + \frac{C}{x+1} + \frac{D}{x-2}$$

then $x^3 - 3x^2 + 1 \equiv (Ax + B)(x+1)(x-2) + C(x-2) + D(x+1)$

Putting $x = -1$: $-3 = -3C$ $\therefore \quad C = 1$

Putting $x = 2$: $-3 = 3D$ $\therefore \quad D = -1$

Putting $x = 0$: $1 = -2B - 2C + D$ $\therefore \quad B = -2$

Comparing coefficients of x^3:

$$1 = A$$

$$\therefore \quad \frac{x^3 - 3x^2 + 1}{x^2 - x - 2} \equiv x - 2 + \frac{1}{x+1} - \frac{1}{x-2}.$$

We now summarise the rules for expressing a rational function in partial fractions.

(1) To a linear factor $(x - a)$ in the denominator, there corresponds a fraction of the form $A/(x - a)$.

(2) To a quadratic factor $(ax^2 + bx + c)$, there corresponds a fraction of the form $(Ax + B)/(ax^2 + bx + c)$.

(3) To a repeated factor $(x - a)^n$, there correspond n fractions of the form $A_1/(x - a), A_2/(x - a)^2, \ldots, A_n/(x - a)^n$.

(4) To a repeated factor $(ax^2 + bx + c)^n$, there correspond n fractions

$$\frac{A_1x + B_1}{(ax^2 + bx + c)}, \frac{A_2x + B_2}{(ax^2 + bx + c)^2}, \ldots, \frac{A_nx + B_n}{(ax^2 + bx + c)^n}.$$

[Note that there are many ways of expressing a rational function as a sum of two or more fractions e.g. $\dfrac{1}{(x+1)(x+2)} \equiv \dfrac{x+1}{x+2} - \dfrac{x}{x+1} \equiv \dfrac{x^2}{x+1} - \dfrac{x^2 + x - 1}{x+2}$.

However, the simplest and most useful form is the expression in partial fractions according to rules (1) to (4) above.]

Exercise 18.6

Express in partial fractions

1. $\dfrac{1}{(x+1)(x+2)}$

2. $\dfrac{4 - x}{(x-1)(x+2)}$

3. $\dfrac{3x-10}{(x-2)(x-4)}$

4. $\dfrac{4x-9}{x(x-3)}$

5. $\dfrac{x+7}{(x-2)(x+1)}$

6. $\dfrac{5x+3}{(x+1)(2x+1)}$

7. $\dfrac{x}{x^2-1}$

8. $\dfrac{5}{6-x-x^2}$

9. $\dfrac{10x-1}{(2x+1)(4x-1)}$

10. $\dfrac{4-2x}{(2x-1)(2x+5)}$

11. $\dfrac{17x+11}{(x-2)(x+3)(x+1)}$

12. $\dfrac{5x^2-12x-5}{(x^2-1)(x-2)}$

13. $\dfrac{3x+1}{(x+1)(x^2+1)}$

14. $\dfrac{x+2}{(2x-1)(x^2+1)}$

15. $\dfrac{3x+1}{x(2x^2+1)}$

16. $\dfrac{x^2-10}{(x^2+3)(2x-1)}$

17. $\dfrac{x^2-13}{(x-1)^2(x+2)}$

18. $\dfrac{3x^2+7x+1}{x^3+2x^2+x}$

19. $\dfrac{2x^2-3x-2}{x^3-x^2}$

20. $\dfrac{x^2+23}{(x+1)^3(x-2)}$

21. $\dfrac{2x^2-5x-5}{(2x^2+5)(4x-5)}$

22. $\dfrac{x+2}{(x-2)(x^2-x+2)}$

23. $\dfrac{x^2+1}{x^2-1}$

24. $\dfrac{x^2}{x^2-x-2}$

25. $\dfrac{x(x-2)}{(3x-1)(x-1)}$

26. $\dfrac{x^3}{x^2-4}$

27. $\dfrac{x^2-x}{(x^2+3)(x^2+2)}$

28. $\dfrac{3x^3+2x^2+2x-3}{(x^2+2)(x+1)^2}$

29. $\dfrac{2x^4-2x^3+x}{(2x-1)^2(x-2)}$

30. $\dfrac{x^6-x^5-4x^2+x}{x^4+3x^2+2}$

31. Express $f(x)=\dfrac{4x+5}{(x+1)(2x+3)}$ in partial fractions. Hence find $f'(x)$ and $f''(x)$.

32. Find the coordinates of the points of inflexion on the curves

(a) $y = \dfrac{2x}{1 - x^2}$, (b) $y = \dfrac{7(x + 1)}{x(x - 7)}$.

Exercise 18.7 (miscellaneous)

1. The functions f, g, h and k are defined by

$$\begin{aligned} f &: x \to \sin^2 3x, && x \in \mathbb{R}, \\ g &: x \to 1/(1 - x), && x \in \mathbb{R}, x \neq 1, \\ h &: x \to \sqrt{(1 - x^2)}, && x \in \mathbb{R}, |x| \leqslant 1, \\ k &: x \to \operatorname{cosec} \tfrac{1}{2}x, && x \in \mathbb{R}, x \neq 2n\pi (n \in \mathbb{Z}). \end{aligned}$$

For each function, state whether it is even, odd or neither. State also whether or not it is periodic, giving the period where appropriate.

2. Sketch the graphs of the following functions. State any values of x at which
(a) there is a discontinuity,
(b) the function is not differentiable.

$$\begin{aligned} f(x) &= |x - 2|, & g(x) &= [\tfrac{1}{2}x], \\ h(x) &= 1/(x^2 - x - 2), & k(x) &= |x^4 - x^3|. \end{aligned}$$

3. The functions f and g are defined by $f : x \to \sin 2x, x \in \mathbb{R}$, $g : x \to \cot x$, $x \in \mathbb{R}$, $x \neq k\pi (k \in \mathbb{Z})$. State the periods of f and g. Find the period of the function $f.g$. On separate axes, sketch the graph of f, g and $f.g$ for the interval $\{x : -\pi < x < \pi, x \neq 0\}$. Find the range of the function $f.g$. **(JMB)**

4. Determine whether each of the three functions, f, g, h defined in $\mathbb{R}$ is an even function, an odd function or neither, given that $f(x) = \pi/2 - \sin(x/2)$, $g(x) = \pi/2 - \cos(x/3)$, $h(x) = \cos(\sin x)$. Determine the period, if it exists, of each of the two functions $G(x) = g(x) - f(x)$, $H(x) = g(x) - h(x) + \sin[f(2x)]$. **(L)**

5. Find the ranges of the given functions f, g, h and k. State whether f^{-1}, g^{-1}, h^{-1}, k^{-1} exist, giving brief reasons for each answer.

$$\begin{aligned} f &: x \to x^2 + 4, && x \in \mathbb{R}, 0 \leqslant x \leqslant 4; \\ g &: x \to x^2 - 4x, && x \in \mathbb{R}, 0 \leqslant x \leqslant 4; \\ h &: x \to 3\cos x + 4\sin x, && x \in \mathbb{R}; \\ k &: x \to 1 - \lg(x + 1), && x \in \mathbb{R}, x \geqslant 0. \end{aligned}$$

6. Sketch the graph of the function f defined by $f(x) = 4x/(1 + x^2)$, x real. Let g denote the function f restricted to the domain $[-a, a]$, so that $g(x) = 4x/(1 + x^2)$, $-a \leqslant x \leqslant a$, where a is the largest positive number such that the function g is one-one. What is the value of a? Obtain an expression for the inverse function g^{-1}, and state the domain and range of g^{-1}. **(W)**

7. The function ϕ is defined by $\phi(x) = x^3 + 2x - 1$, and the inverse function ϕ^{-1} is denoted by ψ. Find the values of $\psi(2)$ and $\psi'(2)$. (W)

8. The domain of the function f is the set $D = \{x : x \in \mathbb{R}, -2 < x < 3\}$. The function $f : D \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 2x - 1 & \text{for} \quad -2 < x \leqslant 1, \\ x^2 & \text{for} \quad 1 < x \leqslant 2, \\ 10 - 3x & \text{for} \quad 2 < x < 3. \end{cases}$$

Find the range of this function and sketch its graph. Explain why there is no inverse function to f. Suggest an interval such that f, restricted to this interval, will have an inverse function. Give an expression for the inverse function in this case. (L)

9. Explain what is meant by the statement "f is a function from a set A to a set B." A function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$\begin{cases} f(x) = \sin 2x, & 0 \leqslant x \leqslant \pi/4, \\ f(x) = 2 - 4x/\pi, & \pi/4 < x \leqslant \pi/2, \\ f \text{ is an odd function, } f \text{ is periodic with period } \pi. \end{cases}$$

Sketch the graph of $y = f(x)$ for $-\pi \leqslant x \leqslant \pi$. Find two different intervals, each of length $\pi/2$, such that f, restricted to either interval, will have an inverse. In each case, give an expression for the inverse function. (L)

10. (i) f_1 is a periodic odd function with a period 2π and f_2 is a periodic even function with a period 3π. Functions F and G are defined by $F(x) = f_2(x) . f_1(x), G(x) = f_2[f_1(x)]$. State whether the functions F and G are
(a) odd, even or neither,
(b) periodic, and if periodic state a period.
Functions f_3 and f_4 are defined by $f_3(x) = x \cos x$, $f_4(x) = \cos(\sin 3x)$. State whether f_3 and f_4 are odd, even or neither and if periodic state a period.
(ii) A mapping $f : A \to \mathbb{R}$, where $A = \{x : x \in \mathbb{R} \text{ and } -\pi/4 \leqslant x \leqslant \pi/4\}$, is defined by $f(x) = \cos x + \sin x$. Find $f^{-1}(1)$. (L)

11. A function f is defined on $\mathbb{R}$ by $f : x \to |x + [x]|$ where $[x]$ indicates the greatest integer less than or equal to x, e.g. $[3] = 3$, $[2{\cdot}4] = 2$, $[-3{\cdot}6] = -4$. Sketch the graph of the function for $-3 \leqslant x \leqslant 3$. What is the range of f? Is the mapping one-one?
The function g is defined by $g : x \to |x + [x]|, x \in \mathbb{R}^+, x \notin \mathbb{Z}^+$. Find the rule and domain of the inverse function g^{-1}. (C)

12. Sketch the curve $y = |x^2 - 5x|$. Find the ranges of values of x for which $|x^2 - 5x| < 6$.

13. Sketch the curves $y = x^2(3 - x)$ and $y^2 = x^2(3 - x)$, showing clearly the behaviour of the curves near the x-axis.

14. Using a sketch graph, or otherwise, solve the inequality $\left|\frac{1}{1+2x}\right| < 1$.

15. Prove that the curves $y = \frac{x}{x+1}$ and $y = \left(\frac{x}{x+1}\right)^2$ intersect at only one point.

Sketch these curves in the same diagram. Label your sketch so that it is immediately clear which curve is which. (O&C)

16. Given that $f(x) \equiv 6x^2 + x - 12$, find the minimum value of $f(x)$ and the values of x for which $f(x) = 0$. Using the same axes, sketch the curves $y = f(x)$ and $y = 1/f(x)$, labelling each clearly. Deduce that there are four values of x for which $[f(x)]^2 = 1$. Find these values, each to two decimal places. (L)

17. The polynomial $P(x)$ leaves a remainder of 2 when divided by $(x - 1)$ and a remainder of 3 when divided by $(x - 2)$. The remainder when $P(x)$ is divided by $(x - 1)(x - 2)$ is $ax + b$. By writing $P(x) \equiv (x - 1)(x - 2)Q(x) + ax + b$, find the values of a and b. Given also that $P(x)$ is a cubic polynomial with coefficient of x^3 equal to unity, and that -1 is a root of the equation $P(x) = 0$, obtain $P(x)$. Show that the equation $P(x) = 0$ has no other real roots. (JMB)

18. Find the remainder when the polynomial $P(x) = x^3 - 5x + 2$ is divided by
(a) $x^2 + 1$,
(b) $(x - 1)^2$.

19. Using the remainder theorem, or otherwise, show that $x + a + b + c$ is a factor of

$$(x + a)(x + b)(x + c) + (b + c)(c + a)(a + b).$$

Hence, or otherwise, solve the equation

$$(x + 2)(x - 3)(x - 1) + 4 = 0. \quad \text{(C)}$$

20. Factorise completely the expressions
(a) $x(y^3 - z^3) + y(z^3 - x^3) + z(x^3 - y^3)$,
(b) $12abc + 4(a^3 + b^3 + c^3) - (a + b + c)^3$.

21. If $f(x)$ is a polynomial in x, show that when $f(x)$ is divided by $x - a$ the remainder is $f(a)$.
When $x^3 + ax^2 + bx + c$ is divided by $x + 3$ the remainder is -26 and when divided by $x^2 - x - 2$ the remainder is 14. Find the values of a, b and c.
(AEB 1978)

22. Show that if a polynomial $f(x)$ is divided by $(x - a)(x - b)$, where $a \neq b$, then the remainder is $\left\{\frac{f(a) - f(b)}{a - b}\right\}x + \left\{\frac{af(b) - bf(a)}{a - b}\right\}$.

23. Solve the equation $x^4 - 4x^3 - 20x^2 + 32 = 0$ given that it has a repeated root.

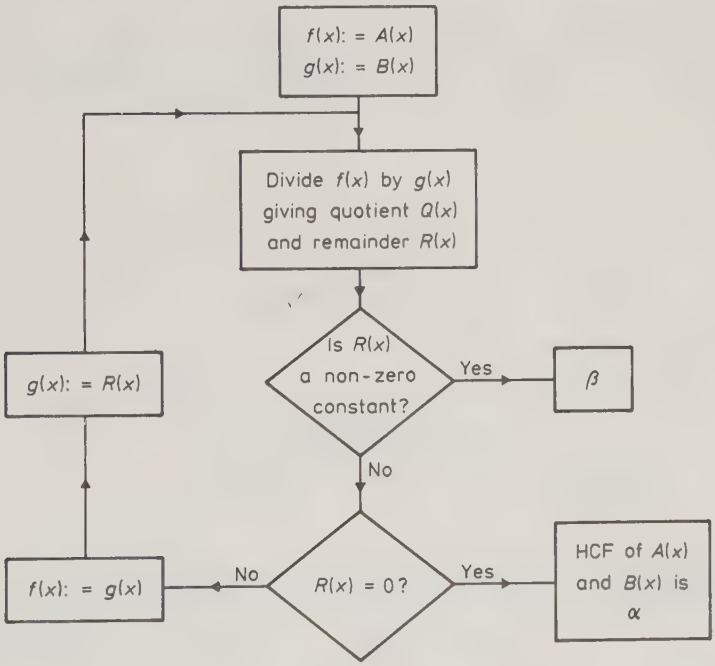
24. Given that the polynomial $f(x) = x^4 - 8x^3 + 10x^2 + p$ has a repeated linear factor, find the possible values of p.

25. The equation $x^3 - px^2 + qx - r = 0$ is such that its roots are in geometric progression. Show that
(a) one root is $\sqrt[3]{r}$,
(b) $p^3r = q^3$.

26. If the expression $2x^2 - xy - y^2 + 2x + ky - 4$ is the product of two linear factors, find the possible values of k.

27. (i) Given that $x + 1$ is a factor of $x^3 + 2x^2 + 3x + c$, find c.

(ii) In the algebraic operation of polynomial division, one polynomial $f(x)$ is divided by another polynomial $g(x)$ of order equal to or lower than that of $f(x)$. Polynomials $Q(x)$ and $R(x)$ are found such that $f(x) = g(x)Q(x) + R(x)$ and $R(x)$ is of lower order than $g(x)$. Carry out the process described in the flow chart when $A(x) = x^3 + 2x^2 + 3x + 2$ and $B(x) = x^2 - x - 2$.



State what should be written in the spaces α and β for the flow chart to be applicable to any two polynomials of suitable degree. (AEB 1980)

28. By considering the equation $yx^2 + (y - 1)x + (y - 1) = 0$, where $y \neq 0$, as a quadratic equation in x, find the range of values of y for which this equation has real roots. State the value of x which satisfies this equation when $y = 0$. Using these results, or otherwise, determine the range of the function $f: x \to \dfrac{x + 1}{x^2 + x + 1}$, where x is real. Sketch the graph of $f(x)$ giving the coordinates of any turning points. Explain why f has no inverse function. (JMB)

29. In each of the following cases find the range of the function f. Sketch the graph of f, showing clearly any asymptotes and turning points.

(a) $f(x) = \dfrac{x-1}{2x+1}$,

(b) $f(x) = \dfrac{4}{(1-x)(3+x)}$,

(c) $f(x) = \dfrac{3(x+1)}{x(x-3)}$,

(d) $f(x) = \dfrac{2x-1}{x^2-4}$.

30. Sketch each of the following curves, showing any stationary points and indicating clearly the form of the curve when $|x|$ becomes large.

(a) $y = x - \dfrac{1}{x+1}$.

(b) $y = \dfrac{x^2}{x-1}$,

(c) $y = \dfrac{x^2}{1+x^2}$,

(d) $y = \dfrac{x-1}{(x+1)^2}$.

31. State the maximum domain and determine the range of the function $f(x) = \dfrac{x^2+2x+3}{x^2-2x-3}$. Sketch the curve $y = \dfrac{x^2+2x+3}{x^2-2x-3}$ indicating clearly any asymptotes. (O)

32. (a) The function f is defined by $f(x) = \dfrac{x}{1+x^2}$. Show that $f'(x) = \dfrac{1-x^2}{(1+x^2)^2}$ and $f''(x) = \dfrac{-6x+2x^3}{(1+x^2)^3}$. Find the local maxima, minima and points of inflexion of f, and sketch the graph of f.

(b) The function g is defined by $g(x) = \dfrac{x^2}{1-x^2}$. Show that $g(x)$ cannot take values between -1 and 0. Sketch the graph of g. (W)

Express in partial fractions:

33. $\dfrac{3x+11}{(3x-2)(2x+3)}$

34. $\dfrac{3x-2}{(x-1)^2(2x-1)}$

35. $\dfrac{x}{x^3-x^2+x-1}$

36. $\dfrac{2}{x^3-2x^2-x+2}$

37. $\dfrac{5+x^2}{3+5x-2x^2}$

38. $\dfrac{3x}{1-x^3}$

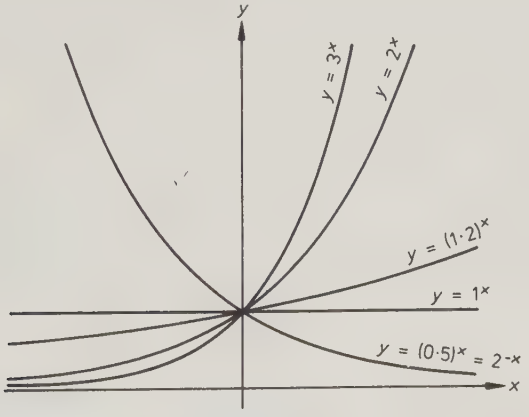
39. $\dfrac{x(x+1)(x-2)}{(x-1)(x+2)}$

40. $\dfrac{7x^2-2x+3}{(x^2-1)^2}$

19 Exponential and logarithmic functions

19.1 Exponential functions

Exponent is another word for index. A function such as 2^x, in which the index is variable, is called *an exponential function.*



The diagram shows the graphs of $y = a^x$ for various positive values of a. We see that each graph lies entirely above the x-axis and cuts the y-axis at the point $(0, 1)$.
We now consider how to differentiate the function a^x.

Let $\quad f(x) = a^x, \quad$ then $\quad f(x + h) = a^{x+h}$

$\therefore \quad f(x + h) - f(x) = a^{x+h} - a^x = a^x(a^h - 1)$

$\therefore \quad f'(x) = \lim_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{a^x(a^h - 1)}{h} = a^x \lim_{h \to 0} \left(\frac{a^h - 1}{h} \right).$$

In particular, $f'(0) = a^0 \lim_{h \to 0} \left(\dfrac{a^h - 1}{h} \right) = \lim_{h \to 0} \left(\dfrac{a^h - 1}{h} \right)$. Since $f'(0)$ is the gradient

of the curve $y = a^x$ at the point (0, 1), it seems reasonable to assume that this limit exists and is equal to some constant k.
Thus, if $f(x) = a^x$, then $f'(x) = ka^x$, where the value of k depends on the value of a. It follows that if $y = a^x$, then $\frac{dy}{dx} = ky$, i.e. the rate of change of y is proportional to y.

Because of this property the theory of exponential functions has many important applications. There are various scientific laws which state that the rate of change of a certain quantity is proportional to the quantity itself. We give here three simple examples of such laws.

(1) *Law of growth.* If a population of simple organisms such as bacteria is allowed to increase without restriction, then the rate of growth is proportional to the size of the population. If n is the number of organisms present at time t, then $\frac{dn}{dt} = kn$, where k is constant.

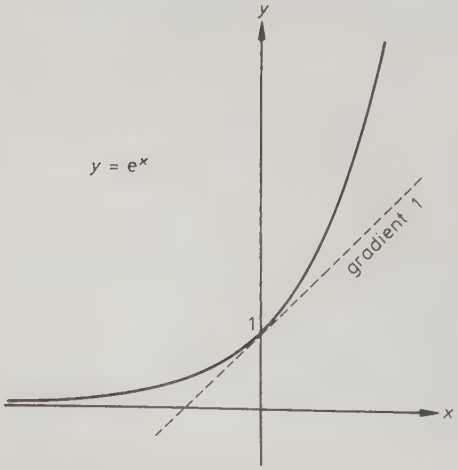
(2) *Law of radio-active decay.* The rate at which a radio-active substance decays is proportional to the quantity of the substance present. If a mass m is present at time t, then $\frac{dm}{dt} = -km$, where k is constant.

(3) *Newton's law of cooling.* If a body is placed in cooler surroundings then its temperature decreases at a rate proportional to the difference between the temperature of the body and that of its surroundings. If T is the temperature difference at time t, then $\frac{dT}{dt} = -kT$, where k is constant.

We arrive at a way of expressing such laws in exponential form by looking again at graphs of the form $y = a^x$. If, as before, the constant k represents the gradient of a particular curve at the point (0, 1), we see that as a increases, k also increases, taking negative values for $0 < a < 1$ and positive values for $a > 1$. The value of a for which $k = 1$ is denoted by e. This value is of special interest because when

$$y = e^x, \quad \frac{dy}{dx} = e^x = y.$$

It can be shown that e is an irrational number approximately equal to 2·718.



The function e^x is called *the exponential function* and is sometimes written $\exp x$. [In numerical work values of e^x are found using a scientific calculator or mathematical tables.]

The result $\frac{d}{dx}(e^x) = e^x$ can be extended as follows.

Let $y = e^u$, where $u = kx$

then $\frac{dy}{du} = e^u$ and $\frac{du}{dx} = k$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = ke^u.$$

Hence, if $y = e^{kx}$, then $\frac{dy}{dx} = ke^{kx} = ky$.

It is now possible to express the scientific laws mentioned earlier in exponential form:

(1) $n = n_0 e^{kt}$, (2) $m = m_0 e^{-kt}$, (3) $T = T_0 e^{-kt}$,

where n_0, m_0 and T_0 are the values of the variables when $t = 0$.

Some useful results to remember:

$$\frac{d}{dx}(e^x) = e^x, \qquad \int e^x \, dx = e^x + c$$

$$\frac{d}{dx}(e^{kx}) = ke^{kx}, \qquad \int e^{kx} \, dx = \frac{1}{k}e^{kx} + c.$$

Example 1 Differentiate e^{x^2+1} with respect to x.

Let $y = e^u$, where $u = x^2 + 1$

$$\frac{dy}{du} = e^u, \qquad \frac{du}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = e^u \times 2x = 2xe^{x^2+1}.$$

[Students thoroughly familiar with the differentiation of composite functions may write:

$$\frac{d}{dx}(e^{x^2+1}) = e^{x^2+1}\frac{d}{dx}(x^2 + 1) = 2xe^{x^2+1}.]$$

Example 2 Find $\int \sin x \, e^{\cos x} \, dx$.

$$\frac{d}{dx}(e^{\cos x}) = e^{\cos x}\frac{d}{dx}(\cos x) = -\sin x \, e^{\cos x}$$

$$\therefore \quad \int \sin x \, e^{\cos x} \, dx = -e^{\cos x} + c.$$

Example 3 Sketch the curve $y = x^2e^{-x}$.

$x = 0 \Leftrightarrow x^2e^{-x} = 0 \Leftrightarrow y = 0$

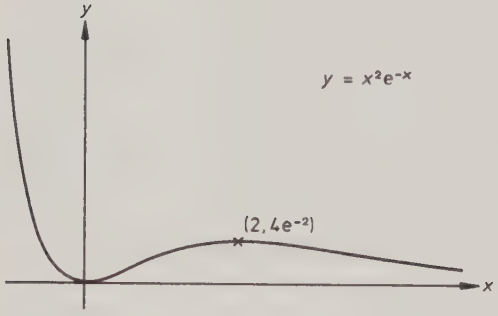
$\therefore$ the curve cuts the x- and y-axes at the origin.
Differentiating (using the product rule),

$$\frac{dy}{dx} = x^2(-e^{-x}) + 2xe^{-x} = (-x^2 + 2x)e^{-x} = x(2 - x)e^{-x}$$

$$\therefore \quad \frac{dy}{dx} = 0 \Leftrightarrow x(2 - x)e^{-x} = 0 \Leftrightarrow x = 0 \text{ or } x = 2.$$

	$x < 0$	$0 < x < 2$	$x > 2$
x	$-$	$+$	$+$
$2 - x$	$+$	$+$	$-$
e^{-x}	$+$	$+$	$+$
$\frac{dy}{dx}$	$-$	$+$	$-$

$\therefore$ when $x = 0$, there is a minimum point $(0, 0)$,
when $x = 2$, there is a maximum point $(2, 4e^{-2})$.



As $x \to -\infty$, $y \to \infty$.
As x increases through positive values, e^x increases at a much greater rate than x^2,
$\therefore$ as $x \to \infty$, $y \to 0$.

[The behaviour of the function x^ne^{-x} as $x \to \infty$ is also considered in *Exercise 22.4*, question 12.]

Exercise 19.1

In questions 1 to 7 differentiate the given functions with respect to x.

1. (a) e^{2x} (b) $5e^{-x}$ (c) e^{3x+5}

2. (a) e^{2x^3} (b) $e^{\sqrt{x}}$ (c) $e^{-1/x}$

3. (a) $e^{\sin x}$ (b) $e^{\cos 2x}$ (c) $e^{4\tan x}$

4. (a) xe^{x^2} (b) x^2e^{5x} (c) $e^{x\cos x}$

5. (a) $e^{2x}\cos 3x$ (b) $e^{-x^2}\sin x$ (c) $(x+1)^3e^{x/2}$

6. (a) $\sqrt{(1-e^{4x})}$ (b) $e^{(e^x)}$ (c) $\sin(e^x)$

7. (a) $\frac{1}{2}(e^x - e^{-x})^2$ (b) $\dfrac{e^x}{1+e^{-x}}$ (c) $\dfrac{e^{\sin^2 x}}{e^{-\cos^2 x}}$

8. Find (a) $\int e^{3x}\,dx$, (b) $\int e^{-x}\,dx$, (c) $\int e^{x+4}\,dx$.

9. Evaluate, leaving your answers in terms of e,

(a) $\int_0^1 6e^{2x}\,dx$ (b) $\int_{-2}^0 e^{(-x/2)}\,dx$, (c) $\int_{-1}^1 \{3/e^{3(x-1)}\}\,dx$.

10. Find dy/dx in terms of x when y is defined implicitly by:
(a) $e^y = x$, (b) $xe^y = 1$, (c) $e^{x+y} = x^2$.

11. If $y = e^x \sin x$, show that $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = 0$.

12. Find the equation of the tangent to the curve $y = e^{3x-5}$ at the point where $x = 2$.

13. Find in terms of e the area enclosed by the curve $y = e^{1-x}$, the axes and the line $x = 1$.

14. By considering the graphs of $y = e^x$ and $y = e^{-x}$, or otherwise, make rough sketches of:
(a) $y = e^x + e^{-x}$, (b) $y = e^x - e^{-x}$, (c) $y = e^x - x$.

In questions 15 to 20 sketch the given curves. Give the coordinates of any turning points and show clearly the behaviour of the curve when $|x|$ is large.

15. $y = xe^{-x}$. 16. $y = (x+1)e^x$. 17. $y = xe^{-x^2/2}$.

18. $y = (x-1)^2e^{2x}$. 19. $y = e^x/x$. 20. $y = e^{1-\cos x}$.

21. A particle moves in a straight line so that after t seconds its distance from a fixed point O is s metres, where $s = t^2e^{2-t}$. Find the distance of the particle from O when it first comes to rest and its acceleration at that point.

22. Find the values of x for which the function $(x^2 - 2x - 1)e^{2x}$ has maximum or minimum values, distinguishing between them.

23. Find the values of x between 0 and 2π for which the function $e^x \cos x$ has maximum or minimum values, distinguishing between them.

24. Find the area of the finite region bounded by the curves $y = e^{2x}$, $y = e^{-x}$ and the line $x = 1$. If this region is rotated completely about the x-axis, find the volume of the solid formed. (AEB 1976)

19.2 Logarithmic functions

If a and x are positive real numbers such that $x = a^p$, then p is the logarithm of x to the base a,

i.e. $$x = a^p \Leftrightarrow \log_a x = p.$$

The laws of logarithms can be summarised as follows:

$$\begin{aligned} \log_a xy &= \log_a x + \log_a y \\ \log_a x/y &= \log_a x - \log_a y \\ \log_a x^n &= n \log_a x \\ (\log_a b)(\log_b c) &= \log_a c \end{aligned}$$

The relationship between logarithms to different bases leads to two useful formulae:

$$\log_a x = \frac{1}{\log_x a}, \quad \log_a x = \frac{\log_b x}{\log_b a}.$$

Now that the exponential function e^x has been defined, we can introduce logarithms to the base e. These are called *natural logarithms* or *Napierian logarithms*†. The natural logarithm of x may be written $\log_e x$, but is more often denoted by $\ln x$.

Thus $$x = e^p \Leftrightarrow \ln x = p.$$

In numerical work values of $\ln x$ are found using a calculator or mathematical tables.

Example 1 Solve the equation $e^x - 2 - 3e^{-x} = 0$.

Letting $y = e^x$,
$$\begin{aligned} & e^x - 2 - 3e^{-x} = 0. \\ \Rightarrow \quad & y - 2 - 3/y = 0 \\ \Rightarrow \quad & y^2 - 2y - 3 = 0 \\ \Rightarrow \quad & (y - 3)(y + 1) = 0 \\ \Rightarrow \quad & y = 3 \quad \text{or} \quad y = -1 \\ \Rightarrow \quad & e^x = 3, \quad \text{since} \quad e^x > 0 \\ \Rightarrow \quad & x = \ln 3 \approx 1{\cdot}1 \end{aligned}$$

To investigate the properties of logarithmic functions we use our knowledge of exponential functions.

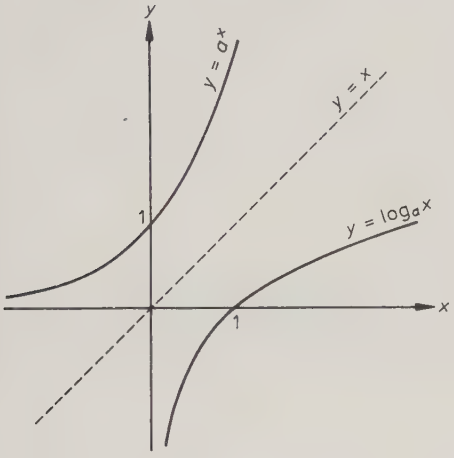
If $f(x) = a^x$, then $y = f^{-1}(x) \Rightarrow f(y) = x \Rightarrow a^y = x \Rightarrow y = \log_a x$.

† *Napier, John* (1550–1617) Scottish mathematician. He was the inventor of logarithms and spent 25 years producing tables. He also invented the calculating apparatus known as "Napier's Bones".

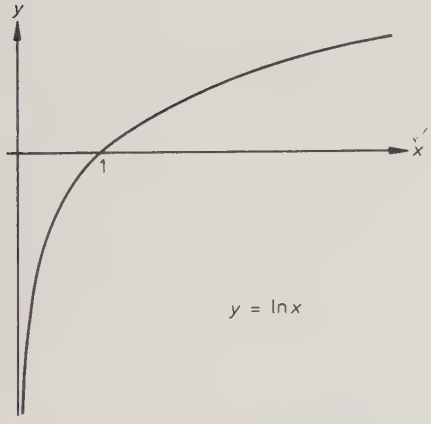
Hence the function $\log_a x$ is the inverse of the function a^x.

Thus $a^{\log_a x} = x$ and $\log_a(a^x) = x$.

It also follows that the graph of $y = \log_a x$ is the reflection in the line $y = x$ of the graph of $y = a^x$.



Since $\log_a x$ is defined only when $x > 0$, there are no points on the graph of $y = \log_a x$ for $x \leqslant 0$.
When $x = 1$, $y = \log_a 1 = 0$
$\therefore$ the graph of $y = \log_a x$ passes through the point $(1, 0)$ for all values of a.



If $y = \ln x$, then $e^y = x$.

Differentiating with respect to x,

$$\frac{d}{dx}(e^y) = 1$$

But $$\frac{d}{dx}(e^y) = \frac{d}{dy}(e^y)\frac{dy}{dx}$$

$$= e^y \frac{dy}{dx} = x\frac{dy}{dx}$$

$\therefore$ $$x\frac{dy}{dx} = 1 \quad \text{i.e.} \quad \frac{dy}{dx} = \frac{1}{x}.$$

Hence, for $x > 0$, $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$ and $\displaystyle\int \frac{1}{x}\,dx = \ln x + c.$

Example 2 Differentiate $\ln(x^2 + 1)$ with respect to x.

Let $y = \ln u$, where $u = x^2 + 1$

$$\frac{dy}{du} = \frac{1}{u} \qquad \frac{du}{dx} = 2x$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{u} \times 2x = \frac{2x}{x^2 + 1}.$$

Some problems may be simplified by using the laws of logarithms.

Example 3 If $y = \ln\left(\dfrac{2x + 1}{1 - 3x}\right)$, find $\dfrac{dy}{dx}$.

$$y = \ln\left(\frac{2x + 1}{1 - 3x}\right) = \ln(2x + 1) - \ln(1 - 3x)$$

$$\therefore \quad \frac{dy}{dx} = \frac{2}{2x + 1} - \frac{(-3)}{1 - 3x} = \frac{2(1 - 3x) + 3(2x + 1)}{(2x + 1)(1 - 3x)}$$

$$= \frac{5}{(2x + 1)(1 - 3x)}$$

Some functions involving variable exponents or complicated products and quotients are differentiated using a process called *logarithmic differentiation*. The next two examples illustrate the method.

Example 4 Differentiate 2^x with respect to x.

Let $y = 2^x$, then $\ln y = \ln(2^x) = x \ln 2$.

Differentiating, $\dfrac{1}{y}\dfrac{dy}{dx} = \ln 2$

$$\therefore \quad \frac{dy}{dx} = y \ln 2 = 2^x \ln 2.$$

Example 5 If $y = \dfrac{x}{\sqrt{(x^2 - 2)}}$, find $\dfrac{dy}{dx}$.

$$\ln y = \ln x - \ln\sqrt{(x^2 - 2)} = \ln x - \tfrac{1}{2}\ln(x^2 - 2)$$

$$\therefore \quad \frac{1}{y}\frac{dy}{dx} = \frac{1}{x} - \frac{1}{2}\left(\frac{2x}{x^2 - 2}\right) = \frac{x^2 - 2 - x^2}{x(x^2 - 2)} = \frac{-2}{x(x^2 - 2)}$$

$$\therefore \quad \frac{dy}{dx} = y\left\{\frac{-2}{x(x^2 - 2)}\right\} = \frac{x}{\sqrt{(x^2 - 2)}}\left\{\frac{-2}{x(x^2 - 2)}\right\} = -\frac{2}{(x^2 - 2)^{3/2}}$$

Example 6 Sketch the curve $y = \dfrac{1}{x}\ln x$.

$\ln x$ is not defined for $x \leqslant 0$,

$\therefore$ the curve lies entirely to the right of the y-axis.

$$y = 0 \Rightarrow \frac{1}{x}\ln x = 0 \Rightarrow x = 1$$

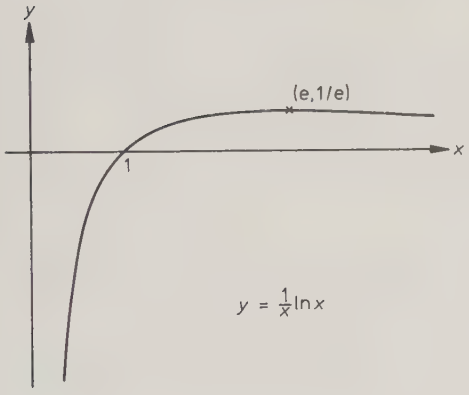
$\therefore$ the curve cuts the x-axis at the point $(1, 0)$.
Differentiating (using the product rule),

$$\frac{dy}{dx} = \frac{1}{x} \times \frac{1}{x} + \left(-\frac{1}{x^2}\right) \ln x = \frac{1}{x^2}(1 - \ln x)$$

$$\therefore \quad \frac{dy}{dx} = 0 \Leftrightarrow 1 - \ln x = 0 \Leftrightarrow x = e$$

When $x < e$, $\frac{dy}{dx} > 0$ and when $x > e$, $\frac{dy}{dx} < 0$,

$\therefore$ the point $(e, 1/e)$ is a maximum point.



As $x \to 0$, $1/x \to \infty$ and $\ln x \to -\infty$
$\therefore$ $y \to -\infty$.
When x is large, the rate of increase of $\ln x$ is very small
$\therefore$ as $x \to \infty$, $y \to 0$.
[The behaviour of the function $\frac{1}{x} \ln x$ as $x \to \infty$ is also considered in *Exercise 22.4*, question 12.]

Exercise 19.2

In questions 1 to 7 differentiate the given functions with respect to x.

1. (a) $\ln 3x$, (b) $\ln(x + 3)$, (c) $\ln(2x - 1)$.

2. (a) $\ln(x^3 + 4)$, (b) $\ln(\sin 2x)$, (c) $\ln(\sec x)$.

3. (a) $x^2 \ln x$, (b) $(\ln x)/x^2$, (c) $\ln(\ln x)$.

4. (a) $\ln(x^3)$, (b) $\ln\sqrt{(4x + 5)}$, (c) $x \ln(1/x)$.

5. (a) $\ln(\sec x + \tan x)$, (b) $\ln(x^2 + 4)^2$, (c) $\log_{10} x$.

6. (a) $\ln\left(\frac{x^2}{3x - 2}\right)$, (b) $\ln\sqrt{\left(\frac{x}{1 + x}\right)}$, (c) $\ln\frac{\cos x}{\sqrt{(1 - x^2)}}$.

7. (a) $e^{2 \ln x}$, (b) $\ln(e^{\tan x})$, (c) $\ln(3x^2 e^{-x})$.

8. Find the equation of the tangent to the curve $y = \ln(3x - 5)$ at the point where $x = 2$.

9. If $x = 1 + \ln t$ and $y = \ln(1 + t)$, find dy/dx in terms of t.

10. By considering the graph of $y = \ln x$, or otherwise, sketch the following:
(a) $y = \ln(-x)$, (b) $y = \ln(1/x)$, (c) $y = \ln(1 + x)$.

11. Given that $\ln 2 = p$ and $\ln 3 = q$, express the following in terms of p and q.
(a) $\ln 12e$, (b) $\log_3 2$, (c) $\log_6 e$, (d) 6.

12. Solve the following equations
(a) $e^{3t} + 2e^t = 3e^{2t}$, (b) $2e^x + 5 = 3e^{-x}$.

13. Sketch the following curves, giving the coordinates of any turning points.
(a) $y = |\ln x|$, (b) $y = 1/\ln x$, (c) $y = (\ln x)^2$.

14. Show that the curve $y = x - \ln x$ has one turning point. Show also that there are no points of inflexion on the curve. Sketch the curve.

15. If $y = \ln(x^2 - 5)$, show that $\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 = 2e^{-y}$.

16. If $y = \sin 2x \ln(\tan x)$, show that $\dfrac{d^2y}{dx^2} + 4y = 4\cot 2x$.

In questions 17 to 20 use logarithmic differentiation to find the derivatives of the given functions.

17. (a) 3^x, (b) 5^{2x}, (c) $10^{\sin x}$.

18. (a) x^x, (b) $x^{\ln x}$, (c) $(\ln x)^x$.

19. (a) $\dfrac{x^2 + 1}{(2x + 1)^2}$, (b) $\dfrac{x^3}{\sqrt{(1 - x)}}$, (c) $\dfrac{(x - 1)^2 e^{4x}}{(x + 1)^2}$

20. (a) $\sqrt{\left\{\dfrac{(x + 2)^3}{1 - 5x}\right\}}$, (b) $\sqrt[3]{\left\{\dfrac{x^2 - 2}{x^2 + 4}\right\}}$, (c) $\dfrac{\sqrt{(\sin 2x)}}{\cos^3 x}$.

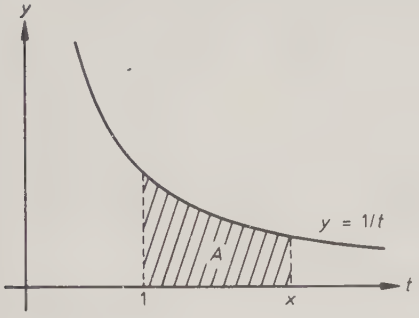
19.3 $\ln x$ defined in integral form

In this section we consider a more rigorous approach to the theory of exponential and logarithmic functions. In §19.1 we assumed with little justification that a^x is a continuous differentiable function defined for all real values of x. However, any elementary definition of a^x is valid only for rational values of x. Similarly, the laws of indices and the rule for differentiating x^n can be proved by elementary methods only for rational values of the exponents. We avoid these difficulties by using definitions of $\ln x$ and $\exp x$ which are valid for both rational and irrational values of x.

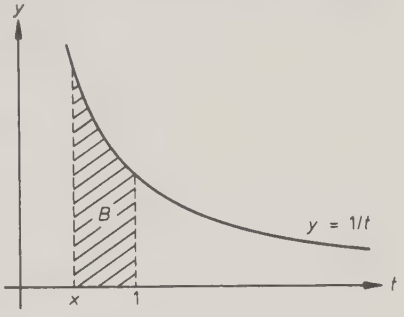
We define $\ln x$ as a definite integral by writing:

$$\ln x = \int_1^x \frac{1}{t}\,dt, \quad \text{where } x \text{ is real and positive.}$$

It follows that when $x = 1$, $\ln x = 0$. Other values of $\ln x$ are related to areas under the curve $y = 1/t$ as shown in the diagrams below.



When $x > 1$, $\ln x = A$.



When $0 < x < 1$, $\ln x = -B$.

Thus the domain of the function $\ln x$ is the set of positive real numbers and its range is the set of all real numbers.

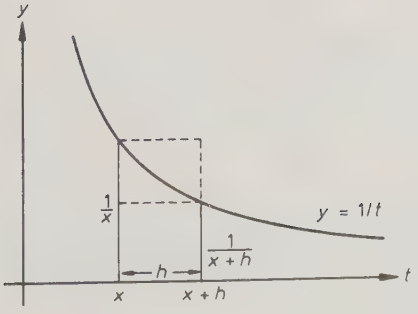
We can now define $\exp x$ for all real values of x by letting the function $\exp x$ be the inverse of the function $\ln x$, so that

$$y = \exp x \Leftrightarrow \ln y = x$$

To differentiate the function $\ln x$, let $f(x) = \ln x$, then

$$f(x + h) - f(x) = \ln(x + h) - \ln x$$

$$= \int_1^{x+h} \frac{1}{t}\,dt - \int_1^x \frac{1}{t}\,dt = \int_x^{x+h} \frac{1}{t}\,dt.$$



From the sketch we see that

$$h \times \frac{1}{x+h} < \int_x^{x+h} \frac{1}{t}\,dt < h \times \frac{1}{x}$$

$$\therefore \quad \frac{1}{x+h} < \frac{f(x+h) - f(x)}{h} < \frac{1}{x}$$

As $h \to 0$, $\dfrac{1}{x+h} \to \dfrac{1}{x}$,

$$\therefore \quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \frac{1}{x}$$

i.e.

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

To differentiate the function $\exp x$, let $y = \exp x$, then

$$\ln y = x \Rightarrow \frac{1}{y}\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = y$$

i.e.
$$\frac{d}{dx}(\exp x) = \exp x$$

We develop the theory further by considering $\ln(x^k)$, where k is any rational number ($k \in \mathbb{Q}$).

$$\frac{d}{dx}\{\ln(x^k)\} = \frac{1}{x^k} \times kx^{k-1} = \frac{k}{x}$$

$$\therefore \quad \frac{d}{dx}\{\ln(x^k) - k\ln x\} = \frac{k}{x} - k \times \frac{1}{x} = 0$$

$\therefore$ the value of $\{\ln(x^k) - k\ln x\}$ must be constant.
When $x = 1$, $\ln(x^k) - k\ln x = \ln 1 - k\ln 1 = 0$,
$\therefore$ the constant value of $\{\ln(x^k) - k\ln x\}$ is zero.

Thus for rational k, $\ln(x^k) = k\ln x$
and hence
$$x^k = \exp(k\ln x) \qquad (1)$$

The constant e is now defined as the real number such that $\ln e = 1$. Substituting $x = e$ in equation (1) we have

$$e^k = \exp(k\ln e) = \exp k, \quad \text{for rational } k.$$

Thus it seems reasonable to give a meaning to the expression e^x when x is irrational by letting $e^x = \exp x$ for all real x.
It follows from this definition that $\log_e x = \ln x$.
More generally, it is consistent with equation (1) to define a^x by writing

$$\boxed{a^x = \exp(x\ln a) \quad \text{for all real } x}$$

It is a consequence of this definition that

$$y = \log_a x \Rightarrow a^y = x \Rightarrow \exp(y\ln a) = x \Rightarrow y\ln a = \ln x$$

$$\therefore \quad \boxed{\log_a x = (\ln x)/(\ln a).}$$

We conclude by summarising the important points in this discussion.

We needed definitions of e^x and $a^x (a > 0)$ valid for irrational as well as rational values of x. We found that using the definitions

$$\ln x = \int_1^x \frac{1}{t}\,dt, \quad y = \exp x \Leftrightarrow \ln y = x,$$

it followed that, for rational values of k,

$$e^k = \exp k, \quad a^k = \exp(k\ln a).$$

Thus suitable definitions of e^x and a^x are

$$e^x = \exp x, \quad a^x = \exp(x \ln a).$$

We deduce from these definitions that

$$\log_e x = \ln x, \quad \log_a x = (\ln x)/(\ln a).$$

These results provide a firmer basis for the work of §19.1 and §19.2.

[Note that the theory of exponential and logarithmic functions defined in terms of $\exp x$ and $\ln x$ is extended in *Exercise 19.3* and in *Exercise 20.6,* question 11.]

Exercise 19.3

1. (a) Use the result $\frac{d}{dx}(\ln x) = \frac{1}{x}$ to prove that

$$\frac{d}{dx}(\ln xy - \ln x - \ln y) = 0.$$

   (b) Deduce that $\ln xy = \ln x + \ln y$.
   (c) By substituting $u = \ln x$, $v = \ln y$, show that

$$(\exp u)(\exp v) = \exp(u + v).$$

2. Use the methods of question 1 to show that
   (a) $\ln(x/y) = \ln x - \ln y$.
   (b) $(\exp u)/(\exp v) = \exp(u - v)$.

3. Use the definition $x^p = \exp(p \ln x)$ or the equivalent statement $\ln(x^p) = p \ln x$ to prove that, for all real values of p,

$$y = x^p \Rightarrow \frac{dy}{dx} = px^{p-1}.$$

19.4 Integration using logarithms

By differentiating logarithmic functions we can obtain various useful integration results.

For $x > 0$, $\frac{d}{dx}(\ln x) = \frac{1}{x}$,

$\therefore$ for $x > 0$, $$\int \frac{1}{x}\,dx = \ln x + c. \tag{1}$$

For $x < 0$, $\frac{d}{dx}\{\ln(-x)\} = \frac{1}{(-x)} \times (-1) = \frac{1}{x}$

$\therefore$ for $x < 0$, $$\int \frac{1}{x}\,dx = \ln(-x) + c. \tag{2}$$

For $ax + b > 0$, $\dfrac{d}{dx}\{\ln(ax + b)\} = \dfrac{1}{ax + b} \times a = \dfrac{a}{ax + b}$

$\therefore$ for $ax + b > 0$, $\displaystyle\int \frac{1}{ax + b}\,dx = \frac{1}{a}\ln(ax + b) + c.$

Example 1 Find $\displaystyle\int \frac{1}{3x - 2}\,dx$ when (a) $3x - 2 > 0$, (b) $3x - 2 < 0$.

(a) $\displaystyle\int \frac{1}{3x - 2}\,dx = \frac{1}{3}\ln(3x - 2) + c$, for $3x - 2 > 0$.

(b) $\displaystyle\int \frac{1}{3x - 2}\,dx = -\int \frac{1}{2 - 3x}\,dx = -\frac{1}{(-3)}\ln(2 - 3x) + c$

$\displaystyle = \frac{1}{3}\ln(2 - 3x) + c$, for $3x - 2 < 0$.

Example 2 Evaluate (a) $\displaystyle\int_5^9 \frac{1}{x - 3}\,dx$, (b) $\displaystyle\int_1^2 \frac{1}{x - 3}\,dx$.

(a) Within the limits of integration $x - 3 > 0$,

$\therefore$ $\displaystyle\int_5^9 \frac{1}{x - 3}\,dx = \Big[\ln(x - 3)\Big]_5^9 = \ln 6 - \ln 2 = \ln(6/2) = \ln 3.$

(b) Within the limits of integration $x - 3 < 0$,

$\therefore$ $\displaystyle\int_1^2 \frac{1}{x - 3}\,dx = -\int_1^2 \frac{1}{3 - x}\,dx = \Big[\ln(3 - x)\Big]_1^2 = \ln 1 - \ln 2 = -\ln 2.$

When there are no stated restrictions on the values of x, results (1) and (2) can be combined in the statement:

$$\int \frac{1}{x}\,dx = \ln|x| + c$$

Similarly

$$\boxed{\int \frac{1}{ax + b}\,dx = \frac{1}{a}\ln|ax + b| + c.}$$

We stress, however, that it should not be necessary to use modulus signs when evaluating definite integrals.

Example 3 Find $\displaystyle\int \frac{x - 1}{x + 1}\,dx$.

$$\int \frac{x - 1}{x + 1}\,dx = \int \left\{1 - \frac{2}{x + 1}\right\} dx = x - 2\ln|x + 1| + c.$$

Example 4 Find $\int_1^3 \frac{2x^2 + 3x}{2x - 1} dx$.

$$\begin{array}{r} x + 2 \\ 2x - 1 \overline{)2x^2 + 3x} \\ \underline{2x^2 - x} \\ 4x \\ \underline{4x - 2} \\ 2 \end{array}$$

By division

$$\frac{2x^2 + 3x}{2x - 1} = x + 2 + \frac{2}{2x - 1}$$

Noting that within the limits of integration $2x - 1 > 0$,

$$\int_1^3 \frac{2x^2 + 3x}{2x - 1} dx = \int_1^3 \left\{ x + 2 + \frac{2}{2x - 1} \right\} dx$$

$$= \left[\tfrac{1}{2}x^2 + 2x + \ln(2x - 1) \right]_1^3$$

$$= (\tfrac{9}{2} + 6 + \ln 5) - (\tfrac{1}{2} + 2 + \ln 1) = 8 + \ln 5.$$

To obtain a general integration formula, let $y = \ln\{f(x)\}$, then

$$\frac{dy}{dx} = \frac{1}{f(x)} \times \frac{d}{dx}\{f(x)\} = \frac{f'(x)}{f(x)}$$

$\therefore$ for $f(x) > 0$, $\int \frac{f'(x)}{f(x)} dx = \ln\{f(x)\} + c$.

When the sign of $f(x)$ is unknown we may write

$$\boxed{\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c.}$$

Example 5 Find $\int \frac{x}{3x^2 + 5} dx$.

If $f(x) = 3x^2 + 5$, then $f'(x) = 6x$,
$\therefore$ noting that $3x^2 + 5 > 0$ for all values of x,

$$\int \frac{x}{3x^2 + 5} dx = \frac{1}{6} \int \frac{6x}{3x^2 + 5} dx = \frac{1}{6} \ln(3x^2 + 5) + c.$$

Example 6 Find $\int \tan x \, dx$.

If $f(x) = \cos x$, then $f'(x) = -\sin x$,

$$\therefore \quad \int \tan x \, dx = -\int \frac{(-\sin x)}{\cos x} dx = -\ln|\cos x| + c$$

$$= \ln \frac{1}{|\cos x|} + c = \ln|\sec x| + c.$$

Exercise 19.4

Find the following indefinite integrals.

1. (a) $\int \frac{1}{2x+7}\,dx$ for $2x+7>0$, (b) $\int \frac{1}{5x-2}\,dx$ for $5x-2>0$.

2. (a) $\int \frac{1}{1-x}\,dx$ for $1-x>0$, (b) $\int \frac{1}{2x-1}\,dx$ for $2x-1<0$.

3. (a) $\int \frac{3}{3x-2}\,dx$ for $3x<2$, (b) $\int \frac{8}{3-4x}\,dx$ for $4x<3$.

4. (a) $\int \frac{1}{x+3}\,dx$, (b) $\int \frac{3}{3-x}\,dx$, (c) $\int \frac{2}{4x+5}\,dx$.

5. (a) $\int \frac{x}{x-1}\,dx$, (b) $\int \frac{x-2}{x-4}\,dx$, (c) $\int \frac{x+1}{2x+1}\,dx$.

6. (a) $\int \frac{x^2-1}{x}\,dx$, (b) $\int \frac{(1-x)^2}{1+x^2}\,dx$, (c) $\int \frac{1-x+x^2}{1-x}\,dx$.

7. (a) $\int \frac{x}{1-x^2}\,dx$, (b) $\int \frac{x^2}{x^3+8}\,dx$, (c) $\int \frac{x+1}{2x^2+4x-1}\,dx$.

8. (a) $\int \frac{\cos x}{1+\sin x}\,dx$, (b) $\int \cot x\,dx$, (c) $\int (1+\tan 2x)\,dx$.

9. (a) $\int \frac{e^x}{1+e^x}\,dx$, (b) $\int \frac{1}{x\ln x}\,dx$, (c) $\int \frac{1}{(1+\sqrt{x})\sqrt{x}}\,dx$.

Evaluate the following definite integrals.

10. (a) $\int_2^4 \frac{5}{x}\,dx$, (b) $\int_3^9 \frac{1}{x-1}\,dx$, (c) $\int_{-4}^0 \frac{1}{1-2x}\,dx$.

11. (a) $\int_0^1 \frac{1}{3x+2}\,dx$, (b) $\int_0^3 \frac{1}{x-4}\,dx$, (c) $\int_3^5 \frac{x}{x+1}\,dx$.

12. (a) $\int_0^{\pi/2} \tan \tfrac{1}{2}x\,dx$, (b) $\int_3^4 \frac{2x-3}{x^2-3x+1}\,dx$, (c) $\int_0^{\ln 2} \frac{e^{-x}}{1+e^{-x}}\,dx$.

13. Evaluate, giving answers correct to 3 significant figures.

(a) $\int_2^4 \frac{x+2}{x^2+4x-7}\,dx$, (b) $\int_{\pi/6}^{\pi/3} \frac{\sec^2 x}{\tan x}\,dx$.

14. Express $\dfrac{1}{2x-1} - \dfrac{2}{3-x}$ as a single fraction and hence evaluate, correct to 3 decimal places, $\displaystyle\int_1^2 \frac{1-x}{(2x-1)(3-x)}\,dx$.

15. Find the area enclosed by the curve $y = 1/(x-2)$ and the line $y = 7 - 2x$.

16. Find the volume of the solid formed by rotating about the x-axis the area bounded by the curve $y = \sqrt{x} + 1/\sqrt{x}$, the x-axis and the lines $x = 1$, $x = 3$.

19.5 Introduction to hyperbolic functions

Hyperbolic sine and cosine are defined as follows:

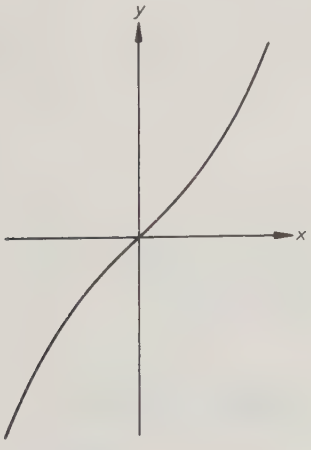
$$\sinh x = \tfrac{1}{2}(e^x - e^{-x}), \quad \cosh x = \tfrac{1}{2}(e^x + e^{-x})$$

The remaining hyperbolic functions can be expressed in terms of $\sinh x$ and $\cosh x$.

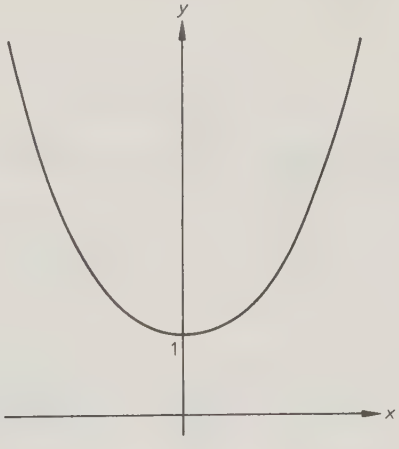
$$\tanh x = \frac{\sinh x}{\cosh x}, \quad \coth x = \frac{\cosh x}{\sinh x}, \quad \operatorname{sech} x = \frac{1}{\cosh x}, \quad \operatorname{cosech} x = \frac{1}{\sinh x}$$

[Note that cosh, sech, cosech, coth are pronounced as they are written, tanh is pronounced 'tanch' and sinh as 'shine' or sometimes 'sinch'.]

The graphs of $y = \sinh x$ and $y = \cosh x$ may be sketched from the graphs of $y = e^x$ and $y = e^{-x}$.



$y = \sinh x$



$y = \cosh x$

$$y = \sinh x = \tfrac{1}{2}(e^x - e^{-x}) \Rightarrow \frac{dy}{dx} = \tfrac{1}{2}(e^x + e^{-x}) = \cosh x$$

$$y = \cosh x = \tfrac{1}{2}(e^x + e^{-x}) \Rightarrow \frac{dy}{dx} = \tfrac{1}{2}(e^x - e^{-x}) = \sinh x$$

$$\therefore \qquad \frac{d}{dx}(\sinh x) = \cosh x, \quad \frac{d}{dx}(\cosh x) = \sinh x.$$

We now establish one of the most useful identities connecting hyperbolic functions.

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \{\tfrac{1}{2}(e^x + e^{-x})\}^2 - \{\tfrac{1}{2}(e^x - e^{-x})\}^2 \\ &= \tfrac{1}{4}\{(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})\} \\ &= \tfrac{1}{4} \times 4 = 1\end{aligned}$$

i.e. $\cosh^2 x - \sinh^2 x = 1.$

Exercise 19.5

1. Differentiate with respect to x
(a) $\sinh 2x$, (b) $\cosh 3x$, (c) $\sinh^2 x$, (d) $\sqrt{(\cosh x)}$.

2. Find (a) $\int \sinh x\,dx$, (b) $\int \cosh 2x\,dx$, (c) $\int \tanh x\,dx$.

3. Differentiate with respect to x
(a) $\tanh x$, (b) $\coth x$, (c) $\operatorname{sech} x$, (d) $\operatorname{cosech} x$.

4. Use the definitions of $\sinh x$ and $\cosh x$ to prove
(a) $\sinh 2x = 2\sinh x \cosh x$, (b) $\cosh 2x = \cosh^2 x + \sinh^2 x$.

5. By substituting $y = e^x$, solve the equations
(a) $3\cosh x - 2\sinh x = 3$, (b) $5\cosh x + 7\sinh x = 1$.

Exercise 19.6 (miscellaneous)

In questions 1 to 3 differentiate the given functions.

1. (a) xe^{3x}, (b) $e^{2x}\ln 4x$, (c) $a^x (a > 0)$.

2. (a) $\sin(\ln x)$, (b) $(1 - x^2)\ln x$, (c) $\ln\left(\tan\frac{1}{2}x\right)$.

3. (a) $\ln\left(\frac{x-3}{3x-1}\right)$, (b) $\ln\left(\frac{1+\sin x}{1-\sin x}\right)$, (c) $\ln\{(x^3)\sqrt{(1-x^2)}\}$.

4. Sketch the given curves, giving the coordinates of any stationary points.
(a) $y = e^x + 4e^{-x}$, (b) $y = x^3 e^{-x}$.

5. Find the value of x for which the curve $y = x^{1/x} (x > 0)$ has a stationary point. State the nature of this point.

6. Show that if $y = e^{4x}\cos 3x$, then d^2y/dx^2 can be expressed in the form $25e^{4x}\cos(3x + \alpha)$. Give the value of $\tan\alpha$.

7. Sketch the curves

(a) $y = \dfrac{x+1}{x-1}$, (b) $y = \left|\dfrac{x+1}{x-1}\right|$, (c) $y = \ln\left|\dfrac{x+1}{x-1}\right|$.

8. Given that $x \ln x \to 0$ as $x \to 0$, sketch the curves

(a) $y = x \ln x$, (b) $y = \dfrac{1}{x} + \ln x$.

9. Find the coordinates of any turning points on the curve $y = e^{-x/2}(x^2 + x - 2)$. Sketch this curve and the curve $y = x^2 + x - 2$ on the same axes.

10. Evaluate (a) $\displaystyle\int_2^6 \frac{1}{2x-3}\,dx$, (b) $\displaystyle\int_0^5 \frac{x}{x^2+5}\,dx$, (c) $\displaystyle\int_{\pi/2}^{\pi} \cot\frac{1}{3}x\,dx$.

11. Find (a) $\displaystyle\int \frac{x}{3x+1}\,dx$, (b) $\displaystyle\int \frac{(x-2)(x+2)}{x^3}\,dx$, (c) $\displaystyle\int \frac{2x^2+1}{x^2+x+1}\,dx$

12. Simplify $\log_e\left[\dfrac{(1+x)e^{-2x}}{1-x}\right]^{1/2}$ and show that its derivative is $x^2/(1-x^2)$. Hence, or otherwise, evaluate dy/dx at $x = 0$ for the function $y = \left[\dfrac{(1+x)e^{-2x}}{1-x}\right]^{1/2}$. (JMB)

13. If $\tan y = \log_e x^2$, show that $x\dfrac{dy}{dx} = 2\cos^2 y$. Hence show that

$$x^2\frac{d^2y}{dx^2} + 2(1 + 2\sin 2y)\cos^2 y = 0.$$

(AEB 1980)

14. If $y = (x - 0{\cdot}5)e^{2x}$, find dy/dx and hence or otherwise calculate, correct to one decimal place, the mean value of xe^{2x} in the interval $0 \leqslant x \leqslant 2{\cdot}5$. (AEB 1980)

15. The coordinates of a point on a curve are given in terms of a parameter θ by $x = a\{\ln(\cot\frac{1}{2}\theta) - \cos\theta\}$, $y = a\sin\theta$, where a is a positive constant and $0 < \theta \leqslant \frac{1}{2}\pi$. Show that, if $\theta \neq \frac{1}{2}\pi$, $\dfrac{dy}{dx} = -\tan\theta$.

The tangent at any point P of the curve meets the x-axis at T. Prove that the length of PT is equal to a. (C)

16. Sketch the region in the x-y plane within which all three of the following inequalities are satisfied:
(i) $y < x + 1$, (ii) $y > (x-1)^2$, (iii) $xy < 2$.
Determine the area of this region. (JMB)

17. Sketch the curve $y = 1 + 2e^{-x}$, showing clearly the behaviour of the curve as $x \to +\infty$. Find the area of the finite region enclosed by the curve and the lines $x = 0$, $x = 1$, $y = 1$. Find also the volume formed when this region is rotated completely about the line $y = 1$. (L)

18. Sketch the curve whose equation is $y + 3 = \dfrac{6}{x - 1}$. Find the coordinates of the points where the line $y + 3x = 9$ intersects the curve and show that the area of the region enclosed between the curve and the line is $\frac{3}{2}(3 - 4\log_e 2)$. Determine the equations of the two tangents to the curve which are parallel to the line. (JMB)

19. Sketch the curve $y = x/(1 + x^2)$, finding its turning points, showing that the origin is a point of inflexion, and indicating the behaviour of y when $|x|$ is large. Calculate the area of the region defined by the inequalities

$$0 \leqslant x \leqslant a, 0 \leqslant y \leqslant x/(1 + x^2),$$

and show that this area has the value 1 when $a = \sqrt{(e^2 - 1)}$. (L)

20. A curve has the equation $y = a/(b + e^{-cx})$, $(a \neq 0, b > 0, c > 0)$. Show that it has one point of inflexion, and that the value of y at the point of inflexion is half the limiting value y as $x \to \infty$. (O)

21. Sketch the curve $y = (x^2 - 7x + 10)/(x - 6)$, indicating clearly any vertical asymptotes and turning values. Determine the finite area bounded by the x-axis and that part of the curve between the points where it crosses the x-axis. (O)

22. Sketch the curve $y = \log_e(x - 2)$. The inner surface of a bowl is of the shape formed by rotating completely about the y-axis that part of the x-axis between $x = 0$ and $x = 3$ and that part of the curve $y = \log_e(x - 2)$ between $y = 0$ and $y = 2$. The bowl is placed with its axis vertical and water is poured in. Calculate the volume of water in the bowl when the bowl is filled to a depth $h(< 2)$. If water is poured into the bowl at a rate of 50 cubic units per second, find the rate at which the water level is rising when the depth of the water is 1·5 units. (AEB 1979)

23. By means of a diagram, or otherwise, show that

$$\frac{1}{r+1} < \int_r^{r+1} \frac{1}{x}\,dx < \frac{1}{r}.$$

Deduce that the sum to N terms of the series $1 + \dfrac{1}{2} + \dfrac{1}{3} + \ldots + \dfrac{1}{r} + \ldots$ lies between $\log_e(1 + N)$ and $1 + \log_e N$, where $N > 1$. Show that the sum of one hundred million terms of this series lies between 18·4 and 19·5. (JMB)

20 Methods of integration

20.1 Standard forms

We first list some important integration results.

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c, \quad (n \neq -1)$$

$$\int \sin x\,dx = -\cos x + c, \qquad \int \cos x\,dx = \sin x + c$$

$$\int \sec^2 x\,dx = \tan x + c, \qquad \int \operatorname{cosec}^2 x\,dx = -\cot x + c$$

$$\int e^x\,dx = e^x + c, \qquad \int \frac{1}{x}\,dx = \ln x + c, \quad (x > 0)$$

When using these standard forms it is worth noting that if we can integrate a function $f(x)$, we should also be able to integrate $f(ax + b)$. For instance, remembering that $\int \cos x\,dx = \sin x + c$, to find $\int \cos(2x - 3)\,dx$ we try differentiating $\sin(2x - 3)$:

$$y = \sin u, \quad \text{where} \quad u = 2x - 3$$

$$\Rightarrow \frac{dy}{du} = \cos u \qquad \frac{du}{dx} = 2$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 2\cos u = 2\cos(2x - 3)$$

Hence $$\int \cos(2x - 3)\,dx = \tfrac{1}{2}\sin(2x - 3) + c.$$

Example 1 Find (a) $\int e^{x/2}\,dx$, (b) $\int (1-2x)^3\,dx$, (c) $\int \frac{1}{(3x+4)^4}\,dx$.

(a) $\frac{d}{dx}\{e^{x/2}\} = e^{x/2} \times \frac{1}{2} = \frac{1}{2}e^{x/2}$

$$\therefore \qquad \int e^{x/2}\,dx = 2e^{x/2} + c.$$

(b) $\frac{d}{dx}\{(1-2x)^4\} = 4(1-2x)^3 \times (-2) = -8(1-2x)^3$

$$\therefore \qquad \int (1-2x)^3\,dx = -\tfrac{1}{8}(1-2x)^4 + c.$$

(c) $\frac{d}{dx}\{(3x+4)^{-3}\} = -3(3x+4)^{-4} \times 3 = -\frac{9}{(3x+4)^4}$

$$\therefore \qquad \int \frac{1}{(3x+4)^4}\,dx = -\frac{1}{9(3x+4)^3} + c.$$

Integrals involving trigonometric functions can often be found using basic trigonometric identities.

$$\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \tan x - x + c.$$

$$\int \cos^2 x\,dx = \int \tfrac{1}{2}(1 + \cos 2x)\,dx = \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x + c.$$

$$\int \sin 3x \cos 2x\,dx = \int \tfrac{1}{2}(\sin 5x + \sin x)\,dx = -\tfrac{1}{10}\cos 5x - \tfrac{1}{2}\cos x + c.$$

In the next two examples we consider integrals of the form

$$\int \{f(x)\}^n f'(x)\,dx.$$

Example 2 Find $\int \sin^2 x \cos x\,dx$ and hence $\int \cos^3 x\,dx$.

[We note that $f(x) = \sin x \Rightarrow f'(x) = \cos x$ and therefore try differentiating $\sin^3 x$.]

$$\frac{d}{dx}(\sin^3 x) = 3\sin^2 x \cos x$$

$$\therefore \qquad \int \sin^2 x \cos x\,dx = \tfrac{1}{3}\sin^3 x + c.$$

Hence
$$\int \cos^3 x\,dx = \int (\cos^2 x)\cos x\,dx$$
$$= \int (1 - \sin^2 x)\cos x\,dx$$
$$= \int (\cos x - \sin^2 x \cos x)\,dx$$
$$= \sin x - \tfrac{1}{3}\sin^3 x + c.$$

Example 3 Find $\int 2x(1 + x^2)^4\,dx$.

[We note that $f(x) = 1 + x^2 \Rightarrow f'(x) = 2x$ and therefore try differentiating $(1 + x^2)^5$.]

$$\frac{d}{dx}\{(1 + x^2)^5\} = 5(1 + x^2)^4 \times 2x$$

$$\therefore \quad \int 2x(1 + x^2)^4\,dx = \tfrac{1}{5}(1 + x^2)^5 + c.$$

Two general results involving logarithms were derived in §19.4:

$$\int \frac{1}{ax + b}\,dx = \frac{1}{a}\ln(ax + b) + c, \quad \text{for} \quad ax + b > 0$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln\{f(x)\} + c, \quad \text{for} \quad f(x) > 0$$

or
$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c.$$

These were used to integrate functions such as:

$$\frac{1}{3x - 2}, \quad \frac{x - 1}{x + 1}, \quad \frac{2x^2 + 3x}{2x - 1}, \quad \frac{x}{3x^2 + 5} \quad \text{and} \quad \tan x.$$

Finally we note that when obtaining an indefinite integral it may be possible to produce several different answers.

(i) $\int (2x + 2)\,dx = x^2 + 2x + c$

or $\int (2x + 2)\,dx = \int 2(x + 1)\,dx = (x + 1)^2 + c.$

(ii) $\int 2\sin x \cos x\,dx = \sin^2 x + c$ or $\int 2\sin x \cos x\,dx = -\cos^2 x + c$

or $\int 2\sin x \cos x\,dx = \int \sin 2x\,dx = -\tfrac{1}{2}\cos 2x + c.$

However, apparently dissimilar results turn out to differ only by a constant. For example, in (ii)

$$\sin^2 x = -\cos^2 x + 1 = -\tfrac{1}{2}\cos 2x + \tfrac{1}{2}.$$

Exercise 20.1

Find the given indefinite integrals.

1. $\int \frac{x-3}{x^3}\,dx$
2. $\int (x^3-1)^2\,dx$
3. $\int (x+1)\sqrt{x}\,dx$
4. $\int \cos 3x\,dx$
5. $\int e^{4x-3}\,dx$
6. $\int \sin(1-2x)\,dx$
7. $\int (1-x)^4\,dx$
8. $\int \frac{1}{(x+4)^2}\,dx$
9. $\int \frac{1}{(2x-5)^3}\,dx$
10. $\int \frac{1}{2x-5}\,dx$
11. $\int \sec^2 5x\,dx$
12. $\int \cot \tfrac{1}{2}x\,dx$
13. $\int \sin^2 x\,dx$
14. $\int \sin x \cos 3x\,dx$
15. $\int \cot^2 x\,dx$
16. $\int \sin x \cos^2 x\,dx$
17. $\int \frac{\sin x}{\cos^2 x}\,dx$
18. $\int \sin^3 x\,dx$
19. $\int \frac{x+1}{x+2}\,dx$
20. $\int \frac{x^2}{x-1}\,dx$
21. $\int \frac{x^3-x}{x^2+5}\,dx$
22. $\int \frac{1}{\sqrt{(5x+3)}}\,dx$
23. $\int \frac{x^3}{1+x^4}\,dx$
24. $\int x^3(1+x^4)^3\,dx$

Evaluate the given definite integrals.

25. $\int_0^1 e^{1-x}\,dx$
26. $\int_{-2/3}^{1/3} (2+3x)^5\,dx$
27. $\int_0^{\pi/4} \cos^3 x \sin x\,dx$
28. $\int_0^{\pi/2} \sin 2x \sin 5x\,dx$
29. $\int_0^{\pi/12} \cos^2 3x\,dx$
30. $\int_0^{\pi/2} \sin^3 x \cos^2 x\,dx$
31. $\int_0^{\pi/3} \tan x \sec^2 x\,dx$
32. $\int_0^{\pi/4} \cos 3x \cos x\,dx$
33. $\int_0^{\pi/2} \tan^2\left(x - \frac{\pi}{4}\right)dx$
34. $\int_0^{\pi/2} \frac{\cos x}{1+\sin x}\,dx$
35. $\int_1^2 \frac{6}{(1-2x)^2}\,dx$
36. $\int_0^{1/2} \frac{x}{1-x^2}\,dx$

37. Show that $\int_0^{\pi/4} (1 + \tan x)^2 \, dx = 1 + \ln 2$.

38. Find $\frac{d}{dx}\{\ln(\sec x + \tan x)\}$ and hence evaluate $\int_0^{\pi/6} \sec x \, dx$.

39. By expressing $\cos x$ in terms of $\cos \frac{1}{2}x$, or otherwise, find

$$\int \frac{1}{1 + \cos x} \, dx.$$

40. Prove that $\sin^4 x = \frac{1}{8}(\cos 4x - 4\cos 2x + 3)$. Hence evaluate

$$\int_0^{\pi} \sin^4 x \, dx.$$

20.2 Use of partial fractions

Most rational functions of the form $P(x)/Q(x)$, where $P(x)$ and $Q(x)$ are polynomials, are more easily integrated when expressed in partial fractions.

As shown in §18.6, a linear factor of the form $(ax + b)$ in the denominator of a rational function leads to a partial fraction $A/(ax + b)$. Such fractions are integrated using the rule:

$$\int \frac{1}{ax + b} \, dx = \frac{1}{a} \ln(ax + b) + c, \quad \text{for} \quad ax + b > 0.$$

Example 1 Find $\int \frac{1}{(x - 2)(x + 1)} \, dx$ for $x > 2$.

Let $$\frac{1}{(x - 2)(x + 1)} \equiv \frac{A}{x - 2} + \frac{B}{x + 1}$$

then $$1 \equiv A(x + 1) + B(x - 2)$$

Putting $x = 2$: $$1 = 3A \quad \therefore \quad A = \tfrac{1}{3}$$

Putting $x = -1$: $$1 = -3B \quad \therefore \quad B = -\tfrac{1}{3}$$

$$\therefore \quad \int \frac{1}{(x - 2)(x + 1)} \, dx = \int \left\{ \frac{1}{3(x - 2)} - \frac{1}{3(x + 1)} \right\} dx$$

$$= \tfrac{1}{3}\ln(x - 2) - \tfrac{1}{3}\ln(x + 1) + c$$

$$= \tfrac{1}{3}\ln\left(\frac{x - 2}{x + 1}\right) + c, \quad \text{for} \quad x > 2.$$

Partial fractions are not needed when integrating rational functions of the form $f'(x)/f(x)$.

Example 2 Find $\displaystyle\int \frac{2x-1}{(x-2)(x+1)}\,dx$.

If $f(x) = (x-2)(x+1) = x^2 - x - 2$, then $f'(x) = 2x - 1$

$$\therefore \quad \int \frac{2x-1}{(x-2)(x+1)}\,dx = \ln|(x-2)(x+1)| + c.$$

A quadratic factor in the denominator of a rational function may give rise to a partial fraction of the form $f'(x)/f(x)$.

Example 3 Find $\displaystyle\int \frac{1-x^2}{x(x^2+1)}\,dx$, for $x > 0$.

Let
$$\frac{1-x^2}{x(x^2+1)} \equiv \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

then
$$1 - x^2 \equiv A(x^2+1) + (Bx+C)x$$

Putting $x = 0$: $\quad 1 = A \quad \therefore \quad A = 1$

Comparing coefficients, x^2: $\quad -1 = A + B \quad \therefore \quad B = -2$

x: $\quad 0 = C \quad \therefore \quad C = 0$

$$\therefore \quad \int \frac{1-x^2}{x(x^2+1)}\,dx = \int \left\{\frac{1}{x} - \frac{2x}{x^2+1}\right\} dx$$

$$= \ln x - \ln(x^2+1) + c$$

$$= \ln\left(\frac{x}{x^2+1}\right) + c, \quad \text{for} \quad x > 0.$$

Example 4 Evaluate $\displaystyle\int_0^1 \frac{1}{(x+1)(x^2+2x+2)}\,dx$.

Let
$$\frac{1}{(x+1)(x^2+2x+2)} \equiv \frac{A}{x+1} + \frac{Bx+C}{x^2+2x+2}$$

then
$$1 \equiv A(x^2+2x+2) + (Bx+C)(x+1)$$

Putting $x = -1$: $\quad 1 = A \quad \therefore \quad A = 1$

Putting $x = 0$: $\quad 1 = 2A + C \quad \therefore \quad C = -1$

Comparing coefficients of x^2: $\quad 0 = A + B \quad \therefore \quad B = -1$

$$\therefore \quad \int_0^1 \frac{1}{(x+1)(x^2+2x+2)}\,dx = \int_0^1 \left\{\frac{1}{x+1} - \frac{x+1}{x^2+2x+2}\right\} dx$$

$$= \int_0^1 \left\{\frac{1}{x+1} - \frac{1}{2}\left(\frac{2x+2}{x^2+2x+2}\right)\right\} dx$$

$$= \Big[\ln(x+1) - \tfrac{1}{2}\ln(x^2+2x+2)\Big]_0^1$$

$$= (\ln 2 - \tfrac{1}{2}\ln 5) - (\ln 1 - \tfrac{1}{2}\ln 2)$$

$$= \tfrac{3}{2}\ln 2 - \tfrac{1}{2}\ln 5$$

$$= \tfrac{1}{2}(3\ln 2 - \ln 5) = \tfrac{1}{2}\ln(8/5).$$

Example 5 Find $\int \frac{x-4}{(x-1)^2(2x+1)}\,dx$.

Let $$\frac{x-4}{(x-1)^2(2x+1)} \equiv \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(2x+1)}$$

then $$x - 4 \equiv A(x-1)(2x+1) + B(2x+1) + C(x-1)^2$$

Putting $x = 1$: $\quad -3 = 3B \quad \therefore \quad B = -1$

Putting $x = -\frac{1}{2}$: $\quad -\frac{9}{2} = \frac{9}{4}C \quad \therefore \quad C = -2$

Comparing coefficients of x^2: $\quad 0 = 2A + C \quad \therefore \quad A = 1$

$$\therefore \int \frac{x-4}{(x-1)^2(2x+1)}\,dx = \int \left\{ \frac{1}{x-1} - \frac{1}{(x-1)^2} - \frac{2}{2x+1} \right\} dx$$

$$= \ln|x-1| + \frac{1}{x-1} - \ln|2x+1| + c$$

$$= \frac{1}{x-1} + \ln\left|\frac{x-1}{2x+1}\right| + c.$$

Exercise 20.2

Find the following indefinite integrals.

1. $\int \frac{3}{(x-1)(x+2)}\,dx$
2. $\int \frac{1}{1-x^2}\,dx$
3. $\int \frac{1}{x(x-3)}\,dx$
4. $\int \frac{x}{x^2-4}\,dx$
5. $\int \frac{4x}{x^2-2x-3}\,dx$
6. $\int \frac{2x-5}{(x-2)(x-3)}\,dx$
7. $\int \frac{3x+5}{(x+1)(x+3)}\,dx$
8. $\int \frac{x-1}{x(2x-1)}\,dx$
9. $\int \frac{2x-11}{x^2-5x+4}\,dx$
10. $\int \frac{x^2}{1-x^2}\,dx$
11. $\int \frac{x-2}{x^2-4x-8}\,dx$
12. $\int \frac{x^3}{x^2-4}\,dx$

13. $\int \frac{2x^2 + 1}{(x + 1)^2(x - 2)}\,dx$

14. $\int \frac{x + 2}{x^2(x - 1)}\,dx$

15. $\int \frac{x}{(x + 3)^2}\,dx$

16. $\int \frac{x^2 + 2x - 1}{(x + 1)(x^2 + 1)}\,dx$

17. $\int \frac{x + 1}{(x^2 + 1)(x - 1)}\,dx$

18. $\int \frac{x}{(x^2 + 3)(x^2 + 5)}\,dx$

19. $\int \frac{4x^2 + x + 4}{(2x + 1)(x^2 + 2)}\,dx$

20. $\int \frac{x^3 - 3x - 4}{x^3 - 1}\,dx$

Evaluate the following definite integrals.

21. $\int_1^2 \frac{x}{(x + 1)(x + 2)}\,dx$

22. $\int_3^4 \frac{5}{x^2 + x - 6}\,dx$

23. $\int_4^5 \frac{2x}{x^2 - 4x + 3}\,dx$

24. $\int_0^{1/3} \frac{3x}{1 - x^2}\,dx$

25. $\int_0^{1/3} \frac{3 + x}{(1 - x)(1 + 3x)}\,dx$

26. $\int_2^3 \frac{1}{x(x - 1)}\,dx$

27. $\int_1^2 \frac{x - 3}{x(x + 1)(x + 3)}\,dx$

28. $\int_0^2 \frac{x^3 + 4x^2 + 3x - 2}{x^2 + 4x + 3}\,dx$

29. $\int_1^2 \frac{1}{x^2(x + 1)}\,dx$

30. $\int_3^4 \frac{x^2 + x - 3}{(x - 2)^2(x - 1)}\,dx$

31. $\int_3^4 \frac{x^3}{(x - 1)(x - 2)}\,dx$

32. $\int_4^5 \frac{2x^2 + 10x + 40}{(x - 3)^2(x + 1)}\,dx$

33. $\int_3^4 \frac{2x + 1}{(x - 2)(x^2 + 1)}\,dx$

34. $\int_1^2 \frac{2(x + 2)^2}{x^2(x^2 + 4)}\,dx$

35. $\int_0^1 \frac{(x - 2)^2}{x^3 + 1}\,dx$

36. $\int_2^3 \frac{x + 3}{(x - 1)(x^2 - x + 2)}\,dx$

37. Find $\int \frac{4}{(x-1)(x+3)}\,dx$ for (a) $x > 1$, (b) $-3 < x < 1$, (c) $x < -3$.

38. Find $\int \frac{x-4}{2x^2-x-3}\,dx$ for (a) $x > 1\frac{1}{2}$, (b) $x < 1\frac{1}{2}$.

20.3 Integration by substitution

The integration methods considered so far rely on the use of differentiation results in reverse. Integrating by substitution is a more systematic procedure, roughly equivalent to reversing the "chain rule" method for differentiating composite functions.

Suppose that as a result of applying the chain rule we have:

$$\frac{dy}{dx} = 10(2x+3)^4$$

We can work "backwards" to an expression for y as follows:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$\therefore$ substituting $u = 2x + 3$ gives $\frac{dy}{dx} = 10u^4$, $\frac{du}{dx} = 2$ and thus

$$10u^4 = \frac{dy}{du} \times 2 \quad \text{i.e.} \quad \frac{dy}{du} = 5u^4.$$

Hence $$y = u^5 + c = (2x+3)^5 + c.$$

The above method can be modified to produce a useful general result.

Let $$\frac{dy}{dx} = F(x), \quad \text{so that} \quad y = \int F(x)\,dx.$$

If, in terms of another variable u, $F(x) = G(u)$,

then $$\frac{dy}{du} = \frac{dy}{dx} \times \frac{dx}{du} = F(x)\frac{dx}{du} = G(u)\frac{dx}{du}$$

$\therefore$ $$y = \int G(u)\frac{dx}{du}\,du$$

i.e.

> if $F(x) = G(u)$, then $\int F(x)\,dx = \int G(u)\frac{dx}{du}\,du.$

Hence, when simplifying an integral $\int F(x)\,dx$ by a change to a new variable u, we

must express $F(x)$ in terms of u and replace dx by $\frac{dx}{du}du$.

[Note that in this context a statement such as $dx = \frac{dx}{du}du$ is often used as shorthand for "dx may be replaced by $\frac{dx}{du}du$".]

There are three different ways of expressing suitable substitutions.

(i) If $f(x) = u$, then $f'(x) = \frac{du}{dx}$,

and $$dx = \frac{dx}{du}du = \frac{1}{f'(x)}du$$

$\therefore$ we write $$f'(x)\,dx = du.$$

(ii) If $x = g(u)$, then $\frac{dx}{du} = g'(u)$,

$\therefore$ we write $$dx = g'(u)du.$$

(iii) If $f(x) = g(u)$, then $f'(x)\frac{dx}{du} = g'(u)$,

$\therefore$ we write $$f'(x)\,dx = g'(u)\,du.$$

Example 1 Find $\int 10(2x + 3)^4\,dx$.

Let $2x + 3 = u$, then $2dx = du$.

$$\therefore \int 10(2x + 3)^4\,dx = \int 5(2x + 3)^4 \times 2\,dx$$

$$= \int 5u^4 du$$

$$= u^5 + c = (2x + 3)^5 + c.$$

Example 2 Find $\int \sin^3 2x \cos 2x\,dx$.

Let $\sin 2x = u$, then $2\cos 2x\,dx = du$.

$$\therefore \int \sin^3 2x \cos 2x\,dx = \int \tfrac{1}{2}\sin^3 2x \times 2\cos 2x\,dx$$

$$= \int \tfrac{1}{2}u^3\,du$$

$$= \tfrac{1}{8}u^4 + c = \tfrac{1}{8}\sin^4 2x + c.$$

Example 3 Find $\int x\sqrt{(2x-1)}\,dx$.

Let $2x-1=u^2$, $2x=u^2+1$

then $2dx=2u\,du$, $x=\frac{1}{2}(u^2+1)$

$\therefore$ $dx=u\,du$

$$\int x\sqrt{(2x-1)}\,dx=\int\tfrac{1}{2}(u^2+1)\times u\times u\,du$$

$$=\frac{1}{2}\int(u^4+u^2)\,du$$

$$=\tfrac{1}{2}(\tfrac{1}{5}u^5+\tfrac{1}{3}u^3)+c$$

$$=\tfrac{1}{30}u^3(3u^2+5)+c$$

$$\therefore\quad\int x\sqrt{(2x-1)}\,dx=\tfrac{1}{30}(2x-1)^{3/2}(3\{2x-1\}+5)+c$$

$$=\tfrac{1}{15}(2x-1)^{3/2}(3x+1)+c.$$

When the method of substitution is applied to a definite integral, the limits of integration must be changed as well as the variable.

Example 4 Evaluate $\int_0^1 4x(3x^2-1)^3\,dx$.

Let $3x^2-1=u$, $x=1\Rightarrow u=2$

then $6x\,dx=du$ $x=0\Rightarrow u=-1$

$$\int_0^1 4x(3x^2-1)^3\,dx=\int_0^1\tfrac{2}{3}(3x^2-1)^3\times 6x\,dx$$

$$=\int_{-1}^{2}\tfrac{2}{3}u^3\,du$$

$$=\left[\tfrac{1}{6}u^4\right]_{-1}^{2}$$

$$=\tfrac{1}{6}\{2^4-(-1)^4\}=\tfrac{15}{6}=2\tfrac{1}{2}.$$

Example 5 Evaluate $\int_0^2\sqrt{(4-x^2)}\,dx$.

$$\int_0^2 \sqrt{(4-x^2)}\,dx$$

$$= \int_0^{\pi/2} 2\cos\theta \times 2\cos\theta\,d\theta$$

$$= \int_0^{\pi/2} 4\cos^2\theta\,d\theta$$

$$= \int_0^{\pi/2} (2 + 2\cos 2\theta)\,d\theta$$

$$= \Big[2\theta + \sin 2\theta\Big]_0^{\pi/2}$$

$$= \pi$$

Let $x = 2\sin\theta$
then $dx = 2\cos\theta\,d\theta$

$4 - x^2 = 4 - 4\sin^2\theta$
$= 4(1 - \sin^2\theta)$
$= 4\cos^2\theta$

$\therefore \sqrt{(4-x^2)} = 2\cos\theta$

$x = 2 \Rightarrow \theta = \frac{1}{2}\pi$
$x = 0 \Rightarrow \theta = 0$

When integrating trigonometric functions one useful substitution is $\tan\frac{1}{2}\theta = t$, where

$$\sin\theta = \frac{2t}{1+t^2}, \quad \cos\theta = \frac{1-t^2}{1+t^2}, \quad \tan\theta = \frac{2t}{1-t^2}$$

Example 6 Evaluate $\displaystyle\int_0^{\pi/2} \frac{4}{3+5\cos\theta}\,d\theta$.

$$\int_0^{\pi/2} \frac{4}{3+5\cos\theta}\,d\theta$$

$$= \int_0^1 \frac{4(1+t^2)}{8-2t^2} \times \frac{2}{1+t^2}\,dt$$

$$= \int_0^1 \frac{4}{4-t^2}\,dt$$

$$= \int_0^1 \left\{\frac{1}{2+t} + \frac{1}{2-t}\right\} dt$$

$$= \Big[\ln(2+t) - \ln(2-t)\Big]_0^1$$

$$= (\ln 3 - \ln 1) - (\ln 2 - \ln 2)$$

$$= \ln 3$$

Let $\tan\frac{1}{2}\theta = t$
then $\frac{1}{2}\sec^2\frac{1}{2}\theta\,d\theta = dt$
$\therefore (1 + \tan^2\frac{1}{2}\theta)\,d\theta = 2dt$
$\therefore d\theta = \dfrac{2}{1+t^2}\,dt$

$$3 + 5\cos\theta = 3 + 5\left(\frac{1-t^2}{1+t^2}\right)$$

$$= \frac{3(1+t^2) + 5(1-t^2)}{1+t^2}$$

$$= \frac{8-2t^2}{1+t^2}$$

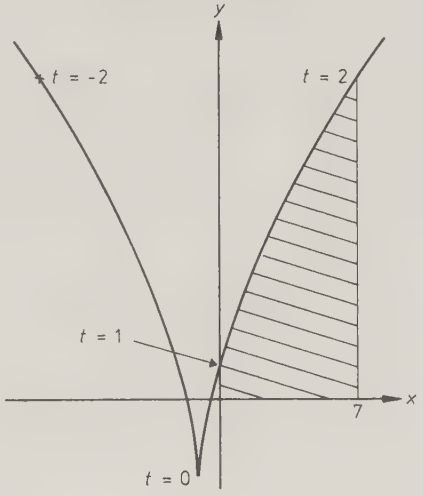
$\theta = \frac{1}{2}\pi \Rightarrow t = 1$
$\theta = 0 \Rightarrow t = 0$

A change of variable may be helpful when finding areas and volumes by integration. For instance, to evaluate $\displaystyle\int_a^b y\,dx$ it may be simpler to work in terms of y rather than x. If x and y are given in parametric form it will usually be best to use the parameter as variable in any integrations.

Example 7 Find the area bounded by the curve $x = t^3 - 1$, $y = 5t^2 - 4$, the x- and y-axes and the line $x = 7$.

As shown in the sketch below, values of x from 0 to 7 are given by values of t from 1 to 2.
Since $x = t^3 - 1$, we have $dx = 3t^2\,dt$.

$$\text{Required area} = \int_0^7 y\,dx$$
$$= \int_1^2 (5t^2 - 4) \times 3t^2\,dt$$
$$= \int_1^2 (15t^4 - 12t^2)\,dt$$
$$= \left[3t^5 - 4t^3\right]_1^2$$
$$= (96 - 32) - (3 - 4) = 65.$$



Exercise 20.3

Use the given substitutions to find the following indefinite integrals.

1. $\int (3x - 2)^3\,dx, 3x - 2 = u.$
2. $\int (1 - 2x)^4\,dx, 1 - 2x = u.$
3. $\int x(1 - x)^5\,dx, 1 - x = u.$
4. $\int 2x^2(x^3 - 1)^3\,dx, x^3 - 1 = u.$
5. $\int \tan^2 x \sec^2 x\,dx, \tan x = u.$
6. $\int \sin x \cos^5 x\,dx, \cos x = u.$
7. $\int x\sqrt{(x^2 + 4)}\,dx, x^2 + 4 = t^2.$
8. $\int x\sqrt{(1 - x)}\,dx, 1 - x = t^2.$
9. $\int \frac{x}{\sqrt{(2x + 1)}}\,dx, 2x + 1 = t^2.$
10. $\int \frac{e^x}{1 + e^x}\,dx, e^x = t.$
11. $\int \cos^3 \theta\,d\theta, \sin \theta = x.$
12. $\int \sin^3 \theta \cos^2 \theta\,d\theta, \cos \theta = x.$
13. $\int \frac{\sin \theta}{1 + \cos 2\theta}\,d\theta, \cos \theta = x.$
14. $\int \frac{\sec^2(\sqrt{x})}{\sqrt{x}}\,dx, \sqrt{x} = \theta.$

Use the given substitutions to find the following definite integrals.

15. $\int_4^5 (x-1)\sqrt{(x-4)}\,dx$, $x-4=u^2$.

16. $\int_0^1 x^2(1-x)^{1/2}\,dx$, $1-x=u^2$.

17. $\int_e^{e^2} \frac{1}{x\ln x}\,dx$, $\ln x = t$.

18. $\int_0^1 \frac{e^x}{(1+e^x)^2}\,dx$, $e^x = t$.

19. $\int_0^2 \frac{1}{\sqrt{(4-x^2)}}\,dx$, $x = 2\sin\theta$.

20. $\int_0^{\sqrt{3}} \frac{1}{9+x^2}\,dx$, $x = 3\tan\theta$.

21. $\int_0^{\pi/2} \sin 2x\sqrt{(\sin x)}\,dx$, $\sin x = u^2$.

22. $\int_0^{1/2} (1-x^2)^{1/2}\,dx$, $x = \sin u$.

23. $\int_0^{\pi/2} \frac{1}{5\cos\theta+4}\,d\theta$, $\tan\frac{1}{2}\theta = t$.

24. $\int_0^{\pi/4} \frac{1}{5\cos^2\theta-1}\,d\theta$, $\tan\theta = t$.

Find the following indefinite integrals.

25. $\int x(2x+3)^4\,dx$

26. $\int x^3(x^2-1)^3\,dx$

27. $\int \sqrt{(1+3x)}\,dx$

28. $\int \frac{x}{\sqrt{(3+x)}}\,dx$

29. $\int \frac{x^2}{(x^3-1)^2}\,dx$

30. $\int \frac{x+1}{(x-1)^3}\,dx$

31. $\int xe^{x^2}\,dx$

32. $\int \frac{1}{x\sqrt{(\ln x)}}\,dx$

33. $\int \frac{e^x}{4-e^{2x}}\,dx$

Find the following definite integrals

34. $\int_{-1}^2 x^2\sqrt{(x^3+1)}\,dx$

35. $\int_0^3 x\sqrt{(9-x^2)}\,dx$

36. $\int_0^3 \sqrt{(9-x^2)}\,dx$

37. $\int_1^2 \frac{x}{\sqrt{(4-x^2)}}\,dx$

38. $\int_1^2 \frac{x^2}{\sqrt{(4-x^2)}}\,dx$

39. $\int_0^{\ln 2} \frac{1}{1+e^x}\,dx$

40. Find the area between the x-axis and the curve $x = 2t^3 - 1$, $y = t^2 - 1$ for $1 \leqslant t \leqslant 2$.

41. Find the area of the finite region bounded by the curve $x = 5t^2$, $y = 2t^3$ and the line $x = 5$. Find also the volume of the solid formed when this region is rotated through π radians about the x-axis.

42. Find the area bounded by the x-axis and the curve $x = \theta - \sin\theta$, $y = 1 - \cos\theta$ for $0 \leqslant \theta \leqslant 2\pi$.

43. Using the substitution $\tan\frac{1}{2}x = t$, or otherwise, show that

$$\int_{\pi/3}^{\pi/2} \frac{1}{1 + \sin x + \cos x}\,dx = 0{\cdot}237$$

correct to 3 significant figures.

44. Using the substitution $x = 2\cos^2\theta + 5\sin^2\theta$, or otherwise, evaluate

$$\int_2^{7/2} \frac{1}{\sqrt{\{(x-2)(5-x)\}}}\,dx.$$

45. Using the substitution $\tan x = t$, or otherwise, show that

$$\int \frac{1}{1 - 2\sin^2 x}\,dx = \tfrac{1}{2}\ln\{\tan(x + \tfrac{1}{4}\pi)\} + c.$$

20.4 Integration by parts

A further general method of integration is obtained by considering the formula for differentiating a product. If u and v are functions of x, then

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrating,

$$uv = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx$$

Rearranging,

$$\boxed{\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx} \qquad (1)$$

This rule provides a way of expressing the integral of a product in terms of a second integral, which may be easier to find. The method is called *integration by parts.*

Example 1 Find $\int x\cos x\,dx$.

In rule (1), let $u = x$, $\dfrac{dv}{dx} = \cos x$,

$$\frac{du}{dx} = 1, \quad v = \sin x,$$

$$\therefore \int x\cos x\,dx = x\sin x - \int(\sin x \times 1)\,dx$$

$$= x\sin x - \int \sin x\,dx = x\sin x + \cos x + c.$$

[Note that it is not necessary to introduce an arbitrary constant when integrating $\frac{dv}{dx}$ to give v, since rule (1) is valid for any function v such that $\frac{dv}{dx} = \cos x$.]

Example 2 Find $\int x \ln x \, dx$.

In rule (1), let $u = \ln x$, $\frac{dv}{dx} = x$,

$$\frac{du}{dx} = \frac{1}{x}, \qquad v = \tfrac{1}{2}x^2,$$

$$\therefore \int x \ln x \, dx = \tfrac{1}{2}x^2 \ln x - \int \left(\tfrac{1}{2}x^2 \times \frac{1}{x}\right) dx$$

$$= \tfrac{1}{2}x^2 \ln x - \int \tfrac{1}{2}x \, dx = \tfrac{1}{2}x^2 \ln x - \tfrac{1}{4}x^2 + c.$$

For definite integrals the rule for integration by parts becomes

$$\boxed{\int_a^b u \frac{dv}{dx} dx = \Big[uv\Big]_a^b - \int_a^b v \frac{du}{dx} dx} \tag{2}$$

Example 3 Evaluate $\int_0^1 x^2 e^x \, dx$.

In rule (2), let $u = x^2$, $\frac{dv}{dx} = e^x$,

$$\frac{du}{dx} = 2x, \qquad v = e^x,$$

$$\therefore \int_0^1 x^2 e^x \, dx = \Big[x^2 e^x\Big]_0^1 - \int_0^1 (e^x \times 2x) \, dx$$

$$= e - 2\int_0^1 x e^x \, dx.$$

In rule (2), let $u = x$, $\frac{dv}{dx} = e^x$,

$$\frac{du}{dx} = 1, \qquad v = e^x,$$

$$\therefore \int_0^1 x e^x \, dx = \Big[x e^x\Big]_0^1 - \int_0^1 e^x \, dx$$

$$= e - \Big[e^x\Big]_0^1 = e - (e - 1) = 1.$$

Hence $\int_0^1 x^2 e^x \, dx = e - 2\int_0^1 xe^x \, dx = e - 2.$

Integration by parts is also used to integrate functions which are not products.

Example 4 Find $\int \ln x \, dx$.

In rule (1), let $u = \ln x, \quad \frac{dv}{dx} = 1,$

$$\frac{du}{dx} = \frac{1}{x}, \qquad v = x,$$

$$\therefore \quad \int \ln x \, dx = x \ln x - \int \left(x \times \frac{1}{x} \right) dx$$

$$= x \ln x - \int dx = x \ln x - x + c.$$

The next example illustrates a further way in which integration by parts may be used.

Example 5 Find $\int e^x \cos x \, dx$.

Integrating by parts,

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx \tag{i}$$

Integrating by parts,

$$\int e^x \sin x \, dx = e^x(-\cos x) - \int (-\cos x) e^x \, dx$$

$$= -e^x \cos x + \int e^x \cos x \, dx$$

Substituting this expression for $\int e^x \sin x \, dx$ in (i),

$$\int e^x \cos x \, dx = e^x \sin x - \left(-e^x \cos x + \int e^x \cos x \, dx \right)$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

$$\therefore \quad 2\int e^x \cos x \, dx = e^x \sin x + e^x \cos x + k$$

Hence $\int e^x \cos x \, dx = \frac{1}{2} e^x (\sin x + \cos x) + c.$

Exercise 20.4

Find the following indefinite integrals.

1. $\int x \sin x \, dx$
2. $\int x \cos \frac{1}{2}x \, dx$
3. $\int xe^{-x} \, dx$
4. $\int x \ln 2x \, dx$
5. $\int xe^{2x} \, dx$
6. $\int x \sec^2 x \, dx$
7. $\int x \sin x \cos x \, dx$
8. $\int x \tan^2 x \, dx$
9. $\int x \cos^2 x \, dx$
10. $\int x^2 \cos x \, dx$
11. $\int x^3 e^x dx$
12. $\int x^3 \ln x \, dx$
13. $\int x(\ln x)^2 \, dx$
14. $\int \ln(x - 1) \, dx$
15. $\int e^{-x} \sin x \, dx$

Find the following definite integrals.

16. $\int_0^{\pi} x \cos x \, dx$
17. $\int_{-1}^{1} xe^x \, dx$
18. $\int_0^{\pi/3} x \sin 3x \, dx$
19. $\int_1^e \ln x \, dx$
20. $\int_0^{\pi} x \sin^2 x \, dx$
21. $\int_1^2 x^2 \ln x \, dx$
22. $\int_2^3 x^2 e^{-x} \, dx$
23. $\int_1^e \frac{1}{x^3} \ln x \, dx$
24. $\int_0^{\pi/2} x^2 \sin x \, dx$
25. $\int_0^{\pi} e^x \sin x \, dx$
26. $\int_1^2 x^2 e^{2x} \, dx$
27. $\int_e^{e^2} \ln(\sqrt{x}) \, dx$

28. Find $\int x(x + 1)^4 \, dx$ using (a) integration by parts, (b) the substitution $x + 1 = t$.

29. Find, using integration by parts or otherwise,

(a) $\int x\sqrt{(x - 1)} \, dx$, (b) $\int \frac{x}{(x + 2)^3} dx$.

30. Use integration by parts to show that

(a) $2\int \cos^2 x \, dx = \cos x \sin x + x + c$,

(b) $3\int \cos^3 x \, dx = \cos^2 x \sin x + 2 \sin x + c$.

20.5 Use of inverse trigonometric functions

The inverse trigonometric functions $\sin^{-1}x$ and $\tan^{-1}x$ are defined as follows:

$$y = \sin^{-1}x \Rightarrow \sin y = x \quad \text{and} \quad -\tfrac{1}{2}\pi \leqslant y \leqslant \tfrac{1}{2}\pi$$
$$y = \tan^{-1}x \Rightarrow \tan y = x \quad \text{and} \quad -\tfrac{1}{2}\pi < y < \tfrac{1}{2}\pi$$

In §11.6 two important integration results were obtained by differentiating $\sin^{-1}\frac{x}{a}$ and $\tan^{-1}\frac{x}{a}$.

$$\int \frac{1}{\sqrt{(a^2 - x^2)}}\,dx = \sin^{-1}\frac{x}{a} + c, \quad \int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c.$$

[These integrals can also be found by substituting $x = a\sin\theta$ and $x = a\tan\theta$ respectively.]

Example 1 Evaluate (a) $\int_0^1 \frac{1}{\sqrt{(4 - x^2)}}\,dx$, (b) $\int_0^2 \frac{1}{4 + x^2}\,dx$.

(a) $$\int_0^1 \frac{1}{\sqrt{(4 - x^2)}}\,dx = \left[\sin^{-1}\frac{x}{2}\right]_0^1 = \sin^{-1}\frac{1}{2} - \sin^{-1}0 = \frac{\pi}{6}.$$

(b) $$\int_0^2 \frac{1}{4 + x^2}\,dx = \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^2 = \frac{1}{2}\tan^{-1}1 - \frac{1}{2}\tan^{-1}0 = \frac{\pi}{8}.$$

Example 2 Find (a) $\int \frac{1}{\sqrt{(9 - 4x^2)}}\,dx$, (b) $\int \frac{1}{25 + 16x^2}\,dx$.

(a) Let $2x = u$, then $4x^2 = u^2$ and $2dx = du$,

$$\therefore \int \frac{1}{\sqrt{(9 - 4x^2)}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{(9 - u^2)}}\,du$$
$$= \frac{1}{2}\sin^{-1}\frac{u}{3} + c.$$
$$= \frac{1}{2}\sin^{-1}\frac{2x}{3} + c.$$

(b) Let $4x = u$, then $16x^2 = u^2$ and $4dx = du$,

$$\therefore \int \frac{1}{25 + 16x^2}\,dx = \frac{1}{4}\int \frac{1}{25 + u^2}\,du$$
$$= \frac{1}{4} \times \frac{1}{5}\tan^{-1}\frac{u}{5} + c$$
$$= \frac{1}{20}\tan^{-1}\frac{4x}{5} + c.$$

The method of Example 2 can be extended to integrals of the form

$$\int \frac{1}{\sqrt{(ax^2+bx+c)}}\,dx, \quad \int \frac{1}{ax^2+bx+c}\,dx.$$

Example 3 Find $\displaystyle\int \frac{1}{x^2-2x+2}\,dx$.

$$x^2-2x+2 = (x^2-2x+1)+1 = (x-1)^2+1$$

$$\therefore \int \frac{1}{x^2-2x+2}\,dx = \int \frac{1}{(x-1)^2+1}\,dx$$

$$= \int \frac{1}{u^2+1}\,du \qquad \text{letting } x-1=u,\ dx=du$$

$$= \tan^{-1}u + c$$

$$= \tan^{-1}(x-1) + c.$$

The remaining examples illustrate further uses of inverse trigonometric functions.

Example 4 Find $\displaystyle\int \frac{x+3}{x^2+9}\,dx$.

$$\int \frac{x+3}{x^2+9}\,dx = \int \left\{\frac{x}{x^2+9} + \frac{3}{x^2+9}\right\} dx = \frac{1}{2}\int \frac{2x}{x^2+9}\,dx + 3\int \frac{1}{x^2+9}\,dx$$

$$= \frac{1}{2}\ln(x^2+9) + \tan^{-1}\frac{x}{3} + c.$$

Example 5 Evaluate $\displaystyle\int_0^1 \frac{3-x}{(x+1)(x^2+1)}\,dx$.

Let $$\frac{3-x}{(x+1)(x^2+1)} \equiv \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

then $$3-x \equiv A(x^2+1) + (Bx+C)(x+1)$$

Putting $x=-1$: $4 = 2A$ $\quad\therefore\quad A = 2$

Putting $x=0$: $3 = A + C$ $\quad\therefore\quad C = 1$

Comparing coefficients of x^2:

$$0 = A + B \quad\therefore\quad B = -2$$

$$\therefore \int_0^1 \frac{3-x}{(x+1)(x^2+1)}\,dx = \int_0^1 \left\{\frac{2}{x+1} + \frac{(-2x+1)}{x^2+1}\right\} dx$$

$$= \int_0^1 \left\{\frac{2}{x+1} - \frac{2x}{x^2+1} + \frac{1}{x^2+1}\right\} dx$$

$$= \Big[2\ln(x+1) - \ln(x^2+1) + \tan^{-1}x\Big]_0^1$$

$$= (2\ln 2 - \ln 2 + \tan^{-1}1) - (2\ln 1 - \ln 1 + \tan^{-1}0)$$

$$= \ln 2 + \tfrac{1}{4}\pi$$

Example 6 Find $\int \frac{2x^2}{\sqrt{(1-x^2)}}\,dx$.

$$\int \frac{2x^2}{\sqrt{(1-x^2)}}\,dx$$

$$= \int \frac{2\sin^2\theta}{\cos\theta} \times \cos\theta\,d\theta$$

$$= \int (1-\cos 2\theta)\,d\theta$$

$$= \theta - \tfrac{1}{2}\sin 2\theta + c$$

$$= \theta - \sin\theta\cos\theta + c$$

$$= \sin^{-1}x - x\sqrt{(1-x^2)} + c.$$

Let $x = \sin\theta$
then $dx = \cos\theta\,d\theta$

$$1 - x^2 = 1 - \sin^2\theta$$
$$= \cos^2\theta$$
$$\therefore \quad \sqrt{(1-x^2)} = \cos\theta$$

Exercise 20.5

Find the following indefinite integrals.

1. $\int \frac{1}{\sqrt{(16-x^2)}}\,dx$
2. $\int \frac{1}{x^2+25}\,dx$
3. $\int \frac{1}{\sqrt{(1-4x^2)}}\,dx$
4. $\int \frac{x+1}{x^2+1}\,dx$
5. $\int \frac{1}{\sqrt{(4-25x^2)}}\,dx$
6. $\int \frac{1}{4+9x^2}\,dx$
7. $\int \frac{x^2-x}{(x+1)(x^2+1)}\,dx$
8. $\int \frac{4}{(1-x)(1+x^2)}\,dx$
9. $\int \frac{6(x-3)}{x(x^2+9)}\,dx$
10. $\int \frac{5x^2+4x-20}{(x+2)(x^2+4)}\,dx$
11. $\int \frac{1}{x^2+2x+5}\,dx$
12. $\int \frac{1}{\sqrt{(5+4x-x^2)}}\,dx$

Evaluate the following definite integrals.

13. $\int_0^{\sqrt{3}} \frac{1}{x^2+9}\,dx$
14. $\int_0^5 \frac{1}{\sqrt{(25-x^2)}}\,dx$
15. $\int_0^{1/2} \frac{1}{4x^2+1}\,dx$
16. $\int_{-2}^{2} \frac{x+2}{x^2+4}\,dx$
17. $\int_0^1 \frac{4}{(x+1)(x^2+1)}\,dx$
18. $\int_3^4 \frac{1}{x^2-6x+10}\,dx$
19. $\int_{-1/2}^{1/2} \frac{1}{\sqrt{(3-4x-4x^2)}}\,dx$

Use appropriate substitutions to find the following indefinite integrals.

29. $\int \frac{e^x + e^{2x}}{1 + e^{2x}}\,dx$

21. $\int \frac{1}{e^x + e^{-x}}\,dx$

22. $\int \sqrt{(1 - x^2)}\,dx$

23. $\int \frac{1 - x}{\sqrt{(1 - x^2)}}\,dx$

Use the given substitutions to find the following indefinite integrals.

24. $\int \frac{2x}{x^2 - 2x + 10}\,dx,\ x = u + 1.$

25. $\int \frac{x}{x^4 + 2x^2 + 2}\,dx,\ x^2 = u - 1.$

26. $\int \frac{1}{x\sqrt{(x^2 - 1)}}\,dx,\ x = \frac{1}{u}.$

27. Evaluate correct to 3 significant figures

(a) $\int_1^3 \frac{4 - 3x}{(2x - 1)(x^2 + 1)}\,dx,$

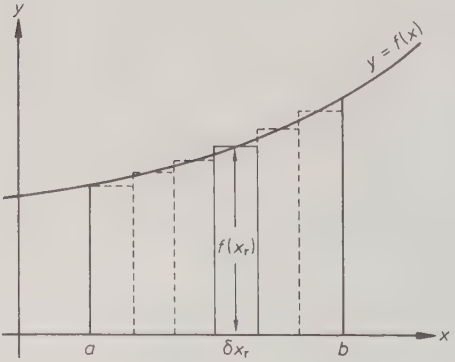
(b) $\int_0^1 \frac{7x + 2}{(x + 1)^2(x^2 + 4)}\,dx$

28. Use integration by parts to show that

$$\int \sin^{-1} x\,dx = x \sin^{-1} x + \sqrt{(1 - x^2)} + c.$$

20.6 Properties of the definite integral

The definite integral is defined formally as the limit of a sum by considering a function $f(x)$ in the interval $a \leqslant x \leqslant b$.



Suppose that this interval is divided into n subintervals of lengths $\delta x_1, \delta x_2, \ldots, \delta x_n$. If n is increased in such a way that the values of $\delta x_1, \delta x_2, \ldots, \delta x_n$ tend to zero, then

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \left\{ \sum_{r=1}^{n} f(x_r)\,\delta x_r \right\},$$

where x_r lies in the interval of length δx_r.

This definition may be expressed more simply as

$$\int_a^b y\,dx = \lim_{\delta x \to 0} \sum y\,\delta x.$$

In general the value of this integral represents the area bounded by the curve $y = f(x)$, the x-axis, the lines $x = a$ and $x = b$. We must remember, however, that areas above the x-axis are positive and areas below the x-axis are negative.

The most important properties of the definite integral are as follows:

$$\int_a^b f(x)\,dx + \int_a^b g(x)\,dx = \int_a^b \{f(x) + g(x)\}\,dx.$$

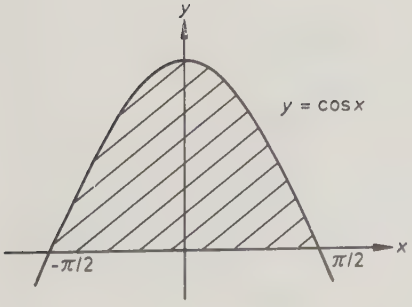
$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx, \quad \text{where } k \text{ is a constant.}$$

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx,$$

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx.$$

If $f(x)$ is an *even function* then the curve $y = f(x)$ will be symmetrical about the y-axis. Hence the area under the curve between $x = -a$ and $x = 0$ is equal to the area between $x = 0$ and $x = a$.

$\therefore$ if $f(x)$ is an even function $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx.$



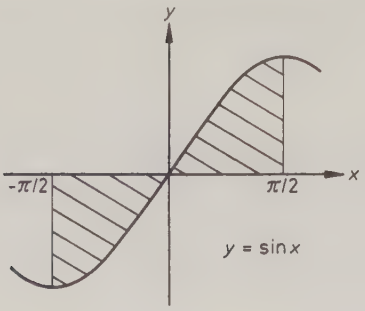
For instance, if $f(x) = \cos x$,
$f(-x) = \cos(-x) = \cos x = f(x)$

$$\therefore \int_{-\pi/2}^{\pi/2} \cos x\,dx = 2\int_0^{\pi/2} \cos x\,dx$$

$$= 2\Big[\sin x\Big]_0^{\pi/2} = 2.$$

If $f(x)$ is an *odd function*, the areas bounded by the curve $y = f(x)$ and the x-axis between $x = -a$ and $x = 0$ and between $x = 0$ and $x = a$ are numerically equal but opposite in sign,

$\therefore$ if $f(x)$ is an odd function $\displaystyle\int_{-a}^{a} f(x)\,dx = 0.$



For instance, if $f(x) = \sin x$,
$f(-x) = \sin(-x) = -\sin x = -f(x)$

$$\therefore \qquad \int_{-\pi/2}^{\pi/2} \sin x \, dx = 0.$$

The symmetries of the sine function can also be used when integrating functions of the form $xf(\sin x)$. We illustrate the method by a simple example.

Example 1 Evaluate $I = \int_0^{\pi} x \sin x \, dx$.

$$I = -\int_{\pi}^{0} (\pi - u)\sin(\pi - u)\, du \qquad \text{Let} \quad x = \pi - u$$
$$\text{then} \quad dx = -du$$
$$= \int_0^{\pi} (\pi - u)\sin u \, du \qquad x = \pi \Rightarrow u = 0$$
$$x = 0 \Rightarrow u = \pi$$
$$= \int_0^{\pi} \pi \sin u \, du - \int_0^{\pi} u \sin u \, du$$
$$= \pi \int_0^{\pi} \sin u \, du - I$$

$$\therefore \quad I = \tfrac{1}{2}\pi \int_0^{\pi} \sin u \, du = \tfrac{1}{2}\pi \Big[-\cos u \Big]_0^{\pi} = \tfrac{1}{2}\pi\{1 - (-1)\} = \pi.$$

Finally we consider definite integrals with *infinite limits*. One type of infinite integral is defined as follows:

$$\int_0^{\infty} f(x)\, dx = \lim_{t \to \infty} \int_0^{t} f(x)\, dx,$$

provided that this limit exists.

Example 2 Evaluate $\int_0^{\infty} e^{-x}\, dx$.

$$\int_0^{t} e^{-x}\, dx = \Big[-e^{-x} \Big]_0^{t} = -e^{-t} + 1$$

$$\therefore \quad \int_0^{\infty} e^{-x}\, dx = \lim_{t \to \infty} (-e^{-t} + 1) = 1.$$

Exercise 20.6

1. Evaluate the following integrals using the properties of odd and even functions.

(a) $\int_{-1}^{1} x^4\,dx$, (b) $\int_{-3}^{3} x(x^2-1)\,dx$, (c) $\int_{-\pi}^{\pi} \sin^2 x\,dx$,

(d) $\int_{-\pi/3}^{\pi/3} \frac{1}{1+\cos 2x}\,dx$, (e) $\int_{-4}^{4} \frac{x}{x^2-25}\,dx$, (f) $\int_{-1}^{1} \frac{x\cos x}{1+\tan^2 x}\,dx$.

2. Sketch the curve $y = (x^2-1)(x-3)$. Hence evaluate $\int_{-7}^{9} (x^2-1)(x-3)\,dx$.

3. Using sketch graphs, or otherwise, evaluate

(a) $\int_{-2}^{2} [x]\,dx$, (b) $\int_{-2}^{2} |x|\,dx$, (c) $\int_{-2}^{2} (x+|x|)\,dx$.

4. Evaluate (a) $\int_{0}^{3} (x-2)\,dx$, (b) $\left|\int_{0}^{3} (x-2)\,dx\right|$, (c) $\int_{0}^{3} |x-2|\,dx$.

5. Evaluate (a) $\int_{0}^{3} x(x-2)\,dx$, (b) $\int_{0}^{3} |x(x-2)|\,dx$.

6. Prove that $\int_{0}^{a} f(x)\,dx = \int_{0}^{a} f(a-x)\,dx$. Hence, or otherwise, show that

(a) $\int_{0}^{\pi} \frac{x\sin x}{4-\cos^2 x}\,dx = \frac{1}{4}\pi \ln 3$, (b) $\int_{0}^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \frac{1}{4}\pi^2$.

7. Evaluate $\int_{0}^{1} e^x\,dx$. Hence show that

$$\lim_{n\to\infty} \frac{1}{n}\left\{e^{1/n} + e^{2/n} + e^{3/n} + \ldots + e\right\} = e - 1.$$

8. Evaluate $\int_{0}^{1} \frac{x\,dx}{\sqrt{(1+x)}}$. By considering integration as the limit of a sum, or otherwise, show that

$$\lim_{n\to\infty} \frac{1}{n\sqrt{n}}\left\{\frac{1}{\sqrt{(n+1)}} + \frac{2}{\sqrt{(n+2)}} + \ldots + \frac{n}{\sqrt{(2n)}}\right\} = \frac{2}{3}(2-\sqrt{2}). \quad \text{(JMB)}$$

9. Explain what is wrong with the following mathematical arguments and evaluate the integrals I and J correctly:

(i) Let $I = \int_{0}^{2} \frac{1}{x^2-2x+2}\,dx$. Then putting $x - 1 = \cot\theta$ we have

$$I = \int_{\cot^{-1}(1)}^{\cot^{-1}(-1)} \frac{-\operatorname{cosec}^2\theta}{\cot^2\theta + 1}\,d\theta = -\int_{-\pi/4}^{\pi/4} d\theta = -\tfrac{1}{2}\pi.$$

(ii) Let $J = \int_0^{\pi} e^{|\sin x|}|\cos x|\,dx$. Then putting $u = |\sin x|$ we have $du = |\cos x|\,dx$ and so $J = \int_0^0 e^u\,du = 0$. (C)

10. Evaluate (a) $\int_0^{\infty} e^{-x/2}\,dx$, (b) $\int_1^{\infty} \frac{1}{x^2}\,dx$, (c) $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$.

11. Use appropriate substitutions to show that

(i) $\int_1^{ak} \frac{1}{t}\,dt = k\int_1^{a} \frac{1}{u}\,du$, (ii) $\int_a^{ab} \frac{1}{t}\,dt = \int_1^{b} \frac{1}{u}\,du$.

Defining $\ln x$ as $\int_1^x \frac{1}{t}\,dt$, deduce that

(iii) $\ln(a^k) = k\ln a$, (iv) $\ln(ab) = \ln a + \ln b$.

Exercise 20.7 (miscellaneous)

In questions 1 to 15 evaluate the given definite integrals.

1. $\int_0^4 \frac{x}{2x+1}\,dx$ 2. $\int_{-1/2}^0 \frac{x}{(x+1)^2}\,dx$ 3. $\int_1^2 \frac{1}{4x^2-1}\,dx$

4. $\int_{-1}^4 \frac{1}{\sqrt{(3x+4)}}\,dx$ 5. $\int_2^3 \frac{x^2+1}{x(x-1)}\,dx$

6. $\int_0^2 (4-x^2)\sqrt{x}\,dx$ 7. $\int_0^2 x\sqrt{(4-x^2)}\,dx$

8. $\int_0^{\pi} (\sin x + \cos x)^2\,dx$ 9. $\int_0^{\pi/3} 2\sin 2x\cos x\,dx$

10. $\int_0^{\pi/4} \tan^3 x\,dx$ 11. $\int_0^{\pi/4} \sin^3 x\cos^3 x\,dx$

12. $\int_{\ln 2}^{\ln 5} \frac{2}{e^x - e^{-x}}\,dx$ 13. $\int_0^{\pi/4} \frac{\sin 2x}{1+\cos^2 x}\,dx$

14. $\int_0^1 x^2\sqrt{(1-x^2)}\,dx$ 15. $\int_0^1 2^x\,dx$

16. Find $\int \frac{2x-1}{(x+2)(x-3)^2}\,dx$, given that $x > 3$.

17. Show that $\int_0^3 \frac{x+3}{x^2+3}\,dx = \ln 2 + \frac{\pi\sqrt{3}}{3}$.

In questions 18 to 21 use the given substitutions to evaluate the definite integrals.

18. $\int_{\sqrt{2}}^2 \frac{1}{x\sqrt{(x^2-1)}}\,dx$, $x = \sec\theta$.

19. $\int_0^{1/2} \sqrt{\left(\frac{x}{1-x}\right)}\,dx$, $x = \sin^2\theta$.

20. $\int_0^1 \frac{1}{(1+x^2)}\,dx$, $x = \tan\theta$.

21. $\int_0^{\pi/6} \sec x\,dx$, $\sin x = u$.

22. Use integration by parts and the result of question 21 to show that

$$\int_0^{\pi/6} \sec^3 x\,dx = \tfrac{1}{3} + \tfrac{1}{4}\ln 3.$$

23. Show that the curve $y = \frac{3}{(2x+1)(1-x)}$ has only one turning point. Find the coordinates of this point and determine its nature. Sketch the curve. Find the area of the region enclosed by the curve and the line $y = 3$. (JMB)

24. Find the coordinates of the turning point of the curve $y = (x^2-4)/(x+1)^2$ and ascertain the nature of this turning point. Calculate the area of the region enclosed by the curve, the y-axis and the x-axis between $x = 0$ and $x = 2$. (AEB 1978)

25. If $f(x) = \frac{x^2+x+2}{(x^2+1)(x-1)}$ express $f(x)$ in partial fractions and hence evaluate $\int_2^3 f(x)\,dx$ and $\int_{-3}^{-2} f(x)\,dx$ each correct to 2 places of decimals. (SU)

26. The region enclosed by the loop of the curve $y^2(3-x) = x^2(3+x)$ is rotated about the x-axis through four right angles. Show that the volume of the solid of revolution is $18\pi(3\ln 2 - 2)$. (C)

27. (i) Given that $0 < a < b$, sketch the graph of $y = |x-a|$ for $-b \leqslant x \leqslant b$. Hence, or otherwise, find $\int_{-b}^{b} |x-a|\,dx$.

(ii) Find the value of $\int_1^2 \log_{10} x\,dx$ leaving your answer in terms of logarithms.

(iii) Using the substitution $x = \frac{1}{2}\sin\theta$, or otherwise, evaluate $\int_0^{1/4} \sqrt{(1-4x^2)}\,dx$.

(L)

28. Sketch the curve given in terms of a parameter t by $x = 2a + at^2$, $y = 2at$ where a is a positive constant. Find the equation of the tangent and of the normal to the curve at the point where $t = 1$. Determine the area of the finite region bounded by the curve and the straight line joining the points at which $t = \pm 1$. (L)

29. A curve is defined parametrically by the equations $x = \cos t$, $y = \sin^3 t$, $-\pi \leqslant t < \pi$. Show that the curve is symmetrical about each of the coordinate axes. Find dy/dx in terms of t and deduce the equation of the tangent to the curve at the point with parameter t_1. Sketch the curve. Show that the area, A, of the region enclosed by the curve is given by $A = 4\int_0^{\pi/2} \sin^4 t\, dt$ and use the relation $2\sin^2\theta = 1 - \cos 2\theta$ to deduce that $A = 3\pi/4$. (L)

30. The curve whose equation is $y = (1 - x)e^x$ meets the x-axis at A and the y-axis at B. The region bounded by the arc AB of the curve and the line segments OA and OB, where O is the origin, is rotated through a complete revolution about the x-axis. Show that the volume swept out is $\frac{1}{4}\pi(e^2 - 5)$. (JMB)

31. Obtain $\int x^2 \cos 2x\, dx$ and hence prove that $\int_0^{\pi} x^2 \cos 2x\, dx = \frac{1}{2}\pi$. Find the area of the region enclosed by the curve $y = x \sin x$, for $0 \leqslant x \leqslant \pi$, and the x-axis. The region is rotated through 2π radians about the x-axis. Find the volume of the solid of revolution thus generated. From your results, or otherwise, find the volume of the solid of revolution generated by rotation of the same region through 2π radians about the line $y = -\frac{1}{8}$. (O&C)

32. Express $y = \dfrac{2x^2 + 3x + 5}{(x + 1)(x^2 + 3)}$ in partial fractions and hence show that $\dfrac{dy}{dx} = -\dfrac{2}{3}$ when $x = 0$. Evaluate $\int_0^{\sqrt{3}} y\, dx$ and state the mean value of y in the interval $0 \leqslant x \leqslant \sqrt{3}$. (AEB 1979)

33. Determine the values of p and q for which $x^2 - 4x + 5 \equiv (x - p)^2 + q$ and hence evaluate $\int_2^3 \dfrac{1}{x^2 - 4x + 5}\, dx$. Calculate also $\int_2^3 \dfrac{2x - 4}{x^2 - 4x + 5}\, dx$ and deduce, or find otherwise, the value of $\int_2^3 \dfrac{2x}{x^2 - 4x + 5}\, dx$. (JMB)

34. (a) Using the formula for $\cos(A + B)$, express $\cos\theta$ in terms of $t = \tan\frac{1}{2}\theta$.

(b) Express $\dfrac{10(1 + t^2)}{(1 - t^2)(1 + 9t^2)}$ in partial fractions.

(c) With the help of (a) and (b), and using the substitution $t = \tan\frac{1}{2}\theta$, show that $\int_0^{\pi/3} \dfrac{5\, d\theta}{5\cos\theta - 4\cos^2\theta} = \log_e(2 + \sqrt{3}) + \frac{8}{9}\pi$. (W)

35. (a) Using integration by parts, or otherwise, evaluate

$$\int \log_e x \, dx \quad \text{and} \quad \int (\log_e x)^2 \, dx.$$

(b) Sketch the graph of the function $\log_e x$. Let A and C denote the points $(1, 0)$ and $(e, 1)$ on the graph, and let B denote the point $(e, 0)$. The region bounded by the graph and the lines, AB, BC is rotated about the x-axis; find the volume of the solid so formed. Show that the volume of the solid formed by rotating the same region about the y-axis is $\frac{1}{2}\pi(e^2 + 1)$. (W)

36. Evaluate (i) $\displaystyle\int_0^{\pi/2} \frac{\sin^3\theta}{2 - \sin^2\theta} d\theta$, (ii) $\displaystyle\int_0^{\sqrt{2}} xe^{-x^2} dx$.

37. Let $I = \displaystyle\int_0^{\pi} \frac{x \, dx}{1 + \sin x}$. Show, by means of the substitution $y = \pi - x$ that $I = \pi \displaystyle\int_0^{\pi/2} \frac{dx}{1 + \sin x}$. Hence, by means of the substitution $t = \tan\frac{1}{2}x$, or otherwise, evaluate I. (JMB)

38. Draw the circle $x^2 + y^2 = a^2$ and, on the same diagram, the line $x = a/2$. Find by integration the smaller area between the line and the circumference of the circle and the volume generated when this area is rotated through 4 right-angles about the x-axis. (SU)

39. The area A in the first quadrant bounded by the curve $(2 - x)y^2 = x$, the x-axis and the line $x = 1$ is rotated completely about the x-axis to form a solid of volume V. Find: (i) the volume V, (ii) the area A. (AEB 1976)

40. Find the area of the region enclosed between the curves $y = x^2/2a$ and $y = a - x^4/2a^3$ $(a > 0)$. Find also the volume of the solid obtained by rotating this region through two right angles about the y-axis. (O)

41. Sketch the curve $y^2 = x^2(4 - x^2)$. Find the area enclosed by one loop of the curve. Find also the volume of the solid generated when this loop is rotated through two right angles about the x-axis.

42. The domain of the function defined by $f(x) = e^{-x}\sin x$ is the set of real numbers in the interval $0 \leqslant x \leqslant 4\pi$. Show that stationary values of this function occur for values of x for which $\tan x = 1$. Determine the nature of these stationary values. Sketch the graph of the function. By integrating by parts twice, or otherwise, evaluate $\displaystyle\int_0^{\pi} f(x) \, dx$. Show that the ratio of $f(a + 2\pi)$ to $f(a)$ is independent of the value of a, and hence write down the value of $\displaystyle\int_{2\pi}^{3\pi} f(x) \, dx$. (JMB)

43. By interpreting $\int_a^b f(x)\,dx$ as an area, show that

(i) $\int_0^{\pi} \sin^5 x \cos^5 x\,dx = 0$, (ii) $\int_0^{\pi} e^{2x} \cos x\,dx < 0$,

(iii) for $n > 1$, $\dfrac{1}{n+1}\left(\dfrac{\pi}{4}\right)^{n+1} < \int_0^{\pi/4} \tan^n x\,dx < \dfrac{\pi}{8}$. (JMB)

44. (a) By considering $\int x^{1/2}\,dx$ between suitable limits, or otherwise, prove that $\frac{2}{3}n^{3/2} < \sqrt{1} + \sqrt{2} + \cdots + \sqrt{n} < \frac{2}{3}\{(n+1)^{3/2} - 1\}$.

(b) By considering an appropriate integral, prove that

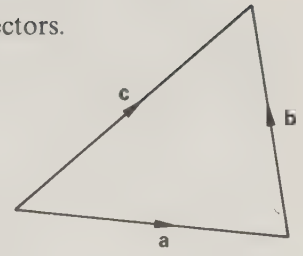
$$\frac{1}{k}\left\{1 - \frac{1}{(n+1)^k}\right\} < \sum_{r=1}^{n} \frac{1}{r^{k+1}} < \frac{1}{k}\left\{k + 1 - \frac{1}{n^k}\right\}, \text{ where } k > 0. \quad \text{(O\&C)}$$

21 Vectors and scalar product

21.1 Properties of vectors

We begin by reviewing some of the basic properties of vectors.

(1) The sum or resultant of two vectors may be found using the triangle law for vector addition



e.g. $\mathbf{a} + \mathbf{b} = \mathbf{c}$.

Vector addition is both commutative and associative

i.e. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$, $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$.

(2) For any scalar λ, the vector $\lambda\mathbf{a}$ is parallel to $\mathbf{a}$ and has magnitude $|\lambda|$ times the magnitude of $\mathbf{a}$.

It can be shown that $\lambda(\mu\mathbf{a}) = (\lambda\mu)\mathbf{a}$,

$(\lambda + \mu)\mathbf{a} = \lambda\mathbf{a} + \mu\mathbf{a}$, $\lambda(\mathbf{a} + \mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b}$.

(3) Given two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$, which are not parallel, any vector $\mathbf{r}$ in the plane of $\mathbf{a}$ and $\mathbf{b}$ can be expressed in the form $\lambda\mathbf{a} + \mu\mathbf{b}$. The vectors $\lambda\mathbf{a}$ and $\mu\mathbf{b}$ are the *component vectors* of $\mathbf{r}$ in the directions of *base vectors* $\mathbf{a}$ and $\mathbf{b}$.

(i) If vectors $\lambda\mathbf{a} + \mu\mathbf{b}$ and $l\mathbf{a} + m\mathbf{b}$ are equal, then $\lambda = l$ and $\mu = m$.

(ii) If vectors $\lambda\mathbf{a} + \mu\mathbf{b}$ and $l\mathbf{a} + m\mathbf{b}$ are parallel, then for some scalar t, $\lambda\mathbf{a} + \mu\mathbf{b} = t(l\mathbf{a} + m\mathbf{b})$ and thus $\dfrac{\lambda}{l} = \dfrac{\mu}{m} = t$.

(4) Given three non-zero vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, which are not coplanar, any vector $\mathbf{r}$ can be expressed in the form $\lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c}$.

(i) If vectors $\lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c}$ and $l\mathbf{a} + m\mathbf{b} + n\mathbf{c}$ are equal, then $\lambda = l$, $\mu = m$ and $\nu = n$.

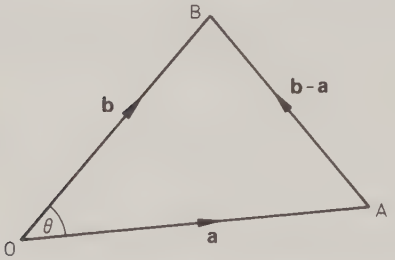
(ii) If vectors $\lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c}$ and $l\mathbf{a} + m\mathbf{b} + n\mathbf{c}$ are parallel, then for some scalar t, $\lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c} = t(l\mathbf{a} + m\mathbf{b} + n\mathbf{c})$ and thus $\dfrac{\lambda}{l} = \dfrac{\mu}{m} = \dfrac{\nu}{n} = t$.

(5) If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, where $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ are unit vectors in the directions of the x-, y- and z-axes respectively then,

(i) the magnitude of $\mathbf{r}$ is given by $r = \sqrt{(x^2 + y^2 + z^2)}$,

(ii) the unit vector in the direction of $\mathbf{r}$ is $\hat{\mathbf{r}} = \frac{x}{r}\mathbf{i} + \frac{y}{r}\mathbf{j} + \frac{z}{r}\mathbf{k}$.

(6)



If the position vectors of points A and B with respect to O are $\mathbf{a}$ and $\mathbf{b}$ respectively, then $\mathbf{OA} = \mathbf{a}$, $\mathbf{OB} = \mathbf{b}$ and $\mathbf{AB} = \mathbf{b} - \mathbf{a}$.

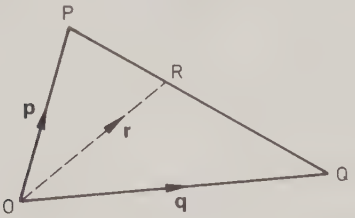
(i) The distance between A and B is given by $|\mathbf{b} - \mathbf{a}|$ (or $|\mathbf{a} - \mathbf{b}|$).

(ii) If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, then the cosine rule can be used to express $\cos\theta$ in terms of $|\mathbf{a}|, |\mathbf{b}|$ and $|\mathbf{a} - \mathbf{b}|$.

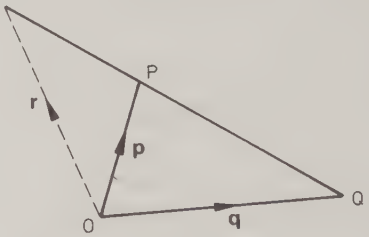
(iii) A point C, with position vector $\mathbf{c}$, lies on the straight line passing through A and B if and only if there is a scalar λ such that $\mathbf{AC} = \lambda\mathbf{AB}$ i.e. $\mathbf{c} - \mathbf{a} = \lambda(\mathbf{b} - \mathbf{a})$.

The fundamental ideas summarised in paragraphs (1) to (6) form the foundation for the work of this chapter.

(i)



(ii)



Let us consider three points P, Q and R such that R divides PQ in the ratio $\mu:\lambda$. We will observe the convention that (i) if λ and μ have the same sign then R lies between P and Q, and (ii) if λ and μ have opposite signs then R does not lie between P and Q.

In both cases $\quad \lambda\mathbf{PR} = \mu\mathbf{RQ}$

$\therefore \quad \lambda(\mathbf{OR} - \mathbf{OP}) = \mu(\mathbf{OQ} - \mathbf{OR})$

$\therefore \quad \lambda\mathbf{OR} - \lambda\mathbf{OP} = \mu\mathbf{OQ} - \mu\mathbf{OR}$

Hence $\quad (\lambda + \mu)\mathbf{OR} = \lambda\mathbf{OP} + \mu\mathbf{OQ}$

This result is called the *ratio theorem* and may also be stated as follows:

> The resultant of the vectors $\lambda\mathbf{OP}$ and $\mu\mathbf{OQ}$ is $(\lambda + \mu)\mathbf{OR}$, where R is the point which divides PQ in the ratio $\mu:\lambda$

If the position vectors of P, Q and R with respect to O are $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ respectively then

$$(\lambda + \mu)\mathbf{r} = \lambda\mathbf{p} + \mu\mathbf{q}.$$

Thus the point R which divides PQ in the ratio $\mu:\lambda$ has position vector

$$\mathbf{r} = \frac{\lambda\mathbf{p} + \mu\mathbf{q}}{\lambda + \mu}$$

In particular, if R is the *mid-point* of PQ, then $\mathbf{r} = \frac{1}{2}(\mathbf{p} + \mathbf{q})$.

Example 1 The points A and B have position vectors $\mathbf{a} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ respectively. Find the position vectors of the point P which divides AB internally in the ratio 1:4 and the point Q which divides AB externally in the ratio 3:2.

Observing the usual sign convention, P divides AB in the ratio 1:4 and Q divides AB in the ratio $3:-2$.
Thus the position vector of P

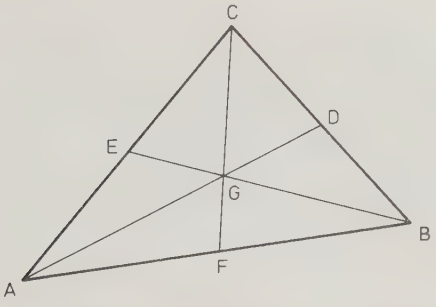
$$= \frac{4\mathbf{a} + \mathbf{b}}{4 + 1} = \frac{4(\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) + (6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})}{5} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}.$$

Similarly, the position vector of Q

$$= \frac{-2\mathbf{a} + 3\mathbf{b}}{-2 + 3} = -2(\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) + 3(6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) = 16\mathbf{i} - 12\mathbf{j} + 13\mathbf{k}.$$

[Note that the same result will be obtained for the position vector of Q if the ratio $-3:2$ is used instead of the ratio $3:-2$.]

Example 2 Prove that the medians of a triangle meet at a point which divides each median in the ratio 2:1.



In $\triangle ABC$ let D, E and F be the mid-points of BC, CA and AB respectively. If the position vectors of A, B and C are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, then the position vectors of D, E and F are $\frac{1}{2}(\mathbf{b} + \mathbf{c}), \frac{1}{2}(\mathbf{c} + \mathbf{a})$ and $\frac{1}{2}(\mathbf{a} + \mathbf{b})$ respectively.

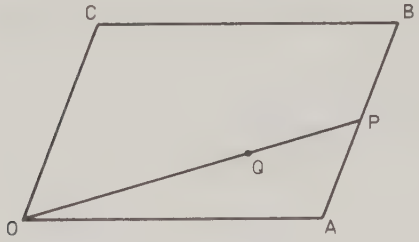
Let G be the point which divides AD in the ratio 2:1, then its position vector is $\dfrac{1 \times \mathbf{a} + 2 \times \frac{1}{2}(\mathbf{b} + \mathbf{c})}{1 + 2}$ i.e. $\frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$.
Since this expression is symmetrical in $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, G must also be the point which divides BE and CF in the ratio 2:1. Hence the medians meet at a point which divides each median in the ratio 2:1.

[As stated in §8.1 the point G with position vector $\frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$ is the centroid of $\triangle ABC$.]

Let us consider again the points P, Q and R with position vectors $\mathbf{p}, \mathbf{q}$ and $\mathbf{r}$ respectively. It follows from the ratio theorem that if $(\lambda + \mu)\mathbf{r} = \lambda\mathbf{p} + \mu\mathbf{q}$, then R is the point which divides PQ in the ratio $\mu:\lambda$. If $\lambda + \mu = 1$, we obtain the following useful result.

> If $\mathbf{r} = \lambda\mathbf{p} + \mu\mathbf{q}$ for scalars λ and μ such that $\lambda + \mu = 1$, then R is the point which divides PQ in the ratio $\mu:\lambda$.

Example 3 In a parallelogram $OABC$, P is the mid-point of AB and Q divides OP in the ratio 2:1. Show that Q lies on AC and find the ratio $AQ:QC$.



Let the position vectors of A, B, C, P and Q with respect to O be $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{p}$ and $\mathbf{q}$ respectively.
P is the mid-point of AB

$$\therefore \quad \mathbf{p} = \tfrac{1}{2}(\mathbf{a} + \mathbf{b})$$

Q divides OP in the ratio 2:1

$$\therefore \quad \mathbf{q} = \tfrac{2}{3}\mathbf{p} = \tfrac{2}{3} \times \tfrac{1}{2}(\mathbf{a} + \mathbf{b}) = \tfrac{1}{3}\mathbf{a} + \tfrac{1}{3}\mathbf{b}$$

However, since $OABC$ is a parallelogram, $\mathbf{b} = \mathbf{a} + \mathbf{c}$,

$$\therefore \quad \mathbf{q} = \tfrac{1}{3}\mathbf{a} + \tfrac{1}{3}(\mathbf{a} + \mathbf{c}) = \tfrac{2}{3}\mathbf{a} + \tfrac{1}{3}\mathbf{c}$$

Hence, by the ratio theorem, Q is the point on AC such that $AQ:QC = 1:2$.

Exercise 21.1

1. The position vectors of the vertices A, B and C of parallelogram $ABCD$ are $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ respectively. Find, in terms of $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$, the position vector of D.

2. The position vectors of the vertices A, B and C of trapezium $ABCD$ are $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ respectively. Given that the sides BC and AD are parallel and that $AD = 2BC$, find the position vector of D in terms of $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$.

3. Find the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$ given that $|\mathbf{a}| = 3$, $|\mathbf{b}| = 8$ and $|\mathbf{a} - \mathbf{b}| = 7$.

4. The angle between the vectors $\mathbf{a}$ and $\mathbf{b}$ is 120°. If $|\mathbf{a}| = 5$ and $|\mathbf{b}| = 3$, find $|\mathbf{a} - \mathbf{b}|$ and $|\mathbf{a} + \mathbf{b}|$.

5. The points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively relative to an origin O, where $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = -4\mathbf{i} + s\mathbf{j} + t\mathbf{k}$. Find the possible values of s and t if (a) the points O, A and B are collinear, (b) $|\mathbf{AB}| = 7$ and $s = 2t$.

6. Given that $\mathbf{a} = 2\mathbf{i} + 6\mathbf{j}$ and $\mathbf{b} = 3\mathbf{i} + t\mathbf{j}$, find the value of t such that (a) $\mathbf{a}$ and $\mathbf{b}$ are parallel, (b) $\mathbf{a}$ and $\mathbf{b}$ are at right angles.

7. If the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, find in terms of $\mathbf{a}$ and $\mathbf{b}$ the position vectors of (a) the point P which divides AB internally in the ratio 5:3, (b) the point Q which divides AB externally in the ratio 3:1, (c) the point R such that A is the mid-point of BR.

8. The points A and B have position vectors $\mathbf{a} = 5\mathbf{i} + 4\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = -\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ respectively. Find the position vectors of (a) the point C on AB such that $AC = 2CB$, (b) the point D on AB produced such that $AD = 2BD$.

9. Write down, in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, the position vectors of the points $A(2, -5, 3)$ and $B(7, 0, -2)$. Hence find the coordinates of the point C which divides AB internally in the ratio 2:3 and the point D which divides AB externally in the ratio 3:8.

10. The vertices A, B and C of a triangle have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. Points P, Q and R have position vectors given respectively by $\mathbf{p} = \mathbf{a} - \mathbf{b} + \mathbf{c}$, $\mathbf{q} = \frac{3}{7}\mathbf{a} + \frac{4}{7}\mathbf{c}$, $\mathbf{r} = 3\mathbf{b} - 2\mathbf{c}$. Describe geometrically the positions of P, Q and R relative to A, B and C.

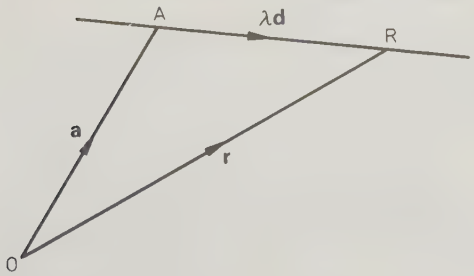
11. The diagonals AC and BD of a quadrilateral $ABCD$ intersect at a point P such that $AP:PC = 2:7$ and $BP:PD = 5:4$. Show that

$$\frac{7\mathbf{a} - 4\mathbf{b}}{7 - 4} = \frac{5\mathbf{d} - 2\mathbf{c}}{5 - 2}$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are the position vectors of A, B, C and D respectively. Explain the geometrical significance of this result.

12. The vertices A, B and C of a triangle have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. The point P divides BC internally in the ratio 3:1, the point Q is the mid-point of CA and the point R divides AB externally in the ratio 1:3. If the position vectors of P, Q and R are $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ respectively, express $\mathbf{r}$ in terms of $\mathbf{p}$ and $\mathbf{q}$. Hence show that R lies on PQ produced and find the ratio $PR:QR$.

21.2 Equations of lines and planes



The diagram shows a straight line which passes through a fixed point A with position vector $\mathbf{a}$ and which is parallel to a given vector $\mathbf{d}$. Let $\mathbf{r}$ be the position vector of a point R on the line.

Since **AR** is parallel to **d**, then, for some scalar λ,

$$\mathbf{AR} = \lambda\mathbf{d}$$
$$\therefore \quad \mathbf{r} - \mathbf{a} = \lambda\mathbf{d}$$
$$\text{i.e.} \quad \mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$$

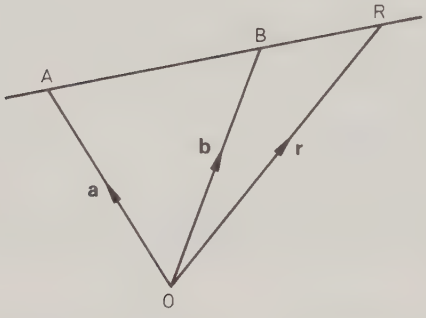
Thus every point on the line has a position vector of the form $\mathbf{a} + \lambda\mathbf{d}$. Moreover, each real value of the parameter λ corresponds to some point on the line.

The equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$ is called the *vector equation* of the line through A parallel to **d**. The vector **d** is a *direction vector* of the line.

Example 1 Find the vector equation of the line which passes through the point with position vector $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and which is parallel to the vector $2\mathbf{j} - \mathbf{k}$.

The vector equation of the given line is

$$\mathbf{r} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k} + \lambda(2\mathbf{j} - \mathbf{k})$$
$$\text{i.e.} \quad \mathbf{r} = 2\mathbf{i} + (2\lambda - 1)\mathbf{j} + (3 - \lambda)\mathbf{k}.$$



Consider now the line through two points A and B with position vectors **a** and **b** respectively. If a point R, with position vector **r**, lies on this line, then for some scalar λ,

$$\mathbf{AR} = \lambda\mathbf{AB}$$
$$\therefore \quad \mathbf{r} - \mathbf{a} = \lambda(\mathbf{b} - \mathbf{a})$$
$$\therefore \quad \mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}).$$

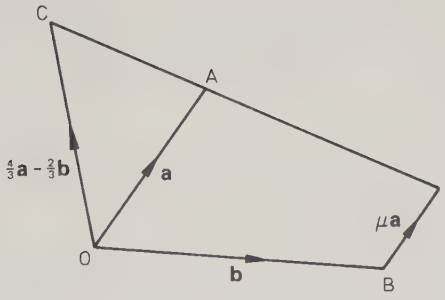
Hence the vector equation of the line through A and B is $\mathbf{r} = (1 - \lambda)\mathbf{a} + \lambda\mathbf{b}$.

Example 2 Find the vector equation of the straight line which passes through the points $A(1, 0, -2)$ and $B(2, 3, -1)$.

The position vectors of A and B are $\mathbf{i} - 2\mathbf{k}$ and $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ respectively.
Hence the vector equation of the line through A and B is

$$\mathbf{r} = (1 - \lambda)(\mathbf{i} - 2\mathbf{k}) + \lambda(2\mathbf{i} + 3\mathbf{j} - \mathbf{k})$$
$$\text{i.e.} \quad \mathbf{r} = (\lambda + 1)\mathbf{i} + 3\lambda\mathbf{j} + (\lambda - 2)\mathbf{k}.$$

Example 3 Three non-collinear points A, B and C have position vectors **a**, **b** and $\frac{4}{3}\mathbf{a} - \frac{2}{3}\mathbf{b}$ respectively, relative to an origin O. Find the position vector of the point D on CA produced, such that BD is parallel to OA.



The vector equation of the line through A and C is

$$\mathbf{r} = (1 - \lambda)\mathbf{a} + \lambda(\tfrac{4}{3}\mathbf{a} - \tfrac{2}{3}\mathbf{b})$$

i.e. $\mathbf{r} = (1 + \frac{1}{3}\lambda)\mathbf{a} - \frac{2}{3}\lambda\mathbf{b}$.

The vector equation of the line through B parallel to OA is

$$\mathbf{r} = \mathbf{b} + \mu\mathbf{a}$$

At the point D where these lines intersect

$$(1 + \tfrac{1}{3}\lambda)\mathbf{a} - \tfrac{2}{3}\lambda\mathbf{b} = \mu\mathbf{a} + \mathbf{b}$$

Since the vectors $\mathbf{a}$ and $\mathbf{b}$ are not parallel, it follows that

$$1 + \tfrac{1}{3}\lambda = \mu \quad \text{and} \quad -\tfrac{2}{3}\lambda = 1$$
$$\therefore \quad \lambda = -\tfrac{3}{2}, \quad \mu = \tfrac{1}{2}$$

Hence the position vector of D is $\frac{1}{2}\mathbf{a} + \mathbf{b}$.

In three dimensional space a pair of vector equations such as $\mathbf{r} = \mathbf{a} + \lambda\mathbf{p}$ and $\mathbf{r} = \mathbf{b} + \mu\mathbf{q}$ may represent

(i) the same line, (ii) parallel lines,
(iii) intersecting lines, (iv) skew lines.

Example 4 Show that the lines l_1 and l_2, with vector equations $\mathbf{r} = \mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} - 3\mathbf{k})$ and $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + \mu(3\mathbf{j} + 5\mathbf{k})$ respectively, intersect, and find the position vector of their point of intersection.

The equations of l_1 and l_2 may be written:

$$\mathbf{r} = \lambda\mathbf{i} - \lambda\mathbf{j} + (1 - 3\lambda)\mathbf{k}$$
$$\mathbf{r} = 2\mathbf{i} + (1 + 3\mu)\mathbf{j} + 5\mu\mathbf{k}$$

At any point of intersection

$$\lambda\mathbf{i} - \lambda\mathbf{j} + (1 - 3\lambda)\mathbf{k} = 2\mathbf{i} + (1 + 3\mu)\mathbf{j} + 5\mu\mathbf{k}$$

Thus

$$\lambda = 2 \tag{1}$$
$$-\lambda = 1 + 3\mu \tag{2}$$
$$1 - 3\lambda = 5\mu \tag{3}$$

From (1) and (2), $\lambda = 2, \mu = -1$.

Since these values also satisfy equation (3), the lines do have a point of intersection. Its position vector, given by substituting $\lambda = 2$ in the equation of l_1, is $2\mathbf{i} - 2\mathbf{j} - 5\mathbf{k}$.

[Note that we can check this result by substituting $\mu = -1$ in the equation of l_2.].

Example 5 Show that the lines l_1 and l_2, with vector equations

$$\mathbf{r} = \lambda\mathbf{i} + (3 - 2\lambda)\mathbf{j} + (2 + \lambda)\mathbf{k} \quad \text{and} \quad \mathbf{r} = (1 - 2\mu)\mathbf{i} + 4\mu\mathbf{j} + (1 - 2\mu)\mathbf{k}$$

respectively, are distinct parallel lines.

The equations of l_1 and l_2 may be written:

$$\mathbf{r} = 3\mathbf{j} + 2\mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$$
$$\mathbf{r} = \mathbf{i} + \mathbf{k} + \mu(-2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$$

Thus l_1 passes through a point A with position vector $\mathbf{a} = 3\mathbf{j} + 2\mathbf{k}$ and is parallel to the vector $\mathbf{p} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$.
Similarly l_2 passes through a point B with position vector $\mathbf{b} = \mathbf{i} + \mathbf{k}$ and is parallel to the vector $\mathbf{q} = -2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.
Since $\mathbf{q} = -2\mathbf{p}$, the vectors $\mathbf{p}$ and $\mathbf{q}$ are parallel.
Hence the lines l_1 and l_2 are parallel or coincident.
Since $\mathbf{AB} = \mathbf{b} - \mathbf{a} = \mathbf{i} - 3\mathbf{j} - \mathbf{k}$, which is not a scalar multiple of $\mathbf{p}$, the line through A and B is not parallel to $\mathbf{p}$.
Hence the lines l_1 and l_2 must be distinct parallel lines.

Example 6 Show that the lines with vector equations $\mathbf{r} = 2\lambda\mathbf{i} - 3\mathbf{j} + (\lambda - 2)\mathbf{k}$ and $\mathbf{r} = (\mu + 1)\mathbf{i} + (2 - \mu)\mathbf{j} + (2\mu - 5)\mathbf{k}$ do not intersect.

At any point of intersection

$$2\lambda\mathbf{i} - 3\mathbf{j} + (\lambda - 2)\mathbf{k} = (\mu + 1)\mathbf{i} + (2 - \mu)\mathbf{j} + (2\mu - 5)\mathbf{k}$$

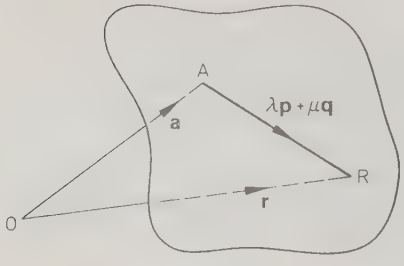
Thus
$$2\lambda = \mu + 1 \tag{1}$$
$$-3 = 2 - \mu \tag{2}$$
$$\lambda - 2 = 2\mu - 5 \tag{3}$$

From (2), $\mu = 5$
Using (1), $2\lambda = 5 + 1 \quad \therefore \quad \lambda = 3$

Testing these values in (3):

$$\lambda - 2 = 1, \quad 2\mu - 5 = 5$$

$\therefore$ no values of λ and μ can be found which satisfy all three equations.
Hence the given lines do not intersect.



Consider now a plane which passes through a point A with position vector $\mathbf{a}$ and which contains non-parallel vectors $\mathbf{p}$ and $\mathbf{q}$. Let $\mathbf{r}$ be the position vector of a point R in the plane.

Since **AR** lies in the plane, there are scalars λ and μ such that

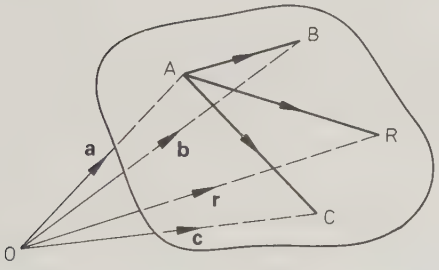
$$\mathbf{AR} = \lambda\mathbf{p} + \mu\mathbf{q}$$
$$\therefore \quad \mathbf{r} - \mathbf{a} = \lambda\mathbf{p} + \mu\mathbf{q}$$
$$\text{i.e.} \quad \mathbf{r} = \mathbf{a} + \lambda\mathbf{p} + \mu\mathbf{q}.$$

This equation is called the *vector equation* of the plane in *parametric form.*

Example 7 Write down the vector equation of the plane through the point with position vector $\mathbf{i} - 2\mathbf{k}$ which is parallel to the vectors $\mathbf{i} + \mathbf{j}$ and $\mathbf{k}$.

The vector equation of the given plane is

$$\mathbf{r} = \mathbf{i} - 2\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j}) + \mu\mathbf{k}$$
$$\text{i.e.} \quad \mathbf{r} = (\lambda + 1)\mathbf{i} + \lambda\mathbf{j} + (\mu - 2)\mathbf{k}.$$



Suppose that **r** is the position vector of a point R which lies in the plane containing three non-collinear points A, B and C with position vectors **a**, **b** and **c** respectively.

Since **AR** lies in the plane containing **AB** and **AC**, there are scalars λ and μ such that

$$\mathbf{AR} = \lambda\mathbf{AB} + \mu\mathbf{AC}$$
$$\therefore \quad \mathbf{r} - \mathbf{a} = \lambda(\mathbf{b} - \mathbf{a}) + \mu(\mathbf{c} - \mathbf{a})$$
$$\text{i.e.} \quad \mathbf{r} = (1 - \lambda - \mu)\mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$$

This is the vector equation of the plane through given points A, B and C.

Example 8 Find a vector equation of the plane through the points $A(1, 0, 0)$, $B(2, -6, 1)$ and $C(-3, 0, 4)$. Hence find the coordinates of the point of intersection of the plane ABC and the line with vector equation $\mathbf{r} = 2t\mathbf{i} + (1 - 5t)\mathbf{j} + (t - 2)\mathbf{k}$.

The points A, B and C have position vectors $\mathbf{i}, 2\mathbf{i} - 6\mathbf{j} + \mathbf{k}$ and $-3\mathbf{i} + 4\mathbf{k}$ respectively.

$\therefore$ the vector equation of the plane ABC is

$$\mathbf{r} = (1 - \lambda - \mu)\mathbf{i} + \lambda(2\mathbf{i} - 6\mathbf{j} + \mathbf{k}) + \mu(-3\mathbf{i} + 4\mathbf{k})$$
$$\text{i.e.} \quad \mathbf{r} = (1 + \lambda - 4\mu)\mathbf{i} - 6\lambda\mathbf{j} + (\lambda + 4\mu)\mathbf{k}.$$

The point of intersection of this plane with the line $\mathbf{r} = 2t\mathbf{i} + (1 - 5t)\mathbf{j} + (t - 2)\mathbf{k}$ is given by

$$1 + \lambda - 4\mu = 2t \qquad (1)$$
$$-6\lambda \qquad = -5t + 1 \qquad (2)$$
$$\lambda + 4\mu = t - 2 \qquad (3)$$

Adding (1) to (3), $1 + 2\lambda = 3t - 2$
$\therefore \quad 2\lambda = 3t - 3$ (4)
Substituting in (2), $-3(3t - 3) = -5t + 1$
i.e. $-9t + 9 = -5t + 1 \quad \therefore \quad t = 2$

Substituting in the equation of the line, $\mathbf{r} = 4\mathbf{i} - 9\mathbf{j}$.
Hence the coordinates of the point of intersection are $(4, -9, 0)$.

Exercise 21.2

1. Find the vector equation of the line with direction vector **d** which passes through the point with position vector **a** given that
(a) $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}, \mathbf{d} = 3\mathbf{i} - \mathbf{k}$,
(b) $\mathbf{a} = 4\mathbf{i} - 3\mathbf{k}, \mathbf{d} = \mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$.

2. Find the vector equation of the line which passes through the points with
(a) position vectors $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $-2\mathbf{i} + \mathbf{j} + \mathbf{k}$,
(b) position vectors $\mathbf{i} + 5\mathbf{j}$ and $3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$,
(c) coordinates $(0, 6, -8)$ and $(5, -7, 2)$,
(d) coordinates $(0, 0, 0)$ and $(5, -2, 3)$.

3. The vertices A, B and C of a triangle have position vectors **a**, **b** and **c** respectively. The point P divides BC internally in the ratio 2:1 and the point Q divides AC externally in the ratio 2:1. Find vector equations for the lines AP and BQ. Hence find the position vector of the point of intersection of the lines AP and BQ.

4. The position vectors of points A and B with respect to an origin O are **a** and **b** respectively. P is the mid-point of OA and $APBQ$ is a parallelogram. The lines AB and OQ intersect at T. Find the position vectors of P, Q and T.

5. The position vectors relative to an origin O of the points A and B are **a** and **b** respectively. The point P on AB is such that $3AP = PB$. The point Q on OP produced is such that AQ is parallel to OB. Find the position vector of the point of intersection of the lines OA and BQ.

6. The position vectors of four points A, B, C, D are $\mathbf{a} = 5\mathbf{j}, \mathbf{b} = -6\mathbf{i} + \mathbf{j}, \mathbf{c} = 6\mathbf{i} - \mathbf{j}$, $\mathbf{d} = -3\mathbf{i} + 5\mathbf{j}$ respectively. The lines AB and CD intersect at a point X. Find the ratio $AX:XB$.

7. Decide whether the following pairs of straight lines are intersecting, parallel or skew. If the lines intersect give the position vector of their point of intersection.
(a) $\mathbf{r} = (1 - \lambda)\mathbf{i} + 3\lambda\mathbf{j} + (2\lambda + 5)\mathbf{k}$,
$\mathbf{r} = 2\mu\mathbf{i} + (\mu - 4)\mathbf{j} + 3\mathbf{k}$.
(b) $\mathbf{r} = -7\mathbf{i} + 4\mathbf{j} + 3\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{k})$,
$\mathbf{r} = 5\mathbf{i} + 8\mathbf{j} + \mathbf{k} + \mu(-4\mathbf{i} + 3\mathbf{k})$.
(c) $\mathbf{r} = 2\lambda\mathbf{i} + 3\mathbf{j} - (2\lambda + 5)\mathbf{k}$,
$\mathbf{r} = (2 - 3\mu)\mathbf{i} - 4\mathbf{j} + 3\mu\mathbf{k}$.
(d) $\mathbf{r} = 2\lambda\mathbf{i} + (4\lambda + 1)\mathbf{j} + (5 - 3\lambda)\mathbf{k}$,
$\mathbf{r} = (\mu + 4)\mathbf{i} + (5\mu + 3)\mathbf{j} + (\mu - 6)\mathbf{k}$.

8. Decide which of the points A, B and C, with position vectors $\mathbf{a} = \mathbf{i} - 2\mathbf{j}$, $\mathbf{b} = 3\mathbf{i} - \mathbf{j} - \mathbf{k}$ and $\mathbf{c} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$ respectively, lie on the following lines and planes.
(a) $\mathbf{r} = (2\lambda + 5)\mathbf{i} + \lambda\mathbf{j} - (\lambda + 2)\mathbf{k}$,
(b) $\mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$,
(c) $\mathbf{r} = (2\lambda - \mu - 2)\mathbf{i} + (1 - \lambda + 2\mu)\mathbf{j} + \lambda\mathbf{k}$,
(d) $\mathbf{r} = -3\mathbf{i} - \mathbf{j} + 4\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} - \mathbf{k}) + \mu(3\mathbf{j} + 2\mathbf{k})$.

9. Write down in parametric form the vector equations of the planes through the given points parallel to the given pairs of vectors.
(a) $(1, -2, 0)$; $\mathbf{i} + 3\mathbf{j}$ and $-\mathbf{j} + 2\mathbf{k}$,
(b) the origin; $2\mathbf{i} - \mathbf{j}$ and $-\mathbf{i} + 2\mathbf{j} - 7\mathbf{k}$,
(c) $(3, 1, -1)$; $\mathbf{j}$ and $\mathbf{i} + \mathbf{j} + \mathbf{k}$.

10. Find a vector equation for the plane passing through the points with position vectors $2\mathbf{k}, \mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $5\mathbf{i} + 2\mathbf{j}$.

11. Write down direction vectors for the lines l_1 and l_2 with equations $\mathbf{r} = (2 - \lambda)\mathbf{i} + 5\mathbf{j} + 3\lambda\mathbf{k}$ and $\mathbf{r} = 4\mathbf{i} + 5\mu\mathbf{j} - 2\mathbf{k}$ respectively. By writing down the position vector of a particular point on l_1, find a vector equation for the plane which contains l_1 and is parallel to l_2.

12. Find a vector equation for the plane through the points $A(1, 0, -2)$ and $B(3, -1, 1)$ which is parallel to the line with vector equation

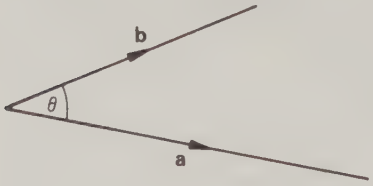
$$\mathbf{r} = 3\mathbf{i} + (2\lambda - 1)\mathbf{j} + (5 - \lambda)\mathbf{k}.$$

Hence find the coordinates of the point of intersection of this plane and the line $\mathbf{r} = \mu\mathbf{i} + (5 - \mu)\mathbf{j} + (2\mu - 7)\mathbf{k}$.

13. The position vectors of three non-collinear points A, B and C, with respect to an origin O, are $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ respectively. Given that O does not lie in the plane ABC, show that
(a) if the point P with position vector $\mathbf{p} = \lambda\mathbf{a} + \mu\mathbf{b}$ lies on the line AB, then $\lambda + \mu = 1$;
(b) if the point Q with position vector $\mathbf{q} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$ lies in the plane ABC, then $\alpha + \beta + \gamma = 1$.

14. OAD, OBE, OCF are three distinct straight lines. The position vectors with respect to O of the points A, B, C, D, E, F are $\mathbf{a}, \mathbf{b}, \mathbf{c}, p\mathbf{a}, q\mathbf{b}, r\mathbf{c}$ respectively, where p, q, r are scalars, no two of which are equal. Show that any point on BC has position vector given parametrically by $(1 - t)\mathbf{b} + t\mathbf{c}$. The intersections of the pairs of lines BC, EF; CA, FD; AB, DE (produced if necessary) are the points L, M, N respectively. Show that the position vector of L is $[q(1 - r)\mathbf{b} + r(q - 1)\mathbf{c}]/(q - r)$ and find the position vectors of M and N. Taking the values $p = 4, q = 3, r = 2$, show that L, M, N are collinear points. (C)

21.3 Scalar product



The *scalar product* (or dot product) of two vectors **a** and **b** is defined as the scalar $ab\cos\theta$, where θ is the angle between **a** and **b**.

The scalar product is denoted by **a . b**, which is read "**a** dot **b**".

Thus $$\mathbf{a}.\mathbf{b} = ab\cos\theta$$

The scalar product **a . a** is sometimes written as $\mathbf{a}^2$.

[Note that although a product of three real numbers a, b, c can be evaluated by writing $abc = (ab)c$, it is not possible to define a scalar product of three vectors in a similar way. Since **a . b** is a scalar, the expressions **(a . b) . c** and **a . b . c** are meaningless. Similarly, since **a . a** is a scalar, the expression $\mathbf{a}^3$ cannot be defined as **(a . a) . a**.]

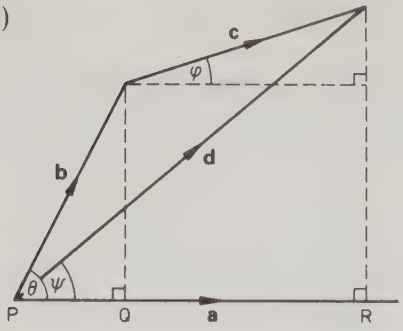
The commutative, associative and distributive properties of multiplication of real numbers can be used to establish various properties of the scalar product.

(1) $\mathbf{a}.\mathbf{b} = ab\cos\theta = ba\cos\theta = \mathbf{b}.\mathbf{a}$
Hence the scalar product is *commutative*.

(2) Since we cannot form products such as **(a . b) . c** there is no true associative law for the scalar product. However, for any scalar λ,

$$\lambda(\mathbf{a}.\mathbf{b}) = (\lambda\mathbf{a}).\mathbf{b} = \mathbf{a}.(\lambda\mathbf{b}) = \lambda ab\cos\theta.$$

(3)



For vectors **a**, **b**, **c** and **d** shown in the diagram,

$$\mathbf{a}.(\mathbf{b}+\mathbf{c}) = \mathbf{a}.\mathbf{d}$$

Since $PR = PQ + QR$

$$d\cos\psi = b\cos\theta + c\cos\phi$$

$$\therefore \quad ad\cos\psi = ab\cos\theta + ac\cos\phi$$

$$\therefore \quad \mathbf{a}.\mathbf{d} = \mathbf{a}.\mathbf{b} + \mathbf{a}.\mathbf{c}$$

$$\therefore \quad \mathbf{a}.(\mathbf{b}+\mathbf{c}) = \mathbf{a}.\mathbf{b} + \mathbf{a}.\mathbf{c}$$

Hence the scalar product is *distributive* over vector addition.

The definition of the scalar product gives rise to further useful properties.

(4) For any vector **a**, $\mathbf{a}.\mathbf{a} = a^2\cos 0^\circ = a^2$.

(5) If non-zero vectors **a** and **b** are perpendicular, then

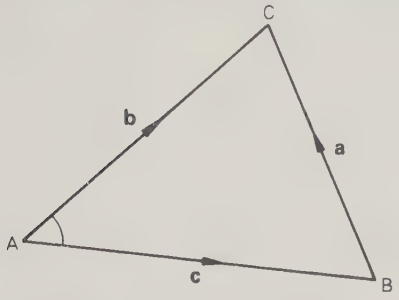
$$\mathbf{a}.\mathbf{b} = ab\cos 90^\circ = 0.$$

Conversely, if non-zero vectors **a** and **b** are such that $\mathbf{a}.\mathbf{b} = 0$, then **a** is perpendicular to **b**.

These results can be summarised as follows:
(1) $\mathbf{a}.\mathbf{b} = \mathbf{b}.\mathbf{a}$, (2) $\lambda(\mathbf{a}.\mathbf{b}) = (\lambda\mathbf{a}).\mathbf{b} = \mathbf{a}.(\lambda\mathbf{b})$,
(3) $\mathbf{a}.(\mathbf{b}+\mathbf{c}) = \mathbf{a}.\mathbf{b} + \mathbf{a}.\mathbf{c}$, (4) $\mathbf{a}.\mathbf{a} = a^2$.
(5) $\mathbf{a}.\mathbf{b} = 0 \Leftrightarrow \mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$ or $\mathbf{a}$ is perpendicular to $\mathbf{b}$.

The scalar product has important applications in geometrical proofs.

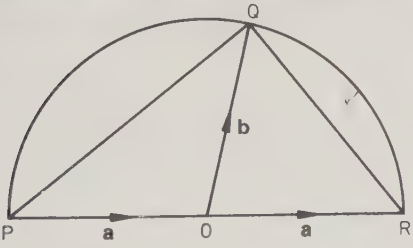
Example 1 Prove the cosine rule in the form
$$a^2 = b^2 + c^2 - 2bc\cos A.$$



In $\triangle ABC$, let $\mathbf{a} = \vec{BC}$, $\mathbf{b} = \vec{AC}$ and $\mathbf{c} = \vec{AB}$, then

$$\begin{aligned} a^2 &= \mathbf{a}.\mathbf{a} = (\mathbf{b}-\mathbf{c}).(\mathbf{b}-\mathbf{c}) \\ &= \mathbf{b}.(\mathbf{b}-\mathbf{c}) - \mathbf{c}.(\mathbf{b}-\mathbf{c}) \\ &= \mathbf{b}.\mathbf{b} - \mathbf{b}.\mathbf{c} - \mathbf{c}.\mathbf{b} + \mathbf{c}.\mathbf{c} \\ &= b^2 + c^2 - 2\mathbf{b}.\mathbf{c} \\ \therefore\quad a^2 &= b^2 + c^2 - 2bc\cos A. \end{aligned}$$

Example 2 Show that any angle inscribed in a semi-circle is a right angle.



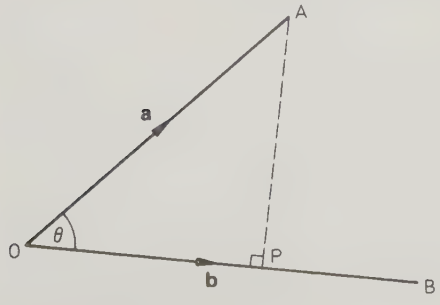
In the diagram, PR is a diameter of a circle with centre O. Let $\vec{PO} = \vec{OR} = \mathbf{a}$ and let Q be a point on the circle such that $\vec{OQ} = \mathbf{b}$, then $\vec{PQ} = \mathbf{a} + \mathbf{b}$, $\vec{QR} = \mathbf{a} - \mathbf{b}$.

$$\begin{aligned} \therefore\quad \vec{PQ}.\vec{QR} &= (\mathbf{a}+\mathbf{b}).(\mathbf{a}-\mathbf{b}) \\ &= \mathbf{a}.\mathbf{a} - \mathbf{a}.\mathbf{b} + \mathbf{b}.\mathbf{a} - \mathbf{b}.\mathbf{b} = a^2 - b^2 \end{aligned}$$

Since PO and OQ are radii of the circle, $a = b$,

$$\therefore\quad \vec{PQ}.\vec{QR} = 0$$

Hence $\vec{PQ}$ is perpendicular to $\vec{QR}$ i.e. $\angle PQR = 90°$.

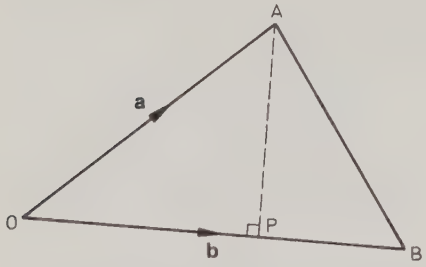


The diagram shows two vectors $\mathbf{a}$ and $\mathbf{b}$ represented by line segments $\vec{OA}$ and $\vec{OB}$ respectively. If the vector $\mathbf{a}$ is resolved along and perpendicular to $\mathbf{b}$, then its component in the direction of $\mathbf{b}$ is $a\cos\theta$.

Thus the component of **a** in the direction of **b** is $\dfrac{\mathbf{a}.\mathbf{b}}{b}$ or $\mathbf{a}.\hat{\mathbf{b}}$, where $\hat{\mathbf{b}}$ is the unit vector in the direction of **b**.

In geometrical work the quantity $\mathbf{a}.\hat{\mathbf{b}}$ is called the *projection* of **a** on **b**. In the diagram the length of this projection is represented by the distance OP.

Example 3 The points A and B have position vectors **a** and **b** with respect to an origin O. Show that the area of triangle OAB is given by $\frac{1}{2}\sqrt{\{a^2b^2 - (\mathbf{a}.\mathbf{b})^2\}}$.



The projection of **a** on **b** is

$$(\mathbf{a}.\mathbf{b})/b,$$

$\therefore$ if P is the foot of the perpendicular from A to OB,

$$OP^2 = (\mathbf{a}.\mathbf{b})^2/b^2$$

Using Pythagoras' theorem in $\triangle OAP$, $OP^2 + AP^2 = OA^2$

$$\therefore \quad AP^2 = OA^2 - OP^2 = a^2 - \frac{(\mathbf{a}.\mathbf{b})^2}{b^2}$$

Hence the area of $\triangle OAB = \frac{1}{2} \times OB \times AP$

$$= \tfrac{1}{2}\sqrt{\left\{b^2 \times \left(a^2 - \frac{(\mathbf{a}.\mathbf{b})^2}{b^2}\right)\right\}}$$
$$= \tfrac{1}{2}\sqrt{\{a^2b^2 - (\mathbf{a}.\mathbf{b})^2\}}.$$

When considering the unit vectors **i**, **j** and **k** the definition of scalar product leads to the results

$$\mathbf{i}.\mathbf{i} = \mathbf{j}.\mathbf{j} = \mathbf{k}.\mathbf{k} = 1, \quad \mathbf{i}.\mathbf{j} = \mathbf{j}.\mathbf{k} = \mathbf{k}.\mathbf{i} = 0$$

Thus for vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$,

$$\begin{aligned}\mathbf{a}.\mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}).(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\ &= a_1b_1\mathbf{i}.\mathbf{i} + a_1b_2\mathbf{i}.\mathbf{j} + a_1b_3\mathbf{i}.\mathbf{k} \\ &\quad + a_2b_1\mathbf{j}.\mathbf{i} + a_2b_2\mathbf{j}.\mathbf{j} + a_2b_3\mathbf{j}.\mathbf{k} \\ &\quad + a_3b_1\mathbf{k}.\mathbf{i} + a_3b_2\mathbf{k}.\mathbf{j} + a_3b_3\mathbf{k}.\mathbf{k}\end{aligned}$$

$\therefore$

$$\boxed{\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3}$$

If the angle between $\mathbf{a}$ and $\mathbf{b}$ is θ, then $\mathbf{a}.\mathbf{b} = ab\cos\theta$.

Thus
$$\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{ab} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{ab}$$

It follows that non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular if and only if
$$a_1b_1 + a_2b_2 + a_3b_3 = 0.$$

Example 4 Find the angle θ between the vectors $\mathbf{a} = 4\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} - 5\mathbf{j} - 4\mathbf{k}$.

$$\mathbf{a}.\mathbf{b} = 4 \times 3 + 5 \times (-5) + 3 \times (-4) = 12 - 25 - 12 = -25$$
$$a = \sqrt{\{4^2 + 5^2 + 3^2\}} = \sqrt{\{16 + 25 + 9\}} = \sqrt{50}$$
$$b = \sqrt{\{3^2 + (-5)^2 + (-4)^2\}} = \sqrt{\{9 + 25 + 16\}} = \sqrt{50}$$
$$\therefore \quad \cos\theta = \frac{\mathbf{a}.\mathbf{b}}{ab} = \frac{-25}{\sqrt{50} \times \sqrt{50}} = -\frac{25}{50} = -\frac{1}{2}$$

Hence the angle between the vectors is 120°.

Example 5 The points A, B and C have position vectors $\mathbf{a} = 5\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{c} = 7\mathbf{i} - 3\mathbf{j} + 10\mathbf{k}$ respectively. Show that $\angle ABC$ is a right angle.

$$\mathbf{BA} = \mathbf{a} - \mathbf{b} = 3\mathbf{i} + 4\mathbf{j} - \mathbf{k}, \quad \mathbf{BC} = \mathbf{c} - \mathbf{b} = 5\mathbf{i} - 2\mathbf{j} + 7\mathbf{k},$$
$$\therefore \quad \mathbf{BA}.\mathbf{BC} = 3 \times 5 + 4 \times (-2) + (-1) \times 7 = 15 - 8 - 7 = 0$$

Hence BA is perpendicular to BC i.e. $\angle ABC$ is a right angle.

Exercise 21.3

1. Use the properties of the scalar product to simplify
(a) $(\mathbf{p} + \mathbf{q}).(\mathbf{p} + \mathbf{q})$, (b) $(\mathbf{p} + \mathbf{q}).(\mathbf{p} - \mathbf{q})$,
(c) $\mathbf{p}.(\mathbf{q} + \mathbf{r}) - \mathbf{p}.(\mathbf{q} - \mathbf{r})$, (d) $|\mathbf{p} - \mathbf{q}|^2$.

2. Given that $\mathbf{x}$ is perpendicular to $\mathbf{y}$, simplify
(a) $\mathbf{x}.(\mathbf{x} + \mathbf{y})$, (b) $(2\mathbf{x} + \mathbf{y}).(\mathbf{x} - 3\mathbf{y})$.

3. Given that $|\mathbf{a}| = 7, |\mathbf{b}| = 4$ and $\mathbf{a}.\mathbf{b} = 8$, calculate $(\mathbf{a} + \mathbf{b}).(\mathbf{a} + \mathbf{b})$ and $(\mathbf{a} - \mathbf{b}).(\mathbf{a} - \mathbf{b})$. Hence find $|\mathbf{a} + \mathbf{b}|$ and $|\mathbf{a} - \mathbf{b}|$.

4. Given that $|\mathbf{x}| = 5, |\mathbf{y}| = 10$ and $\mathbf{x}.\mathbf{y} = 22$, calculate $|\mathbf{x} + \mathbf{y}|$ and $|\mathbf{x} - \mathbf{y}|$.

5. Show that if $\mathbf{a}$ and $\mathbf{b}$ are non-zero parallel vectors, then either $\mathbf{a}.\mathbf{b} = ab$ or $\mathbf{a}.\mathbf{b} = -ab$.

6. Show that if non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are of the same magnitude, then $(\mathbf{a} + \mathbf{b})$ is perpendicular to $(\mathbf{a} - \mathbf{b})$.

7. Show that if $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are non-zero vectors such that $\mathbf{a}.\mathbf{b} = \mathbf{a}.\mathbf{c}$, then either $\mathbf{b} = \mathbf{c}$ or $\mathbf{a}$ is perpendicular to $(\mathbf{b} - \mathbf{c})$.

8. Vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are such that $\mathbf{a}.\mathbf{c} = 3$ and $\mathbf{b}.\mathbf{c} = 4$. Given that the vector $\mathbf{d} = \mathbf{a} + \lambda\mathbf{b}$ is perpendicular to $\mathbf{c}$, find the value of λ.

9. Coplanar vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are such $|\mathbf{a}| = |\mathbf{b}| = 1$, $\mathbf{a}.\mathbf{b} = \frac{1}{2}$, $\mathbf{b}.\mathbf{c} = -\frac{1}{2}$ and $\mathbf{a}.\mathbf{c} = 2$. Find an expression for $\mathbf{c}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

10. Use vector methods to prove Pythagoras' theorem.

11. Use vector methods to prove that the diagonals of a rhombus bisect each other at right angles.

12. In triangle ABC the point D is the mid-point of BC. By letting $\overrightarrow{AB} = \mathbf{x}$ and $\overrightarrow{BD} = \mathbf{y}$, prove Apollonius' theorem, which states that

$$AB^2 + AC^2 = 2(AD^2 + BD^2).$$

13. In triangle ABC the altitudes through A and B intersect at a point H. Show that if $\overrightarrow{HA} = \mathbf{a}$, $\overrightarrow{HB} = \mathbf{b}$ and $\overrightarrow{HC} = \mathbf{c}$ then $\mathbf{a}.\mathbf{b} = \mathbf{a}.\mathbf{c}$ and $\mathbf{b}.\mathbf{a} = \mathbf{b}.\mathbf{c}$. Deduce that the altitudes of a triangle are concurrent.

14. Find the angles between the following pairs of vectors
(a) $8\mathbf{i} - \mathbf{j} + 4\mathbf{k}, 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, (b) $\mathbf{j} + 7\mathbf{k}, -5\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$,
(c) $2\mathbf{i} - 3\mathbf{j} + 8\mathbf{k}, 6\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}$, (d) $\mathbf{i} - \mathbf{j} + \mathbf{k}, \mathbf{i} + \mathbf{k}$.

15. Find the cosine of the angle BAC, given that
(a) the points A, B and C have position vectors $\mathbf{i} - \mathbf{j}, 5\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and $\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ respectively;
(b) the points A, B and C have coordinates $(0, -2, 1), (1, -1, -2)$ and $(-1, 1, 0)$ respectively.

16. Find the component of the vector $\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ in the direction of
(a) $\mathbf{j}$, (b) $-\mathbf{k}$, (c) $-4\mathbf{i} + 3\mathbf{j}$ (d) $6\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$.

17. Given the points $A(1, -5, 3)$, $B(0, 2, -4)$, $C(-1, 1, 0)$ and $D(7, -2, -8)$, find the lengths of the projections of AB on AC and BD.

18. If $\mathbf{a} = \lambda\mathbf{i} + \mu\mathbf{j}$ and $\mathbf{b} = \mu\mathbf{i} - \lambda\mathbf{j}$, show that $\mathbf{a}$ is perpendicular to $\mathbf{b}$. Hence write down vectors perpendicular to
(a) $3\mathbf{i} + 4\mathbf{j}$, (b) $5\mathbf{i} - 2\mathbf{j}$, (c) $-2\mathbf{i} - 3\mathbf{j}$.

19. Points A and B have position vectors $4\mathbf{i} - 5\mathbf{j} - 2\mathbf{k}$ and $8\mathbf{i} - 5\mathbf{j} + 6\mathbf{k}$ respectively relative to an origin O. Find the angles of triangle AOB and also its area.

20. Given the points $A(7, 3, -1)$, $B(1, 5, 2)$ and $C(3, -1, 1)$, find the cosine of $\angle BAC$ and the area of triangle ABC.

21. Show that the lines with vector equations $\mathbf{r} = (1 + 4\lambda)\mathbf{i} + (1 - \lambda)\mathbf{j} + 2\lambda\mathbf{k}$ and $\mathbf{r} = (5 + 3\mu)\mathbf{i} + 2\mu\mathbf{j} + (2 - 5\mu)\mathbf{k}$ cut at right angles and give the position vector of their point of intersection.

22. The points A, B, C and D have coordinates $(3, -2, 0)$, $(-1, 2, 4)$, $(0, 1, -1)$ and $(5, -4, 4)$ respectively. Find (a) the point of intersection of the lines AB and CD, (b) the acute angle between these lines.

23. $ABCDA'B'C'D'$ is a cube with horizontal faces $ABCD$ and $A'B'C'D'$ and vertical edges AA', BB', CC', DD'. Taking $\mathbf{AB} = \mathbf{i}$, $\mathbf{AD} = \mathbf{j}$, $\mathbf{AA'} = \mathbf{k}$, obtain vectors (in terms of $\mathbf{i}, \mathbf{j}, \mathbf{k}$) perpendicular to the planes $ACC'A'$ and $CDA'B'$, and hence find the angle between these two planes. (O&C)

24. A plane p contains the vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. If $\mathbf{c} = \mathbf{i} + \lambda\mathbf{j} + \mu\mathbf{k}$, find the values of λ and μ such that $\mathbf{c}$ is perpendicular to the plane p. If $\mathbf{d} = -3\mathbf{i} + \mathbf{j} - 3\mathbf{k}$, find $\mathbf{c} \cdot \mathbf{d}$. Hence find the sine of the angle between the plane p and the vector $\mathbf{d}$.

25. Prove that if the vectors $\mathbf{a}$ and $\mathbf{b}$ are differentiable functions of a scalar variable t, then $\dfrac{d}{dt}(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot \dfrac{d\mathbf{b}}{dt} + \dfrac{d\mathbf{a}}{dt} \cdot \mathbf{b}$.

21.4 Further lines and planes

As shown in §21.2 the vector equation of the straight line which passes through the point with position vector $\mathbf{a}$ and has direction vector $\mathbf{d}$ is

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}.$$

Similarly the vector equation of the straight line passing through the points with position vectors $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{r} = (1 - \lambda)\mathbf{a} + \lambda\mathbf{b}.$$

The scalar product can be used to find vectors perpendicular to such lines.

Example 1 The points A, B and C have position vectors $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 4\mathbf{k}$, $\mathbf{b} = \mathbf{j} - 4\mathbf{k}$ and $\mathbf{c} = 6\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ respectively. Find the position vector of the point R on BC such that AR is perpendicular to BC. Hence find the perpendicular distance of A from the line BC.

Since R lies on BC its position vector is of the form

$$\mathbf{r} = (1 - \lambda)(\mathbf{j} - 4\mathbf{k}) + \lambda(6\mathbf{i} + 4\mathbf{j} + 5\mathbf{k})$$

i.e. $$\mathbf{r} = 6\lambda\mathbf{i} + (3\lambda + 1)\mathbf{j} + (9\lambda - 4)\mathbf{k} \qquad (1)$$

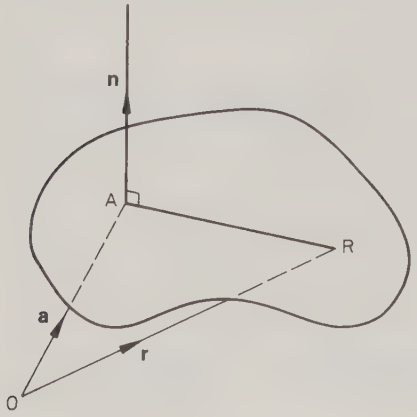
Since AR is perpendicular to BC, $\mathbf{AR}.\mathbf{BC} = 0$.

But $\mathbf{BC} = \mathbf{c} - \mathbf{b} = 6\mathbf{i} + 3\mathbf{j} + 9\mathbf{k}$
and $\mathbf{AR} = \mathbf{r} - \mathbf{a} = (6\lambda - 3)\mathbf{i} + (3\lambda + 2)\mathbf{j} + (9\lambda - 8)\mathbf{k}$ (2)
$\therefore \quad 6(6\lambda - 3) + 3(3\lambda + 2) + 9(9\lambda - 8) = 0$
$\therefore \quad 36\lambda - 18 + 9\lambda + 6 + 81\lambda - 72 = 0$
$\therefore \quad 126\lambda - 84 = 0 \qquad \therefore \quad \lambda = \frac{2}{3}$
From (1), the position vector of R is $4\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$
From (2), $\mathbf{AR} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$
Hence the perpendicular distance of A from BC
$= |\mathbf{AR}| = \sqrt{\{1^2 + 4^2 + (-2)^2\}} = \sqrt{\{1 + 16 + 4\}} = \sqrt{21}$.

The parametric form of the vector equation of the plane which passes through a point A with position vector $\mathbf{a}$ and which contains non-parallel vectors $\mathbf{p}$ and $\mathbf{q}$ is

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{p} + \mu\mathbf{q}.$$

Another form of the equation of a plane is obtained by using the scalar product.



Consider a plane which passes through a point A with position vector $\mathbf{a}$ and which is perpendicular to a vector $\mathbf{n}$. If $\mathbf{r}$ is the position vector of a point R in the plane, then since $\mathbf{AR}$ is perpendicular to $\mathbf{n}$,

$$\mathbf{AR}.\mathbf{n} = 0$$
$$\therefore \quad (\mathbf{r} - \mathbf{a}).\mathbf{n} = 0$$

This is the vector equation of the plane through A perpendicular to $\mathbf{n}$, in scalar product form.

$$(\mathbf{r} - \mathbf{a}).\mathbf{n} = 0 \iff \mathbf{r}.\mathbf{n} - \mathbf{a}.\mathbf{n} = 0$$
$$\iff \mathbf{r}.\mathbf{n} = \mathbf{a}.\mathbf{n}$$
$$\iff \mathbf{r}.\mathbf{n} = p, \quad \text{where} \quad p = \mathbf{a}.\mathbf{n}.$$

Hence the equation of any plane perpendicular to a vector $\mathbf{n}$ may be written in the form $\mathbf{r}.\mathbf{n} = p$.

If the point D with position vector $\mathbf{d}$ is the foot of the perpendicular from the origin to the plane $\mathbf{r}.\mathbf{n} = p$, then the perpendicular distance of the origin from the plane is d.

Since D lies in the plane, $\mathbf{d}.\mathbf{n} = p$.

However, $\mathbf{d}$ is parallel to $\mathbf{n}$,

$\therefore$ either $\mathbf{d}.\mathbf{n} = dn$ or $\mathbf{d}.\mathbf{n} = -dn$.

Hence $dn = |p|$ giving $d = |p|/n$.

Thus the plane $\mathbf{r}.\mathbf{n} = p$ is perpendicular to the vector $\mathbf{n}$ and lies at a distance $\frac{|p|}{n}$ from the origin.

Example 2 Find the vector equation of the plane which passes through the point with position vector $2\mathbf{i} - \mathbf{j} + \mathbf{k}$ and is perpendicular to the vector $3\mathbf{i} + \mathbf{j} - 4\mathbf{k}$.

The vector equation of the given plane is

$$(\mathbf{r} - \{2\mathbf{i} - \mathbf{j} + \mathbf{k}\}).(3\mathbf{i} + \mathbf{j} - 4\mathbf{k}) = 0$$
$$\mathbf{r}.(3\mathbf{i} + \mathbf{j} - 4\mathbf{k}) - (2\mathbf{i} - \mathbf{j} + \mathbf{k}).(3\mathbf{i} + \mathbf{j} - 4\mathbf{k}) = 0$$
$$\mathbf{r}.(3\mathbf{i} + \mathbf{j} - 4\mathbf{k}) - (6 - 1 - 4) = 0$$
i.e $$\mathbf{r}.(3\mathbf{i} + \mathbf{j} - 4\mathbf{k}) = 1.$$

Example 3 Find in scalar product form the equation of the plane

$$\mathbf{r} = (1 + 3\lambda + 2\mu)\mathbf{i} + (1 + \lambda + 4\mu)\mathbf{j} + (\mu - \lambda)\mathbf{k}.$$

Rearranging the given equation, we have

$$\mathbf{r} = \mathbf{i} + \mathbf{j} + \lambda(3\mathbf{i} + \mathbf{j} - \mathbf{k}) + \mu(2\mathbf{i} + 4\mathbf{j} + \mathbf{k})$$

$\therefore$ the given plane passes through the point with position vectors $\mathbf{i} + \mathbf{j}$ and is parallel to the vectors $3\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $2\mathbf{i} + 4\mathbf{j} + \mathbf{k}$.
Let $\mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}$ be a vector perpendicular to $3\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $2\mathbf{i} + 4\mathbf{j} + \mathbf{k}$, then

$$\mathbf{n}.(3\mathbf{i} + \mathbf{j} - \mathbf{k}) = 0 \quad \text{i.e.} \quad 3n_1 + n_2 - n_3 = 0 \tag{1}$$
$$\mathbf{n}.(2\mathbf{i} + 4\mathbf{j} + \mathbf{k}) = 0 \quad \text{i.e.} \quad 2n_1 + 4n_2 + n_3 = 0 \tag{2}$$

Adding (1) to (2): $5n_1 + 5n_2 = 0 \quad \therefore \quad n_2 = -n_1$
Substituting in (1): $3n_1 - n_1 - n_3 = 0 \quad \therefore \quad n_3 = 2n_1$

$\therefore \quad n_1 : n_2 : n_3 = n_1 : -n_1 : 2n_1 = 1 : -1 : 2$

Thus one vector perpendicular to the given plane is $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. Hence the equation of the plane in scalar product form is

$$(\mathbf{r} - \{\mathbf{i} + \mathbf{j}\}).(\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = 0$$
$$\mathbf{r}.(\mathbf{i} - \mathbf{j} + 2\mathbf{k}) - (\mathbf{i} + \mathbf{j}).(\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = 0$$
i.e. $$\mathbf{r}.(\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = 0$$

Example 4 Find the angle between the planes $\mathbf{r}.(2\mathbf{i} - \mathbf{j} - 3\mathbf{k}) = 10$ and $\mathbf{r}.(\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) = 16$.

The given planes are perpendicular to the vectors $\mathbf{n}_1 = 2\mathbf{i} - \mathbf{j} - 3\mathbf{k}$ and $\mathbf{n}_2 = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ respectively.
$\therefore$ the angle θ between the planes is equal to the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$.

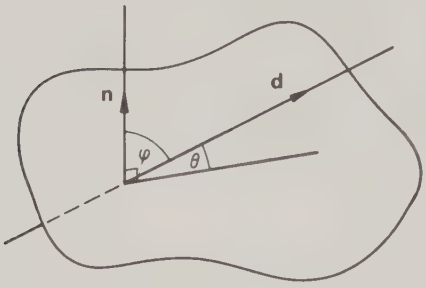
$$\mathbf{n}_1 . \mathbf{n}_2 = 2 \times 1 + (-1) \times 3 + (-3) \times (-2) = 2 - 3 + 6 = 5$$
$$n_1 = \sqrt{\{2^2 + (-1)^2 + (-3)^2\}} = \sqrt{\{4 + 1 + 9\}} = \sqrt{14}$$
$$n_2 = \sqrt{\{1^2 + 3^2 + (-2)^2\}} = \sqrt{\{1 + 9 + 4\}} = \sqrt{14}$$

$$\therefore \quad \cos\theta = \frac{\mathbf{n}_1 . \mathbf{n}_2}{n_1 n_2} = \frac{5}{\sqrt{14} \times \sqrt{14}} = \frac{5}{14}$$

Hence the angle between the planes is 69·1°.

Example 5 Find the angle between the line $\mathbf{r} = 3\mathbf{k} + \lambda(7\mathbf{i} - \mathbf{j} + 4\mathbf{k})$ and the plane $\mathbf{r}.(2\mathbf{i} - 5\mathbf{j} - 2\mathbf{k}) = 8$.

The given line is parallel to the vector $\mathbf{d} = 7\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ and the given plane is perpendicular to the vector $\mathbf{n} = 2\mathbf{i} - 5\mathbf{j} - 2\mathbf{k}$.



If the angle between the line and the plane is θ and the angle between $\mathbf{d}$ and $\mathbf{n}$ is ϕ then

$$\theta + \phi = 90°,$$

$$\therefore \quad \sin\theta = \cos\phi = \frac{\mathbf{d}.\mathbf{n}}{dn}$$

$$\mathbf{d}.\mathbf{n} = 7 \times 2 + (-1) \times (-5) + 4 \times (-2) = 14 + 5 - 8 = 11$$
$$d = \sqrt{\{7^2 + (-1)^2 + 4^2\}} = \sqrt{\{49 + 1 + 16\}} = \sqrt{66}$$
$$n = \sqrt{\{2^2 + (-5)^2 + (-2)^2\}} = \sqrt{\{4 + 25 + 4\}} = \sqrt{33}$$

$$\therefore \quad \sin\theta = \frac{11}{\sqrt{66} \times \sqrt{33}} = \frac{11}{33\sqrt{2}} = \frac{1}{3\sqrt{2}}$$

Hence the angle between the line and the plane is 13·6°.

Example 6 Find the position vector of the point of intersection of the line $\mathbf{r} = (2\mathbf{i} - \mathbf{k}) + \lambda(\mathbf{i} + 3\mathbf{j})$ and the plane $\mathbf{r}.(5\mathbf{i} - \mathbf{j} + 7\mathbf{k}) = 9$.

At the point of intersection

$$\{(2\mathbf{i} - \mathbf{k}) + \lambda(\mathbf{i} + 3\mathbf{j})\}.(5\mathbf{i} - \mathbf{j} + 7\mathbf{k}) = 9$$
$$\therefore \quad (2\mathbf{i} - \mathbf{k}).(5\mathbf{i} - \mathbf{j} + 7\mathbf{k}) + \lambda(\mathbf{i} + 3\mathbf{j}).(5\mathbf{i} - \mathbf{j} + 7\mathbf{k}) = 9$$
$$\therefore \quad (10 + 0 - 7) + \lambda(5 - 3 + 0) = 9$$
$$\therefore \quad 3 + 2\lambda = 9$$
$$\therefore \quad \lambda = 3$$

Hence the position vector of the point of intersection is

$$2\mathbf{i} - \mathbf{k} + 3(\mathbf{i} + 3\mathbf{j}) \quad \text{i.e.} \quad 5\mathbf{i} + 9\mathbf{j} - \mathbf{k}.$$

By writing $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ we can express the equations of lines and planes in Cartesian form.

Consider first the straight line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$, which passes through the point with position vector $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and is parallel to the vector $\mathbf{d} = d_1\mathbf{i} + d_2\mathbf{j} + d_3\mathbf{k}$.

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{d} \Leftrightarrow \left.\begin{array}{l} x = a_1 + \lambda d_1 \\ y = a_2 + \lambda d_2 \\ z = a_3 + \lambda d_3 \end{array}\right\} \text{ These are the } \textit{parametric equations} \text{ of the line.}$$

Assuming that d_1, d_2 and d_3 are non-zero, we have

$$\frac{x - a_1}{d_1} = \lambda, \quad \frac{y - a_2}{d_2} = \lambda, \quad \frac{z - a_3}{d_3} = \lambda$$

$$\therefore \qquad \frac{x - a_1}{d_1} = \frac{y - a_2}{d_2} = \frac{z - a_3}{d_3}$$

These are the *Cartesian equations* of the line.
Any line parallel to the vector $d_1\mathbf{i} + d_2\mathbf{j} + d_3\mathbf{k}$ is said to have *direction ratios* $d_1 : d_2 : d_3$.

Thus the equations $\dfrac{x - a_1}{d_1} = \dfrac{y - a_2}{d_2} = \dfrac{z - a_3}{d_3}$ represent the straight line through the point (a_1, a_2, a_3) with direction ratios $d_1 : d_2 : d_3$.

By convention the Cartesian equations of a straight line may be given in the above form for all values of d_1, d_2, d_3 including zero. For instance, the equations $\dfrac{x - 2}{3} = \dfrac{y + 1}{0} = \dfrac{z}{-2}$ represent the straight line through the point $(2, -1, 0)$, with direction ratios $3 : 0 : -2$ i.e. in a direction parallel to the vector $3\mathbf{i} - 2\mathbf{k}$.

Consider now the plane which passes through the point with position vector $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and is perpendicular to the vector $\mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}$. Writing $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, its vector equation $(\mathbf{r} - \mathbf{a}).\mathbf{n} = 0$ becomes

$$\{(x - a_1)\mathbf{i} + (y - a_2)\mathbf{j} + (z - a_3)\mathbf{k}\}.(n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}) = 0.$$

Hence the Cartesian equation of this plane is

$$n_1(x - a_1) + n_2(x - a_2) + n_3(x - a_3) = 0.$$

Similarly the vector equation $\mathbf{r}.\mathbf{n} = p$ gives rise to the Cartesian form

$$n_1x + n_2y + n_3z - p = 0.$$

For instance, the Cartesian equation of the plane $\mathbf{r}.(3\mathbf{i} - \mathbf{j} + 4\mathbf{k}) = 5$ is $3x - y + 4z - 5 = 0$.
It follows that the equation $Ax + By + Cz + D = 0$ is the Cartesian equation of a plane perpendicular to the vector $A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$.

Thus the equation $Ax + By + Cz + D = 0$ represents a plane, the direction ratios of the normal to the plane being $A : B : C$.

Exercise 21.4

1. The equation of a line l is $\mathbf{r} = \lambda\mathbf{i} - (\lambda + 4)\mathbf{j} + \mathbf{k}$. Find the position vector with respect to the origin O of the point A on l, given that OA is perpendicular to l. Hence find the perpendicular distance of O from the line l.

2. The position vectors of points A and B are $3\mathbf{i} - 8\mathbf{j} + \mathbf{k}$ and $4\mathbf{j} - 2\mathbf{k}$ respectively. Find the position vector of the foot of the perpendicular from the origin O to AB. Hence find the area of the triangle OAB.

3. The points A and B have position vectors $2\mathbf{j} + 7\mathbf{k}$ and $5\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ respectively relative to an origin O. Find the position vector of the point R on AB such that OR is perpendicular to AB. Hence find the position vector of the reflection of O in AB.

4. A point A has position vector $2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$ and a line l has equation $\mathbf{r} = \mathbf{i} - \mathbf{j} + 7\mathbf{k} + \lambda(4\mathbf{i} - \mathbf{k})$. Find the position vector of a point R on l such that AR is perpendicular to l. Hence find the perpendicular distance of A from the line l.

5. Find, in scalar product form, the equation of the plane through the point with position vector $\mathbf{a}$ and perpendicular to the vector $\mathbf{n}$, given that
(a) $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\mathbf{n} = 4\mathbf{i} - \mathbf{k}$, (b) $\mathbf{a} = 3\mathbf{j} - 2\mathbf{k}$, $\mathbf{n} = \mathbf{i} - \mathbf{j} - 2\mathbf{k}$.

6. Find, in the form $\mathbf{r}.\mathbf{n} = p$, the equations of the planes through the given points and perpendicular to the given vectors
(a) $(0, 0, 1)$, $\mathbf{k}$, (b) $(2, -1, 1)$, $3\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.

7. Find the perpendicular distances from the origin of the following planes
(a) $\mathbf{r}.(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) = 15$, (b) $\mathbf{r}.\mathbf{i} = 3$,
(c) $\mathbf{r}.(3\mathbf{i} - 4\mathbf{k}) = 20$, (b) $\mathbf{r}.(\mathbf{i} + \mathbf{j} + \mathbf{k}) = 0$.

8. Find a vector which is perpendicular to both $\mathbf{i} + 2\mathbf{k}$ and $3\mathbf{i} + \mathbf{j} + \mathbf{k}$. Hence find the vector equation in scalar product form of the plane parallel to these vectors which passes through the point with position vector $\mathbf{i} - \mathbf{j}$.

9. Find in scalar product form the equation of the plane

$$\mathbf{r} = (3 + 3\lambda + 4\mu)\mathbf{i} + (\lambda + 2\mu)\mathbf{j} + (1 + 3\lambda + 5\mu)\mathbf{k}.$$

10. Find in scalar product form the equation of the plane containing the lines

$$\mathbf{r} = (\lambda + 1)\mathbf{i} - 2\mathbf{j} + 3\mathbf{k} \text{ and } \mathbf{r} = 3\mu\mathbf{i} + (2\mu - 4)\mathbf{j} + (\mu + 2)\mathbf{k}.$$

11. Find the acute angles between the following pairs of planes.
(a) $\mathbf{r}.(\mathbf{i} - \mathbf{j}) = 4$, $\mathbf{r}.(\mathbf{j} + \mathbf{k}) = 1$,
(b) $\mathbf{r}.(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) = 5$, $\mathbf{r}.\mathbf{j} = 0$,
(c) $\mathbf{r}.(\mathbf{i} + \mathbf{j} + \mathbf{k}) = 1$, $\mathbf{r}.(\mathbf{i} - \mathbf{j} + \mathbf{k}) = 0$.

12. In each of the following cases find the sine of the angle between the given line and plane.
(a) $\mathbf{r} = \mathbf{i} - 3\mathbf{j} + \lambda(2\mathbf{i} - \mathbf{j} - \mathbf{k})$, $\mathbf{r}.(\mathbf{i} - 2\mathbf{j} - 7\mathbf{k}) = 10$,
(b) $\mathbf{r} = (2 + \lambda)\mathbf{i} - 3\mathbf{j} + (1 - \lambda)\mathbf{k}$, $\mathbf{r}.(4\mathbf{i} + \mathbf{j} - \mathbf{k}) = 6$,
(c) $\mathbf{r} = 3\lambda\mathbf{i} + 2\lambda\mathbf{j} - 6\lambda\mathbf{k}$, $\mathbf{r}.(4\mathbf{i} - 3\mathbf{k}) = 20$.

13. Find the position vectors of points of intersection of the line $\mathbf{r} = \lambda\mathbf{i} - 2\mathbf{j} + (2\lambda - 1)\mathbf{k}$ with the planes $\mathbf{r}.(2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) = 5$ and $\mathbf{r}.(\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}) = -2$.

14. Find the vector equation of the line through the point A with position vector $5\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ which is perpendicular to the plane $\mathbf{r}.(3\mathbf{i} - 4\mathbf{j} + \mathbf{k}) = 5$. Hence find the position vector of the foot of the perpendicular from A to this plane.

15. Find the Cartesian equations of the straight line which
(a) has vector equation $\mathbf{r} = \mathbf{i} - 2\mathbf{j} + \mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$,
(b) has vector equation $\mathbf{r} = \lambda\mathbf{i} + (2 - \lambda)\mathbf{j} + (3 + 2\lambda)\mathbf{k}$,
(c) passes through the point $(2, -3, 1)$ and has direction ratios $5: -6: 3$,
(d) passes through the point $(5, 0, -4)$ and has direction ratios $1: 3: 0$.

16. Write down the Cartesian equations of the planes
(a) $\mathbf{r}.(2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) = 5$, (b) $\mathbf{r}.(3\mathbf{i} + \mathbf{j} - 5\mathbf{k}) = 9$.

17. Find the Cartesian equation of the plane which passes through the point with position vector $\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and is perpendicular to the vector $5\mathbf{i} + \mathbf{j} + \mathbf{k}$.

18. Find the Cartesian equation of a plane which passes through the point $(4, 0, -1)$, given that the normal to the plane has direction ratios $2: -1: 5$.

19. The points A, B and C have position vectors $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{i} + 2\mathbf{j}$ and $4\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ respectively. Find a vector perpendicular to both **AB** and **AC**. Hence find the equation of the plane containing A, B and C in (a) scalar product form, (b) Cartesian form.

20. Find the Cartesian equation of the plane containing the points $A(2, -1, 1)$, $B(1, -2, 0)$ and $C(-3, 6, -1)$.

21. Find the acute angle between

(a) the lines $\dfrac{x-1}{1} = \dfrac{y+2}{-2} = \dfrac{z}{2}$, $\dfrac{x+3}{3} = \dfrac{y+1}{0} = \dfrac{z-2}{-4}$;

(b) the planes $x - 2y - 3z = 1$, $2x - 4y + z = 3$;

(c) the line $\dfrac{x}{4} = \dfrac{y-1}{-1} = \dfrac{z+3}{-5}$, the plane $x - 2y + 4z + 3 = 0$.

22. The points A and B are fixed points with position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively relative to an origin O. If R is a variable point with position vector $\mathbf{r}$, describe

the locus of R in each of the following cases

(a) $\mathbf{r}.\mathbf{a} = 0$, (b) $(\mathbf{r} - \mathbf{a}).\mathbf{a} = 0$, (c) $(\mathbf{r} - \mathbf{a}).(\mathbf{r} - \mathbf{a}) = 1$,
(d) $\mathbf{r}.\mathbf{a} = \mathbf{r}.\mathbf{b}$, (e) $(\mathbf{r} - \mathbf{a}).\mathbf{r} = 0$, (f) $(\mathbf{r} - \mathbf{a}).(\mathbf{r} - \mathbf{b}) = 0$.

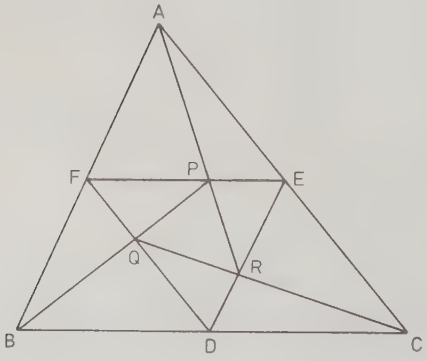
Exercise 21.5 (miscellaneous)

1. A cube $OPQRABCD$ has OA, PB, QC and RD perpendicular to the base $OPQR$; M is the centre of the face $ABCD$. The position vectors of A, B, C relative to O are $\mathbf{a}, \mathbf{b}, \mathbf{c}$ respectively. Find in terms of $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ the vectors $\mathbf{AB}, \mathbf{OM}, \mathbf{OD}$. If $|\mathbf{OP}| = 2$, express $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{OM}, \mathbf{OD}$ in terms of unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ along $\mathbf{OP}, \mathbf{OR}$, $\mathbf{OA}$ respectively. Find the cosine of the angle MOC. (L)

2. Points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively with respect to a given origin O. Prove that the position vector of the point P dividing the line segment AB in the ratio λ to μ is $(\mu\mathbf{a} + \lambda\mathbf{b})/(\lambda + \mu)$. The position vectors of A, B, C and D, the vertices of a plane quadrilateral, are $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ respectively. Find the position vectors of P, Q, R, S, T and U, the mid-points of AB, BC, CD, DA, AC and BD respectively. Hence show that the point with position vector $\frac{1}{4}(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d})$ lies on each of the lines PR, QS and TU and that these lines bisect each other. If $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}, \mathbf{b} = 6\mathbf{i} + 5\mathbf{j}, \mathbf{c} = 8\mathbf{i} - \mathbf{j}$ and $\mathbf{d} = \mathbf{j}$, where $\mathbf{i}$ and $\mathbf{j}$ are orthogonal unit vectors, show that the acute angle between AC and BD is $\cos^{-1}(5/13)$. Find a unit vector orthogonal to the line TU. (W)

3. The points A, B and C have position vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ with respect to an origin O. The point R in the plane ABC has position vector $\mathbf{r}$ where $\mathbf{r} = \frac{1}{2}(\frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}) + \frac{1}{2}\mathbf{c}$. Use the ratio theorem to obtain a geometrical description of the position of R with reference to the points A, B and C. Illustrate your answer with a diagram. By writing $\mathbf{r}$ in the form $\lambda\mathbf{a} + \mu(m\mathbf{b} + n\mathbf{c})$ where $\lambda + \mu = 1$, obtain an alternative description of the position of R. Given that the point S has position vector $\frac{2}{3}\mathbf{a} + \frac{1}{12}\mathbf{b} + \frac{1}{4}\mathbf{c}$, show that S bisects the line segment AR. (JMB)

4.



The diagram shows a triangle ABC, with D, E, F as the mid-points of the sides. The points P, Q, R are chosen on EF, FD, DE so that RP, PQ, QR pass through A, B, C. If the position vectors of A, B, C are $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and if

$$\frac{PF}{EP} = \frac{QD}{FQ} = \frac{RE}{DR} = \lambda,$$

find the position vectors of P, Q, R and hence the value of λ. (O)

5. The position vectors of two points A and B relative to an origin O are $\mathbf{a}$ and $\mathbf{b}$ respectively. Show that the vector equation of the line bisecting angle AOB is $\mathbf{r} = \lambda(\hat{\mathbf{a}} + \hat{\mathbf{b}})$, where $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ are unit vectors in the directions of $\mathbf{a}$ and $\mathbf{b}$. Hence prove that the bisector of an interior angle of a triangle divides the opposite side into segments proportional to the adjacent sides.

6. Three lines l_1, l_2 and l_3 have vector equations $\mathbf{r} = 2\mathbf{i} - \mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 3\mathbf{j} + 3\mathbf{k})$, $\mathbf{r} = \mathbf{i} - 3\mathbf{j} + \mu(2\mathbf{i} + 5\mathbf{j} + 4\mathbf{k})$ and $\mathbf{r} = 3\mathbf{i} + \nu(\mathbf{j} + 2\mathbf{k})$ respectively. Find the position vector of the point of intersection of l_1 and l_2. Hence show that the three lines are concurrent. Write down, in parametric form, a vector equation of the plane containing l_2 and l_3. Hence show that the three lines are coplanar.

7. Two planes p_1 and p_2 have vector equations $\mathbf{r} = (1 - \lambda)\mathbf{i} + (2\lambda + \mu)\mathbf{j} + (\mu - 1)\mathbf{k}$ and $\mathbf{r} = (s - t)\mathbf{i} + (2s - 3)\mathbf{j} + t\mathbf{k}$ respectively. Show that if a point in p_1 also lies in p_2, then $\mu = 4\lambda + 3$. Hence find a vector equation of the line of intersection of the two planes.

8. Given non-zero vectors $\mathbf{a}$ and $\mathbf{b}$, show that $|\mathbf{a} + \mathbf{b}| = |\mathbf{a} - \mathbf{b}|$ if and only if $\mathbf{a}$ is perpendicular to $\mathbf{b}$. Deduce that a parallelogram is a rectangle if and only if its diagonals are equal in length.

9. Use vector methods to find the acute angle between two diagonals of a cube.

10. A tetrahedron $OABC$ with vertex O at the origin is such that $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$. Show that the line segments joining the mid-points of opposite edges bisect one another. Given that two pairs of opposite edges are perpendicular prove that $\mathbf{a}.\mathbf{b} = \mathbf{b}.\mathbf{c} = \mathbf{c}.\mathbf{a}$ and show that the third pair of opposite edges is also perpendicular. Prove also that, in this case, $OA^2 + BC^2 = OB^2 + AC^2$. (L)

11. ABC is a given triangle, and the centre of the circle through A, B, C is O. Relative to O, the position vectors of A, B, C are $\mathbf{a}, \mathbf{b}, \mathbf{c}$. If H is the point with position vector $\mathbf{a} + \mathbf{b} + \mathbf{c}$, prove that AH, BH, CH are perpendicular to BC, CA, AB respectively. If A', B', C' are the mid-points of BC, CA, AB respectively, prove that AA', BB' and CC' are concurrent in G, and find the position vector of G. Prove that O, G, H lie on a straight line. (O)

12. Let $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}, \mathbf{b} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$. Given that $\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$ and that $\mathbf{c}$ is perpendicular to $\mathbf{a}$, find the ratio of λ to μ. Let A, B be the points with position vectors $\mathbf{a}, \mathbf{b}$ respectively with respect to an origin O. Write down, in terms of $\mathbf{a}$ and $\mathbf{b}$, a vector equation of the line l through A, in the plane of O, A and B, which is perpendicular to OA. Find the position vector of P, the point of intersection of l and OB. (JMB)

13. The position vectors of A, B, C are $\mathbf{i}, 2\mathbf{j}, 3\mathbf{k}$ respectively. The perpendicular from the origin O to the plane ABC meets the plane at the point D. Find (i) the vector equation of the plane ABC, (ii) the vector equation of the line OD, (iii) the position vector of the point D. (AEB 1980)

14. A straight line passes through the point $(2, 0, -1)$ and is parallel to the vector $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$. The feet of the perpendiculars from the points $A(2, -2, 1)$ and $B(1, 7, 0)$ to this line are P and Q. Find (i) the coordinates of P and Q, (ii) the angles APB and AQB, (iii) the Cartesian equations of the planes PAQ and PBQ, (iv) the angle between the planes PAQ and PBQ.

15. Interpret geometrically the equations $\mathbf{r} \cdot \mathbf{n} = p$ and $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ where $\mathbf{n}, \mathbf{a}$ and $\mathbf{b}$ are constant vectors, p is a constant scalar and t is a parameter which can take all real values. Find the value of t in terms of $\mathbf{a}, \mathbf{b}, \mathbf{n}$ and p if the value of $\mathbf{r}$ given by the second equation also satisfies the first. Explain the geometrical significance of the point given by this value of t, giving special attention to the cases when (i) $\mathbf{b} \cdot \mathbf{n} = 0$ and (ii) $\mathbf{b} = \mathbf{n}$.

The point A has coordinates $(1, 1, 2)$. Determine the coordinates of the point A' which is the reflection of A in the plane $2x + 2y + z = 9$. (AEB 1979)

16. The lines l_1 and l_2 have equations $\mathbf{r}_1 = (2\lambda + 3)\mathbf{i} + (2 - 3\lambda)\mathbf{j} + \lambda\mathbf{k}$ and $\mathbf{r}_2 = (\mu - 4)\mathbf{i} + (2\mu + 1)\mathbf{j} + (4\mu - 3)\mathbf{k}$ respectively. Show that l_1 is perpendicular to l_2. Find the values of λ and μ such that the vector $\mathbf{r}_1 - \mathbf{r}_2$ is perpendicular to both l_1 and l_2. Hence find the position vectors of the ends of the common perpendicular to l_1 and l_2. Find also the length of this common perpendicular.

17. A pyramid has a square base $OABC$ and vertex V. The position vectors of A, B, C, V referred to O as origin are given by $\mathbf{OA} = 2\mathbf{i}, \mathbf{OB} = 2\mathbf{i} + 2\mathbf{j}, \mathbf{OC} = 2\mathbf{j}$, $\mathbf{OV} = \mathbf{i} + \mathbf{j} + 3\mathbf{k}$. (i) Express $\mathbf{AV}$ in terms of $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$. (ii) Using scalar products, or otherwise, find a vector $\mathbf{x}$ which is perpendicular to both $\mathbf{OV}$ and $\mathbf{AV}$. (iii) Calculate the angle between the vector $\mathbf{x}$, found in (ii), and $\mathbf{VB}$, giving your answer to the nearest degree. (iv) Write down the acute angle between VB and the plane OVA. (C)

22 Series and expansions

22.1 Summation of finite series

There are various standard results that can be used to find the sums of finite series. Two of these are the formula for the sums of arithmetic and geometric progressions obtained in Chapter 13.

> The sum to n terms of the A.P. with first term a and common difference d is $\frac{1}{2}n\{2a + (n-1)d\}$.
> The sum to n terms of the G.P. with first term a and common ratio r is $a(1-r^n)/(1-r)$.

The series $\sum_{r=1}^{n} r = 1 + 2 + 3 + \ldots + n$ is an arithmetic progression with first term 1 and common difference 1.

$$\therefore \qquad \sum_{r=1}^{n} r = \tfrac{1}{2}n\{2 + (n-1)\} = \tfrac{1}{2}n(n+1).$$

The series $\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2$ can be summed using the identity

$$(r+1)^3 - r^3 \equiv 3r^2 + 3r + 1.$$

Substituting $r = 1, 2, 3, \ldots, n$, we have

$$\begin{aligned}
2^3 - 1^3 &= 3.1^2 + 3.1 + 1 \\
3^3 - 2^3 &= 3.2^2 + 3.2 + 1 \\
4^3 - 3^3 &= 3.3^2 + 3.3 + 1 \\
&\ldots\ldots\ldots\ldots \\
(n+1)^3 - n^3 &= 3n^2 + 3n + 1
\end{aligned}$$

Adding together all n equations, we find that most of the terms on the left may be

cancelled out to give:

$$(n+1)^3 - 1^3 = 3\sum_{r=1}^{n} r^2 + 3\sum_{r=1}^{n} r + n$$

$$\therefore \quad 3\sum_{r=1}^{n} r^2 = (n+1)^3 - 1^3 - 3\sum_{r=1}^{n} r - n$$

$$= (n+1)^3 - 1 - \tfrac{3}{2}n(n+1) - n$$
$$= n^3 + \tfrac{3}{2}n^2 + \tfrac{1}{2}n = \tfrac{1}{2}n(2n^2 + 3n + 1)$$

Hence $$\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1).$$

By a similar method it can be shown that

$$\sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2 = \left(\sum_{r=1}^{n} r\right)^2$$

To summarise, we now have three further important results:

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1), \quad \sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1), \quad \sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2$$

Example 1 Find the sum of the first n terms of the series

$$1.3.5 + 2.4.6 + 3.5.7 + \ldots.$$

The rth term of the series $= r(r+2)(r+4) = r^3 + 6r^2 + 8r$

$\therefore$ the sum of the first n terms

$$= \sum_{r=1}^{n} (r^3 + 6r^2 + 8r)$$

$$= \sum_{r=1}^{n} r^3 + 6\sum_{r=1}^{n} r^2 + 8\sum_{r=1}^{n} r$$

$$= \tfrac{1}{4}n^2(n+1)^2 + 6.\tfrac{1}{6}n(n+1)(2n+1) + 8.\tfrac{1}{2}n(n+1)$$
$$= \tfrac{1}{4}n(n+1)\{n(n+1) + 4(2n+1) + 16\}$$
$$= \tfrac{1}{4}n(n+1)(n^2 + n + 8n + 4 + 16)$$
$$= \tfrac{1}{4}n(n+1)(n^2 + 9n + 20)$$
$$= \tfrac{1}{4}n(n+1)(n+4)(n+5)$$

Example 2 Find the sum of the series $(n+1)^2 + (n+2)^2 + (n+3)^2 + \ldots + (2n)^2$.

The sum of the series $= \sum_{r=1}^{2n} r^2 - \sum_{r=1}^{n} r^2$

$$= \tfrac{1}{6}.2n(2n+1)(4n+1) - \tfrac{1}{6}n(n+1)(2n+1)$$
$$= \tfrac{1}{6}n(2n+1)\{2(4n+1) - (n+1)\}$$
$$= \tfrac{1}{6}n(2n+1)(7n+1).$$

Example 3 Find the sum of the series $1^3 + 3^3 + 5^3 + \ldots + (2n-1)^3$.

$$1^3 + 2^3 + 3^3 + \ldots + (2n)^3 = \sum_{r=1}^{2n} r^3$$

$$= \tfrac{1}{4}(2n)^2(2n+1)^2$$

$$= n^2(2n+1)^2$$

$$2^3 + 4^3 + 6^3 + \ldots + (2n)^3 = 2^3(1^3 + 2^3 + 3^3 + \ldots + n^3)$$

$$= 8\sum_{r=1}^{n} r^3$$

$$= 2n^2(n+1)^2$$

$$\therefore \quad 1^3 + 3^3 + 5^3 + \ldots + (2n-1)^3$$
$$= n^2(2n+1)^2 - 2n^2(n+1)^2$$
$$= n^2\{(2n+1)^2 - 2(n+1)^2\} = n^2(2n^2-1)$$

In the next example we use the *method of differences* to find the sum of the given series. We express each term of the series as a difference, in such a way that most terms of the series cancel each other out. A similar process was used earlier in the section to show that $\sum_{r=1}^{n} \{(r+1)^3 - r^3\} = (n+1)^3 - 1^3$.

Example 4 Show that $\dfrac{1}{r} - \dfrac{1}{r+1} \equiv \dfrac{1}{r(r+1)}$. Hence evaluate $\sum_{r=1}^{n} \dfrac{1}{r(r+1)}$.

$$\frac{1}{r} - \frac{1}{r+1} \equiv \frac{(r+1)-1}{r(r+1)} \equiv \frac{1}{r(r+1)}$$

$$\therefore \quad \sum_{r=1}^{n} \frac{1}{r(r+1)} = \sum_{r=1}^{n} \left(\frac{1}{r} - \frac{1}{r+1}\right)$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots$$

$$\ldots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

[Note that as $n \to \infty, \dfrac{1}{n+1} \to 0, \quad \therefore \quad 1 - \dfrac{1}{n+1} \to 1$. Thus the sum to infinity of this series is 1.]

We can extend the method of Example 4 to other series by using partial fractions.

Example 5 Find

$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{1}{1.2.3} + \frac{1}{2.3.4.} + \ldots + \frac{1}{n(n+1)(n+2)}.$$

Let $\dfrac{1}{r(r+1)(r+2)} \equiv \dfrac{A}{r} + \dfrac{B}{r+1} + \dfrac{C}{r+2}$

then $1 \equiv A(r+1)(r+2) + Br(r+2) + Cr(r+1)$

Putting $r = 0$: $1 = 2A \quad \therefore \quad A = \frac{1}{2}$

Putting $r = -1$: $1 = -B \quad \therefore \quad B = -1$

Putting $r = -2$: $1 = 2C \quad \therefore \quad C = \frac{1}{2}$

Hence $\dfrac{1}{r(r+1)(r+2)} \equiv \dfrac{1}{2r} - \dfrac{1}{r+1} + \dfrac{1}{2(r+2)}$

$$\therefore \sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{1}{2.1} - \frac{1}{2} + \cancel{\frac{1}{2.3}}$$
$$+ \frac{1}{2.2} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{2.4}}$$
$$+ \cancel{\frac{1}{2.3}} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{2.5}}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+ \cancel{\frac{1}{2(n-1)}} - \cancel{\frac{1}{n}} + \frac{1}{2(n+1)}$$
$$+ \cancel{\frac{1}{2n}} - \frac{1}{n+1} + \frac{1}{2(n+2)}$$

$$\therefore \sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{1}{2.1} - \frac{1}{2} + \frac{1}{2.2} + \frac{1}{2(n+1)} - \frac{1}{n+1} + \frac{1}{2(n+2)}$$
$$= \frac{1}{4} - \frac{1}{2(n+1)} + \frac{1}{2(n+2)}$$

[Note that the results obtained in this section can also be proved by the method of mathematical induction, as shown in §17.4.]

Exercise 22.1

1. Use standard results to find the sums of the following series.

(a) $\sum_{r=1}^{n} (2r+1)$ (b) $\sum_{r=1}^{n} (r-1)(r+1)$ (c) $\sum_{r=1}^{n} 2^r$

(d) $\sum_{r=1}^{n} r^2(r+1)$ (e) $\sum_{r=1}^{n} 3^{1-r}$ (f) $\sum_{r=1}^{n} (r+4)^3$

2. Use partial fractions to find the sums of the following series.

(a) $\sum_{r=1}^{n} \dfrac{1}{(r+1)(r+2)}$ (b) $\sum_{r=1}^{n} \dfrac{1}{r(r+2)}$ (c) $\sum_{r=1}^{n} \dfrac{1}{(r+1)(r+2)(r+3)}$

3. Find the sums of the following series (i) to $2n$ terms, (ii) to $(2n + 1)$ terms.
(a) $1 - 1 + 1 - 1 + \ldots$ (b) $1 - 2 + 3 - 4 + \ldots$

4. Show that $r(r + 1)(r + 2) - (r - 1)r(r + 1) \equiv 3r(r + 1)$.

Hence find the sum of the series $\sum_{r=1}^{n} r(r + 1)$.

5. Use the method of question 4 to show that

$$\sum_{r=1}^{n} r(r + 1)(r + 2) = \tfrac{1}{4}n(n + 1)(n + 2)(n + 3).$$

6. Show that $(r + 1)^4 - r^4 \equiv 4r^3 + 6r^2 + 4r + 1$. Use this identity and the standard formulae for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to prove that $\sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n + 1)^2$.

7. Use the method of question 6 to find $\sum_{r=1}^{n} r^4$.

8. Use the method of differences to find the sum to n terms of the series $1(1!) + 2(2!) + 3(3!) + \ldots$.

[Hint: express each term in the form $p! - q!$.]

9. Find the sum of (a) the integers from $2n$ to $3n$ inclusive, (b) the squares of the integers from $2n$ to $3n$ inclusive.

10. Find the sum of (a) the first n odd positive integers, (b) the squares of the first n odd positive integers.

11. Find the sums of the following series
(a) $21^3 + 22^3 + 23^3 + \ldots + 35^3$,
(b) $1^3 + 3^3 + 5^3 + \ldots + 25^3$,
(c) $(n + 1)^3 + (n + 2)^3 + (n + 3)^3 + \ldots + (2n)^3$,
(d) $1^3 - 2^3 + 3^3 - 4^3 + \ldots + (2n - 1)^3 - (2n)^3$.

In questions 12 to 19 write down an expression for the rth term of the given series and hence find the sum of the first n terms of the series.

12. $1.4 + 2.5 + 3.6 + \ldots$

13. $2^2 + 5^2 + 8^2 + 11^2 + \ldots$

14. $1.2 + 3.4 + 5.6 + \ldots$

15. $1.2.5 + 2.3.6 + 3.4.7 + \ldots$

16. $\dfrac{1}{1.4} + \dfrac{1}{2.5} + \dfrac{1}{3.6} + \ldots$

17. $\dfrac{1}{1.3} + \dfrac{1}{3.5} + \dfrac{1}{5.7} + \ldots$

18. $\dfrac{1}{1.3.5} + \dfrac{1}{3.5.7} + \dfrac{1}{5.7.9} + \ldots$

19. $\dfrac{1}{2.3.4} + \dfrac{2}{3.4.5} + \dfrac{3}{4.5.6} + \ldots$

20. Find the sum of the series $1 + x + x^2 + \ldots + x^n$, where $x \neq 1$. By differentiating your result, find the sum of the series $1 + 2x + 3x^2 + \ldots + nx^{n-1}$ for $x \neq 1$.

21. Find the sum $S = x + 2x^2 + 3x^3 + \ldots + nx^n$, where $x \neq 1$, by considering the expression $S - xS$.

22. Find the sum $S = 1 - 3a + 5a^2 - \ldots + (2n + 1)(-a)^n$, where $a \neq -1$, by considering the expression $(1 + a)S$.

22.2 Maclaurin's theorem

Under certain conditions a function $f(x)$ can be expressed as an infinite series of ascending powers of x. Assuming that such a series exists, we may write

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots + a_rx^r + \ldots$$

Differentiating with respect to x, we have

$$\begin{aligned}
f'(x) &= a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots + ra_rx^{r-1} + \ldots \\
f''(x) &= 2a_2 + 3.2a_3x + 4.3a_4x^2 + \ldots + r(r-1)a_rx^{r-2} + \ldots \\
f'''(x) &= 3.2a_3 + 4.3.2a_4x + \ldots + r(r-1)(r-2)a_rx^{r-3} + \ldots \\
&\ldots\ldots\ldots

sum $f(x)$ for all values of x. However, for many functions, Maclaurin's theorem holds only within a restricted range of values of x.

We now use Maclaurin's theorem to obtain various important expansions.

Let $f(x) = (1 + x)^n$, where n is a real number,

then $f'(x) = n(1 + x)^{n-1}$,

$f''(x) = n(n - 1)(1 + x)^{n-2}$,

$f'''(x) = n(n - 1)(n - 2)(1 + x)^{n-3}$,

.....

$f^{(r)}(x) = n(n - 1)(n - 2)\ldots(n - r + 1)(1 + x)^{n-r}$,

$\therefore$ $f(0) = 1$, $f'(0) = n$, $f''(0) = n(n - 1)$, $f'''(0) = n(n - 1)(n - 2)$

and $f^{(r)}(0) = n(n - 1)(n - 2)\ldots(n - r + 1)$.

Thus

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r + \ldots$$

This result was introduced in §13.7 as an extension of the binomial theorem. When n is a positive integer the expansion is valid for all values of x. It can be shown that, for other values of n, the result holds only when $|x| < 1$, i.e. when $-1 < x < 1$.

If

$f(x) = e^x$,	then	$f(0) = 1$
$f'(x) = e^x$,		$f'(0) = 1$
$f''(x) = e^x$,		$f''(0) = 1$
.....		.....
$f^{(r)}(x) = e^x$,		$f^{(r)}(0) = 1$

Thus

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^r}{r!} + \ldots$$

This expansion is valid for all values of x.

The function $\ln x$ is not defined for $x = 0$ and therefore has no Maclaurin series. However, an expansion of $\ln(1 + x)$ can be obtained.

If

$f(x) = \ln(1 + x)$,		then	$f(0) = \ln 1 = 0$
$f'(x) = (1 + x)^{-1}$,			$f'(0) = 1$
$f''(x) = -(1 + x)^{-2}$,			$f''(0) = -1$
$f'''(x) = 2(1 + x)^{-3}$,			$f'''(0) = 2$
$f^{(4)}(x) = -3.2(1 + x)^{-4}$,			$f^{(4)}(0) = -3!$
.....			.....
$f^{(r)}(x) = (-1)^{r+1}(r - 1)!(1 + x)^{-r}$,			$f^{(r)}(0) = (-1)^{r+1}(r - 1)!$

Thus $\ln(1 + x) = 0 + x - \frac{x^2}{2!} + \frac{2x^3}{3!} - \frac{3!x^4}{4!} + \ldots + \frac{(-1)^{r+1}(r-1)!x^r}{r!} + \ldots$

i.e. $$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^{r+1}\frac{x^r}{r} + \ldots$$

This result is valid for $-1 < x \leqslant 1$.

If

	then
$f(x) = \sin x,$	$f(0) = 0$
$f'(x) = \cos x,$	$f'(0) = 1$
$f''(x) = -\sin x,$	$f''(0) = 0$
$f'''(x) = -\cos x,$	$f'''(0) = -1$
$f^{(4)}(x) = \sin x,$	$f^{(4)}(0) = 0$
$f^{(5)}(x) = \cos x,$	$f^{(5)}(0) = 1$
.....	.....

Since we can now see a pattern emerging, we may write

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

A similar result can be obtained for $\cos x$:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

The expansions of $\sin x$ and $\cos x$ are valid for all values of x (in radians).

Exercise 22.2

In questions 1 to 9 use Maclaurin's theorem to expand the given function in ascending powers of x as far as the term in x^4.

1. $\cos x$
2. $\tan x$
3. $\sec x$
4. $\ln(\cos x)$
5. $\sin^2 x$
6. $\ln(1 + \sin x)$
7. $\tan^{-1} x$
8. $e^{\sin x}$
9. $\sin^{-1} x$

10. If $y = e^x \sin x$, show that $\dfrac{d^2y}{dx^2} = 2\left(\dfrac{dy}{dx} - y\right)$. By further differentiation of this result, find the Maclaurin expansion of the function $e^x \sin x$ as far as the term in x^6.

22.3 The binomial series

The *binomial series* is the expansion of $(1 + x)^n$ in ascending powers of x, as follows:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$
$$\dots + \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}x^r + \dots$$

where n is a real number and $|x| < 1$.

Simple applications of this expansion were discussed in §13.7. We now consider further uses of the series.

First we recall that the expression $(a+x)^n$ cannot be expanded directly using the binomial series. Provided that $|x| < |a|$, we write

$$(a+x)^n = a^n\left(1+\frac{x}{a}\right)^n = a^n\left\{1 + n\left(\frac{x}{a}\right) + \frac{n(n-1)}{2!}\left(\frac{x}{a}\right)^2 + \dots\right\}$$

Example 1 Expand $(8+3x)^{1/3}$ in ascending powers of x as far as the term in x^3, stating the values of x for which the expansion is valid. Hence obtain an approximate value for $\sqrt[3]{8{\cdot}72}$.

$$(8+3x)^{1/3} = 8^{1/3}\left(1+\frac{3}{8}x\right)^{1/3}$$
$$= 2\left\{1 + \frac{1}{3}\left(\frac{3}{8}x\right) + \frac{1}{2!}\times\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{3}{8}x\right)^2 + \frac{1}{3!}\times\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)\left(\frac{3}{8}x\right)^3 - \dots\right\}$$
$$= 2\left\{1 + \frac{1}{8}x - \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{2}{3}\cdot\frac{3^2}{8^2}x^2 + \frac{1}{6}\cdot\frac{1}{3}\cdot\frac{2}{3}\cdot\frac{5}{3}\cdot\frac{3^3}{8^3}x^3 - \dots\right\}$$
$$= 2\left\{1 + \frac{1}{8}x - \frac{1}{8^2}x^2 + \frac{5}{3.8^3}x^3 - \dots\right\}$$
$$= 2 + \frac{1}{4}x - \frac{1}{32}x^2 + \frac{5}{768}x^3 - \dots$$

The expansion is valid for $|\frac{3}{8}x| < 1$ i.e. for $|x| < \frac{8}{3}$.
Substituting $x = 0{\cdot}24$, we have

$$(8{\cdot}72)^{1/3} = 2 + \tfrac{1}{4}(0{\cdot}24) - \tfrac{1}{32}(0{\cdot}24)^2 + \tfrac{5}{768}(0{\cdot}24)^3 - \dots$$
$$= 2 + 0{\cdot}06 - 0{\cdot}0018 + 0{\cdot}00009 - \dots$$
$$\approx 2{\cdot}05829.$$

Hence the approximate value of $\sqrt[3]{8{\cdot}72}$ is 2·05829.

[To decide whether it is reasonable to give this result to 5 decimal places, we may consider the sum, S_n, of the first n terms in the expansion, for $n = 1, 2, 3, \dots$.

$$S_1 = 2$$
$$S_2 = 2{\cdot}06$$
$$S_3 = 2{\cdot}0582$$
$$S_4 = 2{\cdot}05829$$
$$S_5 = 2{\cdot}0582846$$

Each term in the expansion of $(8{\cdot}72)^{1/3}$ is smaller than the previous term but opposite in sign. It follows that $\sqrt[3]{8{\cdot}72}$ lies between S_4 and S_5. Thus the difference between 2·05829 and the true value of $\sqrt[3]{8{\cdot}72}$ must be less than 1×10^{-5}.]

Example 2 Given that the first three terms in the expansion in ascending powers of x of $(1 - 8x)^{1/4}$ are the same as the first three terms in the expansion of $(1 + ax)/(1 + bx)$, find the values of a and b. Hence find an approximation to $(0{\cdot}6)^{1/4}$ in the form p/q, where p and q are integers.

$$(1 - 8x)^{1/4} = 1 + \frac{1}{4}(-8x) + \frac{1}{2!} \times \frac{1}{4}\left(\frac{1}{4} - 1\right)(-8x)^2 + \dots$$

$$= 1 - 2x - \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{4}\cdot 8^2x^2 - \dots$$

$$= 1 - 2x - 6x^2 - \dots$$

$$\frac{1 + ax}{1 + bx} = (1 + ax)(1 + bx)^{-1}$$

$$= (1 + ax)\left\{1 + (-1)bx + \frac{1}{2!}(-1)(-2)b^2x^2 + \dots\right\}$$

$$= (1 + ax)(1 - bx + b^2x^2 + \dots)$$
$$= 1 + ax - bx - abx^2 + b^2x^2 + \dots$$
$$= 1 + (a - b)x + (b^2 - ab)x^2 + \dots$$

Since the first three terms of the expansions are the same.

$$a - b = -2, \quad b^2 - ab = -6$$
i.e. $$b - a = 2, \quad b(b - a) = -6$$

Hence $b = -3$ and $a = -5$.

Thus
$$(1 - 8x)^{1/4} \approx \frac{1 - 5x}{1 - 3x}.$$

Substituting $x = 0{\cdot}05$, we have

$$(1 - 0{\cdot}4)^{1/4} \approx \frac{1 - 0{\cdot}25}{1 - 0{\cdot}15} = \frac{0{\cdot}75}{0{\cdot}85}.$$

Hence $(0{\cdot}6)^{1/4}$ is approximately equal to 15/17.

Example 3 Assuming that x is so small that terms in x^3 and higher powers may be neglected, find a quadratic approximation to $\sqrt{\left(\frac{1 - x}{1 + 2x}\right)}$.

$$\sqrt{\left(\frac{1-x}{1+2x}\right)} = (1-x)^{1/2}(1+2x)^{-1/2}$$

$$(1-x)^{1/2} = 1 + \frac{1}{2}(-x) + \frac{1}{2!} \times \frac{1}{2}\left(-\frac{1}{2}\right)(-x)^2 + \dots$$

$$= 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \dots$$

$$(1+2x)^{-1/2} = 1 + \left(-\frac{1}{2}\right)(2x) + \frac{1}{2!} \times \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(2x)^2 + \dots$$

$$= 1 - x + \frac{3}{2}x^2 - \dots$$

$\therefore$ neglecting terms in x^3 and higher powers,

$$\sqrt{\left(\frac{1-x}{1+2x}\right)} \approx \left(1 - \frac{1}{2}x - \frac{1}{8}x^2\right)\left(1 - x + \frac{3}{2}x^2\right)$$

$$\approx 1 - x + \frac{3}{2}x^2 - \frac{1}{2}x + \frac{1}{2}x^2 - \frac{1}{8}x^2$$

Hence $\sqrt{\left(\frac{1-x}{1+2x}\right)} \approx 1 - \frac{3}{2}x + \frac{15}{8}x^2.$

Example 4 Expand $\frac{1}{1+x+2x^2}$ in ascending powers of x up to and including the term in x^3.

$$\frac{1}{1+y} = (1+y)^{-1}$$

$$= 1 + (-1)y + \frac{(-1)(-2)}{2!}y^2 + \frac{(-1)(-2)(-3)}{3!}y^3 + \dots$$

$$= 1 - y + y^2 - y^3 + \dots$$

Substituting $y = x + 2x^2$, we obtain

$$\frac{1}{1+x+2x^2} = 1 - (x+2x^2) + (x+2x^2)^2 - (x+2x^2)^3 + \dots$$

$$= 1 - x - 2x^2 + x^2 + 4x^3 - x^3 + \dots$$

$$= 1 - x - x^2 + 3x^3 + \dots$$

Suppose that we now wish to expand $\frac{1}{(1-x)(1+2x)}$ in ascending powers of x. We could use the method of Example 4:

$$\frac{1}{(1-x)(1+2x)} = (1 + x - 2x^2)^{-1} = 1 - (x - 2x^2) + (x - 2x^2)^2 - \dots$$

or we could write:

$$\frac{1}{(1-x)(1+2x)} = (1-x)^{-1}(1+2x)^{-1}$$
$$= (1+x+x^2+\ldots)(1-2x+4x^2-\ldots)$$

We may quickly obtain the first few terms of the expansion using either of these methods. However, if a general formula for the term in x^r is required, we must adopt a different approach. This further method involves the use of partial fractions. (See §18.6)

Example 5 Expand $\dfrac{1}{(1-x)(1+2x)}$ in ascending powers of x, giving the first three terms and the term in x^r. State the values of x for which the expansion is valid.

$$(1-x)^{-1} = 1 + (-1)(-x) + \frac{(-1)(-2)}{2!}(-x)^2 + \ldots + \frac{(-1)(-2)\ldots(-r)}{r!}(-x)^r + \ldots$$

Thus $(1-x)^{-1} = 1 + x + x^2 + \ldots + x^r + \ldots$

Similarly $(1+2x)^{-1} = 1 - 2x + 4x^2 + \ldots + (-2)^r x^r + \ldots$

Let $\dfrac{1}{(1-x)(1+2x)} \equiv \dfrac{A}{1-x} + \dfrac{B}{1+2x}$

then $1 \equiv A(1+2x) + B(1-x)$

Putting $x = 1$: $1 = 3A \quad \therefore \quad A = \frac{1}{3}$

Putting $x = -\frac{1}{2}$: $1 = \frac{3}{2}B \quad \therefore \quad B = \frac{2}{3}$

$$\therefore \quad \frac{1}{(1-x)(1+2x)} = \tfrac{1}{3}(1-x)^{-1} + \tfrac{2}{3}(1+2x)^{-1}$$
$$= \tfrac{1}{3}(1+x+x^2+\ldots+x^r+\ldots) + \tfrac{2}{3}(1-2x+4x^2+\ldots+(-2)^r x^r+\ldots)$$
$$= 1 - x + 3x^2 + \ldots + \{\tfrac{1}{3} + \tfrac{2}{3}(-2)^r\}x^r + \ldots$$
$$= 1 - x + 3x^2 + \ldots + \tfrac{1}{3}\{1-(-2)^{r+1}\}x^r + \ldots$$

The expansion of $(1-x)^{-1}$ is valid for $|x| < 1$ and the expansion of $(1+2x)^{-1}$ is valid for $|2x| < 1$, i.e. $|x| < \frac{1}{2}$.

Thus the expansion of $\dfrac{1}{(1-x)(1+2x)}$ is valid when $|x| < \frac{1}{2}$.

Exercise 22.3

In questions 1 to 9 expand the given function in ascending powers of x up to and including the term in x^3.

1. $(1+x)^{-3}$ 2. $(1+2x)^{1/2}$ 3. $(4+x)^{-3/2}$

4. $\dfrac{1+2x}{1-2x}$ 5. $\dfrac{1-x}{(2+x)^2}$ 6. $\dfrac{1-2x}{\sqrt{(1-4x)}}$

7. $\dfrac{1}{1+x+x^2}$ 8. $\dfrac{1}{\sqrt{(1+x-2x^2)}}$ 9. $\sqrt[3]{(1+x+3x^2)}$

In questions 10 to 15 assume that x is so small that terms in x^3 and higher powers of x may be neglected. Hence find a quadratic approximation to the given function, stating the values of x for which your answer is valid.

10. $\sqrt{\left(\dfrac{1-x}{1+x}\right)}$ 11. $\sqrt[3]{\left(\dfrac{8+x}{1-3x}\right)}$ 12. $\dfrac{(1+4x)^{1/4}}{(1+5x)^{1/5}}$

13. $\dfrac{1}{(1+x)(3-x)}$ 14. $\dfrac{1}{(1-x)(1+2x)^2}$ 15. $\dfrac{1}{(1-2x)\sqrt{(1-x)}}$

In questions 16 to 21 use partial fractions to find the first five non-zero terms in the expansion of the given function in ascending powers of x. State the values of x for which the expansion is valid.

16. $\dfrac{3-5x}{(1-3x)(1+x)}$ 17. $\dfrac{4x}{(1-x)(3+x)}$ 18. $\dfrac{1+x}{(1+x^2)(1-x)}$

19. $\dfrac{4-x}{(1-x)^2(1+2x)}$ 20. $\dfrac{2}{(1+x)(1+x^2)}$ 21. $\dfrac{8(2x-1)}{(x-2)^2(x^2+2)}$

In questions 22 to 24 expand the given function in ascending powers of x, giving the first three non-zero terms and the term in x^r.

22. $\dfrac{3}{(1+x)(1-2x)}$ 23. $\dfrac{4x}{(1-x)(1+3x)}$ 24. $\dfrac{4+3x}{(2-x)(1+2x)}$

25. Expand $(1+2x)^{1/4}$ in ascending powers of x as far as the term in x^3, stating the values of x for which the expansion is valid. Hence obtain approximate values for (a) $\sqrt[4]{1{\cdot}4}$, (b) $\sqrt[4]{1{\cdot}08}$. Obtain the values of $\sqrt[4]{1{\cdot}4}$ and $\sqrt[4]{1{\cdot}08}$ by calculator, (using the function $x^{1/y}$ if available). Explain why the error in answer (a) is greater than the error in answer (b).

26. Expand $(a+2x)\sqrt{(1-4x)} + \dfrac{b}{(1-x)^2}$ in ascending powers of x as far as the term in x^3. Given that the coefficients of x and x^2 are zero, find the first term in the expansion and the coefficient of x^3.

27. Given that the first three terms in the expansion in ascending powers of x of $(1+x+x^2)^n$ are the same as the first three terms in the expansion of $\left(\dfrac{1+ax}{1-3ax}\right)^3$,

find the non-zero values of n and a. Show that the coefficients of x^3 in the two expansions differ by 7·5.

28. If x is so small that x^3 and higher powers of x may be neglected, find the values of a and b such that

$$\sqrt{(1 + 4x)} \approx \frac{1 + ax}{1 + bx}.$$

By letting $x = 0{\cdot}04$ find an approximation to $\sqrt{29}$ in the form p/q, where p and q are integers.

22.4 Exponential and logarithmic series

As shown in §22.2 the expansion in ascending powers of x of the exponential function e^x is as follows:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^r}{r!} + \ldots \text{ for all real values of } x.$$

Example 1 Expand e^{2x} in ascending powers of x giving the first three terms and the term in x^r.

$$e^{2x} = 1 + (2x) + \frac{(2x)^2}{2!} + \ldots + \frac{(2x)^r}{r!} + \ldots$$

$$= 1 + 2x + 2x^2 + \ldots + \frac{2^r}{r!}x^r + \ldots$$

Example 2 Find the first three terms in the expansion of e^{x+x^2} in ascending powers of x.

$$e^{x+x^2} = 1 + (x + x^2) + \frac{(x + x^2)^2}{2!} + \ldots$$

$$= 1 + x + x^2 + \frac{1}{2}x^2 + \text{higher powers of } x$$

$$= 1 + x + \frac{3}{2}x^2 + \ldots$$

Example 3 Expand e^{2-x} in ascending powers of x as far as the term in x^3.

$$e^{2-x} = e^2 \times e^{-x} = e^2\left\{1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \ldots\right\}$$

$$= e^2\left\{1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \ldots\right\}$$

Example 4 Find the coefficient of x^r in the expansion of $(x^2 + 2x + 3)e^x$ in ascending powers of x.

Considering the terms $\ldots + \dfrac{x^{r-2}}{(r-2)!} + \dfrac{x^{r-1}}{(r-1)!} + \dfrac{x^r}{r!} + \ldots$ in the expansion of e^x, we find that the terms in x^r in the expansion of $(x^2 + 2x + 3)e^x$ must be

$$\ldots + x^2 \times \frac{x^{r-2}}{(r-2)!} + 2x \times \frac{x^{r-1}}{(r-1)!} + 3 \times \frac{x^r}{r!} + \ldots$$

Hence the coefficient of x^r in the expansion

$$= \frac{1}{(r-2)!} + \frac{2}{(r-1)!} + \frac{3}{r!}$$

$$= \frac{r(r-1)}{r!} + \frac{2r}{r!} + \frac{3}{r!}$$

$$= \frac{r^2 - r + 2r + 3}{r!}$$

$$= \frac{r^2 + r + 3}{r!}$$

In §22.2 we also obtained the logarithmic series:

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{r+1}\frac{x^r}{r} + \ldots \text{ where } -1 < x \leqslant 1.$$

Replacing x by $-x$ in this expansion gives:

$$\ln(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots - \frac{x^r}{r} - \ldots \text{ where } -1 \leqslant x < 1.$$

When these series are used to find numerical values of logarithms, large numbers of terms are needed to produce accurate results. A much better expansion for this purpose is given by:

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1 + x) - \ln(1 - x)$$

$$= \left(x - \frac{x^2}{2} + \frac{x^2}{3} - \ldots\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots\right)$$

i.e.

$$\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right)$$

We demonstrate the advantages of using this series rather than the expansion of $\ln(1 + x)$ by calculating a sequence of approximate values for $\ln 1{\cdot}5$.

	$\ln(1 + x)$, $x = \frac{1}{2}$	$\ln\left(\frac{1+x}{1-x}\right)$, $x = \frac{1}{5}$
1 term	0·5	0·4
2 terms	0·375	0·40533333
3 terms	0·41666667	0·40546133
4 terms	0·40104167	0·40546499
5 terms	0·40729167	0·40546510
6 terms	0·40468750	0·40546511

Using the figures on the left we can give $\ln 1{\cdot}5$ to 1 decimal place, but not to 2. However, the figures on the right give a result correct to 8 decimal places.

We now use the laws of logarithms to obtain some further expansions.

Example 5 Expand $\ln(2 + 3x)$ in ascending powers of x as far as the term in x^3. State the values of x for which the expansion is valid.

$$\ln(2 + 3x) = \ln 2\left(1 + \frac{3}{2}x\right)$$

$$= \ln 2 + \ln\left(1 + \frac{3}{2}x\right)$$

$$= \ln 2 + \left(\frac{3}{2}x\right) - \frac{1}{2}\left(\frac{3}{2}x\right)^2 + \frac{1}{3}\left(\frac{3}{2}x\right)^3 - \ldots$$

$$= \ln 2 + \frac{3}{2}x - \frac{9}{8}x^2 + \frac{9}{8}x^3 - \ldots$$

The expansion is valid for $-1 < \frac{3}{2}x \leqslant 1$ i.e. $-\frac{2}{3} < x \leqslant \frac{2}{3}$.

Example 6 Expand $\ln\dfrac{1+x}{\sqrt{(1-2x)}}$ in ascending powers of x as far as the term in x^3, stating the values of x for which the expansion is valid.

$$\ln\frac{1+x}{\sqrt{(1-2x)}} = \ln(1 + x) - \frac{1}{2}\ln(1 - 2x)$$

$$= \left\{x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \ldots\right\}$$

$$-\frac{1}{2}\left\{(-2x) - \frac{1}{2}(-2x)^2 + \frac{1}{3}(-2x)^3 - \ldots\right\}$$

$$= \left\{x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \ldots\right\} + \left\{x + x^2 + \frac{4}{3}x^3 + \ldots\right\}$$

$$= 2x + \frac{1}{2}x^2 + \frac{5}{3}x^3 + \ldots$$

The expansion of $\ln(1+x)$ is valid for $-1 < x \leqslant 1$ and the expansion of $\ln(1-2x)$ is valid for $-1 < -2x \leqslant 1$, i.e. $-\frac{1}{2} \leqslant x < \frac{1}{2}$.

Hence the expansion of $\ln\dfrac{1+x}{\sqrt{(1-2x)}}$ is valid for $-\frac{1}{2} \leqslant x < \frac{1}{2}$.

Example 7 Assuming that x is so small that x^3 and higher powers may be neglected, obtain a quadratic approximation to the function

$$\frac{1+e^x}{1+\ln(1+x)}.$$

$$1+e^x = 1+\left(1+x+\frac{x^2}{2!}+\ldots\right) \approx 2+x+\frac{1}{2}x^2$$

$$1+\ln(1+x) = 1+\left(x-\frac{x^2}{2}+\ldots\right) \approx 1+x-\frac{1}{2}x^2$$

Using the binomial expansion,

$$(1+y)^{-1} = 1+(-1)y+\frac{(-1)(-2)}{2!}y^2+\ldots = 1-y+y^2-\ldots$$

Substituting $y = x-\frac{1}{2}x^2$,

$$\frac{1}{1+\ln(1+x)} \approx \left\{1+\left(x-\frac{1}{2}x^2\right)\right\}^{-1}$$

$$= 1-\left(x-\frac{1}{2}x^2\right)+\left(x-\frac{1}{2}x^2\right)^2-\ldots$$

$$= 1-x+\frac{1}{2}x^2+x^2+\ldots$$

$$\approx 1-x+\frac{3}{2}x^2$$

$$\therefore \quad \frac{1+e^x}{1+\ln(1+x)} \approx \left(2+x+\frac{1}{2}x^2\right)\left(1-x+\frac{3}{2}x^2\right)$$

$$\approx 2-2x+3x^2+x-x^2+\frac{1}{2}x^2$$

$$\text{i.e.} \quad \frac{1+e^x}{1+\ln(1+x)} \approx 2-x+\frac{5}{2}x^2.$$

Exercise 22.4

1. Find the first three non-zero terms and the general terms in the expansions in ascending powers of x of:

(a) e^{-x} (b) e^{3x} (c) e^{x^2} (d) e^{1+x}

2. Find the first four terms in the expansions of:
(a) $(1 + 2x)e^{2x}$ (b) $(1 - x)^2e^{-x}$ (c) $e^{(x^2-x)}$

3. Use the exponential series to calculate the values of e and $1/e$ correct to 4 decimal places.

4. Find the first three non-zero terms and the general terms in the expansions of the following functions, stating the values of x for which the expansions are valid:
(a) $\ln(1 + 3x)$ (b) $\ln(1 - \frac{1}{2}x)$ (c) $\ln(3 + x)$ (d) $\ln(e - x)$

5. Find the first four terms in the expansions of:

(a) $\ln\left(\dfrac{1 + x}{1 + x^2}\right)$ (b) $\ln(1 + x - 2x^2)$ (c) $\ln\dfrac{(1 - x)^2}{\sqrt{(1 + 4x)}}$

6. Use the expansion of $\ln\left(\dfrac{1 + x}{1 - x}\right)$ to calculate the value of $\ln 3$ correct to 4 decimal places.

7. Find the coefficient of x^r in the expansions of each of the following functions in ascending powers of x:
(a) $(1 - x)e^x$ (b) $(1 + x - 3x^2)e^{2x}$ (c) $(1 + x)^2e^{x/2}$

8. Given that x is so small that terms in x^3 and higher powers of x may be neglected, find quadratic approximations to the following functions:
(a) $e^x \ln(1 + x)$ (b) $(1 + e^{-x})^2$ (c) $(1 + x)^8e^{-x/2}$

(d) $\dfrac{e^{3x} - e^x}{e^{2x}}$ (e) $\dfrac{1 - \ln(1 - x)}{1 - x}$ (f) $\dfrac{\ln(1 - 2x)}{1 - e^x}$

9. Show that, if $n > 1$, $\ln\left(\dfrac{n + 1}{n - 1}\right) = 2\left(\dfrac{1}{n} + \dfrac{1}{3n^3} + \dfrac{1}{5n^5} + \ldots\right)$. Hence calculate the value of $\ln 2$ correct to 4 decimal places.

10. The expansion of $(1 - x)^n$ in ascending powers of x is the same as the expansion of e^{ax+x^2} as far as the term in x^2. Find the values of a and n.

11. The expansion of $(3x + ax^2)/(3 + bx)$ in ascending powers of x is the same as the expansion of $\ln(1 + x)$ as far as the term in x^3. Find the values of a and b. Hence find a rational approximation to $\ln 1{\cdot}2$.

12. (a) Use the expansion of e^x to show that if n is a positive integer, then $x^ne^{-x} \to 0$ as $x \to \infty$.
(b) By letting $x = e^y$, show that $\dfrac{1}{x}\ln x \to 0$ as $x \to \infty$.

13. Show that, if $n > 1$, $\ln\left(1+\frac{1}{n}\right)^n = 1 - \frac{1}{2n} + \frac{1}{3n^2} - \ldots$

Use the expansion of e^x to deduce that

$$\left(1+\frac{1}{n}\right)^n = e\left(1 - \frac{1}{2n} + \frac{11}{24n^2} - \ldots\right).$$

Deduce that, as $n \to \infty$, $\left(1+\frac{1}{n}\right)^n \to e$.

22.5 The expansions of sin x and cos x

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots, \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

for all real values of x.
The expansions of $\sin x$ and $\cos x$ are used in the same ways as the various series discussed in previous sections.

Example 1 Expand $e^{\sin x}$ in ascending powers of x, neglecting terms in x^4 and higher powers of x.

$$\sin x = x - \frac{x^3}{3!} + \ldots = x - \frac{1}{6}x^3 + \ldots$$

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots = 1 + y + \frac{1}{2}y^2 + \frac{1}{6}x^3 + \ldots$$

Substituting $y = x - \frac{1}{6}x^3$,

$$e^{\sin x} = 1 + \left(x - \frac{1}{6}x^3\right) + \frac{1}{2}\left(x - \frac{1}{6}x^3\right)^2 + \frac{1}{6}\left(x - \frac{1}{6}x^3\right)^3 + \ldots$$

$$= 1 + x - \frac{1}{6}x^3 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \ldots$$

$$= 1 + x + \frac{1}{2}x^2 + \ldots$$

Example 2 Expand $\sec x$ in ascending powers of x, as far as the term in x^4.

$$\sec x = (\cos x)^{-1} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right)^{-1} = \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \ldots\right)^{-1}$$

Using the binomial expansion, $(1+y)^{-1} = 1 - y + y^2 - \ldots$

Thus, substituting $y = -\frac{x^2}{2} + \frac{x^4}{24}$,

$$\sec x = 1 - \left(-\frac{x^2}{2} + \frac{x^4}{24}\right) + \left(-\frac{x^2}{2} + \frac{x^4}{24}\right)^2 - \dots$$

$$= 1 + \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^4}{4} + \dots$$

$$= 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \dots$$

Exercise 22.5

1. Evaluate, correct to 4 decimal places,
(a) $\sin 0{\cdot}3$ (b) $\cos 0{\cdot}4$ (c) $\cos 2$ (d) $\sin 1$

2. Find the first three non-zero terms in the expansions of the following functions in ascending powers of x:
(a) $\sin 3x$ (b) $\cos 2x$ (c) $\ln(1 + \sin x)$
(d) $e^{\cos x}$ (e) $x \operatorname{cosec} x$ (f) $\tan x$

3. Find approximations to the following functions, given that powers of x higher than the fourth are negligible:
(a) $\sin x(x + \cos x)$ (b) $\sin\{\ln(1 + 2x)\}$
(c) $\cos(\sin 2x) - e^{-2x^2}$ (d) $e^{2x}\cos^2 x$

Exercise 22.6 (miscellaneous)

1. Find the sums of the following series
(a) $\sum_{r=2}^{n} r(r-1)$ (b) $\sum_{r=0}^{n} (2r+1)(2r+3)$
(c) $\sum_{r=n}^{2n} \frac{1}{r(r+1)}$ (d) $\sum_{r=2}^{n} \frac{1}{r^2 - 1}$

2. Find the sum to n terms of each of the following series
(a) $1.2^2 + 2.3^2 + 3.4^2 + \dots$ (b) $1.2.3 + 3.4.5 + 5.6.7 + \dots$

3. Find the sum of all possible products of the form rs^2, where r and s are distinct positive integers such that $1 \leqslant r, s \leqslant n$.

4. The sum of the first n terms of the series $1 + 2x^2 + 3x^4 + \dots$, where $x^2 \neq 1$, is S. Find an expression for S by considering $S - x^2 S$.

5. Using the remainder theorem, or otherwise, factorise $x^3 + 6x^2 + 11x + 6$. Express $\dfrac{4x+6}{x^3 + 6x^2 + 11x + 6}$ in partial fractions. Hence show that

$$\sum_{n=0}^{18} \frac{4n+6}{n^3 + 6n^2 + 11n + 6} = 2\frac{43}{140}.$$ (L)

6. (a) Find the sum of the first n terms of the series whose rth term is $2^r + 2r - 1$.
(b) If x is so small that terms in x^n, $n \geqslant 3$, can be neglected and $\dfrac{3 + ax}{3 + bx} = (1 - x)^{1/3}$, find the values of a and b. Hence, without the use of tables, find an approximation in the form p/q, where p and q are integers, for $\sqrt[3]{0{\cdot}96}$. (AEB 1979)

7. Find the expansions of $\left(1 + \dfrac{1}{x^2}\right)^{1/2}$ and $\left(1 - \dfrac{1}{x^2}\right)^{1/2}$ in ascending powers of $1/x^2$ up to and including the term in $1/x^4$. Hence, or otherwise, show that, if the positive number x is very large compared with 1, then $(x^2 + 1)^{1/2} - (x^2 + 1)^{1/2} \approx \dfrac{1}{x}$. (C)

8. Expand $\sqrt{(4 - x)}$ as a series in ascending powers of x up to and including the terms in x^2. If terms in x^n, $n \geqslant 3$, can be neglected, find the quadratic approximation to $\sqrt{\left(\dfrac{4 - x}{1 - 2x}\right)}$. State the range of values of x for which this approximation is valid. (AEB 1980)

9. Expand the function $(2 - x)\sqrt{(1 + 2x + 2x^2)}$ in ascending powers of x as far as the term in x^3.

10. Use partial fractions to expand the following functions in ascending powers of x, giving the first three non-zero terms and the term in x^r. State the values of x for which the expansions are valid.

(a) $\dfrac{3x}{(1 + 2x)(1 - x)}$ (b) $\dfrac{x}{(x - 3)(x - 2)}$ (c) $\dfrac{1 - x - x^2}{(1 - 3x)(1 - 2x)(1 - x)}$

11. Expand the following functions in ascending powers of x, as far as the term in x^4. State the values of x for which the expansions are valid.

(a) $\dfrac{1 - 2x}{\sqrt{(1 + 2x^2)}}$ (b) $\dfrac{1 + 2x^2}{(1 - 2x)^2}$ (c) $\dfrac{3}{(1 + 2x^2)(1 - 2x)}$

12. (a) Expand $e^{x/2} \log_e(1 + x)$ in ascending powers of x as far as the term in x^4 and hence show that, for certain values of x to be stated, $e^{x/2} \log_e(1 + x) + e^{-x/2} \log_e(1 - x) = ax^4 + \ldots$ and give the value of a.
(b) Expand $(1 + x)^{-1/2}$ in ascending powers of x, giving the first four terms and the general term. Hence, without using tables or calculator, obtain the value of $1/\sqrt{101}$, correct to 6 decimal places, showing all working. (SU)

13. (a) Draw rough sketches of the following graphs, showing quite clearly their general shape, and where they cross any axis:
(i) $y = e^x$ (ii) $y = e^{-x}$ (iii) $y = \log_e x$ (iv) $y = \log_e(1 + x)$

(b) Expand $\log_e \dfrac{(1+2x)^2}{1-3x}$ in ascending powers of x, as far as the term in x^4, giving the range of values of x for which the expansion is valid. What is the nth term?

(c) Prove that the coefficient of x^n in the expansion of $e^{2x}(x^2+3x-7)$ is $\dfrac{2^{n-2}(n^2+5n-28)}{n!}$ (SU)

14. Obtain the series expansion of $\dfrac{e^{2x}}{1+x}$ in ascending powers of x up to and including the term in x^3. Using these terms find an approximation to the value of $\displaystyle\int_0^{1/4} \frac{e^{2x}dx}{1+x}$, giving your answer correct to three places of decimals. (O)

15. State the series expansion in ascending powers of x, including the general term, of $\log_e(1+x)$ for $|x|<1$. Show that

(i) $\log_e\left(\dfrac{1+x}{1-x}\right) = 2\left[x+\dfrac{1}{3}x^3+\dfrac{1}{5}x^5+\ldots\right]$,

(ii) $\log_e m = 2\left[\left(\dfrac{m-1}{m+1}\right)+\dfrac{1}{3}\left(\dfrac{m-1}{m+1}\right)^3+\dfrac{1}{5}\left(\dfrac{m-1}{m+1}\right)^5+\ldots\right]$.

Hence evaluate $\log_e 2$ to four decimal places, showing all your working.

If $\log_e\left(\dfrac{1+x}{1-x}\right)-\left(\dfrac{e^{2x}-1}{e^x}\right) = ax^3+bx^5$, when terms in x^n, $n \geqslant 6$, are neglected, find the values of a and b. (AEB 1978)

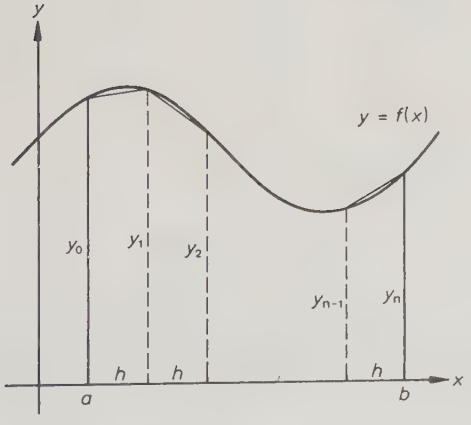
16. Given that $1-px+x^2 \equiv (1-\alpha x)(1-\beta x)$, obtain expressions for $\alpha^2+\beta^2$ and $\alpha^3+\beta^3$ in terms of p. Hence, or otherwise, find the first three terms in the expansion of $\log_e(1-px+x^2)$ in ascending powers of x. (JMB)

17. Find the values of positive constants a, b and c given that the expansions of $e^{-ax}\sin 2x$ and $\cos bx \ln(1+cx)$ in ascending powers of x are the same as far as the term in x^3.

23 Some numerical methods

23.1 Numerical integration

When a function $f(x)$ is difficult to integrate, it may be possible to estimate the value of the definite integral $\int_a^b f(x)\,dx$ by numerical methods. Such methods are also useful when, instead of a formula for $f(x)$, we have a set of values obtained experimentally.



The integral $\int_a^b f(x)\,dx$ is represented by the area under the curve $y = f(x)$ between $x = a$ and $x = b$. Suppose that the area is divided into n strips of width h, as shown in the diagram. The ordinates $y_0, y_1, y_2, \ldots, y_n$ represent the values of $f(x)$ at $x_0 = a, x_1 = a + h, x_2 = a + 2h, \ldots, x_n = b$.

An approximate value for the area is obtained by regarding each strip as a trapezium, so that

$$\int_a^b f(x)\,dx \approx \tfrac{1}{2}h(y_0 + y_1) + \tfrac{1}{2}h(y_1 + y_2) + \ldots + \tfrac{1}{2}h(y_{n-1} + y_n)$$

i.e.

$$\int_a^b f(x)\,dx \approx \tfrac{1}{2}h(y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n)$$

Thus we have the *trapezium rule* for n strips ($n + 1$ ordinates)

$$\int_a^b f(x)\,dx \approx \tfrac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\},$$

where $h = (b - a)/n$.

Example 1 Use the trapezium rule with 5 ordinates to find an approximate value for $\int_0^2 \frac{1}{1 + x^2}\,dx$.

Using 5 ordinates in the interval from $x = 0$ to $x = 2$ gives 4 strips of width $h = 0{\cdot}5$.

n	x_n	y_n (rounded to 5 d.p.)
0	0	1
1	0·5	0·8
2	1	0·5
3	1·5	0·307 69
4	2	0·2
		1·2
		3·215 38
		4·415 38

1·607 69
× 2
3·215 38

By the trapezium rule,

$$\int_0^2 \frac{1}{1 + x^2}\,dx \approx \tfrac{1}{2}h\{(y_0 + y_4) + 2(y_1 + y_2 + y_3)\}$$
$$\approx \tfrac{1}{2} \times 0{\cdot}5 \times 4{\cdot}415\,38$$
$$\approx 1{\cdot}104$$

This integral can also be evaluated directly as follows:

$$\int_0^2 \frac{1}{1 + x^2}\,dx = \Big[\tan^{-1} x\Big]_0^2 = \tan^{-1} 2 = 1{\cdot}107\,15 \text{ (to 5 d.p.)}$$

Thus the error in the value obtained using the trapezium rule is less than 0·5%. The error may be reduced by increasing the number of ordinates used. For instance, we can obtain a better approximation by using 11 ordinates i.e. 10 intervals of width $h = 0{\cdot}2$.

n	x_n	y_n (rounded to 5 d.p.)
0	0	1
1	0·2	0·961 54
2	0·4	0·862 07
3	0·6	0·735 29
4	0·8	0·609 76
5	1	0·5
6	1·2	0·409 84
7	1·4	0·337 84

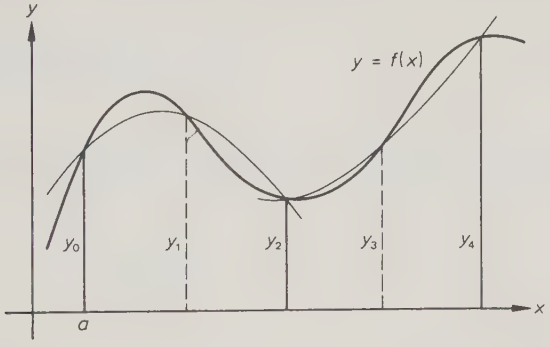
8	1·6		0·280 90
9	1·8		0·235 85
10	2	0·2	
		1·2	4·933 09
		9·866 18	× 2
		11·066 18	9·866 18

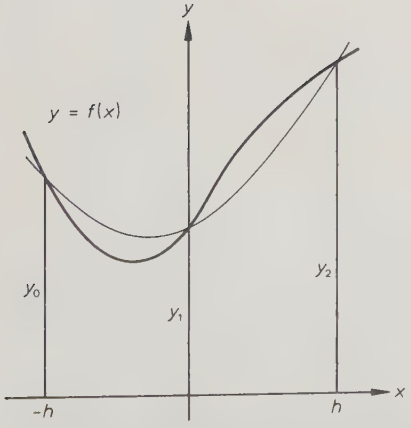
By the trapezium rule, $\int_0^2 \frac{1}{1+x^2}\,dx \approx \frac{1}{2}h\{(y_0 + y_{10}) + 2(y_1 + y_2 + \ldots + y_9)\}$

$$\approx \tfrac{1}{2} \times 0{\cdot}2 \times 11{\cdot}066\,18$$
$$\approx 1{\cdot}1066$$

Using 21 ordinates the value 1·1070 is obtained. However, rounding errors in the values of y_n will limit further improvements in accuracy.

Evaluating the integral $\int_a^b f(x)\,dx$ by the trapezium rule is equivalent to replacing the curve $y = f(x)$ by a series of straight lines joining the tops of the ordinates $y_0, y_1, y_2, \ldots$ A better estimate of the area under the curve is usually obtained using *Simpson's rule.*† In this case the curve $y = f(x)$ is regarded as a series of parabolic arcs joining the tops of the ordinates y_0, y_1 and y_2; y_2, y_3 and y_4; ....





We consider first the integral $\int_{-h}^{h} f(x)\,dx$, letting y_0, y_1, y_2 be the ordinates at $x_0 = -h, x_1 = 0$ and $x_2 = h$. If $y = px^2 + qx + r$ is the parabola which joins the tops of the ordinates y_0, y_1, y_2, then

$$y_0 = ph^2 - qh + r,$$
$$y_1 = r,$$
$$y_2 = ph^2 + qh + r.$$

†*Simpson, Thomas* (1710–1761) English mathematician known as 'the oracle of Nuneaton'. The rule which bears his name was used by earlier mathematicians, but Simpson was credited with its discovery when he published it in 1743.

The area under this parabola between $x = -h$ and $x = h$

$$= \int_{-h}^{h} (px^2 + qx + r)\,dx = \left[\tfrac{1}{3}px^3 + \tfrac{1}{2}qx^2 + rx\right]_{-h}^{h}$$

$$= (\tfrac{1}{3}ph^3 + \tfrac{1}{2}qh^2 + rh) - (-\tfrac{1}{3}ph^3 + \tfrac{1}{2}qh^2 - rh)$$

$$= \tfrac{2}{3}ph^3 + 2rh$$

$$= \tfrac{1}{3}h(2ph^2 + 6r)$$

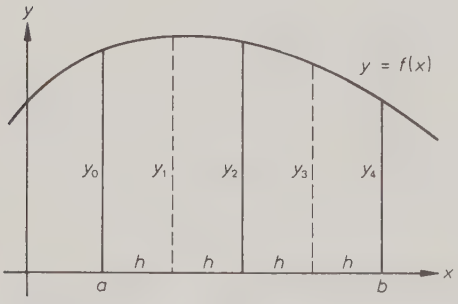
$$= \tfrac{1}{3}h(y_0 + 4y_1 + y_2)$$

Thus $$\int_{-h}^{h} f(x)\,dx \approx \tfrac{1}{3}h(y_0 + 4y_1 + y_2).$$

Since this result depends only on the values of the ordinates y_0, y_1, y_2 and the interval h between them, we may also write:

$$\int_a^b f(x)\,dx \approx \tfrac{1}{3}h(y_0 + 4y_1 + y_2)$$

where $h = \frac{1}{2}(b - a)$ and $y_0 = f(a)$, $y_1 = f(a + h)$, $y_2 = f(b)$.
This is Simpson's rule in its simplest form. The rule for 2 strips (3 ordinates) can be extended to any even number of strips (or any odd number of ordinates.)



In the case of 4 strips, as shown in the diagram, we apply the basic rule twice:

$$\int_a^b f(x)\,dx \approx \tfrac{1}{3}h(y_0 + 4y_1 + y_2) + \tfrac{1}{3}h(y_2 + 4y_3 + y_4)$$

Thus Simpson's rule for 4 strips (5 ordinates) is

$$\int_a^b f(x)\,dx \approx \tfrac{1}{3}h(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4)$$

More generally, Simpson's rule states that:

$$\int_a^b f(x)\,dx \approx \tfrac{1}{3}h(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \ldots + 4y_{n-1} + y_n)$$

i.e.
$$\int_a^b f(x)\,dx \approx \tfrac{1}{3}h\{(y_0 + y_n) + 4(y_1 + y_3 + \ldots) + 2(y_2 + y_4 + \ldots)\}$$
where n is even and $h = (b - a)/n$.

Example 2 Use Simpson's rule with 5 ordinates to find an approximate value for $\int_0^2 \frac{1}{1+x^2}\,dx$.

Using 5 ordinates in the interval from $x = 0$ to $x = 2$ gives 4 strips of width $h = 0{\cdot}5$.

n	x_n	y_n (rounded to 5 d.p.)		
0	0	1		
1	0·5		0·8	
2	1			0·5
3	1·5		0·307 69	
4	2	0·2		
		1·2	1·107 69	0·5
		4·430 76	× 4	× 2
		1·0	4·430 76	1·0
		6·630 76		

By Simpson's rule,

$$\int_0^2 \frac{1}{1+x^2}\,dx \approx \tfrac{1}{3}h\{(y_0 + y_4) + 4(y_1 + y_3) + 2y_2\}$$

$$\approx \tfrac{1}{3} \times 0{\cdot}5 \times 6{\cdot}630\,76$$
$$\approx 1{\cdot}105$$

This is closer to the true value of the integral than the result obtained in Example 1 using the trapezium rule. Accuracy can again be improved by increasing the number of ordinates to 11.

n	x_n	y_n (rounded to 5 d.p.)		
0	0	1		
1	0·2		0·961 54	
2	0·4			0·862 07
3	0·6		0.735 29	
4	0·8			0·609 76
5	1		0·5	
6	1·2			0·409 84
7	1·4		0·337 84	
8	1·6			0·280 90
9	1·8		0·235 85	
10	2	0·2		
		1·2	2·770 52	2·162 57
		11·082 08	× 4	× 2
		4·325 14	11·082 08	4·325 14
		16·607 22		

By Simpson's rule,

$$\int_0^2 \frac{1}{1+x^2}\,dx \approx \tfrac{1}{3}h\{(y_0 + y_{10}) + 4(y_1 + y_3 + \ldots) + 2(y_2 + y_4 + \ldots)\}$$
$$\approx \tfrac{1}{3} \times 0{\cdot}2 \times 16{\cdot}607\,22$$
$$\approx 1{\cdot}1071$$

[Unless more accurate values of y_n can be obtained, there is little point in making further increases in the number of ordinates.]

Under certain conditions an approximate value for $\int_a^b f(x)\,dx$ can be obtained by expanding $f(x)$ in ascending powers of x, then integrating term by term. Clearly the method cannot be used unless the expansion of $f(x)$ is valid throughout the interval $a \leqslant x \leqslant b$.

Since the binomial series for $(1 + x^2)^{-1}$ is valid only when $|x| < 1$, this expansion will not produce a good approximation to the value of $\int_0^2 \frac{1}{1+x^2}\,dx$. We consider instead $\int_0^{0.4} \frac{1}{1+x^2}\,dx$. Using the binomial expansion,

$$(1 + x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \cdots$$

$$\therefore \quad \int_0^{0.4} \frac{1}{1+x^2}\,dx \approx \int_0^{0.4} (1 - x^2 + x^4 - x^6 + \ldots)\,dx$$
$$\approx \left[x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\right]_0^{0.4}$$
$$\approx 0{\cdot}4 - \frac{(0{\cdot}4)^3}{3} + \frac{(0{\cdot}4)^5}{5} - \frac{(0{\cdot}4)^7}{7}$$
$$\approx 0{\cdot}3805.$$

Exercise 23.1

Use the trapezium rule to obtain approximate values of the following integrals, giving your answers to 3 decimal places.

1. $\int_1^3 \frac{1}{1+x}\,dx$, 5 ordinates.
2. $\int_0^3 \sqrt{(1 + x^2)}\,dx$, 4 ordinates.
3. $\int_0^2 10^x\,dx$, 5 strips.
4. $\int_1^8 \lg x\,dx$, 7 strips.
5. $\int_0^{\pi/6} \tan x\,dx$, 7 ordinates.
6. $\int_0^4 xe^{-x}\,dx$, 5 ordinates.

Questions 7 to 12. Use sketches to decide whether the results obtained in questions 1 to 6 are greater than or less than the true values of the given integrals.

Use Simpson's rule to obtain approximate values of the following integrals, giving your answers to 4 decimal places.

13. $\int_0^1 \frac{1}{1+x^3}$, 5 ordinates.

14. $\int_0^{\frac{1}{2}} \sqrt{(1-x^2)}\,dx$, 4 strips.

15. $\int_0^{\pi/3} \sqrt{(\sin x)}\,dx$, 6 strips.

16. $\int_0^4 xe^{-x}\,dx$, 5 ordinates.

17. $\int_1^2 x \lg x\,dx$, 11 ordinates.

18. $\int_0^2 \sqrt{(1+x^3)}\,dx$, 11 ordinates.

19. Expand $(1+x^2)^{1/2}$ in ascending powers of x as far as the term in x^8. Hence find an approximate value for the integral $\int_0^{0.5} (1+x^2)^{1/2}\,dx$. Using the trapezium rule with 6 ordinates, obtain a second estimate for the value of this integral. Give your answers to 4 decimal places.

20. Use Simpson's rule with 5 ordinates to estimate the value of $\int_0^{0.4} \ln(1+x)dx$. By expanding $\ln(1+x)$ in ascending powers of x as far as the term in x^3 and integrating term by term, obtain a second estimate for the value of this integral. Give your answers to 4 decimal places.

21. Values of a continuous function $f(x)$ were obtained experimentally as follows:

x	1	1·5	2	2·5	3
$f(x)$	8·01	6·02	4·69	3·80	3·27

Estimate the value of $\int_1^3 f(x)\,dx$ using (a) the trapezium rule, (b) Simpson's rule.

22. A car of mass 1000 kg starts from rest and accelerates continuously along a straight road for 20 seconds. After t seconds the resultant force acting on the car is P newtons. The table below gives values of P at intervals of 5 seconds.

t	0	5	10	15	20
P	800	740	590	460	360

Use the trapezium rule to estimate (a) the speed of the car after 10 seconds, (b) the speed of the car after 20 seconds. Use the trapezium rule and your answers to (a) and (b) to estimate the distance travelled in the first 20 seconds.

23. A particle moves in a straight line under the action of a variable force of magnitude F newtons. The value of F after the particle has travelled a distance of s metres is given in the table below.

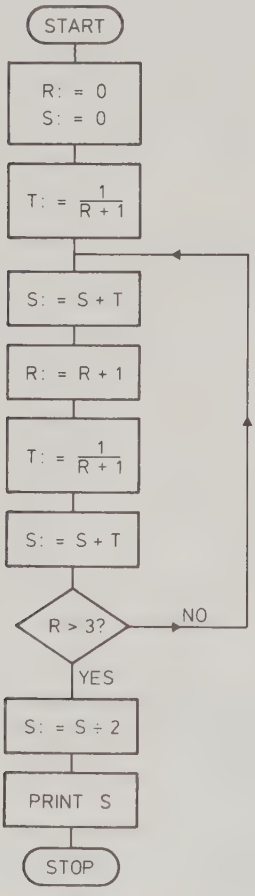
s	0	1	2	3	4	5	6
F	54	61	67	72	76	79	81

Estimate the work done when the particle has travelled a distance of 6 metres using (a) the trapezium rule, (b) Simpson's rule.

24. Construct a flow diagram to evaluate $\int_0^1 e^{-x^2}\,dx$ using the trapezium rule with 6 ordinates. Test your procedure using a pocket calculator.

25. Construct a flow diagram to evaluate $\int_2^3 \sqrt{(x^2-4)}\,dx$ using Simpson's rule with 11 ordinates. Test your procedure using a pocket calculator.

26.



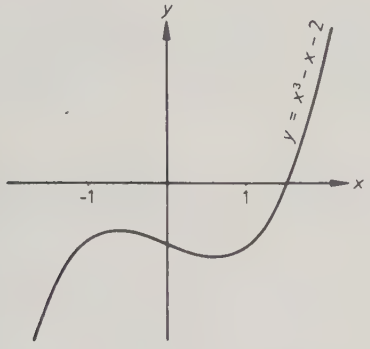
The following flow diagram is designed for the calculation of an approximate value of $\int_0^4 \frac{1}{1+x}\,dx$ using the trapezium rule. Carry out the procedure indicated, tabulating the successive value of R, S and T in simplified fractional form as you proceed. State the printed value of S giving two decimal places in your answers.

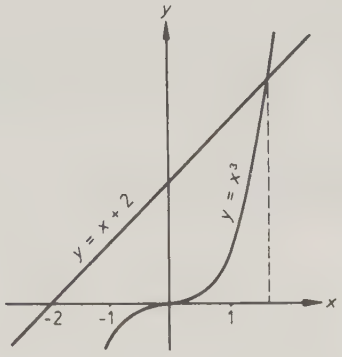
By referring to a sketch graph of $y = \frac{1}{x+1}$ for $0 \leqslant x \leqslant 4$, give a reason why the printed value of S will be an over-estimate of the value of the integral. (JMB)

27. Show that the formula $\int_0^{2h} f(x)\,dx \approx \frac{1}{3}h\{f(0) + 4f(h) + f(2h)\}$ is exact when $f(x)$ is a cubic polynomial in x. Use the formula given above to obtain an approximate value of I where $I = \int_0^{\frac{1}{2}} \frac{1}{\sqrt{(1-x^2)}}\,dx$, giving your result to three decimal places. Show, by direct integration, that $I = \pi/6$. (L)

23.2 Approximate solutions to equations

The roots of the equation $f(x) = 0$ are the values of x at which the curve $y = f(x)$ cuts the x-axis. Thus approximate values for the roots can be obtained by drawing the graph of $y = f(x)$. However, it is sometimes better to estimate the roots by rearranging the equation and finding the points of intersection of two graphs.

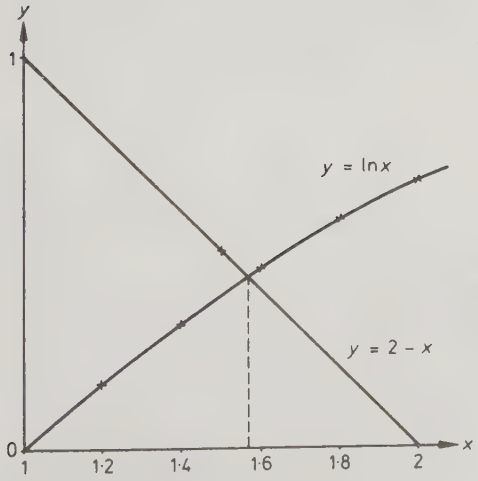




Consider, for instance, the equation $x^3 - x - 2 = 0$. An accurate graph of $y = x^3 - x - 2$ shows that the equation has one real root approximately equal to 1·5. Rearranging the equation as $x^3 = x + 2$, we see that we can also find this root by drawing the graphs of $y = x^3$ and $y = x + 2$.

Example 1 By drawing the graphs of $y = \ln x$ and $y = 2 - x$ in the interval $1 \leqslant x \leqslant 2$, find the solution of the equation $\ln x + x - 2 = 0$, correct to 1 decimal place.

x	1·0	1·2	1·4	1·6	1·8	2·0
$\ln x$	0	0·182	0·336	0·470	0·588	0·693



x	1	1·5	2
$2 - x$	1	0·5	0

At the point of intersection of the graphs of $y = \ln x$ and $y = 2 - x$ the approximate value of x is 1·6.

Hence if $\ln x + x - 2 = 0$, then $x \approx 1{\cdot}6$.

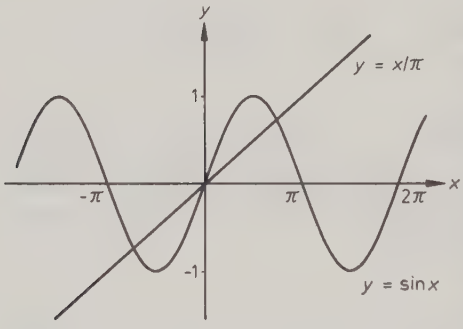
We can check that this result is accurate to one decimal place as follows:

If $x = 1{\cdot}55, \ln x \approx 0{\cdot}438$ and $2 - x = 0{\cdot}45$ $\therefore$ $\ln x < 2 - x$.
If $x = 1{\cdot}65, \ln x \approx 0{\cdot}501$ and $2 - x = 0{\cdot}35$ $\therefore$ $\ln x > 2 - x$.

Hence $\ln x = 2 - x$ for some value of x between 1·55 and 1·65 i.e. when $x = 1{\cdot}6$ correct to 1 decimal place.

Since drawing an accurate graph is rather time-consuming, it is often quicker to use a sketch to find the approximate locations of the roots of an equation, then to improve the accuracy of these rough estimates by numerical methods. First we consider the use of a sketch to determine the number of real roots of an equation.

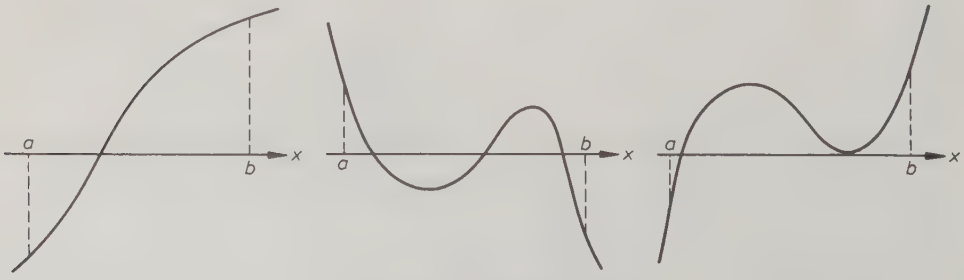
Example 2 Show graphically that the equation $\pi \sin x = x$ has three real roots.



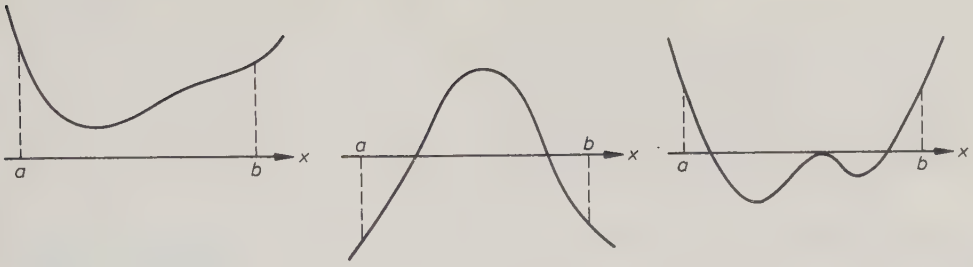
The roots of the equation $\pi \sin x = x$ are the x-coordinates of the points of intersection of the graphs of $y = \sin x$ and $y = x/\pi$. The curve $y = \sin x$ always lies between the lines $y = -1$ and $y = 1$. The straight line $y = x/\pi$ passes through the origin and the points $(-\pi, -1)$, $(\pi, 1)$. Thus, as shown in the sketch, the graphs intersect at three points. Hence the equation $\pi \sin x = x$ has three real roots.

Roots of the equation $f(x) = 0$ can be located by considering the sign changes of the function $f(x)$. Each of the following diagrams shows a function $f(x)$ which is continuous between $x = a$ and $x = b$. We see that if $f(a)$ and $f(b)$ are *opposite* in sign then, in the interval $a < x < b$, the equation $f(x) = 0$ must have at least one root, but may have any *odd* number of roots. If $f(a)$ and $f(b)$ have the *same* sign then, in the interval $a < x < b$, the equation $f(x) = 0$ may have no roots, but could have any *even* number of roots.

[Some of the roots may be coincident.]

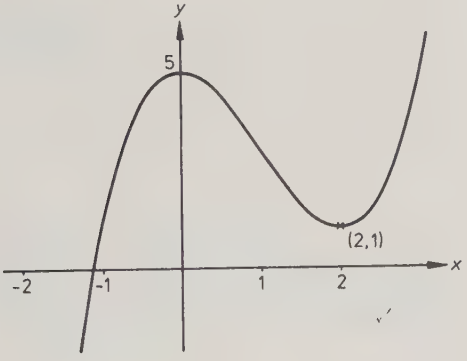


$f(a)$ and $f(b)$ opposite in sign



$f(a)$ and $f(b)$ have the same sign

Example 3 Show that the equation $x^3 - 3x^2 + 5 = 0$ has only one real root and that this root lies between -2 and -1.



Let $y = x^3 - 3x^2 + 5$,

then $\dfrac{dy}{dx} = 3x^2 - 6x = 3x(x - 2)$

and $\dfrac{d^2y}{dx^2} = 6x - 6 = 6(x - 1)$

$\therefore$ $\dfrac{dy}{dx} = 0$ when $x = 0$ or $x = 2$.

When $x = 0$, $y = 5$ and $\dfrac{d^2y}{dx^2} < 0$.

When $x = 2, y = 1$ and $\dfrac{d^2y}{dx^2} > 0$.

Hence on the curve $y = x^3 - 3x^2 + 5$ there is a maximum point at $(0, 5)$ and a minimum point at $(2, 1)$.
Since both turning points are above the x-axis, if follows that the equation $x^3 - 3x^2 + 5 = 0$ has only one real root.
When $x = -2$, $y = -15$ and when $x = -1$, $y = 1$
$\therefore$ y changes sign in the interval $-2 < x < -1$.
Hence the root of the given equation lies between -2 and -1.

Example 4 Show that the equation $x^3 + x - 16 = 0$ has only one real root α and find the integer n such that $n \leqslant \alpha < n + 1$.

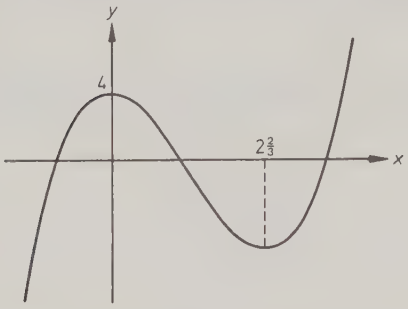
Let $f(x) = x^3 + x - 16$, then $f'(x) = 3x^2 + 1$.
Since $f'(x)$ is always positive, the curve $y = f(x)$ has no turning points and cannot cut the x-axis more than once. For large negative values of x, $f(x)$ is negative and for large positive values of x, $f(x)$ is positive,
$\therefore$ the curve $y = f(x)$ cuts the x-axis exactly once.
Hence the equation $x^3 + x - 16 = 0$ has only one real root α.
When $x \leqslant 0$, $f(x) < 0$ $\therefore$ $\alpha > 0$.

x	1	2	3
$f(x)$	-14	-6	14

The table shows that $f(x)$ changes sign between $x = 2$ and $x = 3$. Thus

$$n \leqslant \alpha < n + 1 \quad \text{for} \quad n = 2.$$

Example 5 Show that the equation $x^3 - 4x^2 + 4 = 0$ has one negative root and two positive roots. Find pairs of successive integers between which these roots lie.



Let $f(x) = x^3 - 4x^2 + 4$ and consider the curve $y = f(x)$.

$$\frac{dy}{dx} = 3x^2 - 8x = x(3x - 8)$$

$$\frac{d^2y}{dx^2} = 6x - 8$$

$$\therefore \quad \frac{dy}{dx} = 0 \text{ when } x = 0 \text{ or } x = 2\tfrac{2}{3}$$

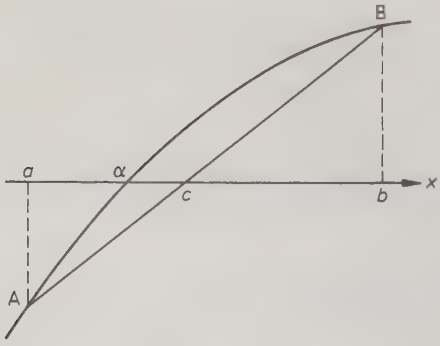
When $x = 0$, $y = 4$ and $\dfrac{d^2y}{dx^2} < 0$. When $x = 2\frac{2}{3}$, $y \approx -5\frac{1}{2}$ and $\dfrac{d^2y}{dx^2} > 0$.

Hence at $x = 0$ there is a maximum point above the x-axis and at $x = 2\frac{2}{3}$ there is a minimum point below the x-axis. It follows that the equation $x^3 - 4x^2 + 4 = 0$ must have one negative root and two positive roots. Testing various integral values of x, we have:

x	-1	0	1	2	3	4
$f(x)$	-1	4	1	-4	-5	4

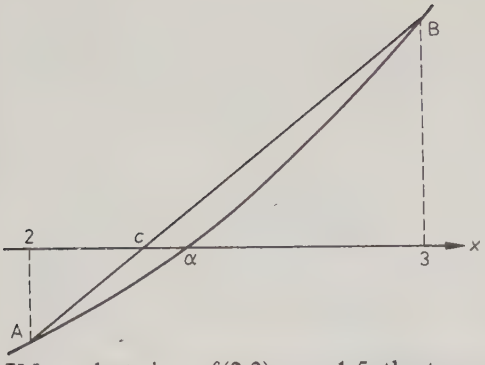
Using the sign changes shown in the table, we deduce that the roots of the given equation lie between -1 and 0, 1 and 2, 3 and 4.

When it has been shown that an equation $f(x) = 0$ has a root α in the interval $a < x < b$, then *linear interpolation* can be used to estimate the value of α.



Let us suppose that A and B are the points on the curve $y = f(x)$ at which $x = a$ and $x = b$ respectively. If the straight line joining A to B cuts the x-axis at $(c, 0)$, then c is taken as an approximate value of the root α.

We illustrate the method by letting $f(x) = x^3 + x - 16$. As shown in Example 4, the equation $x^3 + x - 16 = 0$ has a root α in the interval $2 < x < 3$ and the



curve $y = x^3 + x - 16$ passes through the points $A(2, -6)$ and $B(3, 14)$. If the straight line joining A to B cuts the x-axis at $(c, 0)$, then using similar triangles, we find that c divides the interval from $x = 2$ to $x = 3$ in the ratio 6:14

Hence, by linear interpolation,

$$\alpha \approx 2 + \frac{6}{6 + 14} = 2{\cdot}3.$$

[Note that since $f(2{\cdot}3) \approx -1{\cdot}5$, the true value of α lies between 2·3 and 3.]

If the equation $f(x) = 0$ is known to have a root so small that, for values of x close to the root, x^3 and higher powers of x are negligible, then the root can be estimated using a *quadratic approximation* to $f(x)$.

Example 6 Given that the equation $(1 - x)e^x = 0{\cdot}7(2x + 1)$ has a small root, use a quadratic approximation to estimate its value.

$$(1 - x)e^x = (1 - x)\left(1 + x + \frac{x^2}{2!} + \dots\right)$$

$$= 1 + x + \tfrac{1}{2}x^2 - x - x^2 + \dots$$

$$\therefore \quad (1 - x)e^x \approx 1 - \tfrac{1}{2}x^2.$$

Hence the small root of the equation $(1 - x)e^x = 0{\cdot}7(2x + 1)$ is approximately equal to the small root of the equation $1 - \frac{1}{2}x^2 = 0{\cdot}7(2x + 1)$.
Multiplying both sides of this equation by 10, we have

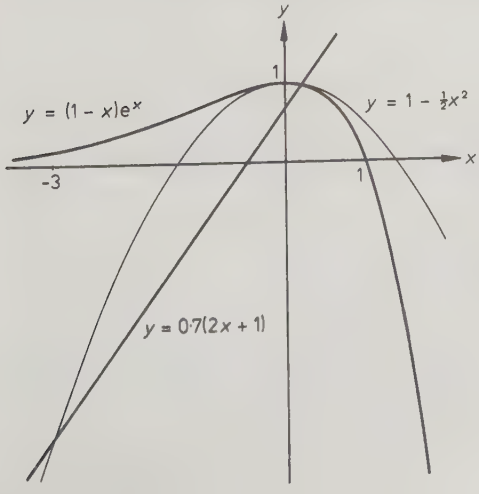
$$10 - 5x^2 = 7(2x + 1)$$

i.e. $5x^2 + 14x - 3 = 0$

$$(5x - 1)(x + 3) = 0$$

$$\therefore \quad x = 0{\cdot}2 \quad \text{or} \quad x = -3.$$

Hence the approximate value of the small root of the given equation is 0·2.



[We see from the sketch that the approximation $(1 - x)e^x \approx 1 - \frac{1}{2}x^2$ is valid for only a small range of values of x. Thus the second root, $x = -3$, obtained using this approximation has no real significance.]

Exercise 23.2

In questions 1 to 10 obtain the roots of the given equations by drawing suitable accurate graphs. Give your answers correct to one decimal place.

1. $x^2 - 2x - 2 = 0$
2. $x^3 + x - 1 = 0$
3. $x^3 - 3x + 3 = 0$
4. $x^3 - 2x^2 + 1 = 0$
5. $x^3 + x^2 - 2x - 1 = 0$
6. $e^x = 3 - x$
7. $2\sin x = x$
8. $4\cos x = x$
9. $\frac{4}{x^2} = 2x + 1$
10. $(x - 2)\ln x = x - 1$

In questions 11 to 14 use a sketch to find the number of real roots of the given equation.

11. $e^{-x} = x - 1$
12. $2e^x = 5x + 1$
13. $\cos 2x = 1 - x$
14. $\sin \pi x = \frac{1}{2}(x - 1)$

In questions 15 to 22 determine the number of real roots of the given equation and find pairs of successive integers between which the roots lie.

15. $x^3 - 6x^2 + 36 = 0$
16. $x^3 + 5x - 10 = 0$
17. $x^3 - 12x + 12 = 0$
18. $x^3 + 2x^2 - 4x - 4 = 0$
19. $2x^3 + 4x - 15 = 0$
20. $x^3 - x - 3 = 0$
21. $e^x = x + 3$
22. $\sin x = \ln x$

23. Show that the equation $x^3 - 7x - 12 = 0$ has one real root and show that this root lies between 3 and 4. Use linear interpolation to estimate the root, giving your answer to one decimal place.

24. Show that the equation $x^4 - 4x - 4 = 0$ has one negative root α and one positive root β. Show that $-1 < \alpha < 0$ and that $1 < \beta < 2$. Use linear interpolation to estimate the values of α and β, giving your answers correct to one decimal place.

25. Show that the equation $x^4 + 3x^3 + x^2 - 9 = 0$ has one negative root α and one positive root β. Find α and the integer k such that $k < \beta < k + 1$. Use linear interpolation to find an approximate value for β, giving your answer correct to one decimal place.

In questions 26 to 31 use linear interpolation between suitable successive integers to obtain estimates for the roots of the given equations, giving your answers correct to one decimal place.

26. $2x^3 - 25 = 0$

27. $x^3 + 3x^2 - 14x - 10 = 0$

28. $4x^3 - 9x^2 + 3 = 0$

29. $x^3 + 3x^2 - 4x + 3 = 0$

30. $\ln x = x^2 - 6x + 8$

31. $6\cos x + x = 0$

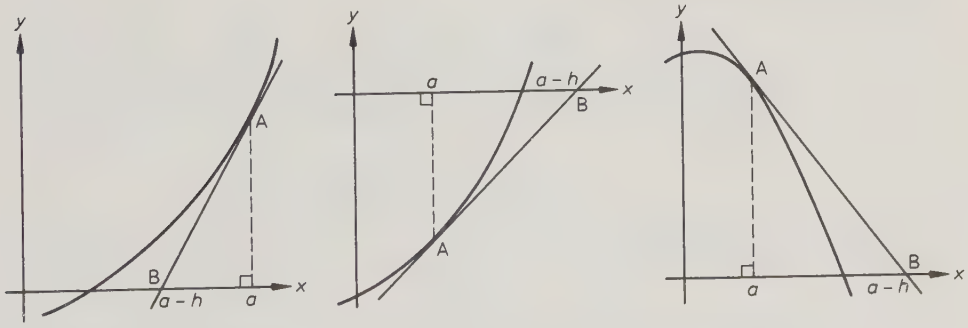
32. Draw an accurate graph of $y = 2x^3 - 5x + 2$ for values of x between -2 and $+2$. Estimate the roots of the equation $2x^3 - 5x + 2 = 0$ (a) from your graph, (b) by linear interpolation between suitable successive integers. Why does linear interpolation give poor approximations to the roots in this case?

33. Each of the following equations has a root so small that, for values of x close to the root, x^3 and higher powers of x are negligible. Use quadratic approximations to estimate these roots.

(a) $2(1 - x)^5 = 18x + 5$
(b) $8\ln(1 + x) = 3x + 1$
(c) $10(1 - \cos x) = 2 - 3x$
(d) $5\sqrt[3]{(1 + 3x)} = 14x + 3$

23.3 The Newton–Raphson method

If $x = a$ is an approximate value of a root of an equation $f(x) = 0$, then the tangent to the curve $y = f(x)$ at $x = a$ can be used to find a second approximation to the root. As shown in the diagrams, the x-coordinate of the point of intersection of this tangent with the x-axis is usually closer to the true root than the first approximation $x = a$.



Let A be the point $(a, f(a))$ on the curve $y = f(x)$ and let the point of intersection of the tangent at A with the x-axis be the point B with coordinates $(a - h, 0)$, then:

$$\text{gradient of } AB = \frac{f(a) - 0}{a - (a - h)} = \frac{f(a)}{h}.$$

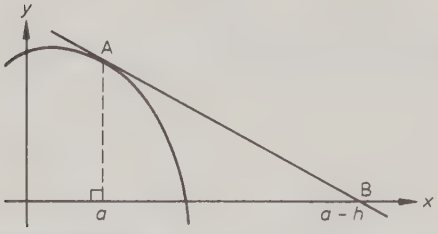
However, since AB is the tangent to the curve $y = f(x)$ at $x = a$, its gradient is also equal to $f'(a)$,

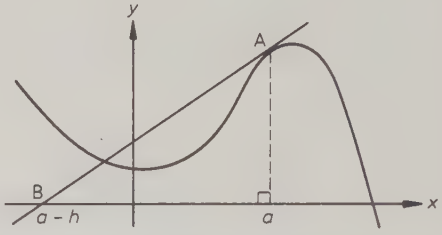
$$\therefore \qquad f'(a) = \frac{f(a)}{h} \quad \text{which gives} \quad h = \frac{f(a)}{f'(a)}.$$

Thus the tangent at A cuts the x-axis where $x = a - \dfrac{f(a)}{f'(a)}$. This result forms the basis of the procedure known as the *Newton-Raphson*† *method* (or simply Newton's method).

If $x = a$ is a first approximation to a root of the equation $f(x) = 0$, then a second approximation to the root is given by $x = a - \dfrac{f(a)}{f'(a)}$.

The following diagrams show that the method may fail to produce a better approximation to the root when the value of $f'(a)$ is small or when the first approximation to the root is poor.





Example 1 Taking $x = 2{\cdot}3$ as a first approximation to the positive root of the equation $3 \sin x = x$, use the Newton-Raphson method to find a second approximation to this root, giving your answer to 2 decimal places.

Let $f(x) = 3 \sin x - x$, then $f'(x) = 3 \cos x - 1$.
Working to 4 decimal places,

$$f(2{\cdot}3) \approx -0{\cdot}0629 \quad \text{and} \quad f'(2{\cdot}3) \approx -2{\cdot}9988.$$

By the Newton-Raphson method, if $x = 2{\cdot}3$ is a first approximation to the positive root of the equation $f(x) = 0$, then a second approximation is given by

$$x = 2{\cdot}3 - \frac{f(2{\cdot}3)}{f'(2{\cdot}3)} \approx 2{\cdot}3 - \frac{0{\cdot}0629}{2{\cdot}9988} \approx 2{\cdot}28.$$

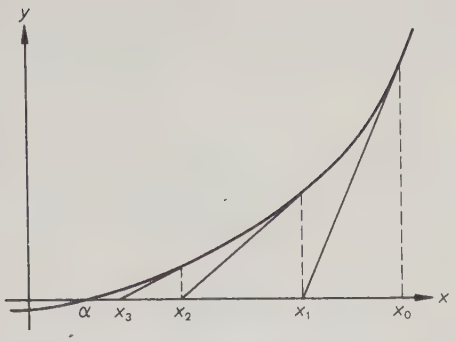
Hence the second approximation to the positive root of the equation $3 \sin x = x$ is $x = 2{\cdot}28$.

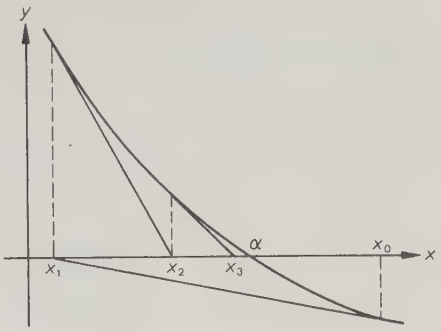
If x_0 is a first approximation to a root α of the equation $f(x) = 0$, then the Newton-Raphson method can be used to produce a sequence of approximations $x_1, x_2, x_3, \ldots$ by writing:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

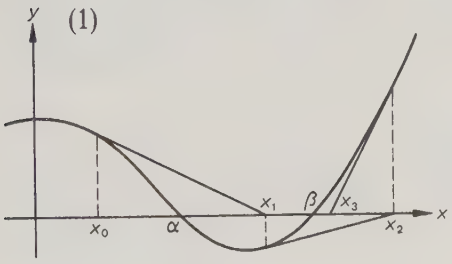
The following diagrams show that, in general, the sequence $x_0, x_1, x_2, x_3, \ldots$ converges quickly to the required root.

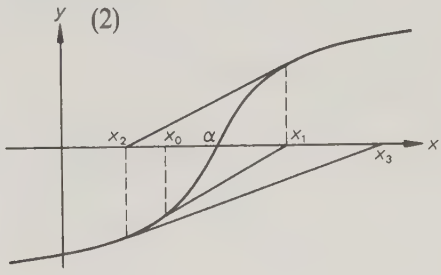
†*Raphson, Joseph* (1648–1715) English mathematician. His account of the Newton-Raphson method was published in 1690 and was based on earlier work by Newton. Raphson produced many other mathematical works including a *History of Fluxions* (1715).





The next two diagrams show that in certain circumstances the method fails. In (1) the sequence $x_0, x_1, x_2, x_3, \ldots$ converges to a different root β and in (2) the sequence diverges.





Example 2 The equation $x^3 - 3x^2 - 1 = 0$ has one real root. Starting with $x = 3$, use two iterations of Newton's method to find an approximate value for this root, correct to 3 decimal places.

Let $f(x) = x^3 - 3x^2 - 1$, then $f'(x) = 3x^2 - 6x$.
Using Newton's method and working to 5 decimal places,

let $x_0 = 3$, then $f(x_0) = -1$, $f'(x_0) = 9$

$$\therefore \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3 - \frac{(-1)}{9} \approx 3{\cdot}11111$$

Let $x_1 = 3{\cdot}11$, then $f(x_1) \approx 0{\cdot}063\,93$, $f'(x_1) = 10{\cdot}3563$

$$\therefore \quad x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 3{\cdot}11 - \frac{0{\cdot}063\,93}{10{\cdot}3563} \approx 3{\cdot}103\,83$$

Hence an approximate value for the root is 3·104.

Checking the accuracy of this result we find that

$$f(3{\cdot}1035) \approx -0{\cdot}003 \quad \text{and} \quad f(3{\cdot}1045) \approx 0{\cdot}007,$$

$\therefore$ the root is 3·104 correct to 3 decimal places.

[Note that when evaluating a function $f(x)$ by calculator it is usually best to store the value of x in the memory. If $f(x)$ is a polynomial more accurate results will be obtained by using nested multiplication as described in §4.4.]

To demonstrate how rapidly the Newton-Raphson process converges we now obtain further approximations to the root of the equation $x^3 - 3x^2 - 1 = 0$, giving the results in the form of a table.

n	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1}
0	3	-1	9	3·111 1111
1	3·11	0·063 931	10·3563	3·103 8269
2	3·1038	$-0·000\,0350$	10·277 9233	3·103 8034
3	3·103 8034	$-0·000\,000\,03$	10·277 9662	3·103 8034

When using the Newton-Raphson process it is usually reasonable to assume that if $|x_{n+1} - x_n| < \varepsilon$, then the error in the approximation x_{n+1} is less than ε.

In the given table the values of x_3 and x_4 agree to 7 decimal places. It follows that $|x_4 - x_3| < 1 \times 10^{-7}$ and thus the error in the approximation x_4 is less than 1×10^{-7}.

Closer examination of the results shows that, allowing for rounding errors in the calculations,

$$|x_4 - x_3| = |f(x_3)/f'(x_3)| < 1 \times 10^{-8}$$

$$\therefore \qquad 3·103\,8034 < x_4 < 3·103\,803\,41$$

Thus the true value of the root is unlikely to exceed 3·103 803 42. Therefore $x = 3·103\,8034$ correct to 7 decimal places.

Exercise 23.3

In questions 1 to 8 use the Newton-Raphson method to find a second approximation to a root of the given equation taking the stated value of a as a first approximation. Give your answers to 2 decimal places.

1. $x^3 - 4x^2 + 4 = 0, a = 1.$
2. $x^3 - 3x^2 + 5 = 0, a = -1.$
3. $x^3 - x + 2 = 0, a = -1·5.$
4. $x^3 + x - 16 = 0, a = 2·3.$
5. $\ln x + x - 2 = 0, a = 1·6.$
6. $e^x + x - 3 = 0, a = 0·8.$
7. $2\sin x = x, a = -2.$
8. $3\cos x = x, a = 1·2.$

In questions 9 to 14 use two iterations of the Newton-Raphson method to find an approximate value for a root of the given equation, taking the stated value of x_0 as a first approximation. Give your answers to 3 decimal places.

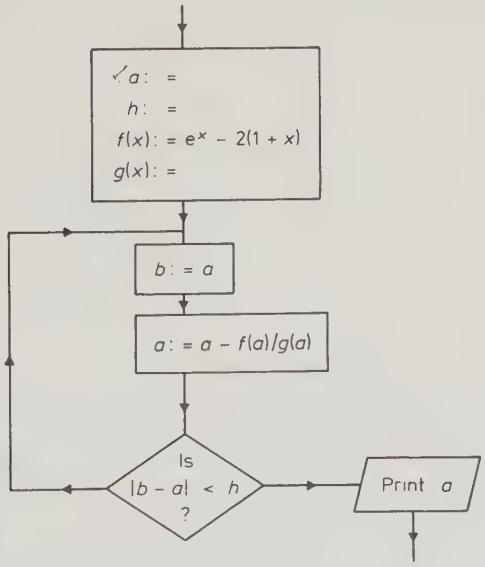
9. $x^3 - 5x - 8 = 0, x_0 = 3.$
10. $x^3 - 9x^2 + 20 = 0, x_0 = 2.$
11. $4x^3 + 6x^2 - 5x - 6 = 0, x_0 = -1.$
12. $x^3 + x + 8 = 0, x_0 = -2.$
13. $e^{-x} = x - 1, x_0 = 1·2.$
14. $\sin x = \ln x, x_0 = 2·3.$

15. Show that the equation $x^3 + 2x - 1 = 0$ has only one real root and that this root lies between 0 and 1. Find the approximate value of this root correct to 3 decimal places.

16. Show that the equation $x^3 - 12x - 6 = 0$ has one positive and two negative roots. Use linear interpolation between suitable successive integers to find a first approximation to the positive root. Hence find the approximate value of this root correct to 4 decimal places.

17. Give, on the same diagram, a sketch of the graph of $y = 3e^{x/2}$ and of the graph of $y = 4x + 6$. State the number of roots of the equation $3e^{x/2} = 4x + 6$. Taking 4 as a first approximation to one root α of this equation, find a second approximation to α, giving three significant figures in your answer and showing clearly how this answer has been obtained. Using a suitable integer as first approximation to another root β of the equation, find a second approximation to β, again giving three significant figures in your answer and showing clearly how this answer has been determined. (C)

18. Show that the equation $e^x = 2(1 + x)$ has only one positive root. Find the nearest integer to this root. Rewrite the flow chart shown below and complete it to specify a process which will find the root of the above equation, with an error of less than 0·01, starting from the integer nearest to the root. Evaluate the root to this degree of accuracy.



(AEB 1978)

23.4 Other iterative processes

An *iterative process* is a repetitive procedure designed to produce a sequence of approximations to some numerical quantity. The Newton-Raphson method described in the previous section is one example of such a process.

Interval bisection is another iterative process used to obtain approximations to a root α of an equation $f(x) = 0$. To use this method it is necessary to locate an interval containing the root, i.e. to find values x_0 and x_1 such that $x_0 < \alpha < x_1$. The length of this interval is halved by taking $x_2 = \frac{1}{2}(x_0 + x_1)$ and then determining whether α lies between x_0 and x_2 or between x_2 and x_1. The new interval is then bisected to produce a further approximation x_3. The bisection process is repeated until the interval containing α is small enough to ensure a result to the required degree of accuracy. We will see in the following example that this method converges much more slowly than the Newton-Raphson method.

Example 1 Use interval bisection to obtain a sequence of approximations to the positive root of the equation $x^2 - 3 = 0$. Hence evaluate $\sqrt{3}$ correct to 2 decimal places.

Let $f(x) = x^2 - 3$, then the positive root of the equation $f(x) = 0$ is $\sqrt{3}$.
Since $f(1) < 0$ and $f(2) > 0$, we have $1 < \sqrt{3} < 2$.
Taking $x_0 = 1$ and $x_1 = 2$ as first approximations to $\sqrt{3}$,

let $x_2 = \frac{1}{2}(x_0 + x_1) = \frac{1}{2}(1 + 2) = 1{\cdot}5$, then $f(x_2) < 0$

$\therefore \quad 1{\cdot}5 < \sqrt{3} < 2$

Let $x_3 = \frac{1}{2}(1{\cdot}5 + 2) = 1{\cdot}75$, then $f(x_3) > 0$

$\therefore \quad 1{\cdot}5 < \sqrt{3} < 1{\cdot}75$

Let $x_4 = \frac{1}{2}(1{\cdot}5 + 1{\cdot}75) = 1{\cdot}625$, then $f(x_4) < 0$

$\therefore \quad 1{\cdot}625 < \sqrt{3} < 1{\cdot}75$

Let $x_5 = \frac{1}{2}(1{\cdot}625 + 1{\cdot}75) = 1{\cdot}6875$, then $f(x_5) < 0$

$\therefore \quad 1{\cdot}6875 < \sqrt{3} < 1{\cdot}75$

Let $x_6 = \frac{1}{2}(1{\cdot}6875 + 1{\cdot}75) = 1{\cdot}71875$, then $f(x_6) < 0$

$\therefore \quad 1{\cdot}7187 < \sqrt{3} < 1{\cdot}75$

Let $x_7 = \frac{1}{2}(1{\cdot}7187 + 1{\cdot}75) = 1{\cdot}73435$, then $f(x_7) < 0$

$\therefore \quad 1{\cdot}7187 < \sqrt{3} < 1{\cdot}7344$

Let $x_8 = \frac{1}{2}(1{\cdot}7187 + 1{\cdot}7344) = 1{\cdot}72655$, then $f(x_8) < 0$

$\therefore \quad 1{\cdot}7265 < \sqrt{3} < 1{\cdot}7344$

Hence $\sqrt{3} = 1{\cdot}73$ correct to 2 decimal places.

In general, an iterative process is based on a rule or *algorithm* for obtaining an approximation x_{n+1} from earlier approximations $x_0, x_1, x_2, \ldots, x_n$. In many cases this rule can be expressed as an iterative formula giving x_{n+1} in terms of x_n. In section §23.3 we showed that the Newton-Raphson method can be used to produce such a formula. We now discuss an alternative method.

Let us consider the general iterative formula

$$x_{n+1} = f(x_n)$$

If the sequence $x_0, x_1, x_2, x_3, \ldots$ converges to a finite limit α, then as $n \to \infty$, $x_{n+1} \to \alpha$ and $f(x_n) \to f(\alpha)$

giving $$\alpha = f(\alpha).$$

Thus, if the iteration $x_{n+1} = f(x_n)$ is convergent, it will produce a sequence of approximations to a root of the equation $x = f(x)$.

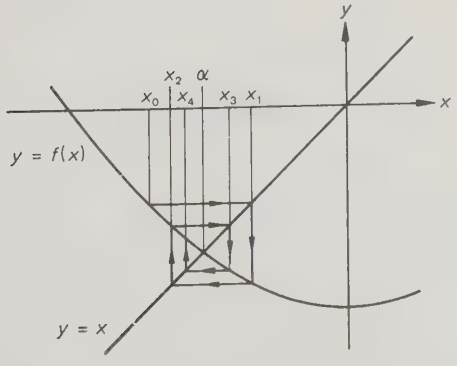
When solving an equation we may be able to obtain a suitable iterative formula by writing the equation in the form $x = f(x)$. For any particular equation several different rearrangements will be possible, but not all the corresponding iterative processes will converge to the required root. It is difficult to predict which rearrangements will be most successful. However, it can be shown that, in general, the iteration $x_{n+1} = f(x_n)$ converges to a root α, provided that (i) x_0 is a fairly good approximation to α, and that (ii) $-1 < f'(\alpha) < 1$.

To illustrate the various possibilities we consider the equation $x^2 - 4x - 8 = 0$. This has one root α between -2 and -1 and another root β between 5 and 6. We will attempt to calculate these roots using iterative formulae based on the rearrangements:

(a) $x = \frac{1}{4}x^2 - 2$, (b) $x = \dfrac{8}{x-4}$, (c) $x = 2\sqrt{(x+2)}$.

(a) The following table shows the sequences obtained using the formula $x_{n+1} = \frac{1}{4}x_n^2 - 2$, taking x_0 as -2, -1, 5 and 6, then working to 3 decimal places.

x_0	-2	-1	5	6
x_1	-1	$-1{\cdot}75$	4·25	7
x_2	$-1{\cdot}75$	$-1{\cdot}234$	2·516	10·25
x_3	$-1{\cdot}234$	$-1{\cdot}619$	$-0{\cdot}417$	24·266
x_4	$-1{\cdot}619$	$-1{\cdot}345$	$-1{\cdot}957$	145·210
x_5	$-1{\cdot}345$	$-1{\cdot}548$	$-1{\cdot}043$	⋮
x_6	$-1{\cdot}548$	$-1{\cdot}401$	$-1{\cdot}728$	⋮



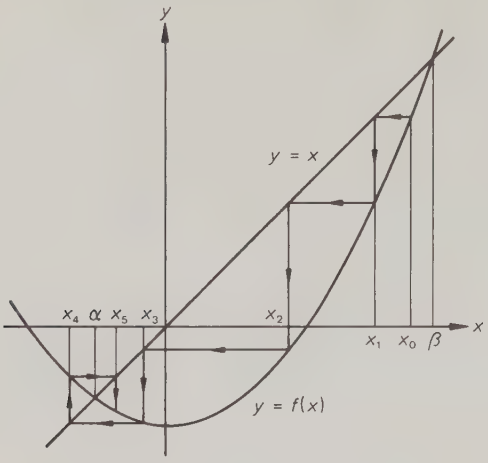
$x_0 = -2$, $f(x) = \frac{1}{4}x^2 - 2$

The sketch graphs show that three of the sequences converge slowly to the root α and that the other sequence diverges. Writing $f(x) = \frac{1}{4}x^2 - 2$, we find that

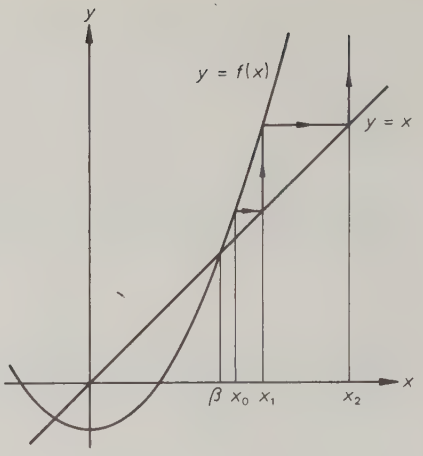
$$f'(\alpha) \approx f'(-1{\cdot}5) = -0{\cdot}75$$

and $$f'(\beta) \approx f'(5{\cdot}5) = 2{\cdot}75.$$

These results tend to confirm that the rate of convergence depends on the value of $f'(x)$.



$x_0 = 5, f(x) = \frac{1}{4}x^2 - 2$

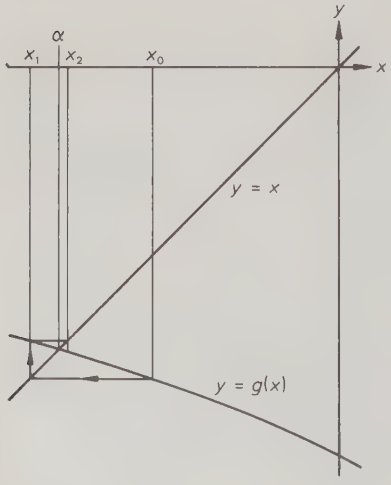


$x_0 = 6, f(x) = \frac{1}{4}x^2 - 2$

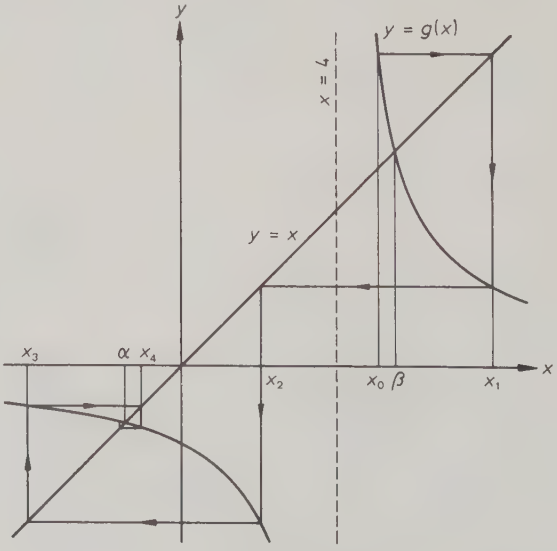
(b) Next we work with the formula $x_{n+1} = \dfrac{8}{x_n - 4}$

x_0	-2	-1	5	6
x_1	$-1{\cdot}333$	$-1{\cdot}6$	8	4
x_2	$-1{\cdot}500$	$-1{\cdot}429$	2	—
x_3	$-1{\cdot}455$	$-1{\cdot}474$	-4	
x_4	$-1{\cdot}467$	$-1{\cdot}461$	-1	
x_5	$-1{\cdot}463$	$-1{\cdot}465$	$-1{\cdot}6$	
x_6	$-1{\cdot}464$	$-1{\cdot}464$	$-1{\cdot}429$	

In this case three of the sequences converge fairly quickly to the root α. In the remaining sequence the value of x_2 is undefined. Writing $g(x) = 8/(x - 4)$, we have $g'(\alpha) \approx -0{\cdot}25$ and $g'(\beta) \approx -4$.



$x_0 = -1, g(x) = 8/(x - 4)$

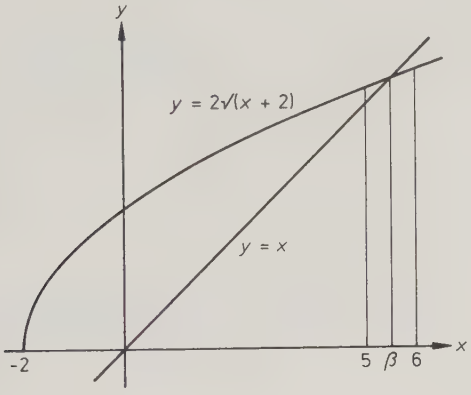


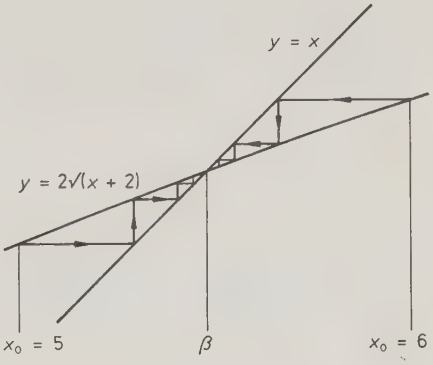
$x_0 = 5, g(x) = 8/(x - 4)$

(c) Since the rearrangement $x = 2\sqrt{(x+2)}$ is not valid for the negative root α of the original equation, we hope to use the formula $x_{n+1} = 2\sqrt{(x_n + 2)}$ to obtain the positive root β.

x_0	5	6
x_1	5·292	5·657
x_2	5·401	5·534
x_3	5·441	5·490
x_4	5·456	5·474
x_5	5·461	5·468
x_6	5·463	5·466

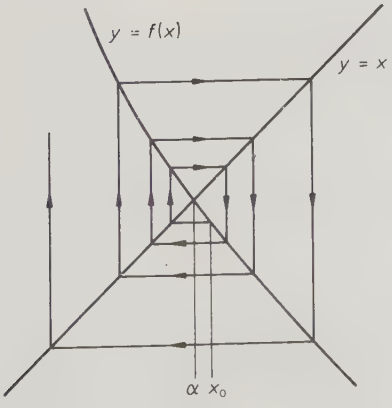
Both sequences converge to the root β, one from below, the other from above. Letting $h(x) = 2\sqrt{(x+2)}$, we find that $h'(\beta) \approx 0{\cdot}37$.





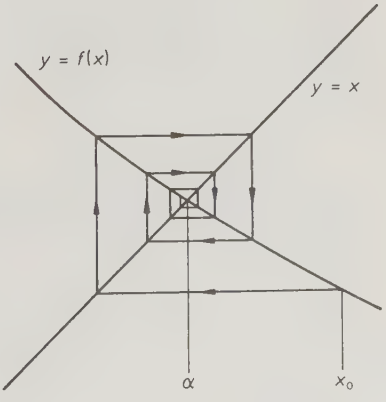
Returning to the general formula $x_{n+1} = f(x_n)$, we conclude that there are four ways in which the sequence of approximations $x_0, x_1, x_2, \ldots$ may behave close to a root α of the equation $x = f(x)$.

(1)



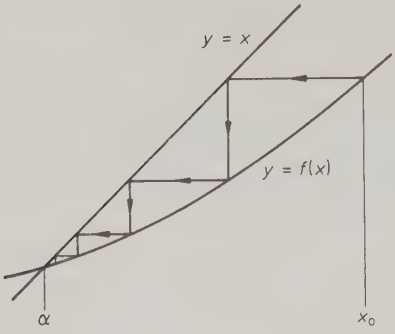
$f'(\alpha) < -1$

(2)



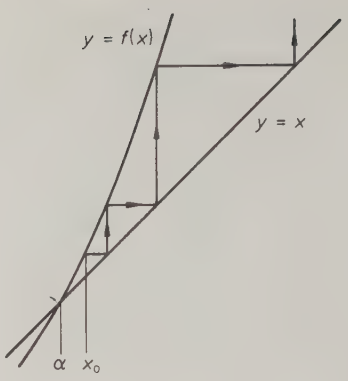
$-1 < f'(\alpha) < 0$

(3)



$0 < f'(\alpha) < 1$

(4)



$f'(\alpha) > 1$

[Note that in cases (1) and (4) the sequence may later converge to a different root of the equation $x = f(x)$.]

Exercise 23.4

1. Use the method of interval bisection to find the positive roots of the following equations correct to 2 decimal places.
(a) $x^2 - 5 = 0$, (b) $x^3 - 2 = 0$, (c) $x^3 - 4x^2 + 2 = 0$.

2. The iterative formula $x_{n+1} = \frac{1}{2}(x_n + a/x_n)$ gives successive approximations to the square root of a. Use this formula and the given values of x_0 to calculate correct to 3 decimal places.
(a) $\sqrt{2}, x_0 = 1$, (b) $\sqrt{10}, x_0 = 3$, (c) $\sqrt{13}, x_0 = 4$.

3. The iterative formula $x_{r+1} = \frac{1}{2}(x_r + N/x_r^2)$ can be used to find an approximate cube root of a number N. Letting $x_0 = 2$, find $\sqrt[3]{5}$ correct to 3 significant figures.

4. By considering the function $f(x) = N - 1/x$ and using the Newton-Raphson method, establish the formula $x_{r+1} = x_r(2 - Nx_r)$ for obtaining successive approximations to the reciprocal of N. Taking $x_0 = 1$, find the reciprocal of 1·3 correct to 3 significant figures.

5. Show that the equation $x = \ln(8 - x)$ has a root between 1 and 2. Calculate this root correct to 3 decimal places (a) using the formula $x_{n+1} = \ln(8 - x_n)$, (b) using the Newton-Raphson method.

6. Show that the equation $x = \cos x - 3$ has a root between -4 and -3. Use an iterative method to calculate this root correct to 3 decimal places.

7. Use the formula $x_{r+1} = (1 - x_r^2)/5$, where $x_0 = 0$, to find the positive root of the equation $x^2 + 5x - 1 = 0$, correct to 4 decimal places. Discuss with the aid of

sketch graphs the behaviour of the sequence $x_0, x_1, x_2, \ldots$ when $x_0 = -6$ and when $x_0 = -5$.

8. Show graphically, or otherwise, that the equation $x^3 - x - 1 = 0$ has only one real root and find the integer n such that the root α satisfies $n < \alpha < n + 1$. An iterative process for finding this root is defined by

$$x_1 = 1, \quad x_{m+1} = (x_m + 1)^{1/3} \quad \text{for all} \quad m \in \mathbb{N}.$$

Obtain, to 3 places of decimals, the values of x_2 and x_3. Show, on a sketch graph, the line $y = x$ and the curve $y = (x + 1)^{1/3}$, indicating on this graph the relation between x_1, x_2, x_3 and the root α. (L)

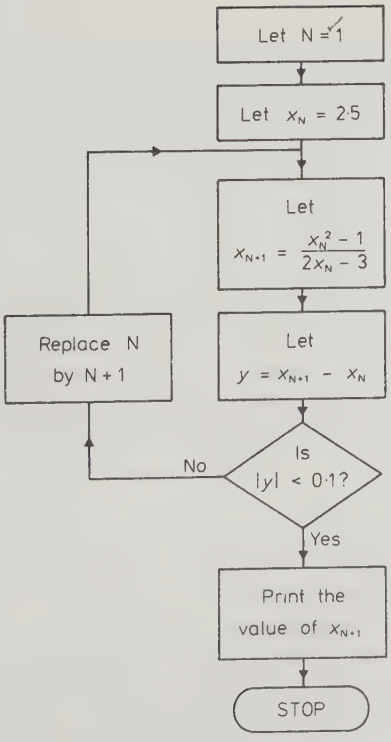
9. Verify by sketching graphs of $y = x$ and $y = 2(1 + 1/x)$ that, in general, the sequence defined by $x_{n+1} = 2(1 + 1/x_n)$ converges to the root $1 + \sqrt{3}$ of the equation $x^2 - 2x - 2 = 0$. Hence find $\sqrt{3}$ to 4 significant figures, taking a suitable integer as first approximation.

10. Demonstrate graphically that neither

$$x_{n+1} = \tfrac{2}{5}(4 - x_n^3) \quad \text{nor} \quad x_{n+1} = (8 - 5x_n)/2x_n^2$$

can be used to find the real root of the equation $2x^3 + 5x - 8 = 0$. Derive a suitable iteration formula and hence obtain the root to 3 decimal places.

11.



The flow diagram shown represents the steps involved in finding an estimate of a solution of the equation $x^2 - 3x + 1 = 0$, using an iterative method. Follow the steps to their conclusion and obtain the estimate of the root. Indicate clearly everything that is to be printed and work to three decimal places. Indicate the purpose of the decision box.

(L)

12. The equation $f(x) = 0$ is known to have precisely one root in the interval $a \leqslant x \leqslant b$. Construct a flow diagram giving an algorithm to estimate this root, using the method of interval bisection. The following are given: the numbers a and b (with $a < b$), the function f, and the positive number δ which is the maximum permissible magnitude of the error in the root. Demonstrate the working of your flow diagram by applying it to the equation $\cos x - x = 0$ with $a = 0$, $b = 1$ and $\delta = 0{\cdot}1$, giving your answer correct to one decimal place. (C)

Exercise 23.5 (miscellaneous)

1. Values of a continuous function f were found experimentally as given below.

t	0	0·3	0·6	0·9	1·2	1·5	1·8
$f(t)$	2·72	3·00	3·32		4·06	4·48	4·95

Use linear interpolation to estimate $f(0{\cdot}9)$. Then use Simpson's rule with seven ordinates to estimate $\int_0^{1\cdot8} f(t)\,dt$, tabulating your working and giving your answer to two places of decimals. (JMB)

2. The following pairs of values of x and y satisfy approximately a relation of the form $y = ax^n$, where a and n are integers. By plotting the graph of $\lg y$ against $\lg x$ find the values of the integers a and n.

x	0·7	0·9	1·1	1·3	1·5
y	1·37	2·92	5·32	8·80	13·50

Estimate the value of the integral $\int_{0\cdot7}^{1\cdot5} y\,dx$ (a) by Simpson's rule, using five ordinates and clearly indicating your method, (b) by using the relation $y = ax^n$ with the values found for a and n. (L)

3. Find an approximate value of $\int_0^{1/2} \sqrt{(1 - x^2)}\,dx$ by using the trapezoidal rule with intervals of 0·1. Show, by working out the integral by substitution, that the magnitude of the error in your approximation is less than 0·001. Draw a diagram to illustrate the reason for this error, having regard to its sign. (AEB 1979)

4. If $f(x) = a + bx + cx^2 + kx^3$, where a, b, c, k are constants, show that $\int_{-h}^{h} f(x)\,dx = \frac{1}{3}h\{f(-h) + 4f(0) + f(h)\}$. Using Simpson's rule with 5 ordinates, obtain an approximate value, to 3 decimal places, for $\int_0^{0\cdot4} \sqrt{(1 + x^2)}\,dx$. Check your result by expanding $\sqrt{(1 + x^2)}$ in ascending powers of x as far as the term in x^6 and integrating term by term. (O&C)

5. Plot the graph of $y = x^3$ for x from -3 to $+3$. Use the graph to obtain solutions to the following equations (i) $x^3 - 2x - 5 = 0$, (ii) $2x^3 - 4x + 1 = 0$. By writing the equation $5x^3 + 2x^2 - 1 = 0$ in terms of $1/x$, obtain its solution, correct to one decimal place. (AEB 1979)

6. Sketch the graph of $y = \tan x$ for values of x between 0 and 2π. Use your sketch graph, together with additional straight line graphs, to solve the following problems:
(i) If $a > 0$ and $b > 0$, find the number of roots, between 0 and 2π, of the equation $\tan x = ax + b$.
(ii) If m can take all real values, find the set of values of m such that the equation $\tan x = mx + \pi$ has exactly three roots lying in the interval $0 < x < 2\pi$. (C)

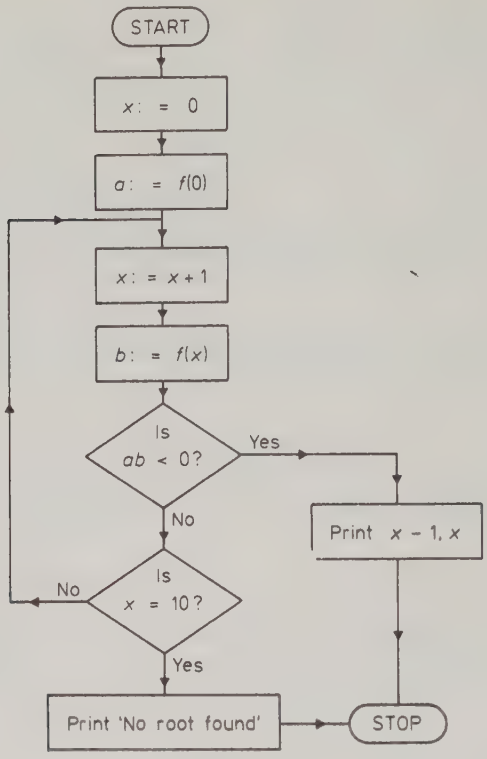
7. Sketch the graph of $y = x^2e^{-x}$ giving the coordinates of the turning points. Explain, from your graph, why, if $0 < k < 4/e^2$, the equation $ke^x = x^2$ has three real roots. (O&C)

8. By considering the graph of $y = x^3 - 3px - q$, where p and q are real constants, show that the condition for the equation $x^3 - 3px - q = 0$ to have three distinct real roots is $4p^3 > q^2$. Find the additional condition necessary for two of the roots to be positive. (JMB)

9. The least positive root of the equation $x^3 - 10x^2 + 12x + 16 = 0$ is denoted by α. Use linear interpolation between two suitable successive integers to obtain an estimate for α, giving your answer to one decimal place. Will your answer be an underestimate or overestimate of the exact root? Explain your reasoning. (O&C)

10. Sketch the graph of $y = \cos x$ for $0 \leqslant x \leqslant 4\pi$. Use your sketch to find the number of positive roots of the equation $\cos x = x/10$. Prove that the equation of the tangent to the curve $y = \cos x$ at $(\frac{1}{2}\pi, 0)$ is $y = \frac{1}{2}\pi - x$. By using $\frac{1}{2}\pi - x$ as an approximation for $\cos x$ (valid for x near $\frac{1}{2}\pi$), calculate an estimate of the smallest positive root of the equation $\cos x = x/10$, giving your answer to 2 decimal places. Show on a new sketch the part of the curve $y = \cos x$ near $x = \frac{1}{2}\pi$ and the tangent at $x = \frac{1}{2}\pi$. Hence determine whether the estimate of the root calculated above is larger or smaller than the correct value. (C)

11. The given flow diagram is designed to search for a root of the equation $f(x) = 0$ in the range $0 < x < 10$. (i) If x reaches 10 explain why the message printed has to be "No root found" and not "There is no root between 0 and 10". (ii) Work through the flow diagram in the case where $f(x) = e^x - 3x - 4$. (iii) Extend the flow diagram so that, when a root is located between two consecutive integers, a further search is carried out which will narrow the range in which the root lies to 0·1.



(C)

12. Using the same axes draw accurate graphs of $y = \ln x$ and $y = 3 - x$ in the interval $1 \leqslant x \leqslant 4$. Deduce that the equation $x + \ln x - 3 = 0$ has a root near $x = 2{\cdot}2$. Clearly showing your method, obtain alternative approximations to the root of the equation (a) by linear interpolation between $x = 2{\cdot}2$ and $x = 2{\cdot}3$, (b) by one application of the Newton-Raphson procedure using $x = 2{\cdot}2$ as the initial value. (L)

13. Given that $P(x) = x^3 - 6x^2 + 9x + a$ for all real x, find the values of x for which $P'(x) = 0$. Determine the values of the constant a such that the equation $P(x) = 0$ has a repeated root. By sketching graphs of $y = P(x)$ for these values of a, or otherwise, find the set of values of a for which $P(x) = 0$ has only one real root. Given now that $a = -5$, and that 4 is an approximation to the root of the equation $P(x) = 0$, use Newton's method once to obtain another approximation to the root, giving your answer correct to one decimal place. (JMB)

14. Use linear interpolation to find an approximation x_1 to the root of the equation $f(x) = 0$ lying between 3 and 4, where $f(x) \equiv x^4 - 4x^3 + x + 12$. Use one application of the Newton-Raphson process applied to $f(x)$ starting with $x = 4$ to find another approximation x_2 to the same root. Use one application of the same process starting with $x = 3$ to find another approximation x_3 to the same root. Correct to 2 decimal places the root is $x = 3{\cdot}69$. Indicate on a diagram how each of the approximations is related to the graph of $f(x)$ for $3 \leqslant x \leqslant 4$. (C)

15. Explain, with the use of diagrams, the use of Newton's method for obtaining the numerical solutions of an equation. Draw a diagram to show how the method can break down.

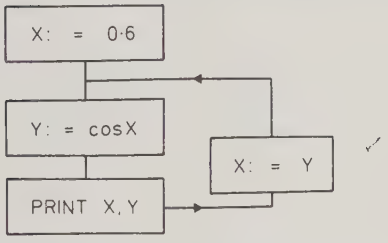
Obtain the real factors, with integer coefficients, of $x^4 - 4x^3 + 3x^2 - 4x + 12$. Hence or otherwise obtain the greater real root of the equation

$$x^4 - 4x^3 + 3x^2 - 4x + 11 = 0,$$

correct to 3 decimal places. (AEB 1979)

16. Show that the equation $x^3 + 3x - 3 = 0$ has only one real root and that it lies between $x = 0{\cdot}8$ and $x = 1$. Obtain approximations to the root (a) by performing one application of the Newton-Raphson procedure using $x = 0{\cdot}8$ as the first approximation, (b) by performing two iterations, using the procedure defined by $x_{n+1} = (3 - x_n^3)/3$, and starting with $x = 0{\cdot}8$. (L)

17. Using the same axes and scales, draw graphs of $y = \cos x$ and $y = x$ for the interval $0 \leqslant x \leqslant 1{\cdot}4$. (A scale of 10 cm to a unit on each axis is recommended.) On your graph mark as A, B, C and D the first four points whose coordinates are printed when the following section of a flow diagram is executed.



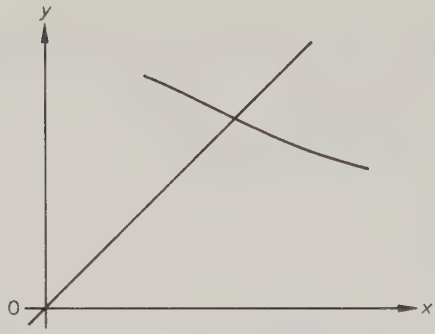
The procedure indicated in the flow diagram is an iterative method of solving a particular equation. Write down this equation and obtain from your graph an approximation solution to it. (JMB)

18. Given that $f(x) = x^3 + x^2 + 2x - 2$, show that the equation $f(x) = 0$ has only one root and that this root lies between 0 and 1. Use the iterative procedure

$$x_{n+1} = \frac{2}{x_n^2 + x_n + 2}, \quad x_1 = 1,$$

to obtain x_2 and x_3. By means of a sketch, illustrate how the successive values of x_n converge to the desired root. (L)

19. A solution of the equation $x = \phi(x)$ is to be attempted by using the iteration $x_{n+1} = \phi(x_n)$, starting with an initial estimate x_1. Draw sketch graphs showing $y = x$ and $y = \phi(x)$ to illustrate the following possibilities regarding convergence towards, or divergence from, the root $x = \alpha$. (i) $x_1 > \alpha$ and successive iterates steadily decrease, with the value α as a limit. (ii) $x_1 > \alpha$ and successive iterates are alternately less than α and greater than α, but approach α as a limit. (iii) $x_1 > \alpha$ and successive iterates get steadily larger. Use an iterative method to find a non-zero root of the equation $x = \tan^{-1}(2x)$ correct to 2 significant figures. (C)

20. 

The equation $\sin x = \cos^{-1} x$ has a root in the neighbourhood of $x = \frac{1}{4}\pi$, and the iteration $x_{n+1} = \cos(\sin x_n)$ is to be employed to calculate this root. The diagram shows the appropriate parts of the graphs of $y = x$ and $y = \cos(\sin x)$. Explain with reference to this diagram why successive iterates are alternately greater than and less than the root. Use the given iteration to determine the root correct to 2 decimal places, showing clearly that you have achieved this degree of accuracy. Construct a flow diagram to show how the iteration may be continued until successive iterates differ by less than a given quantity δ. (C)

21. Show that the equation $x^3 - 3x + 1 = 0$ has one negative root and two positive roots. Rearrange the equation in the form $x = F(x)$ in such a way that the iteration $x_{n+1} = F(x_n)$ will converge to the larger positive root, showing clearly that the condition for convergence is satisfied by your function F. Show by means of a sketch how the iterates converge to the root if the initial estimate x_1 is taken to be 1. Use your iteration to calculate the root correct to two decimal places, taking 1·5 as your initial approximation. (C)

24 Differential equations

24.1 Differential equations and their solution curves

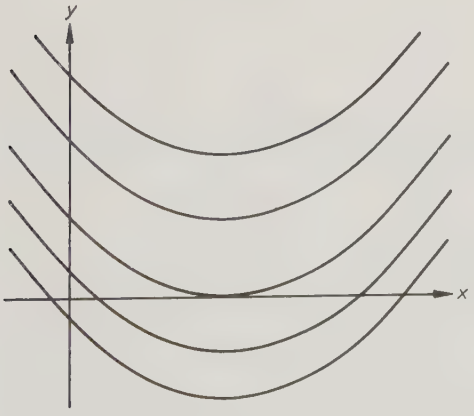
An equation connecting a function and one or more of its derivatives is called a *differential equation*. We shall see in §24.4 that many physical laws, such as the laws of natural growth or decay, can be expressed as differential equations.

The *order* of a differential equation is the order of the highest derivative which appears in it. For instance, the equations $\frac{dx}{dt} = -4x$ and $\left(\frac{dy}{dx}\right)^2 = x + y$ are first order equations and the equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} = e^{2x}$ is a second order equation.

Let us consider a simple differential equation of the first order:

$$\frac{dy}{dx} = x - 1.$$

Any equation connecting x and y which satisfies this differential equation is a solution. Thus $y = \frac{1}{2}x^2 - x$, $y = \frac{1}{2}x^2 - x + 5$ and $y = \frac{1}{2}x^2 - x - 28$ are all solutions. The *general solution* of the differential equation is $y = \frac{1}{2}x^2 - x + c$, where c is an arbitrary constant. Since c can take any real value, this general solution can be represented graphically by an infinite set of curves called the *solution curves* or *integral curves* of the equation. Each member of this "family" of curves represents a particular solution, or *particular integral*, of the differential equation.



The solution curves of $\frac{dy}{dx} = x - 1$.

Given the equation of a family of curves, we can obtain the corresponding differential equation by differentiating and then eliminating any arbitrary constants.

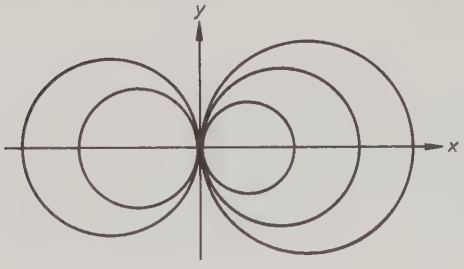
Example 1 Find the differential equation representing the family of curves, $y = Ax^3$, where A is an arbitrary constant.

If $y = Ax^3$, then $\dfrac{dy}{dx} = 3Ax^2$

$\therefore \quad x\dfrac{dy}{dx} = 3Ax^3 = 3y$

Hence the required differential equation is $x\dfrac{dy}{dx} = 3y$.

Example 2 Sketch some of the curves given by the equation $x^2 + y^2 - 2kx = 0$. Obtain the differential equation which represents this family of curves.



The equation $x^2 + y^2 - 2kx = 0$ may be written in the form

$$(x - k)^2 + y^2 = k^2.$$

Hence each curve in the family is a circle, centre $(k, 0)$ and radius $|k|$.

Differentiating the equation $\quad x^2 + y^2 - 2kx = 0$

we obtain $\quad 2x + 2y\dfrac{dy}{dx} - 2k = 0$

$\therefore \quad 2x^2 + 2xy\dfrac{dy}{dx} = x^2 + y^2 \quad (= 2kx)$

Hence this family of circles is represented by the differential equation

$$2xy\frac{dy}{dx} = y^2 - x^2.$$

Example 3 Given that $y = Ae^x + Be^{2x}$, find a differential equation connecting x and y which is independent of the arbitrary constants A and B.

If $y = Ae^x + Be^{2x}$, then $\quad \dfrac{dy}{dx} = Ae^x + 2Be^{2x}$

and $\quad \dfrac{d^2y}{dx^2} = Ae^x + 4Be^{2x}$

$\therefore \quad \dfrac{dy}{dx} - y = Be^{2x}$ and $\dfrac{d^2y}{dx^2} - y = 3Be^{2x}$

$\therefore \quad \dfrac{d^2y}{dx^2} - y = 3\left(\dfrac{dy}{dx} - y\right)$

Hence the required differential equation is

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0.$$

In Examples 1 and 2 equations involving one arbitrary constant gave rise to differential equations of the first order. However, in Example 3 it was necessary to introduce two derivatives to eliminate the two arbitrary constants. This suggests that in general an equation containing n arbitrary constants should lead to an nth order differential equation. Conversely, it seems reasonable to assume that the general solution of a differential equation of order n should contain n arbitrary constants.

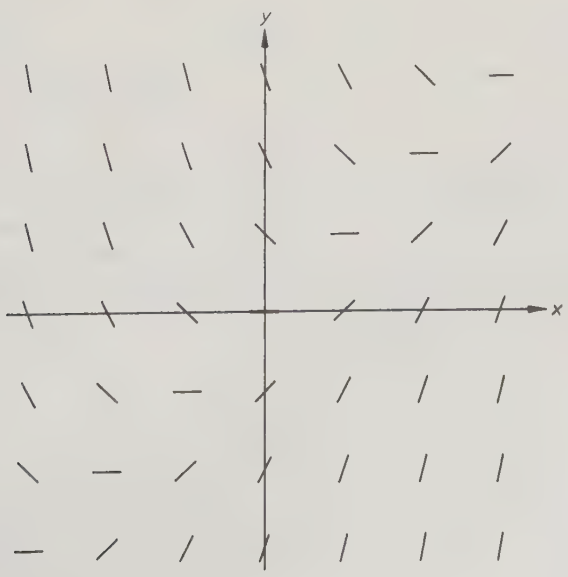
We now consider ways of sketching the solution curves of a differential equation when the general solution of the equation is not available. We take as an example the differential equation

$$\frac{dy}{dx} = x - y.$$

A general impression of the solution curves can be gained by calculating values of dy/dx at various points of the x, y plane, then plotting the directions on a sketch graph.

y \ x	−3	−2	−1	0	1	2	3
3	−6	−5	−4	−3	−2	−1	0
2	−5	−4	−3	−2	−1	0	1
1	−4	−3	−2	−1	0	1	2
0	−3	−2	−1	0	1	2	3
−1	−2	−1	0	1	2	3	4
−2	−1	0	1	2	3	4	5
−3	0	1	2	3	4	5	6

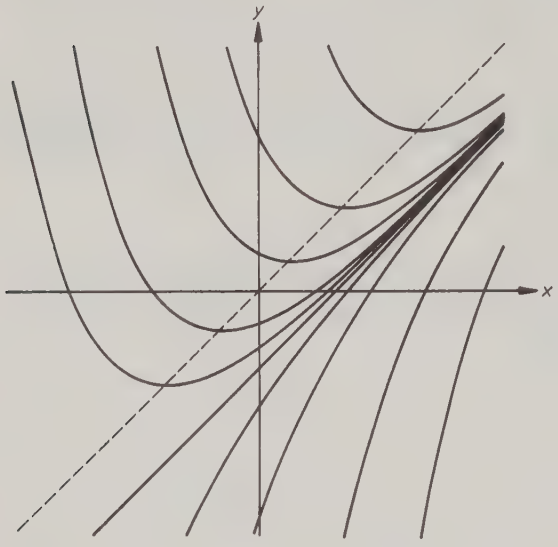
Table showing values of $\frac{dy}{dx}$



Sketch graph showing directions of the solution curves

We see that $\frac{dy}{dx} = 0$ on the line $y = x$. Above this line $\frac{dy}{dx} < 0$ and below the line $\frac{dy}{dx} > 0$. Hence the points on the line $y = x$ are minimum points on the solution curves.

The diagram also suggests that the line $y = x - 1$ could be one of the solution curves. Substituting $y = x - 1$ in the differential equation confirms that this is so.



It is now possible to make a rough sketch of the solution curves.

Exercise 24.1

1. Verify that the following are solutions of the differential equation

$$x\frac{dy}{dx} = x^2 + y,$$

(a) $y = x^2$, (b) $y = x^2 - x$, (c) $y = x^2 + 5x$.

2. Verify that the following are solutions of the differential equation $\frac{d^2y}{dx^2} = 1 - y$,

(a) $y = 1 + 2\cos x$, (b) $y = 1$, (c) $y = 1 - 3\sin x$.

3. Decide which of the following are solutions of the differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = 6y.$$

(a) $y = 3e^{2x}$, (b) $y = 6e^{x}$, (c) $y = 2e^{3x}$, (d) $y = e^{2x} - e^{-3x}$.

4. Given that $y = ax + b$ is a solution of the differential equation $\frac{dy}{dx} = 4x - 2y$, find a and b.

5. Given that $y = ax^2 + bx + c$ is a solution of the differential equation

$$\frac{d^2y}{dx^2} = y - x^2,$$

find a, b and c.

In questions 6 to 11 sketch some of the curves represented by the given equation and obtain a differential equation for y which does not contain the arbitrary constant A.

6. $y = x^3 + A$ 7. $y = x^2 - Ax$ 8. $y = A/x$

9. $y = Ae^{-x}$ 10. $x^2 + y^2 = A^2$ 11. $y^2 = 4Ax$

In questions 12 to 19 form differential equations by eliminating the arbitrary constant A, B and C.

12. $y = x + \frac{A}{x}$ 13. $y = x^2 \ln x + Ax^2$

14. $y^2 = A\cos x$ 15. $y = x^A$

16. $y = A\cos x + B\sin x$ 17. $y = Ae^{-x} + Be^{3x}$

18. $y^2 = Ax^2 + Bx + C$ 19. $y = (Ax + B)e^x + C$

In questions 20 to 25 sketch the solution curves of the given differential equation.

20. $\frac{dy}{dx} = \frac{x}{y}$ 21. $\frac{dy}{dx} = \frac{y}{x}$ 22. $\frac{dy}{dx} = -\frac{x}{2y}$

23. $\frac{dy}{dx} = x + y$ 24. $x\frac{dy}{dx} = x - y$ 25. $2y\frac{dy}{dx} = x + y$

24.2 Elementary methods of solution

Differential equations of the form $\frac{dy}{dx} = f(x)$ may be solved by direct integration.

Example 1 Solve the differential equation $\frac{dy}{dx} = 6x^2 - 5$.

By integration the general solution is $y = 2x^3 - 5x + c$, where c is an arbitrary constant.

Example 2 Solve the differential equation $\cos^2 x \dfrac{dy}{dx} = \sin x$.

Rearranging, we have: $\dfrac{dy}{dx} = \dfrac{\sin x}{\cos^2 x} = \sec x \tan x$

Integrating, we obtain the general solution $y = \sec x + c$, where c is an arbitrary constant.

Sometimes we require a solution which satisfies certain given conditions. We then use these conditions to determine the values of any arbitrary constants in the general solution.

Example 3 Solve the differential equation $e^x \dfrac{dy}{dx} + x = 0$, given that $y = 2$ when $x = 0$.

Rearranging we obtain: $\dfrac{dy}{dx} = -xe^{-x}$

Integrating by parts using the formula

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx,$$

where $u = x$ and $\dfrac{dv}{dx} = -e^{-x}$

$\dfrac{du}{dx} = 1$ $\qquad v = e^{-x}$

$$\int (-xe^{-x})\, dx = xe^{-x} - \int e^{-x} dx$$

$$= xe^{-x} + e^{-x} + c$$

$\therefore$ the general solution of the differential equation is

$$y = xe^{-x} + e^{-x} + c.$$

Since $y = 2$ when $x = 0$, $\qquad 2 = 0 \times 1 + 1 + c \qquad \therefore \quad c = 1$

Hence the required solution is $y = xe^{-x} + e^{-x} + 1$.

When the integration of a differential equation leads to a general solution involving logarithmic functions, any restrictions on the values of x and y should be clearly stated.

Example 4 Solve the differential equation $\dfrac{dy}{dx} = \dfrac{x}{x-1}$, given that $y = 2$ when $x = 2$.

Rearranging: $\dfrac{dy}{dx} = \dfrac{x}{x-1} = 1 + \dfrac{1}{x-1}$

[Since we require a solution valid for $x = 2$, we will assume that x is greater than 1 when integrating.]

For $x > 1$, the general solution of this differential equation is

$$y = x + \ln(x - 1) + c.$$

Since $y = 2$ when $x = 2$, $\quad 2 = 2 + \ln 1 + c \quad \therefore \quad c = 0$
Hence the required solution is $y = x + \ln(x - 1)$.

Second order differential equations of the form $\dfrac{d^2y}{dx^2} = f(x)$ can also be solved by direct integration.

Example 5 Solve the differential equation $\dfrac{d^2y}{dx^2} = 6x^3 + \dfrac{1}{x^3}$

Integrating we have: $\dfrac{dy}{dx} = \dfrac{3}{2}x^4 - \dfrac{1}{2x^2} + A$

Integrating again we obtain the general solution

$$y = \frac{3}{10}x^5 + \frac{1}{2x} + Ax + B,$$

where A and B are arbitrary constants.

Example 6 Solve the differential equation $\dfrac{d^2x}{dt^2} = -\sin t + \cos t$, subject to the conditions that $x = 0, \dfrac{dx}{dt} = 4$ when $t = 0$.

Integrating: $\dfrac{dx}{dt} = \cos t + \sin t + A$

$$x = \sin t - \cos t + At + B$$

Since $\dfrac{dx}{dt} = 4$ when $t = 0$, $4 = 1 + 0 + A \quad \therefore \quad A = 3$
Since $x = 0$ when $t = 0$, $\quad 0 = 0 - 1 + 0 + B \quad \therefore \quad B = 1$
Hence the required solution is $x = \sin t - \cos t + 3t + 1$.

To solve differential equations of the form $\dfrac{dy}{dx} = f(y)$, we use the fact that $\dfrac{dy}{dx} = 1 \Big/ \dfrac{dx}{dy}$.

Example 7 Solve the differential equation $\dfrac{dy}{dx} = y^2$.

The equation may be written as: $\dfrac{dx}{dy} = \dfrac{1}{y^2}$

Integrating with respect to y: $x = -\dfrac{1}{y} + c$

$\therefore$ $\dfrac{1}{y} = c - x$

Hence the general solution is $y = \dfrac{1}{c - x}$.

A substitution is sometimes used to reduce a differential equation to a simpler form.

Example 8 Use the substitution $y = vx$, where v is a function of x, to solve the differential equation $x\dfrac{dy}{dx} = x + y$, given that $y = -1$ when $x = 1$.

If $y = vx$, then using the product rule, $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Substituting in the given differential equation we have,

$$x\left(v + x\frac{dv}{dx}\right) = x + vx$$

i.e. $vx + x^2\dfrac{dv}{dx} = x + vx$

$\therefore$ $\dfrac{dv}{dx} = \dfrac{1}{x}$

For $x > 0$, the general solution of this differential equation is

$v = \ln x + c$

$\therefore$ $y = vx = x(\ln x + c)$

Since $y = -1$ when $x = 1$, $-1 = 1(0 + c)$ $\therefore$ $c = -1$

Hence the required solution is $y = x(\ln x - 1)$.

In Example 8 we again dealt with an integration involving logarithms by considering a restricted set of values of x. However, in some problems it is better to avoid placing restrictions on the values of x and y by using modulus signs, as described in §19.4.

For instance, in the case of the differential equation

$$\frac{dy}{dx} = \frac{1}{x - 5}$$

the general solution may be given as

$$y = \ln|x - 5| + c \qquad (1)$$

Suppose now that we require the particular solution which satisfies the condition $y = -2$ when $x = 6$.
Substituting these values in the general solution (1),

$$-2 = \ln 1 + c \qquad \therefore \quad c = -2$$

Hence $$y = \ln|x - 5| - 2.$$

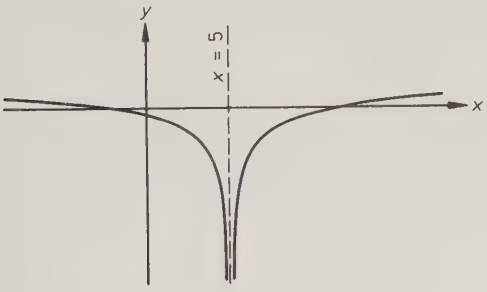
But, using the method of Examples 4 and 8,

for $x > 5$, $$y = \ln(x - 5) + c \tag{2}$$

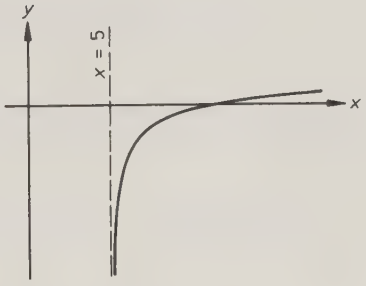
As before, since $y = -2$ when $x = 6$, we have $c = -2$

Hence $$y = \ln(x - 5) - 2.$$

Thus, although equation (1) provides a neat general solution, equation (2) leads to a simpler particular integral.



$y = \ln|x - 5| - 2$



$y = \ln(x - 5) - 2$

The sketch graphs show that solution (1) represents a family of curves with two branches, whereas solution (2) produces single curves lying to the right of the line $x = 5$. A question requiring a particular integral is probably best answered by a solution of type (2).

Exercise 24.2

Solve the following differential equations giving y in terms of x.

1. $\dfrac{dy}{dx} = 3x^2 + 1$

2. $\dfrac{dy}{dx} = \cos \frac{1}{2}x$

3. $x\dfrac{dy}{dx} = x^2 + 1$

4. $(x - 1)\dfrac{dy}{dx} = x + 1$

5. $\dfrac{d^2y}{dx^2} + \cos x = 2x$

6. $e^x\dfrac{d^2y}{dx^2} = 1 - e^{2x}$

7. $\dfrac{dy}{dx} + 2y^2 = 0$

8. $\dfrac{dy}{dx} = \cos^2 y$

9. $\dfrac{dy}{dx} = e^{-y}$

10. $3\dfrac{dy}{dx} = y^4$

Solve the following differential equations, subject to the given conditions.

11. $\dfrac{dy}{dx} = (x + 1)^2$; $y = 0$ when $x = 2$

12. $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{(2x + 3)}}$; $y = 4$ when $x = 3$

13. $\sec x \dfrac{dy}{dx} = x$; $y = 0$ when $x = \pi$

14. $(1 + x^2)\dfrac{dy}{dx} = 2x$; $y = 0$ when $x = -1$

15. $\dfrac{d^2y}{dx^2} = \sin 2x$; $y = 0, \dfrac{dy}{dx} = -\dfrac{1}{4}$ when $x = 0$

16. $x^2\dfrac{d^2y}{dx^2} = 2(x^2 - 1)$; $y = \dfrac{dy}{dx} = 0$ when $x = 1$

17. $\dfrac{dy}{dx} + 2y = 0$; $y = 3$ when $x = 0$

18. $\dfrac{dy}{dx} = y - 1$; $y = 0$ when $x = 0$

19. $\dfrac{dy}{dx} = 1 + y^2$; $y = 1$ when $x = 0$

20. $2\dfrac{dy}{dx} = y^2 - 1$; $y = 2$ when $x = 0$

21. Use the substitution $y = 1/z$ to solve the differential equation $x^2\dfrac{dy}{dx} = y^2$.

22. Use the substitution $y = vx$, where v is a function of x, to solve the differential equation $x\dfrac{dy}{dx} - y = x^2 \cos x$.

23. Use the substitution $y = vx^2$, where v is a function of x, to solve the differential equation $x\dfrac{dy}{dx} - 2y = x$.

24. Use the substitution $dy/dx = p$, where p is a function of x, to solve the differential equation $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 1$, given that $y = \dfrac{dy}{dx} = 2$ when $x = 1$.

25. If $z = xe^y$, where y is a function of x, find an expression for dz/dx. Hence find y in terms of x given that $xe^y \dfrac{dy}{dx} + e^y = 2x$ and that $y = 0$ when $x = 2$.

24.3 Equations with separable variables

If a first order differential equation can be expressed in the form

$$f(y)\frac{dy}{dx} = g(x)$$

then the variables x and y are said to be *separable.*
Integrating both sides of the equation with respect to x:

$$\int f(y)\frac{dy}{dx}\,dx = \int g(x)\,dx$$

As shown in §20.3, $\dfrac{dy}{dx}\,dx$ may be replaced by dy.

Thus $$\int f(y)\,dy = \int g(x)\,dx$$

Example 1 Find the general solution of the equation $2y\dfrac{dy}{dx} = \cos x$.

Integrating both sides of the equation: $\displaystyle\int 2y\,dy = \int \cos x\,dx$

Hence the general solution of the equation is

$$y^2 = \sin x + c.$$

Example 2 Find the general solution of the equation

$$\frac{dy}{dx} = 2xy^2 - y^2.$$

Rearranging: $$\frac{dy}{dx} = (2x - 1)y^2$$

$$\frac{1}{y^2}\frac{dy}{dx} = 2x - 1$$

$$\therefore \quad \int \frac{1}{y^2}\,dy = \int (2x - 1)\,dx \qquad \therefore \quad -\frac{1}{y} = x^2 - x + c$$

Hence the general solution of the equation is

$$y = -1/(x^2 - x + c).$$

Example 3 Find y in terms of x given that $2\dfrac{dy}{dx} = e^{x-2y}$ and that $y = 1$ when $x = 2$.

Rearranging: $$2\frac{dy}{dx} = e^x \times e^{-2y}$$

$$2e^{2y}\frac{dy}{dx} = e^x$$

$\therefore$ $$\int 2e^{2y}\,dy = \int e^x\,dx$$

$\therefore$ $$e^{2y} = e^x + c$$

Since $y = 1$ when $x = 2$, $\quad e^2 = e^2 + c \quad \therefore \quad c = 0$

Thus $$e^{2y} = e^x$$

$\therefore$ $$2y = x$$

Hence $$y = \tfrac{1}{2}x.$$

Example 4 Express y in terms of x given that $\dfrac{dy}{dx} = xy - x$ and that $y = 4$ when $x = 0$.

Rearranging: $$\frac{dy}{dx} = x(y-1)$$

$$\frac{1}{(y-1)}\frac{dy}{dx} = x$$

$\therefore$ $$\int \frac{1}{y-1}\,dy = \int x\,dx$$

$\therefore$ $$\ln|y-1| = \tfrac{1}{2}x^2 + c$$

$\therefore$ $$|y-1| = e^{x^2/2+c}$$

Thus $$y - 1 = Ae^{x^2/2}, \quad \text{where} \quad |A| = e^c$$

Since $y = 4$ when $x = 0$, $3 = Ae^0 = A$

Hence $y - 1 = 3e^{x^2/2}$

i.e. $$y = 1 + 3e^{x^2/2}.$$

Example 5 Find the general solution of the equation

$$(x^2 - 1)\frac{dy}{dx} = xy.$$

Rearranging: $$\frac{1}{y}\frac{dy}{dx} = \frac{x}{x^2-1}$$

$\therefore$ $$\int \frac{1}{y}\,dy = \int \frac{x}{x^2-1}\,dx$$

$$\therefore \quad \ln|y| = \tfrac{1}{2}\ln|x^2 - 1| + c$$
$$\therefore \quad 2\ln|y| - \ln|x^2 - 1| = 2c$$

Using the laws of logarithms:

$$\ln\left|\frac{y^2}{x^2 - 1}\right| = 2c$$

$$\therefore \quad \frac{y^2}{x^2 - 1} = A, \quad \text{where} \quad \ln|A| = 2c$$

Hence the general solution of the equation is $y^2 = A(x^2 - 1)$.

Example 6 Find the general solution of the equation

$$2x\frac{dy}{dx} - (2y + 1)(x + 1) = 0$$

Rearranging:
$$2x\frac{dy}{dx} = (2y + 1)(x + 1)$$

$$\frac{2}{(2y + 1)}\frac{dy}{dx} = 1 + \frac{1}{x}$$

$$\therefore \quad \int \frac{2}{2y + 1}\,dy = \int\left(1 + \frac{1}{x}\right)dx$$

$$\therefore \quad \ln|2y + 1| = x + \ln|x| + c$$
$$\therefore \quad \ln|2y + 1| - \ln|x| = x + c$$

i.e.
$$\ln\left|\frac{2y + 1}{x}\right| = x + c$$

$$\therefore \quad \left|\frac{2y + 1}{x}\right| = e^{x+c}$$

Thus
$$\frac{2y + 1}{x} = Ae^x, \quad \text{where} \quad |A| = e^c$$

$$\therefore \quad 2y + 1 = Axe^x$$

Hence the required general solution is $y = \frac{1}{2}(Axe^x - 1)$.

Example 7 Find y in terms of x given that $x(x - 1)\dfrac{dy}{dx} = y$ and that $y = 1$ when $x = 2$.

Rearranging:
$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x(x - 1)}$$

$$\therefore \qquad \int \frac{1}{y}\,dy = \int \frac{1}{x(x-1)}\,dx$$

$$\therefore \qquad \int \frac{1}{y}\,dy = \int \left\{\frac{1}{x-1} - \frac{1}{x}\right\} dx$$

$$\therefore \qquad \ln|y| = \ln|x-1| - \ln|x| + c$$

i.e. $\ln|y| - \ln|x-1| + \ln|x| = c$

Thus $$\ln\left|\frac{xy}{x-1}\right| = c$$

$$\therefore \qquad \frac{xy}{x-1} = A, \quad \text{where} \quad \ln|A| = c$$

Since $y = 1$ when $x = 2$, $\qquad 2 = A$

Hence $$\frac{xy}{x-1} = 2$$

i.e. $$y = 2\left(1 - \frac{1}{x}\right).$$

Exercise 24.3

In questions 1 to 10 find the general solutions of the given differential equations, expressing y in terms of x in each case.

1. $3y^2\dfrac{dy}{dx} = 2x - 1$

2. $\dfrac{dy}{dx} = 6xy^2$

3. $\dfrac{dy}{dx} = e^y \sin x$

4. $\dfrac{dy}{dx} = e^{x-y}$

5. $\dfrac{dy}{dx} = x \sec y$

6. $\dfrac{dy}{dx} = 3\cos^2 y$

7. $x\dfrac{dy}{dx} = y$

8. $(1-x)\dfrac{dy}{dx} = y$

9. $\dfrac{dy}{dx} = \dfrac{4xy}{x^2+1}$

10. $\dfrac{dy}{dx} = \dfrac{2y}{x^2-1}$

In questions 11 to 20 find expressions for y in terms of x.

11. $x^2\dfrac{dy}{dx} - y^2 = 0$; $y = -1$ when $x = 1$.

12. $\frac{dy}{dx} + 2xy = 0$; $y = 5$ when $x = 0$.

13. $\cot x \frac{dy}{dx} = y$; $y = 2$ when $x = 0$.

14. $\frac{dy}{dx} = xe^{-2y}$; $y = 0$ when $x = 0$.

15. $\frac{dy}{dx} - 2xy = 2x$; $y = 1$ when $x = 0$.

16. $x\frac{dy}{dx} = xy + y$; $y = 1$ when $x = 1$.

17. $(1 + x^3)\frac{dy}{dx} = 3x^2 \tan y$; $y = \frac{1}{2}\pi$ when $x = 0$.

18. $x \cos y \frac{dy}{dx} = 1 + \sin y$; $y = 0$ when $x = 1$.

19. $x\frac{dy}{dx} = 2y(y - 1)$; $y = 2$ when $x = \frac{1}{2}$.

20. $2x\frac{dy}{dx} = 1 - y^2$; $y = 0$ when $x = 1$.

In questions 21 to 24 find the general solutions of the given differential equations, expressing y in terms of x.

21. $(1 - x)\frac{dy}{dx} = xy$

22. $(x^2 - 1)\frac{dy}{dx} = (x^2 + 1)y$

23. $\frac{dy}{dx} = e^x(1 + y^2)$

24. $e^y\frac{dy}{dx} + 2x = 2xe^y$

25. Find y in terms of x given that $e^{2x}y\frac{dy}{dx} + 2x = 0$ and that $y = 1$ when $x = 0$.

26. Find y in terms of x given that $xy\frac{dy}{dx} = \sqrt{(y^2 - 9)}$ and that $y = 5$ when $x = e^4$.

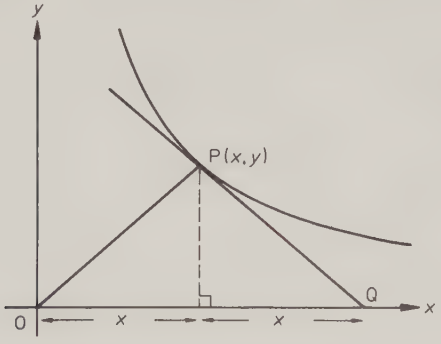
27. Use the substitution $x + y = z$ to obtain the general solution of the differential equation $(x + y - 1)\frac{dy}{dx} = x - y + 1$.

28. Use the substitution $y = vx$, where v is a function of x, to obtain the general solution of the differential equation $xy\dfrac{dy}{dx} = 2x^2 - y^2$.

24.4 Formulation of differential equations

In our first example information about the tangent to a curve is expressed in the form of a differential equation. This equation is then solved to find the equation of the curve.

Example 1 The tangent at any point P on a curve in the first quadrant cuts the x-axis at Q. Given that $OP = PQ$, where O is the origin, and that the point $(1, 4)$ lies on the curve, find the equation of the curve.



Since $OP = PQ$ the triangle OPQ is isosceles. Thus if P is the point (x, y), the coordinate of Q must be $(2x, 0)$.

$\therefore$ the gradient of PQ

$$= \frac{y - 0}{x - 2x} = -\frac{y}{x}$$

However, since PQ is the tangent at P its gradient is also given by dy/dx.

$$\therefore \quad \frac{dy}{dx} = -\frac{y}{x}$$

i.e. $$\frac{1}{y}\frac{dy}{dx} = -\frac{1}{x}$$

$$\therefore \int \frac{1}{y}dy = -\int \frac{1}{x}dx$$

Since the curve lies in the first quadrant, $x > 0$ and $y > 0$

$$\therefore \quad \ln y = -\ln x + c$$

i.e. $$\ln y + \ln x = c$$

$$\therefore \quad \ln xy = c$$

$$\therefore \quad xy = k, \quad \text{where} \quad \ln k = c$$

Since the curve passes through the point $(1, 4)$, $1 \times 4 = k$

Hence the equation of the curve is $xy = 4$.

Many scientific laws can be expressed as differential equations. For instance, as we saw in §19.1, Newton's law of cooling and certain laws of growth and decay lead to

differential equations of the form $dx/dt = \pm kx$, where k is a positive constant. The solutions of such equations are used to make predictions about the behaviour of the variables involved.

Example 2 A liquid is being heated in an oven maintained at a constant temperature of 180°C. It is assumed that the rate of increase in the temperature of the liquid is proportional to $(180 - \theta)$, where θ°C is the temperature of the liquid at time t minutes. If the temperature of the liquid rises from 0°C to 120°C in 5 minutes, find the temperature of the liquid after a further 5 minutes.

The rate of increase in the temperature of the liquid is $d\theta/dt$,

$$\therefore \quad \frac{d\theta}{dt} = k(180 - \theta), \text{ where } k \text{ is constant.}$$

Rearranging, $$\frac{1}{(180 - \theta)}\frac{d\theta}{dt} = k$$

$$\therefore \quad \int \frac{1}{180 - \theta} d\theta = \int k\, dt$$

$$\therefore \quad \int \frac{-1}{180 - \theta} d\theta = -\int k\, dt$$

Thus for $\theta < 180$, $\ln(180 - \theta) = -kt + c$

$$\therefore \quad 180 - \theta = e^{-kt+c}$$

Writing $A = e^c$, $180 - \theta = Ae^{-kt}$

Hence $\theta = 180 - Ae^{-kt}$

Letting $t = 0$ when $\theta = 0$, $0 = 180 - A$

$$\therefore \quad \theta = 180(1 - e^{-kt})$$

Since $\theta = 120$ when $t = 5$, $120 = 180(1 - e^{-5k})$

$$\therefore \quad \tfrac{2}{3} = 1 - e^{-5k}$$

i.e. $e^{-5k} = \frac{1}{3}$

Hence when $t = 10$, $\theta = 180(1 - e^{-10k})$

$$= 180(1 - \{e^{-5k}\}^2)$$

$$= 180(1 - \{\tfrac{1}{3}\}^2) = 160$$

Thus the temperature of the liquid after a further 5 minutes is 160°C.

Exercise 24.4

1. The tangent at any point P on a curve cuts the x-axis at the point Q. Given that $\angle OPQ = 90°$, where O is the origin, and that the point $(1, 2)$ lies on the curve, find the equation of the curve.

2. The tangent at any point P on a curve cuts the x-axis at A and the y-axis at B. Given that $2\overrightarrow{AP} = \overrightarrow{PB}$ and that the curve passes through the point $(1, 1)$, find the equation of the curve.

3. At time t, the rate of increase in the concentration C of micro-organisms in a controlled environment is equal to k times the concentration, where k is a positive constant. When $t = 0, C = C_0$. Write down a differential equation involving C, t and k. Hence find C in terms of C_0, t and k. Find also, in terms of k, the time at which the concentration has increased by 50% from its value at $t = 0$. (L)

4. A body is placed in a room which is kept at a constant temperature. The temperature of the body falls at a rate $k\theta$ C per minute, where k is a constant and θ is the difference between the temperature of the body and that of the room at time t. Express this information in the form of a differential equation and hence show that $\theta = \theta_0 e^{-kt}$, where θ_0 is the temperature difference at time $t = 0$. The temperature of the body falls 5°C in the first minute and 4°C in the second minute. Show that the fall of temperature in the third minute is 3·2°C. (L)

5. A rectangular tank with its base horizontal is filled with water to a depth h at time $t = 0$. Water leaks out of the tank from a small hole in the base at a rate proportional to the square root of the depth of the water. If the depth of water is $\frac{1}{2}h$ at time t, find the further time it will take before the tank is empty. (O)

6. A radioactive substance decays so that the rate of decrease of mass at any time is proportional to the mass present at that time. Denoting by x the mass remaining at time t, write down a differential equation satisfied by x, and show that $x = x_0 e^{-kt}$, where x_0 is the initial mass and k is a constant. The mass is reduced to 4/5 of its initial value in 30 days. Calculate, correct to the nearest day, the time required for the mass to be reduced to half its initial value. A mass of 625 milligrammes of the substance is prepared. Determine the mass which is present 90 days after the preparation. (JMB)

7. According to Newton's law, the rate of cooling of a body in air is proportional to the difference between the temperature T of the body and the temperature T_0 of the air. If the air temperature is kept constant at 20°C and the body cools from 100°C to 60°C in 20 minutes, in what further time will the body cool to 30°C? (O)

8. Two liquids, X and Y, are flowing into a trough at the constant rate of 10 and 20 litres per minute respectively. The liquid in the trough is stirred continuously and pumped out at the rate of 30 litres per minute. Initially the trough contains 200 litres of X and 100 litres of Y. After t minutes the tank contains x litres of X. By considering the change in x in a small interval of time δt, show that

$$\frac{dx}{dt} = 10 - \frac{x}{10}.$$

Hence find an expression for x in terms of t. Find, correct to the nearest litre, the quantity of liquid X in the trough after 10 minutes. After how long, to the nearest second, will there be less of liquid X in the mixture than of liquid Y?

9. A tank being cleaned initially contains 3900 litres of water and 100 kg of dissolved dye. The mixture is kept uniform by stirring. Pure water is run in at the rate

of 130 litres per minute and the mixture is removed at the same rate. If the mass of dissolved dye remaining after t minutes is x kg, show that

$$\frac{dx}{dt} = -\frac{x}{30}.$$

How long will it be before half the dye has been removed? Determine how much dye remains after 30 minutes. (L)

10. In a certain chemical reaction, in which a compound X is formed from a compound Y, the masses of X, Y present at time t are x, y respectively. The sum of the masses of X and Y is a, where a is constant, and at any time the rate at which x is increasing is proportional to the product of the two masses at that time. Show that $\frac{dx}{dt} = kx(a - x)$, where k is constant. If $x = a/5$ at $t = 0$ and $x = a/2$ at $t = \ln 2$, show that $k = 2/a$. Hence find t, correct to three significant figures, when $x = 99a/100$. (O & C)

11. A plant grows in a pot which contains a volume V of soil. At time t the mass of the plant is m and the volume of soil utilised by the roots is αm, where α is constant. The rate of increase of the mass of the plant is proportional to the mass of the plant times the volume of soil not yet utilised by the roots. Obtain a differential equation for m, and verify that it can be written in the form $V\beta\frac{dt}{dm} = \frac{1}{m} + \frac{\alpha}{V - \alpha m}$, where β is a constant. The mass of the plant is initially $V/4\alpha$. Find, in terms of V and β, the time taken for the plant to double its mass. Find also the mass of the plant at time t. (JMB)

12. At any instant, a spherical meteorite is gaining mass because of two effects: (i) mass is condensing onto it at a rate which is proportional to the surface area of the meteorite at that instant, (ii) the gravitational field of the meteorite attracts mass onto itself, the rate being proportional to the meteorite mass at that instant. Assuming that the two effects can be added together and that the meteorite remains spherical and of constant density, show that its radius r at time t satisfies the differential equation $\frac{dr}{dt} = A + Br$, where A and B are constants. If $r = r_0$ at $t = 0$, show that $r = r_0e^{Bt} + \frac{A}{B}(e^{Bt} - 1)$. (L)

24.5 The equations $d^2y/dx^2 \pm n^2y = \text{constant}$

We now consider some second order differential equations which have important applications especially in mechanics.

Type 1: $\dfrac{d^2y}{dx^2} + n^2y = 0$, where n is constant.

Multiplying both sides by $2\dfrac{dy}{dx}$ we obtain:

$$2\frac{dy}{dx}\frac{d^2y}{dx^2} + n^2\left(2y\frac{dy}{dx}\right) = 0.$$

But $$2\frac{dy}{dx}\frac{d^2y}{dx^2} = \frac{d}{dx}\left\{\left(\frac{dy}{dx}\right)^2\right\} \quad \text{and} \quad 2y\frac{dy}{dx} = \frac{d}{dx}(y^2),$$

so integrating both sides with respect to x:

$$\left(\frac{dy}{dx}\right)^2 + n^2y^2 = \text{constant}$$

Letting $y = a$ when $\dfrac{dy}{dx} = 0$, this becomes

$$\left(\frac{dy}{dx}\right)^2 + n^2y^2 = n^2a^2$$

$\therefore$ $$\frac{dy}{dx} = \pm n\sqrt{(a^2 - y^2)}$$

Hence $$\int \frac{1}{\sqrt{(a^2 - y^2)}}\,dy = \pm\int n\,dx$$

$\therefore$ $$\sin^{-1}\frac{y}{a} = \pm(nx + b)$$

$\therefore$ $$\frac{y}{a} = \sin\{\pm(nx + b)\}$$

$\therefore$ $$y = \pm a\sin(nx + b)$$

Since a is an arbitrary constant which may take any real value, positive or negative, there is no loss of generality in writing:

$$y = a\sin(nx + b)$$

Letting $b = c + \frac{1}{2}\pi$, we obtain:

$$y = a\sin(nx + c + \tfrac{1}{2}\pi)$$

i.e. $$y = a\cos(nx + c)$$

But $$a\cos(nx + c) = a\cos nx\cos c - a\sin nx\sin c,$$

thus the solution can also be expressed in the form

$$y = A\cos nx + B\sin nx$$

Thus the general solution of the differential equation

$$\frac{d^2y}{dx^2} + n^2y = 0$$

may be expressed in three possible forms:

$$y = a\sin(nx + b), \quad y = a\cos(nx + c)$$

or $$y = A\cos nx + B\sin nx,$$

where a, b, c, A and B are arbitrary constants.

Example 1 Solve the differential equation $\frac{d^2y}{dx^2} + 9y = 0$ given that $y = 1$ when $x = 0$ and that $y = 2$ when $x = \frac{1}{2}\pi$.

The general solution of the given equation is

$$y = A\cos 3x + B\sin 3x$$

Since $y = 1$ when $x = 0$, $\quad 1 = A + 0$ i.e. $A = 1$

Since $y = 2$ when $x = \frac{1}{2}\pi$, $\quad 2 = 0 - B$ i.e. $B = -2$

Hence the required solution is $y = \cos 3x - 2\sin 3x$.

Example 2 Solve the differential equation $9\frac{d^2y}{dx^2} + 4y = 0$ given that $y = \frac{dy}{dx} = 2$ when $x = 0$.

Rearranging the equation, $\quad \frac{d^2y}{dx^2} + \frac{4}{9}y = 0$

Hence the general solution is

$$y = A\cos\tfrac{2}{3}x + B\sin\tfrac{2}{3}x$$

$$\therefore \quad \frac{dy}{dx} = -\tfrac{2}{3}A\sin\tfrac{2}{3}x + \tfrac{2}{3}B\cos\tfrac{2}{3}x$$

Since $y = 2$ when $x = 0$, $\quad 2 = A$

Since $\frac{dy}{dx} = 2$ when $x = 0$, $\quad 2 = \frac{2}{3}B$

$\therefore \quad A = 2$ and $B = 3$

Hence the required solution is $y = 2\cos\frac{2}{3}x + 3\sin\frac{2}{3}x$.

Type 2: $\frac{d^2y}{dx^2} + n^2y = k$, where n and k are constant.

Rearranging: $$\frac{d^2y}{dx^2} + n^2\left(y - \frac{k}{n^2}\right) = 0$$

Letting $z = y - \frac{k}{n^2}$, we find that $\frac{d^2z}{dx^2} = \frac{d^2y}{dx^2}$

so the equation becomes: $\dfrac{d^2z}{dx^2} + n^2z = 0$

$\therefore$ the general solution is $z = A\cos nx + B\sin nx$

i.e. $y - \dfrac{k}{n^2} = A\cos nx + B\sin nx$

Thus the general solution of the differential equation

$$\frac{d^2y}{dx^2} + n^2y = k$$

is $y = A\cos nx + B\sin nx + k/n^2$,

where A and B are arbitrary constants.

Type 3: $\dfrac{d^2y}{dx^2} - n^2y = 0$, where n is constant.

Since $y = e^{nx}$ is a solution of this equation, we try the substitution $y = e^{nx}v$, where v is a function of x.

Differentiating: $\dfrac{dy}{dx} = e^{nx}\dfrac{dv}{dx} + ne^{nx}v$

$\therefore$ $\dfrac{d^2y}{dx^2} = e^{nx}\dfrac{d^2v}{dx^2} + 2ne^{nx}\dfrac{dv}{dx} + n^2e^{nx}v$

Hence the differential equation becomes

$$e^{nx}\frac{d^2v}{dx^2} + 2ne^{nx}\frac{dv}{dx} + n^2e^{nx}v - n^2e^{nx}v = 0$$

i.e. $\dfrac{d^2v}{dx^2} + 2n\dfrac{dv}{dx} = 0$

Integrating with respect to x,

$$\frac{dv}{dx} + 2nv = a$$

Rearranging: $\dfrac{dv}{dx} = a - 2nv$

$\therefore$ $\dfrac{1}{(a-2nv)}\dfrac{dv}{dx} = 1$

$\therefore$ $\displaystyle\int \frac{-2n}{(a-2nv)}\,dv = \int(-2n)\,dx$

$\therefore$ $\ln|a - 2nv| = -2nx + c$

i.e. $|a - 2nv| = e^{-2nx+c}$

Thus $a - 2nv = ke^{-2nx}$, where $|k| = e^c$.

$\therefore \quad v = \frac{a}{2n} - \frac{k}{2n}e^{-2nx}$

$\therefore \quad y = e^{nx}v = \frac{a}{2n}e^{nx} - \frac{k}{2n}e^{-nx}$

Thus the general solution of the differential equation

$$\frac{d^2y}{dx^2} - n^2y = 0$$

is of the form $y = Ae^{nx} + Be^{-nx}$,

where A and B are arbitrary constants.

Example 3 Solve the differential equation $\frac{d^2y}{dx^2} - 100y = 0$, given that $y = 6$, $\frac{dy}{dx} = 40$ when $x = 0$.

The general solution of the given equation is

$$y = Ae^{10x} + Be^{-10x}$$

$\therefore \quad \frac{dy}{dx} = 10Ae^{10x} - 10Be^{-10x}$

Since $y = 6$ when $x = 0$, $\quad 6 = A + B$

Since $\frac{dy}{dx} = 40$ when $x = 0$, $\quad 40 = 10A - 10B$

$\therefore \quad A = 5$ and $B = 1$

Hence the required solution is $y = 5e^{10x} + e^{-10x}$.

Type 4: $\frac{d^2y}{dx^2} - n^2y = k$, where n and k are constant.

By the method applied to equations of type 2 it can be shown that

$$y + \frac{k}{n^2} = Ae^{nx} + Be^{-nx}$$

Thus the general solution of the differential equation

$$\frac{d^2y}{dx^2} - n^2y = k$$

is $y = Ae^{nx} + Be^{-nx} - \frac{k}{n^2}$,

where A and B are arbitrary constants.

Exercise 24.5

[In this exercise the general solutions of the given differential equations may be quoted without proof.]

Write down the general solutions of the following differential equations:

1. $\frac{d^2y}{dx^2} = -16y$

2. $4\frac{d^2y}{dx^2} + 7y = 0$

3. $\frac{d^2y}{dx^2} + 9y = 18$

4. $\frac{d^2y}{dx^2} = 10 - 10y$

5. $\frac{d^2y}{dx^2} = 36y$

6. $\frac{d^2y}{dx^2} - 8y = 0$

7. $9\frac{d^2y}{dx^2} = 1 + 27y$

8. $\frac{d^2y}{dx^2} - 4y + 1 = 0$

Solve the following differential equations subject to the given conditions.

9. $\frac{d^2y}{dx^2} + 4y = 0$; $y = 4$ when $x = 0$ and $y = 3$ when $x = 3\pi/4$.

10. $\frac{d^2y}{dx^2} - 16y = 0$; $y = 10$ and $\frac{dy}{dx} = -16$ when $x = 0$.

11. $36\frac{d^2y}{dx^2} + y = 0$; $y = 6$ and $\frac{dy}{dx} = 0$ when $x = 0$.

12. $9\frac{d^2y}{dx^2} + 16y = 0$; $y = -5$ when $x = 0$ and $y = 4$ when $x = 3\pi/8$.

13. $4\frac{d^2y}{dx^2} - 25y = 0$; $y = -9$ and $\frac{dy}{dx} = 12\frac{1}{2}$ when $x = 0$.

14. $\frac{d^2y}{dx^2} - 12y = 0$; $y = 4/e^2$ when $x = 1/\sqrt{3}$ and $y = 4$ when $x = 0$.

15. $9\frac{d^2y}{dx^2} - 49y = 28$; $y = 8\frac{3}{7}$ and $\frac{dy}{dx} = -7$ when $x = 0$.

16. $4\frac{d^2y}{dx^2} + 25y = 40$; $y = 5{\cdot}6$ and $\frac{dy}{dx} = 22{\cdot}5$ when $x = 0$.

17. $\frac{d^2y}{dx^2} + 3y = 6$; $y = 2$ when $x = 0$ and $y = 3$ when $x = \pi/2\sqrt{3}$.

18. $16\dfrac{d^2y}{dx^2} - 81y = 243$; $y = -1$ and $\dfrac{dy}{dx} = 0$ when $x = 0$.

19. $\dfrac{d^2y}{dx^2} - 18y = 54$; $y = -3 + \sqrt{2}$ and $\dfrac{dy}{dx} = 6$ when $x = 0$.

20. $\dfrac{d^2y}{dx^2} + 12y = 6$; $\dfrac{dy}{dx} = 24$ when $x = \pi/4\sqrt{3}$ and $\dfrac{dy}{dx} = -18$ when $x = 0$.

Exercise 24.6 (miscellaneous)

1. Sketch some of the straight lines represented by the equation $y = k(2x - k)$, where k is any constant. Obtain the differential equation for this family of lines.

2. Find a second order differential equation of which the general solution is $y = Axe^x + Be^{-x}$, where A and B are arbitrary constants.

3. The differential equation of a family of curves is $\dfrac{dy}{dx} = y - x^2$. Find the equations of the loci of points where $\dfrac{dy}{dx} = 0$ and where $\dfrac{d^2y}{dx^2} = 0$. Hence sketch the integral curves of the differential equation.

4. (a) Find, in any form, the solution of the differential equation $\dfrac{du}{dt} - 7u + 3 = 0$, given that $u = 0$ when $t = 5$.

   (b) Solve the differential equation $\dfrac{dy}{dx} = x\sqrt{(x^2 + 9)}$ given that $y = -9$ when $x = 0$. (C)

5. (a) Solve the differential equation $\dfrac{dx}{dt} = \dfrac{1}{x + 2} - \dfrac{3}{3x + 5}$, given that $x = 1$ when $t = 2$.

   (b) Solve the differential equation $\dfrac{dy}{dx} = \dfrac{2y^2 + 3}{4y}$, given that $y = 1$ when $x = -1$. Give your answer in a form expressing y in terms of x. (C)

6. The temperature y degrees of a body, t minutes after being placed in a certain room, satisfies the differential equation $6\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} = 0$. By using the substitution $z = dy/dt$, or otherwise, find y in terms of t, given that $y = 63$ when $t = 0$ and $y = 36$ when $t = 6 \ln 4$. Find after how many minutes the rate of cooling of the body will have fallen below one degree per minute, giving your answer correct to the nearest minute. How cool does the body get? (O&C)

7. By substituting $z = (1 + x)y$, solve the differential equation

$$\frac{dy}{dx} + \frac{y}{1+x} = \cos x,$$

given that $y = 0$ when $x = 0$. In your answer give y in terms of x.

8. (a) Solve the differential equation $xy\dfrac{dy}{dx} = 1 + y^2$, given that, when $x = 2$, $y = 0$. Find the equation of the tangent to the curve at the point in the first quadrant where $x = 4$.
   (b) If $Ax^2 - By^2 = C$, where A, B and C are constants, prove that

$$xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0. \qquad \text{(SU)}$$

9. (a) Solve the differential equation $dy/dx = \sin x \cos^2 y$, given that, when $x = \pi/2$, $y = \pi/4$.
   (b) A curve passes through the origin and is such that $dy/dx = e^{2x-y}$. Obtain the equation of the curve in the form $y = f(x)$ and give a rough sketch of its graph. (SU)

10. (i) Find y in terms of x given that $(1 + x)\dfrac{dy}{dx} = (1 - x)y$ and that $y = 1$ when $x = 0$.
    (ii) A curve passes through $(2, 2)$ and has gradient at (x, y) given by the differential equation $ye^{y^2}\dfrac{dy}{dx} = e^{2x}$. Find the equation of the curve. Show that the curve also passes through the point $(1, \sqrt{2})$ and sketch the curve. (L)

11. If y is a solution of the differential equation $\dfrac{dy}{dx} = \dfrac{1+y^2}{1+x^2}$ deduce that

$$y = x + k(1 + xy)$$

where k is a constant.

(You may assume that $\tan^{-1} a - \tan^{-1} b = \tan^{-1}\left(\dfrac{a-b}{1+ab}\right) + n\pi$, n an integer.) (W)

12. Obtain the general solution of the differential equation $2x^2\dfrac{dy}{dx} + x^2y^2 = y^2$.
Find the three particular solutions such that (a) when $x = 1$, $y = 1$, (b) when $x = 1$, $y = \frac{2}{3}$, (c) when $x = 1$, $y = \frac{1}{2}$. Show that each of the integral curves given by these particular solutions has a maximum point on the line $x = 1$.
Sketch the three curves on the same axes. (L)

13. In a chemical reaction there are present, at time, t, x kg of substance X and y kg of substance Y, and initially there is 1 kg of X and 2 kg of Y. The variables x and y satisfy the equations $\frac{dx}{dt} = -x^2y, \frac{dy}{dt} = -xy^2$. Find $\frac{dy}{dx}$ in terms of x and y, and express y in terms of x. Hence obtain a differential equation in x and t only, and so find an expression for x in terms of t. (JMB)

14. (a) The equation of the family of all parabolas whose axes lie along the x-axis has the form $y^2 = 4a(x - h)$ where each parabola is specified by values of a and h. Form a differential equation (*not* involving a or h) satisfied by any point (x, y) lying on any one of these parabolas.

(b) The tangent to a curve at the point (x, y) cuts the x-axis at Q. If $OQ = kx$ (where O is the origin of the coordinate system) show that points on the curve satisfy the differential equation

$$\frac{dy}{dx} = \frac{y}{x - kx}.$$

By solving this differential equation show that the equation of the curve has the form $y = Ax^{1/(1-k)}$ where A is an arbitrary constant. (AEB 1975)

15. (i) Obtain the general solution of the differential equation $x \ln x \frac{dy}{dx} = 1$

(a) where $0 < x < 1$ and (b) where $x > 1$.

(ii) Sketch the family of integral curves for the differential equation

$$\frac{dy}{dx} = x(y + 1).$$

(iii) Find the equations of the two curves which pass through the point $(1, 1)$ and which satisfy the differential equation

$$\tan(y - 1)\frac{dy}{dx} = \tan(x - 1). \quad \text{(L)}$$

16. According to Newton's law of cooling, the rate at which a hot object cools is proportional to the difference between the temperature of the object and the temperature of the surrounding air (assumed to be constant). If an object cools from 100° to 80° in 10 minutes, and from 80° to 65° in another 10 minutes, find the temperature of the surrounding air and the temperature of the object after a further 10 minutes. (O)

17. (a) Given that $\frac{dy}{dx} = 9y^2 - 4$ and that $y = 1$ when $x = 2$, find an equation expressing x in terms of y.

(b) In a certain country, the price p of a particular commodity increases with the time t at a rate equal to kp where k is a positive constant. Write down a differential equation expressing this information. Show that if $p = 1$ when $t = 0$ and $p = \alpha$ when $t = 1$, then, at time t, $p = \alpha^t$. (C)

18. (a) Solve the differential equation $\frac{dy}{dx} = 3y + 2$ given that $y = 1$ when $x = 2$, expressing your answer in a form giving y as a function of x.

(b) The rate of decomposition of a radioactive substance is proportional to the mass of the substance remaining. Write down a differential equation which expresses this law, stating the meaning of each symbol used. If one third of the mass is left after 12 years, how much was left after 3 years? (C)

19. Social scientists studying the growth of population in an urban region use a model in which the rate of growth of population is proportional to $P(N - P)$, where P is the current size of the population (regarded as a continuous variable) and N is a constant. Set up a differential equation for this model and find its general solution expressing P as a function of time t. In a particular region, the population at a certain time was 100 000. Fifty years later the population was 200 000 and fifty years after that it was 300 000. Using the above model determine the value of N in this case. What is the significance of N? (C)

20. A water reservoir is in the shape of the surface obtained by rotating a parabola $y = kx^2$ about its vertical axis, $x = 0$. Show that when the central depth of the water in the reservoir is h, the surface area A of the water is proportional to h and the volume V is proportional to h^2. In a dry period the rate of loss of water, due entirely to evaporation, is λA per day. Obtain a differential equation for h after t days and deduce that the depth decreases at a constant rate. If, in addition to the evaporation a constant volume C of water is now used per day, obtain the new differential equation and show that the time to empty the reservoir from a depth H will be $\int_0^H \frac{\pi h\, dh}{Ck + \pi\lambda h}$ days. Verify that when $H = 20$, $\lambda = \frac{1}{2}$, $C = 500$ and $k = 0{\cdot}0025$ this amounts to just under 35 days. (AEB 1979)

21. (i) An electrically charged body loses its charge, Q coulombs, at a rate kQ coulombs per second, where k is a constant. Write down a differential equation involving Q and t, where t seconds is the time since the discharge started. Solve this equation for Q, given that the initial charge was Q_0 coulombs. If $Q_0 = 0{\cdot}001$, and $Q = 0{\cdot}0005$ when $t = 10$, find the value of Q when $t = 20$.

(ii) Solve the differential equation $\frac{d^2y}{dx^2} - 25y = 0$, given that $y = 4, \frac{dy}{dx} = 10$ when $x = 0$. (L)

22. Find the solution of the differential equation $\frac{d^2y}{dx^2} + 9y = 18$, for which y has

a maximum at $(\frac{1}{2}\pi, 6)$. Find the minimum value of y, and the values of x for which $y = 0$. (O)

23. (i) Solve the differential equation $\dfrac{dy}{dx} = \left(\dfrac{x}{y}\right)e^{x+y}$ in the form $f(y) = g(x)$, given that $y = 0$ when $x = 0$.

(ii) Solve the differential equation $\dfrac{d^2y}{dx^2} + 4y = 16$, given that $y = 4$ and $\dfrac{dy}{dx} = -4$ when $x = 0$. For which values of x does the graph of y have points of inflexion? (O)

24. (i) Find the general solution of the differential equation

$$(1 + x)\frac{dy}{dx} = xy.$$

(ii) By using the substitution $z = dy/dx$, or otherwise, solve the differential equation $\dfrac{d^3y}{dx^3} + 4\dfrac{dy}{dx} = 0$, given that y takes the values $0, 0, 1$ for $x = 0, \frac{1}{4}\pi, \frac{1}{2}\pi$ respectively. (O&C)

25. Prove that, if $y = \cos^2(x^2)$, then

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} + 16x^3y = 8x^3.$$

Prove that changing the independent variable in this differential equation from x to t, where $t = x^2$, gives the equation

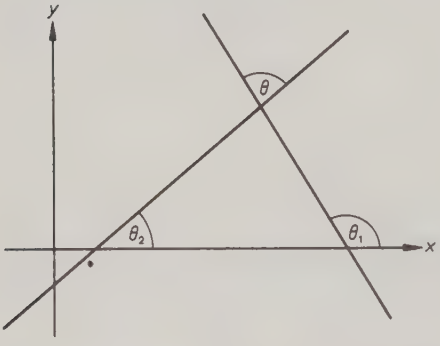
$$\frac{d^2y}{dt^2} + ky = C,$$

where k, C are constants and hence, or otherwise, find the general solution of the first equation in terms of x. What is the general solution of the differential equation

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} - 16x^3y = 8x^3?$$ (O&C)

25 Further coordinate geometry

25.1 Angles, distances and areas



The diagram shows two lines which make angles θ_1 and θ_2 with the positive direction of the x-axis. If the gradients of these lines are m_1 and m_2 respectively, then

$$m_1 = \tan\theta_1, \qquad m_2 = \tan\theta_2.$$

One angle between the lines is θ, where $\theta = \theta_1 - \theta_2$,

$$\therefore \qquad \tan\theta = \tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}.$$

Hence the angle θ between two lines with gradients m_1 and m_2 is given by

$$\boxed{\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}}$$

Example 1 Find the acute angle between the straight lines $3x - y + 2 = 0$ and $x - 2y - 1 = 0$.

The equations of the lines may be re-written as:

$y = 3x + 2$ and $y = \frac{1}{2}x - \frac{1}{2}$.

Hence the gradients of the lines are 3 and $\frac{1}{2}$.

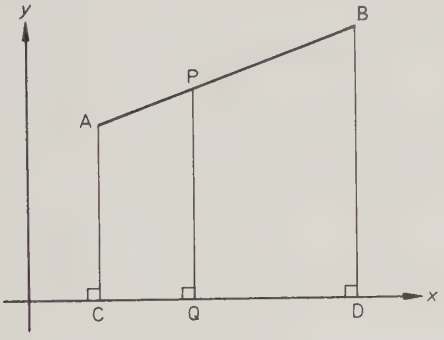
If θ is one angle between the lines, then

$$\tan\theta = \frac{3 - \frac{1}{2}}{1 + 3 \times \frac{1}{2}} = \frac{2\frac{1}{2}}{2\frac{1}{2}} = 1$$

$\therefore$ the acute angle between the lines is 45°.

[Note that the angle between two curves at any point of intersection is defined as the angle between the tangents to the curves at the point.]

Consider now the line joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$. Let P be the point which divides AB in the ratio $\mu:\lambda$. Then if C, D and Q are the feet of the perpendiculars from A, B and P to the x-axis, Q must divide CD in the ratio $\mu:\lambda$. It follows that the x-coordinate of P



$$= x_1 + \frac{\mu}{\lambda + \mu}(x_2 - x_1) = \frac{\lambda x_1 + \mu x_2}{\lambda + \mu}$$

A similar expression is obtained for the y-coordinate of P.

i.e. The point dividing the line AB in the ratio $\mu:\lambda$ has coordinates

$$\left(\frac{\lambda x_1 + \mu x_2}{\lambda + \mu}, \frac{\lambda y_1 + \mu y_2}{\lambda + \mu}\right).$$

This formula can be used when P divides AB externally in a given ratio if μ and λ are taken to have opposite signs.

[We recall that the corresponding formula for the position vector of a point which divides a lines in the ratio $\mu:\lambda$ was obtained in §21.1.]

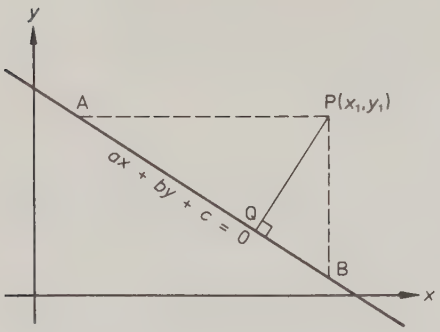
Example 2 If A and B are the points $(2, 1)$ and $(-4, 4)$ respectively, find the coordinates of the point C which divides AB internally in the ratio 2:1 and of the point D which divides AB externally in the ratio 3:2.

The coordinates of C are

$$\left(\frac{1 \times 2 + 2(-4)}{1 + 2}, \frac{1 \times 1 + 2 \times 4}{1 + 2}\right) \quad \text{i.e.} \quad (-2, 3)$$

Regarding D as the point which divides AB in the ratio $3:-2$, the coordinates of D are

$$\left(\frac{(-2)2 + 3(-4)}{-2 + 3}, \frac{(-2)1 + 3 \times 4}{-2 + 3}\right) \quad \text{i.e.} \quad (-16, 10).$$



Next we derive a formula for the perpendicular distance from a point $P(x_1, y_1)$ to a straight line $ax + by + c = 0$. In the diagram A and B are points on the line $ax + by + c = 0$, such that AP is parallel to the x-axis and BP is parallel to the y-axis. The point Q is the foot of the perpendicular from P to the line and so PQ is the required perpendicular distance.

Substituting $y = y_1$ in the equation $ax + by + c = 0$, we find that the x-coordinate of A is $-(by_1 + c)/a$.

$$\therefore \qquad AP = \left| x_1 - \left\{ -\frac{(by_1 + c)}{a} \right\} \right| = \left| \frac{ax_1 + by_1 + c}{a} \right|$$

Similarly $BP = \left| \dfrac{ax_1 + by_1 + c}{b} \right|$

$$\therefore \qquad AB = \sqrt{\{AP^2 + BP^2\}}$$

$$= \sqrt{\left\{ \frac{(ax_1 + by_1 + c)^2}{a^2} + \frac{(ax_1 + by_1 + c)^2}{b^2} \right\}}$$

$$= \sqrt{\left\{ \frac{(ax_1 + by_1 + c)^2(a^2 + b^2)}{a^2 b^2} \right\}}$$

$$= \left| \frac{(ax_1 + by_1 + c)\sqrt{(a^2 + b^2)}}{ab} \right|$$

However, the area of $\triangle APB = \frac{1}{2} \times PQ \times AB = \frac{1}{2} \times AP \times BP$

$$\therefore \qquad PQ = \frac{AP \times BP}{AB}$$

$$= \left| \frac{(ax_1 + by_1 + c)^2}{ab} \times \frac{ab}{(ax_1 + by_1 + c)\sqrt{(a^2 + b^2)}} \right|$$

$$= \left| \frac{ax_1 + by_1 + c}{\sqrt{(a^2 + b^2)}} \right|$$

i.e. The perpendicular distance from the point (x_1, y_1) to the line $ax + by + c = 0$ is $\left| \dfrac{ax_1 + by_1 + c}{\sqrt{(a^2 + b^2)}} \right|$

Example 3 Find the distance of the point $(3, -5)$ from the line $2x - y = 1$.

The distance of the point $(3, -5)$ from the line $2x - y - 1 = 0$ is

$$\left|\frac{2 \times 3 - (-5) - 1}{\sqrt{\{2^2 + (-1)^2\}}}\right| = \frac{10}{\sqrt{5}} = 2\sqrt{5}.$$

Example 4 Find the locus of points equidistant from the lines $y = 2x$ and $2x + 4y - 3 = 0$.

Let $P(x_1, y_1)$ be a point equidistant from the lines $2x - y = 0$ and $2x + 4y - 3 = 0$, then

$$\left|\frac{2x_1 - y_1}{\sqrt{\{2^2 + (-1)^2\}}}\right| = \left|\frac{2x_1 + 4y_1 - 3}{\sqrt{\{2^2 + 4^2\}}}\right|$$

$$\therefore \quad \frac{1}{\sqrt{5}}|2x_1 - y_1| = \frac{1}{\sqrt{20}}|2x_1 + 4y_1 - 3|$$

$$\therefore \quad 2|2x_1 - y_1| = |2x_1 + 4y_1 - 3|$$

Thus either $\quad 2(2x_1 - y_1) = 2x_1 + 4y_1 - 3$

$$4x_1 - 2y_1 = 2x_1 + 4y_1 - 3$$

$$2x_1 - 6y_1 + 3 = 0$$

or $\quad 2(2x_1 - y_1) = -(2x_1 + 4y_1 - 3)$

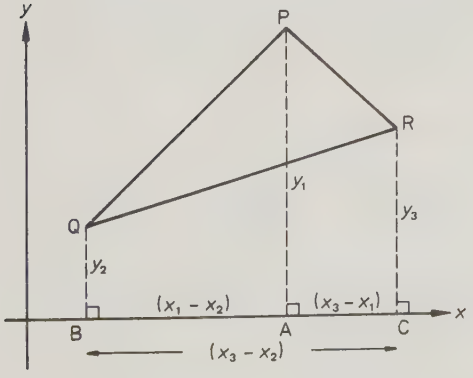
$$4x_1 - 2y_1 = -2x_1 - 4y_1 + 3$$

$$6x_1 + 2y_1 - 3 = 0$$

Hence the locus of all points P equidistant from the given lines is the pair of straight lines

$$2x - 6y + 3 = 0, \qquad 6x + 2y - 3 = 0.$$

It follows that these lines are the bisectors of the angles between the given lines.



Finally we consider the area of the triangle whose vertices are $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$.

As shown in the diagram, the area of ΔPQR = area $PQBA$ + area $PACR$ − area $QBCR$.

Since each of these areas is a trapezium,

$$\text{area of } \Delta PQR = \tfrac{1}{2}(x_1 - x_2)(y_1 + y_2) + \tfrac{1}{2}(x_3 - x_1)(y_1 + y_3) - \tfrac{1}{2}(x_3 - x_2)(y_2 + y_3)$$

$$= \tfrac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}$$

It is found that, although this result holds for any points P, Q and R, it sometimes gives a negative value for the area. We must therefore include modulus signs in the general formula.

$$\text{Area of triangle} = \tfrac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$$

[Note that this formula is rarely needed in elementary work. Problems on area usually involve right-angled triangles, isosceles triangles or other triangles whose heights and bases are easily calculated.]

Exercise 25.1

1. Find the tangent of the acute angle between the following pairs of lines

(a) $y = 2x - 3$, $y = 3x + 1$
(b) $3x - 2y + 1 = 0$, $2x - y - 7 = 0$
(c) $x + 3y - 2 = 0$, $5x - 6y + 2 = 0$
(d) $7x + 4y + 5 = 0$, $y = 0$
(e) $y = 4x$, $2x - 8y - 5 = 0$
(f) $4x + 3y + 2 = 0$, $x = 0$.

2. Find the coordinates of the points which divide the lines joining the given pairs of points internally in the stated ratio.

(a) $(7, 3)$, $(1, -6)$; 1:2
(b) $(2, -4)$, $(-8, 1)$; 2:3
(c) $(0, 6)$, $(2\frac{1}{2}, -4)$; 4:1
(d) $(-3, 5)$, $(1, -1)$; 5:3

3. Find the coordinates of the points which divide the lines joining the given pairs of points externally in the stated ratio.

(a) $(5, -1)$, $(-2, 3)$; 1:2
(b) $(2, -3)$, $(0, 5)$; 2:3
(c) $(4, 3)$, $(-2, 0)$; 4:1
(d) $(-3, 4)$, $(1, -1)$; 5:3

4. Find the distances between the given points and the given lines.

(a) $(1, 2)$, $4x - 3y - 3 = 0$
(b) $(-3, -1)$, $5x + 12y + 1 = 0$
(c) $(2, 5)$, $x - y + 5 = 0$
(d) $(0, 0)$, $2x + 3y = 13$
(e) $(-6, 10)$, $x = 3$
(f) $(7, -4)$, $3y - x - 1 = 0$

5. Find the areas of the triangles with the following vertices:

(a) $(2, 3)$, $(5, -1)$, $(2, -1)$
(b) $(-3, 4)$, $(5, 1)$, $(-3, -2)$
(c) $(-1, 3)$, $(2, -3)$, $(4, 5)$
(d) $(3, -1)$, $(-2, 0)$, $(1, 4)$
(e) $(5, 7)$, $(-4, -11)$, $(0, -3)$
(f) $(-2, 6)$, $(3, -7)$, $(4, 5)$

6. Find the equations of the bisectors of the angles between the given pairs of lines

(a) $3x - y - 3 = 0$, $x + 3y + 1 = 0$
(b) $2x + 3y = 2$, $3x + 2y = 3$
(c) $3x + 6y - 1 = 0$, $x - 2y + 1 = 0$
(d) $y = 7x - 1$, $y = x + 5$

7. Find the angles of the triangle PQR with vertices $P(1, 4)$, $Q(3, -2)$ and $R(5, 2)$.

8. Find the angles of the triangle ABC with vertices $A(1, -3)$, $B(4, 6)$ and $C(-1, 1)$.

9. A triangle has vertices $A(-4, 1)$, $B(3, 0)$ and $C(1, 2)$. If the internal bisector of $\angle B$ meets AC at P, use the fact that $AP : PC = AB : BC$ to find the coordinates of P.

10. A triangle has vertices $P(1, -2)$, $Q(5, 1)$ and $R(6, 10)$. Find the coordinates of the point on QR which is equidistant from the sides PQ and PR.

11. Find the tangent of the acute angle between the parabola $y^2 = 4x$ and the circle $x^2 + y^2 = 5$ at their points of intersection.

12. The curves $y = x^2$ and $y^2 = 8x$ intersect at the origin and at a point A. Find the angle between the curves at A.

13. Find the incentre of the triangle with vertices $(-1, 4)$, $(2, -2)$ and $(7, 8)$.

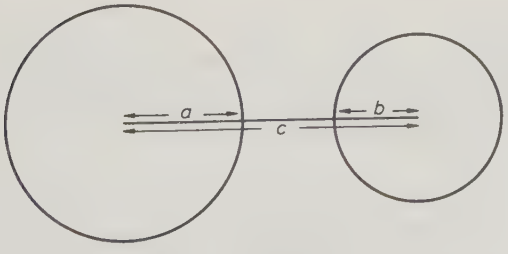
14. Find the incentre of the triangle whose sides have equations $x - y + 1 = 0$, $x + 7y + 9 = 0$ and $7x - y - 2 = 0$.

15. The points $A(-8, 9)$ and $C(1, 2)$ are opposite vertices of a parallelogram $ABCD$. The sides BC, CD of the parallelogram lie along the lines $x + 7y - 15 = 0$, $x - y + 1 = 0$, respectively. Calculate (i) the coordinates of D, (ii) the tangent of the acute angle between the diagonals of the parallelogram, (iii) the length of the perpendicular from A to the side CD, (iv) the area of the parallelogram. (JMB)

25.2 Further work on circles

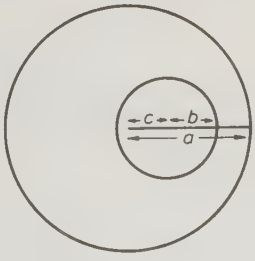
When a problem involves two circles, of radii a and b, the relative positions of the circles can be determined by finding the distance c between their centres. Assuming that $a > b$, there are various different possibilities.

(1)



$c > a + b$

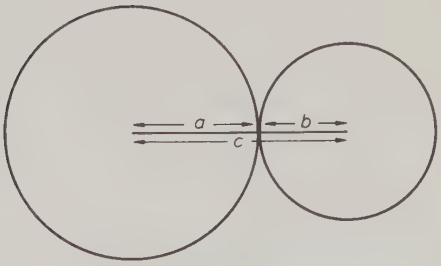
(2)



$c < a - b$

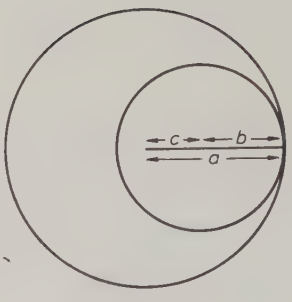
In diagrams (1) and (2) the circles do not intersect.

(3)



$c = a + b$

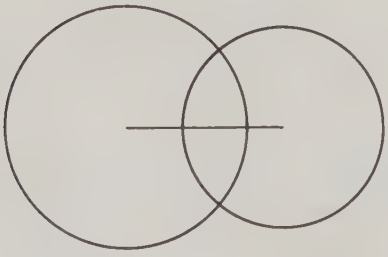
(4)



$c = a - b$

In diagrams (3) and (4) the circles touch either externally or internally.

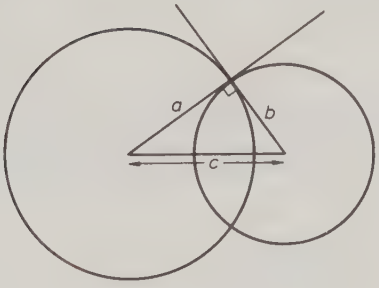
(5)



In diagram (5) the circles have two distinct points of intersection.

$a - b < c < a + b$

(6)



Two circles which cut at right-angles are called *orthogonal* circles. Diagram (6) shows two such circles.

$c^2 = a^2 + b^2$

The *common chord* of two circles is the line joining their points of intersection. The equation of this chord can be obtained without finding the coordinates of the points of intersection.

Example 1 The circles $x^2 + y^2 + 4x - 3y + 1 = 0$ and $x^2 + y^2 + x - y - 2 = 0$ intersect at the points A and B. Find the equation of the common chord AB.

$$x^2 + y^2 + 4x - 3y + 1 = 0 \quad (1)$$
$$x^2 + y^2 + x - y - 2 = 0 \quad (2)$$

Subtracting (2) from (1) we obtain:

$$3x - 2y + 3 = 0$$

Since the coordinates of A and B satisfy equations (1) and (2), they must also satisfy this new equation, which represents a straight line.
Hence $3x - 2y + 3 = 0$ is the equation of AB.

We now extend the method of Example 1 by considering two intersecting circles with equations:

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{(i)}$$
$$x^2 + y^2 + 2Gx + 2Fy + C = 0 \quad \text{(ii)}$$

From these equations we can form a new equation:

$$x^2 + y^2 + 2gx + 2fy + c + k(x^2 + y^2 + 2Gx + 2Fy + C) = 0 \quad \text{(iii)}$$

i.e. $$(1 + k)x^2 + (1 + k)y^2 + 2(g + kG)x + 2(f + kF)y + c + kC = 0$$

This is the equation of a circle for all values of k, except $k = -1$. Moreover, any pair of values of x and y which satisfy equations (i) and (ii) must also satisfy equation (iii). Hence, in general, equation (iii) represents a circle passing through the points of intersection of circles (i) and (ii). When $k = -1$ equation (iii) represents a straight line through the points of intersection i.e. the common chord of the circles.

Example 2 The circles $x^2 + y^2 + 3x - y - 5 = 0$ and $x^2 + y^2 - 2x + y - 1 = 0$ intersect at points A and B. Find the equation of the circle which passes through the origin and the points A and B.

Any circle which passes through A and B has an equation of the form:

$$x^2 + y^2 + 3x - y - 5 + k(x^2 + y^2 - 2x + y - 1) = 0$$

If this circle also passes through the origin,

$$-5 + k(-1) = 0 \quad \text{i.e.} \quad k = -5$$

Hence the equation of the circle which passes through the origin and the points A and B is

$$x^2 + y^2 + 3x - y - 5 - 5(x^2 + y^2 - 2x + y - 1) = 0$$

i.e. $$4x^2 + 4y^2 - 13x + 6y = 0.$$

Exercise 25.2

1. Two circles have equations $x^2 + y^2 - 2x - 6y - 54 = 0$ and $x^2 + y^2 - 8x + 2y + 13 = 0$.

Show that one circle lies entirely inside the other.

2. Prove that the circles whose equations are $x^2 + y^2 - 2x - 2y - 2 = 0$, $x^2 + y^2 - 6x - 10y + 33 = 0$ lie entirely outside one another.

3. A circle C_1 has equation $x^2 + y^2 - 32x - 24y + 300 = 0$ and a second circle C_2 has as diameter the line joining the points (8, 0) and (0, 6). Show that the circles C_1 and C_2 touch externally.

4. A circle C has equation $x^2 + y^2 - 4x + 2y = 40$. Find the equations of the circles with centre $(3, 1)$ which touch circle C. Find also the coordinates of the points of contact.

5. Show that the circles $x^2 + y^2 - 2x - 2y - 2 = 0$ and
$$x^2 - y^2 - 8x - 10y + 32 = 0$$
touch externally and find the coordinates of the point of contact.

6. Find the equations of the two circles with centres $A(1, 1)$ and $B(9, 7)$ which have equal radii and touch each other externally. Find also the equations of the common tangents to these circles.

7. Two circles with centres $(3, -5)$ and $(-6, 7)$ pass through the point $(4, 2)$. Find the equations of these circles. Find also the equation and the length of their common chord.

8. Prove that the circles whose equations are $x^2 + y^2 - 4y - 5 = 0$, $x^2 + y^2 - 8x + 2y + 1 = 0$ cut orthogonally and find the equation of the common chord.

9. Find the equation of the circle with centre $(3, -2)$ and radius 5 units. Find also the equation of the circle with centre $(-7, 3)$ which intersects the original circle at right angles.

10. Show that any circle which passes through the points $(1, 0)$ and $(-1, 0)$ has an equation of the form $x^2 + y^2 - 2\lambda y - 1 = 0$. Prove also that if the circles given by $\lambda = \lambda_1$ and $\lambda = \lambda_2$ cut orthogonally then $\lambda_1 \lambda_2 = -1$.

11. Show that if the circles $x^2 + y^2 + 2gx + 2fy + c = 0$ and
$$x^2 + y^2 + 2Gx + 2Fy + C = 0$$
cut orthogonally, then $2gG + 2fF = c + C$.

12. The circles $x^2 + y^2 - 2x = 0$ and $x^2 + y^2 + 4x - 6y - 3 = 0$ intersect at the points A and B. Find (a) the equation of AB, (b) the equation of the circle which passes through A, B and the point $(1, 2)$.

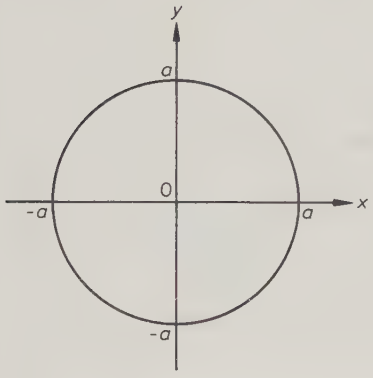
13. Given that the circles $x^2 + y^2 - 3y = 0$ and $x^2 + y^2 + 5x - 8y + 5 = 0$ intersect at P and Q, find the equation of the circle which passes through P, Q and (a) the point $(1, 1)$, (b) the origin, (c) touches the x-axis.

14. The circle $x^2 + y^2 + 3x - 5y - 4 = 0$ and the straight line $y = 2x + 5$ intersect at the points A and B. Find the equation of the circle which passes through A, B and (a) the point $(3, 1)$, (b) the origin, (c) has its centre on the y-axis.

15. Show that the circles $x^2 + y^2 + 8x + 2y + 8 = 0$, $x^2 + y^2 - 2x + 2y - 2 = 0$ have three common tangents, and find their equations. (C)

25.3 Loci using parametric forms

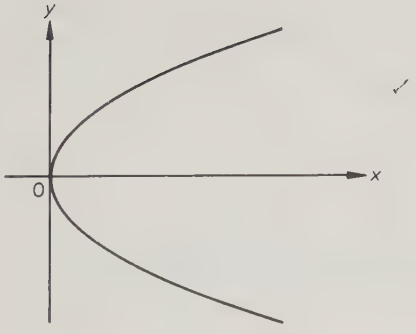
Many problems in coordinate geometry are best solved using the parametric equations of curves. The basic techniques involved were introduced earlier and used to investigate the properties of various curves including the parabola $y^2 = 4ax$. We now consider further applications of these methods. As clear diagrams are often useful in this type of work, we begin by giving sketches of several important curves together with the most commonly used parametric forms of their equations.



The circle $x^2 + y^2 = a^2$.
Parametric equations:

$$x = a\cos\theta,$$
$$y = a\sin\theta.$$

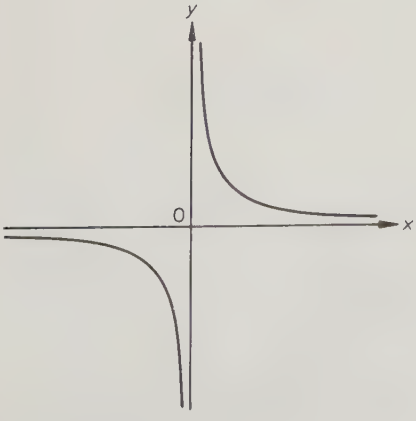
This is the circle with centre O and radius a.



The parabola $y^2 = 4ax$.
Parametric equations:

$$x = at^2,$$
$$y = 2at.$$

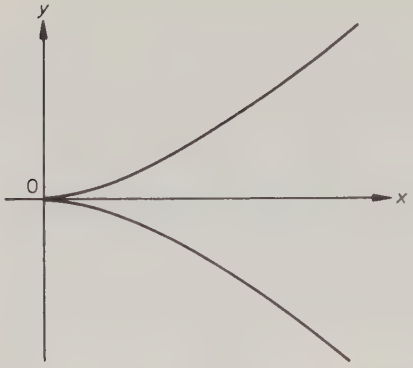
This parabola has vertex O and is symmetrical about the x-axis.



The rectangular hyperbola $xy = c^2$.
Parametric equations:

$$x = ct,$$
$$y = c/t.$$

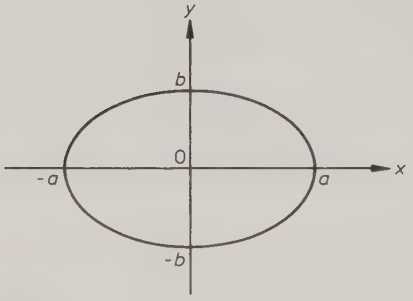
The x- and y-axes are asymptotes to the curve and the curve is symmetrical about the lines $y = \pm x$.



The semi-cubical parabola $ay^2 = x^3$.
Parametric equations:

$$x = at^2,$$
$$y = at^3.$$

The curve has a *cusp* at O and is symmetrical about the x-axis.



The ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.
Parametric equations:

$$x = a\cos\theta,$$
$$y = b\sin\theta.$$

The curve is symmetrical about the x- and y-axes.

Example 1 Find the equation of the chord of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ joining the points $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$.

$$\text{Gradient of } PQ = \frac{b\sin\theta - b\sin\phi}{a\cos\theta - a\cos\phi}$$

$$= \frac{2b\cos\tfrac{1}{2}(\theta+\phi)\sin\tfrac{1}{2}(\theta-\phi)}{-2a\sin\tfrac{1}{2}(\theta+\phi)\sin\tfrac{1}{2}(\theta-\phi)} = -\frac{b\cos\tfrac{1}{2}(\theta+\phi)}{a\sin\tfrac{1}{2}(\theta+\phi)}$$

Thus the equation of the chord PQ is

$$y - b\sin\theta = -\frac{b\cos\tfrac{1}{2}(\theta+\phi)}{a\sin\tfrac{1}{2}(\theta+\phi)}(x - a\cos\theta)$$

i.e. $ay\sin\tfrac{1}{2}(\theta+\phi) - ab\sin\theta\sin\tfrac{1}{2}(\theta+\phi)$

$$= -bx\cos\tfrac{1}{2}(\theta+\phi) + ab\cos\theta\cos\tfrac{1}{2}(\theta+\phi)$$

$\therefore\quad bx\cos\tfrac{1}{2}(\theta+\phi) + ay\sin\tfrac{1}{2}(\theta+\phi)$

$$= ab\{\cos\theta\cos\tfrac{1}{2}(\theta+\phi) + \sin\theta\sin\tfrac{1}{2}(\theta+\phi)\}$$

$$= ab\cos\{\theta - \tfrac{1}{2}(\theta+\phi)\} = ab\cos\tfrac{1}{2}(\theta-\phi).$$

Hence the equation of the chord PQ may be written:

$$\frac{x\cos\tfrac{1}{2}(\theta+\phi)}{a} + \frac{y\sin\tfrac{1}{2}(\theta+\phi)}{b} = \cos\tfrac{1}{2}(\theta-\phi).$$

Example 2 Find the equations of the tangent and the normal to the curve $xy = c^2$ at the point $P(ct, c/t)$. Given that the normal at P meets the curve again at Q, find the coordinates of Q. If the tangent at P meets the y-axis at R, find the equation of the locus of the mid-point M of PR.

The parametric equations of the curve are

$$x = ct, \qquad y = \frac{c}{t}$$

Differentiating with respect to t we have

$$\frac{dx}{dt} = c, \qquad \frac{dy}{dt} = -\frac{c}{t^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \left(-\frac{c}{t^2}\right) \Big/ c = -\frac{1}{t^2}$$

Hence the gradients of the tangent and the normal to the curve at P are $-1/t^2$ and t^2 respectively.

$\therefore$ the equation of the tangent at P is

$$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$$

i.e. $$t^2y - ct = -x + ct$$

i.e. $$x + t^2y = 2ct \qquad (1)$$

The equation of the normal at P is

$$y - \frac{c}{t} = t^2(x - ct)$$

i.e. $$ty - c = t^3x - ct^4$$

i.e. $$t^3x - ty = c(t^4 - 1) \qquad (2)$$

Let $(cu, c/u)$ be a point which lies on the curve $xy = c^2$ and on the normal at P, then from equation (2)

$$t^3 \times cu - t \times \frac{c}{u} = c(t^4 - 1)$$

$$\therefore \quad t^3u^2 + (1 - t^4)u - t = 0$$

$$\therefore \quad (u - t)(t^3u + 1) = 0$$

$\therefore$ either $u = t$ or $u = -1/t^3$

Since the solution $u = t$ gives the point P, the remaining solution $u = -1/t^3$ must give the point Q.

Hence the coordinates of Q are $(-c/t^3, -ct^3)$.

Substituting $x = 0$ in equation (1) we have,

$$t^2y = 2ct \quad \text{i.e.} \quad y = 2c/t$$

$\therefore$ the coordinates of the point R are $(0, 2c/t)$.

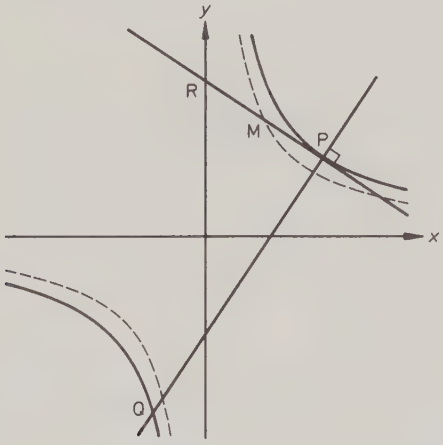
Thus the coordinates of the mid-point M of PR are

$$\left(\frac{1}{2}\{ct + 0\}, \frac{1}{2}\left\{\frac{c}{t} + \frac{2c}{t}\right\}\right) \quad \text{i.e.} \quad \left(\frac{1}{2}ct, \frac{3c}{2t}\right)$$

Hence as t varies the parametric equations of the locus of M are

$$x = \frac{1}{2}ct, \qquad y = \frac{3c}{2t}$$

Eliminating t from these equations we obtain the Cartesian equation of the locus, namely $xy = \frac{3}{4}c^2$.



As shown in the diagram the locus of M is another rectangular hyperbola.

Exercise 25.3

1. Write down parametric coordinates for a point on each of the following curves and sketch the curves.

(a) $y^2 = 12x$, (b) $xy = 9$, (c) $y^2 = x^3$,
(d) $4xy = 25$, (e) $4y^2 = x^3$, (f) $y^2 = 10x$.

2. Write down parametric coordinates for a point on each of the following curves and sketch the curves.

(a) $\frac{x^2}{9} + \frac{y^2}{4} = 1$, (b) $x^2 + y^2 = 9$, (c) $\frac{x^2}{5} + y^2 = 1$,
(d) $4x^2 + 9y^2 = 4$, (e) $9x^2 + y^2 = 9$, (f) $x^2 + (y - 1)^2 = 25$.

3. Find the equations of the tangent and normal to the curve $xy = 16$ at the point $(4t, 4/t)$.

4. Find the equations of the tangent and normal to the curve $x = t$, $y = 1/t$ at the point where $t = 2$.

5. Find the equations of the tangent and normal to the curve $x = 2\cos\theta$, $y = \sin\theta$ at the point where $\theta = \frac{1}{3}\pi$.

6. P is the point $(5\cos\theta, 4\sin\theta)$ on the curve $x^2/25 + y^2/16 = 1$. Given that S and S' are the points with coordinates $(-3, 0)$ and $(3, 0)$, show that the value of $PS + PS'$ is independent of θ.

7. Find the equations of the tangents to the curve $xy = c^2$ which have gradient -4 and write down the coordinates of their points of contact.

8. The points $P(cp, c/p)$, $Q(cq, c/q)$ and $R(cr, c/r)$ lie on the curve $xy = c^2$. Show that if $\angle RPQ$ is a right angle, then $p^2qr = -1$. Prove further that QR is perpendicular to the tangent at P to the curve.

9. The parametric equations of a curve are $x = 2t$, $y = 2/t$. The tangent to the curve at $P(2t, 2/t)$ meets the x-axis at A; the normal to the curve at P meets the y-axis at B. The point Q divides AB in the ratio 3:1. Find the parametric equations of the locus of Q as P moves on the given curve. (JMB)

10. Let P and Q be points on the rectangular hyperbola $xy = a^2$, with coordinates $(ap, a/p)$ and $(aq, a/q)$. Obtain the equation of the line PQ, and deduce, or obtain otherwise, the equation of the tangent at P to the hyperbola. Show that, if the line PQ meets the coordinate axes at A and B, then the mid-point M of AB is the mid-point of PQ also. Obtain the coordinates of the point T where the tangents at P and Q meet, and show that the line MT passes through the origin. (W)

11. Prove that the normal to the hyperbola $xy = c^2$ at the point $P(ct, c/t)$ has equation $y = t^2x + \dfrac{c}{t} - ct^3$. If the normal at P meets the line $y = x$ at N, and O is the origin, show that $OP = PN$ provided that $t^2 \neq 1$. The tangent to the hyperbola at P meets the line $y = x$ at T. Prove that $OT \,.\, ON = 4c^2$. (O&C)

12. The normal at the point $P(ct, c/t)$ on the rectangular hyperbola $xy = c^2$ meets the curve again at Q. If the normal meets the x-axis at A and the y-axis at B, show that the mid-point of AB is also the mid-point of PQ.

13. Show that the tangent at $P(cp, c/p)$ to the rectangular hyperbola $xy = c^2$ has the equation $p^2y + x = 2cp$. The perpendicular from the origin to this tangent meets it at N, and meets the hyperbola again at Q and R. Prove that (i) the angle QPR is a right angle, (ii) as p varies, the point N lies on the curve whose equation is $(x^2 + y^2)^2 = 4c^2xy$. (C)

14. The tangents at the points $P(cp, c/p)$ and $Q(cq, c/q)$ on the rectangular hyperbolar $xy = c^2$ intersect at the point R. Given that R lies on the rectangular hyperbola $xy = \frac{1}{2}c^2$, find the equation of the locus of the mid-point M of PQ as p and q vary.

15. Find the equation of the tangent to the curve $ay^2 = x^3$ at the point (at^2, at^3) and prove that, apart from one exceptional case, the tangent meets the curve again. Find the coordinates of the point of intersection. What is the exceptional case? (O&C)

16. The tangents to the curve $9y^2 = x^3$ at the points $P(9p^2, 9p^3)$ and $Q(9q^2, 9q^3)$ intersect at the point R. Find the coordinates of R. If p and q vary in such a way that $\angle PRQ$ is always a right angle, find the equation of the locus of R.

17. Show that the equation of the tangent to the curve $x = a\cos t$, $y = b\sin t$ at the point $P(a\cos p, b\sin p)$ is $\frac{x}{a}\cos p + \frac{y}{b}\sin p = 1$. This tangent meets the curve $x = 2a\cos\theta$, $y = 2b\sin\theta$ at the points Q and R, which are given by $\theta = q$ and $\theta = r$ respectively. Show that p differs from each of q and r by $\pi/3$. (JMB)

18. Show that if the tangents to the ellipse $x^2/a + y^2/b = 1$ at the points $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$ intersect at the point R, then the coordinates of R are $\left(\frac{a\cos\frac{1}{2}(\theta+\phi)}{\cos\frac{1}{2}(\theta-\phi)}, \frac{b\sin\frac{1}{2}(\theta+\phi)}{\cos\frac{1}{2}(\theta-\phi)}\right)$. If P and Q move on the ellipse in such a way that $\phi = \theta + \frac{1}{2}\pi$, find the equation of the locus of R.

19. Find the equation of the normal to the curve $x^2 - y^2 = 1$ at the point $P(\sec\theta, \tan\theta)$, where $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$. If this normal cuts the x-axis at A and the y-axis at B, show that P is the centre of the circle which passes through A, B and the origin O. If the line OP cuts this circle again at the point Q, find the equation of the locus of Q as θ varies.

20. (a) The points $A(d, 0)$, $B(-d, 0)$ and $P(d\cos\theta, d\sin\theta)$ lie on the circle $x^2 + y^2 = d^2$. Show that the equation of the tangent to the circle at P is $x\cos\theta + y\sin\theta = d$. Show that the sum of the perpendicular distances from A and B to this tangent is independent of θ, and calculate the product of these distances.
(b) The point $P(p, p^3)$ lies on the curve $y = x^3$. Show that the equation of the tangent to the curve at P is $y - 3p^2x + 2p^3 = 0$. Show that when $p = 3$ this tangent passes through the point $A(7/3, 9)$, and find the other two values of p for which the tangent passes through A. (W)

21. A curve has the parametric equations $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, where $0 \leqslant \theta \leqslant \pi$. A point P of the curve has parameter ϕ, where $\phi \neq 0$.

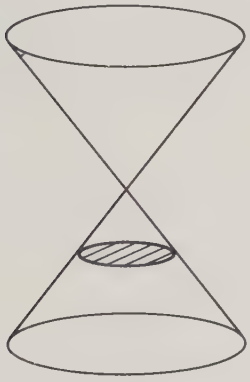
(i) Show that, at P, $\frac{dy}{dx} = \frac{\sin\phi}{1 - \cos\phi}$.

(ii) The normal to the curve at P meets the x-axis at G, and O is the origin. Show that $OG = a\phi$.

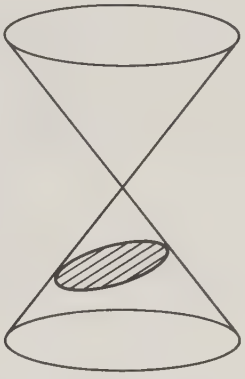
(iii) The tangent to the curve at P meets at K the line through G parallel to the y-axis. Show that $GK = 2a$. (C)

25.4 Conic sections

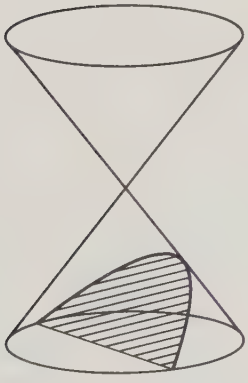
Some of the curves we have been studying arise naturally as cross-sections of a circular cone. The following diagrams show the sections produced when a double cone of semi-vertical angle θ is cut by a plane which does not pass through the vertex. The nature of the curve obtained in each case is determined by the value of the angle α between the plane and the axis of the cone, where $0 \leqslant \alpha \leqslant \frac{1}{2}\pi$.



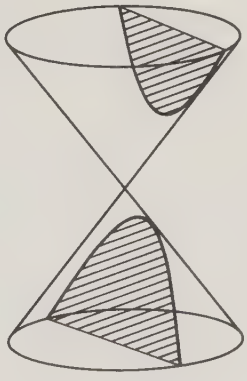
Circle, $\alpha = \frac{1}{2}\pi$



Ellipse, $\alpha > \theta$



Parabola, $\alpha = \theta$

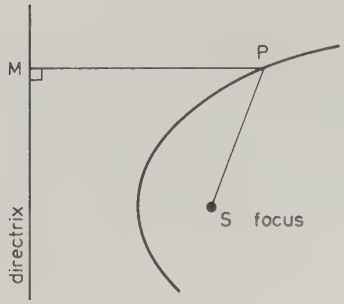


Hyperbola, $\alpha < \theta$

The complete set of conics includes the sections produced by planes through the vertex of the double cone, namely:

for $\alpha < \theta$, a pair of straight lines;
for $\alpha = \theta$, a single straight line;
for $\alpha > \theta$, a single point.

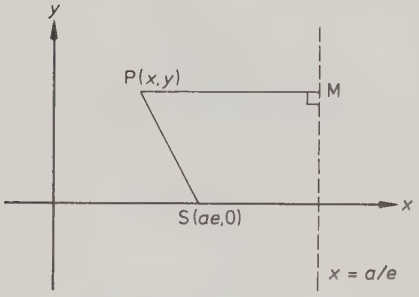
It can be shown that the definition of a parabola as a conic section is consistent with the locus definition used in §15.4. We now consider the corresponding locus definitions of the ellipse and hyperbola.



Suppose that P is a point which moves so that its distance from a fixed point S, the *focus*, is a constant multiple of its distance from a fixed line called the *directrix*. If M is the foot of the perpendicular from P to the directrix then

$$PS = ePM$$

where e is a positive constant called the *eccentricity*. The nature of the locus of P depends on the value taken by e, as follows:



for an ellipse, $0 < e < 1$;
for a parabola, $e = 1$;
for a hyperbola, $e > 1$.

The standard form of the equation of the ellipse is obtained by taking the point $(ae, 0)$ as focus and the line $x = a/e$ as directrix.

$$PS^2 = e^2PM^2$$

$$\therefore \quad (x - ae)^2 + y^2 = e^2\left(\frac{a}{e} - x\right)^2$$

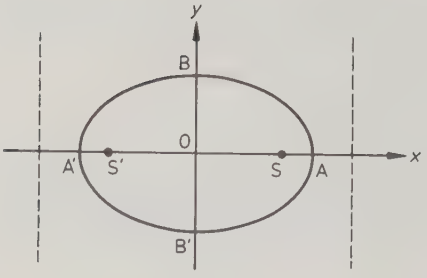
$$x^2 - 2aex + a^2e^2 + y^2 = a^2 - 2aex + e^2x^2$$

$$x^2(1 - e^2) + y^2 = a^2(1 - e^2)$$

i.e.
$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$$

Hence the equation of the ellipse is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \text{where} \quad b^2 = a^2(1 - e^2).$$



From this equation we see that the ellipse is symmetrical about both x- and y-axes. It cuts the x-axis at the points $A(a, 0)$ and $A'(-a, 0)$. It cuts the y-axis at the points $B(b, 0)$ and $B'(-b, 0)$. AA' is called the *major axis* and BB' the *minor axis*. The origin O is the *centre*

of the ellipse and any chord through O is a *diameter*. From symmetry of the curve it is clear that the same locus would have been produced using the point $S'(-ae, 0)$ as focus and the line $x = -a/e$ as directrix. Thus the ellipse is said to have foci $(\pm ae, 0)$ and directrices $x = \pm ae$.

[Note that in terms of a and b the coordinates of the foci are $(\pm\sqrt{\{a^2 - b^2\}}, 0)$ and the equations of the directrices are $x = \pm a^2/\sqrt{(a^2 - b^2)}$.]

Example 1 Find the eccentricity, the foci and the directrices of the ellipse $\frac{x^2}{9} + y^2 = 1$.

For the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we have $b^2 = a^2(1 - e^2)$.

Substituting $a = 3$, $b = 1$ we find that for the given ellipse,

$$1 = 9(1 - e^2)$$
$$\therefore \quad 1 = 9 - 9e^2$$
$$\therefore \quad e^2 = 8/9$$

Hence the eccentricity of the ellipse is $\frac{2}{3}\sqrt{2}$. Thus the foci are the points $(\pm 2\sqrt{2}, 0)$ and the directrices are the lines $x = \pm\frac{9}{4}\sqrt{2}$.

The standard form of the equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad \text{where} \quad b^2 = a^2(e^2 - 1).$$

Using the methods applied to the ellipse it can be shown that this hyperbola has foci $(\pm ae, 0)$ and directrices $x = \pm a/e$.

From the equation we see that the hyperbola is also symmetrical about both x- and y-axes. It cuts the x-axis at the points $A(a, 0)$ and $A'(-a, 0)$, but does not cut the y-axis.

Rearranging the equation we have:

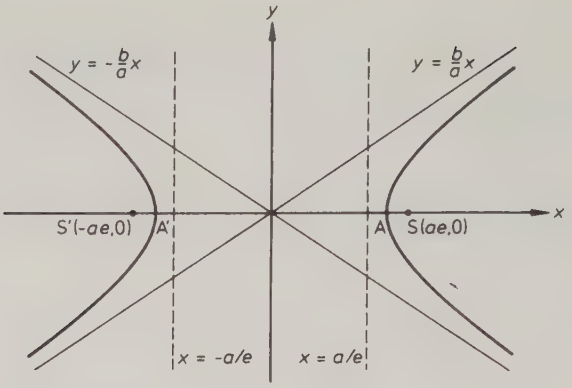
$$\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1$$

$$\therefore \quad \frac{y^2}{x^2} = \frac{b^2}{a^2} - \frac{b^2}{x^2}$$

$$\therefore \quad \text{as } |x| \to \infty, \quad \frac{y^2}{x^2} \to \frac{b^2}{a^2} \quad \text{and} \quad \frac{y}{x} \to \pm\frac{b}{a}$$

i.e. as $|x|$ increases the curve approaches the lines $y = \pm\frac{b}{a}x$.

Hence the lines $y = \pm\frac{b}{a}x$ are asymptotes to the hyperbola.



A hyperbola with perpendicular asymptotes is called a *rectangular hyperbola.* Since the lines $y = \pm\frac{b}{a}x$ are perpendicular when $a = b$, the equation of a rectangular hyperbola is of the form

$$x^2 - y^2 = a^2.$$

When the asymptotes are used as x- and y-axes the equation of the rectangular hyperbola takes the more familiar form $xy = c^2$.

Exercise 25.4

1. Find the eccentricities, the foci and the directrices of the following ellipses.

(a) $\frac{x^2}{25} + \frac{y^2}{9} = 1$, (b) $16x^2 + 25y^2 = 100$, (c) $\frac{x^2}{4} + y^2 = 1$,

(d) $\frac{x^2}{9} + \frac{y^2}{5} = 1$, (e) $\frac{x^2}{4} + \frac{y^2}{9} = 1$, (f) $4x^2 + y^2 = 1$.

2. Find the eccentricities, the foci, the directrices and the asymptotes of the following hyperbolas.

(a) $\frac{x^2}{16} - \frac{y^2}{9} = 1$, (b) $x^2 - y^2 = 4$, (c) $x^2 - 4y^2 = 4$

3. Find the eccentricities of the following ellipses

(a) $x = 2\cos\theta$, $y = \sin\theta$, (b) $x = 5\cos\theta$, $y = 3\sin\theta$,
(c) $x = 2\cos\theta$, $y = 3\sin\theta$, (d) $x = 5\sin\theta$, $y = 3\cos\theta$.

4. Find in the form $x^2/a^2 + y^2/b^2 = 1$ the equations of the ellipses with (a) eccentricity 1/2, foci $(\pm 2, 0)$, (b) eccentricity 3/5, foci $(\pm 9, 0)$.

5. Use the locus definition of the ellipse to find the equation of the ellipse with eccentricity $\frac{2}{3}$, focus $(2, 1)$ and directrix $x = -\frac{1}{2}$.

6. Find the equations of the tangent and the normal to each of the following ellipses at the given point:

(a) $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1; (3, 0)$, (b) $\dfrac{x^2}{8} + \dfrac{y^2}{2} = 1; (-2, 1)$,

(c) $9x^2 + 16y^2 = 40; (2, \frac{1}{2})$, (d) $4x^2 + 5y^2 = 120; (-5, -2)$.

7. Find the equations of the tangents at the point (x_1, y_1) to the following curves:

(a) $4x^2 + 9y^2 = 36$, (b) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, (c) $\dfrac{x^2}{3} - \dfrac{y^2}{2} = 1$.

8. Use the locus definition of the hyperbola to find the equation of the hyperbola with eccentricity $\sqrt{2}$, focus $(2k, 2k)$ and directrix $x + y = k$.

9. The line $y = mx + c$ is a tangent to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1 = 0$, $(a > b > 0)$. Show that $c^2 = a^2m^2 + b^2$. The perpendicular distances from the points $(\sqrt{(a^2 - b^2)}, 0)$ and $(-\sqrt{(a^2 - b^2)}, 0)$ to any tangent to the ellipse are p_1, p_2. Show that $p_1p_2 = b^2$. (C)

10. A line with gradient m is drawn through the fixed point $C(h, 0)$, where $0 < h < a$, to meet the ellipse $x^2/a^2 + y^2/b^2 = 1$ at the points P and Q. Prove that the mid-point R of PQ has the coordinates $\left(\dfrac{a^2hm^2}{a^2m^2 + b^2}, \dfrac{-b^2hm}{a^2m^2 + b^2}\right)$. Show that, as m varies, R always lies on the curve whose equation is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{hx}{a^2}$. (C)

11. The ellipse $x^2/a^2 + y^2/b^2 = 1$ intersects the positive x-axis at A and the positive y-axis at B. Determine the equation of the perpendicular bisector of AB. (i) Given that this line intersects the x-axis at P and that M is the mid-point of AB, prove that the area of triangle PMA is $b(a^2 + b^2)/8a$. (ii) If $a^2 = 3b^2$, find, in terms of b, the coordinates of the points where the perpendicular bisector of AB intersects the ellipse. (JMB)

12. Prove that the hyperbola H_1, with equation $x^2 - y^2 = a^2$, cuts the hyperbola H_2, with equation $xy = c^2$, at right angles at two points P and Q. If the distance between the tangents to H_1 at P and Q is equal to the distance between the tangents to H_2 at P and Q, find the relation between a and c. (O)

Exercise 25.5 (miscellaneous)

1. A straight line parallel to the line $2x + y = 0$ intersects the x-axis at A and the y-axis at B. The perpendicular bisector of AB cuts the y-axis at C. Prove that the gradient of the line AC is $-\frac{3}{4}$. Find also the tangent of the acute angle between the line AC and the bisector of the angle AOB, where O is the origin. (JMB)

2. The triangle ABC has vertices $A(0, 12)$, $B(-9, 0)$, $C(16, 0)$. Find the equations of the internal bisectors of the angles ABC and ACB. Hence, or otherwise, find the equation of the inscribed circle of the triangle ABC. Find also the equation of the circle passing through A, B and C. (C)

3. Write down the perpendicular distance from the point (a, a) to the line $4x - 3y + 4 = 0$. The circle, with centre (a, a) and radius a, touches the line $4x - 3y + 4 = 0$ at the point P. Find a, and the equation of the normal to the circle at P. Show that P is the point $(1/5, 8/5)$. Show that the equation of the circle which has centre P and which passes through the origin is $5(x^2 + y^2) - 2x - 16y = 0$. (L)

4. Find the centre and the radius of the circle C which passes through the points $(4, 2)$, $(2, 4)$ and $(2, 6)$. If the line $y = mx$ is a tangent to C, obtain the quadratic equation satisfied by m. Hence or otherwise find the equations of the tangents to C which pass through the origin O. Find also (i) the angle between the two tangents, (ii) the equation of the circle which is the reflection of C in the line $y = 3x$. (AEB 1977)

5. Show that the equation of the tangent to the curve $y^2 = 4ax$ at the point $(at^2, 2at)$ is $ty = x + at^2$ and the equation of the tangent to the curve $x^2 = 4by$ at the point $(2bp, bp^2)$ is $y = px - bp^2$. The curves $y^2 = 32x$ and $x^2 = 4y$ intersect at the origin and at A. Find the equation of the common tangent to these curves and the coordinates of the points of contact B and C between the tangent and the curves. Calculate the area of the triangle ABC. (AEB 1976)

6. On the same diagram sketch the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 10x = 0$. The line $ax + by + 1 = 0$ is a tangent to both these circles. State the distances of the centres of the circles from this tangent. Hence, or otherwise, find the possible values of a and b and show that, if 2ϕ is the angle between the common tangents, then $\tan\phi = \frac{3}{4}$. (L)

7. Find the centre and radius of each of the circles C_1 and C_2 whose equations are $x^2 + y^2 - 16y + 32 = 0$ and $x^2 + y^2 - 18x + 2y + 32 = 0$ respectively and show that the circles touch externally. Find the coordinates of their point of contact and show that the common tangent at that point passes through the origin. The other tangents from the origin, one to each circle, are drawn. Find, correct to the nearest degree, the angle between these tangents. (SU)

8. Two circles, C_1 and C_2, have equations $x^2 + y^2 - 4x - 8y - 5 = 0$ and $x^2 + y^2 - 6x - 10y + 9 = 0$, respectively. Find the x-coordinates of the points P and Q at which the line $y = 0$ cuts C_1, and show that this line touches C_2. Find the tangent of the acute angle made by the line $y = 0$ with the tangents to C_1 at P and Q. Show that, for all values of the constant λ, the circle C_3 whose equation is $\lambda(x^2 + y^2 - 4x - 8y - 5) + x^2 + y^2 - 6x - 10y + 9 = 0$ passes through the points of intersection of C_1 and C_2. Find the two possible values of λ for which the line $y = 0$ is a tangent to C_3. (JMB)

9. The circles whose equations are $x^2 + y^2 - x + 6y + 7 = 0$
and $x^2 + y^2 + 2x + 2y - 2 = 0$

intersect at the points A and B. Find (i) the equation of the line AB, (ii) the coordinates of A and B. Show that the two given circles intersect at right angles and obtain the equation of the circle which passes through A and B and which also passes through the centres of the two circles. (AEB 1978)

10. A curve C_1 has equation $2y^2 = x$; a curve C_2 is given by the parametric equations $x = 4t$, $y = 4/t$. Sketch C_1 and C_2 on the same diagram and calculate the coordinates of their point of intersection, P. The tangent to C_1 at P crosses the x-axis at T and meets C_2 again at Q. (a) Show that T is a point of trisection of PQ. (b) Find the area of the finite region bounded by the x-axis, the line PT and the curve C_1. (L)

11. Show that the tangent at the point P, with parameter t, on the curve $x = ct$, $y = c/t$ has equation $x + t^2y = 2ct$. This tangent meets the x-axis in a point Q and the line through P parallel to the x-axis cuts the y-axis in a point R. Show that, for any position of P on the curve, QR is a tangent to the curve with parametric equations $x = ct$, $y = c/(2t)$. (L)

12. Prove that the equation of the normal to the rectangular hyperbola $xy = c^2$ at the point $P(ct, c/t)$ is $ty - t^3x = c(1 - t^4)$. The normal at P and the normal at the point $Q(c/t, ct)$, where $t > 1$, intersect at the point N. Show that $OPNQ$ is a rhombus, where O is the origin. Hence, or otherwise, find the coordinates of N. If the tangents to the hyperbola at P and Q intersect at T, prove that the product of the lengths of OT and ON is independent of t. (JMB)

13. The line of gradient $m(\neq 0)$ through the point $A(a, O)$ is a tangent to the rectangular hyperbola $xy = c^2$ at the point P. Find m in terms of a and c, and show that the coordinates of P are $(\frac{1}{2}a, 2c^2/a)$. The line through A parallel to the y-axis meets the hyperbola at Q, and the line joining Q to the origin O intersects AP at R. Given that OQ and AP are perpendicular to each other, find the numerical value of c^2/a^2 and the numerical value of the ratio $AR:RP$. (JMB)

14. The point P in the first quadrant lies on the curve with parametric equations $x = t^2$, $y = t^3$. The tangent to the curve at P meets the curve again at Q and is normal to the curve at Q. Find the coordinates of P and of Q. (L)

15. Sketch the parabola whose parametric equations are $x = t^2$, $y = 2t$ and on the same diagram sketch the curve with parametric equations $x = 10(1 + \cos\theta)$, $y = 10\sin\theta$. These curves touch at the origin and meet again at two other points A and B. The normals at A and B to the parabola meet at P and the tangents to the other curve at A and B meet at Q. Calculate the length of PQ. (L)

16. Obtain an equation of the tangent, at the point with parameter t, to the curve $\mathscr{C}$ whose parametric equations are given by

$$x = 2\sin^3 t, \quad y = 2\cos^3 t, \quad 0 \leqslant t \leqslant \pi/2.$$

Show that, if the tangent meets the coordinate axes in points R and S, then RS is of constant length. Sketch the curve $\mathscr{C}$. Find the area of the finite region enclosed by the curve $\mathscr{C}$ and the coordinate axes. (L)

17. Sketch the curve whose parametric equations are $x = a\cos\phi$, $y = b\sin\phi$, $0 \leqslant \phi \leqslant 2\pi$, where a and b are positive constants. The point P is given by $\phi = \pi/4$. Find (a) the equation of the tangent to the curve at P, (b) the equation of the normal to the curve at P. By evaluating a suitable integral, calculate the area of the region in the first quadrant between the curve and the coordinate axes. Hence deduce the area of the region enclosed by the curve. (L)

18. Prove that the equation of the tangent at the point (x_1, y_1) on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is $\dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} = 1$. The tangent at the point $(2\cos\theta, \sqrt{3}\sin\theta)$ on the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$ passes through the point $P(2, 1)$. Show that $\sqrt{3}\cos\theta + \sin\theta = \sqrt{3}$. Without using tables, calculator or slide rule, find all the solutions of this equation which are in the range $0° \leqslant \theta < 360°$. Hence obtain the coordinates of the points of contact, Q and R, of the tangents to the ellipse from P. Verify that the line through the origin and the point P passes through the mid-point of the line QR. (JMB)

19. Prove that the equation of the normal at $(\alpha\cos\phi, \beta\sin\phi)$ to the ellipse $\dfrac{x^2}{\alpha^2} + \dfrac{y^2}{\beta^2} = 1$ is $\alpha x\sec\phi - \beta y\operatorname{cosec}\phi = \alpha^2 - \beta^2$. P is the point $(a\cos\theta, b\sin\theta)$ on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. M and N are the feet of the perpendiculars from P to the axes. Find the equation of MN. Prove that, for variable θ, MN is always normal to a fixed concentric ellipse and find the equation of this ellipse. (O&C)

20. Prove that the equation of the tangent to the curve $xy^2 = c^3$ $(c > 0)$ at the point $P(ct^2, c/t)$ is $2t^3y + x = 3ct^2$. Prove that the parameters of the points of contact of the tangents which pass through the point $Q(h, k)$ $(k \neq 0)$ satisfy a cubic equation and, by considering the turning values of this cubic, or otherwise, prove that there are three distinct tangents to the curve which pass through Q if $hk^2 < c^3$

and $h > 0$. State a necessary and sufficient condition on the parameters t_1, t_2, t_3 for the tangents at the corresponding points to be concurrent.

When Q lies on the curve, the tangent at Q cuts the curve again at R; if the parameter of Q is t, determine the parameter of R. Prove that, if the tangents Q_1, Q_2, Q_3 are concurrent and cut the curve again at R_1, R_2, R_3 then the tangents at R_1, R_2, R_3 are concurrent. (O&C)

26 Complex numbers

26.1 Algebraic form

We are already familiar with various types of real number, such as integers, rationals and irrationals. However, since the square of any non-zero real number is positive, there are no solutions within the real number system of the equations such as $x^2 + 2 = 0$ and $x^2 + 2x + 2 = 0$. To deal with these equations we must extend the number system to include the square roots of negative numbers, e.g. $\sqrt{(-1)}$, $-\sqrt{(-5)}$, $\sqrt{(-7/3)}$. Numbers of the form $\pm\sqrt{(-k^2)}$, where k is real, are called *imaginary numbers*. If the number $\sqrt{(-1)}$ is denoted by i, so that $i^2 = -1$, then any imaginary number can be expressed as a multiple of i. For instance:

$$\sqrt{(-4)} = \sqrt{\{4 \times (-1)\}} = \sqrt{4} \times \sqrt{(-1)} = 2i,$$
$$\sqrt{(-7)} = \sqrt{\{7 \times (-1)\}} = \sqrt{7} \times \sqrt{(-1)} = i\sqrt{7}.$$

The set of all numbers of the form $a + ib$, where a and b are real, is called the set of *complex numbers* and denoted by $\mathbb{C}$. The number a is called the *real part* of $a + ib$ and b is the *imaginary part* of $a + ib$. Since $a + ib$ is real when $b = 0$, the set of real numbers $\mathbb{R}$ is a subset of $\mathbb{C}$.

Let us now consider the roots of the quadratic equation $ax^2 + bx + c = 0$, given by $x = \dfrac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$. As we know, when $b^2 - 4ac \geqslant 0$, this equation has real roots, but when $b^2 - 4ac < 0$ the formula for x involves the square root of a negative number. We can write this square root as a multiple of i, thus obtaining the roots of the quadratic equations in the form of complex numbers.

Example 1 Find the roots of the equations (a) $x^2 + 9 = 0$, (b) $x^2 + 2x + 2 = 0$, (c) $2x^2 - 2x + 5 = 0$.

(a) $x^2 + 9 = 0 \Rightarrow x^2 = -9 \Rightarrow x = \pm\sqrt{(-9)} = \pm 3i.$

(b) $x^2 + 2x + 2 = 0 \Rightarrow x = \dfrac{-2 \pm \sqrt{(4-8)}}{2}$

$$= \frac{-2 \pm \sqrt{(-4)}}{2}$$

$$= \frac{-2 \pm 2i}{2} = -1 \pm i.$$

(c) $2x^2 - 2x + 5 = 0 \Rightarrow x = \dfrac{2 \pm \sqrt{(4-40)}}{4}$

$$= \frac{2 \pm \sqrt{(-36)}}{4}$$

$$= \frac{2 \pm 6i}{4} = \frac{1}{2} \pm \frac{3}{2}i.$$

Each of the equations in Example 1 has a pair of roots of the form $p \pm iq$. The numbers $p + iq, p - iq$ are called *conjugate complex numbers* and $p - iq$ is described as the *conjugate* of $p + iq$. Further consideration of the general formula for the roots of a quadratic equation with real coefficients shows that complex roots must always occur in conjugate pairs. Thus if $p + iq$ is one root of a given quadratic equation, then the other root is $p - iq$.

It is assumed that complex numbers can be combined according to the usual laws of algebra, together with the property $i^2 = -1$.

Example 2 Express, in the form $a + ib$, (a) $(2 - 3i) + (-1 + 4i)$, (b) $(2 - 3i) - (-1 + 4i)$, (c) $(2 - 3i)(-1 + 4i)$, (d) $(2 - 3i)(2 + 3i)$.

(a) $(2 - 3i) + (-1 + 4i) = (2 - 1) + (-3 + 4)i = 1 + i,$

(b) $(2 - 3i) - (-1 + 4i) = (2 + 1) + (-3 - 4)i = 3 - 7i,$

(c) $(2 - 3i)(-1 + 4i) = -2 + 8i + 3i - 12i^2$

$= -2 + 8i + 3i + 12 = 10 + 11i,$

(d) $(2 - 3i)(2 + 3i) = 4 + 6i - 6i - 9i^2 = 4 + 9 = 13.$

This last result illustrates the fact that the product of a pair of conjugate complex numbers is real. In general,

$$(a + ib)(a - ib) = a^2 + b^2.$$

This property is used to simplify a quotient of complex numbers.

Example 3 Express $\dfrac{-1 + 2i}{1 + 3i}$ in the form $a + ib$.

Multiplying numerator and denominator by the conjugate of $1 + 3i$:

$$\frac{(-1 + 2i)(1 - 3i)}{(1 + 3i)(1 - 3i)} = \frac{-1 + 3i + 2i - 6i^2}{1 - 3i + 3i - 9i^2} = \frac{-1 + 5i + 6}{1 + 9} = \frac{5 + 5i}{10}$$

$$\therefore \quad \frac{-1 + 2i}{1 + 3i} = \frac{1}{2} + \frac{1}{2}i$$

We next consider two *equal* complex numbers $a + ib$ and $c + id$, where a, b, c, d are real.

$$\begin{aligned} a + ib = c + id &\Rightarrow \quad a - c = i(d - b) \\ &\Rightarrow (a - c)^2 = i^2(d - b)^2 \\ &\Rightarrow (a - c)^2 = -(d - b)^2 \\ &\Rightarrow \quad a - c = d - b = 0 \\ &\Rightarrow \qquad a = c \quad \text{and} \quad b = d \end{aligned}$$

Hence $a + ib$ and $c + id$ are equal if and only if $a = c$ and $b = d$. This means that if two complex numbers are known to be equal, we may equate real and imaginary parts.

Example 4 Find the square roots of the complex number $5 - 12i$, giving your answers in the form $a + ib$.

Let $(a + ib)^2 = 5 - 12i$
then $a^2 - b^2 + 2abi = 5 - 12i$
Equating real and imaginary parts:

$$\begin{aligned} &a^2 - b^2 = 5, \quad 2ab = -12 \\ \text{i.e.} \quad &a^2 - b^2 = 5, \quad ab = -6 \end{aligned}$$

By inspection, either $a = 3, b = -2$ or $a = -3, b = 2$.
$\therefore$ the square roots of $5 - 12i$ are $3 - 2i$ and $-3 + 2i$.

[Note that the equations $a^2 - b^2 = 5$, $ab = -6$ could also be solved by eliminating b to obtain a quadratic equation in a^2.]

It is often convenient to use a single letter, such as z, to represent a complex number. The complex conjugate of z is then denoted by z^* (or sometimes $\bar{z}$). The real and imaginary parts of z are denoted by $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ respectively.

Thus, if $z = x + iy$, $z^* = x - iy$, $\mathrm{Re}(z) = x$ and $\mathrm{Im}(z) = y$.

Exercise 26.1

1. Express the following in terms of i:
(a) $\sqrt{(-25)}$, (b) $\sqrt{(-5)}$, (c) $\sqrt{9} - \sqrt{(-16)}$, (d) $2 - \sqrt{(-18)}$

2. Write down the roots of the following equations:
(a) $x^2 - 4 = 0$, (b) $x^2 + 4 = 0$, (c) $x^2 + 2 = 0$,
(d) $4x^2 + 9 = 0$, (e) $x^3 + 3x = 0$, (f) $x^4 - 1 = 0$.

3. Find the roots of the following equations:
(a) $x^2 - 2x + 2 = 0$, (b) $x^2 - 6x + 10 = 0$, (c) $x^2 - 4x + 13 = 0$,
(d) $4x^2 - 4x + 5 = 0$, (e) $2x^2 + 3x + 2 = 0$, (f) $x^2 - x + 1 = 0$.

4. Write down the conjugates of the following complex numbers:
(a) $3 + 5i$, (b) $4 - 7i$, (c) $3i$, (d) -4, (e) $-1 + i$.

5. Find $z_1 + z_2$ and $z_1 - z_2$ in each of the following cases:
(a) $z_1 = 5 + 8i, z_2 = 2 + i$, (b) $z_1 = 2 + 3i, z_2 = 1 - 4i$,
(c) $z_1 = 3 - i, z_2 = 5 - 3i$, (d) $z_1 = -1 + 2i, z_2 = 3 + 5i$.

6. Express in the form $a + ib$:
(a) $(3 + i)(2 + 3i)$, (b) $(1 - 2i)(5 + i)$, (c) $(2 - i)(3 - 2i)$,
(d) $(3 - 4i)(3 + 4i)$, (e) $(2 - 5i)^2$, (f) $(1 - i)^3$.

7. Simplify (a) i^3, (b) i^5, (c) i^8, (d) $\frac{1}{i^2}$, (e) $\frac{1}{i}$, (f) $\frac{1}{i^3}$.

8. Express in the form $a + ib$:

(a) $\frac{3 + 4i}{1 - 2i}$, (b) $\frac{2i}{3 + i}$, (c) $\frac{3 - 2i}{1 - i}$,

(d) $\frac{\sqrt{3} - i}{\sqrt{3} + i}$, (e) $\frac{1}{(1 + i)^2}$, (f) $\frac{1}{(2 - i)(1 + 2i)}$.

9. Express in the form $a + ib$:

(a) $\frac{1}{2 + 3i} + \frac{1}{2 - 3i}$, (b) $\frac{1}{3 + i} - \frac{1}{1 + 7i}$, (c) $3 + 4i + \frac{25}{3 + 4i}$.

10. Find the values of a and b given that
(a) $(a + ib)(2 - i) = a + 3i$, (b) $(a + i)(1 + ib) = 3b + ia$.

11. Find the square roots of the following complex numbers
(a) $3 - 4i$, (b) $2i$, (c) $8 + 6i$, (d) $21 - 20i$.

12. Find the values of real numbers p and q given that one root of the equation $x^2 + px + q = 0$ is
(a) $2 + i$, (b) $-1 + 3i$, (c) $4i$, (d) $3 - 5i$.

13. Find the complex factors of the following quadratic expressions.
(a) $x^2 + 1$, (b) $x^2 + 25$, (c) $9x^2 + 4$,
(d) $x^2 + 2x + 5$, (e) $x^2 + 6x + 13$, (f) $4x^2 - 12x + 25$.

14. Given that the complex number $a + ib$ is conjugate to its own square, find all possible pairs of values of a and b.

15. Given that $\frac{5}{x + iy} + \frac{2}{1 + 3i} = 1$, where x and y are real, find x and y.

16. If $(1 + i)z - iw + i = iz + (1 - i)w - 3i = 6$, find the complex numbers z, w, expressing each in the form $a + bi$ where a, b are real. (O&C)

17. Express $(6 + 5i)(7 + 2i)$ in the form $a + ib$. Write down $(6 - 5i)(7 - 2i)$ in a similar form. Hence find the prime factors of $32^2 + 47^2$. (JMB)

18. The roots of the quadratic equation $z^2 + pz + q = 0$ are $1 + i$ and $4 + 3i$. Find the complex numbers p and q. It is given that $1 + i$ is also a root of the equation $z^2 + (a + 2i)z + 5 + ib = 0$, where a and b are real. Determine the values of a and b. (JMB)

19. Given that $z = (p + i)^4$, where p is real, find the values of p for which (a) z is real, (b) z is a real multiple of i.

20. Find the two square roots of $5 + 12i$. Hence find the roots of the equation $z^2 - (1 + 6i)z - 10 = 0$.

21. Prove that for any complex number z,

$$\mathrm{Re}(z) = \tfrac{1}{2}(z + z^*) \quad \text{and} \quad \mathrm{Im}(z) = \frac{1}{2i}(z - z^*).$$

26.2 Roots of polynomial equations

In the previous section we saw that a quadratic equation with real coefficients has either real roots or a pair of conjugate complex roots. This result can be extended to polynomial equations of higher degree by considering further the properties of conjugate complex numbers.

For two complex numbers $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, we have

$$\begin{aligned}(z_1 + z_2)^* &= \{(x_1 + iy_1) + (x_2 + iy_2)\}^* \\ &= \{(x_1 + x_2) + i(y_1 + y_2)\}^* \\ &= (x_1 + x_2) - i(y_1 + y_2) \\ &= (x_1 - iy_1) + (x_2 - iy_2) = z_1^* + z_2^* \\ (z_1 z_2)^* &= \{(x_1 + iy_1)(x_2 + iy_2)\}^* \\ &= \{(x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)\}^* \\ &= (x_1 x_2 - y_1 y_2) - i(x_1 y_2 + x_2 y_1) \\ &= (x_1 - iy_1)(x_2 - iy_2) = z_1^* z_2^*\end{aligned}$$

i.e.

$$\boxed{(z_1 + z_2)^* = z_1^* + z_2^*, \qquad (z_1 z_2)^* = z_1^* z_2^*}$$

It follows that $(z^2)^* = (z^*)^2$ and $(z^3)^* = (z^*)^3$.
Thus, if $f(z) = az^3 + bz^2 + cz + d$, where a, b, c, d are real, then

$$\begin{aligned}\{f(z)\}^* &= \{az^3 + bz^2 + cz + d)^* \\ &= (az^3)^* + (bz^2)^* + (cz)^* + d^* \\ &= a(z^3)^* + b(z^2)^* + cz^* + d \\ &= a(z^*)^3 + b(z^*)^2 + cz^* + d = f(z^*)\end{aligned}$$

More generally, it can be shown that $(z^n)^* = (z^*)^n$, where n is a positive integer.

Hence if $P(z)$ is a polynomial of degree n with real coefficients, then

$$\boxed{\{P(z)\}^* = P(z^*)}$$

Suppose now that α is a complex root of the polynomial equation $P(z) = 0$.

$$P(\alpha) = 0 \Leftrightarrow \{P(\alpha)\}^* = \{0\}^* \Leftrightarrow P(\alpha^*) = 0$$

$\therefore$ if α is a root of the equation $P(z) = 0$, its conjugate α^* is also a root.

Thus complex roots of polynomial equations with real coefficients occur in conjugate pairs.

Much of the general theory of polynomial equations is too difficult to discuss here. However, because of its importance, we will state one basic theorem.

Every polynomial equation of degree n has exactly n roots of the form $a + ib$, where a and b are real. Some of these roots may be repeated.

If the polynomial equation $P(z) = 0$ has a real root a, then $P(z)$ has a linear factor $z - a$. Thus, if the equation $P(z) = 0$ has r real roots, then $P(z)$ has r real linear factors.

Similarly, if the equation has a complex root $a + ib$, then since $a - ib$ is also a root, the polynomial $P(z)$ has factors $z - a - ib$ and $z - a + ib$. The product of these factors is $z^2 - 2az + a^2 + b^2$, which is a quadratic factor with real coefficients. Thus, if the equation $P(z) = 0$ has k pairs of conjugate complex roots, then $P(z)$ has k real quadratic factors which cannot be split into real linear factors.

Hence any polynomial with real coefficients is a product of real linear and quadratic factors.

When solving problems it is useful to remember that the formulae for sums and products of roots of polynomial equations remain valid for equations with complex roots.

Example 1 Find the real root of the equation $3z^3 - 10z^2 + 7z + 10 = 0$ given that one root is $2 - i$.

Since $2 - i$ is a root, its complex conjugate $2 + i$ is also a root.

The equation has three roots with sum $-\dfrac{(-10)}{3}$ i.e. $\dfrac{10}{3}$

$\therefore$ the real root of the equation $= \frac{10}{3} - (2 - i + 2 + i) = -\frac{2}{3}$

One cubic equation of special interest is the equation $z^3 = 1$, whose roots are the *cube roots of unity*.

$$z^3 - 1 = 0$$
$$\Rightarrow \quad (z-1)(z^2 + z + 1) = 0$$
$$\Rightarrow \quad \text{either } z - 1 = 0 \quad \text{or} \quad z^2 + z + 1 = 0$$

Hence the complex cube roots of unity are the roots of the equation $z^2 + z + 1 = 0$. Let ω be one of these complex roots, then

$$\omega^3 = 1 \quad \text{and} \quad \omega^2 + \omega + 1 = 0.$$

But $(\omega^2)^3 = \dot{\omega}^6 = (\omega^3)^2 = 1^2 = 1$,
$\therefore$ ω^2 must also be a complex cube root of unity.

Thus the cube roots of 1 are 1, ω, ω^2, where $\omega^3 = 1$ and $\omega^2 + \omega + 1 = 0$.

By solving the equation $z^2 + z + 1 = 0$ we find that the complex cube roots of 1 are $\frac{1}{2}(-1 \pm i\sqrt{3})$. However, when solving problems involving these roots it is often easier to use ω and its properties, rather than the numerical values of the roots.

Example 2 If ω is a complex cube root of unity, find the value of $\omega^3 + \omega^4 + \omega^5$.

Since $\omega^3 = 1$ and $\omega^2 + \omega + 1 = 0$,
$\omega^3 + \omega^4 + \omega^5 = \omega^3(1 + \omega + \omega^2) = 0$.

Example 3 Given that ω is a complex cube root of unity find the equation whose roots are i, $i\omega$, $i\omega^2$.

The roots of the required equation are given by $z = i\alpha$, where α is a cube root of unity.
$\therefore$ $z^3 = (i\alpha)^3 = i^3\alpha^3 = -i$
Hence the equation with roots i, $i\omega$, $i\omega^2$ is $z^3 + i = 0$.

Exercise 26.2

1. Prove that a number z is real if and only if $z = z^*$.

2. Prove that if z is a complex number and k is real, $(kz)^* = kz^*$.

3. If z_1 and z_2 are two complex numbers, prove that

(a) $(z_1 - z_2)^* = z_1^* - z_2^*$, (b) if $z_2 \neq 0$, then $\left(\frac{z_1}{z_2}\right)^* = \frac{z_1^*}{z_2^*}$.

4. Prove by induction that if z is a complex number then $(z^n)^* = (z^*)^n$, where n is a positive integer.

5. Find the real root of the equation $z^3 + z + 10 = 0$ given that one root is $1 - 2i$.

6. Given that $3 + i$ is a root of the equation $z^3 - 3z^2 - 8z + 30 = 0$, find the remaining roots.

7. Given that $1 + i$ is a root of the equation $z^3 - 2z + k = 0$, find the other two roots and the value of the real constant k.

8. Given that $2 - 3i$ is a root of the equation $z^3 + pz^2 + qz + 13 = 0$, find the other two roots and the value of the real constants p and q.

9. Show that $z = i$ is a root of the equation $z^4 + z^3 + z - 1 = 0$. Find the three other roots.

10. Show that $z = -1 + i$ is a root of the equation $z^4 - 2z^3 - z^2 + 2z + 10 = 0$. Find the remaining roots.

11. Given that $1, \omega_1, \omega_2$ are the roots of the equation $z^3 = 1$, express ω_1 and ω_2 in the form $a + ib$. Hence show that
(a) $(\omega_1)^2 = \omega_2$, (b) $(\omega_2)^2 = \omega_1$, (c) $1/\omega_1 = \omega_2$.

12. Express the roots of the equation $z^3 - \alpha^3 = 0$ in terms of α and ω, where ω is a complex cube root of unity. Use your answer to find the roots of the following equations in the form $a + ib$:
(a) $z^3 - 27 = 0$, (b) $z^3 + 8 = 0$, (c) $z^3 - i = 0$.

13. Find the roots of the following equations in the form $a + ib$:
(a) $z^3 - 8 = 0$, (b) $z^3 + 1 = 0$, (c) $(z + 1)^3 = 1$.

14. Evaluate the following expressions given that ω is a complex cube root of unity.
(a) $\omega + \omega^3 + \omega^5$, (b) $\dfrac{1}{\omega^2 + \omega^4}$, (c) $\dfrac{\omega^2}{\omega + \omega^3}$.

15. Given that ω is a complex cube root of unity, show that $\omega^* = \omega^2$. Hence write down in terms of ω the conjugates of
(a) $1 + \omega$, (b) $1 - \omega^2$, (c) $3 + 4\omega + 5\omega^2$.

16. Given that ω is a complex cube root of unity, simplify the expression

$$(a + b + c)(a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c).$$

17. Given that $1, \omega, \omega^2$ are the cube roots of unity, find the equation whose roots are $\dfrac{1}{3}, \dfrac{1}{2 + \omega}, \dfrac{1}{2 + \omega^2}$.

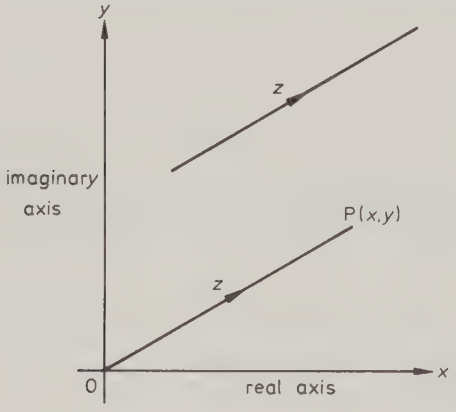
18. Given that ω denotes either one of the non-real roots of the equation $z^3 = 1$, show that (i) $1 + \omega + \omega^2 = 0$, and (ii) the other non-real root is ω^2.

Show that the non-real roots of the equation $\left(\frac{1-u}{u}\right)^3 = 1$ can be expressed in the form $A\omega$ and $B\omega^2$, where A and B are real numbers, and find A and B. **(JMB)**

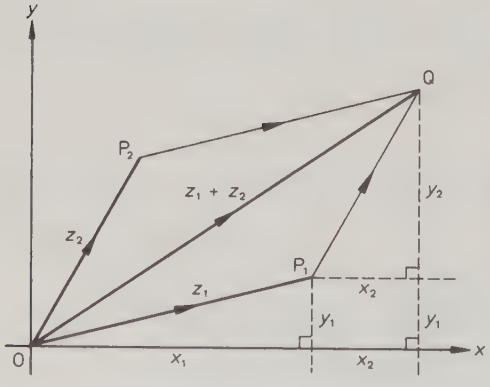
26.3 Geometrical representation

The complex number $z = x + iy$ is completely specified by the ordered pair of real numbers (x, y). Thus to any complex number z there corresponds a unique point $P(x, y)$ in the x, y plane. This suggests that z may be represented by

(1) the point P with coordinates (x, y),
or (2) the vector $\overrightarrow{OP}$, where O is the origin,
or (3) any vector equal to $\overrightarrow{OP}$ in magnitude and direction.



A diagram showing complex numbers represented by points or vectors is called an *Argand*† *diagram.*
Since real numbers are represented along the x-axis and imaginary numbers along the y-axis, in an Argand diagram these axes are often referred to as the *real axis* and the *imaginary axis* respectively.



If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$. Let z_1, z_2 and $z_1 + z_2$ be represented by the vectors $\overrightarrow{OP_1}$, $\overrightarrow{OP_2}$ and $\overrightarrow{OQ}$ respectively. The diagram shows that $\overrightarrow{P_1Q}$ is equal to $\overrightarrow{OP_2}$ in magnitude and direction

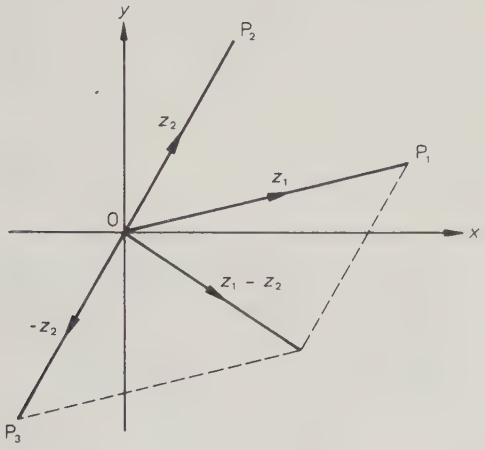
$$\therefore \quad \overrightarrow{OQ} = \overrightarrow{OP_1} + \overrightarrow{P_1Q} = \overrightarrow{OP_1} + \overrightarrow{OP_2}.$$

Thus the sum of two complex numbers z_1 and z_2 is represented in an Argand diagram by the sum of the corresponding vectors $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$.

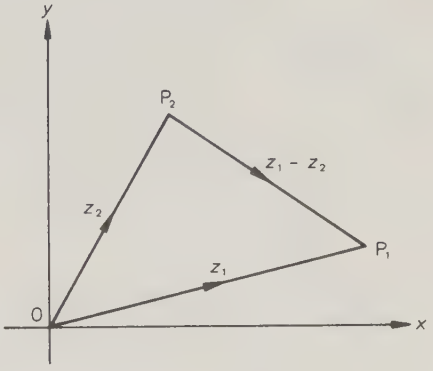
†*Argand, Jean Robert* (1768–1822) Swiss mathematician. He published the geometrical representation of complex numbers in an "Essai" in 1806.

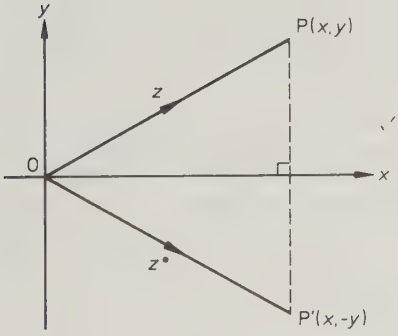
There are two ways of obtaining a geometrical representation of $z_1 - z_2$. If the complex number $(-z_2)$ is represented by the vector $\overrightarrow{OP_3}$, then $z_1 - z_2$ is represented by the sum of the vectors $\overrightarrow{OP_1}$ and $\overrightarrow{OP_3}$, as shown in diagram (i). However, since $\overrightarrow{OP_1} - \overrightarrow{OP_2} = \overrightarrow{P_2P_1}$, $z_1 - z_2$ can also be represented by $\overrightarrow{P_2P_1}$, as shown in diagram (ii).

(i)

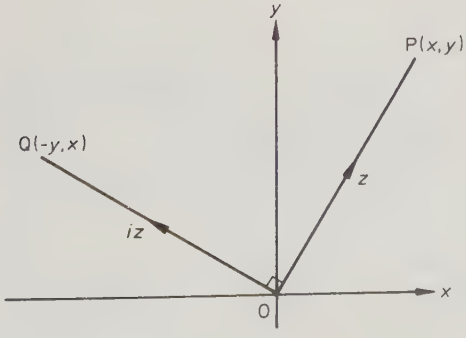


(ii)





Let the complex number $z = x + iy$ and its conjugate $z^* = x - iy$ be represented by $\overrightarrow{OP}$ and $\overrightarrow{OP'}$ respectively. Since the coordinates of P and P' are (x, y) and $(x, -y)$ respectively, P' must be the reflection of P in the real axis.

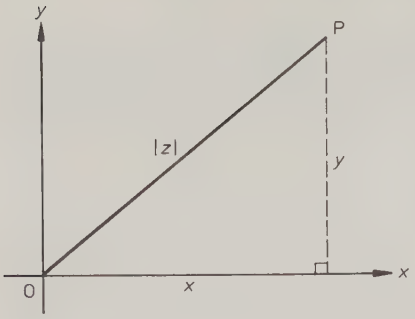


If $z = x + iy$, then

$$iz = i(x + iy) = -y + ix.$$

Thus if z and iz are represented by the vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$, then the coordinates of P and Q are (x, y) and $(-y, x)$ respectively.

Since the gradient of OP is y/x and the gradient of OQ is $-x/y$, OP must be perpendicular to OQ. Hence multiplication of a complex number z by i results in an anti-clockwise rotation of the vector $\overrightarrow{OP}$ through $\frac{1}{2}\pi$ radians.

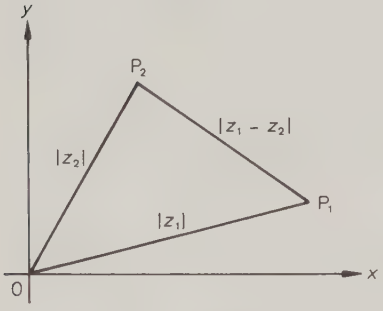


The *modulus* of a complex number z is defined as the length of the corresponding vector $\overrightarrow{OP}$ and is denoted by $|z|$.

Thus if $z = x + iy$, $|z| = \sqrt{(x^2 + y^2)}$.

For instance, $|3 + 4i| = \sqrt{\{3^2 + 4^2\}} = 5$

and $|1 - 2i| = \sqrt{\{1^2 + (-2)^2\}} = \sqrt{5}$.



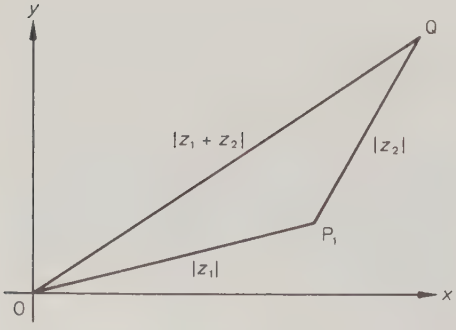
If complex numbers z_1 and z_2 are represented by the vectors $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ respectively, then $z_1 - z_2$ is represented by $\overrightarrow{P_2P_1}$.

$\therefore \quad |z_1 - z_2| = |\overrightarrow{P_2P_1}|$

Thus $|z_1 - z_2|$ is the distance between the points P_1 and P_2.

Example 1 Given that the complex numbers $2 + 5i$ and $3 - 2i$ are represented in an Argand diagram by the points A and B, find the length of AB.

$$AB = |(2 + 5i) - (3 - 2i)|$$
$$= |-1 + 7i| = \sqrt{\{(-1)^2 + 7^2\}} = \sqrt{50} = 5\sqrt{2}.$$



If complex numbers z_1, z_2 and $z_1 + z_2$ are represented by the vectors $\overrightarrow{OP_1}$, $\overrightarrow{OP_2}$ and $\overrightarrow{OQ}$, then as shown earlier, z_2 is also represented by $\overrightarrow{P_1Q}$. We now use triangle OP_1Q to produce a set of results called the *triangle inequalities*.

Since $OQ \leqslant OP_1 + P_1Q$, $\quad |z_1 + z_2| \leqslant |z_1| + |z_2|$

Since $OP_1 \leqslant P_1Q + OQ$, $\quad |z_1| \leqslant |z_2| + |z_1 + z_2|$

$\therefore \quad |z_1| - |z_2| \leqslant |z_1 + z_2|$

Since $P_1Q \leqslant OP_1 + OQ$, $\quad |z_2| \leqslant |z_1| + |z_1 + z_2|$

$\therefore \quad |z_2| - |z_1| \leqslant |z_1 + z_2|$

Combining these three results, we have

$$\boxed{||z_1| - |z_2|| \leqslant |z_1 + z_2| \leqslant |z_1| + |z_2|}$$

[For an alternative method of proof see Exercise 26.4 question 14.]

Exercise 26.3

1. Write down the complex numbers corresponding to the following points in an Argand diagram.
(a) $(3, 2)$, (b) $(-2, 4)$, (c) $(0, 5)$, (d) $(1, -2)$, (e) $(-2, 0)$.

2. Represent the following complex numbers by vectors in an Argand diagram and find the modulus of the complex number in each case.
(a) $4 + 3i$, (b) $2 - 5i$, (c) $3i$, (d) $-1 - i$, (e) -4.

3. Represent the complex number $z = 2 + i$ by a vector in an Argand diagram. Find and represent in the same diagram $-z, z^*, -z^*, iz$ and $-iz$.

4. Represent the complex number $z = 1 + i$ by a vector in an Argand diagram. Find and represent in the same diagram z^2, z^3 and $1/z$.

5. If $z_1 = 1 + 3i$ and $z_2 = 3 - i$, find $z_1 z_2$ and z_1/z_2 in the form $a + ib$. Show on an Argand diagram the vectors representing $z_1, z_2, z_1 z_2$ and z_1/z_2.

6. Find and represent in an Argand diagram the square roots of
(a) -1, (b) $-2i$, (c) $3 + 4i$, (d) $15 + 8i$.

7. In each of the following cases represent z_1, z_2 and $z_1 + z_2$ in an Argand diagram and verify that

$$||z_1| - |z_2|| \leqslant |z_1 + z_2| \leqslant |z_1| + |z_2|.$$

(a) $z_1 = 12 + 5i, z_2 = -3 + 4i$, (b) $z_1 = -1 - i, z_2 = 7 - i$.

8. If $z_1 = 1 + 2i$, find the set of values of z_2 for which
(a) $|z_1 + z_2| = |z_1| + |z_2|$, (b) $|z_1 + z_2| = |z_1| - |z_2|$,
(c) $|z_1 + z_2| = |z_2| - |z_1|$.

9. If z_1 and z_2 are complex numbers, show geometrically that

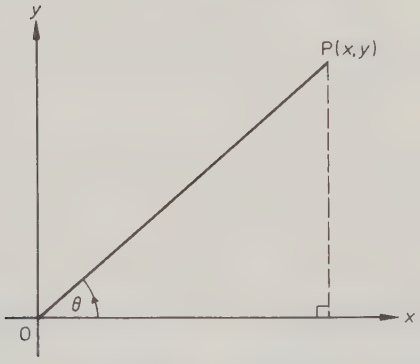
$$||z_1| - |z_2|| \leqslant |z_1 - z_2| \leqslant |z_1| + |z_2|.$$

10. In each of the following cases find the distance between points in the Argand diagram representing the given pair of complex numbers.
(a) $2 + i, 5 - 3i$, (b) $1 - 2i, 3 + 7i$, (c) $2 - 5i, -1 - 3i$.

11. The points A, B, C and D in the Argand diagram correspond to the complex numbers $8 - i, 3 + 11i, -9 + 6i, -4 - 6i$. Prove that $ABCD$ is a square.

12. Given that $z = 1 + i\sqrt{2}$, express in the form $a + ib$ each of the complex numbers $p = z + \frac{1}{z}, q = z - \frac{1}{z}$. In an Argand diagram, P and Q are the points which represent p and q respectively, O is the origin, M is the midpoint of PQ and G is the point on OM such that $OG = \frac{2}{3}OM$. Prove that the angle PGQ is a right angle. **(JMB)**

26.4 Modulus–argument form



Let us suppose that a complex number $z = x + iy$ is represented in an Argand diagram by the vector $\overrightarrow{OP}$, where r is the length of $\overrightarrow{OP}$ and θ is the angle $\overrightarrow{OP}$ makes with the positive direction of the real axis.

From the diagram we see that

$$x = r\cos\theta, \qquad y = r\sin\theta,$$
$$r^2 = x^2 + y^2, \quad \tan\theta = y/x.$$

As we already know, the length r is called the modulus of z and written $|z|$,

$$\therefore \qquad \boxed{|z| = r = \sqrt{(x^2 + y^2)}}$$

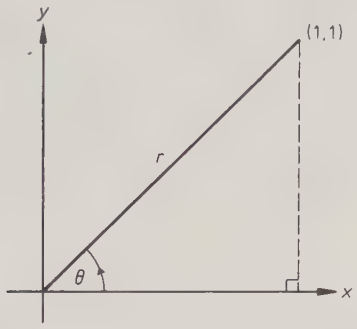
The angle θ is called the *argument* of z (or less commonly, the amplitude), positive values of θ being measured anti-clockwise from the real axis and negative values clockwise. The problem with this definition is that for any given z there is an infinite set of possible values for θ. To avoid confusion we usually work with the value of θ for which $-\pi < \theta \leqslant \pi$. This is called the *principal argument* of z and denoted by arg z.

$$\therefore \qquad \boxed{\arg z = \theta, \quad \text{where} \quad \sin\theta = \frac{x}{r}, \quad \cos\theta = \frac{y}{r} \quad \text{and} \quad -\pi < \theta \leqslant \pi.}$$

In practice the formula $\tan\theta = y/x$ is often used to find the principal argument of a complex number z, despite the fact that it leads to two possible values for θ in the permitted range. The correct value of arg z is then chosen with the aid of a sketch showing the approximate position of the corresponding vector $\overrightarrow{OP}$.

Example 1 Find the moduli and principal arguments of the complex numbers (a) $1 + i$, (b) $-1 - i\sqrt{3}$, (c) -5.

(a)

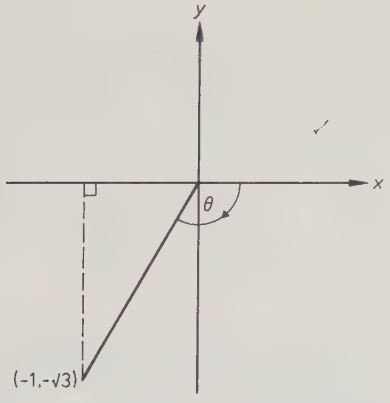


$|1 + i| = \sqrt{(1^2 + 1^2)} = \sqrt{2}$
If $\arg(1 + i) = \theta$,
then $\tan\theta = \frac{1}{1} = 1 = \tan\frac{1}{4}\pi$
Since the point representing $1 + i$ lies in the first quadrant, $0 < \theta < \frac{1}{2}\pi$.

$\therefore \quad \theta = \frac{1}{4}\pi$

Hence $|1 + i| = \sqrt{2}$ and $\arg(1 + i) = \frac{1}{4}\pi$.

(b)



$$|-1 - i\sqrt{3}| = \sqrt{\{(-1)^2 + (-\sqrt{3})^2\}} = \sqrt{\{1 + 3\}} = 2$$

If $\arg(-1 - i\sqrt{3}) = \theta$, then

$$\tan\theta = \frac{-\sqrt{3}}{-1} = \sqrt{3} = \tan\tfrac{1}{3}\pi$$

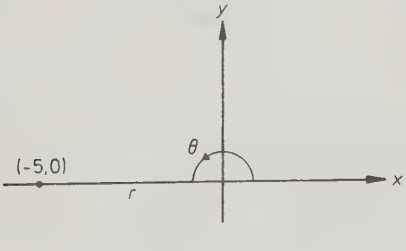
Since the point representing $-1 - i\sqrt{3}$ lies in the third quadrant,

$$-\pi < \theta < -\tfrac{1}{2}\pi.$$

$\therefore \quad \theta = \frac{1}{3}\pi - \pi = -\frac{2}{3}\pi.$

Hence $|-1 - i\sqrt{3}| = 2$ and $\arg(-1 - i\sqrt{3}) = -\frac{2}{3}\pi$.

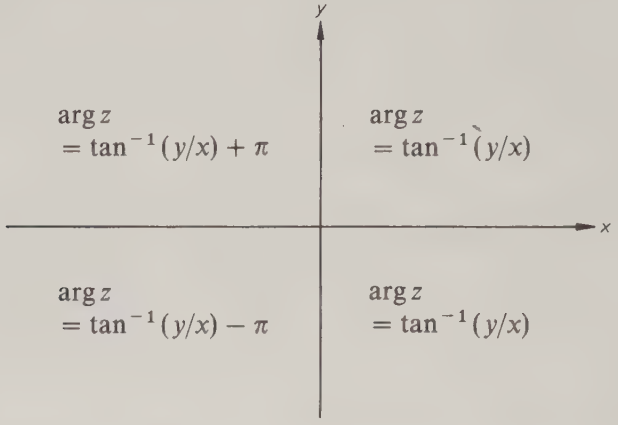
(c)



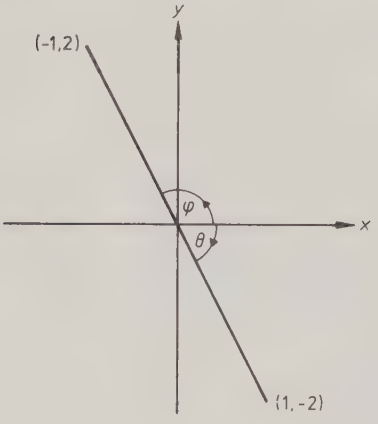
Since the point representing -5 lies on the negative part of the real axis, at a distance of 5 units from the origin, $|-5| = 5$ and $\arg(-5) = \pi$.

When obtaining the argument of a complex number by calculator, care must be taken to use the inverse tangent function properly. Since this function always takes

values between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$, it gives the correct value of arg z only when z is represented by a point in the first or fourth quadrants. When the point representing z lies in the second or third quadrants, it will be necessary to add $\pm\pi$ to the value obtained for $\tan^{-1}(y/x)$.



Example 2 Find in radians the principal arguments of the complex numbers $1 - 2i$ and $-1 + 2i$.



If $\arg(1-2i)=\theta$, then $\tan\theta=\dfrac{-2}{1}=-2$

Since the point $(1, -2)$ lies in the 4th quadrant,

$\arg(1 - 2i) = \tan^{-1}(-2) \approx -1{\cdot}107$

If $\arg(-1 + 2i) = \phi$, then

$$\tan\phi = \frac{2}{-1} = -2$$

Since the point $(-1, 2)$ lies in the 2nd quadrant,
$\arg(-1 + 2i) = \tan^{-1}(-2) + \pi \approx 2{\cdot}034.$

If a complex number $z = x + iy$ has modulus r and argument θ, then $x = r\cos\theta$ and $y = r\sin\theta$

$$\therefore \qquad z = r(\cos\theta + i\sin\theta)$$

This is called the *modulus-argument* form of the number.
Using the results of Example 1 the numbers $1 + i$, $-1 - i\sqrt{3}$ and -5 have expressions in modulus-argument form as follows:

$$1 + i = \sqrt{2}(\cos\tfrac{1}{4}\pi + i\sin\tfrac{1}{4}\pi),$$
$$-1 - i\sqrt{3} = 2\{\cos(-\tfrac{2}{3}\pi) + i\sin(-\tfrac{2}{3}\pi)\},$$
$$-5 = 5(\cos\pi + i\sin\pi).$$

Example 3 Write down the moduli and arguments of (a) $3(\cos 2\alpha + i\sin 2\alpha)$, (b) $\cos\frac{2}{5}\pi + i\sin\frac{2}{5}\pi$, (c) $a(\cos\phi - i\sin\phi)$.

(a) Since $r(\cos\theta + i\sin\theta)$ has modulus r and argument θ, $3(\cos 2\alpha + i\sin 2\alpha)$ has modulus 3 and argument 2α.
(b) Since $\cos\frac{2}{5}\pi + i\sin\frac{2}{5}\pi = 1(\cos\frac{2}{5}\pi + i\sin\frac{2}{5}\pi)$, this number has modulus 1 and argument $\frac{2}{5}\pi$.
(c) Since $a(\cos\phi - i\sin\phi) = a\{\cos(-\phi) + i\sin(-\phi)\}$, its modulus is a and its argument is $(-\phi)$.

We now investigate the general properties of complex numbers in modulus-argument form.
If $z = r(\cos\theta + i\sin\theta)$, then $z^* = r(\cos\theta - i\sin\theta)$.
$\therefore\quad zz^* = r^2(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = r^2(\cos^2\theta + \sin^2\theta) = r^2$.
Since z^* may also be written as $r\{\cos(-\theta) + i\sin(-\theta)\}$, the modulus of z^* is r and its argument is $(-\theta)$. It follows that:

$$zz^* = |z|^2, |z^*| = |z| \text{ and, for } -\pi < \arg z < \pi, \arg z^* = -\arg z.$$

If $z = r(\cos\theta + i\sin\theta)$, then $\dfrac{1}{z} = \dfrac{1}{r(\cos\theta + i\sin\theta)}$

$$\therefore\qquad \frac{1}{z} = \frac{\cos\theta - i\sin\theta}{r(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)}$$

$$= \frac{\cos\theta - i\sin\theta}{r(\cos^2\theta + \sin^2\theta)} = \frac{1}{r}\{\cos(-\theta) + i\sin(-\theta)\}$$

$$\therefore\qquad \left|\frac{1}{z}\right| = \frac{1}{|z|} \text{ and, for } -\pi < \arg z < \pi, \arg\left(\frac{1}{z}\right) = -\arg z.$$

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then

$$\begin{aligned} z_1z_2 &= r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1r_2\{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\} \\ &= r_1r_2\{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\} \end{aligned}$$

Hence z_1z_2 has modulus r_1r_2 and argument $\theta_1 + \theta_2$.

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{r_2(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)} \\ &= \frac{r_1\{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)\}}{r_2(\cos^2\theta_2 + \sin^2\theta_2)} \\ &= \frac{r_1}{r_2}\{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\} \end{aligned}$$

Hence z_1/z_2 has modulus r_1/r_2 and argument $\theta_1 - \theta_2$.

$\therefore$

$$\boxed{|z_1 z_2| = |z_1||z_2| \quad \text{and} \quad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}}$$

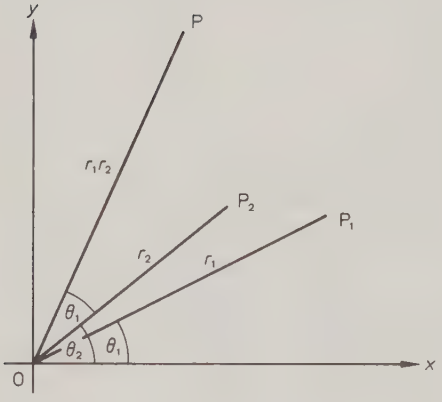
If $\theta_1 = \arg z_1$ and $\theta_2 = \arg z_2$, it follows that $z_1 z_2$ has argument

$$\theta_1 + \theta_2 = \arg z_1 + \arg z_2.$$

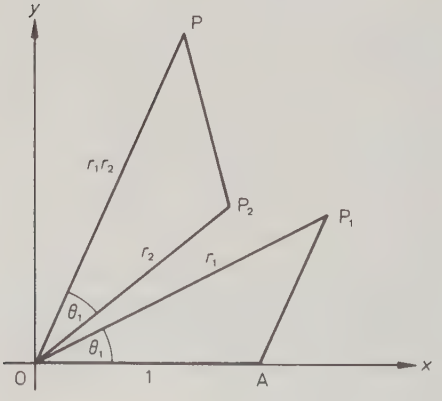
However, since $-\pi < \theta_1 \leqslant \pi$ and $-\pi < \theta_2 \leqslant \pi$, $\theta_1 + \theta_2$ may itself lie outside the interval from $-\pi$ to π. In such cases $\arg z_1 + \arg z_2$ does not give the principal argument of $z_1 z_2$ and it is necessary to add or subtract 2π to obtain $\arg(z_1 z_2)$. Similar adjustments may be needed when finding $\arg(z_1/z_2)$.

$\therefore$

$$\boxed{\begin{aligned} &\arg(z_2 z_2) = \arg z_1 + \arg z_2 \ (\pm 2\pi \text{ if necessary}), \\ &\arg(z_1/z_2) = \arg z_1 - \arg z_2 \ (\pm 2\pi \text{ if necessary}). \end{aligned}}$$



Let z_1 and z_2 be represented in an Argand diagram by the vectors $\overrightarrow{OP}_1$ and $\overrightarrow{OP}_2$ respectively. When z_2 is multiplied by z_1 the vector $\overrightarrow{OP}_2$ is rotated through an angle θ_1 and its length is multiplied by r_1 to produce the vector $\overrightarrow{OP}$ representing $z_1 z_2$.



In this diagram we see that if A is the point $(1, 0)$, then triangles OAP_1 and OP_2P are similar.

[There are similar geometrical representations of the quotient z_1/z_2.]

Example 4 If $z_1 = 6(\cos\frac{5}{12}\pi + i\sin\frac{5}{12}\pi)$ and $z_2 = 3(\cos\frac{1}{4}\pi + i\sin\frac{1}{4}\pi)$, find z_1z_2 and z_1/z_2 in the form $a + ib$.

$$z_1z_2 = 6 \times 3\{\cos(\tfrac{5}{12}\pi + \tfrac{1}{4}\pi) + i\sin(\tfrac{5}{12}\pi + \tfrac{1}{4}\pi)\}$$

$$= 18\{\cos\tfrac{2}{3}\pi + i\sin\tfrac{2}{3}\pi\} = 18\left\{-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right\} = -9 + 9\sqrt{3}i$$

$$\frac{z_1}{z_2} = \frac{6}{3}\{\cos(\tfrac{5}{12}\pi - \tfrac{1}{4}\pi) + i\sin(\tfrac{5}{12}\pi - \tfrac{1}{4}\pi)\}$$

$$= 2\{\cos\tfrac{1}{6}\pi + i\sin\tfrac{1}{6}\pi\} = 2\left\{\frac{\sqrt{3}}{2} + \frac{1}{2}i\right\} = \sqrt{3} + i.$$

Exercise 26.4

In questions 1 to 4 find the modulus and principal argument of each of the given complex numbers. Give the argument in radians, either as a multiple of π or correct to 3 decimal places.

1. (a) $\sqrt{3} + i$ (b) $1 - i$ (c) $4i$ (d) -3

2. (a) $3 - 4i$ (b) $-2 + i$ (c) $-1 - 3i$ (d) $5 - 3i$

3. (a) $\dfrac{10}{\sqrt{3} - i}$ (b) $\dfrac{1}{1 + i\sqrt{3}}$ (c) $\dfrac{2 + 3i}{5 + i}$ (d) $\dfrac{7 - i}{-4 - 3i}$

4. (a) $5(\cos\frac{1}{3}\pi + i\sin\frac{1}{3}\pi)$ (b) $\cos\frac{3}{2}\pi + i\sin\frac{3}{2}\pi$
   (c) $\cos\frac{2}{3}\pi - i\sin\frac{2}{3}\pi$ (d) $-2(\cos\frac{1}{4}\pi + i\sin\frac{1}{4}\pi)$

5. Express the following in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $-\pi < \theta \leqslant \pi$,
   (a) $\sqrt{3} - 3i$ (b) 8 (c) $-2 - 2i$ (d) $-i$

6. If $z = r(\cos\theta + i\sin\theta)$, where $r > 0$ and $0 < \theta < \frac{1}{2}\pi$, find in terms of r and θ the modulus and principal argument of
   (a) $-z$ (b) iz (c) z^2 (d) z/z^*

7. Simplify the following expressions:
   (a) $(\cos\frac{1}{4}\pi + i\sin\frac{1}{4}\pi)(\cos\frac{3}{4}\pi + i\sin\frac{3}{4}\pi)$ (b) $(\cos\frac{5}{12}\pi + i\sin\frac{5}{12}\pi)^2$
   (c) $(\cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi)(\cos\frac{7}{12}\pi + i\sin\frac{7}{12}\pi)$ (d) $(\cos\frac{2}{5}\pi + i\sin\frac{2}{5}\pi)^3$
   (e) $\dfrac{(\cos\frac{1}{3}\pi + i\sin\frac{1}{3}\pi)}{(\cos\frac{5}{6}\pi + i\sin\frac{5}{6}\pi)}$ (f) $\dfrac{(\cos\frac{1}{4}\pi + i\sin\frac{1}{4}\pi)^2}{(\cos\frac{1}{6}\pi + i\sin\frac{1}{6}\pi)}$

8. Find the moduli and principal arguments of w, z, wz and w/z, given that:
   (a) $w = 10i$, $z = 1 + i\sqrt{3}$, (b) $w = -2\sqrt{3} + 2i$, $z = 1 - i$.

9. Given that $z = 2/(1 + \cos 2\theta - i \sin 2\theta)$, where $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$, find expressions for $|z|$ and $\arg z$ in terms of θ.

10. Given that $z = \cos\theta + i\sin\theta$, where $0 < \theta < \pi$, find the moduli and principal arguments of $1 - z$, $z - 1$, $1/(1 - z)$, $z/(z - 1)$.

11. If $z_1 = 1 + i\sqrt{3}$ and $z_2 = 2i$, find $|z_1|$ and $\arg z_1$, $|z_2|$ and $\arg z_2$. Using an Argand diagram, deduce that $\arg(z_1 + z_2) = 5\pi/12$. Hence show that $\tan(5\pi/12) = 2 + \sqrt{3}$.

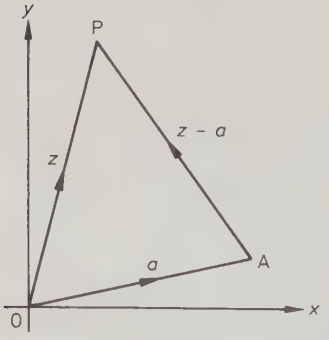
12. Two non-zero complex numbers z_1 and z_2 are such that $|z_1 + z_2| = |z_1 - z_2|$. Represent $z_1, z_2, z_1 + z_2$ and $z_1 - z_2$ by vectors on an Argand diagram. Hence, or otherwise, find the possible values of $\arg(z_1/z_2)$. **(JMB)**

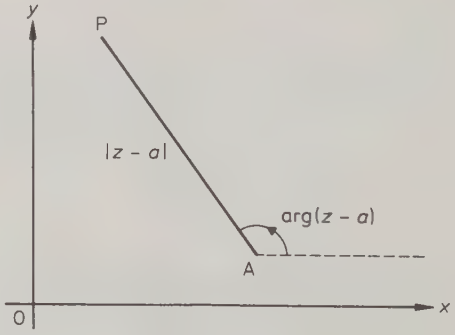
13. (a) Express the complex number $z = \dfrac{1 + i}{3 - 2i}$ in the form $a + bi$ $(a, b \in \mathbb{R})$, and find the argument of z, giving your answer in radians correct to 3 significiant figures. Write down the argument of z^2, and find the exact value of $|z^2|$.
(b) The conjugate of the non-zero complex number z is denoted by z^*. In an Argand diagram with origin O, the point P represents z and Q represents $1/z^*$. Prove that O, P and Q are collinear, and find the ratio $OP{:}OQ$ in terms of $|z|$. **(C)**

14. Show that, for any complex number z, (a) $zz^* = |z|^2$, (b) $z + z^* = 2\,\mathrm{Re}(z)$, (c) $\mathrm{Re}(z) \leqslant |z|$. Hence show that
(i) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1 z_2^*)$,
(ii) $|z_1 + z_2| \leqslant |z_1| + |z_2|$.

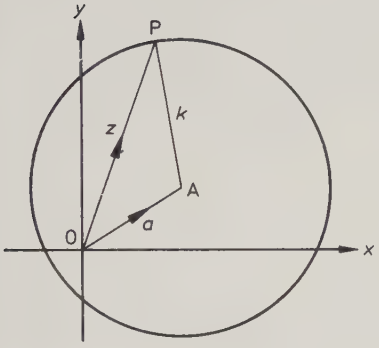
26.5 Loci in the Argand diagram

Let us consider again a complex number $z = x + iy$ represented by the vector $\overrightarrow{OP}$, where P is the point (x, y). If z varies subject to some given condition, then the corresponding set of points in the Argand diagram is called the *locus* of P. Since many loci can be defined using either distances or angles, equations of loci in the Argand diagram often involve moduli or arguments of complex variables.



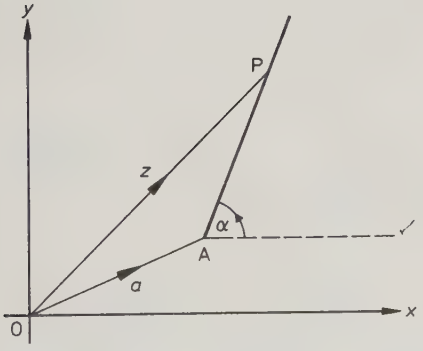


If a is constant complex number represented by a fixed vector $\overrightarrow{OA}$, then $z - a$ is represented by $\overrightarrow{AP}$. Hence $|z - a|$ is represented by the distance AP and $\arg(z - a)$ is the angle $\overrightarrow{AP}$ makes the positive direction of the real axis.



It follows that if $|z - a| = k$, where k is a positive constant, then the distance of P from A is constant. Thus

> $|z - a| = k$ is the equation of the circle with centre at the point A and radius k.



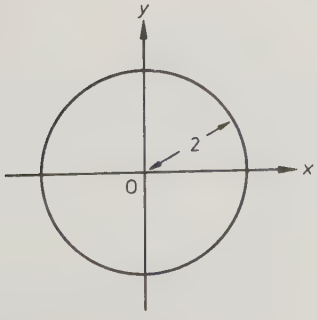
However, if $\arg(z - a) = \alpha$, where α is constant and $-\pi < \alpha \leqslant \pi$, then the direction of $\overrightarrow{AP}$ is constant. Thus

> $\arg(z - a) = \alpha$ is the equation of a *half-line* with end-point A, inclined at an angle α to the real axis.

Example 1 Sketch the loci defined by the equations (a) $|z| = 2$, (b) $|z - 3 + 2i| = 5$.

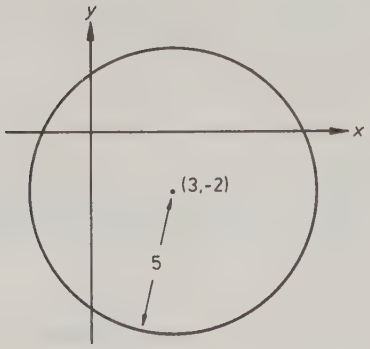
Let z be represented in the Argand diagram by the point P.

(a)



If $|z| = 2$, then the distance of P from the origin is 2 units.
Hence the locus of P is a circle with centre the origin and radius 2.

(b)

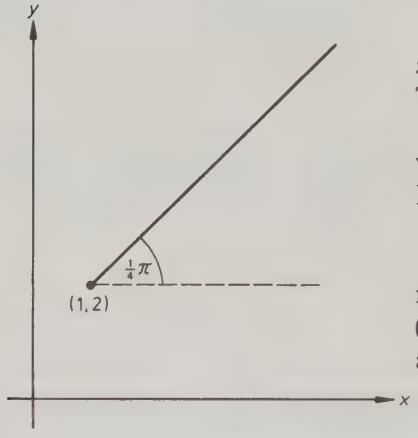


If $|z - 3 + 2i| = 5$,
then $|z - (3 - 2i)| = 5$,
i.e. the distance of P from the point representing $3 - 2i$ is 5 units.
Hence the locus of P is a circle with centre $(3, -2)$ and radius 5.

Example 2 Sketch the loci defined by the equations (a) $\arg(z - 1 - 2i) = \frac{1}{4}\pi$, (b) $\arg(z + 2) = -\frac{2}{3}\pi$.

Let z be represented in the Argand diagram by the point P.

(a)

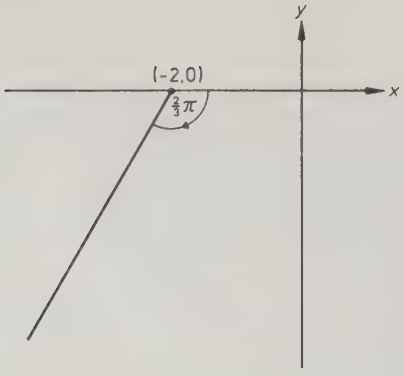


$z - 1 - 2i = z - (1 + 2i)$
Thus, if A is the point representing $1 + 2i$, $\arg(z - 1 - 2i)$ is the angle which $\overrightarrow{AP}$ makes with the real axis.
Hence the equation

$$\arg(z - 1 - 2i) = \tfrac{1}{4}\pi$$

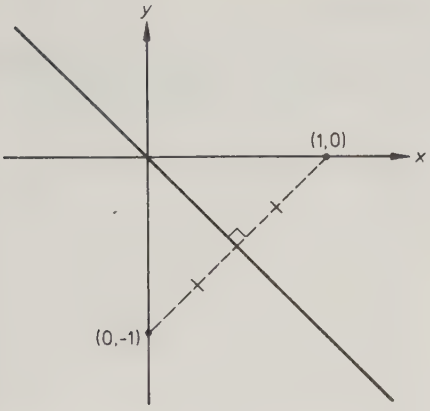
represents the half-line with end-point $(1, 2)$, inclined at an angle $\frac{1}{4}\pi$ to the real axis.

(b)



Using similar arguments the equation $\arg(z + 2) = -\frac{2}{3}\pi$ represents the half-line with end-point $(-2, 0)$, inclined to the real axis at an angle $\frac{2}{3}\pi$, measured clockwise.

Example 3 Sketch the locus of the point $P(x, y)$ representing the complex number $z = x + iy$, given that $|z - 1| = |z + i|$. Write down the Cartesian equation of the locus.



$|z - 1|$ and $|z + i|$ are the distances between the point P and the points representing 1 and $-i$ respectively. Hence the equation

$$|z - 1| = |z + i|$$

represents the locus of points equidistant from $(1, 0)$ and $(0, -1)$ i.e. the perpendicular bisector of the line joining the points $(1, 0)$ and $(0, -1)$. From the sketch it is clear that the Cartesian equation of the locus is $y = -x$.

Example 4 Find the Cartesian equation of the locus of the point $P(x, y)$ representing the complex number z, given that $2|z - 3i| = |z|$. Show that the locus is a circle, giving its centre and radius.

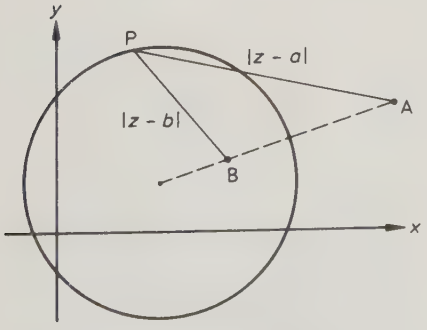
$$\begin{aligned} 2|z - 3i| = |z| &\Rightarrow 2|x + iy - 3i| = |x + iy| \\ &\Rightarrow 2|x + i(y - 3)| = |x + iy| \\ &\Rightarrow 4\{x^2 + (y - 3)^2\} = x^2 + y^2 \\ &\Rightarrow 4x^2 + 4y^2 - 24y + 36 = x^2 + y^2 \\ &\Rightarrow 3x^2 + 3y^2 - 24y + 36 = 0 \end{aligned}$$

Hence the Cartesian equation of the locus is

$$x^2 + y^2 - 8y + 12 = 0$$

Rearranging: $x^2 + (y - 4)^2 = 4$

This equation represents a circle with centre $(0, 4)$ and radius 2.



In general, if a point P moves so that the ratio of its distances from two fixed points A and B is constant, then the locus of P is a circle. This locus is referred to as Apollonius' circle and is represented in the Argand diagram by an equation of the form

$$|z - a| = k|z - b| \quad \text{or} \quad \left|\frac{z - a}{z - b}\right| = k, \quad (k \neq 1).$$

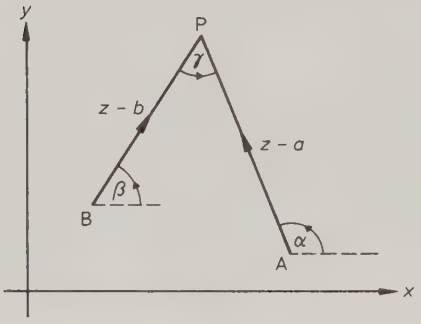
As shown in Example 3, when $k = 1$, the equation represents the perpendicular bisector of the line joining A to B.

Equations involving $\arg\left(\frac{z-a}{z-b}\right)$ are more difficult to interpret.

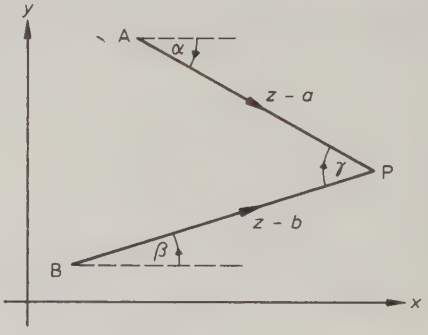
If $\arg(z-a)=\alpha$, $\arg(z-b)=\beta$ and $\arg\left(\frac{z-a}{z-b}\right)=\gamma$,

then $$\gamma=\alpha-\beta(\pm 2\pi \text{ if necessary}).$$

Thus γ is the angle which the vector $\overrightarrow{AP}$ makes with the vector $\overrightarrow{BP}$. If the turn from $\overrightarrow{BP}$ to $\overrightarrow{AP}$ is anti-clockwise then γ is positive; if the turn is clockwise then γ is negative.



$$\arg\left(\frac{z-a}{z-b}\right)>0$$

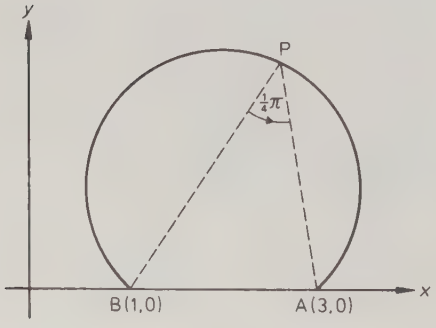


$$\arg\left(\frac{z-a}{z-b}\right)<0$$

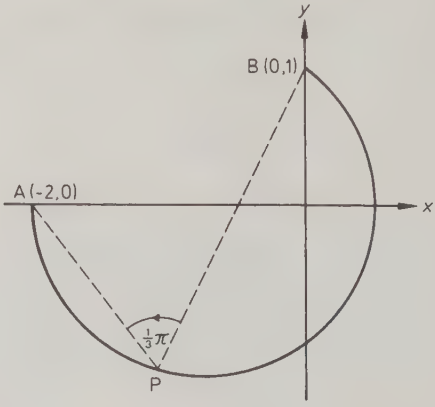
It follows that the equation $\arg\left(\frac{z-a}{z-b}\right)=\lambda$, where λ is constant, represents a circular arc with end-points A and B.

For instance, if $\arg\left(\frac{z-3}{z-1}\right)=\frac{1}{4}\pi$, then the locus of P is a circular arc with end-points $A(3,0)$ and $B(1,0)$, such that $\angle APB=\frac{1}{4}\pi$.

Similarly, if $\arg\left(\frac{z+2}{z-i}\right)=\frac{1}{3}\pi$, then the locus of P is a circular arc with end-points $A(-2,0)$ and $B(0,1)$, such that $\angle APB=\frac{1}{3}\pi$. Since in both cases the given arguments are positive, the arcs must be drawn so that the turn from $\overrightarrow{BP}$ to $\overrightarrow{AP}$ is anti-clockwise.



$$\arg\left(\frac{z-3}{z-1}\right)=\tfrac{1}{4}\pi$$



$$\arg\left(\frac{z+2}{z-i}\right)=\tfrac{1}{3}\pi$$

We use a worked example to illustrate the use of equations involving parameters to represent loci in the Argand diagram.

Example 5 If z is represented in an Argand diagram by the point P and $z - 6i = \lambda iz$, where λ is real, find the Cartesian equation of the locus of P as λ varies.

Let $z = x + iy$, then the equation of the locus becomes

$$x + iy - 6i = \lambda i(x + iy)$$

i.e. $$x + i(y - 6) = -\lambda y + i\lambda x$$

Equating real and imaginary parts,

$$x = -\lambda y, \qquad y - 6 = \lambda x$$

$\therefore$ provided that $x \neq 0$ and $y \neq 0$, $$\frac{x}{-y} = \frac{y - 6}{x} = \lambda$$

i.e. $$x^2 = -y^2 + 6y$$

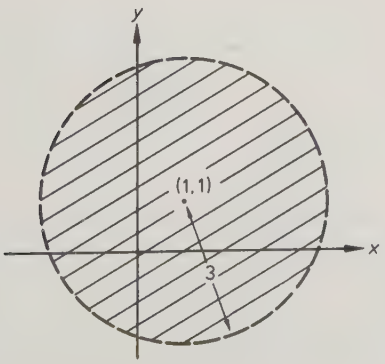
Hence the Cartesian equation of the locus of P is $x^2 + y^2 - 6y = 0$, which represents a circle with centre $(0, 3)$ and radius 3.

[Investigating the excluded values $x = 0, y = 0$, we find that there is no finite value of λ such that $z = 0$ satisfies the equation $z - 6i = \lambda iz$. This means that strictly speaking the origin $(0, 0)$ is not part of the locus.]

Inequalities involving the modulus or argument of a complex variable can be used to represent regions of an Argand diagram.

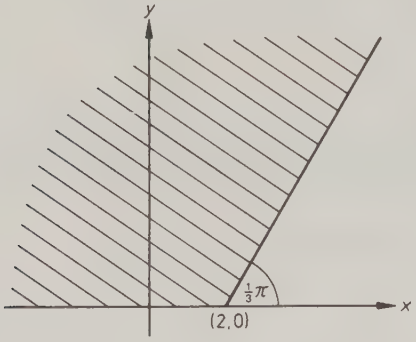
Example 6 Shade in separate Argand diagrams the regions in which
(a) $|z - 1 - i| < 3$, (b) $\frac{1}{3}\pi \leqslant \arg(z - 2) \leqslant \pi$.

(a)



Since $z - 1 - i = z - (1 + i)$, the equation $|z - 1 - i| = 3$ represents a circle with centre $(1, 1)$ and radius 3. Hence the inequality $|z - 1 - i| < 3$ represents the interior of this circle, but not the circle itself.

(b)



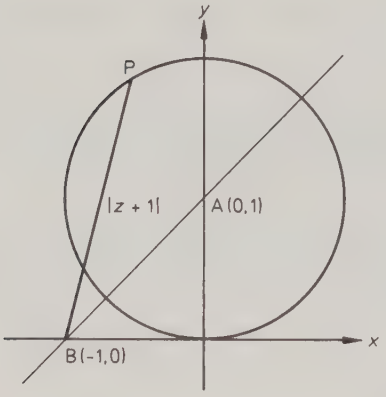
The equations $\arg(z-2)=\frac{1}{3}\pi$ and $\arg(z-2)=\pi$ represent half-lines with end-point (2,0). Hence the inequality

$$\tfrac{1}{3}\pi \leqslant \arg(z-2) \leqslant \pi$$

represents these two lines and the region between them.

A geometrical approach is sometimes helpful when solving problems.

Example 7 Given that z is a complex number which varies such that $|z-i|=1$, find the greatest and least values of $|z+1|$.



Let z be represented in an Argand diagram by the point P. Given that $|z-i|=1$, the locus of P is a circle with centre $A(0,1)$ and radius 1. Since $|z+1|$ is the distance of P from the point $B(-1,0)$, its greatest and least values occur when P lies on the straight line through A and B. As $AB=\sqrt{2}$ and the radius of the circle is 1, these greatest and least values must be $\sqrt{2}+1$ and $\sqrt{2}-1$.

Exercise 26.5

In questions 1 to 11 sketch in separate Argand diagrams the loci defined by the given equations.

1. (a) $|z|=5$, (b) $|z-2|=3$.

2. (a) $|z-3i|=2$, (b) $|z+4|=4$.

3. (a) $|z-1+i|=1$, (b) $|z-2-3i|=4$.

4. (a) $|z+1+2i|=3$, (b) $|z+3-4i|=5$.

5. (a) $\arg z = \frac{1}{4}\pi$, (b) $\arg(z - i) = \frac{1}{3}\pi$.

6. (a) $\arg(z + 1 - 3i) = -\frac{1}{6}\pi$, (b) $\arg(z - 3 + 2i) = \pi$.

7. (a) $\arg(z + 2 + i) = \frac{1}{2}\pi$, (b) $\arg(z - 1 - i) = -\frac{3}{4}\pi$.

8. (a) $|z + 1| = |z - 3|$, (b) $|z| = |z - 6i|$.

9. (a) $\left|\frac{z - i}{z - 1}\right| = 1$, (b) $\left|\frac{z - 2 - 3i}{z + 2 + i}\right| = 1$.

10. (a) $\arg\left(\frac{z - 1}{z + 1}\right) = \frac{1}{3}\pi$, (b) $\arg\left(\frac{z + 1}{z - 1}\right) = \frac{1}{3}\pi$.

11. (a) $\arg\left(\frac{z - 3}{z - 2i}\right) = \frac{1}{4}\pi$, (b) $\arg\left(\frac{z}{z - 4 + 2i}\right) = \frac{1}{2}\pi$.

In questions 12 to 24 find the Cartesian equation of the locus of the point P representing the complex number z. Sketch the locus of P in each case.

12. $2|z + 1| = |z - 2|$

13. $|z + 4i| = 3|z - 4|$

14. $\left|\frac{z}{z - 4}\right| = 5$

15. $\left|\frac{z + i}{z - 5 - 2i}\right| = 1$

16. $\left|\frac{z}{z + 6}\right| = 5$

17. $\left|\frac{z - 1}{z + 1 - i}\right| = \frac{2}{3}$

18. $z - 5 = \lambda i(z + 5)$, where λ is a real parameter.

19. $\frac{z + 2i}{z - 2} = \lambda i$, where λ is a real parameter.

20. $z = 3i + \lambda(2 + 5i)$, where λ is a real parameter.

21. $\mathrm{Im}(z^2) = 2$

22. $\mathrm{Re}(z^2) = 1$

23. $\mathrm{Re}\left(z - \frac{1}{z}\right) = 0$

24. $\mathrm{Im}\left(z + \frac{9}{z}\right) = 0$.

In questions 25 to 34 shade in separate Argand diagrams the regions represented by the given inequalities. Indicate in each case whether the boundaries are to be included in the region.

25. $|z| > 5$

26. $1 \leqslant |z| \leqslant 2$

27. $|z - i| \leqslant 3$

28. $|z - 4 + 3i| < 4$

29. $0 \leqslant \arg z \leqslant \frac{1}{3}\pi$

30. $\frac{1}{4}\pi < \arg z < \frac{3}{4}\pi$

31. $-\frac{1}{6}\pi < \arg(z - 1) < \frac{1}{6}\pi$

32. $-\frac{1}{2}\pi \leqslant \arg(z + i) \leqslant \frac{2}{3}\pi$

33. $|z| > |z + 2|$

34. $|z + i| \leqslant |z - 3i|$

35. Represent each of the following loci in an Argand diagram.
(a) $\arg(z - 1) = \arg(z + 1)$, (b) $\arg z = \arg(z - 1 - i)$,
(c) $\arg(z - 2) = \pi + \arg z$, (d) $\arg(z - 1) = \pi + \arg(z - i)$.

36. Find the least value of $|z + 4|$ for which
(a) $\mathrm{Re}(z) = 5$, (b) $\mathrm{Im}(z) = 3$, (c) $|z| = 1$, (d) $\arg z = \frac{1}{4}\pi$.

37. Given that the complex number z varies such that $|z - 7| = 3$, find the greatest and least values of $|z - i|$.

38. Given that the complex numbers w and z vary subject to the conditions $|w - 12| = 7$ and $|z - 5i| = 4$, find the greatest and least values of $|w - z|$.

39. In an Argand diagram, the point P represents the complex number z, where $z = x + iy$. Given that $z + 2 = \lambda i(z + 8)$, where λ is a real parameter, find the Cartesian equation of the locus of P as λ varies. If also $z = \mu(4 + 3i)$, where μ is real, prove that there is only one possible position for P. (JMB)

40. (i) Represent on the same Argand diagram the loci given by the equations $|z - 3| = 3$ and $|z| = |z - 2|$. Obtain the complex numbers corresponding to the points of intersection of these loci. (ii) Find a complex number z whose argument is $\pi/4$ and which satisfies the equation $|z + 2 + i| = |z - 4 + i|$. (L)

26.6 De Moivre's theorem for an integral index

In §26.4 it was shown that

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2).$$

Writing $\theta_1 = \theta_2 = \theta$, we find that

$$(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$$
$$\therefore \quad (\cos\theta + i\sin\theta)^3 = (\cos 2\theta + i\sin 2\theta)(\cos\theta + i\sin\theta)$$
$$\text{Thus} \quad (\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$$

These results suggest a general formula,

$$\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta}$$

This statement, known as *De Moivre's*† *theorem*, may be proved for positive integral values of n by the method of induction.
Assuming that the theorem holds for $n = k$,

$$(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$$

$$\therefore \quad (\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)$$
$$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$$
$$= \cos(k\theta + \theta) + i\sin(k\theta + \theta)$$
$$= \cos(k+1)\theta + i\sin(k+1)\theta$$

$\therefore$ if the theorem holds for $n = k$, it also holds for $n = k + 1$.

When $n = 1$, $\quad (\cos\theta + i\sin\theta)^n = \cos\theta + i\sin\theta$
and $\quad \cos n\theta + i\sin n\theta = \cos\theta + i\sin\theta$

$\therefore$ the theorem holds for $n = 1$.
Hence, by induction, De Moivre's theorem holds when n is a positive integer.

Example 1 Find the value of $(\cos\frac{1}{4}\pi + i\sin\frac{1}{4}\pi)^{12}$.

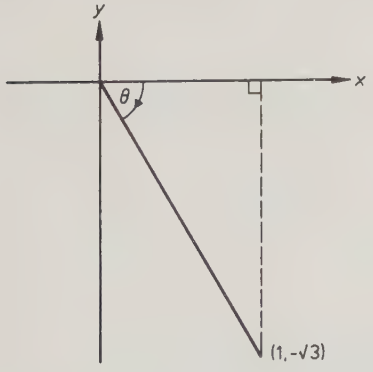
By De Moivre's theorem,

$$(\cos\tfrac{1}{4}\pi + i\sin\tfrac{1}{4}\pi)^{12} = \cos(12 \times \tfrac{1}{4}\pi) + i\sin(12 \times \tfrac{1}{4}\pi)$$
$$= \cos 3\pi + i\sin 3\pi = -1.$$

Example 2 Express $(1 - i\sqrt{3})^4$ in the form $a + ib$.

$$|1 - i\sqrt{3}| = \sqrt{\{1^2 + (-\sqrt{3})^2\}} = \sqrt{\{1 + 3\}} = 2$$

If $\arg(1 - i\sqrt{3}) = \theta$, then $\tan\theta = \dfrac{-\sqrt{3}}{1} = -\sqrt{3}$



Since the point representing $1 - i\sqrt{3}$ lies in the fourth quadrant, $\arg(1 - i\sqrt{3}) = -\frac{1}{3}\pi$.

$\therefore \quad 1 - i\sqrt{3} = 2\{\cos(-\frac{1}{3}\pi) + i\sin(-\frac{1}{3}\pi)\}$

By De Moivre's theorem,

$$(1 - i\sqrt{3})^4 = 2^4\{\cos(-\tfrac{1}{3}\pi) + i\sin(-\tfrac{1}{3}\pi)\}^4$$
$$= 16\{\cos(-\tfrac{4}{3}\pi) + i\sin(-\tfrac{4}{3}\pi)\} = 16\left\{-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right\}$$

† *De Moivre, Abraham* (1667–1754) French mathematician of Huguenot extraction who settled in England, and was a friend of Newton. He published his theorem in *Miscellanea Analytica* in 1730.

Hence $(1 - i\sqrt{3})^4 = -8 + 8\sqrt{3}i$.

To prove De Moivre's theorem when n is a negative integer, let $n = -m$ where m is a positive integer, then

$$\begin{aligned}(\cos\theta + i\sin\theta)^n &= (\cos\theta + i\sin\theta)^{-m} \\ &= \frac{1}{(\cos\theta + i\sin\theta)^m}\end{aligned}$$

Applying De Moivre's theorem for positive integral index,

$$\begin{aligned}(\cos\theta + i\sin\theta)^n &= \frac{1}{\cos m\theta + i\sin m\theta} \\ &= \cos m\theta - i\sin m\theta \\ &= \cos(-m\theta) + i\sin(-m\theta) \\ &= \cos n\theta + i\sin n\theta\end{aligned}$$

Hence De Moivre's theorem holds for all integral values of n.

Example 3 Evaluate $\dfrac{1}{(1 - i\sqrt{3})^3}$.

As shown in Example 2, $1 - i\sqrt{3} = 2\{\cos(-\frac{1}{3}\pi) + i\sin(-\frac{1}{3}\pi)\}$
By De Moivre's theorem,

$$\begin{aligned}(1 - i\sqrt{3})^{-3} &= 2^{-3}\{\cos(-\tfrac{1}{3}\pi) + i\sin(-\tfrac{1}{3}\pi)\}^{-3} \\ &= \tfrac{1}{8}\{\cos[(-3)(-\tfrac{1}{3}\pi)] + i\sin[(-3)(-\tfrac{1}{3}\pi)]\} \\ &= \tfrac{1}{8}\{\cos\pi + i\sin\pi\}\end{aligned}$$

Hence $\dfrac{1}{(1 - i\sqrt{3})^3} = -\frac{1}{8}$

De Moivre's theorem may be used to obtain certain types of trigonometric identity.

Example 4 Express $\cos 4\theta$ in terms of $\cos\theta$ and $\tan 4\theta$ in terms of $\tan\theta$.

By De Moivre's theorem, $\cos 4\theta + i\sin 4\theta = (\cos\theta + i\sin\theta)^4$

Using the binomial expansion,

$$\begin{aligned}\cos 4\theta + i\sin 4\theta &= \cos^4\theta + 4\cos^3\theta(i\sin\theta) \\ &\quad + 6\cos^2\theta(i\sin\theta)^2 + 4\cos\theta(i\sin\theta)^3 + (i\sin\theta)^4 \\ &= (\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta) \\ &\quad + i(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)\end{aligned}$$

Equating real and imaginary parts,

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta \qquad (1)$$
$$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta \qquad (2)$$

From (1): $\cos 4\theta = \cos^4\theta - 6\cos^2\theta(1 - \cos^2\theta) + (1 - \cos^2\theta)^2$
$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$

$\therefore \quad \cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$

Using (1) and (2): $\tan 4\theta = \dfrac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}$

Dividing numerator and denominator by $\cos^4\theta$,

$$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$

Expressions for powers of $\sin\theta$ and $\cos\theta$ in terms of sines and cosines of multiples of θ can be derived using the following results.

If $z = \cos\theta + i\sin\theta$, then $\dfrac{1}{z} = \cos\theta - i\sin\theta$

$\therefore \quad z + \dfrac{1}{z} = 2\cos\theta \quad$ and $\quad z - \dfrac{1}{z} = 2i\sin\theta$

By De Moivre's theorem,

$$z^n = \cos n\theta + i\sin n\theta, \quad \text{so that} \quad \frac{1}{z^n} = \cos n\theta - i\sin n\theta$$

$$\boxed{z^n + \frac{1}{z^n} = 2\cos n\theta \quad \text{and} \quad z^n - \frac{1}{z^n} = 2i\sin n\theta}$$

Example 5 Express $\cos^4\theta$ in terms of cosines of multiples of θ.

$$\left(z + \frac{1}{z}\right)^4 = z^4 + 4z^3.\frac{1}{z} + 6z^2.\frac{1}{z^2} + 4z.\frac{1}{z^3} + \frac{1}{z^4}$$

$$= z^4 + 4z^2 + 6 + \frac{4}{z^2} + \frac{1}{z^4}$$

$$= \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6$$

If $z = \cos\theta + i\sin\theta$, $z^n + \dfrac{1}{z^n} = 2\cos n\theta$

$\therefore \quad (2\cos\theta)^4 = 2\cos 4\theta + 4 \times 2\cos 2\theta + 6$

i.e. $\quad 16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$

Hence $\quad \cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$

Exercise 26.6

1. Use De Moivre's theorem to simplify the following:

(a) $\left(\cos\dfrac{\pi}{5} + i\sin\dfrac{\pi}{5}\right)^{10}$ (b) $\left(\cos\dfrac{\pi}{8} + i\sin\dfrac{\pi}{8}\right)^{12}$

(c) $\left(\cos\frac{\pi}{9}+i\sin\frac{\pi}{9}\right)^{-3}$ (d) $\left\{\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right\}^{-4}$

2. Express $\sqrt{3}+i$ in modulus-argument form. Hence find $(\sqrt{3}+i)^{10}$ and $1/(\sqrt{3}+i)^7$ in the form $a+ib$.

3. Express $-1+i$ in modulus-argument form. Hence show that $(-1+i)^{16}$ is real and that $1/(-1+i)^6$ is purely imaginary, giving the value of each.

4. Simplify the following expressions:

(a) $\dfrac{\left(\cos\frac{2\pi}{7}-i\sin\frac{2\pi}{7}\right)^3}{\left(\cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}\right)^4}$ (b) $\dfrac{\left(\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}\right)^8}{\left(\cos\frac{3\pi}{5}-i\sin\frac{3\pi}{5}\right)^3}$

5. Find expressions for $\cos 3\theta$ in terms of $\cos\theta$, $\sin 3\theta$ in terms of $\sin\theta$ and $\tan 3\theta$ in terms of $\tan\theta$.

6. Express $\sin 5\theta$ and $\cos 5\theta/\cos\theta$ in terms of $\sin\theta$.

7. Prove that $\tan 5\theta = \dfrac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$. By considering the equation $\tan 5\theta = 0$, show that $\tan^2(\pi/5) = 5 - 2\sqrt{5}$.

8. Find expressions for $\cos 6\theta$ and $\sin 6\theta/\sin\theta$ in terms of $\cos\theta$ and for $\tan 6\theta$ in terms of $\tan\theta$.

9. Express in terms of cosines of multiples of θ:
(a) $\cos^5\theta$ (b) $\cos^7\theta$ (c) $\sin^2\theta\cos^3\theta$

10. Express in terms of sines of multiples of θ:
(a) $\sin^3\theta$ (b) $\sin^7\theta$ (c) $\cos^4\theta\sin^3\theta$

11. Prove that $\cos^6\theta + \sin^6\theta = \frac{1}{8}(3\cos 4\theta + 5)$.

12. Evaluate (a) $\displaystyle\int_0^{\pi}\sin^4\theta\,d\theta$, (b) $\displaystyle\int_0^{\pi/2}\cos^4\theta\sin^2\theta\,d\theta$.

Exercise 26.7 (miscellaneous)

1. Let $z = x + iy$ be any non-zero complex number. Express $\dfrac{1}{z}$ in the form $u + iv$. Given that $z + \dfrac{1}{z} = k$ with k *real*, prove that either $y = 0$ or $x^2 + y^2 = 1$. Show (i) that if $y = 0$ then $|k| \geqslant 2$, (ii) that if $x^2 + y^2 = 1$ then $|k| \leqslant 2$. **(JMB)**

2. In the quadratic equation $x^2 + (p + iq)x + 3i = 0$, p and q are real. Given that the sum of the squares of the roots is 8, find all possible pairs of values of p and q. (JMB)

3. Express the complex number $z_1 = \dfrac{11 + 2i}{3 - 4i}$ in the form $x + iy$ where x and y are real. Given that $z_2 = 2 - 5i$, find the distance between the points in the Argand diagram which represent z_1 and z_2. Determine the real numbers α, β such that $\alpha z_1 + \beta z_2 = -4 + i$. (JMB)

4. (i) Find two complex numbers z satisfying the equation $z^2 = -8 - 6i$.
   (ii) Solve the equation $z^2 - (3 - i)z + 4 = 0$ and represent the solutions on an Argand diagram by vectors $\vec{OA}$ and $\vec{OB}$, where O is the origin. Show that triangle OAB is right-angled.

5. If z and w are complex numbers, show that

$$|z - w|^2 + |z + w|^2 = 2\{|z|^2 + |w|^2\}.$$

Interpret your results geometrically. (AEB 1977)

6. A regular octagon is inscribed in the circle $|z| = 1$ in the complex plane and one of its vertices represents the number $\dfrac{1}{\sqrt{2}}(1 + i)$. Find the numbers represented by the other vertices. (AEB 1977)

7. (i) Two complex numbers z_1 and z_2 each have arguments between 0 and π. If $z_1 z_2 = i - \sqrt{3}$ and $z_1/z_2 = 2i$ find the values of z_1 and z_2 giving the modulus and argument of each.
   (ii) Obtain in the form $a + ib$ the solutions of the equation $z^2 - 2z + 5 = 0$, and represent the solutions on an Argand diagram by the points A and B.

The equation $z^2 - 2pz + q = 0$ is such that p and q are real, and its solutions in the Argand diagram are represented by the points C and D. Find in the simplest form the algebraic relation satisfied by p and q in each of the following cases:
(a) $p^2 < q$, $p \neq 1$ and A, B, C, D are the vertices of a rectangle;
(b) $p^2 > q$ and $\angle CAD = \frac{1}{2}\pi$. (O&C)

8. (a) If $-\pi < \arg z_1 + \arg z_2 \leqslant \pi$ show that $\arg(z_1 z_2) = \arg z_1 + \arg z_2$. The complex numbers $a = 4\sqrt{3} + 2i$ and $b = \sqrt{3} + 7i$ are represented in the Argand diagram by points A and B respectively. O is the origin. Show that triangle OAB is equilateral and find the complex number c which the point C represents where $OABC$ is a rhombus. Calculate $|c|$ and $\arg c$. (You may leave answers in surd form.)

   (b) z is a complex number such that $z = \dfrac{p}{2 - i} + \dfrac{q}{1 + 3i}$ where p and q are real. If $\arg z = \pi/2$ and $|z| = 7$ find the values of p and q. (SU)

9. (i) Given that $z_1 = 3 + 4i$ and $z_2 = -1 + 2i$, represent z_1, z_2, $(z_1 + z_2)$ and $(z_2 - z_1)$ by vectors in the Argand diagram. Express $(z_1 + z_2)/(z_2 - z_1)$ in the form $a + ib$, where a and b are real. Find the magnitude of the angle between the vectors representing $(z_1 + z_2)$ and $(z_2 - z_1)$.
   (ii) One root of the equation $z^3 - 6z^2 + 13z + k = 0$, where k is real, is $z = 2 + i$. Find the other roots and the value of k. (L)

10. (a) Show that $(1 + 3i)^3 = -(26 + 18i)$.
    (b) Find the three roots z_1, z_2, z_3 of the equation $z^3 = -1$.
    (c) Find, in the form $a + ib$, the three roots z'_1, z'_2, z'_3 of the equation $z^3 = 26 + 18i$.
    (d) Indicate in the same Argand diagram the points represented by z_r and z'_r for $r = 1, 2, 3$, and prove that the roots of the equations may be paired so that $|z_1 - z'_1| = |z_2 - z'_2| = |z_3 - z'_3| = 3$. (O&C)

11. Write down, or obtain, the non-real cube roots of unity, ω_1 and ω_2, in the form $a + ib$, where a and b are real. A regular hexagon is drawn in an Argand diagram such that two adjacent vertices represent ω_1 and ω_2, respectively, and the centre of the circumscribing circle of the hexagon is the point $(1, 0)$. Determine, in the form $a + ib$, the complex numbers represented by the other four vertices of the hexagon and find the product of these four complex numbers. (JMB)

12. A complex number ω is such that $\omega^3 = 1$ and $\omega \neq 1$. Show that

(i) $\omega^2 + \omega + 1 = 0$, (ii) $(x + a + b)(x + \omega a + \omega^2 b)(x + \omega^2 a + \omega b)$

is real for real x, a and b, and simplify this product. Hence, or otherwise, find the three roots of the equation $x^3 - 6x + 6 = 0$, giving your answers in terms of ω, ω^2 and cube roots of integers. (JMB)

13. (i) Find, without the use of tables, the two square roots of $5 - 12i$ in the form $x + iy$, where x and y are real.
    (ii) Represent on an Argand diagram the loci $|z - 2| = 2$ and $|z - 4| = 2$. Calculate the complex numbers corresponding to the points of intersection of these loci. (L)

14. (i) Given that $(1 + 5i)p - 2q = 3 + 7i$, find p and q when (a) p and q are real, (b) p and q are conjugate complex numbers.
    (ii) Shade on the Argand diagram the region for which $3\pi/4 < \arg z < \pi$ and $0 < |z| < 1$. Choose a point in the region and label it A. If A represents the complex number z, label clearly the points B, C, D and E which represent $-z$, iz, $z + 1$ and z^2 respectively. (L)

15. (i) Show that $z = 1 + i$ is a root of the equation $z^4 + 3z^2 - 6z + 10 = 0$. Find the other roots of this equation.
    (ii) Sketch the curve in the Argand diagram defined by $|z - 1| = 1$, $\operatorname{Im} z \geqslant 0$. Find the value of z at the point P in which this curve is cut by the line $|z - 1| = |z - 2|$. Find also the value of $\arg z$ and $\arg(z - 2)$ at P. (L)

16. (i) If $z = 1 + i\sqrt{3}$, find $|z|$ and $|z^5|$, and also the values of $\arg z$ and $\arg(z^5)$ lying between $-\pi$ and π. Show that $\mathrm{Re}(z^5) = 16$ and find the value of $\mathrm{Im}(z^5)$.

(ii) Draw the line $|z| = |z - 4|$ and the half-line $\arg(z - i) = \pi/4$ in the Argand diagram. Hence find the complex number that satisfies both equations. (L)

17. (i) Without using tables, simplify $\dfrac{\left(\cos\dfrac{\pi}{9} + i\sin\dfrac{\pi}{9}\right)^4}{\left(\cos\dfrac{\pi}{9} - i\sin\dfrac{\pi}{9}\right)^5}$.

(ii) Express $z_1 = \dfrac{7 + 4i}{3 - 2i}$ in the form $p + qi$, where p and q are real.

Sketch in an Argand diagram the locus of points representing complex numbers z such that $|z - z_1| = \sqrt{5}$. Find the greatest value of $|z|$ subject to this condition. (L)

18. (i) Given that $z = 1 - i$, find the values of $r(>0)$ and θ, $-\pi < \theta \leqslant \pi$, such that $z = r(\cos\theta + i\sin\theta)$. Hence, or otherwise, find $1/z$ and z^6, expressing your answers in the form $p + iq$, where $p, q \in \mathbb{R}$.

(ii) Sketch on an Argand diagram the set of points corresponding to the set A, where $A = \{z : z \in \mathbb{C},\ \arg(z - i) = \pi/4\}$. Show that the set of points corresponding to the set B, where $B = \{z : z \in \mathbb{C},\ |z + 7i| = 2|z - 1|\}$, forms a circle in the Argand diagram. If the centre of this circle represents the numbers z_1, show that $z_1 \in A$. (L)

19. Use De Moivre's theorem to show that

$$\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta.$$

20. (i) If $(1 + 3i)z_1 = 5(1 + i)$, express z_1 and z_1^2 in the form $x + iy$, where x and y are real. Sketch in an Argand diagram the circle $|z - z_1| = |z_1|$ giving the coordinates of its centre.

(ii) If $z = \cos\theta + i\sin\theta$. show that

$$z - \frac{1}{z} = 2i\sin\theta. \qquad z^n - \frac{1}{z^n} = 2i\sin n\theta.$$

Hence, or otherwise, show that

$$16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta. \qquad \text{(L)}$$

21. (i) Given that $(1 + i)^n = x + iy$. where x and y are real and n is an integer, prove that $x^2 + y^2 = 2^n$.

(ii) Given that $\left|\dfrac{z - 1}{z + 1}\right| = 2$, find the cartesian equation of the locus of z and represent the locus by a sketch in the Argand diagram. Shade the region for

which the inequalities $\left|\frac{z-1}{z+1}\right| > 2$ and $0 < \arg z < 3\pi/4$ are both satisfied. (L)

22. (i) Given that x and y are real, find the values of x and y which satisfy the equation $\frac{2y+4i}{2x+y} - \frac{y}{x-i} = 0$.

(ii) Given that $z = x + iy$, where x and y are real, (a) show that, when $\operatorname{Im}\left(\frac{z+i}{z+2}\right) = 0$, the point (x, y) lies on a straight line, (b) Show that, when $\operatorname{Re}\left(\frac{z+i}{z+2}\right) = 0$, the point (x, y) lies on a circle with centre $(-1, -\frac{1}{2})$ and radius $\frac{1}{2}\sqrt{5}$. (L)

23. (i) Find $|z|$ and $\arg z$ for each of the complex numbers z given by (a) $12 - 5i$, (b) $(1 + 2i)/(2 - i)$, giving the argument in degrees (to the nearest degree) such that $-180° < \arg z \leqslant 180°$.

(ii) By expressing $\sqrt{3} - i$ in modulus-argument form, or otherwise, find the least positive integer n such that $(\sqrt{3} - i)^n$ is real and positive.

(iii) The point P in the Argand diagram lies outside or on the circle of radius 4 with centre at $(-1, -1)$. Write down in modulus form the condition satisfied by the complex number z represented by the point P. (L)

24. Sketch the circle C with Cartesian equation $x^2 + (y - 1)^2 = 1$. The point P, representing the non-zero complex number z, lies on C. Express $|z|$ in terms of θ, the argument of z. Given that $z' = 1/z$, find the modulus and argument of z' in terms of θ. Show that, whatever the position of P on the circle C, the point P' representing z' lies on a certain line, the equation of which is to be determined. (JMB)

25. (a) The sum of the infinite series $1 + z + z^2 + z^3 + \ldots$ for values of z such that $|z| < 1$ is $1/(1 - z)$. By substituting $z = \frac{1}{2}(\cos\theta + i\sin\theta)$ in this result and using De Moivre's theorem, or otherwise, prove that

$$\frac{1}{2}\sin\theta + \frac{1}{2^2}\sin 2\theta + \ldots + \frac{1}{2^n}\sin n\theta + \ldots = \frac{2\sin\theta}{5 - 4\cos\theta}.$$

(b) Two variable complex numbers z and w are such that $|z - i| = |z - 1|$ and $|w - 4| = 2$. Show on the same Argand diagram the loci of the points representing z and w. Mark on the diagram the points A and B representing z and w respectively for which $|z - w|$ has its least value and find this value. (O&C)

26. The point representing the complex number z in an Argand diagram describes the circle $|z - 1| = 1$. Show that $z = 1 + \cos\theta + i\sin\theta$, where $-\pi < \theta \leqslant \pi$, and deduce that the point representing the complex number $1/z$ describes a straight line. Find the modulus and argument of z and hence, or otherwise, express

$(1 + \cos\theta + i\sin\theta)^n$ in the form $x + iy$, where n is a positive integer, and x and y are real. By writing $\cos\theta + i\sin\theta = \omega$, and using the binomial expansion of $(1 + \omega)^n$, prove that

$$1 + \binom{n}{1}\cos\theta + \binom{n}{2}\cos 2\theta + \ldots + \cos n\theta \equiv \left[2\cos\left(\frac{\theta}{2}\right)\right]^n \cos\left(\frac{n\theta}{2}\right). \quad \text{(L)}$$

Formulae for reference

Logarithms

$$x = a^p \iff \log_a x = p; \quad \log_a x = \log_b x / \log_b a.$$

Series

Sum of A.P. $= \frac{1}{2}n\{2a + (n-1)d\}$, sum of G.P. $= \dfrac{a(1-r^n)}{1-r}$

For $|r| < 1$, sum to infinity of G.P. $= a/(1-r)$

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + b^n,$$

where n is a positive integer.

$$\sum_1^n r = \tfrac{1}{2}n(n+1); \quad \sum_1^n r^2 = \tfrac{1}{6}n(n+1)(2n+1); \quad \sum_1^n r^3 = \tfrac{1}{4}n^2(n+1)^2$$

Trigonometry

$$\cos^2\theta + \sin^2\theta = 1; \quad 1 + \tan^2\theta = \sec^2\theta; \quad \cot^2\theta + 1 = \operatorname{cosec}^2\theta.$$

$$\sin(A+B) = \sin A\cos B + \cos A\sin B \qquad \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B \qquad \tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$

$\sin 2A = 2\sin A\cos A$; $\tan 2A = 2\tan A/(1-\tan^2 A)$
$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$
$\cos^2 A = \frac{1}{2}(1+\cos 2A)$, $\sin^2 A = \frac{1}{2}(1-\cos 2A)$

$$\sin\theta = \frac{2t}{1+t^2}, \quad \cos\theta = \frac{1-t^2}{1+t^2}, \quad \tan\theta = \frac{2t}{1-t^2}, \quad \text{where} \quad t = \tan\tfrac{1}{2}\theta.$$

$$2\sin A\cos B = \sin(A+B) + \sin(A-B)$$
$$2\cos A\cos B = \cos(A+B) + \cos(A-B)$$
$$2\sin A\sin B = -\{\cos(A+B) - \cos(A-B)\}$$
$$\sin X + \sin Y = 2\sin\tfrac{1}{2}(X+Y)\cos\tfrac{1}{2}(X-Y)$$
$$\sin X - \sin Y = 2\cos\tfrac{1}{2}(X+Y)\sin\tfrac{1}{2}(X-Y)$$
$$\cos X + \cos Y = 2\cos\tfrac{1}{2}(X+Y)\cos\tfrac{1}{2}(X-Y)$$
$$\cos X - \cos Y = -2\sin\tfrac{1}{2}(X+Y)\sin\tfrac{1}{2}(X-Y)$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R, \quad a^2 = b^2 + c^2 - 2bc\cos A$$

Area of $\triangle ABC = \frac{1}{2}bc\sin A = \sqrt{\{s(s-a)(s-b)(s-c)\}}$,
where $s = \frac{1}{2}(a+b+c)$.
Length of circular arc $= r\theta$, area of sector $= \frac{1}{2}r^2\theta$ (θ in radians)

Calculus

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}, \qquad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

y	$\sin x$	$\cos x$	$\tan x$	$\cot x$	$\sec x$	$\operatorname{cosec} x$
dy/dx	$\cos x$	$-\sin x$	$\sec^2 x$	$-\operatorname{cosec}^2 x$	$\sec x\tan x$	$-\operatorname{cosec} x\cot x$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int e^{kx}dx = \frac{1}{k}e^{kx} + c, \qquad \int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + c,$$

$$\int \frac{dx}{\sqrt{(a^2-x^2)}} = \sin^{-1}\frac{x}{a} + c, \quad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c.$$

If $\dfrac{d^2y}{dx^2} + n^2y = k$, then $y = A\cos nx + B\sin nx + \dfrac{k}{n^2}$.

If $\dfrac{d^2y}{dx^2} - n^2y = k$, then $y = Ae^{nx} + Be^{-nx} - \dfrac{k}{n^2}$.

Expansions

$$(1+x)^n = 1 + nx + \frac{1}{2!}n(n-1)x^2 + \ldots + \binom{n}{r}x^r + \ldots \quad (|x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^r}{r!} + \ldots \quad \text{(all } x\text{)}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + \frac{(-1)^{r+1}x^r}{r} + \ldots \quad (-1 < x \leqslant 1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots, \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

Maclaurin's series: $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$

Hyperbolic functions

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}), \quad \sinh x = \tfrac{1}{2}(e^x - e^{-x})$$

$$\cosh^2 x - \sinh^2 x = 1.$$

Numerical methods

Newton-Raphson process: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

Trapezium rule:

$$\int_a^b f(x)\,dx \approx \tfrac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\}$$

Simpson's rule:

$$\int_a^b f(x)\,dx \approx \tfrac{1}{3}h\{(y_0 + y_n) + 4(y_1 + y_3 + \ldots) + 2(y_2 + y_4 + \ldots)\}$$

Coordinate geometry

Point dividing line joining (x_1, y_2) to (x_2, y_2) in ratio $\mu : \lambda$ is

$$\left(\frac{\lambda x_1 + \mu x_2}{\lambda + \mu}, \quad \frac{\lambda y_1 + \mu y_2}{\lambda + \mu}\right)$$

Distance of (x_1, y_1) from $ax + by + c = 0$ is $\dfrac{|ax_1 + by_1 + c|}{\sqrt{(a^2 + b^2)}}$

Area of triangle $= \frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$

Circle: $(x - h)^2 + (y - k)^2 = a^2$, centre (h, k), radius a;
$x^2 + y^2 + 2gx + 2fy + c = 0$, centre $(-g, -f)$, radius $\sqrt{(f^2 + g^2 - c)}$.

Rectangular hyperbola: $xy = c^2$, $(ct, c/t)$

Semicubical parabola: $ay^2 = x^3$, (at^2, at^3)

Parabola: $y^2 = 4ax$, $(at^2, 2at)$ focus $(a, 0)$, directrix $x = -a$.

Ellipse: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $(a\cos\theta, b\sin\theta)$, where $b^2 = a^2(1 - e^2)$, $e < 1$, foci $(\pm ae, 0)$ or $(\pm\sqrt{\{a^2 - b^2\}}, 0)$, directrices $x = \pm a/e$ or $\pm a^2/\sqrt{(a^2 - b^2)}$

Hyperbola: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, $(a\sec\theta, b\tan\theta)$, where $b^2 = a^2(e^2 - 1)$, $e > 1$, foci $(\pm ae, 0)$, directrices $x = \pm a/e$, asymptotes $y = \pm bx/a$

Answers

Exercise 1.1 (pp. 2–3)

1. (a) True, (b) False, (c) True, (d) True,
   (e) False, (f) False, (g) True, (h) False.
2. $A = C$; $B \subset A$, $B \subset C$, $E \subset D$.
3. $\varnothing \subset X, \varnothing \subset Y, \varnothing \subset P, \varnothing \subset Q, X \subset Y$; $\varnothing \in Q, 0 \in X, 0 \in Y, X \in P$.
4. (a) $p \Rightarrow q$ (b) $p \Leftrightarrow q$ (c) $p \Rightarrow q$ (d) $p \not\Rightarrow q$.
5. (a) $X \subset Z$ (b) $X = Z$.
6. $A = \varnothing, B = \{2\}, C = \{2,3\}, D = \{1,2\}$.
7. (a) $(x,f), (x,g), (x,h)$, (b) $(x,p), (x,q)$, (c) $(p,x), (q,x)$, (d) (x,x).
8. $A = \{1,2,3\}, B = \{1,2\}$.
9. (a) $\{0,2\}$, (b) $\mathscr{E}$, (c) $\{2\}$, (d) $\varnothing$.
10. (a) True, (b) True only if $\mathscr{E} = \{\text{integers}\}$, (c) True, (d) False. A could be a subset of C.

Exercise 1.2 (pp. 4–5)

1. (a) $2, 0, -2$, (b) $2, 0, -2, 0\dot{7}, 7{\cdot}3$, (c) $\sqrt[3]{2}, \pi$ (d) $2, 0, -2, 0\dot{7}. \sqrt[3]{2}, 7{\cdot}3$, (e) π.
2. (a) $7, -1$, (b) $7, -1, 4/3, 3{\cdot}142$, (c) $\pi+2, \sqrt{\pi}, \sqrt{3}+2$,
   (d) $7, -1, 4/3, 3{\cdot}142, \sqrt{3}+2$, (e) $\pi+2, \sqrt{\pi}$.
3. (i) (a) $\{4\}$ (b) $\{\ \}$ (c) $\{1\}$ (d) $\{\ \}$
   (ii) (a) $\{4, -1\}$ (b) $\{\ \}$ (c) $\{1\}$ (d) $\{\ \}$
   (iii) (a) $\{4, -1\}$ (b) $\{\ \}$ (c) $\{1, \frac{1}{2}\}$ (d) $\{\ \}$
   (iv) (a) $\{4, -1\}$ (b) $\{-\sqrt{5}, \sqrt{5}\}$ (c) $\{1, \frac{1}{2}\}$ (d) $\{\ \}$.
4. $\frac{1}{2}, \frac{7}{16}, \frac{1}{25}, \frac{1}{625}, \frac{5}{64}, \frac{51}{75}$.
5. (a) $\frac{5}{9}$, (b) $\frac{1}{2}$, (c) $\frac{3}{11}$, (d) $\frac{1}{27}$, (e) $\frac{1}{7}$.

Exercise 1.3 (pp. 6–7)

1. $-5, -\frac{1}{4}, 2, -k, n$.

2. $\frac{1}{2}, -3, -1, 5/4, x$.
3. (a) not commutative, (b) not associative.
4. (a) commutative, (b) associative.
5. (a) commutative, (b) not associative.
6. (a) 4, (b) 9, (c) 2, (d) -2, (e) -3, (f) 0 or 2.

Exercise 1.4 (p. 8)

2. (a) $x < 7$, (b) $x \geqslant 5$, (c) $x < -1$, (d) $x \leqslant 4$, (e) $x < -2$, (f) $x < -\frac{3}{4}$.
3. (a) $x < 1$, (b) $x \geqslant 8$, (c) $x < -3$, (d) $x \geqslant 7$, (e) $x \leqslant -5$ or $x \geqslant 5$, (f) $-6 < x < 6$.
4. (a) $-\frac{1}{2} < x < \frac{1}{2}$, (b) $-2 \leqslant x \leqslant 0$, (c) $1 < x < 2$, (d) $-\frac{1}{2} < x \leqslant 1$, (e) $-1 < x < 3$, (f) $-10 \leqslant x \leqslant 4$.
5. (a) False, (b) True, (c) False, (d) True, (e) False, (f) True, (g) False, (h) True, (i) False, (j) True.

Exercise 1.5 (pp. 11–12)

1. (a) 49, (b) 1, (c) 1/5, (d) 6, (e) 2, (f) 1/8, (g) $3\frac{1}{2}$, (h) $\frac{1}{3}$.
2. (a) 8, (b) $\frac{2}{3}$, (c) 16, (d) $\frac{1}{3}$, (e) $2\frac{7}{9}$, (f) $1\frac{1}{2}$, (g) 1, (h) 8.
3. (a) $3\sqrt{6}$, (b) $7\sqrt{2}$, (c) $6\sqrt{2}$, (d) $2\sqrt{17}$, (e) 18, (f) 2, (g) $\sqrt{3}$, (h) $\sqrt{2}$.
4. (a) $2\sqrt{5}$, (b) $5\sqrt{5}$, (c) $7\sqrt{2}$, (d) $-\sqrt{7}$.
5. (a) $\sqrt{18}$, (b) $\sqrt{125}$, (c) $\sqrt{x^2y}$, (d) $\sqrt{4a^4b}$.
6. (a) $2\sqrt{5}$, (b) $\dfrac{3\sqrt{7}+14}{7}$, (c) $\sqrt{15}-\sqrt{5}$, (d) $\dfrac{7+3\sqrt{5}}{4}$, (e) $\dfrac{17+13\sqrt{2}}{7}$, (f) $5+2\sqrt{6}$.
7. (a) $1\frac{1}{2}$, (b) $3\frac{1}{2}$, (c) -3.
8. (a) -5, (b) 1, (c) -1, (d) 2.
9. (a) 0, 3, (b) 0, 2, (c) 1, 2, (d) -1, 2, (e) 1, 2, (f) $\frac{1}{2}, -\frac{1}{2}$.
10. (a) $\sqrt{(x^4-x)}$, (b) $\sqrt{(1+x^2)}$, (c) $x(x-2)^{1/2}$, (d) $2x(2x-1)^{-1/2}$.

Exercise 1.6 (pp. 14–15)

1. (a) $\log_2 16 = 4$, (b) $\log_3 (1/9) = -2$, (c) $\log_{16} 4 = \frac{1}{2}$.
2. (a) $64 = 4^3$, (b) $0{\cdot}2 = 5^{-1}$, (c) $27 = 9^{1{\cdot}5}$.
3. (a) 3, (b) -2, (c) 0, (d) 1, (e) $\frac{1}{2}$.
4. (a) 2, (b) $\frac{1}{2}$, (c) -1, (d) $1\frac{1}{2}$, (e) 0.
5. (a) 4, (b) $\frac{1}{3}$, (c) $-\frac{1}{2}$, (d) $\frac{1}{2}$, (e) $-\frac{2}{3}$, (f) -3, (g) $1\frac{1}{2}$, (h) -2.
6. (a) $2\log x + \log y$, (b) $\frac{1}{2}\log x + \frac{1}{2}\log y$, (c) $4\log x - 3\log y$.
7. (a) $\log A + 2\log B - 3\log C$, (b) $\log A + \frac{1}{2}\log B + 2\log C$, (c) $\frac{3}{2}\log A + \log B + \frac{1}{2}\log C$.
8. (a) $\log 4$, (b) $\log 80$, (c) $\log \frac{3}{4}$, (d) $\log \frac{1}{2}$.
9. (a) 5, (b) $1\frac{1}{2}$, (c) 2.
10. (a) $1+2p+q$, (b) $5p-q$, (c) $p/(p+q)$.
11. (a) $1+2t$, (b) $1/t$, (c) $\frac{1}{2}t$, (d) $-1/t$.

12. (a) 4, (b) 1/5.
13. (a) 3, (b) 1.
14. (a) 9, 1/9, (b) 2, 64, (c) $\frac{1}{2}$, 64, (d) 5, 25.
15. $x = 4$, $y = \frac{1}{2}$.

Exercise 1.7 (pp. 15–16)

1. (a) False, (b) True, (c) False, (d) True.
2. (a) False, (b) True, (c) True, (d) False.
3. (a) True, (b) False, (c) False, (d) True.
4. (a) True, (b) False, (c) False, (d) False.
5. (a) {irrational numbers}, (b) {non-negative numbers}, (c) $\{x : x > 4\}$, (d) $\{0\}$, (e) $\mathbb{R}$, (f) $\{x : x^2 \geqslant 1\}$.
6. (a) commutative, (b) not associative.
8. (a) $x < -3$, (b) $x \leqslant -8$, (c) $\frac{1}{2} < x \leqslant 2$, (d) $x > 3$ or $x < -3$, (e) $-7 < x < 3$, (f) $x \leqslant 1$ or $x \geqslant 5$.
9. $2\sqrt{11}, 3\sqrt{5}, 4\sqrt{3}, 7, 5\sqrt{2}$.
10. (a) $1\frac{1}{2}$, (b) 10, (c) $\sqrt[3]{2}$.
11. (a) $3+2\sqrt{2}$, (b) $17-12\sqrt{2}$, (c) $1-\sqrt{2}$, $-1+\sqrt{2}$.
12. $x = 2$, $y = 0$.
13. (a) $-1{\cdot}89$, (b) $0{\cdot}63$, (c) $0{\cdot}26$ or $0{\cdot}37$.
14. (a) 16, (b) 10, (c) $\frac{1}{3}$, (d) 2.
15. (a) -3, (b) 2, (c) 0.
16. (a) 4, 1/16, (b) 1/9.

Exercise 2.1 (pp. 18–19)

1. (a) Yes, (b) no, (c) no.
2. (a) Yes, (b) no, (c) yes.
3. (a) 4, 10, 4, (b) many-to-one, (c) no, (d) yes.
4. (a) 8, 38, 602, (b) one-one, (c) yes, (d) no.
5. (a) 13, 343, 90301, (b) one-one, (c) yes, (d) no.
6. (a) 7, 20, 352, (b) many-to-one, (c) yes, (d) no.
7. $\mathbb{R}$ (a) yes, (b) yes, (c) yes.
8. $\{2\}$ (a) no, (b) no, (c) no.
9. $\mathbb{R}$ (a) yes, (b) yes, (c) yes.
10. $\{x \in \mathbb{R} : x \geqslant 0\}$ (a) no, (b) no, (c) no.
11. $\mathbb{Z}$ (a) yes, (b) yes, (c) yes.
12. $\mathbb{Z}$ (a) yes, (b) yes, (c) yes.
13. $\{x \in \mathbb{R} : x > 0\}$ (a) yes, (b) no, (c) no.
14. $\{x \in \mathbb{R} : x > 1\}$ (a) no, (b) yes, (c) no.
15. $\{1, 2, 4, 8, 16, \ldots\}$ (a) no, (b) no, (c) no.

Exercise 2.2 (p. 21)

1. $y = 2-x$.
2. $y = 2x+1$.
3. $y = 4-x^2$.
4. $y = x^3-2x^2$.
5. $y = 1/x^2$.
6. $y = 1/(1-x)$.

7. Yes; $\mathbb{R}, \mathbb{R}$; one-one.
8. Yes; $\mathbb{R}, \mathbb{R}$; one-one.
9. Yes; $\mathbb{R}, \mathbb{R}$; one-one.
10. Yes; $\mathbb{R}, \{y \in \mathbb{R} : y \geqslant 4\}$; many-to-one.
11. No.
12. Yes; $\mathbb{R}, \{y \in \mathbb{R} : y \leqslant 1\}$; many-to-one.
13. Yes; $\mathbb{R} - \{0\}, \mathbb{R} - \{0\}$; many-to-one.
14. Yes; $\mathbb{R} - \{1\}, \mathbb{R} - \{0\}$; one-one.
15. Yes; $\mathbb{R} - \{4\}, \mathbb{R} - \{0\}$; one-one.
16. No.
17. Yes; $\{x \in \mathbb{R} : x \geqslant 0\}, \{y \in \mathbb{R} : y \geqslant 0\}$; one-one.
18. Yes; $\mathbb{R}, \{y \in \mathbb{R} : y \geqslant 0\}$; many-to-one.

Exercise 2.3 (pp. 24–25)

1. (a) $(0, -1), (1, 0)$, (b) $(0, 5), (-2\frac{1}{2}, 0)$, (c) $(0, 0), (-3, 0)$, (d) $(0, -2), (-2, 0), (\frac{1}{2}, 0)$, (e) $(0, -1)$, (f) $(0, \frac{1}{2}), (3, 0)$.
2. (a) $(-3, 6)$, (b) $(3, 1)$, (c) $(2, 3), (-5, 115)$, (d) $(-2, -7), (4, 35), (-1, -5)$, (e) $(-2, 2), (-1, -4)$, (f) $(7, -6), (0, 1)$.
3. (a) 15, (b) 5, (c) 13, (d) 10, (e) 17, (f) 25, (g) $\sqrt{13}$, (h) $3\sqrt{5}$.
4. (a) $(6, 2\frac{1}{2})$, (b) $(-5, -6\frac{1}{2})$, (c) $(\frac{1}{2}, 8)$, (d) $(8, 3)$, (e) $(0, -2\frac{1}{2})$, (f) $(-\frac{1}{2}, 8)$, (g) $(6, -2\frac{1}{2})$, (h) $(-\frac{1}{2}, 0)$.
5. (a) $(5, 0)$, (b) $(0, -6)$, (c) $(1, -6)$, (d) $(2, 1)$.
6. (a) $(9, -8)$, (b) $(0, 1)$, (c) $(5, 8)$, (d) $(2, -4)$.
7. (a) isosceles, (b) neither, (c) right-angled, (d) both.
9. 5.
10. 50.
11. $(9, 4)$; 126.
12. $(-4, 1)$; 80.
13. $S(7, 0)$, $T(3, -6)$.
14. 2; $(10, 5)$, 10.
15. 20.

Exercise 2.4 (pp. 27–28)

1. (a) $\frac{3}{4}$, (b) -7, (c) -2, (d) $3/7$, (e) 5, (f) 1.
2. (a) $-4/3$, (b) $1/7$, (c) $\frac{1}{2}$, (d) $-7/3$, (e) $-1/5$, (f) -1.
3. (a) No, (b) yes, (c) yes, (d) no.
6. (a) -1, (b) -5, (c) 3 or -4.
7. (a) $(4, 0)$, (b) $(9, 0)$, (c) $(-1, 0)$.
8. $2\sqrt{10}, \sqrt{10}, \sqrt{10}$; 15 sq. units.
10. $0, -1$; $7\frac{1}{2}$ sq. units, 10 sq. units.
11. 123 sq. units.

Exercise 2.5 (pp. 31–32)

1. (a) $y = 4x + 11$, (b) $x + 2y + 4 = 0$, (c) $3x + y = 11$,

(d) $2x-3y-11=0$, (e) $5x+12y=0$, (f) $x-ty+t^2=0$.

2. (a) $y=x-7$, (b) $2x+y=2$, (c) $y=7$, (d) $3x-4y=0$, (e) $3x-7y=15$, (f) $x=6$, (g) $(p+q)y=2x+2pq$, (h) $x+pqy=p+q$.
3. (a) $5/2, 9/2$, (b) $7/4, -13/4$, (c) $-1/4, -19/8$, (d) $1/2, -17/6$, (e) $0, -5/2$, (f) $-5/9, 3$.
4. (a) $(1,1)$, (b) $(3,2)$, (c) $(0,-3)$, (d) none, (e) $(2,1)$, (f) $(-\frac{2}{3}, \frac{1}{3})$.
5. (a) perpendicular, (b) perpendicular, (c) neither, (d) parallel, (e) perpendicular, (f) neither.
6. (a) $y=2x+5, x+2y=5$, (b) $3x+2y=4, 3y-2x=6$, (c) $x=3, y+4=0$, (d) $x-3y=1, 3x+y+7=0$.
7. $(1,-2)$.
8. $(2,0)$.
9. $y=3x+5$.
10. $y=x-5$.
12. (a) $2y+x=13, y=2x+9$, (b) $(-1,7)$.
13. $(2,-4)$.
14. $B(3,-1), D(1,1)$.
15. $(-3,7)$.
16. $(-1,3)$, $y=7x+10$.

Exercise 2.6 (p. 34)

3. $y>-2$.
4. $x<3$.
5. (a) $-1\frac{1}{2}<x<6$, (b) $0<y<3$, (c) $0<x+y<9$.
6. (a) $-14/3<x<0$, (b) $0<y\leqslant 5$, (c) $-14/3<x+y<17/4$.

Exercise 2.7 (pp. 34–35)

1. (a) $\{y\in\mathbb{R}: y\geqslant -4\}$; neither,
   (b) $\{y\in\mathbb{Q}: y=1/n, n\in\mathbb{N}\}$; one-one,
   (c) $\mathbb{R}$; onto.
2. $-2\frac{1}{4}\leqslant y\leqslant 10$; $\{x\in\mathbb{R}: -3\leqslant x\leqslant 0 \text{ or } 3\leqslant x\leqslant 6\}$.
3. $A(-4,0), B(0,2)$; $2\sqrt{5}$.
6. $(10,-1), (2,15), (-6,-1)$.
7. (a) $y=2x+1$, (b) $9x+5y-2=0$, (c) $4x-3y=11$, (d) $5x+3y=3$.
8. $A(2,-5), B(1,-2), C(4,13)$.
9. $(3,\frac{2}{3})$.
10. $(6,-4)$; 50 sq. units.
11. $(5,1), (-1,7), (-1,-3)$.
12. (a) $\frac{1}{2}, 8\frac{1}{2}, -3$, (b) 25 sq. units.
13. (a) $(\frac{1}{2}\{h+3\}, \frac{1}{2}\{k+10\})$, (b) $(k-10)/(h-3)$.
14. (a) $(6,2)$, (b) $x+7y=45$.
16. (a) 6, (b) $16x-12y=35$.
17. (a) $2x+3y=1$, (b) $3\sqrt{13}$.
18. (a) $(2,6)$, (b) 20 sq. units.
19. $(0,2)$, 5.
20. $(2,5), (5,3), (6,1)$; $19k/(2+3k)$.

Exercise 3.1 (p. 39)

1. (a) Min, -9, (b) Max, 1, (c) Min, 7/4, (d) Min, 4, (e) Min, -12, (f) Min, 0, (g) Max, 25/12, (h) Max, $-25/3$.
2. (a) $\{y \in \mathbb{R} : y \geqslant -5\}$, (b) $\{y \in \mathbb{R} : y \geqslant -9\}$, (c) $\{y \in \mathbb{R} : y \geqslant \frac{1}{2}\}$, (d) $\{y \in \mathbb{R} : y \leqslant 12\frac{1}{4}\}$.
3. $(x-2)^2+4$.
6. (a) $0 < x < 2$, (b) $1 \leqslant x \leqslant 5$, (c) $x \leqslant -2$ or $x \geqslant 2$, (d) $x < 1$ or $x > 3$, (e) $-1 < x < \frac{1}{3}$. (f) $x \leqslant -3$ or $x \geqslant -\frac{1}{2}$.
7. -7; $c > 12$.
8. $k \geqslant 9/4$.
9. (a) $-3, 8$, (b) $\frac{1}{2}, -5$, (c) $1, 3$.
10. 8.

Exercise 3.2 (pp. 41–42)

1. (a) $\frac{1}{2}(3 \pm \sqrt{5})$, (b) $\frac{1}{4}(-3 \pm \sqrt{17})$, (c) $1 \pm \sqrt{2}$, (d) no real roots, (e) $-3/2, 9/2$, (f) $\frac{1}{2}(3 \pm \sqrt{3})$.
2. (a) $3, -2$, (b) -1.
3. (a) $p < -5$ or $p > 0$, (b) $-2 < p < 2$, (c) $p < 1$ or $p > 9$, (d) all real $p \neq 0$.
4. (a) $a < -1$ or $a > 0$, (b) all real $a \neq 0$, (c) $0 < a < 3$, (d) $a > 1$.
5. $3, -1$.
7. 3.
8. $0 < k < 1$.
9. $k \leqslant -6$ or $2 \leqslant k < 3$.

Exercise 3.3 (pp. 44–45)

1. (a) $5/2, -3/2$, (b) $1, -6$, (c) $0, -5/3$, (d) $-p, -p$, (e) $7, 5$, (f) $\frac{3}{4}, -\frac{1}{4}$.
2. (a) $x^2-3x+2=0$, (b) $4x^2+2x+3=0$, (c) $x^2-4=0$, (d) $14x^2-6x-1=0$, (e) $x^2-(p-q)x+p+q=0$, (f) $abx^2-a^2x+1=0$.
3. 9.
4. 4, 6.
6. (i) 3, (ii) -2, (iii) 13, (iv) 45.
7. (i) 4, (ii) 24, (iii) $2\sqrt{6}$, (iv) $-4\sqrt{6}$.
8. (a) 17/4, (b) 33/4, (c) 25/8, (d) $-1/4$, (e) 17/16, (f) 25/32.
9. 3; $x^2-5x+3=0$.
10. $x^2 \pm 11x+28=0$.
11. (a) $x^2-12x+18=0$, (b) $x^2-10x+23=0$, (c) $x^2-16x+56=0$.
12. (a) $x^2+(2k-1)x+k^2=0$, (b) $x^2-x+k=0$, (c) $x^2+3x+(k+2)=0$.
13. (a) $cx^2+bx+a=0$, (b) $a^2x^2+4ac-b^2=0$, (c) $b^2x^2-b^2x+ac=0$.
14. $2+2p-q$, $2\frac{1}{2}+2p-q$.
15. 0, 1.

Exercise 3.4 (p. 47)

1. (a) (1,2), (4,17), (b) (2, −1) repeated (i.e. tangent), (c) (1, −2), (2, −1), (d) none, (e) (2, −3), (−5/4, −19/2), (f) (−5, 8/3).
2. 4, −3; −3, 4.
3. 4, 6; −4, −6.
4. 3, −2.
5. 3, −1; −2, −2.
6. 1, 2 (repeated).
7. $3, \frac{1}{2}$; 5/3, 5/2.
8. 5, 1.
9. 1, −2; −2, 1.
10. $1, -1; \frac{1}{4}, \frac{1}{2}$.
11. 6, −5; −5, 6.
12. −1, 5/3; 5/2, −2/3.
13. 2, 1; −2, −1; $\sqrt{3}, 2/\sqrt{3}$; $-\sqrt{3}, -2/\sqrt{3}$.
14. 5/2, −3/2; −3/2, 5/2.
15. −1, 4 (repeated).
16. −2, 0; −2, −2; 2/3, 2/3; 2/3, −8/3.
17. $\pm\sqrt{3}, 0$; 2, −1; −2, 1.
18. 0, 3 (repeated); $1+2\sqrt{2}, -1-\sqrt{2}$; $1-2\sqrt{2}, -1+\sqrt{2}$.

Exercise 3.5 (pp. 50–51)

1. (a) $2x^3+3x^2+5x+14$, (b) $6x^4-13x^3+16x^2-17x+3$, (c) $4x^4+11x^3-10x^2+5x-2$, (d) $5x^5-7x^3+3x^2+2x-3$, (e) $9x^5-4x^3+18x^2-8$, (f) x^6-1.
2. (a) −16, −7, (b) 7, 5, (c) −5, 3, (d) −4, 5.
3. (a) $\dfrac{3}{(1-x)(2x-5)}$, (b) $\dfrac{x+2}{x^2-2x+4}$, (c) x^2+x+1, (d) $\dfrac{2x+1}{2x-3}$.
4. (a) 19, (b) 112, (c) 15, (d) −16, (e) a^2, (f) 0.
5. (a) $3x-5$; 10, (b) $2x^2+x+4$; 13, (c) x^2-3x+9; −30, (d) x^4-2x^2+3x+1; 0, (e) x^2-x; $5x-5$, (f) $2x^2-x+7$; $-2x+7$.
6. (a) $(x-1)^2(2x+1)$, (b) $(x+1)(x-2)(3x+1)$, (c) $(x+3)(x-3)(x^2+8)$, (d) $x(x^4+x^2+1)$, (e) $(x+2)(2x-1)(2x-3)$, (f) $(x+1)(x-2)(2x+1)(2x-1)$.
7. 2/3, 7/3.
8. 3, −6.
9. −5, 7.
10. 3, −5.
11. 3, −2.
12. 0, 6, −6.
13. $a = 3 : 3(x-3)(x+1)^2$; $a = -3 : -3(x+3)(x-1)^2$.
14. (a) 10, (b) 3.
15. (a) $(2x-1)(2x-3)(2x-5)$, (b) $(2x+1)^2(3x-2)$, (c) $(3x+2)(x^2-x+2)$.

Exercise 3.6 (pp. 53–54)

1. −1, 2, 7.
2. 1, −2, −3.
3. 4, −1 (repeated).
4. −2, −2/3, −5/2.
5. $\pm\frac{1}{2}$.
6. $\pm 3, \pm\sqrt{2}$.
7. $-2, 1\pm\sqrt{3}$.
8. 3.
9. $-\frac{1}{2}, \pm\sqrt{3}$.
10. $2, -1, 2\pm\sqrt{2}$.
11. 3.
12. 4.
13. 1, 5.
14. −4.

15. $-7, -6; 3, -2, -1.$ 16. $4; 2/3, -3/2.$
17. (a) $-3/2, 6,$ (b) $-5/2, 0,$ (c) $-4/3, 2/3,$ (d) $0, 6.$
18. $-8; a = -28, b = -32.$ 19. $\frac{1}{2}; p = -9, q = 12.$
20. (a) $6, -4,$ (b) $1/2, 3/2,$ (c) $0, 0,$ (d) $3/2, 1/2.$
21. 1 (repeated), $-3, -\frac{1}{3}.$ 22. $2, \frac{1}{2}.$

Exercise 3.7 (pp. 54–55)

1. (a) Min $(-1, -1),$ (b) Min $(-\frac{1}{4}, -3\frac{1}{8}),$ (c) Max $(\frac{1}{2}, 4),$ (d) Min $(3, 1).$
2. $x > -1.$ 3. $5, -2, \frac{1}{2}(3 \pm \sqrt{5}).$
4. (a) $0, 3,$ (b) $4, -1.$
5. $x^2 + (2q - p - p^2)x + q(p + q + 1) = 0.$
6. $3x^2 + 10x - 9 = 0.$ 7. $p = -3, q = 19.$
8. (a) $p = 7, q = -3\frac{1}{2}$ or $p = -8, q = 4.$
9. (b) $k \leqslant 0$ or $k \geqslant 3,$ (c) $k \geqslant 3.$
11. $(1\frac{1}{2}, 8).$ 12. $(k + 2, 2k\{k + 2\}).$
13. $2, -1; 3/5, 2/5.$
14. $1, 2; -1, -2; 1/2, 5/2; -1/2, -5/2.$
15. $-13, 4; (x + 1), (x + 2), (2x - 1).$
16. $-2, 2, -8.$
17. $2x^2 + 5x - 3; x = -3$ or $\frac{1}{2}.$
18. $n(n - 1)(n + 1)(n + 2).$
20. $-2 \leqslant c \leqslant 2.$
21. (a) $1, 4, -1 \pm \sqrt{5}, 1 \pm \sqrt{5},$ (b) $\frac{1}{4}b^2(b^4 + 6a^2b^2 - 3a^4).$
22. $b = 1 - a; ax^2 + (1 - a)x + 1.$

Exercise 4.1 (pp. 58–59)

1. $44{\cdot}72 < z < 47{\cdot}52; z = 46 \pm 2.$
2. $92{\cdot}885 < z < 95{\cdot}625; z = 94 \pm 2.$
3. $0{\cdot}3883 < z < 0{\cdot}3900; z = 0{\cdot}389 \pm 0{\cdot}001.$
4. $4{\cdot}000 < z < 4{\cdot}243; z = 4{\cdot}1 \pm 0{\cdot}2.$
5. (a) 10·65, 10·73; 10·7, (b) 10·69, 10·77; 11.
6. (a) 1·691, 1·703; 1·7, (b) 1·690, 1·702; 1·7.
7. (a) 20, (b) 0·0720, (c) 14·3, (d) 12·7.

Exercise 4.2 (pp. 62–63)

1. (a) $f^{-1} : x \to x/3,$ (b) $f^{-1} : x \to 2x,$ (c) $f^{-1} : x \to x + 1.$
   (d) $f^{-1} : x \to 1 - x,$ (e) $f^{-1} : x \to \frac{1}{2}(x - 3),$ (f) $f^{-1} : x \to \frac{1}{4}(x + 4).$
2. (a) $y = \frac{1}{2}x,$ (b) $y = 2/x,$ (c) $y = 2 - x,$ (d) $y = x + 2,$
   (e) $y = \log_2 x,$ (f) $y = 10^x + 2.$
4. (a) 1·172, (b) 64·82, (c) 19 840, (d) 6·463.
5. (a) 1·404, (b) 2·096, (c) 0·4307.
6. (a) 3·170, (b) 0·3920, (c) 0·5377.
8. (a) 1·292, (b) 1·295, (c) 1·613, (d) 4·962.
9. (a) 8·432, (b) 1·453.
10. 1·921.

Exercise 4.3 (pp. 67–68)

1. (a) 1·2, (b) 3·36.
2. (a) REAL AND DISTINCT; 5, −3, (b) REAL AND EQUAL; −0·5, (c) NO REAL ROOTS, (d) REAL AND DISTINCT; 8, −0·6.

Exercise 4.4 (pp. 70–71)

1. (a) 12, −3, (b) 3, 6, (c) 16, 40.
2. (a) 10·52, 5·080, (b) 4·696, 20·84, (c) 18·10, 4·102, (d) 93·16, −5·533.
3. (a) 9·487, (b) 1·225, (c) 0·5435.
4. (a) 0·3832, (b) 3·889.
5. (a) 5·477, (b) 35·36.
6. (a) 1·556, (b) 9·595. (c) 9·595
7. (a) 255, (b) −4058, (c) 4·335.
8. (a) 3·922, (b) 2·900, (c) 6·423.

Exercise 4.5 (pp. 75–76)

1. $y = 0{\cdot}30x + 0{\cdot}60$.
2. $R = 88 + 2{\cdot}3V$.
3. $E = 0{\cdot}054L + 0{\cdot}53$.
4. $\theta = 7{\cdot}6 - 0{\cdot}32T$.
5. 16.
6. $-2{\cdot}7, 11$.
7. 1·5, 10.
8. 0·65, 1·4.
9. 2·5, 4·8.
10. 5·0, −2·0.

Exercise 4.6 (pp. 77–79)

1. 180·12 cm², 187·92 cm²; 184 ± 4 cm².
2. (a) 37 cm², (b) 37·4 cm², (c) 37·4 cm².
3. (a) $f^{-1} : x \to \frac{1}{2}(1 - x)$, (b) $g^{-1} : x \to \log_5 x$, (c) $h^{-1} : x \to -1/x$.
4. 1·277.
5. 0, 1·262.
6. 3·78.
7. $z = 6p - 800$.
8. (a) $x = v^2, y = u^2$, (b) $x = u, y = v/u$, (c) $x = \lg v, y = \lg u$, (d) $x = u, y = \lg v$.
9. 0·25, 3.
10. 20, 0·50.
11. 0·411.
12. LET $S = 0$, LET $C = N$, $C = 0$?, LET $S = SX$; −1.
13. 5, 12, 13; 9, 12, 15.

Exercise 5.1 (pp.82–83)

1. (a) 9, (b) 0, (c) 10, (d) 73.
2. (a) $\frac{1}{3}$, (b) $1\frac{1}{2}$, (c) $-\frac{1}{2}$, (d) 4.
3. (a) 3, (b) 1, (c) 2, (d) 1.
4. (a) 2, (b) 2.
5. As $x \to 1$ from above, $f(x) \to 0$; as $x \to 1$ from below, $f(x) \to 1$.

6. As $x \to 0$ from above, $f(x) \to \infty$; as $x \to 0$ from below, $f(x) \to -\infty$.
7. (a) $f(x) \to 0$, (b) $g(x) \to \infty$, (c) as $x \to k$ from above, $h(x) \to \infty$; as $x \to k$ from below, $h(x) \to -\infty$.
8. (a) 0, (b) 0, (c) 2, (d) 2.
9. (a) -5, (b) 3/2, (c) 1, (d) $\frac{1}{2}$.

Exercise 5.2 (pp. 85–86)

1. $-1, -0{\cdot}5, -0{\cdot}1, -0{\cdot}01, -0{\cdot}001$.
2. $1, 0, -0{\cdot}8, -0{\cdot}98, -0{\cdot}998$.
3. $2+h$; 2.
4. (a) 6, (b) -3, (c) 1, (d) 3.
5. $6x+3h$; $6x$.
6. (a) $2x-1$, (b) $-1/x^2$, (c) $3x^2$.
7. $8a+4h$; $8a$.
8. (a) 3; 3, (b) $12-10a-5h$; $12-10a$.

Exercise 5.3 (p. 88)

1. $2x+\delta x$; $2x$.
2. (a) $-2x-\delta x$; $-2x$, (b) $2x+2+\delta x$; $2x+2$, (c) $\dfrac{-2}{x(x+\delta x)}$; $-\dfrac{2}{x^2}$.
3. (a) $4x$, (b) $10x$, (c) $4x$.
4. (a) 1, (b) -1, (c) $1+2x$.
5. (a) 3, (b) $\frac{1}{2}$, (c) 0.
6. (a) $-3x^2$, (b) $6x^2$, (c) $4x^3$.
7. (a) $1+\dfrac{1}{x^2}$, (b) $-\dfrac{2}{x^3}$, (c) $-\dfrac{3}{x^4}$.

Exercise 5.4 (pp. 92–93)

1. (a) $5x^4$, (b) $12x^{11}$, (c) $12x^3$, (d) 0.
2. (a) 2, (b) $6x-4$, (c) $6x^2-42x^5$.
3. (a) $4-2x$, (b) $\dfrac{3}{2}x^2+\dfrac{2}{3}x$, (c) $5-16x^3$.
4. (a) $4x+3$, (b) $18x-6$, (c) $3x^2+8x+4$.
5. (a) $-2x^{-3}$, (b) $-4/x^5$, (c) $-9/x^{10}$, (d) $-6/x^4$.
6. (a) $2x+\dfrac{1}{x^2}$, (b) $4x^3-\dfrac{8}{x^3}$, (c) $-\dfrac{1}{x^3}+\dfrac{1}{x^4}$.
7. (a) $1-\dfrac{1}{x^2}$, (b) $\dfrac{15}{x^4}-\dfrac{4}{x^3}$, (c) $1+\dfrac{2}{x^2}$.
8. (a) $3x^2(x^3+1)$, 0, -84, (b) $9(x+1)^2$, 9, 9.
9. (a) $2-\dfrac{2}{x^3}$, $1\frac{3}{4}$, 4, (b) $2x-\dfrac{2}{x^3}$, $3\frac{3}{4}$, 0.
10. (a) 5, (b) 25, (c) 2, (d) $\frac{1}{4}$.

11. (a) $y = x-4$, (b) $y = 11x+7$, (c) $3x+y = 2$,
(d) $8y+121 = 0$, (e) $y = 4x-3$, (f) $16y = 2x-11$.
12. (a) $10t\,\mathrm{m\,s^{-1}}$, (b) $30\,\mathrm{m\,s^{-1}}$, $50\,\mathrm{m\,s^{-1}}$.
13. (a) $27-6t\,\mathrm{m\,s^{-1}}$, (b) $9\,\mathrm{m\,s^{-1}}$, $-3\,\mathrm{m\,s^{-1}}$.
14. 4π.
15. -2.
16. 3, -4.
17. $2, \frac{1}{2}$.
18. (a) $\frac{1}{3}x^{-2/3}$, (b) $\frac{3}{2}x^{1/2}$, (c) $-\frac{1}{4}x^{-5/4}$, (d) $\frac{1}{5}x^{-4/5} = 1/(5\sqrt[5]{x^4})$,
(e) $-\frac{1}{2}x^{-3/2} = -1/(2\sqrt{x^3})$, (f) $\frac{2}{3}x^{-2/3} = 2/(3\sqrt[3]{x^2})$.
19. (a) $\frac{1}{2}x^{-1/2} - \frac{3}{2}x^{-5/2}$, (b) $3x^{1/2} - \frac{1}{2}x^{-1/2}$, (c) $1 + \frac{1}{\sqrt{x}}$,
(d) $\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{x^3}}$, (e) $\frac{5}{2}x\sqrt{x} + \frac{3}{2}\sqrt{x}$, (f) $\frac{3}{2}\sqrt{x} + \frac{3}{2\sqrt{x}} - \frac{3}{2x\sqrt{x}} - \frac{3}{2x^2\sqrt{x}}$.
20. $4y = 5x+4$.

Exercise 5.5 (p. 99)

1. $(1, -4)$ min.
2. $(-\frac{3}{4}, 3\frac{1}{8})$ max.
3. $(-\frac{1}{2}, -2\frac{1}{4})$ min.
4. $(\frac{1}{2}, 2)$ min.
5. $(0,0)$ max, $(2, -4)$ min.
6. $(0,1)$ inflexion.
7. $(2, -8)$ inflexion.
8. $(-2,0)$ max, $(1, -27)$ min.
9. $(-1/3, -11/27)$ min, $(\frac{1}{2}, \frac{3}{4})$ max.
10. $(-1\frac{1}{4}, 15\frac{3}{16})$ max, $(\frac{1}{2}, -6\frac{1}{4})$ min.
11. $(-4, 256)$ max, $(0,0)$ min.
12. $(0,0)$ inflexion, $(1,1)$ max.
13. None.
14. None.
15. $(\frac{2}{3}, 3\frac{2}{3})$ max, $(2\frac{2}{3}, -32\frac{1}{3})$ min.
16. $(-1,4)$ min, $(0,6)$ max, $(2\frac{1}{2}, -17\frac{7}{16})$ min.
17. $(-1\frac{1}{2}, -12)$ max, $(1\frac{1}{2}, 12)$ min.
18. $(2, 3)$ min.
19. $(0, -64)$ inflexion, $(3, 17)$ inflexion.
20. $(\frac{1}{4}, -\frac{1}{4})$ min.
21. $(0, 21)$.
22. $-3, -9, 12$.
23. (i) $a^2 > 3b$, (ii) $a^2 = 3b$, (iii) $a^2 < 3b$.
24. $x = 0$, $y = x$.

Exercise 5.6 (pp. 103–104)

1. 6.
2. $30x-2$.
3. $12x^2-12x+\frac{2}{x^3}$.
4. $-\frac{8}{x^3} - \frac{24}{x^4}$.
5. $-\frac{1}{4}x^{-3/2} + \frac{3}{4}x^{-5/2}$.
6. $\frac{3}{2}x^{-1/2} + \frac{1}{4}x^{-3/2} - \frac{9}{2}x^{-5/2}$.
7. $(-1, 2)$ max; $(1, -2)$ min.
8. $(-1,0)$ min; $(0,1)$ max; $(1,0)$ min.
9. $(0,0)$ inflexion; $(1\frac{1}{2}, -1\frac{11}{16})$ min.
10. $(-\frac{1}{2}, -7)$ max; $(\frac{1}{2}, 1)$ min.
11. $(1,2)$ max; $(2,1)$ min.

12. $(-1,0)$ min; $(0,5)$ max; $(2,-27)$ min.
13. (a) 4, -17, 50, (b) 3, 0, 6.
14. $(-1,-27)$ min; $(2,0)$ inflexion.
15. $(1,0)$, 3.
16. (a) $v = 18t-9t^2$, $a = 18-18t$, (b) $0\,\mathrm{m\,s^{-1}}$, $18\,\mathrm{m\,s^{-2}}$,
    (c) $-27\,\mathrm{m\,s^{-1}}$, $-36\,\mathrm{m\,s^{-2}}$.
17. (a) $v = 24t^2-66t+27$, $a = 48t-66$, (b) $27\,\mathrm{m\,s^{-1}}$, $-66\,\mathrm{m\,s^{-2}}$,
    (c) $45\,\mathrm{m\,s^{-1}}$, $78\,\mathrm{m\,s^{-2}}$.
18. 2, -8, 5.
19. (a) $\left(\frac{1}{\sqrt[3]{2}}, \frac{3}{2}\sqrt[3]{2}\right)$ min, $(-1,0)$ inflexion.
    (b) $(2,\frac{1}{4})$ max, $(3,\frac{2}{9})$ inflexion.
    (c) $\left(-\sqrt{3}, -\frac{2}{9}\sqrt{3}\right)$ min, $\left(\sqrt{3}, \frac{2}{9}\sqrt{3}\right)$ max, $\left(-\sqrt{6}, -\frac{5}{36}\sqrt{6}\right)$ inflexion,
    $\left(\sqrt{6}, \frac{5}{36}\sqrt{6}\right)$ inflexion.
20. $(0,0)$.

Exercise 5.7 (pp. 104–105)

1. (a) 2, (b) -4, (c) 1.
2. (a) 0, (b) 2, (c) 0.
3. (b) 4.
4. (a) $3(x+1)^2$, (b) $-(x+1)^{-2}$.
5. $\frac{1}{2}x^{-1/2}$.
6. (a) $6(3x+1)$, (b) $1-\frac{6}{x^2}$, (c) $\frac{3}{2}\sqrt{x}-\frac{1}{\sqrt{x}}$.
7. $y = 4(x+1)$, $y = -3x$, $y = 12(x-3)$.
9. (a) $(0,-4)$ min, (b) $\left(-\frac{2}{3}\sqrt{3}, \frac{16}{9}\sqrt{3}\right)$ max; $\left(\frac{2}{3}\sqrt{3}, -\frac{16}{9}\sqrt{3}\right)$ min,
    (c) $(-\sqrt{2},-4)$ min; $(0,0)$ max; $(\sqrt{2},-4)$ min, (d) no turning points,
    (e) no turning points, (f) $\left(2\sqrt{3}, \frac{1}{9}\sqrt{3}\right)$ max; $\left(-2\sqrt{3}, \frac{-1}{9}\sqrt{3}\right)$ min.
10. $(-2,-3\frac{3}{4})$ max; $(1,3)$ inflexion.
11. (a) $v = 7+10t-6t^2\,\mathrm{m\,s^{-1}}$, $a = 10-12t\,\mathrm{m\,s^{-2}}$,
    (b) $7\,\mathrm{m\,s^{-1}}$, $10\,\mathrm{m\,s^{-2}}$; $3\,\mathrm{m\,s^{-1}}$, $-14\,\mathrm{m\,s^{-2}}$.
14. $-b/3a$.

Exercise 6.1 (pp. 108–109)

3. (a) $\frac{1}{3}x^3+c$, (b) $-\frac{1}{x}+c$, (c) $2x^4+c$, (d) $-\frac{3}{x^2}+c$.
4. x^3+2x^2+c.
5. $\frac{1}{6}x^6-x+c$.
6. $\frac{4}{3}x^3+2x^2+x+c$.
7. $3x^2+\frac{1}{12x^2}+c$.

8. $\frac{1}{3}x^3+2x-\frac{1}{x}+c.$ 9. $-\frac{1}{x}+\frac{1}{3x^3}+c.$

10. $\frac{3}{4}x^{4/3}+\frac{3}{2}x^{2/3}+c.$ 11. $\frac{2}{3}x^{3/2}-2x^{1/2}+c.$

12. $\frac{2}{5}x^{5/2}+2x^{-1/2}+c.$

13. (a) $t^3+7t^2-24t+c$, (b) $-\frac{4}{t}-\frac{2}{t^2}-\frac{1}{3t^3}+c$,
(c) $\frac{4}{5}t^{5/2}-\frac{10}{3}t^{3/2}+4t^{1/2}+c.$

14. $x^3-4x^2+5x-2.$ 15. $-\frac{3}{4}.$

16. $y=x^3-4x-1.$ 17. $y=6+4x-2x^2.$

18. $2\sqrt{h^3}-\sqrt{h}-4.$ 19. $x^3-5x^2+8x-3.$

20. $-5.$

Exercise 6.2 (pp. 113–114)

1. (a) $3x^2+x+c$, (b) $3x^3-3x^2+x+c$, (c) $\frac{4}{3}x^3+\frac{1}{2}x^2+c.$

2. (a) $3x+\frac{1}{x}+c$, (b) $\frac{1}{3}x^3-2x-\frac{1}{x}+c$, (c) $-\frac{3}{x}-\frac{1}{2x^2}+c.$

3. (a) $\frac{8}{3}x^{3/2}+c$, (b) $\frac{2}{3}x^{3/2}-2x^{1/2}+c$, (c) $3x^{4/3}+6x^{2/3}+c.$

4. (a) $t^3+\frac{1}{t}+c$, (b) $\frac{1}{2}u^2-\frac{2}{5}u^{5/2}+c$, (c) $s-\frac{1}{2s^2}+c.$

5. (a) 6, (b) 22, (c) $82\frac{2}{3}$.

6. (a) $4\frac{5}{6}$, (b) $2\frac{7}{24}$, (c) 54.

7. (a) -36, (b) 60, (c) $6\frac{1}{2}$.

8. (a) $1\frac{1}{3}$, (b) $2\frac{1}{4}$, (c) 8, (d) 38.

9. (a) $4\frac{1}{2}$, (b) $6\frac{3}{4}$, (c) $-5\frac{1}{3}$, (d) $-9\frac{3}{5}$.

10. (a) $4\frac{1}{2}$, (b) 36, (c) $1\frac{1}{3}$, (d) 1/12, (e) $\frac{1}{3}$, (f) $\frac{3}{4}$.

11. $6\frac{2}{3}$.

12. (0,0), (1,1); $\frac{1}{3}$.

Exercise 6.3 (p. 117)

1. (a) $21\frac{1}{3}$, (b) $13\frac{1}{2}$, (c) $3\frac{1}{12}$, (d) $1\frac{1}{3}$.

2. 36. 3. $10\frac{2}{3}$.

4. $20\frac{1}{4}$. 5. 1.

6. (a) $\frac{1}{2}$, (b) $1\frac{1}{3}$, (c) $10\frac{2}{3}$.

7. (a) 4, (b) $5\frac{5}{24}$, (c) $2\frac{2}{3}$.

Exercise 6.4 (pp. 119–120)

1. (a) 1220π, (b) $47\pi/10$, (c) 9π, (d) $405\pi/14$, (e) $377\pi/6$, (f) $57\pi/8$.

2. (a) $512\pi/15$, (b) $81\pi/35$.

3. (a) $15\pi/4$, (b) 36π.
4. (a) $32\pi/5$, (b) $7\pi/3$, (c) $\pi/2$, (d) $96\pi/5$.
5. (a) $202\pi/5$, (b) 20π.
6. (a) $9\pi/2$, (b) $53\pi/15$.
7. $\frac{1}{3}\pi r^2 h$.
8. $\frac{4}{3}\pi a^3$.
9. $32\pi/3$.
10. $8\pi/7$.

Exercise 6.5 (p. 123)

1. $28\pi/15$.
2. $16\pi/15$.
3. (a) 24 m, (b) 21 m, (c) $14\frac{1}{3}$ m, (d) 47 m, (e) 30 m, (f) 41 m.
4. $64\pi/15$.
5. 8π.
6. $\frac{1}{3}a^2 h\ \text{cm}^3$.
7. $960\sqrt{3}\ \text{cm}^3$.

Exercise 6.6 (p. 125)

1. (a) $22\ \text{m s}^{-1}$, (b) $78\ \text{m s}^{-2}$.
2. (a) $3/4\ \text{m s}^{-1}$, (b) $4/5\ \text{m s}^{-2}$.
3. (a) 1, (b) $8\frac{2}{3}$, (c) -9, (d) 16.
4. $85\pi/6$.
5. 8.
6. (a) 93, (b) 1/64, (c) 496/15.

Exercise 6.7 (pp. 125–127)

1. (a) $y = 2x - x^2 + c$, (b) $y = x^{1/2} + c$, (c) $y = x - \dfrac{1}{2x^2} + c$.
2. (a) $\frac{2}{3}x^6 - \frac{1}{4}x^4 + c$, (b) $4x + \dfrac{3}{x} + c$, (c) $\frac{1}{2}x^2 - \dfrac{1}{2x^2} + c$.
3. (a) $4t - 6t^2 + 3t^3 + c$, (b) $x^5 + \dfrac{1}{15x^3} + c$, (c) $\frac{3}{2}u^{2/3} + c$.
5. $y = x^2 + \dfrac{2}{x} - 3$.
6. (a) $-\frac{11}{24}$, (b) $4\frac{1}{4}$.
7. 1.
8. $2\frac{2}{3}$.
9. $2\frac{1}{2}$.
10. $1\frac{1}{3}$.
11. $y = 2x - 2$; $1\frac{1}{3}$.
12. $4\frac{1}{2}$.
13. $\frac{1}{3}$.
14. 2.
15. (a) 24π, (b) 12π.
16. (a) $52\pi/5$, (b) 18π, (c) $76\pi/15$, (d) $10\pi/3$.
17. (a) $4\pi/3$, (b) $16\pi/15$.
19. $\pm\left(\dfrac{3}{2\sqrt{x}} - \dfrac{3}{2}\sqrt{x}\right)$; (a) 1, (b) 0; $27\pi/4$.
20. 3.
21. 10.
22. (a) $41\frac{23}{25}$ m, (b) 101 m.
24. $\frac{2}{3}\pi a^2$.

Exercise 7.1 (pp. 132–133)

1. (a) $-\cos 15°$, (b) $-\sin 35°$, (c) $-\tan 50°$, (d) $\sin 68°$,
   (e) $\tan 85°$, (f) $\cos 42°$, (g) $-\sin 15°$, (h) $-\cos 57°$.
2. (a) $-\tan 56°$, (b) $\sec 23°$, (c) $\cot 68°$, (d) $-\sin 18°$,
   (e) $-\operatorname{cosec} 80°$, (f) $-\cot 15°$, (g) $-\cos 6°$, (h) $-\sec 81°$.
3. (a) $-320°, -220°, 40°, 140°, 400°, 500°$,
   (b) $-350°, -10°, 10°, 350°, 370°, 710°$,
   (c) $-285°, -105°, 75°, 255°, 435°, 615°$,
   (d) $-180°, 180°, 540°$.
4. (a) $25°, 335°$, (b) $50°, 130°$, (c) $42{\cdot}3°, 222{\cdot}3°$,
   (d) $243{\cdot}5°, 296{\cdot}5°$.
5. (a) $\sqrt{3}/2$, (b) $\sqrt{3}$, (c) $-\sqrt{3}/2$, (d) -1, (e) $-1/\sqrt{2}$, (f) $\sqrt{3}/2$,
   (g) -2, (h) $2/\sqrt{3}$.
6. (a) 0, (b) $-\sqrt{3}$, (c) 0, (d) -2, (e) 0, (f) -1, (g) $-\sqrt{2}$, (h) -1.
7. (a) $\theta = 90°, 270°$, $\phi = 0°, 180°, 360°$,
   (b) $\theta = 0°, 180°, 360°$, $\phi = 90°, 270°$,
   (c) $\theta = 90°$, $\phi = 0°, 360°$,
   (d) $\theta = 90°$, $\phi = 180°$.
8. (a) $0° < \theta < 45°$, (b) $30° < \theta < 90°$, (c) $36° < \theta < 72°$,
   (d) $0° < \theta < 15°, 30° < \theta < 45°, 60° < \theta < 75°$.

Exercise 7.2 (pp. 135–136)

1. (a) $17{\cdot}5°, 162{\cdot}5°$, (b) $56{\cdot}3°, 236{\cdot}3°$, (c) $136{\cdot}9°, 223{\cdot}1°$,
   (d) $218{\cdot}3°, 321{\cdot}7°$.
2. (a) $-53{\cdot}1°, 53{\cdot}1°$, (b) $53{\cdot}1°, 126{\cdot}9°$, (c) $-31{\cdot}8°, 148{\cdot}2°$,
   (d) $-110°, 110°$.
3. (a) $45°, 135°$, (b) $150°, 330°$, (c) $45°, 225°$, (d) $120°, 240°$,
   (e) $180°$, (f) $30°, 150°$.
4. (a) $36{\cdot}9°, 216{\cdot}9°$, (b) $135°, 315°$, (c) $48{\cdot}6°, 131{\cdot}4°$
   (d) $48{\cdot}2°, 311{\cdot}8°$.
5. (a) $106{\cdot}1°, 233{\cdot}9°$, (b) $41{\cdot}9°, 198{\cdot}1°$, (c) $97{\cdot}2°, 342{\cdot}8°$,
   (d) $119{\cdot}2°, 299{\cdot}2°$, (e) $17{\cdot}5°, 107{\cdot}5°, 197{\cdot}5°, 287{\cdot}5°$,
   (f) $210{\cdot}4°$, (g) $112{\cdot}2°, 247{\cdot}8°$, (h) $90°, 270°$
6. (a) $-75°, 15°, 45°$, (b) $-90°, -45°, 0°, 45°, 90°$,
   (c) $-81°, -45°, -9°, 27°, 63°$, (d) $-80°, -40°, 40°, 80°$.
7. (a) $60°, 120°, 240°, 300°$, (b) $60°, 90°, 270°, 300°$,
   (c) $0°, 78{\cdot}5°, 180°, 281{\cdot}5°, 360°$, (d) $270°$,
   (e) $0°, 30°, 150°, 180°, 360°$, (f) $56{\cdot}3°, 90°, 236{\cdot}3°, 270°$.
8. (a) $-135°, -26{\cdot}6°, 45°, 153{\cdot}4°$, (b) $-165{\cdot}5°, -14{\cdot}5°$,
   (c) $-60°, 0°, 60°$, (d) $-111{\cdot}8°, -45°, 68{\cdot}2°, 135°$.
9. (a) $x = 30°, y = 150°$; $x = 120°, y = 60°$,
   (b) $x = -45°, y = -165°$; $x = 75°, y = -45°$; $x = 135°, y = 15°$.

Exercise 7.3 (pp. 139–140)

2. (a) $\sin\theta$, (b) $\cos\theta$, (c) $\sin\theta$, (d) $-\cos\theta$, (e) $-\cos\theta$, (f) $-\cos\theta$.

3. (a) $-\cot\theta$, (b) $\tan\theta$, (c) $\cot\theta$, (d) $-\tan\theta$, (e) $\tan\theta$, (f) $-\cot\theta$.
4. 3/5, 3/4, 5/4.
5. 8/17, −17/15, −15/8.
6. −24/25, −7/25, −25/24.
7. 5/13, −12/13, 12/5.
8. (a) $\cos\theta$, (b) $\tan\theta$, (c) $\sin\theta$.
9. (a) 1, (b) $\sin^4\theta$, (c) $\cos\theta+\sin\theta$, (d) $1+\cos\theta$.
11. (a) $x^2+y^2=1$, (b) $(x-1)^2/4+(y-1)^2/9=1$,
    (c) $x^2-y^2=1$, (d) $x^2+y^2=2$.
12. $t=2/5$, $\theta=126{\cdot}9°$ or $t=-2/5$, $\theta=306{\cdot}9°$.
13. 199·5°, 340·5°.
14. 41·4°, 120°, 240°, 318·6°.
15. 60°, 300°.
16. 51·3°, 128·7°.
17. 70·5°, 120°, 240°, 289·5°.
18. 19·5°, 30°, 150°, 160·5°.
19. 26·6°, 116·6°, 206·6°, 296·6°.
20. 18·4°, 135°, 198·4°, 315°.
21. 63·4°, 135°, 243·4°, 315°.
22. 45°, 76°, 225°, 256°.
23. $(\sin x+1)^2-2$; 2, −2.
24. $\{y : y \geqslant 1\}$.

Exercise 7.4 (pp. 143–144)

1. (a) $\frac{1}{4}(\sqrt{6}+\sqrt{2})$, (b) $\frac{1}{4}(\sqrt{2}-\sqrt{6})$, (c) $\sqrt{3}-2$, (d) $2-\sqrt{3}$.
2. (a) $\frac{1}{2}$, (b) $\frac{1}{2}$, (c) 0, (d) 1.
3. (a) $\sqrt{3}$, (b) $1/\sqrt{3}$, (c) $\frac{1}{2}\sqrt{6}$, (d) $\sqrt{2}$.
4. 56/65, −33/65.
5. $-29/35$, $-8\sqrt{6}/35$.
6. 225°.
7. $4\frac{1}{2}$.
8. (a) $2\sin A\cos B$, (b) $2\cos A\cos B$, (c) $\cos B$.
9. 12°, 60°, 84°, 132°, 156°.
10. −135°, 45°.
11. (a) 26·6°, 108·4°, 206·6°, 288·4°,
    (b) 40·9°, 220·9°, (c) 0°, 180°, 360°, (d) 135°, 315°.
12. $\dfrac{\cot A\cot B-1}{\cot A+\cot B}$.

Exercise 7.5 (pp. 146–147)

1. 24/25. −7/25, 44/125.
2. 1/9, $-4\sqrt{5}$, −79/81.
3. 4/5, 4/3, $5\frac{1}{2}$.
4. $-3/4$, $-\sqrt{7}/3$, $2\sqrt{2}$.
5. $-1/\sqrt{5}$, $2/\sqrt{5}$, $-2/5\sqrt{5}$.
6. (a) $1+1/\sqrt{2}$, (b) $\sqrt{3}/2$, (c) 1/4, (d) $1/\sqrt{2}$.
7. (a) 0°, 60°, 180°, (b) 0°, 120°, (c) 0°, 60°, 120°, 180°,
   (d) 0°, 45°, 135°, 180°.
16. (a) 120°, 240°, (b) 41·8°, 138·2°, 270°,
    (c) 45°, 135°, 225°, 315°, (d) 0°, 104·5°, 180°, 255·5°, 360°.
17. (a) 135°, 161·6°, 315°, 341·6°,
    (b) 23·2°, 135°, 203·2°, 315°.
18. (a) −126·9°, 43·6°, (b) −90°, 53·1°, (c) −143·1°, 90°,
    (d) −36·9°, 112·6°.
19. 45°, 135°, 225°, 315°.
20. $4\cos^3\theta-3\cos\theta$; 90°, 120°, 240°, 270°.

Exercise 7.6 (pp. 149–150)

1. (a) $\sin 3x + \sin x$, (b) $\cos 8\theta + \cos 2\theta$, (c) $3(\cos A - \cos 5A)$,
   (d) $\frac{1}{2}(\sin 6t - \sin 2t)$, (e) $\cos 2A + \cos 2B$, (f) $\frac{1}{4}(1 + 2\sin 2x)$.
2. (a) $1 + \sqrt{3}$, (b) $5\sqrt{2}$, (c) $2 + \sqrt{3}$, (d) $3 + 2\sqrt{2}$.
3. (a) 0°, 180°, 360°, (b) 60°, 120°, 240°, 300°,
   (c) 0°, 60°, 180°, 240°, 360°,
   (d) 111·3°, 168·7°, 291·3°, 348·7°.
4. (a) $2\sin 4x \cos 3x$, (b) $2\cos 4A \cos A$, (c) $2\cos\theta \sin\alpha$,
   (d) $-2\sin 2x \sin x$.
5. (a) $\cot 5\theta$, (b) $-\tan 3x$, (c) $\cot 2A$, (d) $-\tan^2\theta$.
6. (a) $\sqrt{3}$, (b) $1/\sqrt{3}$.
7. (a) 0°, 90°, 180°, (b) 0°, 30°, 90°, 150°, 180°,
   (c) 30°, 120°, 150°, (d) $22\frac{1}{2}°$, 45°, $112\frac{1}{2}°$.
8. (a) 0°, 90°, 120°, 180°, (b) 15°, 30°, 75°, 90°, 150°,
   (c) 0°, 54° 126°.

Exercise 7.7 (pp. 151–152)

1. (a) $\sqrt{2}\cos(\theta + 45°)$; $\sqrt{2}$, $\theta = -45°, 315°$; $-\sqrt{2}$, $\theta = -225°, 135°$,
   (b) $2\cos(\theta + 30°)$; 2, $\theta = -30°, 330°$; -2, $\theta = -210°, 150°$,
   (c) $13\cos(\theta - 67{\cdot}4°)$; 13, $\theta = -292{\cdot}6°$, 67·4°; -13, $\theta = -112{\cdot}6°$, 247·4°,
   (d) $\sqrt{5}\cos(\theta - 26{\cdot}6°)$; $\sqrt{5}$, $\theta = -333{\cdot}4°, 26{\cdot}6°$; $-\sqrt{5}$, $\theta = -153{\cdot}4°$, 206·6°.
2. (a) 41·6°, 244·7°, (b) 26·2°, 110·2°, (c) 180°, 313·6°,
   (d) 98°, 205·9°.
3. $\sqrt{89}\sin(\theta - 58°)$; (a) 97·5°, 198·5°, (b) 90°, 206°.
4. (a) 10·9°, 231°, (b) 257·6°, 349·8°.

Exercise 7.8 (pp. 152–154)

1. (a) 70°, 110°, (b) 115°, 155°, 295°, 335°, (c) 145°, 215°,
   (d) 25°, 85°, 145°, 205°, 265°, 325°.
2. (a) 0·8, (b) −0·6, (c) −0·8, (d) −0·6, (e) −0·75, (f) 0·75.
3. (a) 56·3°, 236·3°, (b) 48·2°, 120°, 240°, 311·8°,
   (c) 0°, 53·1°, 180°, 306·9°, 360°, (d) 23·6°, 156·4°.
4. (a) $2k^2 - 1$, (b) $(4k^2 - 1)\sqrt{(1 - k^2)}$, (c) $\sqrt{\left(\frac{1 + k}{2}\right)}$. (d) $\sqrt{\left(\frac{1 - k}{1 + k}\right)}$.
5. (a) $(x - 1)^2 + 4y^2 = 1$, (b) $xy = 1$, (c) $y = 2x^2$,
   (d) $y^2 = 4x(2 - x)(x - 1)^2$.
6. (a) 10·9°, 100·9°, (b) 11·8°, 78·2°,
   (c) 0°, 45°, 135°, 180°,
   (d) 0°, 60°, 90°, 120°, 180°.
7. $56{\cdot}3° \leqslant x \leqslant 236{\cdot}3°$. 8. 8.
9. (a) max 1, min −1, (b) max 2, min 0, (c) max 2, min −2,
   (d) max 4, min 0.

11. (a) 79·7°, 153·4°, (b) $\left(\frac{\lambda+1}{\lambda-1}\right)\tan\alpha$; 120°, 300°.
12. (a) 45°, 120°, 135°, 225°, 240°, 315°,
    (b) $5\cos(x - 36{\cdot}9°)$; 103·3°, 330·4°,
    (c) 90°, 270°; 0°, 180°, 360°.
13. $\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$; 0°, 19·1°, 160·9°, 180°.
14. $\frac{2t}{1-t^2}, \frac{1+t^2}{1-t^2}$; 18°.
15. (a) $25\sin(x - 73{\cdot}7°)$; 110·6°, 216·9°, (b) 60°, 60°.
16. 90°, 323·1°; 53·1°, 180°. 18. (b) 130·2°, 342·4°.
19. 36·9°; −36·9°, 143·1°; 0°, −73·7°.
20. (a) 49·8°, 130·2°, (b) 26·6°, 108·4°, (c) 30°, 90°, 150°,
    (d) $22\frac{1}{2}°$, 30°, $67\frac{1}{2}°$, $112\frac{1}{2}°$, 150°, $157\frac{1}{2}°$.
21. 36·9°, 61·9°.
22. (a) 30°, (b) 180°, (c) −45°, (d) 30°, (e) 90°, (f) 90° or −90°,
    (g) 45°, (h) 45°.
23. $k = 2$; $-\sqrt{2}, \frac{1}{2}(\sqrt{2} \pm \sqrt{6})$.
24. (a) 0°, 48·6°, 131·4°, 180°, 270°,
    (b) $R = \sqrt{(2\lambda^2 - 2\lambda + 1)}$, $\tan\phi = (1-\lambda)/\lambda$; min $-\sqrt{(2\lambda^2 - 2\lambda + 1)}$.

Exercise 8.1 (p. 157)

1. $\angle B = \angle A + \angle C$.
2. (a) $x + y = 180°$, (b) $z = 2y - 180°$.
3. 5:12.
6. A is the incentre of $\triangle DEF$ when $\angle A$ is obtuse.

Exercise 8.2 (pp. 162–163)

1. $C = 70°$, $a = 4{\cdot}61$, $b = 4{\cdot}08$.
2. $B = 61°$, $a = 6{\cdot}69$, $c = 8{\cdot}70$.
3. $A = 33{\cdot}6°$, $B = 95{\cdot}2°$, $c = 7{\cdot}04$.
4. $A = 26{\cdot}5°$, $C = 108{\cdot}5°$, $b = 3{\cdot}80$.
5. $A = 40{\cdot}8°$, $B = 60{\cdot}6°$, $C = 78{\cdot}6°$.
6. $A = 36{\cdot}9°$, $B = 90°$, $C = 53{\cdot}1°$.
7. $B = 67{\cdot}3°$, $C = 45{\cdot}4°$, $c = 7.72$.
8. $B = 35{\cdot}6°$, $C = 23{\cdot}4°$, $a = 10{\cdot}2$.
9. $A = 103°$, $b = 2{\cdot}57$, $c = 5{\cdot}31$.
10. $A = 87{\cdot}1°$, $C = 37{\cdot}9°$, $a = 14{\cdot}6$.
11. $A = 112{\cdot}4°$, $B = 29{\cdot}5°$, $C = 38°$.
12. $A = 60°$, $B = 90°$, $a = 5{\cdot}20$.
13. 4·83, 1·86.
14. 15·9, 3·97.
15. 27·5 cm².
16. 30°; 14 cm.
17. 4·07 cm, 11·5 cm.
18. 4·21 cm, 2·26 cm.
21. (a) 82·9°, 45·1°, (b) 115·9°, 24·1°.
22. $\triangle = rs$; $r = \sqrt{3} = 1{\cdot}73$.

Exercise 8.3 (p. 164)

1. 13·4 km; 098°.
2. 14·1 km; S 23·9° E.
3. 19 min 15 sec.
4. 6·55 cm.
5. 13·5.
6. 20·2 m.
7. 1350 m.
8. 71·1 cm^2.
9. 10 cm; 36·9°.
10. 2/3, 7 cm.

Exercise 8.4 (pp. 167–168)

1. (a) 17·2°, (b) 44·4°, (c) 18·1°.
2. (a) 44·4°, (b) 63°, (c) 90°.
3. (a) 58°, (b) 53·9°, (c) 63·4°.
4. (a) 27°, (b) 32·7°.
5. (a) 53°, (b) 61·9°, (c) 64·8°, (d) 90°.
6. (a) 49·2°, (b) 65·4°.
7. (a) $a\sqrt{2}$, (b) 70·5°, (c) 109·5°.
8. (a) 1/3, (b) $2\sqrt{6}$.
9. (a) 22·5°, (b) 39·7°, (c) 112·9°, (d) 117·8°.
10. (a) 73°, (b) 83·3°, (c) 70·6°, (d) 96°.

Exercise 8.5 (pp. 170–171)

1. 76·7°, 474 cm^2.
2. 93·7 m, 56·5°.
3. (a) 27·6°, (b) 53·4°.
4. 14·7°; 13·4° or 87·4°.
5. 060·5°, 38·2°.
6. 051·4°, 125·7°.
7. 20·4°, 43·2°.
8. 2, 26·6°, 18·4°.
10. 9·17 m, 15·6 m; 19·1°, 26·6°.
11. 6·6.

Exercise 8.6 (pp. 172–174)

1. 33·6°, 7·54 cm.
3. 62·6°, 91·1°, 153·7°; 7·70 cm.
5. (i) 2; $c = \frac{1}{2}(5\sqrt{3} \pm \sqrt{11})$, (ii) none, (iii) 1; $c = \sqrt{3} + 2\sqrt{2}$, (iv) 1; $c = 2\sqrt{2} - \sqrt{3}$.
9. $\tan^2 \frac{1}{2}A = \dfrac{(s-b)(s-c)}{s(s-a)}$; area $= \sqrt{\{s(s-a)(s-b)(s-c)\}}$.
10. 124°, 28°, 28°.
11. S2θ° W.
12. 32·3 km h^{-1}.
15. (i) $\frac{1}{2}a\sqrt{10}$, (ii) $a\sqrt{3}$, (iii) 90°, (iv) $\frac{1}{3}\sqrt{2}$.
16. (i) 90°, (ii) 30°, (iii) 45°.
17. 60°, 41·4°; 49·1°.
20. 60 cm; 688 cm^2.
21. $a\sqrt{6}$.
22. (i) $2a\sqrt{5}$, (ii) $-1/5$, (iii) $4/\sqrt{30}$.
23. (i) 836 m, (ii) 042·9°, (iii) 13·8°.

Exercise 9.1 (p. 177)

1. (a) **PS**, (b) **AE**, (c) **AB**, (d) **0**, (e) **MR**, (f) **PS**.
2. (a) false, (b) true, (c) true, (d) true, (e) false, (f) true.
3. (a) true, (b) false, (c) true, (d) true, (e) false, (f) false.
4. $\mathbf{x}+\mathbf{y}$, $\mathbf{y}+\mathbf{z}$, $\mathbf{x}+\mathbf{y}+\mathbf{z}$.
5. $\mathbf{a}+\mathbf{d}$, $\mathbf{d}-\mathbf{b}$, $\mathbf{d}-\mathbf{b}-\mathbf{c}$, $\mathbf{a}+\mathbf{d}-\mathbf{c}$.

Exercise 9.2 (pp. 180–181)

1. (a) 3 km E, (b) 10 km N, (c) 1 km N 30° E, (d) 1 km S.
2. (a) $2\sqrt{2}$ km NE, (b) $2\sqrt{2}$ km NE, (c) $2\sqrt{2}$ km SE, (d) $2\sqrt{2}$ km NW.
3. (a) 1 km S 30° E, (b) $\sqrt{3}$ km S, (c) $\sqrt{3}$ km S 60° E, (d) $\sqrt{3}$ km N 60° E.
4. $\mathbf{p}+\mathbf{q}$, $\mathbf{q}-\mathbf{p}$, $\frac{1}{2}(\mathbf{p}+\mathbf{q})$, $\frac{1}{2}(\mathbf{q}-\mathbf{p})$.
5. $-\mathbf{a}$, $\mathbf{a}-\mathbf{b}$, $2\mathbf{b}$, $2\mathbf{b}-\mathbf{a}$.
6. $2\mathbf{a}+\mathbf{b}$, $\mathbf{a}+\mathbf{b}$.
7. $2\mathbf{a}-\mathbf{b}$, $\mathbf{a}+\mathbf{b}$, $\mathbf{b}-\mathbf{a}$, $\frac{1}{2}\mathbf{b}$.
8. (a) $2\mathbf{x}+\mathbf{y}$, (b) $\mathbf{u}-2\mathbf{w}$, (c) $3\mathbf{p}+7\mathbf{q}$.
16. $\mathbf{a}$ must be a positive multiple of $\mathbf{b}$.

Exercise 9.3 (pp. 183–184)

1. (a) $2\mathbf{b}$, (b) $-3\mathbf{a}$, (c) $2\mathbf{b}-6\mathbf{a}$, (d) $\mathbf{b}-\mathbf{a}$, (e) $\mathbf{a}$, (f) $2\mathbf{b}-\mathbf{a}$.
2. (a) $\mathbf{z}$, (b) $\mathbf{x}$, (c) $-\mathbf{x}$, (d) $\mathbf{z}$, (e) $-\mathbf{x}+\mathbf{z}$, (f) $\mathbf{y}-\mathbf{z}$.
3. (a) $-\mathbf{x}+\mathbf{y}$, (b) $\mathbf{x}+\mathbf{z}$, (c) $-\mathbf{x}+\mathbf{y}-\mathbf{z}$, (d) $2\mathbf{x}-\mathbf{y}$,
   (e) $2\mathbf{x}-\mathbf{y}+2\mathbf{z}$, (f) $-\mathbf{y}+2\mathbf{z}$.
4. (a) $5\mathbf{i}+3\mathbf{j}$, (b) $-\mathbf{i}+6\mathbf{j}$, (c) $-4\mathbf{i}+2\mathbf{j}$, (d) $2\mathbf{i}-\mathbf{j}$,
   (e) $3\mathbf{i}+4\mathbf{j}$, (f) $\mathbf{i}+5\mathbf{j}$.
5. (a) $-4\mathbf{i}+5\mathbf{j}+2\mathbf{k}$, (b) $3\mathbf{j}+2\mathbf{k}$, (c) $\mathbf{i}-\mathbf{j}+\mathbf{k}$, (d) $\mathbf{i}+2\mathbf{j}+3\mathbf{k}$.
6. (a) 13, (b) $\sqrt{5}$, (c) 10, (d) 3, (e) 7, (f) $5\sqrt{3}$.
7. (a) $\frac{8}{9}\mathbf{i}-\frac{4}{9}\mathbf{j}-\frac{1}{9}\mathbf{k}$, (b) $\frac{8}{9}\mathbf{i}-\frac{4}{9}\mathbf{j}-\frac{1}{9}\mathbf{k}$, (c) $-\frac{8}{9}\mathbf{i}+\frac{4}{9}\mathbf{j}+\frac{1}{9}\mathbf{k}$.
8. (a) 5, (b) 3, (c) 6.
9. (a) $\sqrt{29}$, (b) $\sqrt{29}$, (c) 5.
10. (a) $\frac{1}{\sqrt{2}}\mathbf{i}-\frac{1}{\sqrt{2}}\mathbf{j}$, (b) $-\frac{7}{9}\mathbf{i}+\frac{4}{9}\mathbf{j}+\frac{4}{9}\mathbf{k}$, (c) $\mathbf{i}$.
11. ± 2.
12. $10 \leqslant |\mathbf{a}+\mathbf{b}| \leqslant 40$.
14. (a) $\frac{1}{2}, \frac{1}{2}$, (b) $-4, -2$.

Exercise 9.4 (pp. 186–188)

1. $\mathbf{b}-\mathbf{a}$, $\mathbf{b}-2\mathbf{a}$; $2\mathbf{b}-2\mathbf{a}$, $2\mathbf{b}-3\mathbf{a}$, $\mathbf{b}-2\mathbf{a}$.
2. $\mathbf{a}-\mathbf{b}$, $\mathbf{c}-\mathbf{b}$, $\mathbf{a}-2\mathbf{b}+\mathbf{c}$; $2\mathbf{a}-3\mathbf{b}+2\mathbf{c}$.
3. $\frac{1}{4}(\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d})$.
4. $2\mathbf{i}+3\mathbf{j}$, $5\mathbf{i}-\mathbf{j}$, $4\mathbf{i}+4\mathbf{j}$; $-\mathbf{i}+5\mathbf{j}$; $\mathbf{i}+8\mathbf{j}$, $(1, 8)$.
5. (a) 6, $-\frac{1}{3}\mathbf{i}+\frac{2}{3}\mathbf{j}+\frac{2}{3}\mathbf{k}$, $2\mathbf{i}-3\mathbf{j}+\mathbf{k}$,
   (b) 22, $\frac{6}{11}\mathbf{i}-\frac{9}{11}\mathbf{j}-\frac{2}{11}\mathbf{k}$, $-\mathbf{i}+4\mathbf{j}-2\mathbf{k}$,

(c) $\sqrt{10}, \dfrac{3}{\sqrt{10}}\mathbf{j} - \dfrac{1}{\sqrt{10}}\mathbf{k}, \mathbf{i} + \dfrac{1}{2}\mathbf{j} + \dfrac{5}{2}\mathbf{k}$,

(d) $3\sqrt{6}, -\dfrac{2}{\sqrt{6}}\mathbf{i} - \dfrac{1}{\sqrt{6}}\mathbf{j} - \dfrac{1}{\sqrt{6}}\mathbf{k}, 2\mathbf{i} + \dfrac{5}{2}\mathbf{j} - \dfrac{1}{2}\mathbf{k}$.

6. $2\mathbf{i}-\mathbf{j}+5\mathbf{k}$, $2\mathbf{j}-\mathbf{k}$, $-2\mathbf{i}+4\mathbf{j}+3\mathbf{k}$; 7, $(-1,3,1)$; $(0,1,9)$.
7. $\mathbf{XY} = 4\mathbf{i}+2\mathbf{j}-2\mathbf{k}$, $\mathbf{XZ} = -2\mathbf{i}-\mathbf{j}+\mathbf{k}$.
9. 50.
10. (a) 1, (b) -1, (c) -2, (d) 4/3.
11. $\frac{1}{2}(\mathbf{p}+\mathbf{q})$, $\frac{1}{3}(\mathbf{p}+\mathbf{q})$, $-\frac{2}{3}\mathbf{p}+\frac{1}{3}\mathbf{q}$; $(1-\frac{2}{3}k)\mathbf{p}+\frac{1}{3}k\mathbf{q}$.
12. $\dfrac{2}{9}(\mathbf{a}+\mathbf{b})$; $\dfrac{1}{2}, \dfrac{9}{4}$; $\dfrac{1}{2}(\mathbf{a}+\mathbf{b})$.
13. $\dfrac{1}{7}\mathbf{a} + \dfrac{4}{7}\mathbf{b}$; $\dfrac{1}{5}\mathbf{a} + \dfrac{4}{5}\mathbf{b}$.

Exercise 9.5 (pp. 188–189)

4. (ii) $\frac{1}{3}\mathbf{PQ}+\frac{2}{3}\mathbf{PS}$, $-\frac{2}{3}\mathbf{PQ}+\frac{2}{3}\mathbf{PS}$, $-\frac{1}{3}\mathbf{PQ}+\frac{1}{3}\mathbf{PS}$.
5. $\mathbf{a}+\mathbf{b}+\mathbf{c}$, $\mathbf{b}+\mathbf{c}-\mathbf{a}$, $\mathbf{a}+\mathbf{c}-\mathbf{b}$, $\mathbf{a}+\mathbf{b}-\mathbf{c}$.
6. (a) $\mathbf{b}-\mathbf{a}$, (b) $3\mathbf{b}-2\mathbf{a}$, (c) $3\mathbf{b}-\mathbf{c}$, (d) $\mathbf{a} - \dfrac{3}{2}\mathbf{b}$, (e) $-\mathbf{a} - \dfrac{1}{2}\mathbf{b}$,

(f) $\mathbf{a} + \dfrac{3}{2}\mathbf{b}$.

7. (a) $2\sqrt{6}, 5\sqrt{2}, 20$, (b) $\frac{2}{3}\mathbf{i}+\frac{1}{3}\mathbf{j}-\frac{2}{3}\mathbf{k}$.
8. (a) $6\sqrt{3}$, (b) $\dfrac{1}{9}\sqrt{3}\mathbf{i} - \dfrac{5}{9}\sqrt{3}\mathbf{j} + \dfrac{1}{9}\sqrt{3}\mathbf{k}$, (c) $-2\mathbf{j}+3\mathbf{k}$.
9. 1, 4.
11. (a) $-\dfrac{1}{\sqrt{2}}\mathbf{i} + \dfrac{1}{\sqrt{2}}\mathbf{j}$, (b) $\dfrac{3}{2}\sqrt{29}$.
12. $\mathbf{a}+k(2\mathbf{b}-\mathbf{a})$; 3, $6\mathbf{b}-2\mathbf{a}$.
13. $\frac{1}{3}\mathbf{a}+\frac{1}{3}\mathbf{b}$.
14. $(4, -7, 8)$: 54 sq. units.
15. $s+t = 1$.
16. $\frac{1}{3}\mathbf{p}+\frac{1}{3}\mathbf{q}$; $\frac{1}{3}\mathbf{a}+\frac{1}{3}\mathbf{b}+\frac{1}{3}\mathbf{c}$.

Exercise 10.1 (pp. 192–193)

1. (a) $2x^2-1$, (b) $4x^2+4x$, (c) x^4-2x^2, (d) $4x+3$.
2. (a) $2x^2-2$, (b) $4x-4$, (c) $4x^2-4x+1$, (d) x^4-1.
3. $x+2$.
4. $4-x^2$.
5. (a) $\sqrt{u}$, (b) $u^5/32$, (c) $-1/u$, (d) $\frac{1}{2}u^8(u+1)$.
6. $6(x+3)^5$.
7. $8(2x-1)^3$.
8. $-21(5-3x)^6$.
9. $-6(3x-2)^{-3}$.
10. $15x^2(x^3-1)^4$.
11. $8x(4-x^2)^{-5}$.

12. $\frac{3}{2}(1+3x)^{-1/2}$.
13. $-2(6x+1)^{-4/3}$.
14. $15x(3x^2-1)^{3/2}$.
15. $-3\left(1+\frac{1}{x^2}\right)\left(x-\frac{1}{x}\right)^{-4}$.
16. $\frac{x}{\sqrt{(1-x^2)^3}}$.
17. $\frac{-1}{\sqrt{x(1+\sqrt{x})^3}}$.
18. -12; $12x+y+11=0$.
19. $3x+8y=22$.
20. $1\,\mathrm{m\,s^{-1}}$, $-2\,\mathrm{m\,s^{-2}}$.
21. $1-4/(t+1)^2$, $3\,\mathrm{m\,s^{-1}}$.
22. (a) $(6x+1)(x+1)^4$, (b) $2/(x+1)^2$, (c) $\frac{1}{2}(x+2)(x+1)^{-3/2}$.
23. (a) $\frac{4(2x-3)^3}{\sqrt{\{(2x-3)^4-1\}}}$, (b) $\frac{-4x}{\{4+\sqrt{(4x^2+1)}\}^2\sqrt{(4x^2+1)}}$.
24. $(1,256)$ max, $(5,0)$ min.
25. $(1,1)$ max.

Exercise 10.2 (pp. 195–196)

1. $3/y$.
2. $-x/y$.
3. $-4x/y$.
4. $2x/3y^2$.
5. $1/(y-1)$.
6. $(x^2-1)/2y$.
7. $\frac{2-2x}{1+2y}$.
8. $\frac{1}{(x+y)^2}-1$.
9. 2.
10. $\frac{1}{2}$, $-\frac{1}{2}$.
11. $3x+4y+5=0$.
12. $4y=5x-9$.
13. $-5/3$, -1.
14. $(3,9)$, $(3,-1)$.

Exercise 10.3 (pp. 199–201)

1. $5x^4+12x^3-1$.
2. $3x^2-7$.
3. $12x+15x^2-75x^4$.
4. $4x^3+4/x^3$.
5. $x(5x+6)(x+3)^2$.
6. $(18x-1)(3x-1)^4$.
7. $(x+1)(7x+9)(x+2)^4$.
8. $-2(3+7x)(2x+3)^2(1-x)^3$.
9. $\frac{x(2+3x^2)}{\sqrt{(1+x^2)}}$.
10. $\frac{2-9x^2-5x^3}{2\sqrt{(1-x^3)}}$.
11. $\frac{4}{(x+2)^2}$.
12. $\frac{2x(x+1)}{(2x+1)^2}$.
13. $\frac{11}{(2-3x)^2}$.
14. $\frac{x(2-3x-x^3)}{(x^3+1)^2}$.
15. $\frac{6x}{(x^2+1)^2}$.
16. $\frac{1+x^2}{(1-x^2)^2}$.
17. $\frac{4(x+1)}{(x+2)^3}$.
18. $\frac{-x(2+x)}{(x-1)^4}$.
19. $-\frac{8x(2x+1)}{(4x+3)^4}$.
20. $\frac{1}{(1-x^2)^{3/2}}$.

21. $\dfrac{x^2(3-4x^2)}{(1-2x^2)^{3/2}}$.
22. $\dfrac{x-4}{(x+1)^2\sqrt{(x^2+4)}}$.
23. $\dfrac{2(x-1)}{(2-x)^3}$.
24. $\dfrac{-1}{\sqrt{\{2x(x-1)^3\}}}$.
25. $\dfrac{x^2-1}{2x\sqrt{(x+x^3)}}$.
26. (a) $\dfrac{2x-y}{x-2y}$, (b) $-\dfrac{2xy}{x^2+3y^2}$, (c) $\dfrac{2x-6y+3}{6x-2y+2}$.
27. $-5/12$.
28. $4y = 3x+14$.
29. (a) $\dfrac{2(3x^2-1)}{(x^2+1)^3}$, (b) $\dfrac{2}{(x-1)^3}$, (c) $\dfrac{1}{(x^2+1)^{3/2}}$.
30. (a) $\dfrac{2x(x-1)(3x^2-3x+2)}{(2x-1)^2}$, (b) $\dfrac{x^2+3x-2}{2(x+1)^2\sqrt{(x-1)}}$,
(c) $\dfrac{3(x-1)}{2\sqrt{\{(x+1)^3(x^3+1)\}}}$.
32. $-3, -12$.
33. $1/12, -1/32$.

Exercise 10.4 (pp. 203–204)

1. Min. 0 when $x = 2\frac{1}{2}$.
2. Min. -27 when $x = -1$.
3. Min. 0 when $x = 0$ and 3; max. 16 when $x = 1$.
4. Min. -1 when $x = -1$; max. $\frac{1}{2}$ when $x = 2$.
5. Min. -3 when $x = -1$; max. 1 when $x = 1$.
6. Min. -1 when $x = 0$.
7. 17, -7.
8. $24\frac{4}{9}$, 12.
9. $0 < y \leqslant 4$; $(-1,3)$, $(1,3)$.
10. $-2\sqrt{3} \leqslant y \leqslant 2\sqrt{3}$; $(-3,-3)$, $(0,0)$, $(3,3)$.
11. 8.
12. 3.
13. 40 m.
14. 108 cm^2.
15. $V = 2\pi h(R^2-h^2)$.
16. $\frac{1}{3}a$.
17. $6-\sqrt{3}$.
18. $\sqrt{\left(\dfrac{S^3p^2}{8\pi(1+p)^3}\right)}$; 2.

Exercise 10.5 (pp. 205–206)

1. 60 cm^2/s.
2. 4/25 cm/s.
3. 4 cm/s.
4. -6.
5. 2000.
6. -50.
7. $(s^4-1)/s^3$ m s^{-2}.
8. $-2/(2s+1)^3$ m s^{-2}.
9. 4 cm^2/s.
10. (a) 6π cm^3/s, (b) 2π cm^2/s.
11. 0·01 m/s.
12. $\frac{1}{3}$ cm/s.

Exercise 10.6 (pp. 207–208)

1. 0·3.
2. 0·01.
3. $\pi/50$.
4. $1{\cdot}2\pi\,\text{cm}^2$.
5. (a) $7{\cdot}5\pi\,\text{cm}^3$, (b) $2{\cdot}7\pi\,\text{cm}^3$.
6. 2%.
7. 6%, 4%.
8. $2\frac{1}{2}$%.
9. 1% decrease.
10. 3; (a) 8·03, (b) 7·94.
11. (a) 24·24, (b) 23·52.
12. (a) 2·001, (b) 0·198, (c) 4·996, (d) 1·02.

Exercise 10.7 (pp. 208–210)

1. (a) $\dfrac{6(x-1)}{(5+2x-x^2)^4}$, (b) $\dfrac{3x^2}{\sqrt{(2x^3+5)}}$.
3. $2x+y=5$.
4. $\pm\frac{3}{4}$.
5. $5x-8y+22=0$.
6. (a) $2(9x^2+12x+1)(x^2+1)^3$, (b) $\dfrac{5x^2-2x-2}{\sqrt{(2x-1)}}$,
   (c) $\dfrac{5x-7}{(x+1)^3}$, (d) $-\dfrac{1}{\sqrt{\{(x+3)^3(x+5)\}}}$.
7. $\dfrac{4x}{\sqrt{(1+4x^2)}}$, $\dfrac{4}{\sqrt{(1+4x^2)^3}}$,
8. $\dfrac{x(2x^2-y^2)}{y(x^2-2y^2)}$.
9. 1, 6.
11. (0, 0) max, (2, −108) min, (5, 0) inflexion.
12. $(1,\frac{1}{2})$ max, $(-1,-\frac{1}{2})$ min; (0, 0) inflexion, $(\sqrt{3},\frac{1}{4}\sqrt{3})$ inflexion, $(-\sqrt{3},-\frac{1}{4}\sqrt{3})$ inflexion.
13. (i) $g'(x+g(a))$. (ii) $g'(x).g'(a+g(x))$, (iii) $2xg'(x^2)$.
15. $20/(\pi+2)$.
16. (a) $25\,\text{cm}^2$, (b) $8\sqrt{5}\,\text{cm}$.
17. −2.
18. (a) $4\pi\,\text{cm}^3/\text{s}$, (b) $17\pi/5\,\text{cm}^2/\text{s}$.
19. (a) 4·001, (b) 0·2008, (c) 2·001.
20. (a) 72·01, (b) 68·98.
21. $1\frac{1}{2}$%.
22. 0·08.
23. $64R^3/81$.
24. 0·5 cm/s.
25. $16\frac{1}{2}$ s, $a/27$ cm/s.

Exercise 11.1 (pp. 212–213)

1. (a) 180°, (b) 45°, (c) 30°, (d) 240°.
2. (a) $\pi/2$, (b) $\pi/3$, (c) $5\pi/6$, (d) $7\pi/4$.
3. (a) −1, (b) $-\frac{1}{2}$, (c) 1, (d) $2/\sqrt{3}$.
4. (a) $\pi/4$, (b) $-\pi/3$, (c) $\pi/6$, (d) π.
5. 15 cm, $75\,\text{cm}^2$.
6. $8-2\pi\,\text{cm}^2$.
7. $\frac{1}{2}r^2(2\theta+\sin 2\theta)$, $2r(1+\theta+\cos\theta)$.
8. (a) 1·2, (b) 0·4, (c) $2\pi/3$, (d) $\pi/6$.
9. 6.

10. (a) $\pi/6, 5\pi/6$, (b) $\pi/3, 2\pi/3, 4\pi/3, 5\pi/3$, (c) $\pi/4, 5\pi/4$.
11. (a) $\pi/12, 5\pi/12, 3\pi/4$, (b) $\pi/18, \pi/6, 5\pi/18, \pi/2, 13\pi/18, 5\pi/6, 17\pi/18$.
12. $-\pi < x < -\pi/2, \pi/4 < x < \pi/2, 3\pi/4 < x < 3\pi/2$.
13. 13·7 cm², 29·7 cm.
14. 98·8 cm².
15. 0·403, 13·0 cm².

Exercise 11.2 (p. 216)

1. $2n\pi \pm 2\pi/5$.
2. $n\pi + \pi/3$.
3. $n\pi + (-1)^n\pi/8$.
4. $(2n+1)\pi$.
5. $n\pi - \pi/4$.
6. $n\pi + (-1)^n\pi/6$.
7. $(2n+1)\pi/2$.
8. $(2n+1)\pi/4$.
9. $n\pi, 2n\pi \pm \pi/3$.
10. $2n\pi/3$.
11. $2n\pi \pm \pi/3$.
12. $n\pi + \pi/4$.
13. $n\pi/3$.
14. $(2n+1)\pi/10$.
15. $(2n+1)\pi/8$.
16. $(4n+1)\pi/16, (4n+1)\pi/4$.
17. $360n^\circ + 122{\cdot}3^\circ, 360n^\circ + 20{\cdot}8^\circ$.
18. $360n^\circ + 121{\cdot}9^\circ, 360n^\circ + 1{\cdot}9^\circ$.

Exercise 11.3 (p. 218)

1. (a) 0·0209, (b) 0·0122, (c) 0·000262.
2. (a) 2, (b) 3/2, (c) 4/9.
3. (a) $\frac{\sqrt{3}}{2} + \frac{1}{2}\theta - \frac{\sqrt{3}}{4}\theta^2$, (b) $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\theta - \frac{\sqrt{2}}{4}\theta^2$, (c) $1 - \frac{5}{2}\theta^2$.
4. (a) 1/2, (b) 3, (c) 5/4.
5. (a) 2, (b) 2/3.

Exercise 11.4 (pp. 221–222)

1. (a) $2\cos 2x$, (b) $-12\sin 4x$, (c) $2\sec^2 \frac{1}{2}x$.
2. (a) $-4\cos^3 x \sin x$, (b) $\cos x/2\sqrt{(\sin x)}$, (c) $5\sin 10x$.
3. (a) $3(x+\sin x)^2(1+\cos x)$, (b) $2x\cos(x^2)$, (c) $\frac{\pi}{30}\sec^2(6x^\circ)$.
4. (a) $x^3(4\cos x - x\sin x)$, (b) $8x\tan x + (4x^2+1)\sec^2 x$,
   (c) $x(2\sin 3x + 3x\cos 3x)$.
5. (a) $-\operatorname{cosec}(x+1)\cot(x+1)$, (b) $4\operatorname{cosec}^2(1-2x)$,
   (c) $3\sec(3x-4)\tan(3x-4)$.
6. (a) $-\sin x \sec^2(\cos x)$, (b) $\frac{1}{2\sqrt{x}}\sec(1+\sqrt{x})\tan(1+\sqrt{x})$,
   (c) $\frac{1}{x^2}\operatorname{cosec}^2\left(\frac{1}{x}\right)$.
7. (a) $\cos x\cos 2x - 2\sin x\sin 2x$, (b) $3\cos^2 x\cos 4x$, (c) $\cos x$.
8. (a) $\frac{x\cos x - 2\sin x}{x^3}$, (b) $\frac{\sin x + \cos x}{(\cos x - \sin x)^2}$, (c) $-\frac{3}{1+\sin 3x}$.
9. $(\pi/2, 5)$ max; $(3\pi/2, -3)$ min.

10. $(\pi/3, \pi/3-\sqrt{3})$ min; $(5\pi/3, 5\pi/3+\sqrt{3})$ max.
11. $(3\pi/4, -1)$ min; $(7\pi/4, -1)$ min.
12. $(0,0)$ inflexion; $(\pi/3, 3\sqrt{3}/16)$ max; $(2\pi/3, -3\sqrt{3}/16)$ min; $(\pi,0)$ inflexion; $(4\pi/3, 3\sqrt{3}/16)$ max; $(5\pi/3, -3\sqrt{3}/16)$ min; $(2\pi,0)$ inflexion.
13. (a) $\pi/3$, (b) 10 m, (c) 30 m s^{-1}, at O.
14. $4500k\pi$ cm^3/s.
15. (a) 2·5 cm^2/s, (b) $\frac{1}{2}\sqrt{3}+1$ cm/s.
16. (a) 0·860, (b) 1·0105, (c) 0·870.
17. $\frac{1}{3}$; $20 < P \leqslant 10(1+\sqrt{10})$.
18. (a) 8 m s^{-2}, (b) $3\sqrt{3}$ m s^{-1}.

Exercise 11.5 (p. 224)

1. (a) $-\frac{1}{3}\cos 3x+c$, (b) $2\sin\frac{1}{2}x+c$, (c) $2\tan 2x+c$.
2. (a) $\frac{1}{2}x+\frac{1}{4}\sin 2x+c$, (b) $-\cot x-x+c$, (c) $\frac{1}{3}\sin^3 x+c$.
3. (a) $-\frac{1}{4}\cos^4 x+c$, (b) $\frac{1}{4}\cos 2x-\frac{1}{8}\cos 4x+c$, (c) $\frac{1}{6}\sin 3x+\frac{1}{2}\sin x+c$.
4. (a) $\frac{1}{2}$, (b) $(\pi-2)/24$, (c) $\frac{1}{3}$.
5. (a) $\sqrt{3}-\pi/3$, (b) 3/10, (c) 1/4.
6. (a) 1, (b) $1-\pi/2$, (c) 2/3.
7. $\frac{1}{3}\tan^3 x-\tan x+x+c$.
8. $-2\cos^2 x+$ constant.
9. $\frac{3}{8}+\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x$; $(3\pi+8)/32$.
10. 1.
11. $11\pi^2$.
12. $4/3\pi$.

Exercise 11.6 (p. 227)

1. $\dfrac{1}{\sqrt{(4-x^2)}}$.
2. $\dfrac{3}{1+9x^2}$.
3. $-\dfrac{1}{\sqrt{(-x-x^2)}}$.
4. $\dfrac{x}{1+x^2}+\tan^{-1}x$.
5. -1.
6. $1-\dfrac{x\sin^{-1}x}{\sqrt{(1-x^2)}}$.
7. $-\dfrac{1}{x\sqrt{(x^2-1)}}$.
8. $\dfrac{2-8x\tan^{-1}2x}{(1+4x^2)^2}$.
9. $\dfrac{1}{1+x^2}$.
10. $\sin^{-1}\dfrac{x}{5}+c$.
11. $\dfrac{1}{3}\tan^{-1}\dfrac{x}{3}+c$.
12. $\dfrac{1}{2}\sin^{-1}2x+c$.
13. $\dfrac{1}{3}\tan^{-1}3x+c$.
14. $\dfrac{1}{3}\sin^{-1}\dfrac{3x}{4}+c$.
15. $\dfrac{1}{6}\tan^{-1}\dfrac{2x}{3}+c$.
16. $\pi/6$.
17. $\pi/8$.
18. $\pi/9$.
19. $\pi/15$.
20. $\pi/72$.
21. $\pi/8$.

Exercise 11.7 (pp. 228–230)

1. $(2+\sqrt{3})\pi x/6$, $(5\pi-6\sqrt{3})x^2/24$.
4. (a) $0, \pi/2, \pi, 3\pi/2, 2\pi$, (b) $0, 2\pi/3, \pi, 4\pi/3, 2\pi$.
5. (a) $(4n+1)\pi/6$, (b) $(2n+1)\pi/6$, $n\pi \pm \pi/3$.
6. $360n^\circ + 12{\cdot}4^\circ$, $360n^\circ - 79{\cdot}8^\circ$.
7. $\frac{1}{2}\cos\alpha$.
9. $\cos x$, $-\sin x$; $-\operatorname{cosec}^2 x$, $-\operatorname{cosec} x \cot x$.
10. (a) $2\sin(4x-10)$, (b) $4x^3(\tan 4x + x\sec^2 4x)$,
    (c) $\dfrac{2\sec x}{(\sec x - \tan x)^2}$, (d) $\dfrac{\cos x}{(\cos 2x)^{3/2}}$.
11. (a) $(\pi/6, 2\sqrt{3})$ max; $(5\pi/6, -2\sqrt{3})$ min; $(3\pi/2, 0)$ inflexion,
    (b) $(0,0)$ inflexion; $(\pi/2, 4)$ max; $(\pi, 0)$ inflexion;
    $(3\pi/2, -4)$ min; $(2\pi, 0)$ inflexion.
12. (a) $\frac{1}{2}\sin(2x-1)+c$, (b) $\frac{1}{2}x - \frac{1}{2}\sin x + c$,
    (c) $\frac{1}{2}\tan^2 x + c$, (d) $-\dfrac{1}{10}\cos 5x - \dfrac{1}{6}\cos 3x + c$.
13. (a) $21/128$, (b) $(3\pi+4)/48$.
14. $x\sin x + \cos x + c$; $2\cos x + 2x\sin x - x^2\cos x + c$.
15. $\pi - \sqrt{3}$.
16. $\frac{3}{4}\pi^2$.
17. $\tan x + \sec x + c$; $4(\sqrt{3}-1)/\pi$.
18. (a) $\dfrac{(\sin^{-1}x)\sqrt{(1-x^2)} - x}{(\sin^{-1}x)^2\sqrt{(1-x^2)}}$, (b) $\dfrac{\sec x \tan x}{1+\sec^2 x}$, (c) $-\dfrac{x}{\sqrt{(1-x^2)}}$.
19. (a) $\sin^{-1}\dfrac{x}{3} + c$, (b) $\dfrac{1}{15}\tan^{-1}\dfrac{3x}{5} + c$.
20. (a) $\pi/3\sqrt{3}$, (b) $\pi/12$.
21. (ii) $\dfrac{2+p^2}{2(1+p^2)}$.
24. $\dfrac{1}{x^2+2ax+a^2+1}$; (a) $\tan^{-1}(x+2)+c$, (b) $\dfrac{1}{2}\tan^{-1}\left(\dfrac{x+1}{2}\right)+c$.

Exercise 12.1 (pp. 233–234)

1. (a) 5040, (b) 600, (c) 1680, (d) 120, (e) 4, (f) 6720.
2. (a) 5!, (b) 10!/6!, (c) $n!/(n-3)!$.
3. 720.
4. 40 320.
5. 20 160.
6. 151 200.
7. 13 800.
8. (a) 240, (b) 600.
9. (a) 120, (b) 432.
10. (a) 1000, (b) 720, (c) 990.
11. (a) 720, (b) 20 160, (c) 60, (d) 50 400.
12. 60, 30.
13. 10 080.
14. 720.
15. 120, 72.
16. 6720; (a) 720, (b) 2400.
17. (a) 48, (b) 180.
18. 96.
19. (a) 168, (b) 2016, (c) 20; 2520.
20. 9.

Exercise 12.2 (p. 236)

1. (a) 35, (b) 5, (c) 6, (d) 1.
3. (a) 210, (b) 1140, (c) 1287, (d) 12.
4. 1260.
5. (a) 462, (b) 5775, (c) 10 395.
6. 15 840.
7. 38, 16.
8. (a) 2277, (b) 2541.
9. 238.
10. 79, 24.
11. (a) 90, (b) 7560.
12. (a) 15 015, (b) 37 037.

Exercise 12.3 (pp. 239–240)

1. $\frac{1}{6}, \frac{1}{2}, \frac{2}{3}$.
2. $\frac{1}{2}, \frac{1}{13}, \frac{3}{13}$.
3. $\frac{1}{8}, \frac{1}{2}, \frac{3}{4}$.
4. $\frac{1}{21}, \frac{2}{7}, \frac{10}{21}$.
5. (a) $\frac{1}{4}$, (b) $\frac{5}{9}$, (c) $\frac{7}{36}$.
6. (a) $\frac{1}{14}$, (b) $\frac{3}{28}$, (c) $\frac{3}{7}$, (d) $\frac{4}{7}$.
7. (a) 0·8, (b) 0·07, (c) 0·1, (d) 0·43.
8. 0·32.
9. (a) $\frac{1}{9}$, (b) $\frac{2}{3}$.
10. (a) $\frac{1}{16}$, (b) $\frac{3}{8}$, (c) $\frac{5}{8}$, (d) $\frac{7}{8}$.
11. (a) $\frac{3}{10}$, (b) $\frac{3}{10}$, (c) $\frac{3}{10}$, (d) $\frac{1}{5}$.
12. $\frac{11!}{3!2!}$; (a) $\frac{2}{11}$, (b) $\frac{1}{77}$.
13. (a) 0·204, (b) 0·052, (c) 0·398.
14. $\frac{21}{55}, \frac{27}{55}, \frac{27}{220}, \frac{1}{220}$.
15. (a) 0·108, (b) 0·374.
16. (a) $\frac{1}{42}$, (b) $\frac{1}{7}$, (c) $\frac{2}{3}$, (d) $\frac{1}{2}$.

Exercise 12.4 (p. 241)

1. (a) 1260, (b) 8232.
2. 30 240; (a) 840, (b) 10 080, (c) 13 440.
3. 63.
4. 20.
5. 280, 36.
6. (a) 243, (b) 21.
7. (a) $\frac{1}{7}$, (b) $\frac{3}{35}$.
8. (a) (i) $\frac{5}{18}$, (ii) $\frac{1}{3}$, (b) $\frac{12}{35}$, (c) 294.

9. (a) $\frac{6}{77}$, (b) 0, (c) $\frac{2}{15}$, (d) $\frac{2}{33}$.
10. $4{\cdot}12 \times 10^{-6}$.

Exercise 13.1 (pp. 243–244)

1. 25, 30; $5n$.
2. 16, 19; $3n+1$.
3. $\frac{1}{5}, \frac{1}{6}; \frac{1}{n}$.
4. $\frac{5}{6}, \frac{6}{7}; \frac{n}{n+1}$.
5. 16, 32; 2^{n-1}.
6. 35, 48; n^2-1.
7. $\frac{4}{3}, \frac{8}{3}; \frac{1}{3}.2^{n-3}$.
8. $\frac{1}{30}, \frac{1}{42}; \frac{1}{n(n+1)}$.
9. 5, -6; $(-1)^{n-1}n$.
10. 4, 17; $10+(-1)^{n+1}n$.
11. 11, 21, 35; $14n-7$ $(n \geqslant 2)$.
12. -4, -46, -130; $-21n^2+21n-4$.
13. 2, 2, 4; 2^{n-1} $(n \geqslant 2)$.
14. $1, -\frac{1}{2}, -\frac{1}{6}; -\frac{1}{n(n-1)}$ $(n \geqslant 2)$.
15. 1, 8, 27; n^3.
16. 1, 4, 9; n^2.
17. 88 200.
18. 22 100, 20 825.

Exercise 13.2 (pp. 246–247)

1. (a) 31, $4n+3$, (b) -17, $25-7n$, (c) 37, $2n-9$, (d) $10\frac{1}{2}$, $\frac{1}{2}(n+5)$.
2. (a) 1325, (b) 1188, (c) 88, (d) $193\frac{1}{2}$.
3. (a) 904, (b) 1105, (c) -80, (d) 152.
4. 1512.
5. 9072.
7. 2600, $6\frac{1}{4}$.
8. 7, 4, 1; -333.
9. 607, 825.
10. 8.
11. 4, 16.
12. -17, 3.
14. 6, 11, 16.
15. (a) 15, (b) -12, (c) $\frac{5}{27}$, (d) $\lg 9$.
16. 8, 11, 14.
17. 3, 10, 17.
18. 4, 6.
19. 20, 190.
20. 7.

Exercise 13.3 (pp. 249–250)

1. (a) $64, 2^{n-2}$, (b) $\frac{2}{3}, 2.3^{5-n}$, (c) $\frac{25}{32}, 200(-4)^{1-n}$, (d) $-\frac{81}{16}, -\left(\frac{3}{2}\right)^{n-3}$.
2. (a) $9, 127\frac{3}{4}$, (b) $9, 42\frac{3}{4}$, (c) 6, 1249·92, (d) $7, 14\frac{15}{32}$.
3. (a) 111·1111, (b) $\frac{182}{243}$, (c) $1-(-2)^n$, (d) $a^p(a^{3k}-1)/(a^3-1)$.
4. $\frac{3}{4}$.
5. $-5, 2\frac{1}{2}$.
6. (a) 9, (b) $\frac{1}{9}$, (c) 10^{15}.
7. 13.
8. $-53\frac{1}{3}, -385\frac{5}{6}$.
9. $-40, -8$.
10. $182; \frac{1}{2}, -1$.
11. 2.
12. 12.
13. 2, 6.
14. $(3^n-1)/24$; 8.
15. 10.

Exercise 13.4 (p. 252)

1. (a) 9, (b) $\frac{2}{3}$, (c) $11\frac{1}{9}$, (d) 27.
2. (a) 6/11, (b) 8/111, (c) 31/54.
3. 25·6.
4. $-\frac{2}{3}$, 81.
5. $6, \frac{1}{4}, 8$.
6. (a) $\frac{1}{1-3x}, |x| < \frac{1}{3}$, (b) $\frac{2}{2+x}, |x| < 2$,
   (c) $\frac{2}{1+2x}, |x| < \frac{1}{2}$, (d) $\frac{3x}{3-x}, |x| < 3$.
7. $x < 0, -3$.
8. $243\left\{1-\left(\frac{5}{9}\right)^n\right\}; 243; 8$.

Exercise 13.5 (pp. 254–255)

1. (a) $4+14+36+76 = 130$, (b) $50+29+6 = 85$,
   (c) $11+15+19+23+27+31 = 126$,
   (d) $120+60+40+30+24+20 = 294$,
   (e) $\sin\frac{\pi}{3} + \sin\frac{2\pi}{3} + \sin\pi + \sin\frac{4\pi}{3} + \sin\frac{5\pi}{3} + \sin 2\pi = 0$,
   (f) $3-5+9-17+33-65 = -42$.

2. (a) $\sum_1^{12}(2r+3)$, (b) $\sum_1^{16} r^3$, (c) $\sum_1^{40}(-1)^r(r+1)$,
(d) $\sum_1^{7} 360(-\frac{1}{2})^{r-1}$, (e) $\sum_1^{10}\frac{2r-1}{2r}$, (f) $\sum_1^{12}\frac{r^2}{3r+1}$.
3. (a) 940, (b) 16, (c) -1275, (d) 855, (e) 2, (f) 1/90.

Exercise 13.6 (p. 258)

1. (a) $x^4+4x^3y+6x^2y^2+4xy^3+y^4$,
(b) $a^7-7a^6b+21a^5b^2-35a^4b^3+35a^3b^4-21a^2b^5+7ab^6-b^7$,
(c) $64+192p^2+240p^4+160p^6+60p^8+12p^{10}+p^{12}$,
(d) $32h^5-80h^4k+80h^3k^2-40h^2k^3+10hk^4-k^5$,
(e) $x^3+3x+\frac{3}{x}+\frac{1}{x^3}$,
(f) $z^8-4z^6+7z^4-7z^2+\frac{35}{8}-\frac{7}{4z^2}+\frac{7}{16z^4}-\frac{1}{16z^6}+\frac{1}{256z^8}$.
2. (a) $210x^4$, (b) $16\,128x^2$, (c) $1760a^3b^9$, (d) $-945p^4q^6$, (e) -20, (f) $36/x^3$.
3. $64x^5+\frac{160}{x}+\frac{20}{x^7}$.
4. 14.
5. 15.
6. 2.
7. 8, $-\frac{1}{2}$.
8. 1·13.
9. 30·43168.
10. $1+30x+420x^2+3640x^3$; 1·03042.
11. (a) $1+7x+21x^2+35x^3$, (b) $1+7x+14x^2-7x^3$.
12. $1+16x+136x^2+784x^3$.
13. $1+9x+24x^2$.
14. (a) $-16\,464$, (b) -19, (c) -1760, (d) 26.

Exercise 13.7 (pp. 260–261)

1. $1-2x+3x^2$; $(-1)^r(r+1)x^r$.
2. $1+x+x^2$; x^r.
3. $1-6x+24x^2$; $(-1)^r(r+1)(r+2)2^{r-1}x^r$.
4. $1+\frac{1}{2}x-\frac{1}{8}x^2$; $|x|<1$.
5. $1-x-x^2$; $|x|<\frac{1}{3}$.
6. $1-\frac{1}{4}x+\frac{3}{32}x^2$; $|x|<2$.
7. $\frac{1}{3}-\frac{1}{9}x+\frac{1}{27}x^2$; $|x|<3$.
8. $2-\frac{1}{4}x-\frac{1}{64}x^2$; $|x|<4$.
9. $27-18x+2x^2$; $|x|<\frac{9}{4}$.
10. $1-x^2+\frac{3}{2}x^4-\frac{5}{2}x^6$.
11. $1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3$; 0·9487.
12. $2+\frac{9}{2}x^2+\frac{17}{6}x^3$.
13. $\frac{1}{4}+\frac{1}{4}x+\frac{3}{16}x^2+\frac{1}{8}x^3+\frac{5}{64}x^4$; 0·309.

14. $1+x-\frac{1}{2}x^2+\frac{1}{2}x^3$; 1·732. 15. $1-7x+28x^2-112x^3$; $|x|<\frac{1}{4}$.

16. (a) $1-x+2x^2-3x^3$, (b) $1+\frac{1}{2}x+\frac{7}{8}x^2-\frac{7}{16}x^3$.

Exercise 13.8 (pp. 261–264)

1. $n(n-1)$; 15th.

2. (a) $3, 4\frac{1}{2}, 4\frac{1}{6}, 4+\frac{1}{n(n-1)}$, (b) 2, 4, 18, $n.n!$.

3. (a) 250, (b) 510, (c) $n(n+1)$.

4. b^2. 5. 11, 20.

6. $-2\frac{1}{2}, \frac{1}{2}$; 205. 7. $2\frac{1}{2}$.

8. 29 mm. 9. (i) $1, \frac{2}{3}$, (ii) $\frac{53}{165}$.

10. (i) $1+\frac{1}{4}+\frac{1}{16}+\frac{1}{64}$, (ii) all values of r; $1+r^2$ for $r \neq 0$, 0 for $r=0$.

12. (a) 1 200 000, (b) 72. 13. $-2<x<2$; 6.

14. $1, -1+\frac{1}{2}\sqrt{2}, -1-\frac{1}{2}\sqrt{2}$; (i) $\frac{8}{7}+\frac{2}{7}\sqrt{2}$, (ii) $-35+\frac{45}{2}\sqrt{2}$.

16. (i) $-\frac{1}{3}<x<1, \frac{1}{2}$, (ii) 149.

17. (a) $1+12x+54x^2$, (b) $1-3x-9x^2$.

18. (a) 28/243, (b) -307. 19. 14.

20. 255, 1. 21. $-\frac{3}{2}, -8$; 405.

22. $\frac{5}{2}x+\frac{15}{4}x^2+\frac{65}{8}x^3$.

23. $1+x-x^2+\frac{5}{3}x^3-\frac{10}{3}x^4$; $|x|<\frac{1}{3}$; 0·98990.

24. $1-6x+24x^2-80x^3$, $|x|<\frac{1}{2}$; 0·000008500.

25. (i) $1+\frac{1}{2}x-\frac{1}{8}x^2$, (ii) $1+\frac{1}{2}x+\frac{3}{8}x^2$; $1+x+\frac{1}{2}x^2$; 3·315.

26. (a) u^3+3u, u^5+5u^3+5u,

(b) $1+kx+\frac{1}{2}(3k^2-1)x^2+\frac{k}{2}(5k^2-3)x^3$.

28. 0·015625, 0·140625, 0·421875, 0·421875.

Exercise 14.1 (pp. 266–267)

1. (a) even, (b) odd, (c) odd, (d) even, (e) neither, (f) odd.

3. (a) even, (b) odd, (c) neither, (d) neither, (e) odd, (f) even.

4. (a) odd, (b) even, (c) odd, (d) even, (e) neither, (f) even.

Exercise 14.2 (pp. 269–270)

5. $(0,-1)$; $(0,-1)$.
6. $(0,1)$; $(0,1)$.
7. $(1,-1)$; $(1,-1)$.
8. $(-2,0)$; none.
9. $(2,1)$; $(2,1)$.
10. $(\frac{1}{2},2\frac{1}{4})$; $(\frac{1}{2},\frac{4}{9})$.
11. $(-2,4)$, $(0,0)$; $(-2,\frac{1}{4})$.
12. $(-2,16)$, $(2,-16)$; $(-2,\frac{1}{16})$, $(2,-\frac{1}{16})$.
15. $(1,-4)$; $(1,4)$.
16. $(3,0)$; $(3,0)$.

Exercise 14.4 (pp. 273–274)

1. $x=-2, y=0$.
2. $x=4, y=0$.
3. $x=1\frac{1}{2}, y=0$.
4. $x=-2, y=1$.
5. $x=-2, y=1$.
6. $x=1\frac{1}{2}, y=1$.
7. $x=-2, y=2$.
8. $x=3, y=3$.
9. $x=\dfrac{2}{5}, y=-\dfrac{2}{5}$.
10. $y=\dfrac{x}{x+1}$.
11. $y=\dfrac{1-x}{1+x}$.

Exercise 14.5 (p. 276)

2. (a) $x+4y=12$, (b) $y=x^2+2x+2$, (c) $8x=(y+4)^2$,
   (d) $y=x+2$ $(x\geqslant -1)$, (e) $y=1/(x-2)$, (f) $y=x^2+2x+2$ $(x\geqslant -1)$.
3. (a) $4x^2+9y^2=36$, (b) $25x^2-y^2=25$,
   (c) $4x^2+y^2-8x-2y+1=0$, (d) $5x^2-6xy+2y^2=1$,
   (e) $2x^2+2xy+y^2=1$.
4. (a) $(0,0)$, $(9,27)$, (b) $(-3,-3)$, $(1\frac{1}{2},6)$, (c) $(1,-4)$, $(16,16)$,
   (d) $(7,9)$, $(-\frac{7}{3},\frac{25}{9})$, (e) $(1,0)$, $(\frac{1}{2},\frac{1}{2})$, (f) $(-3,4)$, $(4,-3)$.

Exercise 14.6 (p. 278)

1. t.
2. $2/t$.
3. $1/4t^2$.
4. $1/6(1-2t)$.
5. $-\frac{3}{4}\cot t$.
6. $-\tan t$.
7. $-\frac{1}{2}t^3, \frac{3}{4}t^5$.
8. $\dfrac{1}{t}-t^2, -\dfrac{1}{12t^3}-\dfrac{1}{6}$.
9. $\dfrac{2t+1}{3t^2}, -\dfrac{2(t+1)}{9t^5}$.
10. $\dfrac{t}{t+1}, \dfrac{1}{2(t+1)^3}$.
11. $-2\sin t, -1$.
12. $-2\cot 2t, -4\operatorname{cosec}^3 2t$.
13. 9.
14. $(3t^2+1)y=2tx+(t^2+1)^2$.
15. $y=x\operatorname{cosec}\theta-\cot\theta$.
16. $\dfrac{t-3}{15-9t}$; $(16,9)$.
17. $(2,4)$, $(10,0)$; $(4,12\sqrt{3}-18)$, $(4,-12\sqrt{3}-18)$.

Exercise 14.7 (pp. 279–281)

1. (a) neither, (b) odd, (c) even, (d) even, (e) even, (f) neither.
2. (a) $a = c = 0$, (b) $b = d = 0$, (c) $a = 0, b > 0, c^2 < 4bd$, (or $a = b = c = 0, d \geqslant 0$), (d) $a = 0, b < 0, c^2 < 4bd$ (or $a = b = c = 0, d \leqslant 0$).
3. $f(x) = k/x$.
6. $(-1, 0)$; 1, -3; $1/\sqrt[3]{2}$, minimum.
7. $f(x)$: max $(0, 0)$, min $(\pm\sqrt{2}, 1)$,
   $g(x)$: max $(\pm\sqrt{2}, 1)$, inflexion $\left(\pm\sqrt{\frac{10}{3}}, \frac{21}{25}\right)$.
9. (a) $(1, C)$, $(2, E)$, $(4, A)$, $(5, B)$, (c) $y = \left|\frac{1}{x+1}\right|$ is a possibility.
12. $a > 4$: one; $0 < a < 4$: none; $-4 < a < 0$: two; $a < -4$: one.
13. (a) $9x^2 - 4y^2 = 144$, (b) $x^2 - y^2 = 1$,
    (c) $10x^2 - 2xy + 5y^2 = 49$, (d) $4x^2 - 6\sqrt{3}xy + 9y^2 = 9$.
14. $\frac{3t^2-4}{2t}$; $\left(\frac{8}{3}, \pm\frac{16}{9}\sqrt{3}\right)$.
15. $\frac{t-1}{t+1}$, $(4, 0)$; $x + y = 2$.
16. $(-\frac{1}{2}, -6)$, $(3, 1)$; 2, $y = 2x - 5$ $(x \neq 0)$.
17. $x = \frac{\sqrt{3}}{2}\sin(\theta + 45°)$, $y = \frac{1}{2}\cos(\theta + 45°)$; $4x^2 + 12y^2 = 3$.
18. (a) $\frac{12t-14}{9+18t}$, $\frac{40}{81(1-2t)(1+2t)^3}$.
    (b) $-\cot\left(\frac{3t}{2}\right)$, $-\frac{3}{8}\operatorname{cosec}^3\left(\frac{3t}{2}\right)\sec\left(\frac{t}{2}\right)$.
19. $(1 - \sin\theta - 2\sin^2\theta)/\cos\theta$, $(\frac{1}{2}, \pm\frac{3}{4}\sqrt{3})$, $(1, 0)$; $1{\cdot}3\pi$.

Exercise 15.1 (pp. 284–285)

1. $x^2 + y^2 + 6x - 8y = 0$.
2. $x + y = 4$.
3. $x^2 = 4$.
4. $y = 4x + 13$.
5. $(x - y)(x + y - 2) = 0$.
6. $x^2 = 4(y - 1)$.
7. $xy = 9$.
8. $x^2 + y^2 - 2x - 4y + 1 = 0$.
9. $y^2 = 12x$.
10. $x^2 + y^2 - 12x - 2y + 21 = 0$.
11. $x^2 + y^2 + 2x - 2y = 0$.
12. $3x + y + 3 = 0$; $(-3, 6)$.
13. $y^2 = 2x$.
14. $(\frac{1}{2}t + 1, \frac{1}{2}t^2 - 2)$; $y = 2x^2 - 4x$.
15. (a) $x + y = \frac{1}{2}c$, (b) $xy = \frac{1}{2}k$, (c) $x^2 + y^2 = \frac{1}{4}l^2$.
16. $y = 2x^2$.

Exercise 15.2 (pp. 288–289)

1. (a) $x^2 + y^2 - 6x - 4y - 3 = 0$, (b) $x^2 + y^2 + 2x + 4y + 4 = 0$,
   (c) $x^2 + y^2 = 25$, (d) $x^2 + y^2 - x - 2 = 0$,
   (e) $x^2 + y^2 - 8x + 2y + 14 = 0$, (f) $x^2 + y^2 + 6x - 10y + 14 = 0$.
2. (a) $(1, 3)$, 3, (b) $(0, 0)$, 2, (c) $(-3, -4)$, 5,
   (d) $(2, -1)$, 1, (e) $(0, 0)$, $\frac{1}{2}\sqrt{5}$, (f) $(1\frac{1}{2}, -2\frac{1}{2})$, $\sqrt{5}$.

3. (a) $k > 0$, (b) $k = 1$, (c) No values of k,
   (d) $k = 0$, (e) $k < 10$, (f) $k > 4$ or $k < -4$.
4. (a) $y = 2x+5$, (b) $x+y+3 = 0$, (c) $y = 3x-13$,
   (d) $x+8y = 22$.
5. (a) 5, (b) 7, (c) 6, (d) $2\sqrt{10}$.
6. (a) $x^2+y^2-2x-6y = 0$, (b) $x^2+y^2-6x-4y-7 = 0$,
   (c) $x^2+y^2-2x+4y-15 = 0$, (d) $x^2+y^2+6x-16y+23 = 0$.
7. (a) 24, 2, (b) 12, 2.
8. $x^2+y^2+6x-12y+20 = 0$; (a) $y = 1$, $y = 11$, (b) $x = 2$, $x = -8$.
9. $x^2+y^2-10x-14y+66 = 0$; $(5,7)$, $2\sqrt{2}$.
10. $x^2+y^2-6x-10y+9 = 0$; 27 sq. units.
11. $x^2+y^2-4x-14y+28 = 0$.
12. $(1,2)$, $(9/5, 18/5)$.
13. $x^2+y^2-7x-3y+8 = 0$; $x^2+y^2-8x+2y+4 = 0$.

Exercise 15.3 (pp. 293–294)

1. $y+3x = 19$; $3y = x+17$.
2. $y = 2x-2$; $2y+x = 6$.
3. $x+y = 1$; $y = x+3$.
4. $y = 5x+22$; $5y+x = 32$.
5. $2y = x-13$; $2x+y+4 = 0$.
6. $9y+7x = 16$; $7y = 9x-2$.
7. $10y = 27x-31$; $27y+10x = 165$.
8. $4y = 9x-6$; $9y+4x = 35$.
9. $4y = 3x+5$; $3y+4x = 10$.
10. $4y = x+16$; $y+4x = 72$.
11. (a) $(2,2)$, (b) $(3,-1)$, (c) $(2\frac{1}{2}, 1\frac{1}{2})$.
12. $5x-4y+34 = 0$.
13. $2x-y = 11$, $2x-y+21 = 0$.
14. 1, 4.
15. $y = 3$ at $(-1,3)$, $y = 12x-21$ at $(5,39)$.
16. $y = 0$, $3y = 4x$; $4/\sqrt{5}$.
17. 1, -7.
18. $\pm\sqrt{17}$, $3x+4y = 0$.
19. $y = 3x \pm 10$.
20. $3y = 15 \pm 4x$; 73·7°.
21. $2x \pm 3y = 10$; $(16/5, 6/5)$, $(16/5, -6/5)$.
22. $x+ty = 2t+t^3$; $x^2 = 2y^2(y-1)$.
23. $y = (p+q)x+p+q-pq$, $y = 2px+2p-p^2$; $y = 0$.
24. $(p+q)y = (p^2+pq+q^2)x-p^2q^2$; $2y = 3px-p^3$;
    $p = \frac{2}{3}\sqrt{2}$, $q = -\frac{1}{3}\sqrt{2}$ or $p = -\frac{2}{3}\sqrt{2}$, $q = \frac{1}{3}\sqrt{2}$.

Exercise 15.4 (pp. 296–297)

1. (a) $(1,0)$; $x = -1$, (b) $(3,0)$; $x = -3$, (c) $(-2,0)$; $x = 2$,
   (d) $(\frac{3}{4},0)$; $x = -5/4$, (e) $(\frac{1}{4},0)$; $x = -\frac{1}{4}$, (f) $(0,\frac{1}{4})$; $y = -\frac{1}{4}$,
   (g) $(2,0)$; $x = -2$, (h) $(3,0)$; $x = -3$.
2. (a) $y^2 = 8x$, (b) $y^2 = 20x$, (c) $x^2 = 4y$,
   (d) $y^2+12x = 0$, (e) $(y-1)^2 = 4x$, (f) $x^2 = 4(y-1)$.
3. (a) $y^2 = 4(x-2)$, (b) $(y-1)^2 = 12x$, (c) $(y-1)^2 = 4(x+2)$,
   (d) $x^2-8x+2y+11 = 0$.
4. (a) $(2t^2, 4t)$, (b) $(6t^2, 12t)$, (c) $(-4t^2, -8t)$.

5. (a) $y = 2x+2$, $x+2y = 9$, (b) $x+2y+6 = 0$, $y = 2x-18$,
(c) $2y = x+4$, $2x+y = 12$, (d) $x+y+3 = 0$, $y = x-9$.
6. $y^2 = a(x-a)$. 7. $2y^2+ax = 0$.
9. $mc = a$; $x+y+1 = 0$, $2y = x+4$.
10. $(apq, a(p+q))$; $y^2 = a(2x-a)$.
12. $-p$; $\dfrac{2}{p+q}$; $\left(\dfrac{121a}{9}, \dfrac{22a}{3}\right)$; $5\sqrt{5}a$.
13. $x+2a = 0$. 14. $ty = x+at^2$.
15. (i) $x = 0$, (ii) $2y^2 = 9ax$.

Exercise 15.5 (p. 300)

1. (a) $(2,3)$, $(1,4)$, $(-1,5)$, (b) $(1,1)$, $(0,2)$, $(-2,3)$,
(c) $(0,5)$, $(-1,6)$, $(-3,7)$.
2. (a) $(1,2)$, (b) $(3,-1)$, (c) $(0,1)$, (d) $(-2,-3)$.
3. (a) $(3,0)$, $(4,0)$, $x = 2$, (b) $(0,1)$, $(2,1)$, $x = -2$,
(c) $(-5,0)$, $(-4,0)$, $x = -6$, (d) $(-4,-2)$, $(-3\frac{3}{4},-2)$, $x = -4\frac{1}{4}$,
(e) $(0,1)$, $(0,2)$, $y = 0$, (f) $(1,-1)$, $(1,-\frac{3}{4})$, $y = -1\frac{1}{4}$.
4. (a) $(1,3)$, $(2,3)$, $x = 0$, (b) $(-2,5)$, $(1,5)$, $x = -5$,
(c) $(1,0)$, $(0,0)$, $x = 2$, (d) $(-3,-1)$, $(-3,1)$, $y = -3$.

Exercise 15.6 (pp. 300–304)

1. $(0,-2)$, $\left(\sqrt{\dfrac{3}{2}}, -\dfrac{1}{2}\right)$, $\left(-\sqrt{\dfrac{3}{2}}, -\dfrac{1}{2}\right)$. 2. $c^2 = a^2(1+m^2)$.
3. $y = x+2-\frac{1}{2}\pi$, $x+y = \frac{1}{2}\pi$. 4. $8y = 9x+7$, $8x+9y = 26$.
5. $(1,0)$, 1; $(0,1)$, 1; $\left(\dfrac{2}{1+m^2}, \dfrac{2m}{1+m^2}\right)$, $\left(\dfrac{2m}{1+m^2}, \dfrac{2m^2}{1+m^2}\right)$.
6. $x^2+y^2-10x-8y+16 = 0$; $(10,-6)$, 10, 53°.
7. (i) $(3,4)$, 5, (iv) $y = 2x-2+5\sqrt{5}$, $y = 2x-2-5\sqrt{5}$.
8. (a) $x^2+y^2-6x-6y+5 = 0$, (b) $3x^2+3y^2-6x-26y+3 = 0$.
9. $x\cos\theta+y\sin\theta = a$; $(a(2\cos\theta+1), a(1-\cos\theta-2\cos^2\theta)/\sin\theta)$;
$(0, 2a/\sqrt{3})$, $(0, -2a/\sqrt{3})$.
10. (i) $x^2+y^2-2y-3 = 0$, (ii) $4\pi/3$,
(iii) $x^2+y^2-\sqrt{3}x-3y = 0$; $x^2+y^2+2y-3 = 0$.
11. $x^2+y^2-10x-10y+25 = 0$; $x^2+y^2-34x-34y+289 = 0$;
(i) $(2,9)$, (ii) $x+y = 11$.
12. $x^2+y^2-5x-6y+9 = 0$; $y = 2x+3$.
13. $y\cos\alpha+x\sin\alpha = a\sin\alpha\cos\alpha$. 14. ± 2.
15. $y\sin T+x = 1+2\sin^2 T$; $\frac{1}{4}\pi$.
17. $y = \left(\dfrac{5}{4}x_0^2 - \dfrac{13}{9}\right)x - \dfrac{5}{6}x_0^3$; $\pm\frac{1}{3}\sqrt{5}$, $\pm\frac{2}{3}\sqrt{2}$.
18. $py = x+ap^2$, $y+px = 2ap+ap^3$.
19. $y+tx = 2at+at^3$, $(2a+at^2, 0)$; $\pm\sqrt{3}$, $\pm\sqrt{15}$.
20. (i) $4a$, (ii) $y^2 = 2a(x-4a)$. 21. $(apq, a(p+q))$.
22. $(a(2+p^2+pq+q^2), -apq(p+q))$; $y^2 = a(x-3a)$.

24. $y = mx + a/m$.
25. $py = x + ap^2$.
26. $y = \frac{3}{4}x + 5$, $y = -\frac{3}{4}x - 5$.
27. $(a(1+t^2), at)$; $(a, 0)$.
28. $(a\{2+(p+q)^2 - pq\}, -apq(p+q))$; $y^2 = 16a(x-6a)$; $(6a, 0)$.
31. $y + px = 2p + p^3$; $27y^2 = 4(x-2)^3$, $(2, 0)$.
32. $y = px - p^3$, $y = qx - q^3$.
33. (ii) $(-a\sin^2\theta, -a\cos^3\theta/\sin\theta)$.

Exercise 16.1 (pp. 306–307)

1. (a) True, (b) False, (c) False, (d) True.
2. (a) $x > 0$, (b) all x, (c) $x < 0$, (d) $x < -a$ or $x > -b$.

Exercise 16.2 (pp. 308–309)

1. $-2 < x < 3$.
2. $x \leqslant -3$ or $x \geqslant -\frac{1}{2}$.
3. $1 \leqslant x \leqslant 5$.
4. $x < 1 - \sqrt{6}$ or $x > 1 + \sqrt{6}$.
5. $-2 - \sqrt{7} < x < -2 + \sqrt{7}$.
6. $x \leqslant -2$ or $0 \leqslant x \leqslant 5$.
7. $3 < x < 4$.
8. $x < -2$ or $x > -1$.
9. $2 < x < 3$.
10. $-3 < x < -\frac{1}{2}$.
11. $x < 0$ or $1 < x < 2$.
12. $x < -3$ or $0 < x < \frac{1}{2}$.
13. $-3 < x < -2$ or $x > 1$.
14. $0 < x < 1$ or $x > 3$.
15. $-3 < x < 3$ or $x > 5$.
16. $-1 < x < 1$.
17. $0 < x < 1\frac{1}{2}$ or $3 < x < 4$.
18. $-2 < x < -1$ or $x > 2$.
19. $x < -\frac{1}{2}\pi$ or $0 < x < \frac{1}{2}\pi$.
20. $x > \frac{1}{3}\pi$.

Exercise 16.3 (pp. 311–312)

1. $-5 < x < 3$.
2. $x < \frac{1}{3}$ or $x > 2$.
3. $-1 < x < 0$ or $x > 6$.
4. $x < 0$ or $1 < x < 4$.
5. $0 < x < 3$ or $x > 4$.
6. $0 < x < 2$ or $x > 3$.
7. $x < -3$ or $1 < x < 2$.
8. $-2 < x < 1\frac{1}{2}$ or $x > 5$.
9. $1 < x < 7$.
10. $1 < x < 3$.
11. $x < -4$ or $-2 < x < 2$.
12. $-2 < x < 4$ or $x > 5$.
13. $x < -1$, $1 < x < 2$, $x > 3$.
14. $x < -1\frac{1}{2}$, $-1 < x < 0$, $x > \frac{2}{3}$.
15. $-4 < x < -1$.
16. $-2 < x < 1\frac{1}{2}$ or $x > 5$.
17. $x < 0$ or $x > 2$.
18. $0 < x < 1\frac{1}{2}$ or $3 < x < 4$.

Exercise 16.4 (p. 313)

1. $x < 1$ or $x > 3$.
2. $-8 \leqslant x \leqslant 2$.
3. $-3 < x < -\frac{1}{3}$.
4. $x \leqslant -5$ or $x \geqslant 6$.
5. $x \leqslant \frac{1}{2}$.
6. $x < -7/3$ or $x > -1$.
7. $x \leqslant 0$ or $x \geqslant 6$.
8. $-5/6 < x < 5/2$.
9. $x < 4/3$ or $x > 4$.
10. $-3/2 \leqslant x \leqslant -1$.

11. $x > -3/5$.
12. All real x.
13. $-1 < x < 0$ or $3 < x < 4$.
14. $x < -1, 2 < x < 3, x > 6$.
15. $x < -8$ or $x > -8/3$.
16. $-4 \leqslant x \leqslant -1$ or $1 \leqslant x \leqslant 4$.

Exercise 16.5 (pp. 313–314)

1. $x \leqslant -7$ or $x \geqslant 1$.
2. $1-\sqrt{2} < x < 1+\sqrt{2}$.
3. $x < -1$ or $-\frac{1}{2} < x < 1$.
4. $x < 0$ or $2 < x < 5$.
5. $x < -2$ or $x > 2$.
6. $x < -\sqrt{3}, -\sqrt{2} < x < \sqrt{2}, x > \sqrt{3}$.
7. $x < 0$ or $x > 3$.
8. $-1 < x < -1/\sqrt{2}$ or $1/\sqrt{2} < x < 1$.
9. $-2 < x < -1$ or $x > 4$.
10. $x < -1$ or $x > 2$.
11. $x < -1, -\frac{2}{3} < x < 3, x > 4$.
12. $x < -5$ or $x > 3$.
13. $-\frac{1}{3} \leqslant x \leqslant 7$.
14. $-2 < x < 2$.
15. $\frac{1}{2} < x < 1$.
16. $-6 < x < 1$ or $2 < x < 3$.
17. $x < 0$.
18. $x < 4/3$ or $x > 4$.
19. (a) $-\dfrac{\pi}{3} < x < \dfrac{\pi}{3}, \dfrac{5\pi}{3} < x < \dfrac{7\pi}{3}$,

    (b) $-\dfrac{\pi}{4} \leqslant x \leqslant \dfrac{\pi}{4}, \dfrac{3\pi}{4} \leqslant x \leqslant \dfrac{5\pi}{4}, \dfrac{7\pi}{4} \leqslant x \leqslant \dfrac{9\pi}{4}$,

    (c) $-\dfrac{\pi}{6} < x < \dfrac{\pi}{6}, \dfrac{5\pi}{6} < x < \dfrac{7\pi}{6}, \dfrac{11\pi}{6} < x < \dfrac{13\pi}{6}$.
22. $\dfrac{\pi}{6} < x < \dfrac{\pi}{2}$ or $\dfrac{5\pi}{6} < x < \dfrac{3\pi}{2}$.
23. (i) $x < -1$ or $x > 0$, (ii) $0 < x < 2$, (iii) $x < 1$ or $x > 2$.
24. $-3 < x < -1$ or $x > 2$, (ii) $x < -2$ or $x > 2$.

Exercise 17.1 (pp. 317–318)

1. (a) Most people disapprove of the Knave of Hearts.
   (b) N is not prime.
   (c) Boycott took over as England cricket captain.
2. (a) Not valid. There could be swallow-tail butterflies in Norfolk, but there may not be.
   (b) Not valid. We are not told that absent-minded people do wear odd socks.
   (c) Not valid. *ABCD* could be an isosceles trapezium.
3. Since $x - y = 0$, we cannot divide by $(x - y)$ to obtain $x + y = y$.
4. (1) Either, (2) Either, (3) Either, (4) True, (5) Either, (6) Either, (7) True, (8) Either, (9) Either, (10) Either, (11) Either, (12) True, (13) Either, (14) False, (15) False (16) True (17) False (18) Either.

Exercise 17.2 (pp. 320–321)

1. (a) $p \Rightarrow q, p \Rightarrow r, q \Rightarrow r$ (b) $p \Rightarrow r$ (c) $p \Rightarrow q, p \Rightarrow r$ (d) $p \Leftrightarrow q, p \Leftrightarrow r, q \Leftrightarrow r$
2. (a) $q \not\Rightarrow p, r \not\Rightarrow p, r \not\Rightarrow q$ (b) $p \not\Rightarrow q, q \not\Rightarrow p, q \not\Rightarrow r, r \not\Rightarrow p, r \not\Rightarrow q$ (c) $q \not\Rightarrow p, q \not\Rightarrow r, r \not\Rightarrow p, r \not\Rightarrow q$ (d) none

3. (a) $\sim q \Rightarrow \sim p$, $\sim r \Rightarrow \sim p$, $\sim r \Rightarrow \sim q$ (b) $\sim r \Rightarrow \sim p$ (c) $\sim q \Rightarrow \sim p$, $\sim r \Rightarrow \sim p$ (d) $\sim p \Leftrightarrow \sim q$, $\sim p \Leftrightarrow \sim r$, $\sim q \Leftrightarrow \sim r$
4. (a) $q \Rightarrow p$, $\sim p \Rightarrow \sim q$ (b) $p \Rightarrow \sim q$, $q \Rightarrow \sim p$ (c) $p \Leftrightarrow \sim q$, $\sim p \Leftrightarrow q$ (d) $p \Leftrightarrow \sim q$, $\sim p \Leftrightarrow q$
5. (a) $q \Rightarrow p$ (b) $\sim p \Leftrightarrow q$ (c) $\sim q \not\Rightarrow p$ (d) $\sim p \not\Leftrightarrow \sim q$ (e) $q \Rightarrow \sim p$ (f) $p \Leftrightarrow \sim q$
6. (a) $q \Rightarrow p$ (b) $q \Rightarrow p$ (c) $q \Rightarrow p$ (d) $q \Rightarrow p$ (e) $p \Rightarrow q$ (f) $p \Rightarrow q$
7. $AB = AC$ if $\triangle ABC$ is equilateral.
   $\triangle ABC$ is equilateral only if $AB = AC$.
   $\triangle ABC$ being equilateral is a sufficient condition for $AB = AC$.
   $AB = AC$ is a necessary condition for $\triangle ABC$ to be equilateral.

Exercise 17.3 (p. 323)

1. If $\angle ABC = \angle CAT$, then AT must be a tangent to the circle.
3. (a) If a diameter of a circle is perpendicular to a chord then it bisects that chord; (i) True, (ii) True.
   (b) If $2x^2 - x - 3 = 0$, then $2x - 3 = 0$; (i) True, (ii) False.
   (c) If a pentagon is equilateral, it must be regular; (i) True, (ii) False.
   (d) If x^2 is greater than y^2, then x is greater than y; (i) False, (ii) False.
   (e) If the sum of the digits of a positive integer N is divisible by 9, then N is divisible by 9; (i) True, (ii) True.
   (f) If there are five pairs of socks in a drawer, then any set of six socks taken from the drawer will contain a pair; (i) False, (ii) True.
4. (a) True, (b) False. 5. (a) True, (b) False.
6. (a) False, (b) True.

Exercise 17.4 (pp. 326–327)

9. (a) 2870 (b) 19 270 (c) 11 480 (d) 10 660 10. $\frac{1}{4}$
11. (a) $(n, 2n - 1)$ (b) $(n + 1)y = (2n + 1)x$
12. $\frac{x}{2x + 1}, \frac{x}{3x + 1}, \frac{x}{nx + 1}$

Exercise 17.5 (pp. 327–330)

1. If $(x - 3)^2 = (x + 1)^2$, then $x - 3 = \pm(x + 1)$.
2. Since $\lg(0{\cdot}2) < 0$, $2\lg(0{\cdot}2) < \lg(0{\cdot}2)$.
3. No value of x satisfies the initial statement.
4. $a + b > 0$ implies $a^2 + ab > 0$ only if $a > 0$.
5. The diagram is incorrect. P lies outside $\triangle ABC$.
6. There is no value of n for which a_n is divisible by 6.
7. It is necessary to prove that $r \Rightarrow q$ and that $s \Rightarrow q$.

8. It is necessary to prove that $P(1)$ and $P(2)$ are true.

14. $\begin{pmatrix}1 & 3\\ 0 & 4\end{pmatrix}, \begin{pmatrix}1 & 7\\ 0 & 8\end{pmatrix}, \begin{pmatrix}1 & 2^n-1\\ 0 & 2^n\end{pmatrix}$ 15. 1.

18. (i) False, (ii) True, (iii) False. 19. One example is $f(x) \equiv x$.

20. One possibility is $\mathbf{P} = \begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$, $\mathbf{X} = \begin{pmatrix}1 & 1\\ 3 & 2\end{pmatrix}$; $\begin{pmatrix}-2+2^n.3 & -2+2^{n+1}\\ 3-2^n.3 & 3-2^{n+1}\end{pmatrix}$.

Exercise 18.1 (pp. 335–336)

1. (a) Even, (b) neither, (c) odd. 2. (a) Even, (b) even, (c) odd.
3. (a) Even, (b) odd, (c) neither. 4. (a) Neither, (b) neither, (c) even.
5. (a) Yes, $2\pi/3$; (b) yes, $\pi/2$; (c) yes, 8π.
6. (a) No; (b) yes, 2π; (c) yes, π.
7. (a) Yes, 2π; (b) no; (c) yes, 2π.
8. (a) Yes, π; (b) no; (c) yes, π.
9. f: even, $f(x) \geqslant -1$; g: neither, $\mathbb{R}$; fg: neither, $fg(x) \geqslant -1$; gf: even, $gf(x) \geqslant -1$.
10. f: neither, $\mathbb{R}$; g: odd, $-1 \leqslant g(x) \leqslant 1$, 2π; fg: neither, $\frac{1}{2}(\pi - 1) \leqslant fg(x) \leqslant \frac{1}{2}(\pi + 1)$, 2π; gf: even, $-1 \leqslant gf(x) \leqslant 1$, 4π.
11. f: even, $-1 \leqslant f(x) \leqslant 1$, 2π; g: even, $g(x) \geqslant 0$; fg: even, $-1 \leqslant fg(x) \leqslant 1$; gf: even, $0 \leqslant gf(x) \leqslant 1$, π.
12. f: even, $f(x) \geqslant 0$; g: odd, $\mathbb{R}$; fg: even, $fg(x) \geqslant 0$; gf: even, $gf(x) \geqslant -2$.
17. (a) -1, 1, (b) -1, 1. 18. (a) none, (b) -1, 1.
19. (a) $x \in \mathbb{Z}$, (b) $x \in \mathbb{Z}$. 20. (a) 1, (b) -1, 1. 22. 3.

Exercise 18.2 (pp. 339–340)

7. $(-1, -1)$ min; $(-1, -1)$ max, $x = -2$, $x = 0$, $y = 0$.
8. $(-1, 0)$ min; $x = -1$, $y = 0$. 9. $(-1, 1)$ min; $(-1, 1)$ max, $y = 0$.
10. $(-3, 36)$ max, $(5/3, -400/27)$ min; $(-3, 1/36)$ min, $(5/3, -27/400)$ max, $x = -5$, $x = 0$, $x = 3$, $y = 0$.
11. $x = -2$, $y = 2$; $x = \frac{1}{2}$, $y = \frac{1}{2}$. 12. None, $(3/2, 0)$ min.
13. (0.4) max; $(-2, 0)$ min, $(0, 16)$ max, $(2, 0)$ min.
14. $(2n\pi + \frac{1}{2}\pi, 1)$ max, $(2n\pi - \frac{1}{2}\pi, -1)$ min; $(n\pi, 0)$ min, $(n\pi + \frac{1}{2}\pi, 1)$ max.
15. $x = 1$, $y = -2$; $(0, 0)$ min, $x = 1$, $y = 4$.
16. $(0, 0)$ min, $(2, 4)$ max; $(2, -2)$ min, $(2, 2)$ max.

Exercise 18.3 (pp. 343–344)

1. Yes; $\frac{1}{2}(x - 1)$; $\mathbb{R}$. 2. Yes; $5 - x$; $\mathbb{R}$. 3. No; e.g. -2, 0.
4. Yes; $\sqrt{(x + 2)}$; $\{x \in \mathbb{R} : x \geqslant -2\}$.
5. Yes; $1/(x - 1)$; $\mathbb{R} - \{1\}$.

6. Yes, $9 - 1/x$; $\mathbb{R} - \{0\}$. 7. Yes; $3 - x^2$; $\{x \in \mathbb{R} : x \geqslant 0\}$.
8. No; e.g. $-1, 1$. 9. Yes; $\log_3 x$; $\mathbb{R}^+$. 10. Yes, $10^x - 2$; $\mathbb{R}$.
11. No; e.g. $-1, 1$. 12. Yes; $x/(x-2)$; $\mathbb{R} - \{2\}$.
13. No; e.g. $0, 2\pi$. 14. Yes; $\sin x$; $\{x \in \mathbb{R} : -\frac{1}{2}\pi \leqslant x \leqslant \frac{1}{2}\pi\}$
15. $1 + 4x^2$; $\frac{1}{2}\sqrt{(x-1)}$. 16. $gf : x \to 3 - 9/x$.
17. (a) $f^{-1} : x \to 2 + \sqrt{x}$, $x \geqslant 0$, (b) $f^{-1} : x \to 1 + 10^{x-1}$,
(c) $f^{-1} : x \to \dfrac{1-x}{1+x}$, $x \neq -1$ (d) $f^{-1} : x \to \sqrt{\left(1 + \dfrac{1}{x^2}\right)}$, $x > 0$.
18. $k = 2$, $\{x \in \mathbb{R} : x \geqslant 4\}$. 19. $k = 3$, $\{x \in \mathbb{R} : x \leqslant 9\}$.
20. $k = 4$, $\{x \in \mathbb{R} : x \geqslant 0\}$. 21. $k = 1$, $\{x \in \mathbb{R} : x \geqslant -1\}$.
22. $k = 1$, $\{x \in \mathbb{R} : x \geqslant 2\}$. 23. $k = \frac{1}{4}$, $\{x \in \mathbb{R} : x \geqslant -\frac{1}{4}\}$.
24. (a) No, (b) Yes, (c) Yes, (d) No, (e) No, (f) Yes.

Exercise 18.4 (pp. 346–348)

1. (a) 5, (b) 4, (c) -2, (d) $x - 7$, (e) 0, (f) $-4x + 5$.
2. $5, -12$. 3. $-3x + 4$. 4. $2, -8$. 5. $4, -2, -5$.
6. $3(a+b)(b+c)(c+a)$. 7. $3(x-y)(y-z)(z-x)$.
8. $k = -7$, $(x+1)^2(2x-7)$; $k = 20$, $(x-2)^2(2x+5)$.
9. $(x+2)^2(3x^2 - 4x + 4)$. 10. $-2, 13$; $(x-1)^2(x-6)$.
11. $-3, -3, 12$. 12. $k = -5$: $-1/2, -1/2, 5/2$; $k = 27$: $3/2, 3/2, -3/2$.
13. $-11; -3, -\frac{1}{2}, 2$. 14. $-14; 3, -2, 4/3$. 15. $1, 4, 2$.
16. $1, 3, 1, 6$. 17. $1, -3, 1, 0$. 19. $-3, \frac{1}{4}$. 20. $0, 2$.
21. $(x-a)^2 g'(x) + 2(x-a)g(x) + A$; $f'(a)$, $f(a) - af'(a)$; $k = -1$: $-3, 0, 0$; $k = 1$; $-1, 2, 2$.

Exercise 18.5 (pp. 352–353)

7. $y \leqslant 1/5$ or $y \geqslant 5$; $x = -4$, $x = 1$, $y = 0$; $(-1, 5)$ min, $(11, 1/5)$ max.
8. $-\frac{1}{2} \leqslant y \leqslant 1$; $y = 0$; $(-2, -\frac{1}{2})$ min, $(1, 1)$ max.
9. All real values; $x = -3$, $x = 0$, $y = 0$; no turning points.
10. $y \leqslant 1/9$ or $y \geqslant 1$; $x = -3$, $x = 0$, $y = 0$; $(-1, 1)$ min, $(3, 1/9)$ max.
11. $y \leqslant 1$ or $y \geqslant 9$; $x = 1$, $x = 2$, $y = 0$; $(0, 1)$ max, $(4/3, 9)$ min.
12. $-5/4 \leqslant y \leqslant 5$; $y = 0$; $(4, -5/4)$ min, $(-1, 5)$ max.
13. $-\frac{1}{3} \leqslant y \leqslant 1$; $y = 0$; $(0, 1)$ max, $(2, -\frac{1}{3})$ min.
14. $y \leqslant -1$ or $y \geqslant -1/9$; $x = -5$, $x = -2$, $y = 0$; $(-8, -1/9)$ min, $(-4, -1)$ max.
15. $y \leqslant 0$ or $y \geqslant 4/3$; $x = 1$, $x = 4$, $y = 0$; $(5/2, 4/3)$ min.
16. All real values; $x = 1$, $x = 4$, $y = 0$; no turning points.
20. $x = 3$, $y = x$; $(1, -1)$ max, $(5, 7)$ min.

Exercise 18.6 (pp. 356–358)

1. $\dfrac{1}{x+1}-\dfrac{1}{x+2}$.
2. $\dfrac{1}{x-1}-\dfrac{2}{x+2}$.
3. $\dfrac{2}{x-2}+\dfrac{1}{x-4}$.
4. $\dfrac{3}{x}+\dfrac{1}{x-3}$.
5. $\dfrac{3}{x-2}-\dfrac{2}{x+1}$.
6. $\dfrac{2}{x+1}+\dfrac{1}{2x+1}$.
7. $\dfrac{1}{2(x-1)}+\dfrac{1}{2(x+1)}$.
8. $\dfrac{1}{3+x}+\dfrac{1}{2-x}$.
9. $\dfrac{1}{4x-1}+\dfrac{2}{2x+1}$.
10. $\dfrac{1}{2(2x-1)}-\dfrac{3}{2(2x+5)}$.
11. $\dfrac{3}{x-2}-\dfrac{4}{x+3}+\dfrac{1}{x+1}$.
12. $\dfrac{6}{x-1}+\dfrac{2}{x+1}-\dfrac{3}{x-2}$.
13. $\dfrac{x+2}{x^2+1}-\dfrac{1}{x+1}$.
14. $\dfrac{2}{2x-1}-\dfrac{x}{x^2+1}$.
15. $\dfrac{1}{x}+\dfrac{3-2x}{2x^2+1}$.
16. $\dfrac{2x+1}{x^2+3}-\dfrac{3}{2x-1}$.
17. $\dfrac{2}{x-1}-\dfrac{4}{(x-1)^2}-\dfrac{1}{x+2}$.
18. $\dfrac{1}{x}+\dfrac{2}{x+1}+\dfrac{3}{(x+1)^2}$.
19. $\dfrac{5}{x}+\dfrac{2}{x^2}-\dfrac{3}{x-1}$.
20. $\dfrac{1}{x-2}-\dfrac{1}{x+1}-\dfrac{2}{(x+1)^2}-\dfrac{8}{(x+1)^3}$.
21. $\dfrac{x}{2x^2+5}-\dfrac{1}{4x-5}$.
22. $\dfrac{1}{x-2}-\dfrac{x}{x^2-x+2}$.
23. $1+\dfrac{1}{x-1}-\dfrac{1}{x+1}$.
24. $1+\dfrac{4}{3(x-2)}-\dfrac{1}{3(x+1)}$.
25. $\dfrac{1}{3}+\dfrac{5}{6(3x-1)}-\dfrac{1}{2(x-1)}$.
26. $x+\dfrac{2}{x-2}+\dfrac{2}{x+2}$.
27. $\dfrac{x+3}{x^2+3}-\dfrac{x+2}{x^2+2}$.
28. $\dfrac{2x-1}{x^2+2}+\dfrac{1}{x+1}-\dfrac{2}{(x+1)^2}$.
29. $\dfrac{x}{2}+1-\dfrac{1}{4(2x-1)}-\dfrac{1}{4(2x-1)^2}+\dfrac{2}{x-2}$.
30. $x^2-x-3+\dfrac{3}{x^2+1}+\dfrac{3x}{x^2+2}$.

31. $\frac{2}{2x+3} + \frac{1}{x+1}; -\frac{4}{(2x+3)^2} - \frac{1}{(x+1)^2}; \frac{16}{(2x+3)^3} + \frac{2}{(x+1)^3}$.

32. (a) (0, 0), (b) (−7, −3/7).

Exercise 18.7 (pp. 358–362)

1. f: even, periodic, $\frac{1}{3}\pi$; g: neither, not periodic; h: even, not periodic; k: odd, periodic, 4π.
2. f: none; 2; g: x an even integer; x an even integer; h: $-1, 2; -1, 2$; k: none; 1.
3. $\pi, \pi; \pi$; $0 \leqslant f(x).g(x) < 2$. 4. f: neither; g: even; h: even; $12\pi, 6\pi$.
5. $4 \leqslant f(x) \leqslant 20$, f^{-1} exists; $-4 \leqslant g(x) \leqslant 0$, no inverse; $-5 \leqslant h(x) \leqslant 5$, no inverse; $k(x) \leqslant 1$, k^{-1} exists.
6. $a = 1$; $g^{-1}(x) = (2 - \sqrt{\{4 - x^2\}})/x$, $-2 \leqslant x \leqslant 2$, $-1 \leqslant g^{-1}(x) \leqslant 1$.
7. 1, 1/5.
8. $-5 < f(x) \leqslant 4$; $-2 < x \leqslant 1$: $f^{-1}(y) = \frac{1}{2}(y+1)$, $1 < x \leqslant 2$: $f^{-1}(y) = \sqrt{y}$, $2 \leqslant x < 3$: $f^{-1}(y) = \frac{1}{3}(10 - y)$.
9. $-\frac{1}{4}\pi \leqslant x \leqslant \frac{1}{4}\pi, f^{-1}(y) = \frac{1}{2}\sin^{-1} y$; $\frac{1}{4}\pi \leqslant x \leqslant \frac{3}{4}\pi, f^{-1}(y) = \frac{1}{4}\pi(2 - y)$.
10. (i) F : odd, periodic, 6π; G: even, periodic, 2π; f_3: odd; f_4: even, period $\frac{1}{3}\pi$, (ii) 0.
11. $\{y \in \mathbb{R} : y \geqslant 0, y \neq 2r+1, r \in \mathbb{Z}\}$; not one-one; $g^{-1} : x \to x - \frac{1}{2}[x]$; $\{x \in \mathbb{R}^+ : 2r < x < 2r+1, r \in \mathbb{Z}\}$.
12. $-1 < x < 2, 3 < x < 6$. 14. $x < -1$ or $x > 0$.
16. $-12\frac{1}{24}$; $-3/2, 4/3$; $-1{\cdot}56, -1{\cdot}44, 1{\cdot}27, 1{\cdot}39$.
17. $a = b = 1$; $x^3 - 2x^2 + 3$. 18. (a) $-6x + 2$, (b) $-2x$.
19. $2, \pm\sqrt{5}$.
20. (a) $(x+y+z)(x-y)(y-z)(z-x)$, (b) $3(a-b-c)(b-c-a)(c-a-b)$.
21. $a = -2, b = -1, c = 16$. 23. $-2, -2, 4 \pm 2\sqrt{2}$. 24. $0, -3, 125$.
26. $4, -5$.
27. (i) 2, (ii) HCF of $A(x)$ and $B(x)$ is $8x + 8$; $\alpha : g(x)$, $\beta : A(x)$ and $B(x)$ have no common factor.
28. $-\frac{1}{3} \leqslant y \leqslant 1, x = -1$; $-\frac{1}{3} \leqslant f(x) \leqslant 1$; $(-2, -\frac{1}{3}), (0, 1)$.
29. (a) $\mathbb{R} - \{\frac{1}{2}\}$; $x = -\frac{1}{2}$, $y = \frac{1}{2}$; none, (b) $\{y \in \mathbb{R} : y < 0$ or $y \geqslant 1\}$; $x = -3$, $x = 1$, $y = 0$; $(-1, 1)$, (c) $\{y \in \mathbb{R} : y \leqslant -3$ or $y \geqslant -\frac{1}{3}\}$; $x = 0$, $x = 3$, $y = 0$; $(-3, -\frac{1}{3}), (1, -3)$, (d) $\mathbb{R}$; $x = -2$, $x = 2$, $y = 0$; none.
30. (a) none, (b) (0, 0) max, (2, 4) min, (c) (0, 0) min, (d) (3, 1/8) max.
31. $\{x \in \mathbb{R} : x \neq -1, 3\}$; $\{y \in \mathbb{R} : y \leqslant -1$ or $y \geqslant \frac{1}{2}\}$; $x = -1$, $x = 3$, $y = 1$.
32. (a) max $(1, \frac{1}{2})$, min $(-1, -\frac{1}{2})$; inflexions $(-\sqrt{3}, -\frac{1}{4}\sqrt{3}), (0, 0), (\sqrt{3}, \frac{1}{4}\sqrt{3})$.

33. $\frac{3}{3x-2} - \frac{1}{2x+3}$. 34. $\frac{1}{x-1} + \frac{1}{(x-1)^2} - \frac{2}{2x-1}$.

35. $\frac{1}{2(x-1)} + \frac{1-x}{2(x^2+1)}$. 36. $\frac{1}{3(x+1)} - \frac{1}{x-1} + \frac{2}{3(x-2)}$.

37. $-\frac{1}{2} + \frac{3}{2(2x+1)} + \frac{2}{3-x}$. 38. $\frac{1}{1-x} - \frac{1-x}{1+x+x^2}$.

39. $x - 2 - \dfrac{2}{3(x-1)} + \dfrac{8}{3(x+2)}$.

40. $\dfrac{1}{x-1} + \dfrac{2}{(x-1)^2} - \dfrac{1}{x+1} + \dfrac{3}{(x+1)^2}$.

Exercise 19.1 (pp. 366–368)

1. (a) $2e^{2x}$, (b) $-5e^{-x}$, (c) $3e^{3x+5}$.
2. (a) $6x^2e^{2x^3}$, (b) $\dfrac{1}{2\sqrt{x}}e^{\sqrt{x}}$, (c) $\dfrac{1}{x^2}e^{-1/x}$.
3. (a) $\cos x e^{\sin x}$, (b) $-2\sin 2x e^{\cos 2x}$, (c) $4\sec^2 x e^{4\tan x}$.
4. (a) $(2x^2+1)e^{x^2}$, (b) $(5x+2)xe^{5x}$, (c) $(\cos x - x\sin x)e^{x\cos x}$.
5. (a) $e^{2x}(2\cos 3x - 3\sin 3x)$, (b) $e^{-x^2}(\cos x - 2x\sin x)$,
   (c) $\frac{1}{2}(x+1)^2(x+7)e^{x/2}$.
6. (a) $-2e^{4x}(1-e^{4x})^{-\frac{1}{2}}$, (b) $e^{(x+e^x)}$, (c) $e^x\cos(e^x)$.
7. (a) $e^{2x} - e^{-2x}$, (b) $(2+e^x)/(1+e^{-x})^2$, (c) 0.
8. (a) $\frac{1}{3}e^{3x} + c$, (b) $-e^{-x} + c$, (c) $e^{x+4} + c$.
9. (a) $3(e^2-1)$, (b) $2(e-1)$, (c) $e^6 - 1$.
10. (a) $\dfrac{1}{x}$, (b) $-\dfrac{1}{x}$, (c) $\dfrac{2}{x} - 1$. 12. $y = e(3x-5)$. 13. $e-1$.
15. $(1, e^{-1})$ max. 16. $(-2, -e^{-2})$ min.
17. $(-1, -e^{-\frac{1}{2}})$ min, $(1, e^{-\frac{1}{2}})$ max. 18. $(0,1)$ max, $(1,0)$ min.
19. $(1, e)$ min. 20. $(2k\pi, 1)$ min, $(\{2k+1\}\pi, e^2)$ max, $k \in \mathbb{Z}$.
21. 4 m, -2 m s^{-2}. 22. $x = -1$, max; $x = 2$, min.
23. $x = \pi/4$, max; $x = 5\pi/4$, min.
24. $\frac{1}{2}(e^2 - 3 + 2e^{-1}) \approx 2{\cdot}56$; $\frac{1}{4}\pi(e^4 - 3 + 2e^{-2}) \approx 40{\cdot}7$.

Exercise 19.2 (pp. 371–372)

1. (a) $\dfrac{1}{x}$, (b) $\dfrac{1}{x+3}$, (c) $\dfrac{2}{2x-1}$.
2. (a) $\dfrac{3x^2}{x^3+4}$, (b) $2\cot 2x$, (c) $\tan x$.
3. (a) $x(2\ln x + 1)$, (b) $(1 - 2\ln x)/x^3$, (c) $1/x\ln x$.
4. (a) $\dfrac{3}{x}$, (b) $\dfrac{2}{4x+5}$, (c) $-1 - \ln x$.
5. (a) $\sec x$, (b) $\dfrac{4x}{x^2+4}$, (c) $(\log_{10} e)/x$.

6. (a) $\dfrac{3x-4}{x(3x-2)}$, (b) $\dfrac{1}{2x(x+1)}$, (c) $\dfrac{x}{1-x^2} - \tan x$.

7. (a) $2x$, (b) $\sec^2 x$, (c) $\dfrac{2}{x} - 1$. 8. $y = 3x - 6$. 9. $t/(1+t)$.

11. (a) $2p + q + 1$, (b) p/q, (c) $1/(p+q)$, (d) e^{p+q}
12. (a) $t = 0$ or $\ln 2$, (b) $x = -\ln 2$.
13. (a) none, (b) none, (c) $(1, 0)$ min. 14. $(1, 1)$ min.
17. (a) $3^x \ln 3$, (b) $5^{2x} 2 \ln 5$, (c) $\ln 10 \cos x (10^{\sin x})$.

18. (a) $x^x(1 + \ln x)$, (b) $(2 \ln x)x^{(\ln x - 1)}$, (c) $(\ln x)^x \left\{ \ln(\ln x) + \dfrac{1}{\ln x} \right\}$.

19. (a) $\dfrac{2(x-2)}{(2x+1)^3}$. (b) $\dfrac{(6-5x)x^2}{2(1-x)^{3/2}}$, (c) $\dfrac{4x^2(x-1)e^{4x}}{(x+1)^3}$.

20. (a) $\frac{1}{2}(13 - 10x)\sqrt{\left\{\dfrac{x+2}{(1-5x)^3}\right\}}$, (b) $\dfrac{4x}{(x-2)^{2/3}(x+4)^{4/3}}$.

(c) $\dfrac{4\sin^2 x + 1}{\cos^3 x \sqrt{(\sin 2x)}}$.

Exercise 19.4 (pp. 378–379)

1. (a) $\frac{1}{2}\ln(2x+7) + c$, (b) $\frac{1}{5}\ln(5x-2) + c$.
2. (a) $-\ln(1-x) + c$, (b) $\frac{1}{2}\ln(1-2x) + c$.
3. (a) $\ln(2-3x) + c$, (b) $-2\ln(3-4x) + c$.
4. (a) $\ln|x+3| + c$, (b) $-3\ln|3-x| + c$, (c) $\frac{1}{2}\ln|4x+5| + c$.
5. (a) $x + \ln|x-1| + c$, (b) $x + 2\ln|x-4| + c$, (c) $\frac{1}{2}x + \frac{1}{4}\ln|2x+1| + c$.
6. (a) $\frac{1}{2}x^2 - \ln|x| + c$, (b) $x - \ln(1+x^2) + c$, (c) $-\frac{1}{2}x^2 - \ln|1-x| + c$.
7. (a) $-\frac{1}{2}\ln|1-x^2| + c$, (b) $\frac{1}{3}\ln|x^3+8| + c$, (c) $\frac{1}{4}\ln|2x^2+4x-1| + c$.
8. (a) $\ln(1+\sin x) + c$, (b) $\ln|\sin x| + c$, (c) $x + \frac{1}{2}\ln|\sec 2x| + c$.
9. (a) $\ln(1+e^x) + c$, (b) $\ln|\ln x| + c$, (c) $2\ln(1+\sqrt{x}) + c$.
10. (a) $5\ln 2$, (b) $\ln 4$, (c) $\ln 3$. 11. $\frac{1}{3}\ln\frac{5}{2}$, (b) $-\ln 4$, (c) $2 + \ln\frac{2}{3}$.
12. (a) $\ln 2$, (b) $\ln 5$, (c) $\ln\frac{4}{3}$. 13. (a) 0·805, (b) 1·10.

14. $\dfrac{5(1-x)}{(2x-1)(3-x)}$; $-0{\cdot}167$. 15. $\frac{3}{4} - \ln 2$. 16. $\pi(8 + \ln 3)$.

Exercise 19.5 (p. 380)

1. (a) $2\cosh 2x$, (b) $3\sinh 3x$, (c) $2\sinh x \cosh x$, (d) $\dfrac{\sinh x}{2\sqrt{(\cosh x)}}$.

2. (a) $\cosh x + c$, (b) $\frac{1}{2}\sinh 2x + c$, (c) $\ln(\cosh x) + c$.
3. (a) $\operatorname{sech}^2 x$, (b) $-\operatorname{cosech}^2 x$, (c) $-\operatorname{sech} x \tanh x$, (d) $-\operatorname{cosech} x \coth x$.
5. (a) 0 or $\ln 5$, (b) $-\ln 2$.

Exercise 19.6 (pp. 380–382)

1. (a) $e^{3x}(3x+1)$, (b) $e^{2x}\left(\frac{1}{x}+2\ln 4x\right)$, (c) $a^x \ln a$.

2. (a) $\frac{1}{x}\cos(\ln x)$, (b) $\frac{1}{x}-x-2x\ln x$, (c) $\operatorname{cosec} x$.

3. (a) $\frac{8}{(x-3)(3x-1)}$, (b) $2\sec x$, (c) $\frac{3-4x^2}{1-x^2}$.

4. (a) $(\ln 2, 4)$ min, (b) $(0,0)$ inflexion, $(3, 27e^{-3})$ max.
5. $x = e$, max. 6. 24/7
9. $(-1, -2\sqrt{e})$ min, $(4, 18e^{-2})$ max.
10. (a) $\ln 3$, (b) $\frac{1}{2}\ln 6$, (c) $\frac{3}{2}\ln 3$.
11. (a) $\frac{1}{3}x-\frac{1}{9}\ln|3x+1|+c$, (b) $\ln|x|+\frac{2}{x^2}+c$,

(c) $2x-\ln(x^2+x+1)+c$.
12. $\frac{1}{2}\ln(1+x)-\frac{1}{2}\ln(1-x)-x$; 0. 14. $2xe^{2x}$; 59·5.

16. $\frac{5}{6}+2\ln 2$. 17. $2-\frac{2}{e}$; $2\pi\left(1-\frac{1}{e^2}\right)$.

18. $(2,3), (3,0)$; $y+3x=\pm 6\sqrt{2}$.
19. $(-1, -\frac{1}{2})$ min, $(1, \frac{1}{2})$ max; $\frac{1}{2}\ln(1+a^2)$.
20. Inflexion at $(-\{\ln b\}/c, a/2b)$.
21. Asymptote $x = 6$; (4, 1) max, (8, 9) min; $\frac{15}{2}-8\ln 2$.
22. $\frac{1}{2}\pi(e^{2h}+8e^h+8h-9)$; $50/\{\pi(e^{1\cdot5}+2)^2\} \approx 0\cdot379$.

Exercise 20.1 (pp. 386–387)

1. $\frac{3}{2x^2}-\frac{1}{x}+c$. 2. $\frac{1}{7}x^7-\frac{1}{2}x^4+x+c$.

3. $\frac{2}{5}x^{5/2}+\frac{2}{3}x^{3/2}+c$. 4. $\frac{1}{3}\sin 3x+c$. 5. $\frac{1}{4}e^{(4x-3)}+c$.

6. $\frac{1}{2}\cos(1-2x)+c$. 7. $-\frac{1}{5}(1-x)^5+c$. 8. $-\frac{1}{x+4}+c$.

9. $-\frac{1}{4(2x-5)^2}+c$. 10. $\frac{1}{2}\ln|2x-5|+c$. 11. $\frac{1}{5}\tan 5x+c$.

12. $2\ln|\sin\frac{1}{2}x|+c$. 13. $\frac{1}{2}x-\frac{1}{4}\sin 2x+c$.
14. $\frac{1}{4}\cos 2x-\frac{1}{8}\cos 4x+c$. 15. $-x-\cot x+c$.
16. $-\frac{1}{3}\cos^3 x+c$. 17. $\sec x+c$. 18. $\frac{1}{3}\cos^3 x-\cos x+c$.
19. $x-\ln|x+2|+c$. 20. $x+\frac{1}{2}x^2+\ln|x-1|+c$.
21. $\frac{1}{2}x^2-3\ln(x^2+5)+c$. 22. $\frac{2}{5}\sqrt{(5x+3)}+c$.
23. $\frac{1}{4}\ln(1+x^4)+c$. 24. $\frac{1}{16}(1+x^4)^4+c$. 25. $e-1$.

26. $40\frac{1}{2}$. 27. 3/16. 28. $-2/21$. 29. $(\pi + 2)/24$. 30. 2/15.
31. 3/2. 32. 1/4. 33. $2 - \frac{1}{2}\pi$. 34. $\ln 2$. 35. 2.
36. $\frac{1}{2}\ln(4/3)$. 38. $\sec x$; $\frac{1}{2}\ln 3$. 39. $\tan\frac{1}{2}x + c$. 40. $3\pi/8$.

Exercise 20.2 (pp. 389–391)

1. $\ln\left|\dfrac{x-1}{x+2}\right| + c$. 2. $\frac{1}{2}\ln\left|\dfrac{1+x}{1-x}\right| + c$. 3. $\frac{1}{3}\ln\left|\dfrac{x-3}{x}\right| + c$.

4. $\frac{1}{2}\ln|x^2 - 4| + c$. 5. $\ln|(x-3)^3(x+1)| + c$.

6. $\ln|(x-2)(x-3)| + c$. 7. $\ln|(x+3)^2(x+1)| + c$.

8. $\frac{1}{2}\ln\left|\dfrac{x^2}{2x-1}\right| + c$. 9. $\ln\left|\dfrac{(x-1)^3}{x-4}\right| + c$. 10. $\frac{1}{2}\ln\left|\dfrac{1+x}{1-x}\right| - x + c$.

11. $\frac{1}{2}\ln|x^2 - 4x - 8| + c$. 12. $\frac{1}{2}x^2 + 2\ln|x^2 - 4| + c$.

13. $\ln|(x+1)(x-2)| + \dfrac{1}{x+1} + c$. 14. $3\ln\left|\dfrac{x-1}{x}\right| + \dfrac{2}{x} + c$.

15. $\ln|x+3| + \dfrac{3}{x+3} + c$. 16. $\ln\left|\dfrac{x^2+1}{x+1}\right| + c$.

17. $\frac{1}{2}\ln\dfrac{(x-1)^2}{x^2+1} + c$. 18. $\frac{1}{4}\ln\dfrac{x^2+3}{x^2+5} + c$.

19. $\frac{1}{2}\ln(2x+1)^2(x^2+2) + c$. 20. $x + \ln\dfrac{x^2+x+1}{(x-1)^2} + c$.

21. $\ln\frac{32}{27}$. 22. $\ln\frac{12}{7}$. 23. $\ln 6$. 24. $\frac{3}{2}\ln\frac{9}{8}$. 25. $\frac{1}{3}\ln\frac{27}{2}$.
26. $\ln\frac{4}{3}$. 27. $\ln\frac{9}{10}$. 28. $2 + \ln\frac{5}{9}$. 29. $\frac{1}{2} + \ln\frac{3}{4}$.
30. $\frac{3}{2} + \ln\frac{8}{3}$. 31. $\frac{13}{2} + \ln\frac{512}{3}$. 32. $11 + 2\ln\frac{6}{5}$.
33. $\frac{1}{2}\ln\frac{40}{17}$. 34. $1 + \ln\frac{5}{2}$. 35. $\ln 2$. 36. $\ln\frac{12}{7}$.

37. (a) $\ln\left(\dfrac{x-1}{x+3}\right) + c$, (b) $\ln\left(\dfrac{1-x}{x+3}\right) + c$, (c) $\ln\left(\dfrac{x-1}{x+3}\right) + c$.

38. (a) $\frac{1}{2}\ln\dfrac{(x+1)^2}{2x-3} + c$, (b) $\frac{1}{2}\ln\dfrac{(x+1)^2}{3-2x} + c$.

Exercise 20.3 (pp. 395–397)

1. $\frac{1}{12}(3x-2)^4 + c$. 2. $-\frac{1}{10}(1-2x)^5 + c$.
3. $-\frac{1}{42}(1-x)^6(1+6x) + c$. 4. $\frac{1}{6}(x^3-1)^4 + c$. 5. $\frac{1}{3}\tan^3 x + c$.
6. $-\frac{1}{6}\cos^6 x + c$. 7. $\frac{1}{3}(x^2+4)^{3/2} + c$.

8. $-\frac{2}{15}(1-x)^{3/2}(2+3x)+c$. 9. $\frac{1}{3}(x-1)\sqrt{(2x+1)}+c$.
10. $\ln(1+e^x)+c$. 11. $\sin\theta-\frac{1}{3}\sin^3\theta+c$.
12. $\frac{1}{5}\cos^5\theta-\frac{1}{3}\cos^3\theta+c$. 13. $\frac{1}{2}\sec\theta+c$. 14. $2\tan(\sqrt{x})+c$.
15. 12/5. 16. 16/105. 17. $\ln 2$. 18. $\dfrac{e-1}{2(e+1)}$. 19. $\pi/2$.
20. $\pi/18$. 21. 4/5. 22. $\frac{1}{24}(2\pi+3\sqrt{3})$. 23. $\frac{1}{3}\ln 2$.
24. $\frac{1}{4}\ln 3$. 25. $\frac{1}{120}(2x+3)^5(10x-3)+c$.
26. $\frac{1}{40}(x^2-1)^4(4x^2+1)+c$. 27. $\frac{2}{9}(1+3x)^{3/2}+c$.
28. $\frac{2}{3}(x-6)\sqrt{(x+3)}+c$. 29. $\dfrac{1}{3(1-x^3)}+c$. 30. $-\dfrac{x}{(x-1)^2}+c$.
31. $\frac{1}{2}e^{x^2}+c$. 32. $2\sqrt{(\ln x)}+c$. 33. $\frac{1}{4}\ln\left|\dfrac{2+e^x}{2-e^x}\right|+c$.
34. 6. 35. 9. 36. $9\pi/4$. 37. $\sqrt{3}$. 38. $\frac{1}{6}(4\pi+3\sqrt{3})$.
39. $\ln(4/3)$. 40. $23\frac{1}{3}$. 41. 8; 5π. 42. 3π. 44. $\frac{1}{2}\pi$.

Exercise 20.4 (p. 400)

1. $\sin x-x\cos x+c$. 2. $2x\sin\frac{1}{2}x+4\cos\frac{1}{2}x+c$.
3. $-e^{-x}(x+1)+c$. 4. $\frac{1}{4}x^2(2\ln 2x-1)+c$.
5. $\frac{1}{4}e^{2x}(2x-1)+c$. 6. $x\tan x+\ln(\cos x)+c$.
7. $\frac{1}{8}\sin 2x-\frac{1}{4}x\cos 2x+c$. 8. $x\tan x+\ln(\cos x)-\frac{1}{2}x^2+c$.
9. $\frac{1}{4}x^2+\frac{1}{4}x\sin 2x+\frac{1}{8}\cos 2x+c$. 10. $(x^2-2)\sin x+2x\cos x+c$.
11. $e^x(x^3-3x^2+6x-6)+c$. 12. $\frac{1}{16}x^4(4\ln x-1)+c$.
13. $\frac{1}{2}x^2(\ln x)^2-\frac{1}{2}x^2\ln x+\frac{1}{4}x^2+c$. 14. $(x-1)\ln(x-1)-x+c$.
15. $-\frac{1}{2}e^{-x}(\sin x+\cos x)+c$. 16. -2. 17. $2/e$. 18. $\pi/9$.
19. 1. 20. $\frac{1}{4}\pi^2$. 21. $\frac{8}{3}\ln 2-\frac{7}{9}$. 22. $10e^{-2}-17e^{-3}$.
23. $\frac{1}{4}-\frac{3}{4}e^{-2}$. 24. $\pi-2$. 25. $\frac{1}{2}(e^{\pi}+1)$. 26. $\frac{5}{4}e^4-\frac{1}{4}e^2$.
27. $\frac{1}{2}e^2$. 28. $\frac{1}{30}(x+1)^5(5x-1)+c$.
29. (a) $\frac{2}{15}(x-1)^{3/2}(3x+2)+c$, (b) $-\dfrac{(x+1)}{(x+2)^2}+c$.

Exercise 20.5 (pp. 403–404)

1. $\sin^{-1}\dfrac{x}{4}+c$. 2. $\frac{1}{5}\tan^{-1}\dfrac{x}{5}+c$. 3. $\frac{1}{2}\sin^{-1}2x+c$.
4. $\frac{1}{2}\ln(x^2+1)+\tan^{-1}x+c$. 5. $\frac{1}{5}\sin^{-1}\dfrac{5x}{2}+c$.

6. $\frac{1}{6}\tan^{-1}\frac{3x}{2} + c$. 7 $\ln|x+1| - \tan^{-1}x + c$.

8. $\ln\left(\frac{x^2+1}{(1-x)^2}\right) + 2\tan^{-1}x + c$. 9. $\ln\left(\frac{x^2+9}{x^2}\right) + 2\tan^{-1}\frac{x}{3} + c$.

10. $3\ln(x^2+4) - \ln|x+2| - 4\tan^{-1}\frac{x}{2} + c$. 11. $\frac{1}{2}\tan^{-1}\frac{(x+1)}{2} + c$.

12. $\sin^{-1}\frac{(x-2)}{3} + c$. 13. $\pi/18$. 14. $\pi/2$. 15. $\pi/8$.

16. $\pi/2$. 17. $\ln 2 + \pi/2$. 18. $\pi/4$. 19. $\pi/4$.
20. $\frac{1}{2}\ln(1+e^{2x}) + \tan^{-1}(e^x) + c$. 21. $\tan^{-1}(e^x) + c$.
22. $\frac{1}{2}\sin^{-1}x + \frac{1}{2}x\sqrt{(1-x^2)} + c$. 23. $\sin^{-1}x + \sqrt{(1-x^2)} + c$.

24. $\ln(x^2-2x+10) + \frac{2}{3}\tan^{-1}\frac{(x-1)}{3} + c$. 25. $\frac{1}{2}\tan^{-1}(x^2+1) + c$.

26. $-\sin^{-1}\left(\frac{1}{x}\right) + c$. 27. (a) $-0{\cdot}123$, (b) $0{\cdot}545$.

Exercise 20.6 (pp. 407–408)

1. (a) 2/5, (b) 0, (c) π, (d) $\sqrt{3}$, (e) 0, (f) 0. 2. 0.
3. (a) -2, (b) 4, (c) 4. 4. (a) $-1\frac{1}{2}$, (b) $1\frac{1}{2}$, (c) $2\frac{1}{2}$.
5. (a) 0, (b) 8/3. 7. $e-1$. 8. $\frac{2}{3}(2-\sqrt{2})$.
9. $I = \frac{1}{2}\pi$, $J = 2(e-1)$. 10. (a) 2, (b) 1, (c) π.

Exercise 20.7 (pp. 408–412)

1. $2 - \frac{1}{2}\ln 3$. 2. $\ln 2 - 1$. 3. $\frac{1}{4}\ln\frac{9}{5}$. 4. 2.

5. $1 + \ln\frac{8}{3}$. 6. $\frac{64\sqrt{2}}{21}$. 7. 8/3. 8. π. 9. 7/6.

10. $\frac{1}{2}(1-\ln 2)$. 11. 1/24. 12. $\ln 2$. 13. $\ln(4/3)$. 14. $\pi/16$.

15. $1/(\ln 2)$. 16. $\frac{1}{5}\ln\left(\frac{x-3}{x+2}\right) - \frac{1}{x-3} + c$. 18. $\pi/12$.

19. $(\pi-2)/4$. 20. $(\pi+2)/8$. 21. $\frac{1}{2}\ln 3$.
23. $(\frac{1}{4}, \frac{8}{3})$ min; $\frac{3}{2} - \ln 4$. 24. $(-4, \frac{4}{3})$ max; $2\ln 3$.

25. $\frac{2}{x-1} - \frac{x}{x^2+1}$; $1{\cdot}04$, $-0{\cdot}23$.

27. (i) a^2+b^2, (ii) $2\log_{10}2 - \log_{10}e$, (iii) $(2\pi + 3\sqrt{3})/48$.
28. $y = x - a$, $x + y = 5a$; $8a^2/3$.
29. $-3\sin t\cos t$; $y + 3x\sin t_1\cos t_1 = 3\sin t_1 - 2\sin^3 t_1$.
31. $\frac{1}{2}x^2\sin 2x + \frac{1}{2}x\cos 2x - \frac{1}{4}\sin 2x + c$; π; $\pi^2(2\pi^2-3)/12$; $\pi^4/6$.

32. $\dfrac{1}{x+1}+\dfrac{x+2}{x^2+3}$; $\ln(\sqrt{2}+\sqrt{6})+\dfrac{\pi}{2\sqrt{3}}$; $\dfrac{1}{\sqrt{3}}\ln(\sqrt{2}+\sqrt{6})+\dfrac{\pi}{6}$.

33. $p=2, q=1$; $\frac{1}{4}\pi$; $\ln 2$; $\pi+\ln 2$.

34. (a) $\dfrac{1-t^2}{1+t^2}$ (b) $\dfrac{1}{1-t}+\dfrac{1}{1+t}+\dfrac{8}{1+9t^2}$.

35. (a) $x(\log_e x-1)+c$; $x\{(\log_e x)^2-2\log_e x+2\}+c$, (b) $\pi(e-2)$.

36. (i) $\frac{1}{2}\pi-1$, (ii) $\frac{1}{2}(1-e^{-2})$. 37. π.

38. $\dfrac{a^2}{12}(4\pi-3\sqrt{3})$; $\dfrac{5\pi a^3}{24}$.

39. (i) $\pi(2\ln 2-1)$, (ii) $\frac{1}{2}\pi-1$.

40. $\frac{22}{15}a^2$, $\frac{7}{12}\pi a^3$. 41. $\frac{16}{3}$, $\dfrac{64\pi}{15}$.

42. $x=\dfrac{\pi}{4}, \dfrac{9\pi}{4}$: max, $x=\dfrac{5\pi}{4}, \dfrac{13\pi}{4}$: min; $\frac{1}{2}(1+e^{-\pi})$; $\frac{1}{2}e^{-2\pi}(1+e^{-\pi})$.

Exercise 21.1 (pp. 416–417)

1. $\mathbf{a}-\mathbf{b}+\mathbf{c}$. 2. $\mathbf{a}-2\mathbf{b}+2\mathbf{c}$. 3. 60°. 4. $7, \sqrt{19}$.
5. (a) $s=-2, t=6$, (b) $s=6/5, t=3/5$; $s=-2, t=-1$.
6. (a) 9, (b) -1. 7. (a) $\frac{3}{8}\mathbf{a}+\frac{5}{8}\mathbf{b}$, (b) $-\frac{1}{2}\mathbf{a}+\frac{3}{2}\mathbf{b}$, (c) $2\mathbf{a}-\mathbf{b}$.
8. (a) $\mathbf{i}+2\mathbf{j}-\mathbf{k}$, (b) $-7\mathbf{i}-2\mathbf{j}-5\mathbf{k}$.
9. $2\mathbf{i}-5\mathbf{j}+3\mathbf{k}$, $7\mathbf{i}-2\mathbf{k}$; $(4,-3,1)$, $(-1,-8,6)$.
10. P is a vertex of parallelogram $ABCP$; Q divides AC internally in the ratio 4:3; R divides BC externally in the ratio 2:3.
11. When produced BA and CD meet a point Q such that $BQ:AQ=7:4$ and $CQ:DQ=5:2$.
12. $\mathbf{r}=3\mathbf{q}-2\mathbf{p}$; 3:2.

Exercise 21.2 (pp. 422–423)

1. (a) $\mathbf{r}=(1+3\lambda)\mathbf{i}+2\mathbf{j}-(1+\lambda)\mathbf{k}$, (b) $\mathbf{r}=(\lambda+4)\mathbf{i}-3\lambda\mathbf{j}+3(\lambda-1)\mathbf{k}$.
2. (a) $\mathbf{r}=2(1-2\lambda)\mathbf{i}+(4\lambda-3)\mathbf{j}+\mathbf{k}$, (b) $\mathbf{r}=(1+2\lambda)\mathbf{i}+(5-6\lambda)\mathbf{j}+2\lambda\mathbf{k}$, (c) $\mathbf{r}=5\lambda\mathbf{i}+(6-13\lambda)\mathbf{j}+2(5\lambda-4)\mathbf{k}$, (d) $\mathbf{r}=5\lambda\mathbf{i}-2\lambda\mathbf{j}+3\lambda\mathbf{k}$.
3. $\mathbf{r}=(1-\lambda)\mathbf{a}+\frac{1}{3}\lambda\mathbf{b}+\frac{2}{3}\lambda\mathbf{c}$, $\mathbf{r}=-\mu\mathbf{a}+(1-\mu)\mathbf{b}+2\mu\mathbf{c}$; $-\frac{1}{2}\mathbf{a}+\frac{1}{2}\mathbf{b}+\mathbf{c}$.
4. $\frac{1}{2}\mathbf{a}, \frac{1}{2}\mathbf{a}+\mathbf{b}, \frac{1}{3}\mathbf{a}+\frac{2}{3}\mathbf{b}$. 5. $\frac{3}{2}\mathbf{a}$. 6. 1:3.
7. (a) Intersect at $2\mathbf{i}-3\mathbf{j}+3\mathbf{k}$, (b) skew, (c) parallel, (d) intersect at $6\mathbf{i}+13\mathbf{j}-4\mathbf{k}$.
8. (a) A, B, (b) B, (c) C, (d) A, B, C.

9. (a) $\mathbf{r} = (\lambda + 1)\mathbf{i} + (3\lambda - \mu - 2)\mathbf{j} + 2\mu\mathbf{k}$,
(b) $\mathbf{r} = (2\lambda - \mu)\mathbf{i} + (2\mu - \lambda)\mathbf{j} - 7\mu\mathbf{k}$,
(c) $\mathbf{r} = (\mu + 3)\mathbf{i} + (\lambda + \mu + 1)\mathbf{j} + (\mu - 1)\mathbf{k}$.
10. $\mathbf{r} = (\lambda + 5\mu)\mathbf{i} + (2\mu - 3\lambda)\mathbf{j} + (2 - \lambda - 2\mu)\mathbf{k}$.
11. $-\mathbf{i} + 3\mathbf{k}$, $\mathbf{j}$; $\mathbf{r} = (2 - \lambda)\mathbf{i} + (5 + \mu)\mathbf{j} + 3\lambda\mathbf{k}$.
12. $\mathbf{r} = (2t + 1)\mathbf{i} + (2s - t)\mathbf{j} - (2 + s - 3t)\mathbf{k}$; (5, 0, 3).
14. $[r(1 - p)\mathbf{c} + p(r - 1)\mathbf{a}]/(r - p)$, $[p(1 - q)\mathbf{a} + q(p - 1)\mathbf{b}]/(p - q)$.

Exercise 21.3 (pp. 427–429)

1. (a) $p^2 + 2\mathbf{p}.\mathbf{q} + q^2$, (b) $p^2 - q^2$, (c) $2\mathbf{p}.\mathbf{r}$, (d) $p^2 - 2\mathbf{p}.\mathbf{q} + q^2$.
2. (a) x^2, (b) $2x^2 - 3y^2$. 3. 81, 49; 9, 7. 4. 13, 9. 8. $-3/4$.
9. $3\mathbf{a} - 2\mathbf{b}$. 14. (a) 68·3°, (b) 60°, (c) 90°, (d) 35·3°.
15. (a) 1/3, (b) 5/11. 16. (a) 3, (b) 2, (c) 1, (d) -1. 17. 65/7, 7/9.
18. (a) $4\mathbf{i} - 3\mathbf{j}$, (b) $2\mathbf{i} + 5\mathbf{j}$, (c) $3\mathbf{i} - 2\mathbf{j}$.
19. $\angle A = 90°$, $\angle B = 36{\cdot}9°$, $\angle O = 53{\cdot}1°$; 30 sq. units.
20. $11/21$, $8\sqrt{5}$ sq. units. 21. $5\mathbf{i} + 2\mathbf{k}$.
22. (a) (2, −1, 1), (b) 70·5°.
23. $\mathbf{i} - \mathbf{j}$, $\mathbf{j} + \mathbf{k}$; 60°. 24. −11, −7; 7; 7/57.

Exercise 21.4 (pp. 434–436)

1. $-2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$; 3 units. 2. $\mathbf{i} - \mathbf{k}$; 9 sq. units.
3. $3\mathbf{i} - \mathbf{j} + 4\mathbf{k}$; $6\mathbf{i} - 2\mathbf{j} + 8\mathbf{k}$. 4. $5\mathbf{i} - \mathbf{j} + 6\mathbf{k}$; 13 units.
5. (a) $\mathbf{r}.(4\mathbf{i} - \mathbf{k}) = 3$, (b) $\mathbf{r}.(\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = 1$.
6. (a) $\mathbf{r}.\mathbf{k} = 1$, (b) $\mathbf{r}.(3\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) = 0$.
7. (a) 5, (b) 3, (c) 4, (d) 0.
8. $2\mathbf{i} - 5\mathbf{j} - \mathbf{k}$, $\mathbf{r}.(2\mathbf{i} - 5\mathbf{j} - \mathbf{k}) = 7$. 9. $\mathbf{r}.(\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) = 1$.
10. $\mathbf{r}.(\mathbf{j} - 2\mathbf{k}) = -8$. 11. (a) 60°, (b) 48·2°, (c) 70·5°.
12. (a) 11/18, (b) 5/6, (c) 6/7. 13. $-\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$, $2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$.
14. $\mathbf{r} = (3\lambda + 5)\mathbf{i} - (4\lambda + 3)\mathbf{j} + (\lambda + 4)\mathbf{k}$; $2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$.
15. (a) $\dfrac{x-1}{2} = \dfrac{y+2}{3} = \dfrac{z-1}{4}$, (b) $\dfrac{x}{1} = \dfrac{y-2}{-1} = \dfrac{z-3}{2}$,
(c) $\dfrac{x-2}{5} = \dfrac{y+3}{-6} = \dfrac{z-1}{3}$, (d) $\dfrac{x-5}{1} = \dfrac{y}{3} = \dfrac{z+4}{0}$
16. (a) $2x - 3y + z - 5 = 0$, (b) $3x + y - 5z - 9 = 0$.
17. $5x + y + z - 6 = 0$. 18. $2x - y + 5z - 3 = 0$.
19. $\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$; (a) $\mathbf{r}.(\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}) = -5$, (b) $x - 3y + 2z + 5 = 0$.
20. $3x + y - 4z - 1 = 0$. 21. (a) 70·5°, (b) 65·9°, (c) 28·5°.
22. (a) Plane through O perpendicular to OA,
(b) Plane through A perpendicular to OA,
(c) Sphere with centre A, radius 1 unit,
(d) Plane through O perpendicular to AB,
(e) Sphere with diameter OA,
(f) Sphere with diameter AB.

Exercise 21.5 (pp. 436–438)

1. $\mathbf{b}-\mathbf{a}, \frac{1}{2}(\mathbf{a}+\mathbf{c}), \mathbf{a}-\mathbf{b}+\mathbf{c}; 2\mathbf{k}, 2\mathbf{j}, 2\mathbf{i}+2\mathbf{j}+2\mathbf{k}, \mathbf{i}+\mathbf{j}+2\mathbf{k}, 2\mathbf{j}+2\mathbf{k}; (2\sqrt{2})/3.$
2. $\frac{1}{2}(\mathbf{a}+\mathbf{b}), \frac{1}{2}(\mathbf{b}+\mathbf{c}), \frac{1}{2}(\mathbf{c}+\mathbf{d}), \frac{1}{2}(\mathbf{d}+\mathbf{a}), \frac{1}{2}(\mathbf{a}+\mathbf{c}), \frac{1}{2}(\mathbf{b}+\mathbf{d}); (\mathbf{i}+\mathbf{j})/\sqrt{2}.$
3. Mid-point of XC, where X divides AB in ratio 1:2; $\frac{1}{3}\mathbf{a}+\frac{2}{3}(\frac{1}{4}\mathbf{b}+\frac{3}{4}\mathbf{c})$;
   R divides AY in ratio 2:1, where Y divides BC in ratio 3:1.
4. $\{(1+\lambda)\mathbf{a}+\mathbf{b}+\lambda\mathbf{c}\}/2(1+\lambda), \{\lambda\mathbf{a}+(1+\lambda)\mathbf{b}+\mathbf{c}\}/2(1+\lambda),$
   $\{\mathbf{a}+\lambda\mathbf{b}+(1+\lambda)\mathbf{c}\}/2(1+\lambda); \frac{1}{2}(1+\sqrt{5}).$
6. $3\mathbf{i}+2\mathbf{j}+4\mathbf{k}; \mathbf{r}=3\mathbf{i}+s(2\mathbf{i}+5\mathbf{j}+4\mathbf{k})+t(\mathbf{j}+2\mathbf{k}).$
7. $\mathbf{r}=(1-\lambda)\mathbf{i}+(6\lambda+3)\mathbf{j}+(4\lambda+2)\mathbf{k}.$ 9. 70·5°.
11. $\frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c}).$ 12. $\lambda:\mu=1:6; \mathbf{r}=(1+t)\mathbf{a}+6t\mathbf{b}; -12\mathbf{i}-6\mathbf{j}+6\mathbf{k}.$
13. (i) $\mathbf{r}=(1+\lambda+\mu)\mathbf{i}-2\lambda\mathbf{j}-3\mu\mathbf{k}$ or $\mathbf{r}.(6\mathbf{i}+3\mathbf{j}+2\mathbf{k})=6,$
    (ii) $\mathbf{r}=\lambda(6\mathbf{i}+3\mathbf{j}+2\mathbf{k})$, (iii) $6(6\mathbf{i}+3\mathbf{j}+2\mathbf{k})/49.$
14. (i) $(1,-2,0),(4,4,-3)$, (ii) 90°, 90°,
    (iii) $x-y-z-3=0, x+z=1$, (iv) 90°.
15. $(p-\mathbf{a}.\mathbf{n})/(\mathbf{b}.\mathbf{n}); (2\frac{1}{3}, 2\frac{1}{3}, 2\frac{2}{3}).$
16. $-1, 1; \mathbf{i}+5\mathbf{j}-\mathbf{k}, -3\mathbf{i}+3\mathbf{j}+\mathbf{k}; 2\sqrt{6}.$
17. (i) $-\mathbf{i}+\mathbf{j}+3\mathbf{k}$, (ii) $3\mathbf{j}-\mathbf{k}$, (iii) 55°, (iv) 35°.

Exercise 22.1 (pp. 442–444)

1. (a) $n(n+2)$, (b) $\frac{1}{6}n(n-1)(2n+5)$, (c) $2^{n+1}-2$,
   (d) $\frac{1}{12}n(n+1)(n+2)(3n+1)$, (e) $\frac{3}{2}(1-3^{-n})$,
   (f) $\frac{1}{4}(n+4)^2(n+5)^2-100.$

2. (a) $\dfrac{n}{2(n+2)}$, (b) $\frac{3}{4}-\dfrac{1}{2(n+1)}-\dfrac{1}{2(n+2)}=\dfrac{n(3n+5)}{4(n+1)(n+2)}$,

   (c) $\frac{1}{12}-\dfrac{1}{2(n+2)}+\dfrac{1}{2(n+3)}.$

3. (a) (i) 0, (ii) 1, (b) (i) $-n$, (ii) $n+1$. 4. $\frac{1}{3}n(n+1)(n+2)$
7. $\frac{1}{6}n(n+1)(2n+1)(3n^2+3n-1).$ 8. $(n+1)!-1.$
9. (a) $\frac{5}{2}n(n+1)$, (b) $\frac{1}{6}n(n+1)(38n+1).$
10. (a) n^2, (b) $\frac{1}{3}n(2n-1)(2n+1).$
11. (a) 352 800, (b) 56 953, (c) $\frac{1}{4}n^2(3n+1)(5n+3)$, (d) $-n^2(4n+3).$
12. $r(r+3); \frac{1}{3}n(n+1)(n+5).$
13. $(3r-1)^2; \frac{1}{2}n(6n^2+3n-1).$
14. $2r(2r-1); \frac{1}{3}n(n+1)(4n-1).$
15. $r(r+1)(r+4); \frac{1}{12}n(n+1)(n+2)(3n+17).$

16. $\dfrac{1}{r(r+3)}; \dfrac{11}{18}-\dfrac{1}{3}\left(\dfrac{1}{n+1}+\dfrac{1}{n+2}+\dfrac{1}{n+3}\right).$

17. $\dfrac{1}{(2r-1)(2r+1)}; \dfrac{n}{2n+1}.$

18. $\dfrac{1}{(2r-1)(2r+1)(2r+3)}; \dfrac{1}{12}-\dfrac{1}{8(2n+1)}+\dfrac{1}{8(2n+3)}.$

19. $\dfrac{r}{(r+1)(r+2)(r+3)}; \dfrac{1}{4} + \dfrac{1}{2(n+2)} - \dfrac{3}{2(n+3)}.$

20. $\dfrac{x^{n+1}-1}{x-1}; \dfrac{nx^{n+1} - (n+1)x^n + 1}{(x-1)^2}.$

21. $\dfrac{nx^{n+2} - (n+1)x^{n+1} + x}{(1-x)^2}.$

22. $\{1 - a - (2n+3)(-a)^{n+1} + (2n+1)(-a)^{n+2}\}/(1+a)^2.$

Exercise 22.2 (p. 446)

1. $1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$, 2. $x + \frac{1}{3}x^3$. 3. $1 + \frac{1}{2}x^2 + \frac{5}{24}x^4$.
4. $-\frac{1}{2}x^2 - \frac{1}{12}x^4$. 5. $x^2 - \frac{1}{3}x^4$. 6. $x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{12}x^4$.
7. $x - \frac{1}{3}x^3$. 8. $1 + x + \frac{1}{2}x^2 - \frac{1}{8}x^4$. 9. $x + \frac{1}{6}x^3$.
10. $x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 - \frac{1}{90}x^6$.

Exercise 22.3 (pp. 450–452)

1. $1 - 3x + 6x^2 - 10x^3$. 2. $1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3$.
3. $\frac{1}{8} - \frac{3}{64}x + \frac{15}{1024}x^2 - \frac{35}{8192}x^3$. 4. $1 + 4x + 8x^2 + 16x^3$.
5. $\frac{1}{4} - \frac{1}{2}x + \frac{7}{16}x^2 - \frac{5}{16}x^3$. 6. $1 + 2x^2 + 8x^3$. 7. $1 - x + x^3$.
8. $1 - \frac{1}{2}x + \frac{11}{8}x^2 - \frac{29}{16}x^3$. 9. $1 + \frac{1}{3}x + \frac{8}{9}x^2 - \frac{49}{81}x^3$,
10. $1 - x + \frac{1}{2}x^2, |x| < 1$. 11. $2 + \frac{25}{12}x + \frac{1175}{288}x^2, |x| < \frac{1}{3}$.
12. $1 + \frac{1}{2}x^2, |x| < \frac{1}{5}$. 13. $\frac{1}{3} - \frac{2}{9}x + \frac{7}{27}x^2, |x| < 1$
14. $1 - 3x + 9x^2, |x| < \frac{1}{2}$. 15. $1 + \frac{5}{2}x + \frac{43}{8}x^2, |x| < \frac{1}{2}$.
16. $3 + x + 11x^2 + 25x^3 + 83x^4, |x| < \frac{1}{3}$.
17. $\frac{4}{3}x + \frac{8}{9}x^2 + \frac{28}{27}x^3 + \frac{80}{81}x^4 + \frac{244}{243}x^5, |x| < 1$.
18. $1 + 2x + x^2 + x^4 + 2x^5, |x| < 1$.
19. $4 - x + 12x^2 - 11x^3 + 38x^4, |x| < \frac{1}{2}$.
20. $2 - 2x + 2x^4 - 2x^5 + 2x^8, |x| < 1$.
21. $-1 + x + \frac{7}{4}x^2 + \frac{1}{2}x^3 - \frac{3}{16}x^4, |x| < \sqrt{2}$.
22. $3 + 3x + 9x^2; \{(-1)^r + 2^{r+1}\}x^r$.
23. $4x - 8x^2 + 28x^3; \{1 - (-3)^r\}x^r$.
24. $2 - \frac{3}{2}x + \frac{17}{4}x^2; \{(-2)^r + 2^{-r}\}x^r$.
25. $1 + \frac{1}{2}x - \frac{3}{8}x^2 + \frac{7}{16}x^3; |x| < \frac{1}{2}$; (a) 1·0885, (b) 1·019 428.
26. $(a+b) + 2(1-a+b)x + (-4-2a+3b)x^2 + 4(-1-a+b)x^3$; 13, −8.
27. $n = 6, a = \frac{1}{2}$. 28. $a = 3, b = 1$; 70/13.

Exercise 22.4 (pp. 455–457)

1. (a) $1 - x + \frac{1}{2}x^2; \frac{(-1)^r x^r}{r!}$, (b) $1 + 3x + \frac{9}{2}x^2; \frac{3^r x^r}{r!}$,
   (c) $1 + x^2 + \frac{1}{2}x^4; \frac{x^{2r}}{r!}$, (d) $e + ex + \frac{1}{2}ex^2; \frac{ex^r}{r!}$.
2. (a) $1 + 4x + 6x^2 + \frac{16}{3}x^3$, (b) $1 - 3x + \frac{7}{2}x^2 - \frac{13}{6}x^3$,
   (c) $1 - x + \frac{3}{2}x^2 - \frac{7}{6}x^3$.
3. 2·7183; 0·3679.
4. (a) $3x - \frac{9}{2}x^2 + 9x^3; -\frac{1}{3} < x \leqslant \frac{1}{3}$. (b) $-\frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{24}x^3; -2 \leqslant x < 2$.
   (c) $\ln 2 + \frac{1}{3}x - \frac{1}{18}x^2; -3 < x \leqslant 3$. (d) $1 - \frac{x}{e} - \frac{x^2}{2e^2}; -e \leqslant x < e$.
5. (a) $x - \frac{3}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4$, (b) $x - \frac{5}{2}x^2 + \frac{7}{3}x^3 - \frac{17}{4}x^4$,
   (c) $-4x + 3x^2 - \frac{34}{3}x^3 + \frac{63}{2}x^4$. 6. 1·0986.
7. (a) $(1 - r)/r!$, (b) $(4 + 5r - 3r^2)2^{r-2}/r!$, (c) $(4r^2 + 1)/2^r r!$.
8. (a) $x + \frac{1}{2}x^2$, (b) $4 - 4x + 3x^2$, (c) $1 + \frac{15}{2}x + \frac{193}{8}x^2$, (d) $2x$,
   (e) $1 + 2x + \frac{5}{2}x^2$, (f) $2 + x + \frac{11}{6}x^2$.
9. 0·6931. 10. $a = 2, n = -2$. 11. $a = \frac{1}{2}, b = 2$; 31/170.

Exercise 22.5 (p. 458)

1. (a) 0·2955, (b) 0·9211, (c) −0·4161, (d) 0·8415.
2. (a) $3x - \frac{9}{2}x^3 + \frac{81}{40}x^5$, (b) $1 - 2x^2 + \frac{2}{3}x^4$, (c) $x - \frac{1}{2}x^2 + \frac{1}{6}x^3$,
   (d) $e - \frac{e}{2}x^2 + \frac{e}{6}x^4$, (e) $1 + \frac{1}{6}x^2 + \frac{7}{360}x^4$, (f) $x + \frac{1}{3}x^3 + \frac{2}{15}x^5$.
3. (a) $x + x^2 - \frac{2}{3}x^3 - \frac{1}{6}x^4$, (b) $2x - 2x^2 + \frac{4}{3}x^3$, (c) $\frac{4}{3}x^4$,
   (d) $1 + 2x + x^2 - \frac{2}{3}x^3 - x^4$.

Exercise 22.6 (pp. 458–460)

1. (a) $\frac{1}{3}n(n^2 - 1)$, (b) $\frac{1}{3}(n + 1)(4n^2 + 14n + 9)$, (c) $\frac{n + 1}{n(2n + 1)}$,
   (d) $\frac{3}{4} - \frac{1}{2n} - \frac{1}{2(n + 1)}$.
2. (a) $\frac{1}{12}n(n + 1)(n + 2)(3n + 5)$, (b) $n(n + 1)(2n^2 + 2n - 1)$.
3. $\frac{1}{6}n^2(n + 1)^2(n - 1)$. 4. $\frac{1 - (n + 1)x^{2n} + nx^{2n+2}}{(1 - x^2)^2}$
5. $(x + 1)(x + 2)(x + 3); \frac{1}{x + 1} + \frac{2}{x + 2} - \frac{3}{x + 3}$.

6. (a) $2^{n+1} - 2 + n^2$, (b) $a = -2, b = -1$; 73/74.

7. $1 + \dfrac{1}{2x^2} - \dfrac{1}{8x^4}, 1 - \dfrac{1}{2x^2} - \dfrac{1}{8x^4}$.

8. $2 - \frac{1}{4}x - \frac{1}{64}x^2; 2 + \frac{7}{4}x + \frac{175}{64}x^2, |x| < \frac{1}{2}$. 9. $2 + x - \frac{3}{2}x^3$.

10. (a) $3x - 3x^2 + 9x^3, \{1 - (-2)^r\}x^r, |x| < \frac{1}{2}$,
(b) $\frac{1}{6}x + \frac{5}{36}x^2 + \frac{19}{216}x^3, \left(\dfrac{1}{2^r} - \dfrac{1}{3^r}\right)x^r, |x| < 2$,
(c) $1 + 5x + 18x^2, \frac{1}{2}(5.3^r - 2^{r+1} - 1)x^r, |x| < \frac{1}{3}$.

11. (a) $1 - 2x - x^2 + 2x^3 + \frac{3}{2}x^4; |x| < 1/\sqrt{2}$,
(b) $1 + 4x + 14x^2 + 40x^3 + 104x^4; |x| < \frac{1}{2}$,
(c) $3 + 6x + 6x^2 + 12x^3 + 36x^4; |x| < \frac{1}{2}$.

12. (a) $x + \frac{5}{24}x^3 - \frac{1}{8}x^4$; for $|x| < 1$; $a = -\frac{1}{4}$,
(b) $1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3, \dfrac{1.3.5....(2r-1)}{2^r r!}(-1)^r x^r$; 0·099 504.

13. (a) (i) (0, 1), (ii) (0, 1), (iii) (1, 0), (iv) (0, 0);
(b) $7x + \frac{1}{2}x^2 + \frac{43}{3}x^3 + \frac{49}{4}x^4, -\frac{1}{3} \leqslant x < \frac{1}{3}; \{(-2)^{n+1} + 3^n\}x^n/n$.

14. $1 + x + x^2 + \frac{1}{3}x^3$; 0·287.

15. (i) $x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots + (-1)^{n+1}\dfrac{x^n}{n} + \ldots$; (ii) 0·6931; 1/3, 23/60.

16. $p^2 - 2, p^3 - 3p; -px - \frac{1}{2}(p^2 - 2)x^2 - \frac{1}{3}(p^3 - 3p)x^3$.

17. $1, \sqrt{3}, 2$.

Exercise 23.1 (pp. 466–468)

1. 0·697. 2. 5·731. 3. 45·992. 4. 4·154. 5. 0·144.
6. 0·825. 7. Greater. 8. Greater. 9. Greater.
10. Less. 11. Greater. 12. Less. 13. 0·8358. 14. 0·4783.
15. 0·6807. 16. 0·8945. 17. 0·2763. 18. 3·2413.
19. $1 + 12x^2 + 66x^4 + 220x^6 + 495x^8$; 1·7655; 1·9253.
20. 0·0711; 0·0715.
21. (a) 10·075, (b) 9·99. 22. (a) 7·175 m s^{-1}, (b) 11·85 m s^{-1}; 131 m.
23. (a) 422·5 J. (b) 423 J. 24. 0·7444. 25. 1·4241.
26. 1·68. 27. 0·524.

Exercise 23.2 (pp. 474–475)

1. 2·7, −0·7. 2. 0·7. 3. −2·1. 4. −0·6, 1, 1·6.
5. −1·8, −0·4, 1·2. 6. 0·8. 7. 0, ±1·9. 8. −3·6, −2·1, 1·3.
9. 1·1. 10. 1, 4·2. 11. One. 12. Two. 13. Three.
14. Five. 15. One; −3, −2. 16. One; 1, 2.
17. Three; −4, −3; 1, 2; 2, 3. 18. Three; −3, −2; −1, 0; 1, 2.
19. One; 1, 2. 20. One; 1, 2. 21. Two; −3, −2; 1, 2.

22. One; 2, 3. 23. 3·2. 24. −0·8, 1·6. 25. −3, 1; 1·1.
26. 2·2. 27. −5·2, −0·6, 2·9. 28. −0·2, 0·6, 2·0. 29. −4·1.
30. 1·8, 4·5. 31. −1·3, 1·9, 4·0.
32. (a) −1·75, 0·43, 1·32, (b) −1·6, 0·7, 1·1.
33. (a) −0·1, (b) 0·25, (c) 0·4, (d) 0·2.

Exercise 23.3 (pp. 478–479)

1. 1·20. 2. −1·11. 3. −1·52. 4. 2·39. 5. 1·56.
6. 0·79. 7. −1·90. 8. 1·17. 9. 2·803. 10. 1·650.
11. −0·830. 12. −1·834. 13. 1·278. 14. 2·219. 15. 0·453.
16. 3·6; 3·6913. 17. Two; 3·98, −1·06.
18. 2, a: = 2, h: = 0·01, $g(x)$: = $e^x - 2$; 1·68.

Exercise 23.4 (pp. 484–486)

1. (a) 2·24, (b) 1·26, (c) 0·79, 3·87. 2. (a) 1·414, (b) 3·162, (c) 3·606.
3. 1·71. 4. 0·769. 5. 1·821. 6. −3·794.
7. 0·1926; for $x_0 = -6$ the sequence diverges, for $x_0 = -5$ the sequence converges to the positive root.
8. 1; 1·260, 1·312. 9. 1·732.
10. 1·087 using $x_{n+1} = (4 - \frac{5}{2}x_n)^{1/3}$, $x_{n+1} = \dfrac{8}{2x_n^2 + 5}$ or Newton's method.
11. 2·618; decision box should ensure maximum error of 0·1. 12. 0·7.

Exercise 23.5 (pp. 486–490)

1. 3·69, 6·71. 2. 4, 3; (a) 4·826, (b) 4·8224. 3. 0·4778; 0·4783.
4. 0·410. 5. (i) 2·09, (ii) −1·53, 0·26, 1·27; 0·5.
6. (i) two, (ii) $m < -\frac{1}{2}$. 7. $(0, 0), (2, 4e^{-2})$. 8. $q < 0$.
9. 2·4; underestimate. 10. Three; 1·43, larger. 11. (ii) output "2, 3".
12. (a) 2·208, (b) 2·208. 13. 1, 3; −4, 0; $a < -4$ or $a > 0$; 4·1.
14. 3·43; 3·75; 15. 15. $(x - 2)(x - 3)(x^2 + x + 2)$; 3·065.
16. (a) 0·818, (b) 0·810. 17. 0·74. 18. 0·5, 0·73.
19. 1·2. 20. 0·77. 21. $x = (3x - 1)^{1/3}$; 1·53.

Exercise 24.1 (pp. 494–495)

3. (a) and (d). 4. 2, −1. 5. 1, 0, 2. 6. $\dfrac{dy}{dx} = 3x^2$.

7. $x\dfrac{dy}{dx} = y + x^2$. 8. $x\dfrac{dy}{dx} + y = 0$. 9. $\dfrac{dy}{dx} + y = 0$.

10. $x + y\dfrac{dy}{dx} = 0$. 11. $2x\dfrac{dy}{dx} = y$. 12. $x\dfrac{dy}{dx} = 2x - y$.

13. $x\dfrac{dy}{dx} = 2y + x^2$. 14. $2\dfrac{dy}{dx} + y\tan x = 0$.

15. $x\ln x\dfrac{dy}{dx} = y\ln y$. 16. $\dfrac{d^2y}{dx^2} + y = 0$.

17. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} - 3y = 0$. 18. $3\dfrac{dy}{dx}\left(\dfrac{d^2y}{dx^2}\right) + y\dfrac{d^3y}{dx^3} = 0$.

19. $\dfrac{d^3y}{dx^3} - 2\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 0$.

Exercise 24.2 (pp. 499–501)

1. $y = x^3 + x + c$. 2. $y = 2\sin\frac{1}{2}x + c$. 3. $y = \frac{1}{2}x^2 + \ln|x| + c$.
4. $y = x + \ln(x-1)^2 + c$. 5. $y = \cos x + \frac{1}{3}x^3 + Ax + B$.
6. $y = e^{-x} - e^x + Ax + B$. 7. $y = \dfrac{1}{2x + k}$. 8. $y = \tan^{-1}(x + k)$.
9. $y = \ln|x + k|$. 10. $y = (c - x)^{-1/3}$. 11. $y = \frac{1}{3}(x+1)^3 - 9$.
12. $y = 1 + \sqrt{(2x+3)}$. 13. $y = x\sin x + \cos x + 1$.
14. $y = \ln\frac{1}{2}(1 + x^2)$. 15. $y = \frac{1}{4}(x - \sin 2x)$.
16. $y = 2\ln x + x^2 - 4x + 3$. 17. $y = 3e^{-2x}$. 18. $y = 1 - e^x$.
19. $y = \tan(x + \frac{1}{4}\pi)$. 20. $y = (3 + e^x)/(3 - e^x)$. 21. $y = x/(1 + kx)$.
22. $y = kx + x\sin x$. 23. $y = kx^2 - x$. 24. $y = x - e^{1-x} + 2$.
25. $xe^y\dfrac{dy}{dx} + e^y$; $y = \ln\left(x - \dfrac{2}{x}\right)$.

Exercise 24.3 (pp. 504–506)

1. $y = (x^2 - x + c)^{1/3}$. 2. $y = 1/(c - 3x^2)$. 3. $y = -\ln(\cos x + c)$.
4. $y = \ln(e^x + c)$. 5. $y = \sin^{-1}(\frac{1}{2}x^2 + c)$. 6. $y = \tan^{-1}(3x + c)$.
7. $y = Ax$. 8. $y = A/(1 - x)$. 9. $y = A(x^2 + 1)^2$.
10. $y = A(x-1)/(x+1)$. 11. $y = x/(1 - 2x)$. 12. $y = 5e^{-x^2}$.
13. $y = 2\sec x$. 14. $y = \frac{1}{2}\ln(x^2 + 1)$. 15. $y = 2e^{x^2} - 1$.
16. $y = xe^{x-1}$. 17. $y = \sin^{-1}(1 + x^3)$. 18. $y = \sin^{-1}(x - 1)$.
19. $y = 1/(1 - 2x^2)$. 20. $y = (x-1)/(x+1)$. 21. $y = Ae^{-x}/(1 - x)$.
22. $y = Ae^x(x-1)/(x+1)$. 23. $y = \tan(e^x + c)$. 24. $y = \ln(1 + Ae^{x^2})$.
25. $y = e^{-x}\sqrt{(2x+1)}$. 26. $y = \sqrt{\{(\ln x)^2 + 9\}}$.
27. $x^2 - y^2 - 2xy + 2x + 2y = A$.
28. $x^2(x^2 - y^2) = A$.

Exercise 24.4 (pp. 507–509)

1. $x^2 + y^2 = 5$. 2. $y = 1/x^2$.

3. $\dfrac{dC}{dt} = kC$; $C = C_0e^{kt}$; $t = \dfrac{1}{k}\ln\frac{3}{2}$. 4. $\dfrac{d\theta}{dt} = -k\theta$.

5. $(1 + \sqrt{2})t$. 6. $\dfrac{dx}{dt} = -kx$; 93 days; 320 mg. 7. 40 min.

8. $x = 100(1 + e^{-t/10})$; 137 litres; 6 min 56 sec. 9. $30\ln 2$ min., $100e^{-1}$ kg.

10. 2·99. 11. $t = \dfrac{\ln 3}{V\beta}$; $m = \dfrac{V}{\alpha(1 + 3e^{-V\beta t})}$.

Exercise 24.5 (pp. 514–515)

1. $y = A\cos 4x + B\sin 4x$. 2. $y = A\cos\frac{\sqrt{7}}{2}x + B\sin\frac{\sqrt{7}}{2}x$.
3. $y = 2 + A\cos 3x + B\sin 3x$. 4. $y = 1 + A\cos\sqrt{10}x + B\sin\sqrt{10}x$.
5. $y = Ae^{6x} + Be^{-6x}$. 6. $y = Ae^{2\sqrt{2}x} + Be^{-2\sqrt{2}x}$.
7. $y = -\frac{1}{27} + Ae^{\sqrt{3}x} + Be^{-\sqrt{3}x}$. 8. $y = \frac{1}{4} + Ae^{2x} + Be^{-2x}$.
9. $y = 4\cos 2x - 3\sin 2x$. 10. $y = 3e^{4x} + 7e^{-4x}$. 11. $y = 6\cos\frac{1}{6}x$.
12. $y = 4\sin\frac{4}{3}x - 5\cos\frac{4}{3}x$. 13. $y = -2e^{5x/2} - 7e^{-5x/2}$.
14. $y = 4e^{-2\sqrt{3}x}$. 15. $y = -\frac{4}{7} + 3e^{7x/3} + 6e^{-7x/3}$.
16. $y = \frac{8}{5} + 4\cos\frac{5}{2}x + 9\sin\frac{5}{2}x$. 17. $y = 2 + \sin\sqrt{3}x$.
18. $y = -3 + e^{9x/4} + e^{-9x/4}$. 19. $y = -3 + \sqrt{2}e^{3\sqrt{2}x}$.
20. $y = \frac{1}{2} - 4\sqrt{3}\cos 2\sqrt{3}x - 3\sqrt{3}\sin 2\sqrt{3}x$.

Exercise 24.6 (pp. 515–519)

1. $\left(\dfrac{dy}{dx}\right)^2 - 4x\dfrac{dy}{dx} + 4y = 0$.

2. $(2x + 1)\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} - (2x + 3)y = 0$.

3. $y = x^2$; $y = x^2 + 2x$.
4. (a) $\ln\frac{1}{3}(3 - 7u) = 7(t - 5)$, (b) $3(y + 18) = (x^2 + 9)^{3/2}$.
5. (a) $2x^3 + 11x^2 + 20x = 37 - 2t$, (b) $y = \{\frac{1}{2}(5e^{x+1} - 3)\}^{1/2}$.

6. $y = 36e^{-t/6} + 27$; 11 minutes; 27 degrees. 7. $y = \sin x + \dfrac{\cos x - 1}{1 + x}$.

8. (a) $x^2 - 4y^2 = 4$; $x - y\sqrt{3} = 1$.
9. (a) $\tan y = 1 - \cos x$, (b) $y = \ln\frac{1}{2}(1 + e^{2x})$.
10. (i) $y = e^{-x}(1 + x)^2$, (ii) $y^2 = 2x$.

12. $y = \dfrac{2x}{x^2 + cx + 1}$; (a) $y = \dfrac{2x}{x^2 + 1}$, (b) $y = \dfrac{2x}{x^2 + x + 1}$,

(c) $y = \dfrac{2x}{(x+1)^2}$.

13. $\dfrac{dy}{dx} = \dfrac{y}{x}$, $y = 2x$; $\dfrac{dx}{dt} = -2x^3$, $x = \dfrac{1}{\sqrt{(4t+1)}}$.

14. (a) $y\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 = 0$.

15. (i) (a) $y = A + \ln[\ln(1/x)]$, (b) $y = A + \ln(\ln x)$, (iii) $y = x$, $y = 2 - x$.

16. $20°$, $53\frac{3}{4}°$. 17. (a) $x = 2 + \dfrac{1}{12}\ln\left(\dfrac{5(3y-2)}{3y+2}\right)$, (b) $\dfrac{dp}{dt} = kp$.

18. (a) $y = \frac{1}{3}(5e^{3x-6} - 2)$, (b) $\dfrac{dm}{dt} = -km$; 75·98%.

19. $\dfrac{dP}{dt} = kP(N - P)$, $P = N/(1 + Ae^{-Nkt})$; $N = 400\,000$; N is the limiting size of P.

20. $\dfrac{dh}{dt} = -\lambda$, $\pi h\dfrac{dh}{dt} = -(Ck + \pi\lambda h)$.

21. (i) $\dfrac{dQ}{dt} = -kQ$; $Q = Q_0e^{-kt}$; $Q = 2{\cdot}5 \times 10^{-4}$, (ii) $y = 3e^{5x} + e^{-5x}$.

22. $y = 2 - 4\sin 3x$; -2, $\frac{2}{3}n\pi + \frac{1}{18}\pi$, $\frac{2}{3}n\pi + \frac{5}{18}\pi$.

23. (i) $(y+1)e^{-y} = (1-x)e^x$, (ii) $y = 4 - 2\sin 2x$; $\frac{1}{2}r\pi$, r an integer.

24. (i) $y = Ae^x/(1+x)$, (ii) $y = \frac{1}{2}(1 - \cos 2x - \sin 2x)$.

25. $y = \frac{1}{2} + A\cos(2x^2) + B\sin(2x^2)$; $y = -\frac{1}{2} + Ae^{2x^2} + Be^{-2x^2}$.

Exercise 25.1 (pp. 524–525)

1. (a) 1/7, (b) 1/8, (c) 21/13, (d) 7/4, (e) 15/8, (f) 3/4.
2. (a) (5, 0), (b) (−2, −2), (c) (2, −2), (d) $(-\frac{1}{2}, 1\frac{1}{4})$.
3. (a) (12, −5), (b) (6, −19), (c) (−4, −1), (d) $(7, -8\frac{1}{2})$.
4. (a) 1, (b) 2, (c) $\sqrt{2}$, (d) $\sqrt{13}$, (e) 9, (f) $2\sqrt{10}$.
5. (a) 6, (b) 24, (c) 18, (d) $11\frac{1}{2}$, (e) 0, (f) $36\frac{1}{2}$.
6. (a) $2x + y - 1 = 0$, $x - 2y - 2 = 0$. (b) $x + y - 1 = 0$, $x - y - 1 = 0$.
   (c) $x = -\frac{1}{3}$, $y = \frac{1}{3}$. (d) $2x - y + 4 = 0$, $x + 2y - 13 = 0$.
7. 45°, 45°, 90°. 8. 45°, 26·6°, 108·4°. 9. (−3/7, 12/7).
10. $(5\frac{5}{18}, 3\frac{1}{2})$. 11. 3. 12. 31·0°. 13. (2, 3).
14. $(-\frac{1}{2}, -\frac{1}{2})$. 15. (i) (6, 7), (ii) 16/15, (iii) $8\sqrt{2}$, (iv) 80.

Exercise 25.2 (pp. 527–528)

4. $x^2 + y^2 - 6x - 2y - 10 = 0, (5, 5); x^2 + y^2 - 6x - 2y - 70 = 0, (-1, -7).$
5. $(11/5, 13/5).$
6. $x^2 + y^2 - 2x - 2y - 23 = 0, x^2 + y^2 - 18x - 14y + 105 = 0, 4x + 3y = 32,$ $3x - 4y + 26 = 0, 3x - 4y = 24.$
7. $x^2 + y^2 - 6x + 10y - 16 = 0, x^2 + y^2 + 12x - 14y - 40 = 0;$ $3x - 4y = 4, 10$
8. $4x - 3y = 3.$
9. $x^2 + y^2 - 6x + 4y - 12 = 0; x^2 + y^2 + 14x - 6y - 42 = 0.$
12. (a) $2x - 2y - 1 = 0,$ (b) $x^2 + y^2 - 2y - 1 = 0.$
13. (a) $x^2 + y^2 + x - 4y + 1 = 0,$ (b) $x^2 + y^2 - 3y = 0,$ (c) $x^2 + y^2 + 4x - 7y + 4 = 0.$
14. (a) $x^2 + y^2 + x - 4y - 9 = 0,$ (b) $5x^2 + 5y^2 + 23x - 29y = 0,$ (c) $2x^2 + 2y^2 - 7y - 23 = 0.$
15. $x + 1 = 0, x + 2\sqrt{6}y = 11 - 2\sqrt{6}, x - 2\sqrt{6}y = 11 + 2\sqrt{6}.$

Exercise 25.3 (pp. 532–534)

1. (a) $(3t^2, 6t),$ (b) $(3t, 3/t),$ (c) $(t^2, t^3),$ (d) $(5t/2, 5/2t),$ (e) $(4t^2, 4t^3),$ (f) $(5t^2/2, 5t).$
2. (a) $(3\cos\theta, 2\sin\theta),$ (b) $(3\cos\theta, 3\sin\theta),$ (c) $(\sqrt{5}\cos\theta, \sin\theta),$ (d) $(\cos\theta, \frac{2}{3}\sin\theta),$ (e) $(\cos\theta, 3\sin\theta),$ (f) $(5\cos\theta, 1 + 5\sin\theta).$
3. $x + t^2y = 8t; ty = t^3x + 4 - 4t^4.$ 4. $x + 4y = 4, 2y = 8x - 15.$
5. $x + 2\sqrt{3}y = 4, 4\sqrt{3}x - 2y = 3\sqrt{3}.$
7. $4x + y = 4c$ at $(\frac{1}{2}c, 2c), 4x + y + 4c = 0$ at $(-\frac{1}{2}c, -2c).$
9. $x = t, y = \dfrac{3}{2t} - \dfrac{3t^3}{2}.$
10. $x + pqy = a(p + q), x + p^2y = 2ap; \left(\dfrac{2apq}{p+q}, \dfrac{2a}{p+q}\right).$ 14. $xy = 2c^2.$
15. $2y = 3tx - at^3; \dfrac{at^2}{4}, -\dfrac{at^3}{8}; t = 0.$
16. $(3\{p^2 + pq + q^2\}, \frac{9}{2}pq\{p + q\}); 9y^2 = 12x - 16.$
18. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 2.$ 19. $x\sin\theta + y = 2\tan\theta; x^2 - y^2 = 4.$
20. (a) $d^2\sin^2\theta,$ (b) $-1, 3/2.$

Exercise 25.4 (pp. 538–539)

1. (a) $4/5; (\pm 4, 0); x = \pm 25/4,$ (b) $3/5; (\pm 3/2, 0); x = \pm 25/6,$ (c) $\frac{1}{2}\sqrt{3}; (\pm\sqrt{3}, 0); x = \pm(4\sqrt{3})/3,$ (d) $2/3; (\pm 2, 0); x = \pm 9/2,$ (e) $\frac{1}{3}\sqrt{5}; (0, \pm\sqrt{5}); y = \pm(9\sqrt{5})/5,$ (f) $\frac{1}{2}\sqrt{3}; (0, \pm\frac{1}{2}\sqrt{3}); y = \pm\frac{2}{3}\sqrt{3}.$

2. (a) $5/4; (\pm 5, 0); x = \pm 16/5; y = \pm \frac{5}{4}x$, (b) $\sqrt{2}; (\pm 2\sqrt{2}, 0); x = \pm\sqrt{2}; y = \pm x$, (c) $\frac{1}{2}\sqrt{5}; (\pm\sqrt{5}, 0); x = \pm 4/\sqrt{5}; y = \pm\frac{1}{2}x$.
3. (a) $\frac{1}{2}\sqrt{3}$, (b) $4/5$, (c) $\frac{1}{3}\sqrt{5}$, (d) $4/5$.
4. (a) $\dfrac{x^2}{16} + \dfrac{y^2}{12} = 1$, (b) $\dfrac{x^2}{225} + \dfrac{y^2}{144} = 1$.
5. $5x^2 + 9y^2 - 40x - 18y + 44 = 0$.
6. (a) $x = 3, y = 0$, (b) $2y = x + 4, 2x + y + 3 = 0$, (c) $9x + 4y = 20, 8x - 18y = 7$, (d) $2x + y + 12 = 0, 2y = x + 1$.
7. (a) $4xx_1 + 9yy_1 = 36$, (b) $\dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} = 1$, (c) $\dfrac{xx_1}{3} - \dfrac{yy_1}{2} = 1$.
8. $2xy + 2kx + 2ky = 7k^2$.
11. (i) $2ax - 2by = a^2 - b^2$; (ii) $(0, -b), \left(\dfrac{3\sqrt{3}}{5}b, \dfrac{4}{5}b\right)$.
12. $a^2 = 2c^2$.

Exercise 25.5 (pp. 540–543)

1. 7.
2. $x - 2y + 9 = 0, x + 3y = 16y; x^2 + y^2 - 2x - 10y + 1 = 0$, $x^2 + y^2 - 7x - 144 = 0$.
3. $\frac{1}{5}|a + 4|; a = 1, 3x + 4y = 7$.
4. $(5, 5), \sqrt{10}; 3m^2 - 10m + 3 = 0; y = \frac{1}{3}x, y = 3x$; (i) $53{\cdot}1°$, (ii) $x^2 + y^2 + 2x - 14y + 40 = 0$
5. $2x + y + 4 = 0, (2, -8), (-4, 4); 108$.
6. $2, 5; 3/10, \pm 2/5$.
7. $(0, 8), 4\sqrt{2}; (9, -1), 5\sqrt{2}; (4, 4); 167°$.
8. $-1, 5; 3/4; 0, -8/9$.
9. (i) $3x - 4y = 9$, (ii) $(-1, -3), (\frac{23}{25}, -\frac{39}{25}), 2x^2 + 2y^2 + x + 8y + 5 = 0$.
10. $(8, 2); 16/3$.
12. $\left(ct + \dfrac{c}{t}, ct + \dfrac{c}{t}\right)$.
13. $-4c^2/a^2; 1/2; 2:3$.
14. $\left(\dfrac{8}{9}, \dfrac{16\sqrt{2}}{27}\right), \left(\dfrac{2}{9}, -\dfrac{2\sqrt{2}}{27}\right)$.
15. $8\frac{2}{3}$.
16. $x\cos t + y\sin t = 2\sin t\cos t; 3\pi/8$.
17. (a) $ay + bx = ab\sqrt{2}$, (b) $(ax - by)\sqrt{2} = a^2 - b^2; \frac{1}{4}\pi ab, \pi ab$.
18. $0°, 60°; (2, 0), (1, 3/2)$.
19. $bx\sec\theta + ay\operatorname{cosec}\theta = ab; \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \left(\dfrac{ab}{a^2 - b^2}\right)^2$.
20. $t_1t_2 + t_2t_3 + t_3t_1 = 0; -2t$.

Exercise 26.1 (pp. 546–548)

1. (a) $5i$ (b) $i\sqrt{5}$, (c) $3 - 4i$, (d) $2 - 3\sqrt{2}i$.
2. (a) ± 2, (b) $\pm 2i$, (c) $\pm i\sqrt{2}$, (d) $\pm \frac{3}{2}i$, (e) $0, \pm i\sqrt{3}$, (f) $\pm 1, \pm i$.
3. (a) $1 \pm i$, (b) $3 \pm i$, (c) $2 \pm 3i$, (d) $\frac{1}{2} \pm i$, (e) $\frac{1}{4}(-3 \pm i\sqrt{7})$, (f) $\frac{1}{2}(1 \pm i\sqrt{3})$.
4. (a) $3 - 5i$, (b) $4 + 7i$, (c) $-3i$, (d) -4, (e) $-1 - i$.
5. (a) $7 + 9i, 3 + 7i$, (b) $3 - i, 1 + 7i$, (c) $8 - 4i, -2 + 2i$,
   (d) $2 + 7i, -4 - 3i$.
6. (a) $3 + 11i$, (b) $7 - 9i$, (c) $4 - 7i$, (d) 25, (e) $-21 - 20i$, (f) $-2 - 2i$
7. (a) $-i$, (b) i, (c) 1 (d) -1, (e) $-i$, (f) i.
8. (a) $-1 + 2i$, (b) $\frac{1}{5} + \frac{3}{5}i$, (c) $\frac{5}{2} + \frac{1}{2}i$, (d) $\frac{1}{2} - \frac{\sqrt{3}}{2}i$, (e) $-\frac{1}{2}i$, (f) $\frac{4}{25} - \frac{3}{25}i$.
9. (a) $\frac{4}{13}$, (b) $\frac{7}{25} + \frac{1}{25}i$, (c) 6.
10. (a) $a = -1, b = 1$, (b) $a = 2, b = \frac{1}{2}$.
11. (a) $\pm(2 - i)$, (b) $\pm(1 + i)$, (c) $\pm(3 + i)$, (d) $\pm(5 - 2i)$.
12. (a) $-4, 5$, (b) $2, 10$, (c) $0, 16$, (d) $-6, 34$.
13. (a) $(x + i)(x - i)$, (b) $(x + 5i)(x - 5i)$,
    (c) $(3x + 2i)(3x - 2i)$, (d) $(x + 1 + 2i)(x + 1 - 2i)$,
    (e) $(x + 3 + 2i)(x + 3 - 2i)$, (f) $(2x - 3 + 4i)(2x - 3 - 4i)$.
14. $0, 0; 1, 0; -\frac{1}{2}, \frac{1}{2}\sqrt{3}; -\frac{1}{2}, -\frac{1}{2}\sqrt{3}$. 15. $x = 4, y = -3$.
16. $2 - i, 2 + 3i$. 17. $32 + 47i, 32 - 47i$; 53, 61.
18. $-5 - 4i, 1 + 7i$; $-3, -1$.
19. (a) $0, \pm 1$, (b) $\pm(1 + \sqrt{2}), \pm(1 - \sqrt{2})$. 20. $\pm(3 + 2i), 2 + 4i, -1 + 2i$.

Exercise 26.2 (pp. 550–552)

5. -2. 6. $3 - i, -3$. 7. $1 - i, -2; 4$. 8. $2 + 3i, -1; -3, 9$.
9. $-i, \frac{1}{2}(-1 \pm \sqrt{5})$. 10. $-1 - i, 2 \pm i$. 11. $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$.
12. $\alpha, \alpha\omega, \alpha\omega^2$; (a) $3, -\frac{3}{2} \pm \frac{3\sqrt{3}}{2}i$, (b) $-2, 1 \pm i\sqrt{3}$,
    (c) $-i, \frac{\sqrt{3}}{2} + \frac{1}{2}, -\frac{\sqrt{3}}{2} + \frac{1}{2}i$.
13. (a) $2, -1 \pm i\sqrt{3}$, (b) $-1, \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$, (c) $0, -\frac{3}{2} \pm i\frac{\sqrt{3}}{2}$.
14. (a) 0, (b) -1, (c) -1. 15. (a) $-\omega$, (b) $1 - \omega$, (c) $\omega - 1$.
16. $a^3 + b^3 + c^3 - 3abc$. 17. $9z^3 - 12z^2 + 6z - 1$. 18. $-1, -1$.

Exercise 26.3 (pp. 555–556)

1. (a) $3 + 2i$, (b) $-2 + 4i$, (c) $5i$, (d) $1 - 2i$, (e) -2.

2. (a) 5, (b) $\sqrt{29}$, (c) 3, (d) $\sqrt{2}$, (e) 4.
3. $-2-i, 2-i, -2+i, -1+2i, 1-2i.$ 4. $2i, -2+2i, \frac{1}{2}-\frac{1}{2}i.$
5. $6+8i, i.$ 6. (a) $\pm i$, (b) $\pm(1-i)$, (c) $\pm(2+i)$, (d) $\pm(4+i)$.
8. (a) $k(1+2i), k \geqslant 0$, (b) $k(1+2i), -1 \leqslant k \leqslant 0$, (c) $k(1+2i), k \leqslant -1$.
10. (a) 5, (b) $\sqrt{85}$, (c) $\sqrt{13}$. 12. $\frac{4}{3}+\frac{2\sqrt{2}}{3}i, \frac{2}{3}+\frac{4\sqrt{2}}{3}i.$

Exercise 26.4 (pp. 561–562)

1. (a) 2, $\pi/6$, (b) $\sqrt{2}, -\pi/4$, (c) 4, $\pi/2$, (d) 3, π.
2. (a) 5, $-0{\cdot}927$, (b) $\sqrt{5}, 2{\cdot}678$, (c) $\sqrt{10}, -1{\cdot}893$, (d) $\sqrt{34}, -0{\cdot}540$.
3. (a) 5, $\pi/6$, (b) $1/2, -\pi/3$, (c) $1/\sqrt{2}, \pi/4$, (d) $\sqrt{2}, 3\pi/4$.
4. (a) 5, $\pi/3$, (b) $1, -\pi/2$, (c) $1, -2\pi/3$, (d) $2, -3\pi/4$.
5. (a) $2\sqrt{3}\{\cos(-\pi/3)+i\sin(-\pi/3)\}$, (b) $8\{\cos 0+i\sin 0\}$,
   (c) $2\sqrt{2}\{\cos(-3\pi/4)+i\sin(-3\pi/4)\}$, (d) $1\{\cos(-\pi/2)+i\sin(-\pi/2)\}$
6. (a) $r, \theta-\pi$, (b) $r, \theta+\frac{1}{2}\pi$, (c) $r^2, 2\theta$, (d) $1, 2\theta$.
7. (a) -1, (b) $-\frac{\sqrt{3}}{2}+\frac{1}{2}i$, (c) $-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i$, (d) $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$,
   (e) $-i$, (f) $\frac{1}{2}+\frac{\sqrt{3}}{2}i.$
8. (a) $10, \pi/2; 2, \pi/3; 20, 5\pi/6; 5, \pi/6$,
   (b) $4, 5\pi/6; \sqrt{2}, -\pi/4; 4\sqrt{2}, 7\pi/12; 2\sqrt{2}, -11\pi/12$.
9. $\sec\theta, \theta$.
10. $2\sin\frac{1}{2}\theta, \frac{1}{2}(\theta-\pi); 2\sin\frac{1}{2}\theta, \frac{1}{2}(\theta+\pi); \frac{1}{2}\operatorname{cosec}\frac{1}{2}\theta, \frac{1}{2}(\pi-\theta); \frac{1}{2}\operatorname{cosec}\frac{1}{2}\theta, \frac{1}{2}(\theta-\pi)$.
11. $2, \pi/3; 2, \pi/2.$ 12. $\pm\pi/2.$
13. (a) $\frac{1}{13}+\frac{5}{13}i$, 1·37; 2·75, $\frac{1}{13}$, (b) $|z|^2:1$.

Exercise 26.5 (pp. 568–570)

12. $x^2+y^2+4x=0.$ 13. $x^2+y^2-9x-y+16=0.$ 14. $x=2.$
15. $5x+3y=14.$ 16. $2x^2+2y^2+25x+75=0.$
17. $5x^2+5y^2-26x+8y+1=0.$ 18. $x^2+y^2=25$, excluding $(-5,0)$.
19. $x^2+y^2-2x+2y=0$, excluding $(2,0)$. 20. $5x-2y+6=0.$
21. $xy=1.$ 22. $x^2-y^2=1.$ 23. $x(x^2+y^2-1)=0$, excluding $(0,0)$.
24. $y(x^2+y^2-9)=0$, excluding $(0,0)$. 36. (a) 9, (b) 3, (c) 3, (d) 4.
37. $5\sqrt{2}+3, 5\sqrt{2}-3.$ 38. 24, 2. 39. $x^2+y^2+10x+16=0.$
40. (i) $1\pm i\sqrt{5}$, (ii) $1+i$.

Exercise 26.6 (pp. 573–574)

1. (a) 1, (b) $-i$, (c) $\frac{1}{2}-\frac{\sqrt{3}}{2}i$, (d) $-\frac{1}{2}+\frac{\sqrt{3}}{2}i.$

2. $2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$; $512-512\sqrt{3}i$, $-\frac{\sqrt{3}}{256}+\frac{1}{256}i$.

3. $\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$; 256, $-\frac{1}{8}i$. 4. (a) 1, (b) -1.

5. $4\cos^3\theta-3\cos\theta$, $3\sin\theta-4\sin^3\theta$, $\dfrac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$.

6. $16\sin^5\theta-20\sin^3\theta+5\sin\theta$, $1-12\sin^2\theta+16\sin^4\theta$.

8. $32\cos^6\theta-48\cos^4\theta+18\cos^2\theta-1$, $32\cos^5\theta-32\cos^3\theta+6\cos\theta$,
$\dfrac{6\tan\theta-20\tan^3\theta+6\tan^5\theta}{1-15\tan^2\theta+15\tan^4\theta-\tan^6\theta}$.

9. (a) $\frac{1}{16}(\cos 5\theta+5\cos 3\theta+10\cos\theta)$,
(b) $\frac{1}{64}(\cos 7\theta+7\cos 5\theta+21\cos 3\theta+35\cos\theta)$,
(c) $\frac{1}{16}(2\cos\theta-\cos 3\theta-\cos 5\theta)$.

10. (a) $\frac{1}{4}(3\sin\theta-\sin 3\theta)$, (b) $\frac{1}{64}(35\sin\theta-21\sin 3\theta+7\sin 5\theta-\sin 7\theta)$,
(c) $\frac{1}{64}(3\sin\theta+3\sin 3\theta-\sin 5\theta-\sin 7\theta)$.

12. (a) $3\pi/8$, (b) $\pi/32$.

Exercise 26.7 (pp. 574–579)

1. $\dfrac{x}{x^2+y^2}-\dfrac{y}{x^2+y^2}i$. 2. 3, 1; -3, -1. 3. $1+2i$; $5\sqrt{2}$; -2, -1.

4. (i) $\pm(1-3i)$, (ii) $2-2i$, $1+i$.

5. Sum of squares of diagonals of a parallelogram = sum of squares of sides.

6. ± 1, $\pm i$, $\pm\dfrac{1}{\sqrt{2}}(1-i)$, $-\dfrac{1}{\sqrt{2}}(1+i)$.

7. (i) $-1+i\sqrt{3}$, 2, $2\pi/3$; $\dfrac{\sqrt{3}}{2}+\frac{1}{2}i$, 1, $\pi/6$,

(ii) $1\pm 2i$; (a) $p^2=q-4$, (b) $2p=q+5$.

8. (a) $-3\sqrt{3}+5i$; $2\sqrt{13}$, 2·38 rad, (b) 5, -20.

9. (i) $-1-i$, $3\pi/4$, (ii) $2-i$, 2; -10.

10. (b) -1, $\frac{1}{2}\pm\dfrac{\sqrt{3}}{2}i$, (c) $-1-3i$, $\frac{1}{2}(1-3\sqrt{3})+\frac{1}{2}(3+\sqrt{3})i$,

$\frac{1}{2}(1+3\sqrt{3})+\frac{1}{2}(3-\sqrt{3})i$.

11. $-\frac{1}{2}\pm\dfrac{\sqrt{3}}{2}i$; $1\pm i\sqrt{3}$, $\frac{5}{2}\pm\dfrac{\sqrt{3}}{2}i$; 28.

12. (ii) $x^3-3abx+a^3+b^3$; $-\sqrt[3]{2}-\sqrt[3]{4}$, $-\omega\sqrt[3]{2}-\omega^2\sqrt[3]{4}$, $-\omega^2\sqrt[3]{2}-\omega\sqrt[3]{4}$.

13. (i) $\pm(3-2i)$, (ii) $3\pm i\sqrt{3}$. 14. (i) (a) 7/5, $-4/5$; (b) $2\pm i$.

15. (i) $1-i$, $-1\pm 2i$, (ii) $\frac{1}{2}(3+i\sqrt{3})$; $\pi/6$, $2\pi/3$.

16. (i) 2, 32; $\pi/3$, $-\pi/3$; $-16\sqrt{3}$, (ii) $2 + 3i$. 17. (i) -1, (ii) $1 + 2i$, $2\sqrt{5}$.
18. (i) $\sqrt{2}$, $-\pi/4$; $\frac{1}{2} + \frac{1}{2}i$, $8i$. 20. (i) $2 - i$, $3 - 4i$; $(2, -1)$.
21. (ii) $3x^2 + 3y^2 + 10x + 3 = 0$. 22. (i) $x = 1$, $y = 2$ or $x = -1$, $y = -2$.
23. (i) (a) 13, $-23°$, (b) 1, 90°; (ii) 12; (iii) $|z + 1 + i| \geqslant 4$.
24. $2 \sin\theta$; $\frac{1}{2}\operatorname{cosec}\theta$, $-\theta$; $y = -\frac{1}{2}$. 25. $2\sqrt{2} - 2$.
26. $2\cos\frac{1}{2}\theta$, $\frac{1}{2}\theta$; $\left[2\cos\left(\frac{\theta}{2}\right)\right]^n \cos\left(\frac{n\theta}{2}\right) + i\left[2\cos\left(\frac{\theta}{2}\right)\right]^n \sin\left(\frac{n\theta}{2}\right)$.

Index

Abscissa, 20
Acceleration, 100
Accuracy in numerical work, 56
Addition formulae, 141, 580
Addition of functions, 270
Algebraic numbers, 4
Algorithm, 63, 480
Altitude of a triangle, 156
Amplitude (of a complex number), 556
Angle between, line and plane, 165
  lines, 520
  planes, 166, 429, 431
  skew lines, 167
  vectors, 414, 427
Angle bisector theorem, 156
Angles of depression and elevation, 163
Anti-logarithms, 14
Apollonius of Perga, 157
Apollonius' circle, 157, 565
Approximate methods, integration, 461, 582
  solution of equations, 469, 477
Approximations, to check calculator results, 69
  trigonometric expressions, 217
  using binomial expansion, 257, 259, 447
  using derivative, 206
  using Maclaurin's series, 454
Arbitrary constant, 107
Arc, length of circular, 211, 581
Area, between curve and y-axis, 115
  between two curves, 116
  of a sector, 212, 581
  of a triangle, 161, 581
  under a curve, 109
Argand, Jean Robert, 552
Argand diagram, 552, 562
Argument (of a complex number), 556
Arithmetic mean, 246, 314
Arithmetic progression, 244, 439, 580
Arrangements, 231
Associative law, 5
  for vector addition, 179
Astroid, 276
Asymptotes, 97, 348
  to a hyperbola, 537, 583
Average velocity, 85, 124

Base of a logarithm, 12
Base vectors, 413
Bearings, 163
Bijective function, 18
Binary operation, 5
Binomial series, 259, 445, 446, 582
Binomial theorem, 256, 580
  proof of, 325
Bisection process, 480

Cartesian components, 183
Cartesian coordinates, 20
  in three dimensions, 186
Cartesian equation, 20, 21, 274
  of a plane, 433
Cartesian equations of a line, 433
Cartesian product of sets, 2
Centroid of a triangle, 156, 415
Chain rule, 191
Change of origin, 300
Change of variable, in differential equations, 498
  in integration, 391
Circle, equation of, 285, 286
Circles, in coordinate geometry, 525, 529, 535, 583
  in the Argand diagram, 563, 565
Circle theorems, 155

Circular measure, 211
Circumcircle, 156, 159
Codomain, 17, 331
Coefficient, 48
Coefficients, comparing, 346
Combinations, 234
Common chord, 526
Common logarithms, 14
Commutative law, 5
  for vector addition, 178
Complementary event, 238
Complement of a set, 2
Completing the square, 37, 39
Complex numbers, 544
  geometrical representation, 552
Complex roots of equations, 544, 548
Component vectors, 182, 413
Components, Cartesian, 183
  in given directions, 426
Composite functions, 190, 333
  differentiation of, 191
  sketching of, 267
Conic sections, 535
Conjugate, complex, 545, 546, 553
Constant of integration, 107
Continuous function, 334
Contradiction, proof by, 321
Convergent geometric series, 251
Converse, 322
Coordinates, 20
Cosecant, 129
cosech x, 379
cosh x, 379, 582
Cosine, 128
Cosine rule, 160
Cotangent, 129
coth x, 379
Counter-example, 322
Cube roots of unity, 549
Cubic equations, sum and product of roots, 52, 345
Cubic function, 47
Curve sketching, 93, 102, 266
  see also Sketch graphs
Cusp, 530
Cycloid, 276

Decimals, finite and recurring, 4, 451
Decision box, 65
Deduction, 316, 320
Definite integral, 112
  properties of, 404
Degree of a polynomial, 47
De Moivre, Abraham, 571
De Moivre's theorem, 570
Dependent variable, 21
Derivative, 86, 334
  second, 100
Derived function, 86, 335
Descartes, René, 20
Differences, method of, 441
Differentiable function, 335
Differential equations, 491
  $d^2y/dx^2 \pm n^2y =$ constant, 509, 581
  formulation, 506
  separable variables, 501
  solution curves, 491, 493
  use of a substitution, 498
Differentiation, composite functions, 191
  from first principles, 86
  implicit functions, 194, 199
  inverse trigonometric functions, 225
  parametric equations, 276
  powers of x, 89, 91, 192, 195, 257
  products and quotients, 196, 197
  standard formulae, 581
  trigonometric functions, 221
Dihedral angle, 166
Direction ratios, 433
Direction vector, 418
Directrix, 294, 536
Discontinuity, 115, 334
Displacement, as a vector, 176
  using a definite integral, 122
Distance between two points, 23
  vector form, 185
Distance from a point to a line, 522, 583
Distributive law, 5
  for vectors, 179, 180
Divergent geometric series, 251
Division of a line in a given ratio, 415, 521, 582
Domain, 17, 20, 331
Dot product, 424
Double angle formulae, 144, 581
Dummy variable, 112

e, definition of, 364
Eccentricity, 536
Element, of area, 120
  of volume, 121
Ellipse, 530, 535, 583
Empty set, 2
Equating coefficients, 346
Equating real and imaginary parts, 546
Equation of a graph, 20, 21
Equations, *see* Solution of equations
Equivalent statements, 1, 318
Error, types of, 56, 57
  using derivative, 206
Even function, 265, 331, 405
Event, 237

exp x, 364, 373
Expansions, *see* Series
Exponent, 363
Exponential functions, 363, 374
Exponential series, 445, 452, 582

Factor formulae, 148, 581
Factor theorem, 48, 344
Factorial notation, 231
Fermat, Pierre de, 315
Fibonacci series, 68
Finite sequences and series, 243, 439, 580
Flow diagram, 63
Focal chord, 295
Focus, 294, 536
Free vector, 177
Function, 17, 331, 337
  composite, 190, 267
  defined implicitly, 193
  graph of, 20
  inverse, 59
  odd and even, 265
  one–one, 18
  onto, 18
Function of a function, 190
Fundamental theorem of calculus, 121

General solution, of a differential equation, 491
  of a trigonometric equation, 214
Geometric mean, 249, 314
Geometric progression, 247, 439, 580
  sum to infinity, 251
Gradient, of a straight line, 26
  of a curve, 83
Gradient function, 84, 101, 106
Graph of a function, 20
  see also Sketch graphs
Greatest value, 202
Growth and decay, laws of, 364, 506

Half angle formulae, 145, 581
Half-line in the Argand diagram, 563
Hero of Alexandria, 161
Hero's formula, 161
Hyperbola, 276, 535, 537, 583
  rectangular, 299, 348, 529, 538, 583
Hyperbolic function, 379, 582

Identically equal to, 49
Identity, algebraic, 49
  trigonometric, 138
Identity element, 6
Image, 17, 331
Imaginary axis, 552
Imaginary number, 544
Imaginary part, 544, 546
Implication, 318, 320
Implication signs, 1
Implicit functions, 193
  differentiation of, 194, 199
Incircle, 156, 163
Independent variable, 20
Indices, laws of, 9
  in equations, 10, 62
Induction, proof by, 323
Inductive reasoning, 315
Inequalities, 7, 305
  analytical methods, 309
  graphical approach, 307
  linear, 32
  modulus, 8, 312
  quadratic, 38
  simultaneous, 33
Infinite sequences and series, 243
Infinity, 82
Inflexion, point of, 94, 101
Injective function, 18
Inscribed circle, 156, 163
Integers, 3
Integral, definite, 112
  indefinite, 107, 112
  as an area under a curve, 110
  as the limit of a sum, 109, 121
  properties, 116
Integral curves, 491
Integral part of x, 81
Integrand, 223
Integration, as reverse of differentiation, 107
  by parts, 397
  by substitution, 391
  numerical, 461, 582
  of trigonometric functions, 222
  of powers of x, 107
  standard forms, 383, 581
  using inverse trigonometric functions, 401
  using logarithms, 375
  using partial fractions, 387
Intercept, 29
Intersection, *see* Points of intersection
Interval bisection, 480
Inverse element, 6
Inverse function, 59, 340
Inverse trigonometric functions, 134, 401
  differentiation of, 225
Irrational numbers, 4
Iterative processes, 476, 479
  general formula, 480

Jump, in flow diagram, 65

Latus rectum, 295
Length of circular arc, 211, 581
Limit of a sum, 109, 121
Limits, 80
  of trigonometric expressions, 217
  properties of, 82
Limits of integration, 112
Line, *see* Straight line
Linear factor, 50
Linear form, reduction to, 73
Linear function, 47
Linear inequalities, 32
Linear interpolation, 472
Line of greatest slope, 169
$\ln x$, 368, 373
Location of roots, 470
Loci, in the Argand diagram, 562
  using parametric equations, 274, 282, 529
Logarithmic differentiation, 370
Logarithmic functions, 368
Logarithmic series, 445, 453, 582
Logarithms, 12
  change of base, 13, 61
  common, 14
  laws of, 13, 368, 580
  Napierian, 368
  natural, 368
Long division of polynomials, 49
Loop, in flow diagram, 66

Maclaurin, Colin, 444
Maclaurin's series, 444, 582
Maclaurin's theorem, 444
Magnitude of a vector, 175, 183
Major axis, 536
Many-to-one function, 18
Mapping, 17
Maximum point, 38, 94, 101
Maximum value, 37, 94, 201
Mean value, 124
Median of a triangle, 156
Method of differences, 441
Mid-point, coordinates of, 23
  position vector of, 185
Minimum point, 37, 94, 101
Minimum value, 37, 94, 201
Minor axis, 536
Modulus, 8
  functions, 268, 271
  inequalities, 8, 312
  of a complex number, 554, 556
Modulus-argument form, 558

Napier, John, 368
Napierian logarithms, 368
Natural logarithms, 368
Natural numbers, 3
Necessary condition, 319
Negation, 319
Nested multiplication, 69
Newton's law of cooling, 364, 506
Newton–Raphson method, 475, 582
Normal to a curve, 290
Null set, 2
Null vector, 175
Number systems, 3
Numerical methods, 461, 582

Odd function, 265, 331, 405
One–one correspondence, 18
One–one function, 18, 340
Ordered pair, 2
Order of a differential equation, 491
Ordinate, 20
Origin, 22
  change of, 300
Orthocentre, 156
Orthogonal circles, 526

Parabola, 38, 529, 535, 583
  equation of, 294, 298
  focus and directrix, 294
  parametric form, 295
Parallel lines, 26, 30
Parallelogram law, 178
Parameter, 274
Parametric equations, 274, 529
  differentiation of, 277
Partial fractions, 353
  in integration, 387
  in series expansions, 450
  in summation of series, 441
Particular integral, 491
Pascal, Blaise, 255
Pascal's triangle, 255
Pedal triangle, 157
Percentage error, 207
Periodic function, 129, 332
Period of a function, 129, 332
Permutations, 231
Perpendicular lines, 27, 30
Perpendicular vectors, 424, 427
Plane, Cartesian equation, 433
  vector equations, 421, 430
Planes, angle between, 431
Point dividing line in given ratio, 415, 521, 582
Points of intersection, of curve and coordinate axes, 22
  of line and curve, 45
  of two lines, 30

using parametric equations, 276
Polynomial equations, 52, 345, 548
sum and product of roots, 53
repeated roots, 345
Polynomials, 47, 344
factorisation, 49
long division, 49
nested multiplication, 69
Position vector, 185
Possibility space, 237
Principal argument, 556
Principal values, 134
Probability, 237
Probability mapping, 239
Probability space, 239
Product formulae, 147
Product rule (differentiation), 196
Product set, 2
Program, 63
Progression, arithmetic, 244
geometric, 247
Projection, 165
of a vector, 426
Proof, 315, 321
by contradiction, 321
by induction, 323
Proper fraction, 353
Proper subset, 1
Pythagoras, 138

Quadrant, 130
Quadratic approximation, 448, 455, 473
Quadratic equation, 39
nature of roots, 40
sum and product of roots, 42
symmetrical functions of roots, 43
Quadratic function, 36, 47
Quadratic inequalities, 38
Quotient rule (differentiation), 198

Radian, 269
Range of a function, 17, 21, 331, 349
Raphson, Joseph, 476
Rates of change, 204
Ratio theorem, 414
Rational function, 348
Rational numbers, 3
Rationalisation of fractions, 10
Real and imaginary parts, 544, 546
Real axis, 552
Real numbers, 3
Rectangular hyperbola, 299, 348, 529, 538, 583
Recurring decimals, 4, 251
Reduction of laws to linear form, 73
Relation, 21, 336
Remainder theorem, 49, 344
Repeated factor, 345
Resolution of vectors, 182
Resultant, 176
Roots of equations, location of, 470
Newton–Raphson approximation, 475
polynomial equations, 52
quadratic equations, 40, 42
real and complex, 548
sums and products, 345
Rounded value, 57
Rounding error, 57

Sample space, 237
Sample space diagram, 238
Scalar product, 424
Scalar quantity, 175
Secant, 129
sech x, 379
Sector, area of, 212, 581
Selections, 234
Semi-cubical parabola, 276, 530, 583
Semi-perimeter of triangle, 161
Sequence, 242
Series, 243, 580, 582
binomial, 445
exponential, 445, 452
finite, 439
logarithmic, 445, 452
Maclaurin's, 444
$\sin x$ and $\cos x$, 446, 457
Sets, 1
Sigma notation, 120, 252
Simpson, Thomas, 463
Simpson's rule, 463, 582
Simultaneous equations, non-linear, 45
Simultaneous inequalities, 33
Sine, 128
Sine rule, 158, 159
sinh x, 379, 582
Sketch graphs, 95, 102
composite functions, 267
modulus functions, 268, 271
odd and even functions, 266, 331
parametric equations, 274
periodic functions, 332
rational functions, 348
sums of functions, 270
$y = [x]$, 81
$y = \lvert f(x) \rvert$, 337
$y = 1/f(x)$, 268, 337
$y = \{f(x)\}^2$, 338
$y = (ax + b)/(cx + d)$, 272
$y^2 = f(x)$, 339
Skew lines, 167, 419

Small angle approximations, 217
Small changes or increments, 206
Solid of revolution, 118
Solution curves, 321, 323
Solution of equations, graphical methods, 469
  iterative processes, 476, 479
  linear interpolation, 472
  location of roots, 470
  Newton–Raphson method, 473
  quadratic approximation, 473
Solution of triangle, 158
Solution set, 7
Standard form, 14
Stationary point, 94
Stationary value, 94
Straight line, equation of, 28
  gradient of, 25
Straight line (in three dimensions), Cartesian equations, 433
  vector equation, 418, 429
Straight line laws, 71
Straight line motion, 85, 91, 100, 122
Subset, 1
Sufficient condition, 319
Sum and product formulae, 147, 148, 581
Sum and product of roots, polynomial equations, 53, 345
  quadratic equations, 42
Summation of finite series, 439
Surds, 10, 52
Surjective function, 18
Symmetrical functions of roots, 43
Symmetry of graphs, 331

t formulae, 145, 581
Tangent, 129
Tangent rule, 162
Tangent to a curve, equation of, 91, 290
  gradient of, 85
$\tanh x$, 379
Three-dimensional coordinate system, 182, 186
Transcendental numbers, 4
Translation of a curve, 267, 298
Trapezium rule, 462, 582
Triangle, area of, 161, 426, 523, 581
  properties of, 156
  solution of, 158
Triangle inequalities, 178, 554
Triangle law (vector addition), 178
Trigonometric equations, 133, 212
  general solution of, 214
Trigonometric functions, 129
  $a \cos \theta + b \sin \theta$, 150
  differentiation of, 219
  graphs of, 129, 130
  integration of, 222
  inverse functions, 134, 225
Trigonometric identities, 138, 580, 581
  addition formulae, 142
  double and half angle formulae, 144, 145
  sum and product formulae, 147, 148
  t formulae, 145
Trigonometric ratios, 129
  properties, 137
Truncated value, 57
Truncation error, 57
Turning point, 94, 101

Unit vector, 183
Unity, cube roots of, 549
Universal set, 2

Variables, 20
Vector equation of a line, 418
Vector equation of a plane, parametric form, 421
  scalar product form, 430
Vectors, 175
  addition, 178
  angle between, 414, 427
  basic properties, 413
  multiplication by scalar, 179
  position vectors, 185
  scalar product, 424
  subtraction, 179
  unit vectors, 183
Velocity, 85, 91
Velocity–time graph, 122
Volume of revolution, 117

COALVILLE TECHNICAL COLLEGE
LIBRARY

Advanced Mathematics: A Pure Course provides a complete course in pure mathematics for A level courses which include both pure and applied mathematics. It will also be particularly useful for students studying pure mathematics as a subsidiary subject on a wide variety of courses.

Topics are presented in a logical sequence and are arranged in short chapters to allow flexibility. The text contains many worked examples and numerous exercises providing both routine practice and opportunities to develop understanding.

Advanced Mathematics: An Applied Course (ISBN 0 7135 28206) provides the companion course in applied mathematics.

Advanced Mathematics: Book 1 (ISBN 0 7135 1272 5) and *Advanced Mathematics: Book 2* (ISBN 0 7135 1322 5) integrate the pure and applied chapters from these two books to form a *combined* course in pure mathematics, statistics, and mechanics which includes chapters on numerical methods and probability.

ISBN 0-7135-2821-4

9 780713 528213

UNWIN HYMAN

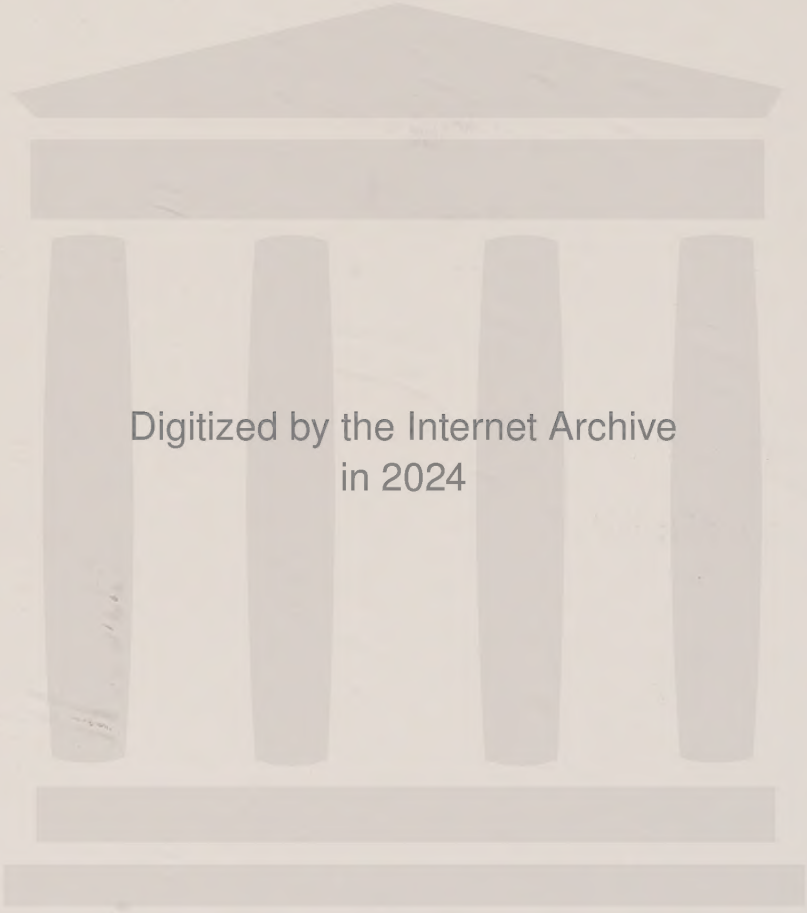
A COMPREHENSIVE COURSE IN APPLIED MATHEMATICS

ADVANCED MATHEMATICS

AN APPLIED COURSE

PERKINS & PERKINS

510



Digitized by the Internet Archive
in 2024

https://archive.org/details/advancedmathemat0000mart

£17.99
4998

ADVANCED MATHEMATICS

AN APPLIED COURSE

Barnfield College Library
T04501

tin and Patricia Perkins

BARNFIELD TECHNOLOGY CENTRE LIBRARY

This book is due for return on or before the last date shown below.

08 DEC 1998
23 FEB 1999
13 APR 1999
-1 SEP 2005
28 APR 2006
29 MAR 2007

7 DAY LOAN

Don Gresswell Ltd., London, N.21 Cat. No. 1207 DG 02242/71

T04501

Published in 1991 by Collins Educational
77–85 Fulham Palace Road, Hammersmith W6 8JB

Reprinted 1993

© Martin and Patricia Perkins 1987
Published in 1987 by Unwin Hyman Ltd

All rights reserved. No part of this publication may be reproduced, stored in a retrieval system, or transmitted in any form or by any means, electronic, mechanical, photocopying, recording or otherwise, without the prior permission.

British Library Cataloguing in Publication Data

Perkins, Martin
Advanced mathematics: an applied course.
1. Mathematics ——— 1961–
I. Title II. Perkins, Patricia
510 QA39.2

ISBN 0 00 322270 5

Printed in Great Britain at the University Press, Cambridge

Preface

This is the second of two books designed to cover most Advanced Level single subject Mathematics syllabuses. This volume provides a thorough treatment of the Mechanics required at this level, together with chapters on probability and elementary statistics. We hope that the inclusion of these additional topics will enable students and teachers to plan a course of study to suit individual syllabus requirements.

We are most grateful for permission to use questions from past G.C.E. Advanced Level examinations. These are acknowledged as follows:

University of London University Entrance and Schools Examination Council (L)
The Associated Examining Board (AEB)
Joint Matriculation Board (JMB)
University of Cambridge Local Examinations Syndicate (C)
Oxford and Cambridge Schools Examination Board (O&C)
Oxford Delegacy of Local Examinations (O)
Southern Universities' Joint Board for School Examinations (SU)
Welsh Joint Education Committee (W)

We must again thank the staff of Messrs. Unwin Hyman, especially Caroline Paines and Antonia Murphy, for both their enthusiasm and their patience. We have also been helped and encouraged by many friends and colleagues.

M.L.P., P.P.

Contents

Preface ii

List of symbols vi

1 Vectors 1
1.1 Introduction to vectors 1
1.2 Cartesian components and position vectors 4

2 Forces and equilibrium 10
2.1 Mathematical models 10
2.2 The action of forces 11
2.3 Resultant forces 14
2.4 Resolution into components 18
2.5 A particle in equilibrium 22

3 Relative motion 31
3.1 Velocity and acceleration as vectors 31
3.2 Relative velocity 35
3.3 Relative displacement 38

4 Velocity and acceleration 46
4.1 Elementary kinematics 46
4.2 Motion with constant acceleration 50
4.3 Projectiles 56
4.4 Differentiation of position vectors 62

5 Newton's laws of motion 68
5.1 Newton's laws 68
5.2 Further applications 72
5.3 Connected particles 75
5.4 Related accelerations 81

LIBRARY

6 Friction; Hooke's law 88
6.1 The laws of friction 88
6.2 Motion on a rough surface 91
6.3 Equilibrium on a rough surface 94
6.4 Angle of friction 96
6.5 Hooke's law 99

7 Work, energy and power 107
7.1 Work and kinetic energy 107
7.2 Conservation of mechanical energy 110
7.3 Work done by a variable force 115
7.4 Power 118

8 Uniform motion in a circle 128
8.1 Angular velocity 128
8.2 Acceleration in circular motion 131
8.3 Further problems 134

9 Coplanar forces; centre of gravity 143
9.1 Parallel forces, moments and couples 143
9.2 Systems of coplanar forces 151
9.3 Centre of gravity; centre of mass 156
9.4 Standard results by integration 164
9.5 Centroids of areas and volumes 172

10 Equilibrium of rigid bodies 180
10.1 Parallel forces in equilibrium 180
10.2 Three force problems 188
10.3 General conditions for equilibrium 197
10.4 Rigid body systems 205
10.5 Light frameworks 211

11 Vector methods 223
11.1 Forces and their lines of action 223
11.2 Use of vectors in kinematics and dynamics 228
11.3 Scalar product and its applications 233

12 Impulse and momentum 243
12.1 The impulse of a force 243
12.2 Conservation of momentum 250
12.3 Work, energy and momentum 253
12.4 Newton's law of restitution 260

13 Further particle dynamics 273
13.1 Acceleration as a function of displacement 273
13.2 Acceleration as a function of velocity 276
13.3 Simple harmonic motion 282

13.4 Forces producing simple harmonic motion 287
13.5 Motion in a vertical circle 295

14 Probability 308
14.1 Permutations and combinations 308
14.2 The language of sets 311
14.3 Elementary theory of probability 317
14.4 Sum and product laws 323
14.5 Conditional probability 328

15 Elementary statistics 337
15.1 Graphical representation of data 337
15.2 Mode, median and mean 345
15.3 Measures of spread 350
15.4 Change of variable 354

16 Probability distributions 362
16.1 Discrete probability distributions 362
16.2 The binomial distribution 369
16.3 Continuous probability distributions 377
16.4 The normal distribution 384
16.5 The Poisson distribution 392
16.6 The algebra of expectation and variance 397

Formulae for reference 415

Statistical tables 418

Answers 420

Index 452

List of symbols

s.f.	significant figures
d.p.	decimal places
$\approx$	is approximately equal to
$<$	is less than
$\leqslant$	is less than or equal to
$>$	is greater than
$\geqslant$	is greater than or equal to
$\lvert x \rvert$	the modulus of x
$[x]$	the integral part of x
e^x, $\exp x$	exponential function of x
$\log_a x$	logarithm to the base a of x
$\ln x$	the natural logarithm of x, $\log_e x$
$\lg x$	the common logarithm of x, $\log_{10} x$
$\sin^{-1}$	inverse function of sin with range $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$
$\cos^{-1}$	inverse function of cos with range $[0, \pi]$
$\tan^{-1}$	inverse function of tan with range $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$
$\lim_{x \to a} f(x)$	limit of $f(x)$ as x tends to a
∞	infinity
δ	small increment
$f'(x)$ or $\frac{dy}{dx}$	first derivative of $y = f(x)$
$f''(x)$ or $\frac{d^2y}{dx^2}$	second derivative of $y = f(x)$
$\dot{x}, \dot{y}, \dot{\theta}, \ldots$	first derivative of $x, y, \theta, \ldots$ with respect to time
$\ddot{x}, \ddot{y}, \ddot{\theta}, \ldots$	second derivative of $x, y, \theta, \ldots$ with respect to time
$\mathbf{PQ}$	vector represented by line segment $\vec{PQ}$
$\lvert\mathbf{PQ}\rvert$, PQ	magnitude of vector $\mathbf{PQ}$
$\lvert\mathbf{a}\rvert$, a	magnitude of vector $\mathbf{a}$

$\hat{\mathbf{a}}$	unit vector in the direction of $\mathbf{a}$
$\mathbf{i}, \mathbf{j}, \mathbf{k}$	unit vectors in the directions of the x-, y- and z-axes
$\mathbf{a} . \mathbf{b}$	scalar product of $\mathbf{a}$ and $\mathbf{b}$
g	acceleration due to gravity (taken as $9{\cdot}8\ \text{m s}^{-2}$ unless otherwise stated)
μ	coefficient of friction
λ	angle of friction or modulus of elasticity
ω	angular velocity
$\varnothing$	empty set
$\mathscr{E}$	universal set
A'	complement of set A
$n(A)$	number of elements in the set A
$\cup$	union
$\cap$	intersection
$\in$	is an element of
$\mathbb{Z}$	set of integers
$\mathbb{Z}^+$	set of positive integers
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	set of real numbers
$n!$	n factorial
${}_nP_r$	number of permutations of r objects chosen from n
${}_nC_r$	number of combinations of r objects chosen from n
$P(A)$	probability of event A
$P(A')$	probability that A does not occur
$P(A \cup B)$	probability of event A or event B (or both)
$P(A \cap B)$	probability of event A and event B
$P(A\|B)$	probability of event A given B
$\sum_{r=1}^{n} f(r)$	$f(1)+f(2)+\ldots+f(n)$
$\binom{n}{r}$	the binomial coefficient: ${}_nC_r = \dfrac{n!}{r!(n-r)!}$ for $n \in \mathbb{Z}^+$, $\dfrac{n(n-1)\ldots(n-r+1)}{r!}$ for $n \in \mathbb{Q}$
$x_1, x_2, \ldots$	observations
$f_1, f_2, \ldots$	frequencies with which the observations $x_1, x_2, \ldots$ occur
$\bar{x}$	mean of observations $x_1, x_2, \ldots$
var (x)	variance of observations $x_1, x_2, \ldots$
s.d.	standard deviation
$X, Y, R, \ldots$	random variables
$x, y, r, \ldots$	values of the random variables $X, Y, R, \ldots$

$p(x)$	value of the probability function $P(X = x)$ of the discrete random variable X
$p_1, p_2, \ldots$	probabilities of the values $x_1, x_2, \ldots$ of the discrete random variable X
$f(x), g(x), \ldots$	value of the probability density function of the continuous random variable X
$F(x), G(x), \ldots$	value of the (cumulative) distribution function $P(X \leqslant x)$ of the continuous random variable X
$E(X), \mu$	expectation (or mean) of the random variable X
$\mathrm{Var}(X), \sigma^2$	variance of the random variable X
$G(t)$	probability generating function for a random variable which takes the values $0, 1, 2, \ldots$
$B(n, p)$	binomial distribution, parameters n and p
$N(\mu, \sigma^2)$	normal distribution, mean μ and variance σ^2
Z	the standardised normal variable with distribution $N(0, 1)$
ϕ	probability density function for $N(0, 1)$
Φ	cumulative distribution function for $N(0, 1)$

The use of the symbols $=$ and $\approx$

In numerical work it is important to distinguish between exact values and approximate values. When giving exact answers we use the symbol $=$. For approximate results we use either the symbol $=$ (together with an appropriate phrase, such as 'by calculator', 'to 3 s.f.'), or the symbol $\approx$. The use of the symbol $\approx$ may show that a result has been obtained by an approximate method, but often indicates that a more accurate result obtained by calculator has been rounded to the given number of significant figures (or, in the case of angles, to the nearest tenth of a degree).

1 Vectors

1.1 Introduction to vectors

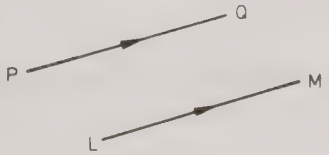
Some physical quantities, such as temperature, may be completely specified by a number referred to some unit of measurement, e.g. 25°C, 350°F. Such quantities, which have magnitude but which are not related to any definite direction in space, are called *scalar quantities*. Similarly, a number representing the magnitude of some physical quantity is called a *scalar*.

When measuring other quantities, such as wind velocity, it is necessary to give a direction as well as a number and a unit of measurement, e.g. 40 km/h from the north-east. These quantities, which have both magnitude and a definite direction in space, are called *vector quantities*. In general, a *vector* may be described as a number associated with a particular direction in space.

Any vector may be represented by a *directed line segment*, whose direction is that of the vector and whose length represents the magnitude of the vector.

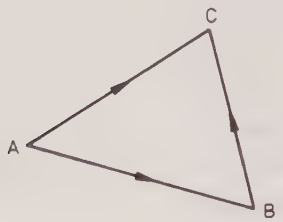
The vector represented here is denoted by **PQ** in bold type (or $\underset{\sim}{PQ}$ in manuscript). The *magnitude* or modulus of the vector is written $|\mathbf{PQ}|$ or PQ. The directed line segment joining P to Q is denoted by $\vec{PQ}$. However, since it is not usually necessary to distinguish between the line segment and the vector it represents, the notation $\vec{PQ}$ is in common use for both vector and line segment.

The magnitude of a vector is usually positive but may be zero, in which case the vector is called the *zero* or *null vector* and written **0** in bold type. The zero vector is the only vector with indeterminate direction.



Two vectors are *equal* if they have the same magnitude and direction. Since PQ and LM are parallel and equal in length, $\mathbf{PQ} = \mathbf{LM}$.

A *displacement* is one of the simplest examples of a vector quantity. For instance, a car journey from a town A to a town B may be represented by a displacement vector **AB**. Its magnitude is the distance between A and B. Its direction is that of the straight line joining A to B.



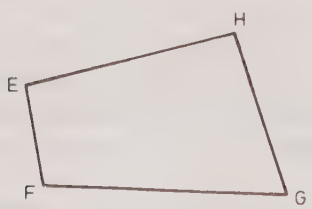
Let us suppose that a man drives from A to B and then from B to C. On another occasion he drives directly from A to C. Since the result of these two journeys is the same, we may write $\mathbf{AB}+\mathbf{BC}=\mathbf{AC}$.

The displacement **AC** is the *sum* or *resultant* of the displacements **AB** and **BC**.

Consider now a journey from A to B, followed by the return journey from B to A. Since the result of these two journeys is a zero displacement, it is reasonable to write, $\mathbf{AB}+\mathbf{BA}=\mathbf{0}$ and $\mathbf{BA}=-\mathbf{AB}$.

All vectors represented by directed line segments may be manipulated in the same way as displacement vectors. For instance, given any three vectors **AB**, **BC** and **AC** represented by the sides of triangle ABC, we may write $\mathbf{AB}+\mathbf{BC}=\mathbf{AC}$. A vector which has the same magnitude as **AB**, but the opposite direction, is denoted by **BA** or $-\mathbf{AB}$.

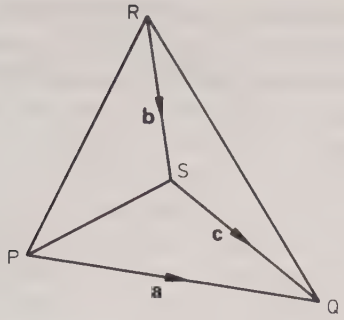
Example 1 Simplify $\mathbf{EF}+\mathbf{FG}-\mathbf{HG}$.



$$\begin{aligned}\mathbf{EF}+\mathbf{FG}-\mathbf{HG} &= \mathbf{EG}-\mathbf{HG}\\ &= \mathbf{EG}+\mathbf{GH}\\ &= \mathbf{EH}.\end{aligned}$$

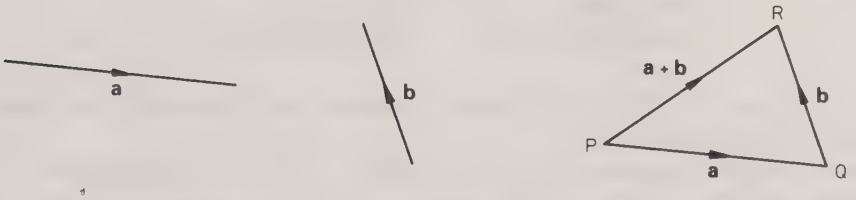
It is often convenient to denote a vector by a single letter such as **a** in bold type (or $\underset{\sim}{a}$ in manuscript). Its magnitude is then written $|\mathbf{a}|$ or a.

Example 2 In the given diagram $\mathbf{PQ}=\mathbf{a}$, $\mathbf{RS}=\mathbf{b}$ and $\mathbf{SQ}=\mathbf{c}$. Find expressions for **RQ**, **PS** and **PR** in terms of **a**, **b** and **c**.



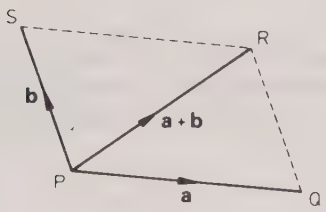
In $\triangle QRS$, $\mathbf{RQ} = \mathbf{RS} + \mathbf{SQ} = \mathbf{b} + \mathbf{c}$
In $\triangle PQS$, $\mathbf{PS} = \mathbf{PQ} + \mathbf{QS} = \mathbf{PQ} - \mathbf{SQ}$
$= \mathbf{a} - \mathbf{c}$
In $\triangle PQR$, $\mathbf{PR} = \mathbf{PQ} + \mathbf{QR} = \mathbf{PQ} - \mathbf{RQ}$
$= \mathbf{a} - (\mathbf{b} + \mathbf{c})$
$= \mathbf{a} - \mathbf{b} - \mathbf{c}$.

More formally, *vector addition* is defined by the *triangle law*, as follows:



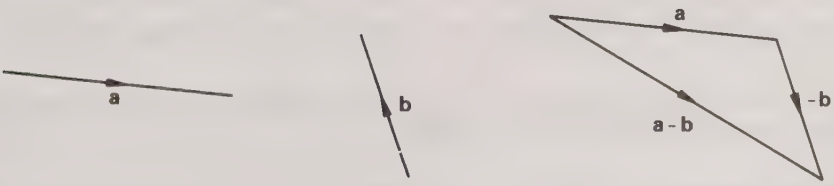
If two vectors **a** and **b** are represented by the sides $\vec{PQ}$ and $\vec{QR}$ of a triangle, then $\mathbf{a} + \mathbf{b}$ is represented by the third side $\vec{PR}$.

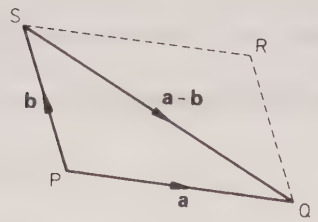
In physics and mechanics it is sometimes more convenient to use the *parallelogram law* for vector addition, which states:



If two vectors **a** and **b** are represented by the sides $\vec{PQ}$ and $\vec{PS}$ of a parallelogram, then $\mathbf{a} + \mathbf{b}$ is represented by the diagonal $\vec{PR}$. [Since the line segments $\vec{PS}$ and $\vec{QR}$ are equal in magnitude and direction, the two addition laws are equivalent.]

The *subtraction* of **b** from **a** may be defined as the addition of $-\mathbf{b}$ to **a**, i.e. $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$.





If the vectors **a** and **b** are represented by the sides $\vec{PQ}$ and $\vec{PS}$ of a parallelogram, then **a** − **b** is represented by the diagonal $\vec{SQ}$.

Exercise 1.1

1. Simplify (a) **PQ** + **QR** + **RS**, (b) **AB** + **DE** + **CD** + **BC**,
(c) **AC** − **BC**, (d) **XY** + **YX**,
(e) **MN** − **RP** − **PN**, (f) **QT** − **QR** + **PR** − **ST**.

2. Given that *ABCDEF* is a regular hexagon, decide whether the following statements are true or false.
(a) **BC** = **EF**, (b) **BF** = **CE**, (c) |**BE**| = |**CF**|, (d) |**AB**| = |**DE**|,
(e) **AC** and **DF** are in the same direction, (f) **AD** is parallel to **EF**.

3. Given that *ABCDPQRS* is a cuboid lettered so that *AP*, *BQ*, *CR* and *DS* are edges, decide whether the following statements are true or false.
(a) **PQ** = **DC**, (b) **BR** = **PD**, (c) |**RA**| = |**BS**|, (d) |**PA**| = |**CR**|,
(e) **DQ** and **AR** are in the same direction, (f) **AQ** is parallel to **CS**.

If **a**, **b** and **c** are displacements of 1 km due east, 2 km due north and 1 km in the direction S 30° W respectively, find the magnitudes and directions of the displacements given in questions 4 to 6. (You may leave your answers in surd form.)

4. (a) 3**a**, (b) 5**b**, (c) −**c**, (d) $-\frac{1}{2}$**b**.

5. (a) 2**a** + **b**, (b) **b** + 2**a**, (c) 2**a** − **b**, (d) **b** − 2**a**.

6. (a) **a** + **c**, (b) **a** + 2**c**, (c) 2**a** + **c**, (d) **a** − **c**.

7. The diagonals of a parallelogram *ABCD* intersect at *M*. If **AB** = **p** and **AD** = **q**, express in terms of **p** and **q** the vectors **AC**, **BD**, **AM** and **MD**.

8. In a regular hexagon *ABCDEF* **AB** = **a** and **BC** = **b**. Find expressions in terms of **a** and **b** for **DE**, **DC**, **AD** and **BD**.

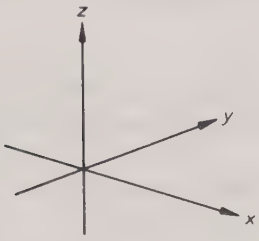
9. In the pentagon *PQRST* **PQ** = **a** and **RS** = **b**. If **PT** = 2**RS** and **TS** = 2**PQ**, find expressions for **PR** and **QR** in terms of **a** and **b**.

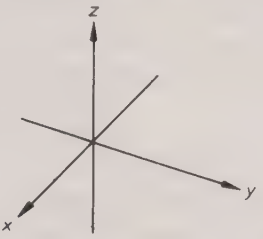
1.2 Cartesian components and position vectors

Resolving a vector into *component vectors* means expressing it as a sum of two or more non-parallel vectors.

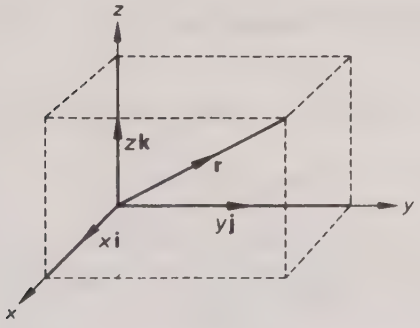
It is often convenient to resolve vectors into component vectors in the directions of the Cartesian coordinate axes. In three-dimensional work the third axis is the z-axis, which is constructed perpendicular to the x- and y-axes to form a *right-handed system*. This means that if we point the thumb of the right hand in the direction of the z-axis, then bend the fingers slightly, they will indicate the direction of a rotation from the x-axis to the y-axis.

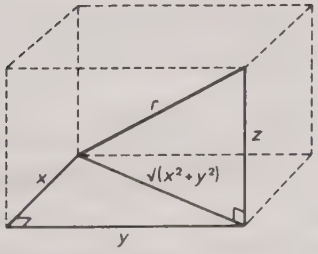
Two standard ways of drawing these axes are shown below.





Any vector with magnitude 1 is called a *unit vector*. The unit vectors in the directions of the x-, y- and z-axes are denoted by **i**, **j** and **k** respectively.





By constructing a cuboid with one diagonal representing **r** and sides parallel to the coordinate axes, we can demonstrate that there is a unique expression for any vector **r** of the form $x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$. The scalars x, y and z are called the *components* of **r** in the directions of **i**, **j** and **k**, or simply the *Cartesian components* of **r**.

Since **i**, **j** and **k** are unit vectors, x, y and z are the lengths of the sides of the cuboid constructed here. Applying Pythagoras' theorem in the two right-angled triangles illustrated, we find that the magnitude of **r** is given by

$$r = \sqrt{(x^2+y^2+z^2)}.$$

Example 1 If $\mathbf{a} = 3\mathbf{i}+2\mathbf{j}+7\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i}+4\mathbf{j}-5\mathbf{k}$, find $|\mathbf{a}|$ and $|\mathbf{a}-\mathbf{b}|$.

$|\mathbf{a}| = \sqrt{(3^2+2^2+7^2)} = \sqrt{(9+4+49)} = \sqrt{62}$

$\mathbf{a}-\mathbf{b} = (3\mathbf{i}+2\mathbf{j}+7\mathbf{k})-(2\mathbf{i}+4\mathbf{j}-5\mathbf{k}) = \mathbf{i}-2\mathbf{j}+12\mathbf{k}$

$\therefore \quad |\mathbf{a}-\mathbf{b}| = \sqrt{(1^2+\{-2\}^2+12^2)} = \sqrt{(1+4+144)} = \sqrt{149}.$

Since a represents the magnitude of a vector $\mathbf{a}$, the vector $\mathbf{a}/a$ has magnitude 1 and the same direction as $\mathbf{a}$. Thus the unit vector in the direction of a given vector $\mathbf{a}$ (sometimes denoted by $\hat{\mathbf{a}}$) may be expressed in the form $\mathbf{a}/a$.

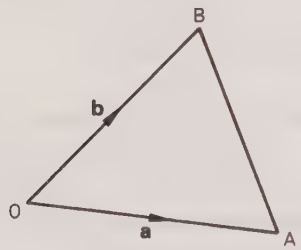
Example 2 Find the unit vector in the direction of $\mathbf{a} = 2\mathbf{i}+2\mathbf{j}-\mathbf{k}$.

$a = \sqrt{(2^2+2^2+\{-1\}^2)} = \sqrt{(4+4+1)} = \sqrt{9} = 3$

$\therefore$ the unit vector in the direction of $\mathbf{a}$

$$= \tfrac{1}{3}(2\mathbf{i}+2\mathbf{j}-\mathbf{k}) = \tfrac{2}{3}\mathbf{i}+\tfrac{2}{3}\mathbf{j}-\tfrac{1}{3}\mathbf{k}.$$

Taking a fixed point O as origin, the position of any point P can be specified by giving the vector $\mathbf{OP}$, which is then called the *position vector* of P.



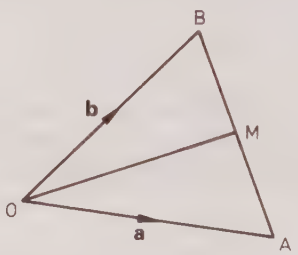
If A and B are two points with position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, then

$$\mathbf{AB} = \mathbf{AO}+\mathbf{OB} = -\mathbf{OA}+\mathbf{OB} = \mathbf{OB}-\mathbf{OA}$$
$$\therefore \quad \mathbf{AB} = \mathbf{b}-\mathbf{a}.$$

Hence the distance between the points A and B is given by $|\mathbf{b}-\mathbf{a}|$.

Consider now the mid-point M of AB.

$$\begin{aligned} & \mathbf{AM} = \mathbf{MB} \\ \Rightarrow\ & \mathbf{OM}-\mathbf{OA} = \mathbf{OB}-\mathbf{OM} \\ \Rightarrow\ & 2\mathbf{OM} = \mathbf{OA}+\mathbf{OB} \\ \Rightarrow\ & \mathbf{OM} = \tfrac{1}{2}(\mathbf{OA}+\mathbf{OB}) \end{aligned}$$



$\therefore$ the position vector of the mid-point of AB is $\frac{1}{2}(\mathbf{a}+\mathbf{b})$.

When working in a Cartesian coordinate system, the point P with coordinates (x, y, z) has position vector $x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors in the directions of the coordinate axes.

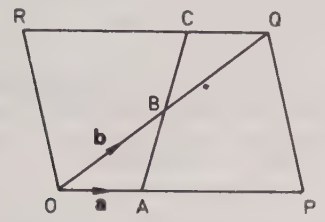
Example 3 The position vectors of points A and B with respect to an origin O are $\mathbf{a} = \mathbf{i}-2\mathbf{j}+7\mathbf{k}$ and $\mathbf{b} = 5\mathbf{i}+\mathbf{j}-5\mathbf{k}$. Find (i) the distance between the origin and the point A, (ii) the distance between the points A and B, (iii) the position vector of the mid-point of AB.

(i) the distance between the origin and the point A
$= |\mathbf{a}| = \sqrt{(1^2+\{-2\}^2+7^2)} = \sqrt{(1+4+49)} = \sqrt{54}.$

(ii) $\mathbf{AB} = \mathbf{b}-\mathbf{a} = (5\mathbf{i}+\mathbf{j}-5\mathbf{k})-(\mathbf{i}-2\mathbf{j}+7\mathbf{k}) = 4\mathbf{i}+3\mathbf{j}-12\mathbf{k}$
$\therefore$ the distance between the points A and B
$= |\mathbf{b}-\mathbf{a}| = |4\mathbf{i}+3\mathbf{j}-12\mathbf{k}| = \sqrt{(16+9+144)} = 13.$

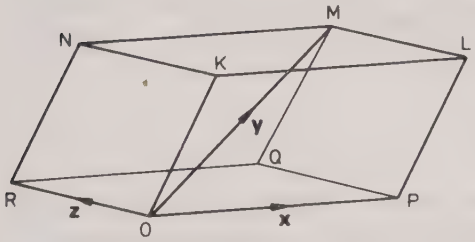
(iii) the position vector of the mid-point of AB
$= \frac{1}{2}(\mathbf{a}+\mathbf{b}) = \frac{1}{2}(1+5)\mathbf{i}+\frac{1}{2}(-2+1)\mathbf{j}+\frac{1}{2}(7-5)\mathbf{k} = 3\mathbf{i}-\frac{1}{2}\mathbf{j}+\mathbf{k}.$

Exercise 1.2



1. In the given diagram $\mathbf{OA} = \mathbf{a}$ and $\mathbf{OB} = \mathbf{b}$. The points P, Q and R are constructed so that $\mathbf{AP} = 2\mathbf{OA}$, $\mathbf{BQ} = \mathbf{OB}$ and $OPQR$ is a parallelogram. Express the following vectors as sums of component vectors in the directions of $\mathbf{a}$ and $\mathbf{b}$

(a) **OQ**, (b) **QR**, (c) **PR**, (d) **AB**, (e) **CQ**, (f) **OC**.



In the parallelepiped shown in the diagram $\mathbf{OP} = \mathbf{x}$, $\mathbf{OM} = \mathbf{y}$ and $\mathbf{OR} = \mathbf{z}$. In questions 2 and 3 resolve the given vectors into component vectors in the directions of of x, y and z.

2. (a) **PQ**. (b) **RQ**, (c) **MN**, (d) **LM**, (e) **PR**, (f) **RM**.

3. (a) **ON**, (b) **KM**, (c) **OK**, (d) **NP**, (e) **KQ**, (f) **LR**.

4. In quadrilateral $ABCD$, $\mathbf{AB} = 2\mathbf{i}-\mathbf{j}$, $\mathbf{BC} = 3\mathbf{i}+4\mathbf{j}$, $\mathbf{AD} = \mathbf{i}+5\mathbf{j}$ and M is the mid-point of CD. Express in terms of $\mathbf{i}$ and $\mathbf{j}$ the vectors
(a) **AC**, (b) **BD**, (c) **CD**, (d) **DM**, (e) **AM**, (f) **BM**.

5. In tetrahedron $PQRS$, $\mathbf{PQ} = 3\mathbf{i}-\mathbf{j}-\mathbf{k}$, $\mathbf{PR} = -\mathbf{i}+4\mathbf{j}+\mathbf{k}$ and $\mathbf{SQ} = 3\mathbf{i}-4\mathbf{j}-3\mathbf{k}$. The point T lies on RS produced such that $RS = ST$. Express in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ the vectors
(a) **QR**, (b) **PS**, (c) **RS**, (d) **PT**.

6. Find the magnitude of each of the following vectors
(a) $5\mathbf{i}-12\mathbf{j}$, (b) $-\mathbf{i}+2\mathbf{k}$, (c) $8\mathbf{j}+6\mathbf{k}$,
(d) $2\mathbf{i}+\mathbf{j}-2\mathbf{k}$, (e) $2\mathbf{i}-6\mathbf{j}+3\mathbf{k}$, (f) $-\mathbf{i}+7\mathbf{j}-5\mathbf{k}$.

7. Find unit vectors in the direction of
(a) $8\mathbf{i}-4\mathbf{j}-\mathbf{k}$, (b) $16\mathbf{i}-8\mathbf{j}-2\mathbf{k}$, (c) $-8\mathbf{i}+4\mathbf{j}+\mathbf{k}$.

In questions 8 to 10 take $\mathbf{a} = \mathbf{i}-2\mathbf{j}+\mathbf{k}$, $\mathbf{b} = 3\mathbf{i}-\mathbf{j}-\mathbf{k}$, $\mathbf{c} = \mathbf{i}+3\mathbf{j}-2\mathbf{k}$.

8. Find (a) $|\mathbf{a}+\mathbf{b}|$, (b) $|\mathbf{a}-\mathbf{b}|$, (c) $|2\mathbf{b}-2\mathbf{a}|$.

9. Find (a) $|\mathbf{b}+\mathbf{c}|$, (b) $|\mathbf{a}+\mathbf{b}+\mathbf{c}|$, (c) $|5\mathbf{a}-\mathbf{b}+3\mathbf{c}|$.

10. Find unit vectors in the directions of
(a) $2\mathbf{b}-\mathbf{c}$, (b) $5\mathbf{a}-5\mathbf{b}+3\mathbf{c}$, (c) $5\mathbf{a}-\mathbf{b}+3\mathbf{c}$.

11. In a regular hexagon $OABCDE$ the position vectors of A and B relative to O are $\mathbf{a}$ and $\mathbf{b}$ respectively. Find expressions in terms of $\mathbf{a}$ and $\mathbf{b}$ for the vectors $\mathbf{AB}$ and $\mathbf{BC}$. Find also the position vectors of the points C, D and E.

12. In a quadrilateral $ABCD$ the point M is the mid-point of the diagonal BD. Given that the position vectors of A, B and C are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively and that $ABCM$ is a parallelogram, find in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ the vectors $\mathbf{BA}$, $\mathbf{BC}$ and $\mathbf{BM}$. Hence find the position vector of the point D.

13. The points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. In each of the following cases, find the distance between A and B, the unit vector in the direction of $\mathbf{AB}$ and the position vector of the mid-point M of AB.
(a) $\mathbf{a} = 3\mathbf{i}-5\mathbf{j}-\mathbf{k}$, $\mathbf{b} = \mathbf{i}-\mathbf{j}+3\mathbf{k}$,
(b) $\mathbf{a} = -7\mathbf{i}+13\mathbf{j}$, $\mathbf{b} = 5\mathbf{i}-5\mathbf{j}-4\mathbf{k}$,
(c) $\mathbf{a} = \mathbf{i}-\mathbf{j}+3\mathbf{k}$, $\mathbf{b} = \mathbf{i}+2\mathbf{j}+2\mathbf{k}$,
(d) $\mathbf{a} = 5\mathbf{i}+4\mathbf{j}+\mathbf{k}$, $\mathbf{b} = -\mathbf{i}+\mathbf{j}-2\mathbf{k}$.

14. The points A, B and C have position vectors $\mathbf{a} = -\mathbf{i}+2\mathbf{j}+3\mathbf{k}$, $\mathbf{b} = 8\mathbf{i}+7\mathbf{j}-9\mathbf{k}$ and $\mathbf{c} = 2\mathbf{i}-3\mathbf{j}-\mathbf{k}$. Prove that the triangle ABC is right-angled and find its area.

Exercise 1.3 (miscellaneous)

1. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are vectors representing the edges of a parallelepiped, find vectors representing the four diagonals.

2. In a tetrahedron $OABC$ $\mathbf{OA} = \mathbf{a}$, $\mathbf{OB} = \mathbf{b}$ and $\mathbf{OC} = \mathbf{c}$. The points P and Q are constructed such that $\mathbf{OA} = \mathbf{AP}$ and $2\mathbf{OB} = \mathbf{BQ}$. The point M is the mid-point of PQ. Find, in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, expressions for (a) $\mathbf{AB}$, (b) $\mathbf{PQ}$, (c) $\mathbf{CQ}$, (d) $\mathbf{QM}$, (e) $\mathbf{MB}$, (f) $\mathbf{OM}$.

3. If $\mathbf{a} = 2\mathbf{i}-4\mathbf{j}+2\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i}+4\mathbf{j}-5\mathbf{k}$, find (a) the magnitudes of $\mathbf{a}$, $\mathbf{b}$ and $5\mathbf{a}+2\mathbf{b}$, (b) the unit vector in the direction of $\mathbf{a}+2\mathbf{b}$.

4. If P and Q are the points with position vectors

$$\mathbf{p} = -\mathbf{i}+3\mathbf{j}+2\mathbf{k} \text{ and } \mathbf{q} = \mathbf{i}-7\mathbf{j}+4\mathbf{k}$$

respectively, find (a) the distance PQ, (b) the unit vector in the direction of PQ, (c) the position vector of the mid-point of PQ.

5. Given that $\mathbf{a} = 6\mathbf{i}+(p-10)\mathbf{j}+(3p-5)\mathbf{k}$ and that $|\mathbf{a}| = 11$, find the possible values of p.

6. Given that the points A, B and C have position vectors $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 3\mathbf{i} - \mathbf{k}$ and $\mathbf{c} = -\mathbf{i} + \mathbf{j} + 4\mathbf{k}$, prove that the triangle ABC is isosceles.

7. In the parallelogram $OPQR$, the position vectors of P and R with respect to O are $6\mathbf{i} + \mathbf{j}$ and $3\mathbf{i} + 4\mathbf{j}$ respectively. M and N are the mid-points of the sides PQ and QR respectively. Find (a) the unit vector in the direction of $\overrightarrow{MN}$, (b) the magnitude of the vector $\overrightarrow{OM}$. (L)

8. In triangle OAB, $\mathbf{OA} = \mathbf{a}$ and $\mathbf{OB} = \mathbf{b}$. Points X, Y and Z are constructed so that $2\mathbf{AX} = 3\mathbf{AB}$, $\mathbf{OB} = \mathbf{BY}$ and Z is the point of intersection of OX and AY. If $\mathbf{AZ} = k\mathbf{AY}$, express $\mathbf{OZ}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and k. Show that $\mathbf{OZ}$ is a scalar multiple of $3\mathbf{b} - \mathbf{a}$. Hence find the value of k and the position vector of Z.

9. In a parallelogram $OABC$, M is the mid-point of AB and P is the point of intersection of OM and AC. If the position vectors of A and B with respect to O as origin are $\mathbf{a}$ and $\mathbf{b}$, find the position vector of P.

10. The points $A(3, -2, 0)$, $B(-2, 5, -4)$ and $C(-1, 0, 4)$ are three vertices of a rhombus. Use vector methods to find the fourth vertex and the area of the rhombus.

2 Forces and equilibrium

2.1 Mathematical models

When applying mathematics to a practical problem, we try to set up what is called a mathematical model. In many cases this will take the form of a set of equations. However, it may also include diagrams, graphs or flow charts. As real situations are often very complicated, a mathematical model is usually a simplification of the original problem. Many scientific laws used in setting up systems of equations represent no more than good approximations to the true relationships between the quantities involved. Some factors governing a situation may be neglected altogether if their effect is likely to be small.

To set up a mathematical model we must first decide what factors are to be taken into account and what mathematical relationships are to be assumed, bearing in mind the degree of accuracy required in the result. We then find ways of giving numerical values to the variables involved, choosing suitable units of measurement.

To illustrate the process we now consider the problem of estimating the amount of petrol which will be consumed during a car journey. For a rough estimate we can use the distance, d miles, to be travelled and an approximate rate of petrol consumption for the car, c miles per gallon, based on past experience. We then assume that the amount of petrol used, P gallons, is proportional to the distance travelled and write $P = d \div c$. Clearly this single equation represents one of the simplest possible models which can be constructed to solve this problem. Many factors have been ignored, such as the load the car will be carrying, the type of road to be used, the traffic and weather conditions.

In elementary mechanics we set up simple models of practical situations in order to study the action of forces on various types of body. In early work we simplify the bodies involved as much as possible and take into account only the most important forces on them. For example, walls, floors and table tops are regarded as flat surfaces. A light rope or cable is assumed to be straight when pulled taut, whereas in practice, unless it is vertical, it hangs in a slight curve. We now list some of the other conventions observed when solving problems in mechanics.

A *particle* is a body of dimensions so small that its position in space can be represented by a single point.

A *lamina* is a flat object whose thickness is negligible but whose area is not.

A *rigid body* is one in which any pair of particles of matter remain a fixed distance apart. This means that rigid bodies are assumed to retain their shape when touching or colliding. For instance, when a sphere rests on a plane surface the area of contact is regarded as a single point.

A *light* string is taken to be weightless. If a body is suspended from one or more light strings, these can be represented by straight lines. Unless otherwise stated strings are assumed *inextensible*, i.e. of constant length.

A *smooth* surface is one which offers negligible frictional resistance to the motion of a body sliding across it. Wheels, pulleys, pivots and joints can also be described as smooth or frictionless.

For most purposes the surface of the earth is assumed to be flat and horizontal. Effects due to air currents and air resistance are usually ignored.

Exercise 2.1

Discuss the setting up of mathematical models in relation to various practical situations. Some problems for consideration are suggested below.

1. To decide which is the quickest way to travel from home to work or college in the rush hour.

2. To estimate the weekly cost of running a car.

3. To budget for the household bills expected in the next three months.

4. To plan orders of crisps, fruit, etc., for a college snack bar.

5. To organise a school time-table.

6. To plan a revision course for candidates taking a mathematics examination.

7. To decide whether buying a domestic appliance such as a freezer or dish-washer will save a family (a) time, (b) money.

8. To evaluate possible ways of saving energy at home or school, such as installing double glazing or switching off unnecessary lights.

2.2 The action of forces

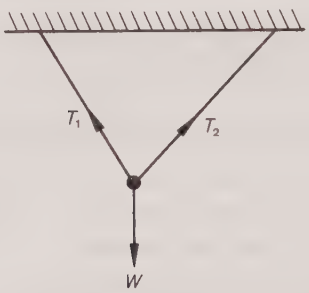
The effect of a *force* is to set a body in motion or to change the speed and direction of its motion. If the forces on a stationary body cancel each other out, then the body is said to be in *equilibrium*. The two main branches of mechanics

are *dynamics*, which is the study of the action of forces on bodies in motion, and *statics*, which is the study of forces in equilibrium. Closely related to dynamics is *kinematics*, which is the study of motion without reference to the forces involved.

A force may be completely specified by stating its magnitude, its direction and either its point of application or its line of action. Thus a force is a vector quantity associated with a particular point or line in space.

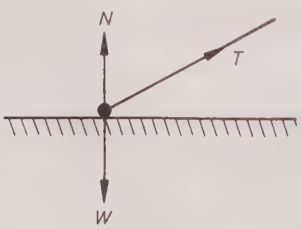
A force which is important in most problems is the gravitational attraction between the earth and any object close to it. This is the force which causes a body to fall to the ground and is called the *weight* of the body. Weight must not be confused with mass, which is not a force. The *mass* of a body is the amount of matter it contains. The weight of an object of fixed mass takes slightly different values at different points on the earth's surface. The gravitational pull on an object is much smaller on the surface of the moon, although its mass is unchanged.

When a body is attached to a string then the force acting along the string is called the *tension* in the string. The tension in a light string is assumed constant throughout its length. It is also possible to produce tension in rods and springs.



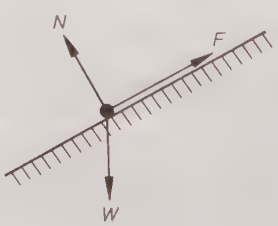
This diagram shows the forces on a particle of weight W, suspended from a beam by two light strings.

Another type of force called a *reaction* must be taken into account when two bodies are in contact. For a body resting on a smooth surface this force is referred to as the *normal reaction*, because its line of action is normal (i.e. perpendicular) to the surface. When two rough surfaces are in contact the total reaction is the sum of a normal reaction and a frictional force. [Friction will be considered further in Chapter 6.]



Shown here are the forces on a particle being pulled by a string across a smooth horizontal surface. The normal reaction is denoted by N.

Consider now a particle at rest on a rough inclined plane. As well as the normal reaction there will be a frictional force opposing the tendency of the particle to slide down the plane.



Force and other basic quantities are now usually measured in SI units, i.e. according to the Système International d'Unités. The SI unit of force is the newton, defined as the force needed to give a mass of 1 kilogramme an acceleration of 1 metre per second per second. [Students not familiar with the unit may find it helpful to remember that a cricket ball weighs approximately 1·5 N and a heavyweight boxer 800 N or more.]

We now give a table showing some of the SI units which are being used in this course.

Quantity	*Name of unit*	*Symbol*
length	metre	m
mass	kilogramme	kg
time	second	s
area	square metre	m^2
volume	cubic metre	m^3
velocity	metre per second	$m\,s^{-1}$ or m/s
acceleration	metre per second squared	$m\,s^{-2}$ or m/s^2
force	newton	N

[Note that 's' should never be added to a symbol to make a plural, because it is the symbol for second. For instance, 3 metres cannot be written as 3 ms because this means 3 milliseconds. Similarly the space in symbols for derived units is important, since without the space the symbol may have a different meaning. For example, $m\,s^{-1}$ means 'metre per second' but ms^{-1} means 'per millisecond'.]

Exercise 2.2

Draw diagrams to show the forces acting on a particle in each of the following situations.

1. At rest on a smooth horizontal plane.

2. At rest suspended from a fixed point by a string.

3. Sliding down a smooth inclined plane.

4. Sliding down a rough inclined plane.

5. Pulled across a rough horizontal plane by a string at an angle of 30° to the plane.

6. Held at rest on a smooth plane inclined at 25° to the horizontal by a string parallel to the plane.

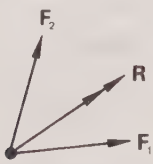
7. Pulled up a rough plane inclined at 25° to the horizontal by a string parallel to the plane.

8. Swinging at the end of a string, when the string is at an angle of 35° to the vertical.

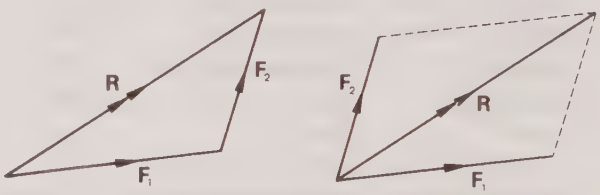
9. Sliding down the smooth inner surface of a hemispherical bowl.

2.3 Resultant forces

Since forces are vector quantities, they can be represented by directed line segments and added using either the triangle law or the parallelogram law as stated in §1.1. Thus, if $\mathbf{F}_1$ and $\mathbf{F}_2$ are two forces acting on a particle, their sum or resultant $\mathbf{R}$ may be represented by either the third side of a vector triangle or the diagonal of a parallelogram. A double arrow is used to emphasise that $\mathbf{R}$ is a resultant force rather than a third force acting on the particle.



Space diagram (showing positions in space)

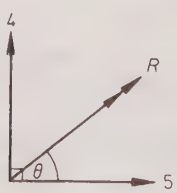


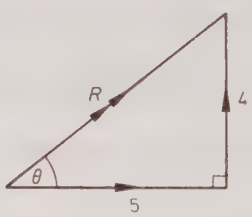
Vector diagrams

The magnitude and direction of the resultant can be found either by accurate scale drawing or by calculation. Although it is usually more convenient to use a vector triangle for this purpose, the parallelogram form of the vector diagram has the advantage that it shows the two forces and their resultant acting at a single point.

Example 1 Find the resultant of two forces of magnitudes 5 N and 4 N acting at right angles.

Let R be the magnitude of the resultant and θ the angle it makes with the 5 N force.





Vector triangle

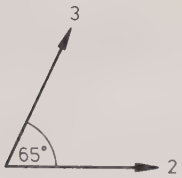
Using Pythagoras' theorem in the vector triangle,

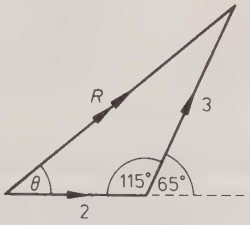
$R^2 = 5^2 + 4^2 = 25 + 16 = 41 \quad \therefore \quad R \approx 6{\cdot}4$

$\tan\theta = 4/5 = 0{\cdot}8 \quad \therefore \quad \theta \approx 38{\cdot}7°$

The resultant is a force of magnitude 6·4 N at an angle 38·7° to the 5 N force.

Example 2 Two forces which act on a particle have magnitudes 2 N and 3 N. If the angle between their directions is 65°, find the resultant force on the particle.





Vector triangle

By the cosine rule, $R^2 = 2^2 + 3^2 - 2.2.3\cos 115°$

$= 13 - 12\cos 115°$

$\therefore \quad R \approx 4{\cdot}25$

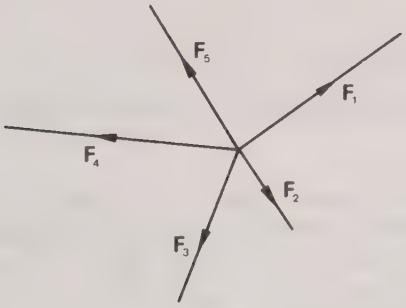
By the sine rule, $\dfrac{\sin\theta}{3} = \dfrac{\sin 115°}{R}$

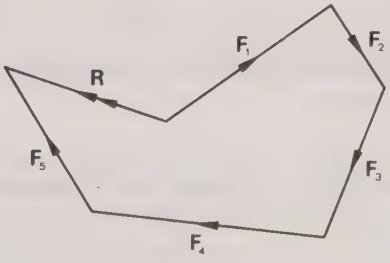
$\therefore \quad \sin\theta = \dfrac{3\sin 115°}{R}$ and $\theta \approx 39{\cdot}8°$

Hence the resultant is a force of magnitude 4·25 N at an angle 39·8° to the 2 N force.

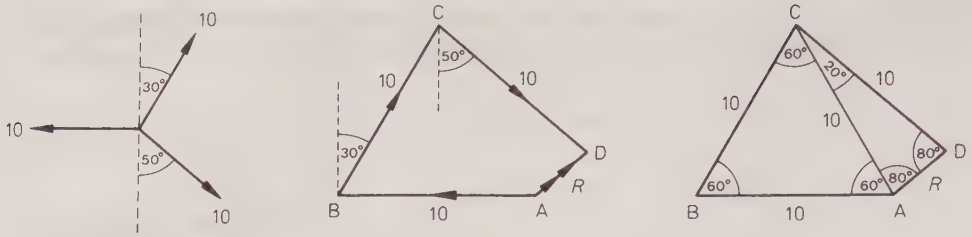
[Note that to achieve maximum accuracy the unrounded value of R should be used to find θ. The given results were obtained using a calculator, the value $R = 4{\cdot}251048811$ being stored in the memory.]

When the resultant of more than two forces is required a vector polygon can be constructed.





Example 3 Three forces, each of magnitude 10 N, act horizontally in the directions due west, N 30° E and S 50° E. Find the magnitude and direction of their resultant.



In the vector diagram, $\angle ABC = 60°$ and $AB = BC = 10$

$\therefore$ $\triangle ABC$ is an equilateral triangle.

Hence in $\triangle ACD$, $\angle ACD = 20°$ and $AC = CD = 10$

$\therefore$ $\triangle ACD$ is an isosceles triangle in which $\angle A = \angle D = 80°$ and $R = 2 . 10 \cos 80° \approx 3{\cdot}47$.

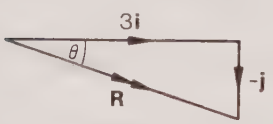
Thus the resultant force has magnitude 3·47 newtons and direction N 50° E.

The resultant of a set of forces can be found more quickly if the forces are expressed in terms of their Cartesian components. (See §1.2.)

Example 4 Find the magnitude and direction of the resultant of the forces represented by $2\mathbf{i} - 5\mathbf{j}$, $3\mathbf{j}$ and $\mathbf{i} + \mathbf{j}$, where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the directions due east and due north respectively.

Let $\mathbf{R}$ be the resultant of the given forces and let θ be the angle between $\mathbf{R}$ and $\mathbf{i}$.

$$\mathbf{R} = (2\mathbf{i} - 5\mathbf{j}) + 3\mathbf{j} + (\mathbf{i} + \mathbf{j}) = (2+1)\mathbf{i} + (-5+3+1)\mathbf{j}$$
$$= 3\mathbf{i} - \mathbf{j}$$



$$R^2 = 3^2 + (-1)^2 = 10$$
$$\therefore \quad R = \sqrt{10} \approx 3{\cdot}16$$
$$\tan\theta = \tfrac{1}{3} \qquad \therefore \quad \theta \approx 18{\cdot}4°$$

Hence the resultant is represented by $3\mathbf{i} - \mathbf{j}$. Its magnitude is 3·16 and its direction S 71·6° E.

Example 5 If $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors parallel to the x-, y- and z-axes, find the magnitude of the resultant of the forces represented by $3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, $\mathbf{i} - 3\mathbf{k}$ and $2\mathbf{i} + 5\mathbf{j} + \mathbf{k}$.

The resultant is given by

$(3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}) + (\mathbf{i} - 3\mathbf{k}) + (2\mathbf{i} + 5\mathbf{j} + \mathbf{k})$, i.e. $6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$

$\therefore$ the magnitude of the resultant is

$$|6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}| = \sqrt{(6^2 + 3^2 + 2^2)} = \sqrt{49} = 7.$$

Example 6 Two forces **P** and **Q** act in the directions of the vectors $4\mathbf{i}+3\mathbf{j}$ and $\mathbf{i}-2\mathbf{j}$ respectively and the magnitude of **P** is 25 N. Given that the magnitude of the resultant of **P** and **Q** is also 25 N, find the magnitude of **Q**.

$|4\mathbf{i}+3\mathbf{j}| = \sqrt{(4^2+3^2)} = \sqrt{25} = 5$

$\therefore$ the force **P** can be represented in magnitude and direction by the vector $5(4\mathbf{i}+3\mathbf{j})$, i.e. $20\mathbf{i}+15\mathbf{j}$.

Since **Q** can be represented by a vector of the form $\lambda(\mathbf{i}-2\mathbf{j})$, we may write

$\mathbf{P}+\mathbf{Q} = (20\mathbf{i}+15\mathbf{j})+\lambda(\mathbf{i}-2\mathbf{j}) = (20+\lambda)\mathbf{i}+(15-2\lambda)\mathbf{j}$

$\therefore$ the magnitude of the resultant of **P** and **Q**

$= \sqrt{\{(20+\lambda)^2+(15-2\lambda)^2\}} = 25$

$\therefore \quad 400+40\lambda+\lambda^2+225-60\lambda+4\lambda^2 = 625$

$5\lambda^2-20\lambda = 0$

$\therefore$ assuming λ is non-zero, $\lambda = 4$

Hence $\mathbf{Q} = 4(\mathbf{i}-2\mathbf{j})$ and $|\mathbf{Q}| = 4\sqrt{\{1^2+(-2)^2\}} = 4\sqrt{5}$.

Thus the magnitude of the force **Q** is $4\sqrt{5}$ N.

Exercise 2.3

1. The resultant of two forces **P** and **Q** is of magnitude R newtons and acts at an angle θ to the direction of **P**. Find R and θ, given that the magnitudes of **P** and **Q** and the angle between their directions are respectively

(a) 12 N, 5 N; 90°, (b) 4 N, 7 N; 90°,
(c) 8 N, 11 N; 120°, (d) 6 N, 10 N; 52°,
(e) 14 N, 9 N; 108°, (f) 2 N, 7 N; 23°.

2. Find the angle between the lines of action of two forces of magnitudes 7 N and 11 N, given that their resultant is of magnitude 8 N.

3. The resultant of two forces **F** and **G** is a force of 20 N acting at an angle of 45° to the force **G**. Given that the magnitude of **G** is 12 N, find the magnitude and direction of **F**.

4. Two forces of magnitudes 7 N and 9 N act at an angle θ where $\sin\theta = 0{\cdot}4$. Find the magnitudes of the two possible resultants.

5. Find the resultant of three horizontal forces whose magnitudes and directions are 16 N in the direction N 70° W, 8 N in the direction N 20° E and 11 N in the direction S 70° E.

6. Four forces, each of magnitude 12 N, act horizontally in the directions N 30° E, S 60° E, S 15° E and S 75° W. Find the magnitude and direction of their resultant.

7. Horizontal forces of magnitudes 12 N, 9 N and 6 N act on a particle in the directions due north, due east and S 60° E respectively. Find the magnitude and direction of their resultant.

8. Find the magnitude and direction of the resultant of the forces represented by each of the following sets of vectors, where **i** and **j** are unit vectors in the directions due east and due north respectively.
(a) $2\mathbf{i}-\mathbf{j}$, $3\mathbf{i}$, $-\mathbf{i}+4\mathbf{j}$,
(b) $5\mathbf{i}-2\mathbf{j}$, $2\mathbf{i}+3\mathbf{j}$, $-\mathbf{j}$,
(c) $7\mathbf{i}+4\mathbf{j}$, $2\mathbf{i}-5\mathbf{j}$, $-3\mathbf{i}-2\mathbf{j}$,
(d) $\mathbf{i}+\mathbf{j}$, $-3\mathbf{i}-2\mathbf{j}$, $-3\mathbf{i}-4\mathbf{j}$.

9. If **i**, **j** and **k** are unit vectors parallel to the x-, y- and z-axes, find the magnitude of the resultant of the forces represented by
(a) $2\mathbf{i}-3\mathbf{j}$, $5\mathbf{i}+\mathbf{j}-3\mathbf{k}$ and $-3\mathbf{i}+4\mathbf{j}-\mathbf{k}$,
(b) $\mathbf{i}+6\mathbf{j}-3\mathbf{k}$, $8\mathbf{i}-\mathbf{j}-5\mathbf{k}$ and $3\mathbf{i}+3\mathbf{j}-\mathbf{k}$,
(c) $2\mathbf{j}-5\mathbf{k}$, $5\mathbf{i}-\mathbf{k}$ and $2\mathbf{i}+3\mathbf{j}+\mathbf{k}$.

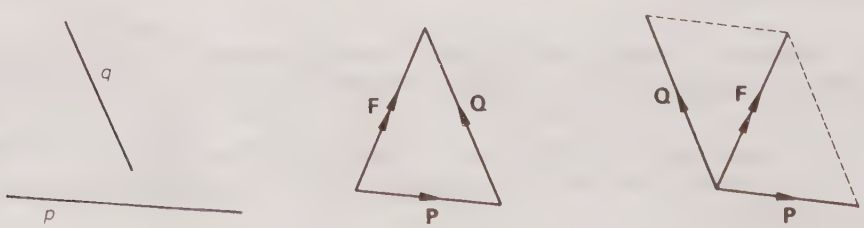
10. Find the magnitude of the resultant of two forces of magnitudes 15 N and $4\sqrt{2}$ N acting in the directions of the vectors $3\mathbf{i}-4\mathbf{j}$ and $\mathbf{i}+\mathbf{j}$ respectively.

11. Two forces, each of magnitude $5P$ newtons, act in the directions of the vectors $7\mathbf{i}+\mathbf{j}$ and $\mathbf{i}-\mathbf{j}$. Show that their resultant acts in the direction of the vector $3\mathbf{i}-\mathbf{j}$. Given that the magnitude of the resultant is 20 newtons, find the value of P in surd form.

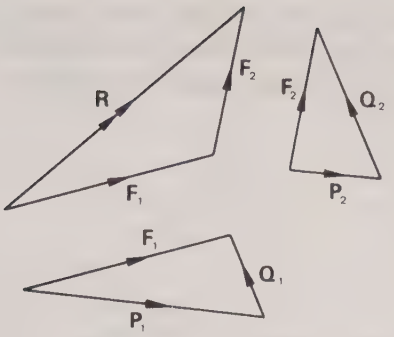
12. Two forces **F** and **G** act parallel to the vectors $3\mathbf{i}+\mathbf{j}$ and $\mathbf{i}+2\mathbf{j}$ respectively. If the resultant is a force of 10 newtons in the direction of the vector $3\mathbf{i}-4\mathbf{j}$, find the magnitudes of **F** and **G** in surd form.

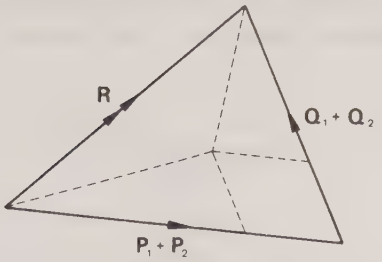
2.4 Resolution into components

If p and q are two non-parallel lines in a plane, then any force **F** acting in that plane can be resolved into component forces **P** and **Q** in directions parallel to p and q by constructing a vector triangle or parallelogram as shown below.



Consider now two forces and their resolution into components parallel to the lines p and q. Let $\mathbf{F}_1 = \mathbf{P}_1 + \mathbf{Q}_1$, $\mathbf{F}_2 = \mathbf{P}_2 + \mathbf{Q}_2$ and let the resultant of $\mathbf{F}_1$ and $\mathbf{F}_2$ be **R**.





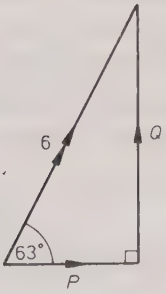
As shown in the diagrams,

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 = (\mathbf{P}_1 + \mathbf{Q}_1) + (\mathbf{P}_2 + \mathbf{Q}_2) = (\mathbf{P}_1 + \mathbf{P}_2) + (\mathbf{Q}_1 + \mathbf{Q}_2).$$

Thus the components of a resultant force are the sums of the corresponding components of the original forces. This property of forces and their components is often used to find resultant forces.

For most practical purposes forces are resolved into two components at right angles.

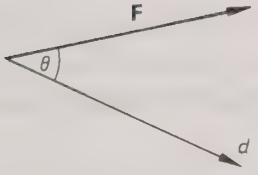
Example 1 Find the magnitudes of the horizontal and vertical components of a force of 6 N acting at an angle of 63° to the horizontal.



Let P and Q newtons be the magnitudes of the required components,

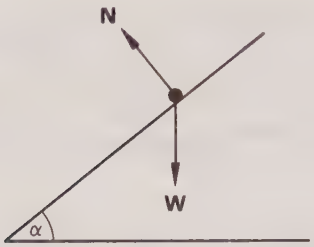
then $P = 6\cos 63° \approx 2{\cdot}72$
and $Q = 6\sin 63° \approx 5{\cdot}35$

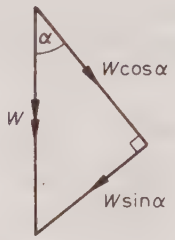
$\therefore$ the magnitudes of the horizontal and vertical components are 2·72 N and 5·35 N respectively.



If a force $\mathbf{F}$ acts at an angle θ to a given direction d, then the quantity $F\cos\theta$ is called the *resolved part* of $\mathbf{F}$ in the direction d, or simply the component of $\mathbf{F}$ in the direction d. If θ is acute, $F\cos\theta$ is positive, but if θ is obtuse, $F\cos\theta$ is negative. Thus the sign of the resolved part indicates whether the direction of the component force is the same as d or opposite to d.

Example 2 A particle is sliding down a smooth plane inclined at an angle α to the horizontal. Find the components of the weight **W** and the normal reaction **N** down the plane and in the direction of an upward perpendicular to the plane.



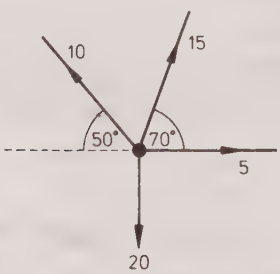


The components of **W** are $W\sin\alpha$ down the plane and $-W\cos\alpha$ along an upward perpendicular to the plane.

Since **N** acts at right angles to the plane, its component down the plane is 0 and its component along an upward perpendicular is N.

The resultant of a set of forces can now be found by resolving in two perpendicular directions.

Example 3 A small package of weight 20 N is attached to three light strings lying in the same vertical plane. If the tensions in the strings are as shown in the given diagram, find the resultant force on the package.



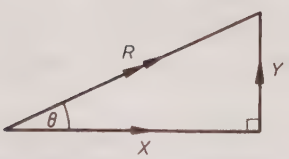
Let the resultant **R** have horizontal and vertical components X and Y. Resolving horizontally and vertically,

$$\rightarrow X = 5 + 15\cos 70° - 10\cos 50°.$$
$$\uparrow Y = 15\sin 70° + 10\sin 50° - 20$$

By calculator $X = 3{\cdot}70242$,
$Y = 1{\cdot}755835$.

$$\therefore \quad R = \sqrt{(X^2 + Y^2)} \approx 4{\cdot}10$$

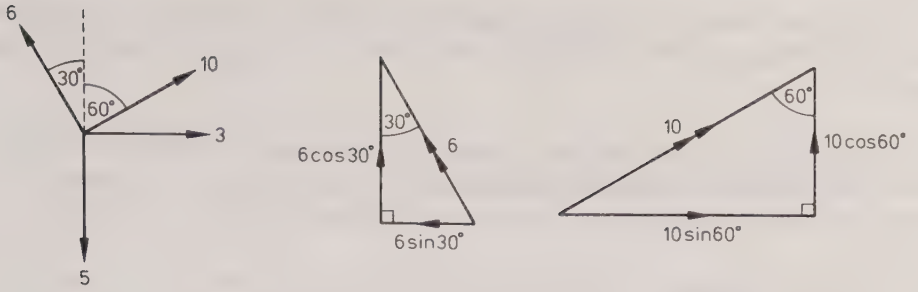
and $\tan\theta = Y/X$, $\theta \approx 25{\cdot}4°$.



Hence the resultant is a force of magnitude 4·10 N at an angle 25·4° to the horizontal.

It is sometimes convenient to express a set of forces in terms of perpendicular unit vectors **i** and **j**.

Example 4 A set of horizontal forces of magnitudes 3 N, 5 N, 6 N and 10 N act on a particle in directions due east, due south, N 30° W and N 60° E respectively. Find the magnitude and direction of the resultant.

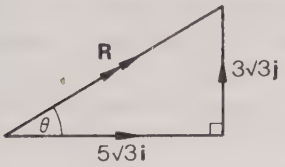


Let **i** and **j** be unit vectors in the directions due east and due north. The given set of forces can then be represented by

$$3\mathbf{i},\ -5\mathbf{j},\ -6\sin 30°\mathbf{i}+6\cos 30°\mathbf{j} \quad \text{and} \quad 10\sin 60°\mathbf{i}+10\cos 60°\mathbf{j}.$$

Hence the resultant **R** is given by

$$\mathbf{R} = 3\mathbf{i}+(-5\mathbf{j})+(-3\mathbf{i}+3\sqrt{3}\mathbf{j})+(5\sqrt{3}\mathbf{i}+5\mathbf{j}) = 5\sqrt{3}\mathbf{i}+3\sqrt{3}\mathbf{j}$$



$$R^2 = (5\sqrt{3})^2+(3\sqrt{3})^2 = 75+27 = 102$$

$$\therefore \quad R \approx 10{\cdot}1$$

$$\tan\theta = \frac{3\sqrt{3}}{5\sqrt{3}} = 0{\cdot}6 \qquad \therefore \quad \theta \approx 31°.$$

The resultant is a force of magnitude 10·1 N and direction N 59° E.

Exercise 2.4

1. Find the magnitudes of the horizontal and vertical components of
(a) a force of 20 N acting at 40° to the horizontal,
(b) a force of 7 N acting at 78° to the horizontal,
(c) a force of 12 N acting at 36° to the vertical.

2. A particle is being pulled up a smooth plane, inclined at an angle α to the horizontal, by a string which makes an acute angle β with the plane. Find the resolved parts of the weight of the particle **W**, the normal reaction **N** and the tension in the string **T** (a) vertically upwards, (b) up the plane, (c) in the direction of a downward perpendicular to the plane.

3. A set of coplanar forces of magnitudes 12, 9, 15 and 16 newtons act on a particle in directions south-west, north, north-west and east respectively. Find the magnitude and direction of their resultant.

4. A particle is being pulled across a smooth horizontal plane by three strings. The directions of the strings are N 60° W, due north and N 30° E. If the tensions in the strings are of magnitude 15 N, 12 N and 10 N respectively, find the resultant force on the particle.

5. Four horizontal forces of equal magnitude act on a particle in the directions whose bearings are 050°, 100°, 250° and 290°. Calculate the bearing of the resultant.

6. *ABCDEF* is a regular hexagon. Forces of magnitude 2, 3, 4, 5 and 6 newtons act in the directions of **AB**, **AC**, **AD**, **AE** and **AF** respectively. If their resultant is a force of R newtons acting at an angle θ to **AB**, find R and θ.

7. *ABCD* is a rectangle in which *AB* is 3 m, *BC* is 4 m. Forces 3 N, 2 N, 10 N, 5 N act in the directions of **AB**, **AD**, **AC**, **BD** respectively. Find their resultant, giving the angle it makes with **AB**.

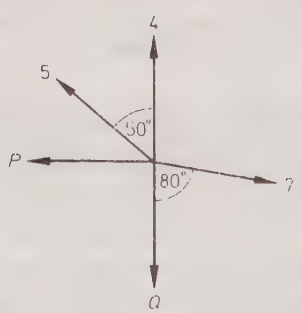
8. *ABCDE* is a regular pentagon. Find the magnitude of the resultant of forces 2 N, 3 N, 1 N, 4 N, 5 N in the directions of **AB**, **BC**, **CD**, **DE**, **EC** respectively.

9. A vertical flagpole of height 24 m is held in position by three straight wires fixed to its top and to points on the horizontal ground. The wires are of lengths 25 m, 30 m and 26 m and their fixing points are at bearings 060°, 150° and 270° respectively from the base of the pole. If the tensions in the wires are each 50 N, find (a) the vertical components of the tensions, (b) the horizontal components of the tensions. Hence find (c) the resultant force downwards caused by the wires, (d) the resultant horizontal force on the flagpole.

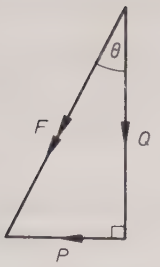
2.5 A particle in equilibrium

Forces on a particle are said to be in *equilibrium* if their resultant is zero. Thus when forces in equilibrium are resolved into components in two fixed directions, the sums of the components in each direction will be zero. This fact provides a basic approach to problem solving.

Example 1 A set of horizontal forces of magnitudes 5, 4 and 7 newtons act on a particle in the directions N 50° W, due north and S 80° E respectively. Find the magnitude and direction of a fourth force which holds the particle in equilibrium.



Let the components in the directions due west and due south of the fourth force **F** be P and Q respectively. Resolving in the directions west and south,



$\leftarrow P + 5\sin 50° - 7\sin 80° = 0$

$\downarrow Q + 7\cos 80° - 5\cos 50° - 4 = 0$

$\therefore \quad P = 7\sin 80° - 5\sin 50°$

$Q = 4 + 5\cos 50° - 7\cos 80°.$

By calculator $P = 3{\cdot}063436$,

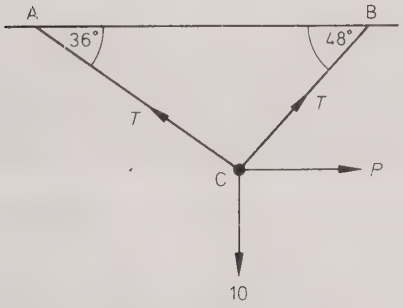
$Q = 5{\cdot}998404$

$\therefore \quad F = \sqrt{(P^2 + Q^2)} \approx 6{\cdot}74$ and $\tan\theta = P/Q, \quad \theta \approx 27{\cdot}1°.$

Hence the fourth force is of magnitude 6·74 newtons in the direction S 27·1° W.

Example 2 A light inextensible string is attached to two points A and B on a horizontal beam. A smooth ring C of weight 10 N, which can move freely on the string, is held in equilibrium by a horizontal force P acting in the vertical plane containing A, B and C. Given that $\angle BAC = 36°$ and $\angle ABC = 48°$, find the magnitude of P and the tension T in the string.

[Note that since the ring is smooth the tension is the same in both parts of the string.]



Resolving horizontally and vertically,

$\rightarrow P + T\cos 48° - T\cos 36° = 0$

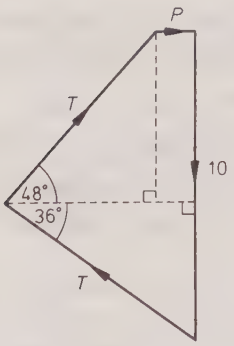
$\uparrow T\sin 48° + T\sin 36° - 10 = 0$

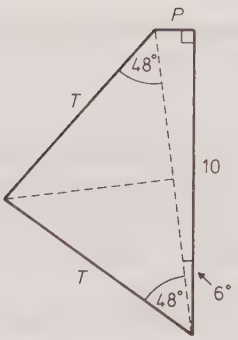
$$\therefore \quad T = \frac{10}{\sin 48° + \sin 36°} \approx 7{\cdot}51$$

and $P = T(\cos 36° - \cos 48°) \approx 1{\cdot}05.$

Hence the magnitude of the horizontal force P is 1·05 N and the tension in the string is 7·51 N.

In §2.3 we saw that forces can be added by constructing a vector polygon. If a set of forces is in equilibrium the length of the side representing their resultant must be zero. Hence a set of forces in equilibrium can be represented by the sides of a closed polygon, often referred to as the *polygon of forces*.





Example 2 could have been solved by drawing a polygon of forces. Using the right-angled triangles shown in the diagram,

$$T\sin 48° + T\sin 36° = 10$$

and $T\cos 48° + P = T\cos 36°$

$\therefore$ as before we find that

$$T = \frac{10}{\sin 48° + \sin 36°},$$

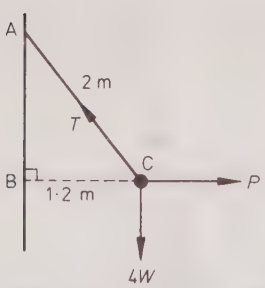
$P = T(\cos 36° - \cos 48°)$.

However, considering instead the angle properties of the figure, we could obtain the equivalent results,

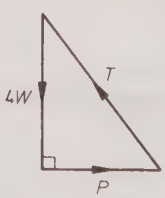
$P = 10\tan 6°$ and
$2T\cos 48°\cos 6° = 10$.

Of special interest are problems involving three forces in equilibrium, because in this case the vector diagram is simply a *triangle of forces.*

Example 3 A particle of weight $4W$ is attached to the end of a light inextensible string of length 2 m. The other end of the string is fixed to a point on a vertical wall. The particle is being held 1·2 m away from the wall by a horizontal force. Express the magnitudes of the horizontal force and the tension in the string in terms of W.



Space diagram



Triangle of forces

Let T be the tension in the string and P the horizontal force.

By Pythagoras' theorem in $\triangle ABC$,

$$AB^2 = AC^2 - BC^2 = 2^2 - (1{\cdot}2)^2 = 2{\cdot}56 \qquad \therefore \quad AB = 1{\cdot}6$$

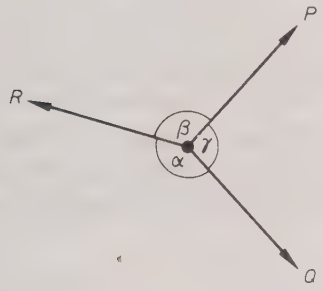
As shown in the diagrams, $\triangle ABC$ is similar to the triangle of forces

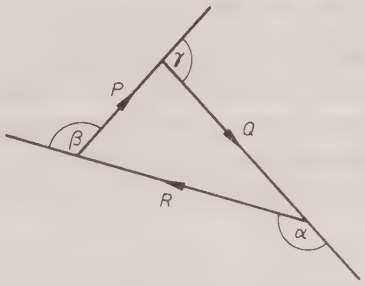
$$\therefore \quad \frac{T}{2} = \frac{P}{1{\cdot}2} = \frac{4W}{1{\cdot}6}.$$

Hence $T = 5W$ and $P = 3W$.

The magnitudes of the horizontal force and the tension in the string are $3W$ and $5W$ respectively.

Consider now three forces of magnitudes P, Q and R acting on a particle. If the angles between the forces are α, β and γ, then the exterior angles of the triangle of forces are also α, β and γ.



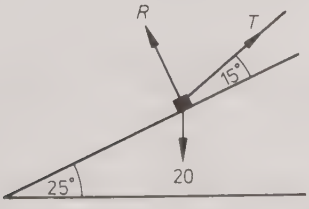


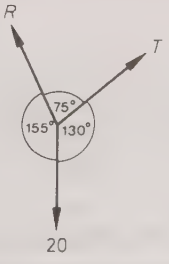
By the sine rule, $$\frac{P}{\sin(180° - \alpha)} = \frac{Q}{\sin(180° - \beta)} = \frac{R}{\sin(180° - \gamma)}$$

$$\therefore \quad \boxed{\frac{P}{\sin\alpha} = \frac{Q}{\sin\beta} = \frac{R}{\sin\gamma}}$$

This result is called *Lami's*† *Theorem*. It is often useful in solving three force problems in which angles rather than lengths are known.

Example 4 A small block of weight 20 N is at rest on a smooth plane inclined at an angle of 25° to the horizontal. The block is being held in position by a light rope. Given that the angle between the rope and the plane is 15°, find the magnitudes of the tension T in the rope and of the reaction R with the plane.





† *Lami, Bernard* (1640–1715) French scholar. He made a substantial contribution to the philosophical and mathematical theories of his time. His work on equilibrium appears in the *Traité de Mécanique* (1679).

By Lami's theorem, $\dfrac{T}{\sin 155°} = \dfrac{R}{\sin 130°} = \dfrac{20}{\sin 75°}$

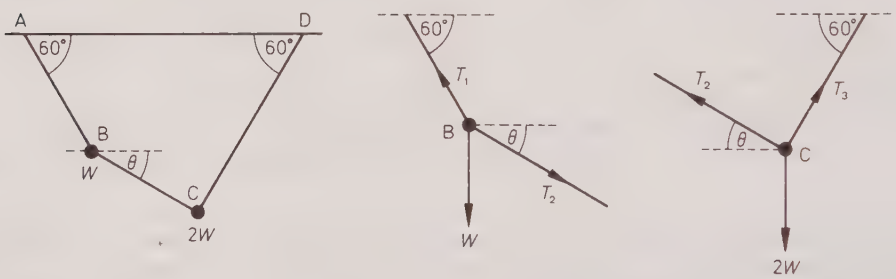
$\therefore \quad T = \dfrac{20 \sin 155°}{\sin 75°} \approx 8{\cdot}75 \quad \text{and} \quad R = 20 \sin 130°/\sin 75° \approx 15{\cdot}9$

The tension in the rope is 8·75 N and the normal reaction with the plane is 15·9 N.

[This problem could, of course, also be solved by resolving the forces parallel and perpendicular to the plane or by drawing a triangle of forces.]

Problems involving more than one particle are often solved by considering the forces on each particle separately.

Example 5 A light inextensible string is suspended from two points A and D in the same horizontal plane. Particles of weight W and $2W$ are attached to the string at points B and C respectively. If AB and CD are inclined at 60° to the horizontal, find in terms of W the tensions T_1, T_2, T_3 in the strings AB, BC and CD respectively. Find also the angle θ, which BC makes with the horizontal.



Resolving the forces on the particle at B

horizontally $\quad \rightarrow T_2 \cos\theta - T_1 \cos 60° = 0 \qquad (1)$

vertically $\quad \uparrow T_1 \sin 60° - T_2 \sin\theta - W = 0 \qquad (2)$

Resolving the forces on the particle at C

horizontally $\quad \rightarrow T_3 \cos 60° - T_2 \cos\theta = 0 \qquad (3)$

vertically $\quad \uparrow T_3 \sin 60° + T_2 \sin\theta - 2W = 0 \qquad (4)$

Adding (1) to (3): $\quad T_3 \cos 60° - T_1 \cos 60° = 0 \quad \therefore \quad T_1 = T_3$

Adding (2) to (4): $\quad T_1 \sin 60° + T_3 \sin 60° - 3W = 0$

$\therefore \quad T_1 + T_3 = 3W/\sin 60° = 2\sqrt{3}W \quad \text{and} \quad T_1 = T_3 = \sqrt{3}W$

Using (1) $\quad T_2 \cos\theta = \sqrt{3}W \,.\cos 60° = \dfrac{\sqrt{3}}{2} W$

Using (2) $\quad T_2 \sin\theta = \sqrt{3}W \,.\sin 60° - W = \dfrac{1}{2} W$

By squaring and adding these results:

$$T_2^2 = T_2^2(\cos^2\theta + \sin^2\theta) = \tfrac{3}{4}W^2 + \tfrac{1}{4}W^2 = W^2$$

Dividing: $\tan\theta = T_2\sin\theta / T_2\cos\theta = \frac{1}{2}W \Big/ \frac{\sqrt{3}}{2}W = 1/\sqrt{3}$

$$\therefore \quad T_2 = W \quad \text{and} \quad \theta = 30°.$$

Hence the tensions in the strings AB, BC and CD are $\sqrt{3}W$, W and $\sqrt{3}W$ respectively and BC is inclined at 30° to the horizontal.

Exercise 2.5

1. A set of horizontal forces of magnitude 20 N, 12 N and 30 N act on a particle in the directions due south, due east and N 40° E respectively. Find the magnitude and direction of a fourth force which holds the particle in equilibrium.

2. A set of four horizontal forces in equilibrium act in directions whose bearings are 020°, 110°, 230° and 320°. If the magnitudes of the forces are 8, 12, P and Q newtons respectively, find the values of P and Q.

3. A smooth ring C of weight 15 N can move freely on a light inextensible string attached to two fixed points A and B in the same horizontal line. A horizontal force of magnitude P holds the ring in equilibrium, such that AC is vertical and $\angle ABC = 30°$. Find the value of P using a polygon of forces.

4. A set of six forces in equilibrium act in the directions of the sides $\vec{AB}$, $\vec{BC}$, $\vec{CD}$, $\vec{DE}$, $\vec{EF}$ and $\vec{FA}$ of a regular hexagon. If the magnitudes of the forces are 7, 4, 10, 5, P and Q newtons respectively, use a polygon of forces to find the values of P and Q.

5. A body of weight 20 newtons is held at rest on a smooth plane inclined at 60° to the horizontal by a force of magnitude P newtons. Find P given that the force is (a) horizontal, (b), vertical, (c) parallel to the plane.

6. One end of a light inextensible string of length 75 cm is fixed to a point on a vertical pole. A particle of weight 12 N is attached to the other end of the string. The particle is held 21 cm away from the pole by a horizontal force. Find the magnitude of this force and the tension in the string.

7. A small object of weight 4 N is suspended from a fixed point by a string. The object is held in equilibrium, with the string at an angle of 25° to the vertical, by a horizontal force. Find the magnitude of this force and the tension in the string.

8. A particle P of weight 50 N is hanging in equilibrium supported by two strings inclined at 20° and 40° to the vertical. Find the tensions in the strings.

9. A particle of weight $10W$ is held at rest on a smooth plane inclined at 30° to the horizontal by a light string. Find the tension in the string and the reaction

between the particle and the plane if (a) the string is parallel to the plane, (b) the string is at an angle of 30° to the plane.

10. A particle of weight 2 N hangs on a string. The particle is pulled to one side by a force of magnitude 3 N acting at an angle of 20° below the horizontal. Find the tension in the string and the inclination of the string to the vertical.

11. A particle P of weight $10W$ is supported by two light inextensible strings attached to fixed points X and Y which lie in the same horizontal plane. Given that $PX = 36$ cm, $PY = 48$ cm and $XY = 60$ cm, find the tension in each string.

12. A light inextensible string passes through a smooth ring fixed at a point P. Particles A and B of weights 10 N and 20 N respectively are attached to the ends of the string. The system is held in equilibrium by a horizontal force acting on the particle A. Find the magnitude of this force. Find also the magnitude and direction of the resultant force on the ring due to the tension in the string.

13. A light inextensible string ABC is fixed at the point A. Two particles each of weight W are attached to the string at B and C. The system is held in equilibrium by a horizontal force of magnitude $2W$ acting on the particle at C. Find (a) the tensions in BC and AB, (b) the inclinations of BC and AB to the vertical.

14. Two smooth straight rods XY and XZ are fixed at 45° to the vertical and at right angles to each other. Two small beads A and B, each of weight W, are free to slide on the rods XY and XZ respectively. The beads are connected by a light inextensible string, to the mid-point of which is attached a particle C of weight $2W$. Find the reaction between the bead A and the rod XY, the tension in the string AC and its inclination to the horizontal when the system is in equilibrium.

15. $ABCD$ is a light inextensible string. The ends A and D are fixed so that the line AD is horizontal. Particles of weight $3W$ and $4W$ are attached to the string at the points B and C respectively. Find the tensions in the strings AB, BC and CD given that (a) $\angle DAB = 45°$ and $\angle ABC = 150°$, (b) $\angle DAB = 45°$ and the strings BC and DC are equally inclined to the horizontal.

Exercise 2.6 (miscellaneous)

1. Find the angle between the lines of action of two forces of magnitudes 15 N and 9 N, given that their resultant is a force of magnitude 20 N.

2. Two forces of magnitude $3P$ and $4P$ newtons act on a particle. The magnitude of their resultant is 20 newtons. When one of the forces is reversed the magnitude of the resultant is halved. Calculate the value of P and the original angle between the forces.

3. Two forces of magnitude 5 N and 3 N act on a particle. Their resultant acts at an angle θ to the direction of the 5 N force. Find the range of possible values of θ.

4. Given that **i** and **j** are unit vectors in the directions due east and due north respectively, find the magnitude and direction of the resultant of the forces represented by the vectors $3\mathbf{i}-\mathbf{j}$, $-2\mathbf{i}$, $5\mathbf{i}+7\mathbf{j}$ and $\mathbf{i}-2\mathbf{j}$.

5. Two concurrent forces **X** and **Y** have resultant **R**. If $X = R$ and the angle between **R** and **Y** is α, prove (a) $Y = 2X\cos\alpha$, (b) the angle between **X** and **Y** is $180° - \alpha$.

6. A force of magnitude 10 N parallel to the vector $4\mathbf{i}+3\mathbf{j}$ is the resultant of two forces parallel respectively to the vectors $2\mathbf{i}+\mathbf{j}$, $\mathbf{i}+\mathbf{j}$. Find the magnitudes of these two forces. (L)

7. The point O is the centre of a regular hexagon $ABCDEF$. Forces of magnitude 7, 3, 10, 9, 6 and 2 newtons act on a particle at O in the directions of **OA**, **OB**, **OC**, **OD**, **OE** and **OF** respectively. Find the magnitude of the resultant and show that it acts in the direction of **AC**.

8. Find the bearing of the resultant of five horizontal forces of equal magnitude which act in the directions whose bearings are 070°, 130°, 160°, 220° and 310°.

9. A set of four forces on a particle are in equilibrium. If **i** and **j** are unit vectors in the directions due east and due north respectively, three of the forces are represented by the vectors $\mathbf{i}+\mathbf{j}$, $3\mathbf{i}-2\mathbf{j}$ and $2\mathbf{i}+3\mathbf{j}$. Find the magnitude and direction of the fourth force.

10. A particle is in equilibrium under the action of a set of five horizontal forces. If four of the forces are of magnitude 6 N, 11 N, 8 N and 15 N acting in the directions N 72° W, N 15° W, N 35° E and S 63° E respectively, find the magnitude and direction of the fifth force.

11. A particle P of weight $10W$ is hanging in equilibrium from two strings PA and PB. If $\angle APB = 90°$ and the tensions in the strings are T and $2T$, find T in terms of W.

12. A particle P of weight 50 N is supported by two light inextensible strings attached to fixed points A and B. Given that A and B lie 2 m apart in the same horizontal plane and that $AP = 2$ m, $PB = 1$ m, find the tension in each string.

13. A small block of weight 20 N is held at rest on a smooth plane inclined at 20° to the horizontal by a string which makes an angle θ with the plane. If the tension in the string is 10 N, find the value of θ and the magnitude of the reaction between the block and the plane.

14. Show that the least force required to keep a particle of weight W in equilibrium on a smooth plane inclined at an angle α to the horizontal is of magnitude $W\sin\alpha$ and parallel to the plane.

15. The ends of a light inextensible string of length 80 cm are fixed to points A and B, which are 48 cm apart. A smooth ring C, of weight 5 N, which can move freely along the string, hangs in equilibrium. Find the tension in the string if (a) AB is horizontal, (b) AB is inclined at 60° to the horizontal.

16. One end of a light inextensible string is attached to a fixed point on a smooth straight wire inclined at 30° to the horizontal. The other end of the string is attached to a light ring which slides freely on the wire. A smooth bead of weight W, which is threaded on the string, is also free to slide. Find, in terms of W, the tension in the string when the system is in equilibrium.

17. A particle of weight 2 N is attached to a light string ABC at the point B. The ends A and C are threaded through two fixed smooth rings at points X and Y respectively, where the line XY makes an angle θ with the horizontal. A particle of weight 3 N is then attached to the string at A and a particle of weight 4 N at C. Show that equilibrium is possible only if Y lies above X and $1/4 < \sin\theta < 11/16$.

18. Two rods OA, OB are fixed in a vertical plane with O uppermost, each rod making an acute angle α with the vertical. Two smooth rings of equal weight W, which can slide one on each rod, are connected by a light inextensible string, upon which slides a third smooth ring of weight $2W$. Show that, in the symmetrical position of equilibrium, the angle between the two straight pieces of the string is 2β, where $\tan\beta = 2\cot\alpha$. Show that the tension in the string is $W\sqrt{(1+4\cot^2\alpha)}$ and that the reaction between the rings and the rods is $2W\operatorname{cosec}\alpha$. (W)

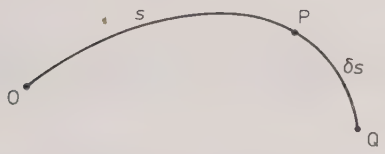
19. Four packages, each of weight 20 N, are fastened to a light rope at points B, C, D and E. The rope is attached to two fixed points A and F, which lie in the same horizontal plane. The points B and C are 1·2 m and 2·2 m respectively below the line AF and $AB = BC = CD = DE = EF$. Find the tension in the rope CD and the length of the rope.

20. Construct a flow chart for a procedure to find the resultant of a set of n horizontal forces, printing out its magnitude and bearing. Assume that the data is provided in the form of a list giving the magnitude x newtons and the bearing $y°$ of each force, where $x > 0$ and $0 \leqslant y < 360$.

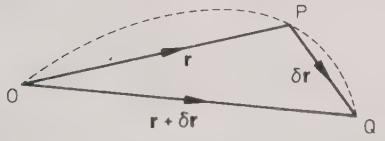
3 Relative motion

LIBRARY

3.1 Velocity and acceleration as vectors

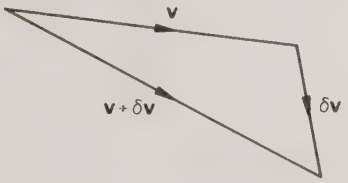


The diagram shows the path of a moving particle. The motion begins at the point O and after time t the particle reaches the point P. The *distance* travelled is the length s of the path OP. The *speed* of the particle is the rate at which it moves along its path. If the particle moves a further distance δs to a point Q in time δt, then its average speed in the motion from P to Q is $\delta s/\delta t$. Hence the speed of the particle as it passes through $P = \lim_{\delta t \to 0} \dfrac{\delta s}{\delta t} = \dfrac{ds}{dt}$.



The *displacement* of the particle from O at time t is the vector $\mathbf{r}$ represented by the line segment $\overrightarrow{OP}$. The *velocity* of the particle is the rate of change of its displacement. Thus the average velocity of the particle in its motion from P to Q can be expressed as $\delta\mathbf{r}/\delta t$. Hence the velocity of the particle as it passes through $P = \lim_{\delta t \to 0} \dfrac{\delta \mathbf{r}}{\delta t} = \dfrac{d\mathbf{r}}{dt}$.

By considering the magnitude and direction of $\overrightarrow{PQ}$ as $\delta t \to 0$, we see that the magnitude of the velocity is the speed of the particle and the direction of the velocity is the direction of motion.



The *acceleration* of a particle is defined as the rate of change of its velocity. If the velocities of a particle at times t and $t+\delta t$ are $\mathbf{v}$ and $\mathbf{v}+\delta\mathbf{v}$ respectively, then the acceleration at time $t = \lim_{\delta t \to 0} \dfrac{\delta \mathbf{v}}{\delta t} = \dfrac{d\mathbf{v}}{dt}$.

[Note that unless a particle is moving in a straight line the magnitude of its acceleration does *not* correspond to the rate of change of its speed and the direction of the acceleration is *not* the same as the direction of motion.]

It is clear from these definitions that distance and speed are scalar quantities, whereas displacement, velocity and acceleration are vector quantities. We now give some examples illustrating the relationships between them. Unless otherwise stated we will assume that the magnitudes of vector quantities are measured in the appropriate SI units.

Example 1 A particle moving with uniform velocity $4\mathbf{i}+3\mathbf{j}$ passes through the point A whose position vector with respect to the origin O is $3\mathbf{i}-\mathbf{j}$. Find (a) the speed of the particle, (b) the distance of the particle from the origin 3 seconds after it passes through A.

(a) The speed of the particle
$= |4\mathbf{i}+3\mathbf{j}| = \sqrt{(4^2+3^2)} = 5.$
(b) After 3 seconds the displacement of the particle from A
$= 3(4\mathbf{i}+3\mathbf{j}) = 12\mathbf{i}+9\mathbf{j}$
$\therefore$ after 3 seconds its position vector
$= (12\mathbf{i}+9\mathbf{j})+(3\mathbf{i}-\mathbf{j}) = 15\mathbf{i}+8\mathbf{j}.$
Hence the distance of the particle from the origin
$= |15\mathbf{i}+8\mathbf{j}| = \sqrt{(15^2+8^2)} = \sqrt{289} = 17.$

Example 2 A particle moves for 2 seconds with velocity $6\mathbf{i}+3\mathbf{j}-2\mathbf{k}$, then for 3 seconds with velocity $\mathbf{i}-2\mathbf{j}-2\mathbf{k}$. Find (a) the average velocity and (b) the average speed of the particle during the motion.

(a) The total displacement of the particle
$= 2(6\mathbf{i}+3\mathbf{j}-2\mathbf{k})+3(\mathbf{i}-2\mathbf{j}-2\mathbf{k}) = 15\mathbf{i}-10\mathbf{k}$
$\therefore$ the average velocity during the 5 seconds of the motion is $3\mathbf{i}-2\mathbf{k}$.
(b) The distance moved in the first 2 seconds
$= 2|6\mathbf{i}+3\mathbf{j}-2\mathbf{k}| = 2\sqrt{(6^2+3^2+2^2)} = 14.$
The distance moved in the next 3 seconds
$= 3|\mathbf{i}-2\mathbf{j}-2\mathbf{k}| = 3\sqrt{(1^2+2^2+2^2)} = 9$
Hence the particle moves 23 m in 5 seconds
$\therefore$ the average speed during the motion is $4{\cdot}6\ \mathrm{m\,s^{-1}}$.

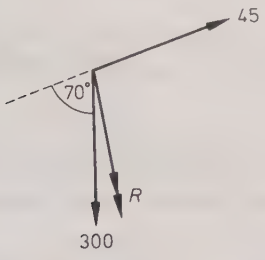
Example 3 A particle moves in a plane with constant acceleration given by $-2\mathbf{i}+\mathbf{j}$. If the initial velocity of the particle is $3\mathbf{i}$, find its speed after 1 second and after 3 seconds.

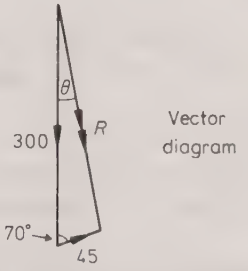
The velocity of the particle after 1 second
$= 3\mathbf{i}+(-2\mathbf{i}+\mathbf{j}) = \mathbf{i}+\mathbf{j}$
$\therefore$ the speed of the particle after 1 second
$= |\mathbf{i}+\mathbf{j}| = \sqrt{(1^2+1^2)} = \sqrt{2}.$

The velocity of the particle after 3 seconds
$= 3\mathbf{i}+3(-2\mathbf{i}+\mathbf{j}) = -3\mathbf{i}+3\mathbf{j}$
$\therefore$ the speed of the particle after 3 seconds
$= |-3\mathbf{i}+3\mathbf{j}| = 3|-\mathbf{i}+\mathbf{j}| = 3\sqrt{2}.$

Resultant velocities and accelerations can be found by the methods used to add forces in Chapter 2.

Example 4 An aircraft is heading due south with a velocity of 300 km/h. It is also being blown by a wind with velocity 45 km/h from the direction S 70° W. Find the velocity of the aircraft relative to the ground.





Vector diagram

Let R km/h be the resultant velocity and θ the angle it makes with due south.

By the cosine rule, $R^2 = 300^2+45^2-2.300.45\cos 70°$

$\therefore \quad R \approx 288$

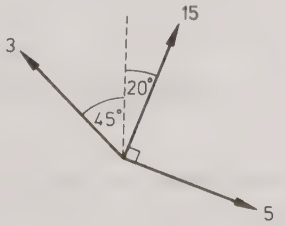
By the sine rule, $\dfrac{\sin\theta}{45} = \dfrac{\sin 70°}{R}$

$\therefore \quad \sin\theta = \dfrac{45\sin 70°}{R}$ and $\theta \approx 8{\cdot}5°.$

Hence the velocity of the aircraft relative to the ground is 288 km/h in the direction S 8·5° E.

Sums of three or more velocities can be found using a vector polygon. However, it is often more convenient to resolve the velocities in suitable directions and then add the components.

Example 5 A ship travelling through the water at 15 km/h is steering a course N 20° E. It is also being carried by a current with velocity 3 km/h to the north west. If a man is walking across the deck of the ship directly from port to starboard at 5 km/h, find his velocity relative to the shore.



Let U and V be the components of the man's velocity along the perpendicular to the course steered by the ship, then resolving in those directions,

$\uparrow \quad U = 15+3\cos 65°$

$\rightarrow \quad V = 5-3\sin 65°$

$R = \sqrt{(U^2 + V^2)} \approx 16{\cdot}4$
$\tan\theta = V/U \quad \therefore \quad \theta \approx 8°$
Hence the man's velocity relative to the shore is 16·4 km/h in the direction N 28° E.

Exercise 3.1

1. A particle starts from a point A with position vector $\mathbf{i}-4\mathbf{j}$ and moves with velocity $12\mathbf{i}+5\mathbf{j}$. Find the speed of the particle and its position vector after 2 seconds.

2. A particle moves from A to B in 5 seconds. If the position vectors of A and B are $7\mathbf{i}-12\mathbf{j}$ and $-3\mathbf{i}+8\mathbf{j}$ respectively, find the average velocity of the particle.

3. A particle starts from the origin and moves with velocity $6\mathbf{i}-3\mathbf{j}+6\mathbf{k}$ for 2 seconds, then with velocity $12\mathbf{i}+9\mathbf{j}$ for 1 second. Find the position vector of the particle. Find also the average velocity and the average speed of the particle during its motion.

4. A particle with position vector $40\mathbf{i}+10\mathbf{j}+20\mathbf{k}$ moves with constant speed $5\,\mathrm{m\,s^{-1}}$ in the direction of the vector $4\mathbf{i}+7\mathbf{j}+4\mathbf{k}$. Find its distance from the origin after 9 seconds.

5. A particle moves with constant acceleration given by the vector $\mathbf{i}+\mathbf{j}$. If the initial velocity of the particle is $2\mathbf{i}+5\mathbf{j}$, find its velocity after 3 seconds. Given that the speed of the particle after k seconds is $15\,\mathrm{m\,s^{-1}}$, find the value of k.

6. The initial velocity of a particle moving with constant acceleration is $2\mathbf{i}-4\mathbf{j}$. After 2 seconds the particle is moving in the opposite direction with double the speed. Find the acceleration of the particle.

7. If $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the directions due east and due north respectively, find the magnitude and direction of the resultant of the velocities represented by each of the following sets of vectors

(a) $5\mathbf{i}+3\mathbf{j}$, $4\mathbf{i}$, $3\mathbf{i}+2\mathbf{j}$,
(b) $\mathbf{i}-2\mathbf{j}$, $3\mathbf{i}-\mathbf{j}$, $2\mathbf{i}+5\mathbf{j}$,
(c) $7\mathbf{i}-5\mathbf{j}$, $-\mathbf{i}+2\mathbf{j}$, $2\mathbf{i}-\mathbf{j}$,
(d) $-2\mathbf{i}+\mathbf{j}$, $\mathbf{i}-6\mathbf{j}$, $-3\mathbf{i}-4\mathbf{j}$.

8. Find the magnitudes and directions of the resultants of the given velocities

(a) $7\,\mathrm{m\,s^{-1}}$ due north and $10\,\mathrm{m\,s^{-1}}$ N 75° E;
(b) 50 km/h N 40° E and 60 km/h S 25° E;
(c) $12\,\mathrm{m\,s^{-1}}$ N 20° E, $6\,\mathrm{m\,s^{-1}}$ due west and $5\,\mathrm{m\,s^{-1}}$ due north;
(d) 30 km/h S 45° W, 50 km/h S 30° E and 70 km/h due east.

9. A stretch of river has straight parallel banks 100 m apart. A boat is being rowed at $4\,m\,s^{-1}$ in a direction perpendicular to the banks. If the river is flowing at $1{\cdot}2\,m\,s^{-1}$, find (a) the magnitude of the resultant velocity of the boat, (b) the distance downstream that the boat is carried as it crosses from one bank to the other.

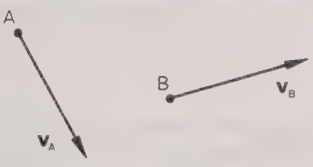
10. A ship travelling through the water at 12 km/h is steering a course S 75° E. It is also being carried by a current with velocity 2 km/h due E. If a passenger is walking across the deck of the ship directly from starboard to port at 4 km/h, find his velocity relative to the shore.

3.2 Relative velocity

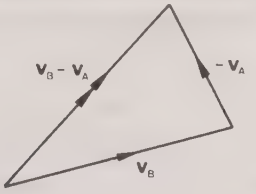
If A and B are two moving objects, then the apparent velocity of B when observed from A is called the velocity of B relative to A, i.e.

velocity of B relative to A = velocity of B − velocity of A.

Denoting the velocities of A and B by $\mathbf{v}_A$ and $\mathbf{v}_B$ respectively, the velocity of B relative to A is $\mathbf{v}_B - \mathbf{v}_A$ and may be found using a vector triangle.

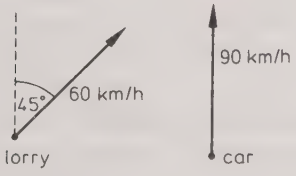


Space diagram

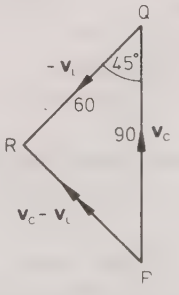


Vector diagram

Example 1 A car is travelling due north on a straight stretch of motorway at 90 km/h. The car is observed by the driver of a lorry travelling north-east on an approach road at 60 km/h. Find the apparent speed of the car.



Space diagram



Vector diagram

If the velocities of the car and the lorry are $\mathbf{v}_C$ and $\mathbf{v}_L$ respectively, then the velocity of the car relative to the lorry is $\mathbf{v}_C - \mathbf{v}_L$.

Using the cosine rule in the vector triangle PQR,

$$PR^2 = 90^2 + 60^2 - 2.90.60\cos 45^\circ \qquad \therefore \quad PR \approx 63{\cdot}7.$$

Hence the apparent speed of the car is 63·7 km/h.

It follows from the definition of relative velocity that for moving objects A and B,

velocity of B = velocity of B relative to A + velocity of A.

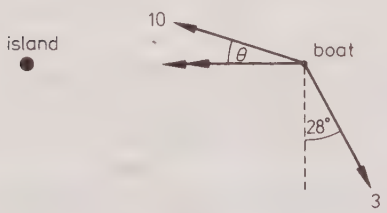
The advantage of using this result when solving problems is that it involves addition rather than subtraction of vectors.

Example 2 A man in a boat wishes to reach a small island due west of his present position. He knows that there is a current with a velocity of 3 km/h in the direction S 28°E. Given that the boat moves at 10 km/h through still water, find the course the man must steer to reach the island.

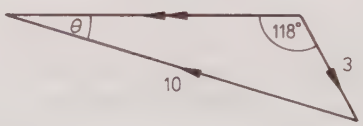
The true velocity of the boat

= velocity of boat relative to water + velocity of current.

If the boat is to reach the island its true velocity must be in the direction due west. Relative to the water the velocity of the boat is of magnitude 10 km/h. Let θ be the angle between these velocities.



Space diagram



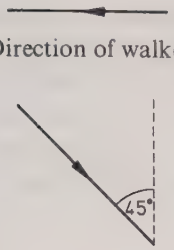
Vector triangle

By the sine rule, $\dfrac{\sin\theta}{3} = \dfrac{\sin 118^\circ}{10}$

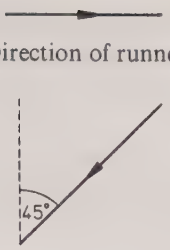
$$\therefore \quad \sin\theta = \frac{3\sin 118^\circ}{10} \quad \text{and} \quad \theta \approx 15{\cdot}4^\circ$$

Hence the man must steer the boat in the direction N 74·6° W.

Example 3 To a man walking due west at 5 km/h the wind appears to be blowing from the north-west. However, to a man running due east at 10 km/h the wind appears to be blowing from the north-east. Find the true velocity of the wind.



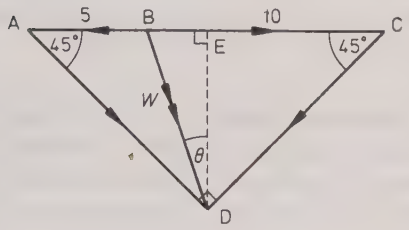
Apparent direction of wind to walker



Apparent direction of wind to runner

True magnitude and direction of wind

Velocity of wind
= velocity of wind relative to walker + velocity of walker
= velocity of wind relative to runner + velocity of runner



Vector diagram

From the symmetry of the diagram,

$$BE = 2{\cdot}5 \quad \text{and} \quad DE = 7{\cdot}5.$$

Using Pythagoras' theorem in $\triangle BDE$,

$$W^2 = (2{\cdot}5)^2 + (7{\cdot}5)^2$$

$$\therefore \quad W \approx 7{\cdot}9$$

$$\tan\theta = \frac{2{\cdot}5}{7{\cdot}5} = \frac{1}{3} \qquad \therefore \quad \theta \approx 18{\cdot}4^\circ$$

Hence the velocity of the wind is 7·9 km/h from the direction N 18·4° W.

Exercise 3.2

1. A man is walking due south at 5 km/h. The wind is blowing from the west at 8 km/h. Find the magnitude and direction of the velocity of the wind relative to the man.

2. A ship is sailing south-east at 20 km/h and a second ship is sailing due west at 25 km/h. Find the magnitude and direction of the velocity of the first ship relative to the second.

3. If particles A and B have velocity vectors $7\mathbf{i} - 5\mathbf{j}$ and $-2\mathbf{i} + 7\mathbf{j}$ respectively, find the magnitude of the velocity of B relative to A and the angle its direction makes with the direction of $\mathbf{i}$.

4. Particles A and B have velocity vectors $3\mathbf{i} - 11\mathbf{j}$ and $5\mathbf{i} + \mathbf{j}$ respectively. The velocity of a particle C relative to A is $-2\mathbf{i} + 7\mathbf{j}$. Find, in vector form, the velocity of C and the velocity of B relative to C.

5. A boat A travels due west at a speed of 30 km/h. The velocity of a boat B relative to A is 14 km/h due south. Find the speed of boat B and the direction in which it is moving.

6. To an observer on a ship P steaming at 20 km/h in the direction due west, a ship Q appears to be steaming at 20 km/h in the direction N 30° W. Find the true velocity of Q.

7. To a cyclist riding due east at 15 km/h the wind appears to be blowing from the north-east at 12 km/h. Find the magnitude and direction of the true velocity of the wind.

8. To the driver of a car travelling due north at 40 km/h the wind appears to be blowing at 50 km/h from the direction N 60° E. The wind velocity remains constant, but the speed of the car is increasing. Find its speed when the wind appears to be blowing from the direction N 30° E.

9. A river with straight parallel banks 400 m apart flows due south at 3·5 km/h. Find the direction in which a boat, travelling at 12·5 km/h relative to the water, must be steered in order to cross the river from east to west along a course perpendicular to the banks. Find also the time taken to make the crossing.

10. A stretch of river has straight parallel banks 120 m apart. A man rowing at $5\,\mathrm{m\,s^{-1}}$ relative to the water and heading directly across the river reaches the other bank 54 m downstream. Find the speed of the current and the direction in which the man would have to row in order to cross the river at right angles to the banks.

11. A helicopter whose speed in still air is 60 km/h is to fly to a village 80 km away in the direction N 40° W. The wind is blowing at 10 km/h from the west. Find the direction in which the helicopter must fly and the time taken to reach the village to the nearest minute.

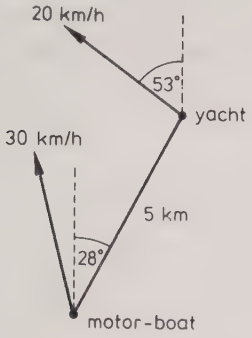
12. A plane flies in a straight line from A to B, where $AB = 580$ km and the bearing of B from A is N 50° E. The speed of the plane in still air is 300 km/h. Given that the wind is blowing from the direction N 40° W and that the flight takes 2 hours, calculate the speed of the wind and the course that should be set for the return journey, if the wind velocity remains unchanged.

3.3 Relative displacement

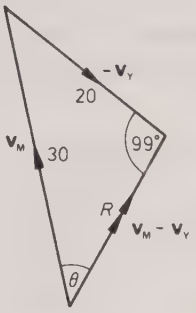
Some problems involve the positions as well as the velocities of objects in motion. In such cases it is usually best to select one of the objects and work with displacements and velocities relative to it.

If A and B are two moving objects then the displacement of B relative to A is given by the displacement vector **AB**. The velocity of B relative to A can be considered to be the rate of change of **AB**. The two objects can meet only if the relative velocity is parallel to the relative displacement.

Example 1 A motor-boat capable of 30 km/h wishes to intercept a yacht 5 km away on a bearing N 28° E. If the yacht is travelling at 20 km/h in the direction N 53° W, what course should the motor-boat take? Find also the time taken to reach the yacht.



Space diagram



Vector diagram

If the velocities of the yacht and the motor-boat are $\mathbf{v}_Y$ and $\mathbf{v}_M$ respectively, then the velocity of the motor-boat relative to the yacht is $\mathbf{v}_M - \mathbf{v}_Y$ and must lie in the direction N 28° E. Let R be the magnitude of this relative velocity and θ the angle it makes with the course of the motor-boat.

By the sine rule, $$\frac{\sin\theta}{20} = \frac{\sin 99^\circ}{30}$$

$$\therefore \quad \sin\theta = \frac{20\sin 99^\circ}{30} \quad \text{and} \quad \theta \approx 41{\cdot}182^\circ.$$

Hence the motor-boat must be steered in the direction N 13·2° W.
The remaining angle in the vector triangle is 39·818°.

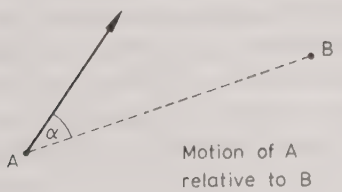
$$\therefore \quad \text{by the sine rule,} \quad \frac{R}{\sin 39{\cdot}818^\circ} = \frac{30}{\sin 99^\circ}$$

$$\therefore \quad R = \frac{30\sin 39{\cdot}818^\circ}{\sin 99^\circ} \approx 19{\cdot}45.$$

Thus on the course N 13·2° W the motor-boat can approach the yacht at a relative speed of 19·45 km/h

$$\therefore \quad \text{the time taken in minutes} = \frac{5}{19{\cdot}45} \times 60 \approx 15{\cdot}4.$$

Hence the motor-boat can reach the yacht in 15·4 minutes.

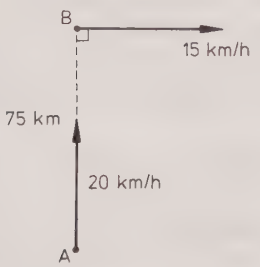


Motion of A relative to B

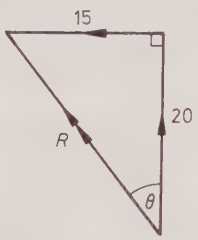
We consider now two objects A and B with given speeds and positions. We assume that the direction of B is such that the objects can never meet. If α is the angle between the velocity of A relative to B and the initial displace-

ment vector **AB**, then A will approach B as closely as possible when α takes its smallest possible value.

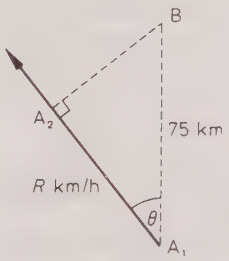
Example 2 A ship A is sailing due north at 20 km/h and a ship B is sailing due east at 15 km/h. At 12.00 hours A is 75 km due south of B. Find the shortest distance between the ships and the time at which they are closest, assuming that their velocities remain unchanged.



Space diagram



Vector diagram



Motion of A relative to B

Let the velocity of A relative to B be R km/h at an angle θ west of north.

Using the vector diagram, $R = \sqrt{(15^2 + 20^2)} = 25$

$\therefore \quad \sin\theta = \dfrac{15}{25} = \dfrac{3}{5}$ and $\cos\theta = \dfrac{20}{25} = \dfrac{4}{5}$.

Let A_1 be the initial position of A and let A_2 be its position when closest to B. By considering the motion of A relative to B, we see that A_2B must be perpendicular to A's relative path

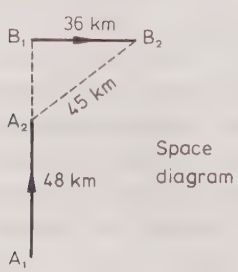
$\therefore$ the shortest distance between the two ships is $75\sin\theta$ km, i.e. 45 km.

Relative to B the apparent distance between A_1 and A_2 is $75\cos\theta$ km, i.e. 60 km.

$\therefore$ the time A takes to move from A_1 to A_2

$$= \frac{\text{relative distance}}{\text{relative speed}} = \frac{60}{25}\text{hours} = 2 \text{ hours } 24 \text{ min.}$$

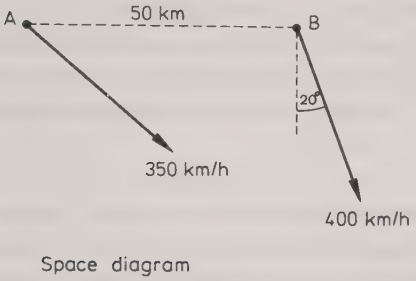
Hence the ships are closest together at 14.24 hours.



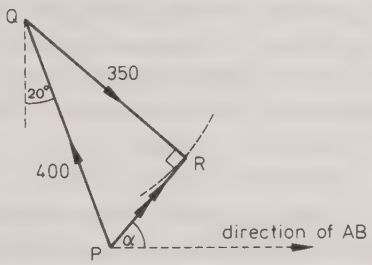
Space diagram

[A further diagram can be drawn to show the true paths of A and B. The true distance A_1A_2 is 48 km and the distance moved by B is 36 km.]

Example 3 An aircraft A flying at a speed of 350 km/h is 50 km due west of another aircraft B. If B is flying at the same height on a course S 20° E at a speed of 400 km/h, find the direction in which A must fly to approach B as closely as possible. Find also the shortest distance between the two aircraft.



Space diagram



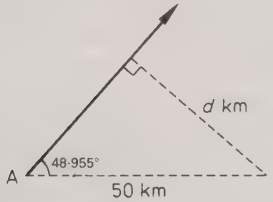
Vector diagram

In the vector diagram $\vec{QR}$ represents the velocity of A, $\vec{QP}$ represents the velocity of B and $\vec{PR}$ represents the velocity of A relative to B. Considering all possible directions for $\vec{QR}$, the smallest possible value of α occurs when $\angle PRQ$ is a right angle.

Hence $\angle PQR = \cos^{-1}(350/400) \approx 28{\cdot}955°$

$\therefore$ A must fly in the direction S 49° E to approach B as closely as possible.

Using the vector diagram, $\alpha = 49°$.



Let d km be the shortest distance between the aircraft, then by considering the direction of A's motion relative to B,

$d = 50 \sin 48{\cdot}955° \approx 37{\cdot}7$

$\therefore$ the shortest distance between the aircraft is 37·7 km.

Exercise 3.3

1. A destroyer must intercept a vessel 40 km away on a bearing of 110°. The vessel is sailing on a course of 200° at a speed of 20 km/h. If the speed of the destroyer is 30 km/h, find, to the nearest degree, the course it must steer. Find also the time taken for the destroyer to reach the vessel.

2. A man walking due north wishes to intercept a cyclist travelling due west at 12 km/h. The bearing of the cyclist from the man is N 60° E. Find the speed at which the man must walk if he does not change his direction. If the man walks at 8 km/h, find the direction in which he must go to intercept the cyclist as quickly as possible. What is the minimum speed at which the man can walk in order to intercept the cyclist?

3. A ship P is sailing due west at 16 km/h and a ship Q is sailing due north at 24 km/h. At a certain instant Q is 10 km due west of P. Find the velocity of P relative to Q and the shortest distance between the ships if they continue on these courses.

4. A cyclist A is riding along a straight level road on a bearing 030° at a steady speed of $5\,\mathrm{m\,s^{-1}}$. A second cyclist B is travelling due east along another straight level road at a steady speed of $6\,\mathrm{m\,s^{-1}}$. Find the magnitude and the direction of the velocity of A relative to B. At a certain instant A is 500 m due south of B. Find the time that elapses before A is due west of B. (JMB)

5. Two cyclists A and B are approaching a road junction at speeds of 20 km/h and 15 km/h respectively. A is 100 m south of the junction riding due north, when B reaches the junction riding due east. Find the velocity of B relative to A. Find also the shortest distance between A and B, assuming their velocities remain unchanged.

6. At 10.00 hours a plane A passes through a point X travelling due south at 360 km/h. At the same instant a plane B is 50 km due west of X travelling at 450 km/h towards X. Find the magnitude and direction of the velocity of B relative to A and the shortest distance between A and B. Find also the time at which the planes are closest and the bearing of B from A at that time.

7. Two planes A and B, flying at the same height, are travelling at 500 km/h in the directions due north and due east respectively. At a certain instant A is 100 km from B on a bearing of 150°. Find the velocity of B relative to A and the shortest distance between the planes if neither changes course.

8. (In this question the units of time and distance are the second and metre respectively.) Two particles A and B start simultaneously from points which have position vectors $3\mathbf{i}+4\mathbf{j}$ and $8\mathbf{i}+4\mathbf{j}$ respectively. Each moves with constant velocity, the velocity of B being $\mathbf{i}+4\mathbf{j}$. (i) If the velocity of A is $5\mathbf{i}+2\mathbf{j}$, show that the least distance between A and B subsequently is $\sqrt{5}$ metres. (ii) If the speed of A is 5 m/s and the particles collide, find the velocity of A. (L)

9. A destroyer sights a ship travelling with constant velocity $5\mathbf{j}$, whose position vector at the time of sighting is $2000(3\mathbf{i}+\mathbf{j})$ relative to the destroyer, distances being measured in m and speeds in $\mathrm{m\,s^{-1}}$. The destroyer immediately begins to move with velocity $k(4\mathbf{i}+3\mathbf{j})$, where k is a constant, in order to intercept the ship. Find k and the time to interception. Find also the distance between the vessels when half the time to interception has elapsed. (O&C)

10. A ship P sailing at 16 km/h is 10 km due south of a second ship Q. If Q is sailing in the direction S 70° E at a speed of 20 km/h, find the direction in which P must sail to approach Q as closely as possible. Find also the shortest distance between the ships.

Exercise 3.4 (miscellaneous)

[1 knot = 1 nautical mile per hour.]

1. During a race between two yachts, A and B, there is a wind of 9 knots blowing from due north. The resultant velocity of A is 6 knots on a bearing of 060°. Find the direction of the wind relative to A. At the same time, the resultant velocity of B is 6 knots on a bearing of 300°. Find, correct to the nearest degree, the direction of the wind relative to B and, in knots correct to one decimal place, the velocity of A relative to B. (L)

2. A yacht A is sailing due east at 9 knots and a second yacht B is sailing on a bearing of 030° at 6 knots. At a certain instant a third yacht C appears to an observer on A to be sailing due south and appears to an observer on B to be sailing on a bearing of 150°. Find the speed of the yacht C and the bearing on which it is sailing. (L)

3. A river flows at 5 m/s from west to east between parallel banks which are at a distance 300 metres apart. A man rows a boat at a speed of 3 m/s in still water. (i) State the direction in which the boat must be steered in order to cross the river from the southern bank to the northern bank in the shortest possible time. Find the time taken and the actual distance covered by the boat for this crossing. (ii) Find the direction in which the boat must be steered in order to cross the river from the southern bank to the northern bank by the shortest possible route. Find the time taken and the actual distance covered by the boat for this crossing. (AEB 1978)

4. A river flows at a constant speed of 5 m/s between straight parallel banks which are 300 m apart. A boat, which has a maximum speed of $3\frac{1}{4}$ m/s in still water, leaves a point A on one bank and sails in a straight line to the opposite bank. Find, graphically or otherwise, the least time the boat can take to reach a point B on the opposite bank where $AB = 500$ m and B is downstream from A. Find also the least time the boat can take to cross the river. Find the time taken to sail from A to B by the slowest boat capable of sailing directly from A to B. (L)

5. A helicopter flies with constant airspeed 100 knots from position A to position B, which is 50 nautical miles north east of A, and then flies back to A. Throughout the whole flight the wind velocity is 30 knots from the west. Find, by drawing or calculation, the course set for each of the two legs of the flight. Find also the total time of flight from A to B and back. (L)

6. A motorist A is travelling in a direction N 10° E at a constant speed of 60 km h^{-1}. The wind is blowing from the direction S 80° E at a constant speed of 25 km h^{-1}. Find the magnitude and direction of the velocity of the wind relative to the motorist. A second motorist B is travelling at a constant speed of 50 km h^{-1}. The velocity of B relative to A is in a direction S 30° W. Find graphically, or otherwise, the two possible directions in which B can be travelling. (C)

7. Three fixed buoys are at the vertices A, B and C of an equilateral triangle of side 12 km. The point B is due north of A, and C is to the east of the line AB. A steady current of speed $3\,\text{km h}^{-1}$ flows from west to east. A boat which has a top speed of $8\,\text{km h}^{-1}$ in still water does the triangular journey $ABCA$ at top speed. Find, graphically or otherwise, the time taken on each leg of the journey, giving your answers to the nearest minute. (C)

8. A helicopter, with a maximum speed in still air of $120\,\text{km h}^{-1}$, is to fly between airports A and B. B is 238 km from A in a direction N θ° W where $\theta = \cos^{-1}\frac{12}{13}$. The helicopter leaves A at 12.00 hours in a wind blowing at $50\,\text{km h}^{-1}$ from the west. What is the earliest possible time of arrival at B, and the corresponding direction in which it must head? At 13.00 hours the helicopter receives a radio message not to proceed but to return to A as soon as possible. Neglecting the time to change course, what is the earliest time it can be back at A? (W)

9. A small boat fitted with an outboard engine is 10 nautical miles due east of a trawler. The trawler is travelling at 15 knots on a course N θ° E $[\theta = \sin^{-1}(24/25)]$. By considering the trawler's velocity components due east and due north, or otherwise, show that the minimum speed necessary for the boat to intercept the trawler is 4·2 knots, and find the corresponding direction in which it must steer. If the boat travels at 9 knots, show that it can steer in either of two directions to reach the trawler and calculate the shorter time needed to do so. (W)

10. [In this question the units of distance and time are the metre and second respectively.] At time $t = 0$ a particle A is at the origin and a particle B is at the point with position vector $10\mathbf{i}$. Each particle is moving with constant velocity, the velocity of B being $-4\mathbf{i} + 3\mathbf{j}$. Subsequently the particles collide. Find (a) the least speed at which A can travel, (b) the velocity of A if the collision occurs at time $t = 2$. A third particle C is moving in the same plane. When the velocity of A is $\mathbf{i} + 3\mathbf{j}$ the velocity of C relative to A is in the direction of the vector $3\mathbf{i} + 2\mathbf{j}$ and relative to B is in the direction of the vector $7\mathbf{i} + 3\mathbf{j}$. Find the speed of C. (L)

11. A motorist A and a cyclist B are travelling along straight roads which cross at right angles at a point O. A is travelling at a constant speed of $48\,\text{km h}^{-1}$ towards the east, and B at $14\,\text{km h}^{-1}$ towards the north. At the moment that B is passing through O, A is 400 m from O and has not yet passed O. Calculate: (i) the velocity of B relative to A in magnitude and direction; (ii) the least distance between car and cycle; (iii) the distances of the car and cycle from O when they are nearest to one another. (SU)

12. A cruiser is moving due east at 30 km/h. Relative to the cruiser a frigate is moving on a course of 210° (S 30° W) at 48 km/h. Using a graphical method, or otherwise, find the magnitude and the direction of the velocity of the frigate relative to a coastguard who is recording the paths of these ships from a lighthouse. At 1300 hours the frigate is 10 km due east of the cruiser. If both ships

maintain their speeds and courses, find the time at which the distance between them is least and their actual distance apart at this instant. Find also the time at which the frigate is due south of the cruiser. (AEB 1977)

13. A ship A whose full speed is 40 km/h is 20 km due west of a ship B which is travelling uniformly with speed 30 km/h in a direction due north. The ship A travels at full speed on a course chosen so as to intercept B as soon as possible. Find the direction of this course and calculate to the nearest minute the time A would take to reach B. When half of this time has elapsed the ship A has engine failure and thereafter proceeds at half speed. Find the course which A should then set in order to approach as close as possible to B, and calculate the distance of closest approach (in kilometres to 2 decimal places). (JMB)

14. A ship A is travelling with constant velocity 20 km/h due east and a ship B has constant velocity 15 km/h in a direction 30° east of north. At noon B is 30 km due south of A. Find the magnitude of the velocity of B relative to A, and show that the direction of this relative velocity is approximately 44° west of north. Find, to the nearest minute, the time at which A and B are closest. At the time of closest approach a boat leaves A to intercept B. Find the least speed at which it must travel. (JMB)

15. Construct a flow chart for a procedure in which the input is the speeds and courses of two ships, A and B, in the form x km/h on a bearing $y°$. Assuming that initially ship A is 20 km due south of ship B, the following output is required:

(a) the time and distance of closest approach or an appropriate message if the ships are not moving closer to each other;

(b) the course A should steer to intercept B as soon as possible and the time of interception;

(c) if interception is not possible, the course A should steer to approach B as closely as possible.

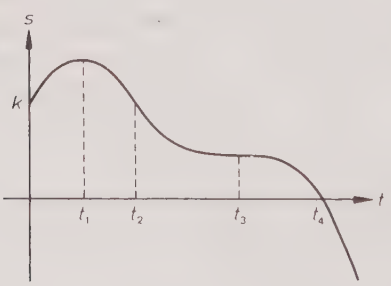
[It may be helpful to consider velocity components due east and due north.]

4 Velocity and acceleration

4.1 Elementary kinematics

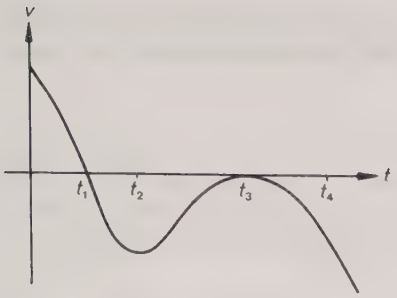
We consider now the motion of a particle in a straight line. Since in this type of motion the displacements, velocities and accelerations are parallel, there is no need to use vector notation. In general the displacement of the particle from a fixed point on the line after t seconds is denoted by s m, the velocity by v m s^{-1} and the acceleration by a m s^{-2}. This means that, $v = \frac{ds}{dt}$ and $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$. Thus if a displacement-time graph is plotted the velocity is represented by the gradient of the curve. When the gradient is positive the motion of the particle is in the positive sense along the straight line and when the gradient is negative motion is in the opposite direction. At stationary points on the curve the velocity is zero and the particle is instantaneously at rest. Similarly the gradient of the velocity-time graph represents the acceleration of the particle.

Consider, for instance, the motion of a particle in a straight line as represented by the graphs given below. The first graph shows the displacement s from a fixed point O at time t and the second is the corresponding velocity-time graph.



At the start of the motion the particle is at a distance k from O, moving away from O with positive velocity. It comes to rest at time t_1, then starts moving towards O. Its speed increases to a maximum value at $t = t_2$, then decreases so that the particle is again at rest when $t = t_3$. It then continues to move with negative velocity as t increases, passing through O at time t_4.

Considering the gradient of the velocity-time graph, we see that a negative acceleration produces the initial



reduction in speed, the change of direction at time t_1, then an increasing speed towards O. The minimum point on the graph at $t = t_2$ corresponds to a maximum speed of the particle in the negative direction. The positive acceleration from $t = t_2$ to $t = t_3$ brings the particle to rest. The motion then continues with velocity and acceleration both negative and hence with increasing speed in the negative direction.

Example 1 A particle is moving in a straight line, so that after t seconds its displacement s metres from a fixed point O is given by $s = 9t + 3t^2 - t^3$. Find (a) the initial displacement, velocity and acceleration of the particle, (b) the time at which the particle is instantaneously at rest, (c) the greatest speed attained by the particle in the first 4 seconds of the motion.

Differentiating, $$v = \frac{ds}{dt} = 9 + 6t - 3t^2$$

$$a = \frac{dv}{dt} = 6 - 6t.$$

(a) When $t = 0$, $s = 0$, $v = 9$ and $a = 6$

$\therefore$ initially the particle is at O moving with velocity $9\,\mathrm{m\,s^{-1}}$ and acceleration $6\,\mathrm{m\,s^{-2}}$ (both in the positive direction along the line of motion).

(b) $v = 0 \Leftrightarrow 9 + 6t - 3t^2 = 0 \Leftrightarrow 3(3 - t)(1 + t) = 0$

$\therefore$ $v = 0$ when $t = 3$ (assuming $t \geqslant 0$).

Hence the particle is instantaneously at rest after 3 seconds.

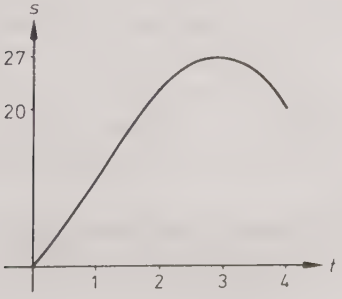
(c) $a = 0 \Leftrightarrow 6 - 6t = 0 \Leftrightarrow t = 1$.

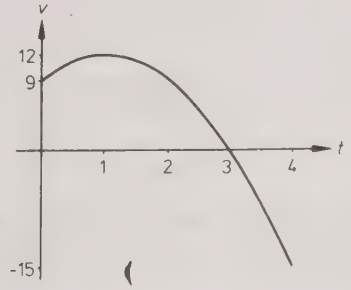
When $t < 1$, $a > 0$ and when $t > 1$, $a < 0$

$\therefore$ when $t = 1$, the velocity reaches a maximum value of $12\,\mathrm{m\,s^{-1}}$.

The velocity then decreases during the remainder of the motion. Since the particle is at rest when $t = 3$, it must move with increasing speed in the negative direction for $t > 3$. When $t = 4$, $v = -15$ $\therefore$ the greatest speed attained by the particle in the first 4 seconds of the motion is $15\,\mathrm{m\,s^{-1}}$.

[These results are illustrated in the sketch graphs below.]





When the acceleration of a particle is represented by a known function of the time t, then expressions for the velocity and displacement can often be found by integration.

Example 2 A vehicle moves in a straight line so that its acceleration after t seconds is given by $a = 2t$. Initially it has a velocity of $20\,\mathrm{m\,s^{-1}}$. Find an expression for the displacement s metres of the vehicle from its starting point.

$$a = \frac{dv}{dt} = 2t$$

Integrating, $v = t^2 + c$

but $v = 20$ when $t = 0$, thus $c = 20$

$\therefore$ $v = \dfrac{ds}{dt} = t^2 + 20.$

Integrating, $s = \frac{1}{3}t^3 + 20t + k$

but $s = 0$ when $t = 0$, thus $k = 0$

$\therefore$ $s = \frac{1}{3}t^3 + 20t.$

The next example shows that the total displacement of a particle in any given interval of time is represented by the area under the velocity-time graph and that this can be found using a definite integral.

Example 3 A particle moves in a straight line so that after t seconds its velocity in $\mathrm{m\,s^{-1}}$ is given by $v = (2t^2 - 25)(t - 1)$. Find (a) the displacement of the particle from its starting point after 3 seconds, (b) the total distance moved by the particle in the first 3 seconds of the motion.

(a) The displacement of the particle from its starting point after 3 seconds (in metres)

$$= \int_0^3 v\,dt = \int_0^3 (2t^2 - 25)(t - 1)\,dt$$

$$= \int_0^3 (2t^3 - 2t^2 - 25t + 25)\,dt$$

$$= \left[\frac{1}{2}t^4 - \frac{2}{3}t^3 - \frac{25}{2}t^2 + 25t\right]_0^3$$

$$= \left(\frac{1}{2}.81 - \frac{2}{3}.27 - \frac{25}{2}.9 + 25.3\right) - 0$$

$$= 40\tfrac{1}{2} - 18 - 112\tfrac{1}{2} + 75 = -15.$$

(b) Since $v = 0$ when $t = 1$, the particle is instantaneously at rest after 1 second.

For $0 < t < 1$, $v > 0$ and for $1 < t < 3$, $v < 0$.

Hence the direction of motion changes once in the first 3 seconds, i.e. when $t = 1$.

$$\int_0^1 v\,dt = \left[\frac{1}{2}t^4 - \frac{2}{3}t^3 - \frac{25}{2}t^2 + 25t\right]_0^1$$

$$= \left(\frac{1}{2} - \frac{2}{3} - \frac{25}{2} + 25\right) - 0 = \frac{1}{2} - \frac{2}{3} - 12\tfrac{1}{2} + 25 = 12\tfrac{1}{3}$$

$$\int_1^3 v\,dt = \left[\frac{1}{2}t^4 - \frac{2}{3}t^3 - \frac{25}{2}t^2 + 25t\right]_1^3$$

$$= \left(\frac{1}{2}.81 - \frac{2}{3}.27 - \frac{25}{2}.9 + 25.3\right) - \left(\frac{1}{2} - \frac{2}{3} - \frac{25}{2} + 25\right)$$

$$= -15 - 12\tfrac{1}{3} = -27\tfrac{1}{3}$$

$\therefore$ the total distance travelled in the first 3 seconds of the motion is $39\frac{2}{3}$ metres.

Exercise 4.1

Each of the questions in this exercise concerns a particle P moving along a straight line such that after t seconds its displacement from a fixed point A is s m, its velocity is $v\,\mathrm{m\,s^{-1}}$ and its acceleration is $a\,\mathrm{m\,s^{-2}}$.

1. Find expressions in terms of t for v and a, given that
(a) $s = 6-5t+3t^3$, (b) $s = (2t-1)(t+2)$.

2. Find expressions in terms of t for v and s, given that
(a) $a = 12t$ and when $t = 0$, $v = 7$ and $s = 0$,
(b) $a = -10$ and when $t = 0$, $v = 20$ and $s = 5$,
(c) $a = 24t(t-1)$ and when $t = 1$, $v = s = 0$.

3. If $v = 2t^3 - 9t^2$, find the values of t for which $a = 0$. Given that $s = 20$ when $t = 0$, find the value of s when $t = 2$.

4. If $v = 12+3t^2$, find the value of v when $a = 12$. Find also the distance travelled in the first two seconds of the motion.

5. The particle P starts from rest at A and moves such that $a = 6-2t$. If the particle later comes to rest at a point B, find the distance AB and the greatest speed attained while moving from A to B.

6. Given that the initial velocity of the particle P is $9\,\mathrm{m\,s^{-1}}$ and that $a = 2t - 10$, find the times when the particle is instantaneously at rest and the maximum speed attained by the particle between these two times.

7. Given that $a = 3t$ and that $v = 5$ when $t = 0$, find the value of v when $t = 4$ and the distance travelled in the first four seconds of the motion.

8. If $s = t^3 - 12t$, find the initial velocity of the particle P and the distance between its positions when $t = 0$ and $t = 4$. Show that P changes direction between $t = 0$ and $t = 4$, then find the total distance travelled in this interval of time.

9. Given that $v = 25 - 4t^2$, calculate (a) the value of t when the particle is instantaneously at rest, (b) the distance travelled by the particle in the third second.

10. Given that $s = t^3 - 2t^2 - 15t$, find (a) the initial velocity and acceleration of P, (b) the times at which P passes through A, (c) the total distance travelled in the first five seconds of the motion.

11. The particle P starts from rest at the point A and is again instantaneously at rest after 3 seconds. If $a = k - 6t$, where k is constant, find (a) expressions for v and s in terms of t, (b) the distance P travels before returning to A.

12. Given that $s = 6t^2 - t^3$, sketch the displacement-time and velocity-time graphs for the motion of the particle P from $t = 0$ to $t = 6$. Find, in the interval $0 \leqslant t \leqslant 6$,
(a) the times at which the particle is at A,
(b) the greatest value of the distance AP,
(c) the total distance travelled,
(d) the greatest speed attained.

13. Repeat question 12 for $s = t^2 - 5t + 4$.

14. Repeat question 12 for $s = 4t^3 - 33t^2 + 54t$.

15. Given that when $t = 1$, $s = 0$ and that when $t > 1$, $v = t^2 + \frac{9}{t^2}$, find the values of a, v and s when $t = 3$. Find also the minimum speed of the particle for $t > 1$.

16. If the particle P starts from rest and moves such that $v = 4t^2(3 - t)$, find (a) the distance travelled by the particle before it next comes to rest, (b) the maximum speed attained by the particle in the first three seconds of its motion, (c) the maximum acceleration in the first two seconds of its motion.

4.2 Motion with constant acceleration

We now obtain general equations for the motion of a particle in a straight line with constant acceleration. These will then be used to solve a variety of problems.

Consider a particle with initial velocity u and constant acceleration a. Let its displacement from its initial position be s and its velocity v at time t.

$$\frac{dv}{dt} = a \quad \text{where } a \text{ is constant}$$

Integrating $v = at + c$

When $t = 0, v = u$ $\therefore \quad c = u$

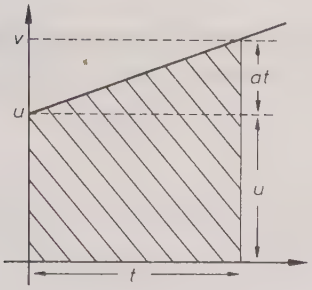
Thus $v = u + at$ (1)

and $\dfrac{ds}{dt} = u + at$

Integrating $s = ut + \frac{1}{2}at^2 + k$

When $t = 0, s = 0$ $\therefore \quad k = 0$

Thus $s = ut + \frac{1}{2}at^2$ (2)



Results (1) and (2) can also be obtained by considering the velocity-time graph for the motion. As shown in the sketch this is a straight line with gradient a. The displacement s is represented by the shaded area under the line.

Rearranging (1) $at = v - u$

Substituting in (2) $s = ut + \frac{1}{2}(v-u)t$

$\therefore \quad s = \frac{1}{2}(u+v)t$ (3)

Hence $2as = (u+v)at = (u+v)(v-u) = v^2 - u^2$

$\therefore \quad v^2 = u^2 + 2as.$ (4)

Eliminating u from equations (1) and (2),

$$s = vt - \tfrac{1}{2}at^2. \quad (5)$$

We now have five equations connecting the quantities s, u, v, a and t. When using them it is advisable to check that the acceleration is constant and that the units of measurement are consistent.

$$\begin{aligned} v &= u + at \\ s &= ut + \tfrac{1}{2}at^2 \\ s &= vt - \tfrac{1}{2}at^2 \\ s &= \tfrac{1}{2}(u+v)t \\ v^2 &= u^2 + 2as. \end{aligned}$$

Example 1 A particle moving in a straight line accelerates uniformly from rest to a speed of $60\,\text{m s}^{-1}$ in 10 seconds. Find the distance travelled in this time.

Given: $u = 0, v = 60, t = 10.$

Using the equation $s = \frac{1}{2}(u+v)t$

$s = \frac{1}{2}(0+60)10 = 300.$

Hence the particle travelled a distance of 300 metres.

Example 2 A train travelling in a straight line is brought to rest in 1 minute with a uniform retardation of $0{\cdot}5\,\mathrm{m\,s^{-2}}$. Find the speed at which the train was travelling in km/h.

Given: $v = 0, a = -0{\cdot}5, t = 60.$

Using the equation $v = u+at$

$0 = u-0{\cdot}5 \times 60 \quad \therefore \quad u = 30.$

Hence the speed of the train $= 30\,\mathrm{m\,s^{-1}}$

$$= \frac{30 \times 60 \times 60}{1000}\,\mathrm{km/h} = 108\,\mathrm{km/h}.$$

Example 3 A particle moving in a straight line with constant acceleration travels 10 m in 2 seconds, then a further 30 m in the following 2 seconds. Find the acceleration of the particle.

When $s = 10, t = 2$ and when $s = 40, t = 4$

$\therefore$ using the equation $s = ut+\frac{1}{2}at^2, \quad 10 = 2u+2a$

$40 = 4u+8a$

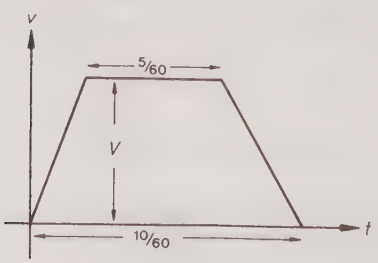
Simplifying $u+a = 5$

$u+2a = 10 \qquad \therefore \quad a = 5.$

Hence the acceleration of the particle is $5\,\mathrm{m\,s^{-2}}$.

Although the constant acceleration formulae are very useful, some problems are better solved by drawing a sketch of the velocity-time graph.

Example 4 A train takes 10 minutes to travel a distance of 12 km between two stations. It starts from rest at the first station, accelerating uniformly until it reaches a speed of V km/h. It travels at this speed for 5 minutes then decelerates uniformly, coming to rest at the second station. Find the value of V.



Consider the velocity-time graph for the motion, taking the units of distance and time as kilometres and hours.

Since the area of the trapezium under the graph represents the distance travelled,

$$\frac{1}{2}\left(\frac{5}{60}+\frac{10}{60}\right)V = 12, \quad \text{i.e. } \frac{1}{8}V = 12.$$

Hence the value of V is 96.

An object falling freely under gravity has a downward acceleration usually denoted by g. The magnitude of this acceleration varies slightly over the surface of the earth. In m s^{-2} the variation is from about 9·78 at the equator to 9·83 at the poles. [On the moon the value of g is 1·62.] Although the pull of gravity on an object decreases as its distance from the earth's surface increases, this decrease can, in general, be regarded as negligible. Thus in most problems g is taken to be a constant with value 9·8 or 10 depending on the accuracy required.

Example 5 A boy standing at the edge of a cliff throws a stone vertically upwards with a velocity of $10\,\text{m s}^{-1}$. If the stone hits the sea below with a velocity of $40\,\text{m s}^{-1}$, find the height of the cliff above sea-level and the maximum height above sea-level reached by the stone. [Take $g = 10$.]

Taking the upward direction as positive, $u = 10$, $v = -40$ and $a = -10$.

Using the equation $v^2 = u^2 + 2as$

$$1600 = 100 + 2(-10)s$$
$$1500 = -20s \qquad \therefore \quad s = -75.$$

Hence the height of the cliff above sea-level is 75 m.

When the stone reaches its maximum height its velocity is zero
$\therefore \quad u = 10$, $v = 0$ and $a = -10$.

Using the equation $v^2 = u^2 + 2as$

$$0 = 100 + 2(-10)s \qquad \therefore \quad s = 5.$$

Hence the maximum height reached by the stone is 5 m above the cliff, i.e. 80 m above sea-level.

Example 6 A particle is projected vertically upwards with a speed of $30\,\text{m s}^{-1}$. After 3 seconds another particle is projected vertically upwards from the same point with a speed of $25\,\text{m s}^{-1}$. Find the height at which the two particles collide. [Take $g = 10$.]

Let us assume that the collision occurs at a height of h m and T seconds after the projection of the second particle, then taking the upward direction as positive,

for the first particle: $s = h$, $u = 30$, $a = -10$, $t = T+3$
for the second particle: $s = h$, $u = 25$, $a = -10$, $t = T$.

Using the equation $s = ut + \frac{1}{2}at^2$ in both cases

$$h = 30(T+3) + \tfrac{1}{2}(-10)(T+3)^2$$
$$= 30T + 90 - 5T^2 - 30T - 45$$

i.e. $h = 45 - 5T^2$ (1)

and $h = 25T + \frac{1}{2}(-10)T^2$

i.e. $h = 25T - 5T^2$. (2)

Hence $25T = 45$ so that $T = 9/5$.

Substituting in (1) $h = 45 - 5\left(\frac{9}{5}\right)^2 = 28{\cdot}8$

$\therefore$ the particles collide at a height of 28·8 metres.

Exercise 4.2

[Take g as $10\,\mathrm{m\,s^{-2}}$.]

In questions 1 to 10 a particle is moving in a straight line with constant acceleration a and initial velocity u. At time t its displacement from its initial position is s and its velocity is v.

1. If $u = 5, a = 2, t = 3$, find v and s.

2. If $v = 7, u = 5, a = 1$, find s and t.

3. If $u = 10, v = -2, t = 4$, find s and a.

4. If $s = 15, t = 5, u = 13$, find v and a.

5. If $s = -5, v = 2, u = -3$, find a and t.

6. If $t = 3, a = -8, s = 36$, find v and u.

7. If $u = 50, s = 300, a = -4$, find t and v.

8. If $a = 0{\cdot}4, t = 15, v = 7$, find u and s.

9. If $v = -18, s = -64, t = 8$, find a and u.

10. If $s = 45, a = 40, v = 60$, find u and t.

11. A car accelerates uniformly from rest to 60 km/h in 30 seconds. Find the distance it travels in this time.

12. A cyclist travels 1·25 km as he accelerates uniformly at a rate of $k\,\mathrm{m\,s^{-2}}$ from a speed of 15 km/h to 30 km/h. Find the value of k.

13. A particle is projected vertically upwards from a point O with a speed of $25\,\mathrm{m\,s^{-1}}$. Find the maximum height reached by the particle and the time that elapses before it returns to O.

14. A boy on a bridge throws a pebble vertically upwards at a speed of $6\,\mathrm{m\,s^{-1}}$. After 2 seconds it hits the water below. Find the speed at which the pebble hits the water and its initial height above the water.

15. The points O, A, B and C lie in a straight line such that $AB = 28\,\mathrm{m}$ and $BC = 72\,\mathrm{m}$. A particle moving with constant acceleration starts from rest at O and passes through A, B and C, its velocities at B and C being $9\,\mathrm{m\,s^{-1}}$ and $15\,\mathrm{m\,s^{-1}}$ respectively. Find the velocity of the particle at A and the time it takes to travel from A to C.

16. A particle moving in a straight line with constant acceleration travels 10 m in 2 seconds, then a further 22 m in the next 2 seconds. Find the further distance travelled in 2 more seconds and the speed of the particle at the end of this 6-second interval of time.

17. A train, being brought to rest with uniform retardation, travels 30 m in 2 seconds, then a further 30 m in 4 seconds. Find the retardation of the train and the additional time it takes to come to rest.

18. A particle is projected vertically upwards from a point A. Given that it rises 15 m in the third second of its motion, find its initial speed and the maximum height above A that it reaches.

19. A bus sets off from a bus station P. It accelerates uniformly for T_1 seconds, covering a distance of 300 m. It travels at a speed of $V\,\mathrm{km/h}$ for T_2 seconds, covering a further distance of 1250 m. It then decelerates uniformly for T_3 seconds, coming to rest at a bus stop Q. If the total time taken is 3 minutes and $2T_1 = 3T_3$, find the distance from P to Q and the values of T_1, T_2, T_3 and V.

20. A train stops at a station A. It then accelerates at $0{\cdot}1\,\mathrm{m\,s^{-2}}$ for 5 minutes, reaching a speed of $V\,\mathrm{m\,s^{-1}}$. It continues at this speed for 12 minutes, then the brakes are applied for 3 minutes, bringing the train to rest with uniform retardation at a station B. Find the value of V and the distance AB.

21. A train stops at two stations 24 km apart. It takes 3 minutes to accelerate uniformly to a speed of $40\,\mathrm{m\,s^{-1}}$ then maintains this speed until it comes to rest with uniform retardation in a distance of 1200 m. Find the time taken for the journey.

22. Two stations A and B are 10 km apart. A train travelling at a constant speed of 144 km/h passes station A at 10.00 hours. At a distance d km from B the brakes are applied, producing a constant retardation of $0{\cdot}4\,\mathrm{m\,s^{-2}}$. If the train comes to rest at station B, find the value of d and the time at which the train reaches B.

23. A particle accelerates from rest at a constant rate of $3\,\mathrm{m\,s^{-2}}$ to a speed of $V\,\mathrm{m\,s^{-1}}$. It continues to move at that speed for a certain time, then decelerates at a constant rate of $1{\cdot}5\,\mathrm{m\,s^{-2}}$. If the total time taken is one minute and the total distance travelled is 1 km, find the value of V.

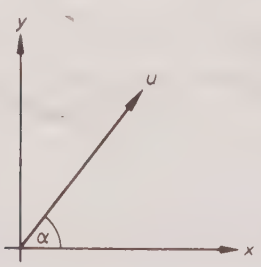
24. A car travelling along a straight level road at a constant speed of 54 km/h passes a second car as it starts to accelerate from rest at a uniform rate of $0{\cdot}5\,\mathrm{m\,s^{-2}}$. Find the time that elapses and the distance covered when the second car draws level with the first.

25. A ball is thrown vertically upwards from a point A with a speed of $20\,\mathrm{m\,s^{-1}}$. At the same instant a second ball is dropped from a point B which is 60 m vertically above A. Find the time which elapses before the two balls meet and their height above A at this instant.

26. A stone is dropped from the top of a tower. After one second another stone is thrown vertically downwards from the same point at a speed of $15\,\mathrm{m\,s^{-1}}$. If the stones reach the ground simultaneously, find the height of the tower.

4.3 Projectiles

Any particle which is given a non-zero initial velocity and then moves freely under gravity may be described as a *projectile*. In elementary work it is assumed that projectiles move in a vertical plane with constant downward acceleration g. The point of projection is taken as the origin O of a Cartesian coordinate system with the x-axis horizontal and the y-axis vertical. If at time t the projectile is at the point (x, y) then its displacement from O has horizontal component x and vertical component y. Thus the horizontal and vertical components of the velocity are dx/dt and dy/dt. Using a dot to indicate differentiation with respect to time these components are usually denoted by $\dot{x}$ and $\dot{y}$. Similarly the horizontal and vertical components of the acceleration are $\ddot{x}$ and $\ddot{y}$ respectively.



Consider a particle projected with initial velocity u at an angle α to the horizontal, the horizontal and vertical components of this velocity being $u\cos\alpha$ and $u\sin\alpha$ respectively. Since the particle has a constant downward acceleration of magnitude g, $\ddot{x} = 0$ and $\ddot{y} = -g$.

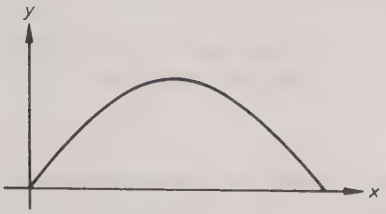
Thus the constant acceleration formulae derived in the previous section can now be applied to both horizontal and vertical motion. Using the equations $v = u + at$ and $s = ut + \frac{1}{2}at^2$,

$$\dot{x} = u\cos\alpha \qquad \dot{y} = u\sin\alpha - gt$$
$$x = ut\cos\alpha \qquad y = ut\sin\alpha - \tfrac{1}{2}gt^2.$$

Most projectile problems can be solved using one or more of these four equations.

The equation of the *trajectory* or path of the particle is obtained by eliminating from the expressions for x and y.

Writing $t = \dfrac{x}{u\cos\alpha}$, $y = u.\dfrac{x}{u\cos\alpha}.\sin\alpha - \dfrac{1}{2}g\left(\dfrac{x}{u\cos\alpha}\right)^2$



$$\therefore \quad y = x\tan\alpha - \frac{gx^2\sec^2\alpha}{2u^2}.$$

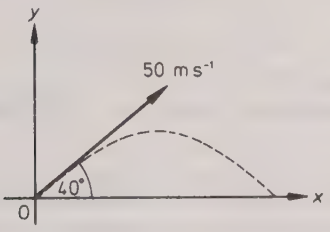
As shown in the sketch this is the equation of a parabola.

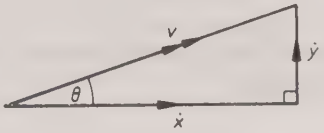
Example 1 A particle is projected from a point O on a horizontal plane with a speed of $50\,\mathrm{m\,s^{-1}}$ at an angle of $40°$ to the horizontal. Find (a) the velocity of the particle after 2 seconds, (b) the greatest height reached by the particle. [Take $g = 9{\cdot}8$.]

The equations for the motion of the particle are

$$\dot{x} = 50\cos 40° \qquad \dot{y} = 50\sin 40° - gt$$
$$x = 50t\cos 40° \qquad y = 50t\sin 40° - \tfrac{1}{2}gt^2.$$

(a)





Let the velocity of the particle after 2 seconds be $v\,\mathrm{m\,s^{-1}}$ at an angle θ to the horizontal.

When $t = 2$, $\dot{x} = 50\cos 40°$ $\dot{y} = 50\sin 40° - 9{\cdot}8 \times 2$

$\therefore \quad v = \sqrt{(\dot{x}^2 + \dot{y}^2)} \approx 40{\cdot}3$

and $\tan\theta = \dot{y}/\dot{x}$, $\theta \approx 18{\cdot}1°$.

[These results were obtained using a calculator, storing the value $\dot{y} = 12{\cdot}5394$ in the memory.]

Thus the velocity after 2 seconds is $40{\cdot}3\,\mathrm{m\,s^{-1}}$ at an angle $18{\cdot}1°$ above the horizontal.

(b) When the particle reaches its greatest height, $\dot{y} = 0$

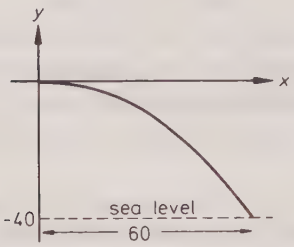
$\therefore \quad 50\sin 40° - gt = 0$ and $t = 50\sin 40°/g$

For this value of t,

$$y = \frac{(50\sin 40°)^2}{g} - \frac{1}{2}g\left(\frac{50\sin 40°}{g}\right)^2 = \frac{(50\sin 40°)^2}{2g} \approx 52{\cdot}7$$

$\therefore$ the greatest height reached by the particle is $52{\cdot}7$ m.

Example 2 A stone is thrown horizontally from the edge of a cliff 40 m above the sea. Given that the stone travels 60 m horizontally before it hits the water, find the time for which it is in the air and its initial speed. [Take $g = 9{\cdot}8$.]



Let the initial speed of the stone be $u\,\mathrm{m\,s^{-1}}$, then after t seconds,

$$x = ut \quad y = -\tfrac{1}{2}gt^2.$$

Let the time for which the stone is in the air be T seconds, then when $t = T$, $x = 60$ and $y = -40$

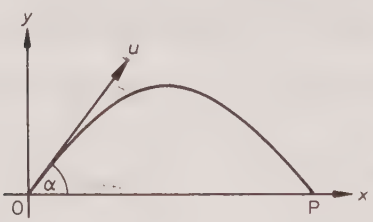
$\therefore \quad 60 = uT$ and $-40 = -\tfrac{1}{2} \times 9{\cdot}8T^2$.

Hence
$$T = \sqrt{\left(\frac{40}{4{\cdot}9}\right)} = \sqrt{\left(\frac{400}{49}\right)} = \frac{20}{7} = 2\frac{6}{7}$$

and
$$u = \frac{60}{T} = 60 \times \frac{7}{20} = 21$$

$\therefore$ the stone is in the air for $2\frac{6}{7}$ seconds and its initial speed is $21\,\mathrm{m\,s^{-1}}$.

If a particle is projected from a point O on a horizontal plane and strikes the plane again at the point P, then the distance OP is called the *range* of the projectile. The time which the particle takes to travel from O to P is called the *time of flight*.



Consider again a particle projected from O with initial velocity u and angle of projection α. Let the range of the particle on the horizontal plane through O be R and the time of flight be T. The horizontal and vertical displacements of the particle at time t are given by

$$x = ut\cos\alpha, \quad y = ut\sin\alpha - \tfrac{1}{2}gt^2$$

When $t = T$, $x = R$ and $y = 0$

$\therefore \quad R = uT\cos\alpha, \quad 0 = uT\sin\alpha - \tfrac{1}{2}gT^2$

Since $T \neq 0$, $u\sin\alpha - \tfrac{1}{2}gT = 0 \qquad \therefore \quad T = 2u\sin\alpha/g$

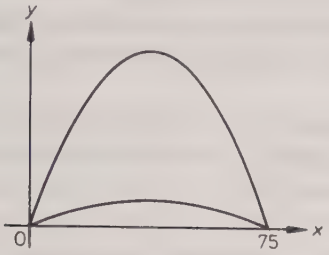
Thus
$$R = u\cos\alpha \cdot \frac{2u\sin\alpha}{g} = \frac{u^2}{g} \cdot 2\sin\alpha\cos\alpha = \frac{u^2\sin 2\alpha}{g}$$

Hence the range of the particle is $u^2\sin 2\alpha/g$ and its time of flight is $2u\sin\alpha/g$.

If u is fixed, but α can be varied, the maximum range of the particle is given by $\sin 2\alpha = 1$, i.e. $\alpha = 45°$.

Hence the particle has a maximum range of u^2/g when projected at 45 to the horizontal.

Example 3 A particle projected from a point O on horizontal ground, with speed $35\,\mathrm{m\,s^{-1}}$ and angle of projection θ, strikes the ground again at a point 75 m from O. Find the two possible values of θ. [Take $g = 9{\cdot}8$.]



Taking O as origin, after t seconds

$$x = 35t\cos\theta$$
$$y = 35t\sin\theta - \tfrac{1}{2}gt^2$$

$\therefore$ $y = 0$ when $35t\sin\theta - \frac{1}{2}gt^2 = 0$
i.e. when $t = 0$ or $t = 70\sin\theta/g$

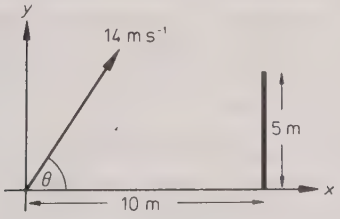
$\therefore$ the time of flight of the particle is $70\sin\theta/g$.
Since the range of the particle is 75 m,

$$75 = 35\cos\theta \cdot \frac{70\sin\theta}{g} = \frac{(35)^2}{9{\cdot}8} \cdot 2\sin\theta\cos\theta$$

Hence
$$\sin 2\theta = 2\sin\theta\cos\theta = \frac{75 \times 9{\cdot}8}{(35)^2} = 0{\cdot}6$$

Since θ is an acute angle, $2\theta = 36{\cdot}87°$ or $143{\cdot}13°$
$\therefore$ the two possible values of θ are $18{\cdot}4°$ and $71{\cdot}6°$.

Example 4 A particle is projected with speed $14\,\mathrm{m\,s^{-1}}$ at an angle θ to the horizontal. Find the values of θ for which the particle just clears an obstacle 5 m high and 10 m from the point of projection. [Take $g = 9{\cdot}8$.]



Taking the point of projection as origin, after t seconds

$$x = 14t\cos\theta$$
$$y = 14t\sin\theta - \tfrac{1}{2}gt^2$$

If the particle just clears the obstacle, then for some value of t, $x = 10$ and $y = 5$

$\therefore$ $10 = 14t\cos\theta$ and $5 = 14t\sin\theta - 4{\cdot}9t^2$.
Eliminating t from these equations,

$$5 = 14 \cdot \frac{10}{14\cos\theta} \cdot \sin\theta - 4{\cdot}9\left(\frac{10}{14\cos\theta}\right)^2$$

$\Leftrightarrow$ $$5 = 10\tan\theta - \frac{5}{2}\sec^2\theta$$

$\Leftrightarrow$ $$2 = 4\tan\theta - (1 + \tan^2\theta)$$

$\Leftrightarrow$ $$\tan^2\theta - 4\tan\theta + 3 = 0$$

$\Leftrightarrow$ $$(\tan\theta - 1)(\tan\theta - 3) = 0.$$

Thus either $\tan\theta = 1$ or $\tan\theta = 3$.
Hence the particle just clears the obstacle if $\theta = 45°$ or $\theta = 71{\cdot}6°$.

Exercise 4.3

[In numerical questions take g as $9{\cdot}8\,\mathrm{m\,s^{-2}}$.]

1. A particle is projected from a point on horizontal ground with speed $24{\cdot}5\,\mathrm{m\,s^{-1}}$ at an angle of 30° to the horizontal. Find the time it takes to reach its greatest height. Find also the time of flight and the range of the particle.

2. A particle is projected horizontally with speed $30\,\mathrm{m\,s^{-1}}$ from a point 19·6 m above horizontal ground. Find the horizontal distance it travels before striking the ground and its velocity as it strikes the ground.

3. A particle is projected from a point O on a horizontal plane with speed $35\,\mathrm{m\,s^{-1}}$ at an angle of elevation θ, where $\tan\theta = 2$. Find the maximum height above the plane of the particle and the equation of its trajectory.

4. A golf ball is given initial velocity $30\,\mathrm{m\,s^{-1}}$ at an angle α to the horizontal, where $\tan\alpha = 4/3$. The ball strikes the ground after T seconds at a horizontal distance of 90 m from the point of projection and k m below it. Find the values of T and k. Find also the velocity of the ball when it strikes the ground.

5. A particle is projected from a point O 24·5 m above a horizontal plane. After 5 seconds it hits the plane at a point whose horizontal distance from O is 100 m. Find the horizontal and vertical components of the initial velocity of the particle and the greatest height it reaches above the plane.

6. A boy throws a ball horizontally from a point 4·9 m above horizontal ground. What is the minimum speed at which the ball must be thrown to clear a fence 2·4 m high at a horizontal distance of 8 m from the point of projection? Find the distance beyond the fence of the point at which the ball strikes the ground if projected at this minimum speed.

7. A particle is projected from a point on a horizontal plane with velocity u and angle of projection α. The range of the particle is R and its time of flight T. Show that the particle reaches its greatest height h at time $\frac{1}{2}T$ and that $h = \frac{1}{4}R\tan\alpha$.

8. A gun fires shells with speed $210\,\mathrm{m\,s^{-1}}$. Find the maximum range of the gun over horizontal ground. Find also the smaller angle of elevation at which a shell should be fired to hit a target at a distance of 3·6 km.

9. A particle is projected from a point O on horizontal ground with an initial velocity whose horizontal and vertical components are $3u$ and $5u\,\mathrm{m\,s^{-1}}$ respectively. Find the equation of the trajectory of the particle. Given that it just clears an obstacle 5 m high and 9 m from O, find the value of u and the distance from O of the point at which the particle strikes the ground.

10. A ball thrown with speed $u\,\mathrm{m\,s^{-1}}$ and angle of projection α just clears the top of a fence 3·6 m high and a horizontal distance of 9·6 m from the point of

projection. Find u and α given that as the ball passes over the fence it is moving in a horizontal direction.

11. A projectile with initial speed $u\,\mathrm{m\,s^{-1}}$ has a maximum range of 80 m on a horizontal plane. Find the value of u. Given that the angle of projection is α, find the set of values of α for which the range is greater than 40 m.

12. The horizontal and vertical components of the initial velocity of a projectile are u and $3u$ respectively. Find in terms of u and g (a) the range of the projectile on the horizontal plane through the point of projection, (b) the times at which the particle is moving in a direction at 45° to the horizontal.

13. A particle is projected with speed u and angle of projection α. If after T seconds the velocity of the particle is perpendicular to its initial velocity, find an expression for T in terms of u, α and g. Comment on the significance of your answer in the cases $\alpha = 0°$ and $\alpha = 90°$.

14. A gun is fired from a vehicle moving along a straight horizontal road at speed U. Relative to the vehicle shells are fired with speed V and elevation θ. Find an expression for the range of the gun along the road ahead of the vehicle.

15. A particle is fired with speed V from a point O of a horizontal plane, the angle of elevation being $\arctan(\frac{2}{3})$. The particle strikes the plane at a point A. Find the distance OA. Show that the same point A could have been reached by firing the particle with the same speed from O at an angle of elevation $\arctan(\frac{3}{2})$. Find the difference in the times of flight for the two trajectories. (O&C)

16. A projectile is fired with initial speed V at an angle θ to the horizontal. It passes through two points A and B at the same height h above the point of projection. Find the speed of the projectile at A, and the time it takes to travel from A to B. Given that $AB = 2h$ and $V^2 = 4gh$, find the angle θ. (L)

17. Prove that two particles projected from the same point at the same time, but with different velocities, can never collide while moving freely under gravity.

18. Two particles are projected with the same speed from a point A, and land at B on the same horizontal level as A. If the greatest heights of the particles are in the ratio 4 : 1, find the angles of projection and the ratio of AB to the maximum range with the same speed of projection.

19. A projectile is fired from a point P with speed $V\,\mathrm{m\,s^{-1}}$ and angle of elevation θ, where $\cos\theta = 0{\cdot}8$. At the same instant a target moving vertically upwards with constant speed $13\,\mathrm{m\,s^{-1}}$ passes through a point Q in the same horizontal plane as P and at a distance of 480 m from P. Given that the projectile hits the target, find the value of V. Find also the speed at which the projectile is travelling when it hits the target.

20. A particle is projected from a point A on horizontal ground such that the horizontal and vertical components of its initial velocity are u_1 and u_2 respectively. At the same instant a particle is projected from a point B at a height h vertically above A. The horizontal and vertical components of its initial velocity are v_1 and v_2. Find the time of flight of the first particle and show that the particles will collide before hitting the ground, provided that $u_1 = v_1$ and $2u_2(u_2 - v_2) > gh$.

4.4 Differentiation of position vectors

In §3.1 it was shown that if a moving particle has position vector $\mathbf{r}$, velocity $\mathbf{v}$ and acceleration $\mathbf{a}$, then

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} \text{ or } \dot{\mathbf{r}} \quad \text{and} \quad \mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} \text{ or } \ddot{\mathbf{r}}.$$

In general, the basic rules for differentiation still hold when vectors are involved.

Thus
$$\frac{d}{dt}\left(\mathbf{r}_1 + \mathbf{r}_2\right) = \frac{d\mathbf{r}_1}{dt} + \frac{d\mathbf{r}_2}{dt},$$

and if $\mathbf{r}$ is of the form $f(t)\mathbf{c}$, where $\mathbf{c}$ is a constant vector,

then
$$\frac{d\mathbf{r}}{dt} = \frac{d}{dt}\left\{f(t)\mathbf{c}\right\} = \frac{df}{dt}\mathbf{c}.$$

Hence if a position vector $\mathbf{r}$ is expressed in terms of the standard set of unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, then

$$\begin{aligned} \mathbf{r} &= x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \\ \mathbf{v} = \dot{\mathbf{r}} &= \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k} \\ \mathbf{a} = \ddot{\mathbf{r}} &= \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k}. \end{aligned}$$

Using these results we can extend some of the earlier work in this chapter to motion in two and three dimensions.

Example 1 A particle moves so that after t seconds its position vector $\mathbf{r}$ is given by $\mathbf{r} = (4t-1)\mathbf{i} + t^2\mathbf{j} + (15t - t^3)\mathbf{k}$. Find vector expressions for its velocity and acceleration at time t. Find also its speed after 3 seconds.

$$\begin{aligned} \mathbf{r} &= (4t-1)\mathbf{i} + t^2\mathbf{j} + (15t - t^3)\mathbf{k} \\ \therefore \quad \mathbf{v} = \dot{\mathbf{r}} &= 4\mathbf{i} + 2t\mathbf{j} + (15 - 3t^2)\mathbf{k} \\ \therefore \quad \mathbf{a} = \ddot{\mathbf{r}} &= 2\mathbf{j} - 6t\mathbf{k}. \end{aligned}$$

Hence the velocity and acceleration at time t are given by

$$\mathbf{v} = 4\mathbf{i} + 2t\mathbf{j} + (15 - 3t^2)\mathbf{k} \quad \text{and} \quad \mathbf{a} = 2\mathbf{j} - 6t\mathbf{k}.$$

When $t = 3$, $\mathbf{v} = 4\mathbf{i}+6\mathbf{j}-12\mathbf{k}$

$\therefore \quad |\mathbf{v}| = \sqrt{\{4^2+6^2+(-12)^2\}} = \sqrt{(16+36+144)} = \sqrt{196} = 14.$

Hence after 3 seconds the speed of the particle is $14\,\mathrm{m\,s^{-1}}$.

Example 2 A particle moves so that at time t its velocity is given by $\mathbf{v} = 10t\mathbf{i}+(3-6t^2)\mathbf{k}$. Given that when $t = 1$ the particle is at the point with position vector $\mathbf{j}+\mathbf{k}$, find an expression for its position vector $\mathbf{r}$ at time t.

$$\mathbf{v} = \dot{\mathbf{r}} = 10t\mathbf{i}+(3-6t^2)\mathbf{k}$$

Integrating $\qquad \mathbf{r} = 5t^2\mathbf{i}+(3t-2t^3)\mathbf{k}+\mathbf{c}$

where $\mathbf{c}$ is a constant vector.

When $t = 1$, $\mathbf{r} = \mathbf{j}+\mathbf{k}$ $\qquad \therefore \quad \mathbf{j}+\mathbf{k} = 5\mathbf{i}+\mathbf{k}+\mathbf{c}$

i.e. $\quad \mathbf{c} = -5\mathbf{i}+\mathbf{j}$

Hence $\qquad \mathbf{r} = 5t^2\mathbf{i}+(3t-2t^3)\mathbf{k}-5\mathbf{i}+\mathbf{j}$

i.e. $\qquad \mathbf{r} = (5t^2-5)\mathbf{i}+\mathbf{j}+(3t-2t^3)\mathbf{k}.$

Consider now a particle with initial velocity $\mathbf{u}$ and constant acceleration $\mathbf{a}$.

$$\frac{d\mathbf{v}}{dt} = \mathbf{a}, \text{ where } \mathbf{a} \text{ is constant}$$

Integrating $\mathbf{v} = \mathbf{a}t+\mathbf{c}$, where $\mathbf{c}$ is constant

When $t = 0$, $\mathbf{v} = \mathbf{u}$ $\quad \therefore \quad \mathbf{c} = \mathbf{u}.$

Thus $\qquad \mathbf{v} = \dfrac{d\mathbf{r}}{dt} = \mathbf{u}+\mathbf{a}t.$

Integrating again and taking the initial position of the particle as origin,

$$\mathbf{r} = \mathbf{u}t+\tfrac{1}{2}\mathbf{a}t^2.$$

Hence $\mathbf{v} = \mathbf{u}+\mathbf{a}t$ $\qquad \mathbf{r} = \mathbf{u}t+\tfrac{1}{2}\mathbf{a}t^2.$

Eliminating $\mathbf{a}$, we also have $\mathbf{r} = \tfrac{1}{2}(\mathbf{u}+\mathbf{v})t$.

These results are the vector forms of the constant acceleration formulae derived in §4.2.

If the acceleration due to gravity is denoted by the constant vector $\mathbf{g}$, then by substituting $\mathbf{a} = \mathbf{g}$ we obtain the vector equations governing the motion of a projectile, namely

$$\mathbf{v} = \mathbf{u}+\mathbf{g}t \quad \text{and} \quad \mathbf{r} = \mathbf{u}t+\tfrac{1}{2}\mathbf{g}t^2.$$

Using the notation of §12.3 and horizontal and vertical unit vectors $\mathbf{i}$ and $\mathbf{j}$, let $\mathbf{r} = x\mathbf{i}+y\mathbf{j}$, $\mathbf{v} = \dot{x}\mathbf{i}+\dot{y}\mathbf{j}$, $\mathbf{u} = (u\cos\alpha)\mathbf{i}+(u\sin\alpha)\mathbf{j}$ and $\mathbf{g} = -g\mathbf{j}$, then

$$\dot{x}\mathbf{i}+\dot{y}\mathbf{j} = (u\cos\alpha)\mathbf{i}+(u\sin\alpha-gt)\mathbf{j}$$

and $\qquad x\mathbf{i}+y\mathbf{j} = (ut\cos\alpha)\mathbf{i}+(ut\sin\alpha-\tfrac{1}{2}gt^2)\mathbf{j}.$

Clearly in this form the vector equations correspond exactly to the results derived in the previous section.

Example 3 A particle is projected from a point O with initial velocity represented by the vector $\mathbf{i}+2\mathbf{j}$. Find vector expressions for its velocity $\mathbf{v}$ and its displacement $\mathbf{r}$ from O at time t. [Take $g = 10$.]

Using the equations $\mathbf{v} = \mathbf{u}+\mathbf{g}t$ and $\mathbf{r} = \mathbf{u}t+\frac{1}{2}\mathbf{g}t^2$,
where $\mathbf{u} = \mathbf{i}+2\mathbf{j}$ and $\mathbf{g} = -10\mathbf{j}$,

$$\mathbf{v} = (\mathbf{i}+2\mathbf{j})-10t\mathbf{j} = \mathbf{i}+(2-10t)\mathbf{j}$$

and
$$\mathbf{r} = (\mathbf{i}+2\mathbf{j})t+\tfrac{1}{2}(-10\mathbf{j})t^2 = t\mathbf{i}+(2t-5t^2)\mathbf{j}.$$

[Note that although vector equations can be used in any projectile problem, in elementary work many students will prefer the methods of §4.3.]

Exercise 4.4

[Use SI units and take $g = 10$.]

1. A particle moves such that its position vector at time t is $\mathbf{r}$. Find vector expressions for its velocity and acceleration at time t given that
(a) $\mathbf{r} = 3t\mathbf{i}-2\mathbf{j}+2t^3\mathbf{k}$, (b) $\mathbf{r} = (t^2+1)\mathbf{i}+t^4\mathbf{j}-6t\mathbf{k}$.

2. A particle moves such that at time t its position vector is $\mathbf{r} = t^2\mathbf{i}+(2-3t)\mathbf{j}+3t^2(t-2)\mathbf{k}$. Find the initial speed of the particle and its speed after 2 seconds.

3. A particle moves such that at time t its velocity is given by $\mathbf{v} = 3\mathbf{i}+4t\mathbf{j}+(1-3t^2)\mathbf{k}$. Given that when $t = 2$ the position vector of the particle is $6(\mathbf{j}-\mathbf{k})$, find an expression for its position vector at time t.

4. A particle moves such that at time t its acceleration is given by $\mathbf{a} = 6t\mathbf{i}-4\mathbf{k}$. Initially the particle has velocity vector $\mathbf{i}+\mathbf{j}$ and position vector $3\mathbf{j}$. Find expressions for its velocity and position vectors at time t.

In questions 5 to 8 the directions of the unit vectors $\mathbf{i}$ and $\mathbf{j}$ are horizontal and vertically upward respectively.

5. A particle is projected from a point O with initial velocity represented by the vector $3\mathbf{i}+2\mathbf{j}$. Find vector expressions for its velocity $\mathbf{v}$ and its displacement $\mathbf{r}$ from O at time t.

6. A particle is projected from a point O on a horizontal plane with initial velocity represented by the vector $40\mathbf{i}+25\mathbf{j}$. Find the greatest height reached by the particle, its range and time of flight.

7. A particle is projected from a point O with speed $20\,\mathrm{m\,s^{-1}}$ at an angle of 60° to the horizontal. Find vector expressions for its velocity $\mathbf{v}$ and its displacement $\mathbf{r}$ from O at time t.

8. A stone is thrown from the top of a cliff with initial velocity vector $8\mathbf{i}-\mathbf{j}$. If the stone hits the sea below after 3 seconds, find the height of the cliff and the horizontal distance travelled by the stone. Find also the speed at which the stone is travelling when it hits the water.

Exercise 4.5 (miscellaneous)

[In numerical questions take g as $10\,\mathrm{m\,s^{-2}}$.]

1. A particle is moving in a straight line such that its displacement from a fixed point after t seconds is s metres, where $s = 2t^3 - 13t^2 + 20t$. Find the total distance travelled and the maximum speed attained by the particle in the first 4 seconds of its motion. Find also the range of values of t for which the acceleration of the particle is negative.

2. A particle moving in a straight line starts from rest at a point A. Its acceleration at time t is $(45 + 12t - 9t^2)\,\mathrm{m\,s^{-2}}$. If the particle comes to rest instantaneously at a point B, find the distance AB and the time the particle takes to reach B. Find also the maximum acceleration and the maximum speed attained in that time.

3. Two particles A and B move along a straight line starting from a point O such that at time t their displacements from O are s_1 and s_2 respectively. If $s_1 = 5t^2(t+7)$ and $s_2 = t^3(t+3)$, find (a) the time at which the particles again meet, (b) an expression for the velocity of A relative to B, (c) the range of values of t for which A is travelling at a greater speed than B.

4. Due to track repairs a train retards uniformly, with retardation $1\,\mathrm{m\,s^{-2}}$, from a speed of $40\,\mathrm{m\,s^{-1}}$ at A to a speed of $10\,\mathrm{m\,s^{-1}}$ at B. The train travels from B to C, a distance of 3·5 km, at a constant speed of $10\,\mathrm{m\,s^{-1}}$ and then accelerates uniformly, with acceleration $0{\cdot}2\,\mathrm{m\,s^{-2}}$ so that its speed at D is $40\,\mathrm{m\,s^{-1}}$. Sketch the velocity-time graph for the journey from A to D, and show that the distance from A to D is 8 km. Show that the journey from A to D takes 330 s more than it would if the train travelled at a constant speed of $40\,\mathrm{m\,s^{-1}}$ from A to D. (C)

5. Two trains, P and Q, travel by the same route from rest at station A to rest at station B. Train P has constant acceleration f for the first third of the time, constant speed for the second third and constant retardation f for the last third of the time. Train Q has constant acceleration f for the first third of the distance, constant speed for the second third and constant retardation f for the last third of the distance. Show that the times taken by the two trains are in the ratio $3\sqrt{3}:5$. (L)

6. A bus starts from rest and moves along a straight road with constant acceleration f until its speed is V; it then continues at constant speed V. When the bus starts, a car is at a distance b behind the bus and is moving in the same direction with constant speed u. Find the distance of the car behind the bus at time t after the bus has started (i) for $0 < t < V/f$, (ii) for $t > V/f$. Show that the car cannot overtake the bus during the period $0 < t < V/f$ unless $u^2 > 2fb$. Find the least distance between the car and the bus in the case when $u^2 < 2fb$ and $u < V$. State briefly what will happen if $u^2 < 2fb$ and $u > V$. (JMB)

7. A man descending in a lift at constant speed $V\,\mathrm{m\,s^{-1}}$ throws a ball vertically upwards with speed $U\,\mathrm{m\,s^{-1}}$ relative to the lift. Find the time that elapses before the ball (a) has zero velocity relative to the lift, (b) returns to the man's hand.

8. A batsman hits a cricket ball with initial speed 25 m/s at an elevation of 50°. Find the greatest height attained, the range on level ground and the time of flight. Show that it will clear the roof of a pavilion 10 m high which is at a distance of 50 m from the batsman. (L)

9. A particle P, projected from a point A on horizontal ground, moves freely under gravity and hits the ground again at B. Referred to A as origin, AB as x-axis and the upward vertical at A as y-axis, the equation of the path of P is $y = x - x^2/40$, where x and y are measured in metres. Calculate (i) the distance AB, (ii) the greatest height above AB attained by P, (iii) the magnitude and the direction of the velocity of P at A, (iv) the time taken by P to reach B from A. Calculate the coordinates of the points on the path of P at the two instants when the speed of P is 15 m/s. (AEB 1978)

10. A particle P is projected from a point A on a horizontal plane with speed V m/s at an angle of elevation α. The particle strikes the horizontal plane again at B, where $AB = 160$ metres. Show that $V^2 \sin 2\alpha = 1600$. A second particle Q is projected from A with the same speed V m/s at an angle of elevation $\alpha - 30°$ and strikes the horizontal plane at C, where C is the mid-point of AB. Find the values of V and α and show that the time of flight of Q is $8 \sin 15°$ seconds. Given that P and Q are projected from A simultaneously, find, at the instant when Q reaches C, the angle between the tangent to the path of P and the horizontal. (AEB 1978)

11. A particle is projected from a point O with a speed of $10\,\mathrm{m\,s^{-1}}$ and angle of elevation θ. Given that the projectile just clears an obstacle of height h at a horizontal distance D from O, express in the form of a quadratic equation in $\tan\theta$ the relationship connecting θ, h and D. Show that provided that $D^2 + 20h < 100$, there are two distinct possible values of θ. Show further that if $h = 0$ the sum of these two values is 90°.

12. A point O lies on a horizontal plane, and the point A is at a height h vertically above O. A particle is projected from A with speed V at an angle α above the horizontal. Taking O as the origin and Oy vertically upwards, show that the equation of the path of the particle can be written in the form $y = h - \dfrac{gx^2}{2V^2} + x\tan\alpha - \dfrac{gx^2}{2V^2}\tan^2\alpha$. The particle hits the plane at the point $B(r, 0)$. In the case when $V^2 = gh$ derive a quadratic equation for $\tan\alpha$ in terms of h and r, and show that $r \leqslant \sqrt{3h}$. For the same value of V show that $r = \sqrt{3h}$ when $\alpha = 30°$. (JMB)

13. A particle is fired from the origin O with speed V at an angle of inclination α above the x-axis, which is horizontal. At the instant when the particle is at the highest point of its path, a second particle is fired from O with speed U at an angle of inclination β above the horizontal so that the particles collide at a point on the x-axis. Show that $\tan\beta = \frac{1}{4}\tan\alpha$. Show also that, if $\tan\alpha = 3$, then $8U^2 = 5V^2$.

14. A and B are points distant a apart on horizontal ground. A ball is thrown from A towards B with velocity u at an angle 2α ($< 90°$) to the horizontal, and simultaneously a second ball is thrown from B towards A with velocity v at an angle α to the horizontal. If the balls collide, show that $v = 2u\cos\alpha$ and hence that they collide after a time $t = a/u(2\cos 2\alpha + 1)$. Show also that (i) the two trajectories would have the same maximum height above AB, (ii) the range of the ball from B would always exceed the range of the ball from A. (W)

15. A particle P is set in motion at a point A and then moves such that at time t its position vector is $\mathbf{r} = 3(t-2)^2\mathbf{i} + 2\mathbf{j} - 4t(t-4)\mathbf{k}$. By considering the vector $\mathbf{AP}$, show that the particle moves in a straight line and find a unit vector parallel to the direction of motion. Find also the time that has elapsed and the position vector of P when it comes instantaneously to rest.

16. [In this question the directions of the unit vectors $\mathbf{i}$ and $\mathbf{j}$ are horizontal and vertically upwards.] A particle projected from the origin O with initial velocity represented by the vector $u\mathbf{i} + v\mathbf{j}$ passes through the points A and B with position vectors $20(\mathbf{i}+\mathbf{j})$ and $25\mathbf{i}$ respectively. Find the values of u and v. Show that a particle projected from O with velocity vector $v\mathbf{i} + u\mathbf{j}$ also passes through B.

5 Newton's laws of motion

5.1 Newton's laws

The theory of elementary dynamics is based on three laws first stated by Newton† in the seventeenth century. Although slight discrepancies can be detected when comparing the predictions of Newtonian mechanics with the true motion of atomic particles travelling at high speed, and with certain astronomical observations (when results based on the theory of relativity are more reliable), Newton's laws can be used to describe the behaviour of moving objects with great accuracy.

Newton's first law: Every body continues in its state of rest or of uniform motion in a straight line unless compelled to change that state by external forces.

This law implies that a force changes the velocity of the body it acts upon. However, if two or more forces act on a body there will be no change in velocity unless the forces have a non-zero resultant. Newton's first law also implies that no force is required to maintain uniform motion, but practical experience tells us that the thrust of an engine is needed to drive a car along a horizontal road with uniform velocity. The explanation of this apparent contradiction is that the resultant force on the car is zero, the thrust of the engine exactly balancing the various forces opposing motion.

Newton's second law: The rate of change of momentum of a body is proportional to the applied force and has the direction of that force.

† *Newton, Sir Isaac* (1642–1727) English scientist and mathematician. He invented, a little before the German mathematician Leibnitz, differential calculus (1665) and integral calculus (1666). He did significant work on the nature of light and the construction of telescopes. However, Newton is chiefly remembered for his statement of the laws of dynamics and his theory of gravitation, which are explained in *Philosophiae Naturalis Principia Mathematica* (1687).

The *momentum* of a body is defined as the product of its mass and its velocity, i.e. the momentum of a particle of mass m moving with velocity $\mathbf{v}$ is the vector quantity $m\mathbf{v}$. Thus for a particle of constant mass m,

$$\text{rate of change of momentum} = \frac{d}{dt}(m\mathbf{v}) = m\frac{d\mathbf{v}}{dt} = m\mathbf{a},$$

where $\mathbf{a}$ is the acceleration of the particle.

Hence Newton's second law states that, for a body of constant mass m, the quantity $m\mathbf{a}$ is proportional to the force $\mathbf{F}$ acting on the body, i.e. $\mathbf{F} = km\mathbf{a}$, where k is constant. The value of k depends on the units of measurement chosen for $\mathbf{F}$, m and $\mathbf{a}$. A newton is defined as the force which gives a mass of 1 kg an acceleration of $1\,\mathrm{m\,s^{-2}}$. Hence when using SI units, $k = 1$ and we obtain the fundamental equation of motion for a body of constant mass

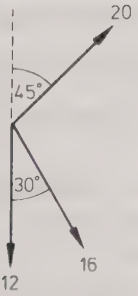
$$\boxed{\mathbf{F} = m\mathbf{a}.}$$

Example 1 A particle of mass 5 kg is moving with an acceleration of $3\,\mathrm{m\,s^{-2}}$. Find the magnitude of the resultant force acting on the particle.

If the resultant force on the particle is F newtons, by Newton's second law, $F = 5 \times 3$.
Hence there is a force of 15 N acting on the particle.

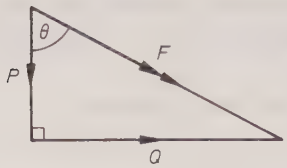
In Chapter 2 various ways of finding the resultant of a set of forces were discussed. Any of these methods, together with Newton's second law, can be used when finding the acceleration of a body moving under the action of two or more forces.

Example 2 A body of mass 8 kg on a smooth horizontal plane is acted upon by horizontal forces of magnitudes 20, 12 and 16 newtons in the directions N 45° E, due south and S 30° E respectively. Find the magnitude and direction of the acceleration of the body.



Let $\mathbf{F}$ be the resultant force on the body acting at an angle θ east of south and let $\mathbf{a}$ be the acceleration produced. If $\mathbf{P}$ and $\mathbf{Q}$ are the components of $\mathbf{F}$ in the directions south and east respectively, then resolving in these directions

$$\downarrow P = 12 + 16\cos 30° - 20\cos 45°$$
$$\rightarrow Q = 16\sin 30° + 20\sin 45°.$$



[By calculator $P = 11{\cdot}71426$, $Q = 22{\cdot}14214$.]

Using Newton's second law, $\mathbf{F} = 8\mathbf{a}$

$$\therefore \quad a = \frac{1}{8}F = \frac{1}{8}\sqrt{(P^2+Q^2)} \approx 3{\cdot}13$$

and $\tan\theta = Q/P$, $\theta \approx 62{\cdot}1°$.

Hence the body moves with an acceleration of $3{\cdot}13\,\mathrm{m\,s^{-2}}$ in the direction S $62{\cdot}1°$ E.

When a body is moving in a straight line under the action of constant forces then its acceleration is constant. Hence the motion is governed by the constant acceleration equations derived in §4.2.

Example 3 A particle of mass 2 kg starting from rest is acted on by a force of 4 N for 5 seconds. Find its acceleration and the speed attained. The particle is then brought to rest in a further 8 seconds by a constant resistance of R N. Find the value of R and the total distance travelled by the particle.

Let $a_1\,\mathrm{m\,s^{-2}}$ and $a_2\,\mathrm{m\,s^{-2}}$ be the accelerations of the particle in the two parts of its motion and let $V\,\mathrm{m\,s^{-1}}$ be the speed attained after 5 seconds.

By Newton's 2nd law, $\quad 4 = 2a_1 \quad \therefore \quad a_1 = 2.$

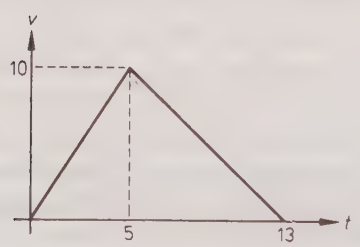
Using the formula $v = u + at$, substituting $u = 0$, $v = V$, $a = 2$, $t = 5$:
$V = 0 + 2 \times 5 = 10.$

Hence the initial acceleration of the particle is $2\,\mathrm{m\,s^{-2}}$ and the speed attained is $10\,\mathrm{m\,s^{-1}}$.

Using the formula $v = u + at$, substituting $u = 10$, $v = 0$, $a = a_2$, $t = 8$
$0 = 10 + 8a_2 \quad \therefore \quad a_2 = -1{\cdot}25.$

By Newton's 2nd law $\quad R = 2 \times 1{\cdot}25 = 2{\cdot}5.$

Hence the resistance is of magnitude 2·5 N.



The total distance travelled is represented by the area under the velocity-time graph for the motion

$\therefore$ total distance travelled

$= \frac{1}{2} \times 10 \times 13\,\mathrm{m} = 65\,\mathrm{m}.$

Newton's third law: To every action there is an equal and opposite reaction.

This means that if a body A exerts a force on a body B, then B exerts an equal and opposite force on A. For instance, consider a body of weight W. The gravitational pull of the earth on the body is W, but Newton's third law tells us that the body also exerts a force of magnitude W on the earth. Similarly, when a body is in contact with a smooth surface, the surface exerts a force called the normal reaction on the body. As stated in Newton's third law, the body exerts an equal and opposite force on the surface. This law will be particularly useful when examining systems involving two or more objects in contact.

Exercise 5.1

[Vector quantities are measured in SI units.]

1. A particle of mass 12 kg is moving with an acceleration of $4\,\mathrm{m\,s^{-2}}$. Find the magnitude of the resultant force on the particle.

2. Find the acceleration of a mass of 45 kg acted upon by a single force of 9 N.

3. A resultant force of 36 N acting on a particle produces an acceleration of $3\,\mathrm{m\,s^{-2}}$. Find the mass of the particle.

4. A particle of mass 8 kg is pulled along a smooth horizontal plane by a horizontal string. Find the tension in the string when the acceleration of the particle is $5\,\mathrm{m\,s^{-2}}$.

5. A particle of mass 15 kg is acted upon by a resultant force represented by the vector $30\mathbf{i}-75\mathbf{j}$. Find, in vector form, the acceleration of the particle.

6. A particle of mass 4 kg is moving with acceleration vector $3\mathbf{i}+\mathbf{j}$. Find a vector representing the resultant force on the particle.

7. Forces of magnitude 5 N and 9 N act at an angle of 50° on a particle of mass 2 kg. Find the magnitude and direction of the acceleration produced.

8. Two forces of magnitude 16 N and 24 N act on a particle of mass 5 kg. Given that an acceleration of $6\,\mathrm{m\,s^{-2}}$ is produced, find the angle between the lines of action of the two forces.

9. Forces represented by the vectors $3\mathbf{i}+\mathbf{j}$, $-5\mathbf{i}+2\mathbf{j}$, $4\mathbf{i}-5\mathbf{j}$ and $2\mathbf{i}-\mathbf{j}$ act on a particle of mass 0·8 kg moving on a smooth horizontal plane. Given that $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the directions due east and due north respectively, find the magnitude and direction of the acceleration of the particle.

10. A body of mass 2 kg on a smooth horizontal plane is acted upon by horizontal forces of magnitudes 9, 4 and 7 newtons in the directions due west, N 20° W and S 75° E respectively. Find the magnitude and direction of the acceleration of the body.

11. A train of mass 300 000 kg travelling at $0{\cdot}3\,\mathrm{m\,s^{-1}}$ is brought to rest at a station by buffers exerting an average force of 50 000 N. Find the time taken to bring the train to rest.

12. A particle of mass 20 kg is brought to rest in a distance of 600 m from a speed of $30\,\mathrm{m\,s^{-1}}$ by a constant retarding force. Find the magnitude of this force.

13. A body is moving in a straight line with a speed of $40\,\mathrm{m\,s^{-1}}$. A retarding force of 25 N reduces its speed to $15\,\mathrm{m\,s^{-1}}$ in 10 seconds. Find the mass of the body.

14. A car of mass 800 kg is moving along a straight horizontal road under the action of a constant resultant force of 1000 N. Find its speed when it has moved 40 m from rest.

15. A particle of mass 5 kg starting from rest is acted on by a force of 12 N for 10 seconds. The particle continues to move with constant velocity for a further 10 seconds. It is then brought to rest by a constant resistance of R N in a further time of 15 seconds. Find the value of R and the total distance travelled.

16. A particle of mass 4 kg starts from rest and moves under the action of a constant force of F newtons for 12 seconds. It is then brought to rest in a further 8 seconds by a constant retarding force of R newtons. If the total distance travelled is 200 m, find the maximum speed attained and the values of F and R.

5.2 Further applications

As stated in §4.2, an object falling freely under gravity has a constant downward acceleration denoted by g. Thus if a body of mass m kg has a weight of W N, then by Newton's second law,

$$W = mg.$$

Hence the force due to the gravitational pull of the earth on a body of mass m kg is of magnitude mg N.

[In this and later chapters the value of g will be taken as $9{\cdot}8\,\mathrm{m\,s^{-2}}$ unless otherwise stated.]

Example 1 A package of mass 5 kg is being lifted by means of a vertical cable. If the tension in the cable is 60 N, find the acceleration of the package.

Let the acceleration of the package be $a\,\mathrm{m\,s^{-2}}$ vertically upwards.
Applying Newton's 2nd law vertically

$$\uparrow 60 - 5g = 5a$$

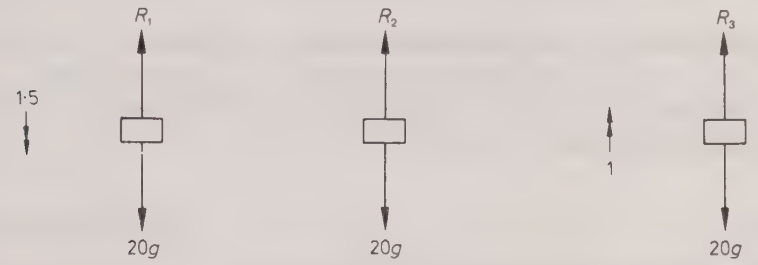
$$\therefore \quad a = 12 - g = 12 - 9{\cdot}8 = 2{\cdot}2.$$

Hence the acceleration of the package is $2{\cdot}2\,\mathrm{m\,s^{-2}}$.

[Note that to avoid confusion in diagrams showing both forces and accelerations, the forces are denoted by single-headed arrows and the accelerations by double-headed arrows.]

Example 2 A lift descends with an acceleration of $1{\cdot}5\,\mathrm{m\,s^{-2}}$, then moves with constant speed until it is retarded at $1\,\mathrm{m\,s^{-2}}$. A suitcase of mass 20 kg stands on the floor of the lift during the journey. Find the magnitude of the force it exerts on the floor of the lift at each stage of the journey.

Let R_1, R_2 and R_3 newtons be the reactions of the floor on the suitcase.

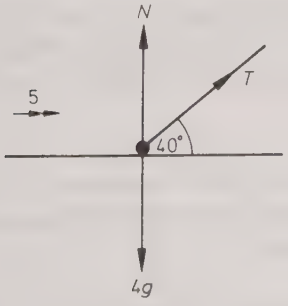


Applying Newton's 2nd law vertically in each case

$$\downarrow 20g - R_1 = 20 \times 1{\cdot}5 \qquad \therefore \quad R_1 = 20g - 30 = 196 - 30 = 166$$
$$\uparrow R_2 - 20g = 0 \qquad \therefore \quad R_2 = 20g = 196$$
$$\uparrow R_3 - 20g = 20 \times 1 \qquad \therefore \quad R_3 = 20g + 20 = 196 + 20 = 216.$$

By Newton's third law, the force exerted by the suitcase on the floor is equal in magnitude to the reaction of the floor on the suitcase. Hence the forces exerted by the suitcase on the floor of the lift are 166, 196 then 216 newtons.

Example 3 A particle of mass 4 kg is being pulled across a smooth horizontal plane by a string inclined at 40° to the horizontal. If the acceleration of the particle is $5\,\mathrm{m\,s^{-2}}$, find the tension in the string and the reaction between the particle and the plane.

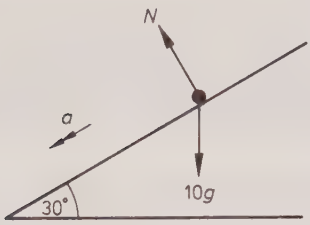


Applying Newton's 2nd law horizontally and vertically

$$\rightarrow T\cos 40^\circ = 4 \times 5$$
$$\therefore \qquad T = 20/\cos 40^\circ \approx 26{\cdot}1$$
$$\uparrow N + T\sin 40^\circ - 4g = 0$$
$$\therefore \quad N = 4 \times 9{\cdot}8 - 20\tan 40^\circ \approx 22{\cdot}4.$$

Hence the tension in the string is 26·1 N and the reaction between the particle and the plane is 22·4 N.

Example 4 A particle of mass 10 kg slides down a smooth plane inclined at 30° to the horizontal. If the particle starts from rest, find its speed after sliding a distance of 5 m.



Let the acceleration of the particle down the plane be $a\,\mathrm{m\,s^{-2}}$.
Applying Newton's 2nd law down the plane

$$\swarrow \quad 10g\sin 30° = 10a$$
$$\therefore \quad a = g\sin 30° = 4{\cdot}9$$

Using the formula $v^2 = u^2 + 2as$, where $u = 0$, $s = 5$, $a = 4{\cdot}9$,

$$v^2 = 0 + 2 \times 4{\cdot}9 \times 5 = 49 \qquad \therefore \quad v = 7$$

Hence the particle attains a speed of $7\,\mathrm{m\,s^{-1}}$.

Exercise 5.2

1. Find the weight in newtons of an object with mass
(a) 10 kg, (b) 1·5 kg, (c) 250 grammes.

2. A particle of mass 2 kg is being lifted by a vertical force of 32 N. Find its acceleration.

3. A package of mass 8 kg is being lowered by means of a vertical cable with a downward acceleration of $2\,\mathrm{m\,s^{-2}}$. Find the tension in the cable.

4. A boy of mass 40 kg is ascending in a lift. Find the force exerted by the floor of the lift on the boy when the lift is (a) accelerating at $1{\cdot}2\,\mathrm{m\,s^{-2}}$, (b) moving with constant speed, (c) decelerating at $0{\cdot}8\,\mathrm{m\,s^{-2}}$.

5. A man descends in a lift carrying a parcel of mass 3 kg. Describe the motion of the lift when the parcel has an apparent weight of (a) 33 N, (b) 27 N, (c) 29·4 N.

6. A box of mass 30 kg is being towed across a smooth horizontal surface by means of a light chain inclined at an angle of 25° to the horizontal. If the box is accelerating at $1\,\mathrm{m\,s^{-2}}$, find the tension in the chain and the reaction between the box and the surface.

7. A particle of mass 6 kg is being pushed across a smooth horizontal plane by a force of 30 N acting downwards at 20° to the vertical. Find the acceleration of the particle and the force the particle exerts on the plane.

8. A particle of mass 0·5 kg is suspended from a string. If the acceleration of the particle is horizontal and of magnitude $2\,\mathrm{m\,s^{-2}}$, find the tension in the string and its angle of inclination to the vertical.

9. A particle of mass 5 kg slides down a smooth plane inclined at an angle of 60° to the horizontal. Find the acceleration of the particle down the plane and the reaction between the particle and the plane.

10. A body of mass 12 kg slides down a smooth plane inclined at an angle of 20° to the horizontal. If it starts from rest, find the time it takes to slide a distance of 10 m.

11. A particle of mass 2 kg is being pulled up a smooth plane inclined at 25° to the horizontal by a string parallel to the plane. Find the tension in the string if the particle is (a) moving with uniform velocity, (b) accelerating at $3\,\mathrm{m\,s^{-2}}$.

12. An object of mass 4 kg is being pushed up a smooth plane inclined at an angle α to the horizontal by a horizontal force of 40 N. If $\sin\alpha = 0{\cdot}6$, find the acceleration of the particle and the reaction between the particle and the plane.

5.3 Connected particles

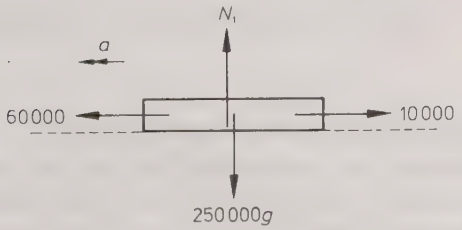
When considering a set of objects which are connected in some way, it is necessary to apply Newton's third law. Any force exerted by one of the objects upon another is matched by an equal and opposite reaction. For instance, in the case of a railway engine pulling a truck, the truck exerts a pull on the engine equal in magnitude to the pull of the engine on the truck. In problems involving such systems it is usually helpful to examine the forces on one or more of the objects separately.

Example 1 An engine of mass 50 tonnes is pulling several trucks of total mass 200 tonnes along a horizontal track. The resistance to the motion of the engine is 60 N per tonne and the resistance to the motion of the trucks is 35 N per tonne. If the tractive force exerted by the engine is 60 kN, find (a) the acceleration, (b) the tension in the coupling between the engine and the trucks.

[1 tonne = 1000 kg, 1 kN = 1000 N.]

Let $a\,\mathrm{m\,s^{-2}}$ be the acceleration and let T N be the tension in the coupling.

(a) Considering the forces on the whole train, the total resistance to motion is $(50 \times 60) + (200 \times 35)$ N, i.e. 10 000 N.



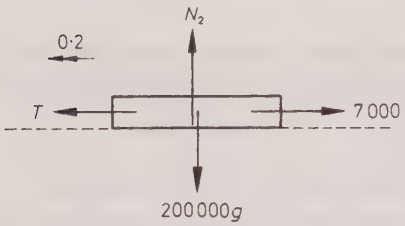
Applying Newton's 2nd law

$\leftarrow 60\,000 - 10\,000 = 250\,000a$

$$\therefore \quad a = \frac{50\,000}{250\,000} = 0{\cdot}2.$$

Hence the acceleration of the train is $0{\cdot}2\,\mathrm{m\,s^{-2}}$.

(b) Considering the forces on the trucks, the resistance to motion is (200×35) N, i.e. 7000 N.



Applying Newton's 2nd law

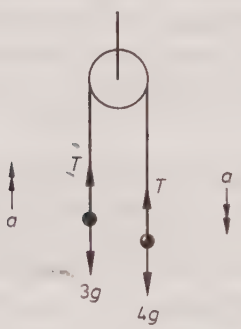
$\leftarrow \quad T - 7000 = 200\,000 \times 0{\cdot}2$

$\therefore \quad T = 40\,000 + 7000 = 47\,000.$

Hence the tension in the coupling is 47 kN.

Each of the remaining examples in this section involves two particles connected by a light inextensible string passing over a smooth fixed light pulley. Under these conditions we can assume that the tension in the string is constant throughout its length.

Example 2 Two particles of mass 3 kg and 4 kg are connected by a light inextensible string passing over a smooth fixed pulley. The particles are released from rest with the strings taut and vertical. Find the acceleration of the particles and the tension in the string.



Let the acceleration be $a\,\mathrm{m\,s^{-2}}$ and the tension T N.
Using Newton's 2nd law

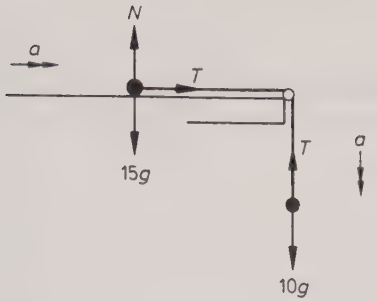
for the 3 kg mass: $T - 3g = 3a$ (1)
for the 4 kg mass: $4g - T = 4a$ (2)

Adding (1) to (2) $\quad g = 7a \quad \therefore \quad a = \frac{1}{7}g = 1{\cdot}4$

Substituting in (1) $\quad T - 3g = \frac{3}{7}g \quad \therefore \quad T = 33{\cdot}6.$

Hence the acceleration of the system is $1{\cdot}4\,\mathrm{m\,s^{-2}}$ and the tension in the string is 33·6 N.

Example 3 A particle of mass 15 kg rests on a smooth horizontal table. It is connected by a light inextensible string passing over a smooth pulley fixed at the edge of the table to a particle of mass 10 kg which hangs freely. Find the acceleration of the system when it is released from rest. Find also the force exerted by the string on the pulley.



Let $a\,\mathrm{m\,s^{-2}}$ be the acceleration and let $T\,\mathrm{N}$ be the tension in the string. Let $F\,\mathrm{N}$ be the force exerted by the string on the pulley.

Using Newton's 2nd law,

horizontally for the 15 kg mass: $T = 15a$ (1)

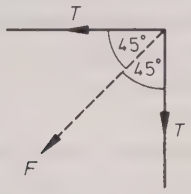
vertically for the 10 kg mass: $10g - T = 10a$ (2)

Adding (1) to (2), $10g = 25a \quad \therefore \quad a = \frac{2}{5}g = 3{\cdot}92.$

Hence the acceleration of the system is $3{\cdot}92\,\mathrm{m\,s^{-2}}$.

Substituting in (1) $T = 15.\frac{2}{5}g = 6g.$

The string exerts on the pulley a force T horizontally and a force T vertically. Since these forces are equal in magnitude, their resultant F acts along the bisector of the angle between the strings.

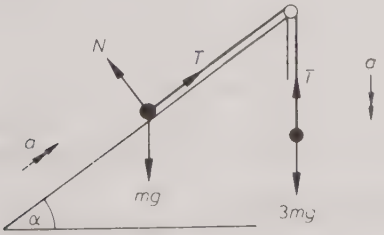


Resolving along this bisector:

$F = 2T\cos 45° = 12g\cos 45° \approx 83{\cdot}2$

Hence the string exerts a force on the pulley of 83·2 N at 45° to the horizontal.

Example 4 A particle of mass m rests on a smooth plane inclined at an angle α to the horizontal where $\sin\alpha = \frac{3}{5}$. It is connected by a light inextensible string passing over a smooth pulley fixed at the top of the plane to a particle of mass 3 m which hangs freely. Find the acceleration of the system when it is released from rest and the tension in the string.



Let a be the acceleration and let T be the tension in the string.

Using Newton's 2nd law,

parallel to the plane for the mass m: $T - mg\sin\alpha = ma$ (1)
vertically for the mass $3m$: $3mg - T = 3ma$ (2)

Adding (1) to (2) $3mg - mg\sin\alpha = 4ma$

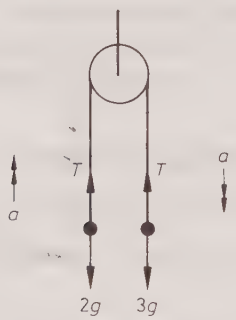
$$\therefore \quad a = \frac{1}{4}g\left(3 - \frac{3}{5}\right) = \frac{3}{5}g.$$

Substituting in (2) $3mg - T = 3m.\frac{3}{5}g$

$$\therefore \quad T = 3mg\left(1 - \frac{3}{5}\right) = \frac{6}{5}mg.$$

Hence the acceleration of the system is $\frac{3}{5}g$ and the tension in the string is $\frac{6}{5}mg$.

Example 5 Two particles of mass 2 kg and 3 kg are connected by a light inextensible string passing over a fixed smooth pulley. Initially the system is at rest with the strings taut and vertical with both particles at a height of 2 m above the ground. When the system is released, find the time which elapses before the 3 kg mass hits the ground and the maximum height reached by the 2 kg mass.



Let $a\,\mathrm{m\,s^{-2}}$ be the acceleration and let $T\,\mathrm{N}$ be the tension in the string.
Using Newton's 2nd law,

for the 2 kg mass: $T - 2g = 2a$ (1)
for the 3 kg mass: $3g - T = 3a$ (2)

(1) + (2) $g = 5a \qquad \therefore \quad a = \frac{1}{5}g.$

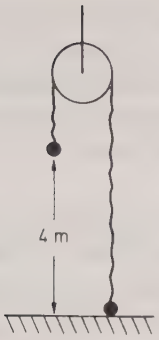
Using the formula $s = ut + \frac{1}{2}at^2$

where $s = 2, u = 0, a = \frac{1}{5}g$

$$2 = 0 + \frac{1}{2}.\frac{1}{5}gt^2$$

$$\therefore \quad t = \sqrt{\left(\frac{20}{g}\right)} = \sqrt{\left(\frac{20}{9{\cdot}8}\right)} = \sqrt{\left(\frac{100}{49}\right)} = \frac{10}{7}.$$

Hence the 3 kg mass hits the ground after $1\frac{3}{7}$ seconds.



position when 3 kg mass hits the ground

Using the formula $v^2 = u^2 + 2as$,

where $u = 0, a = \frac{1}{5}g, s = 2,$

$$v^2 = 0 + 2.\frac{1}{5}g.2 = \frac{4}{5}g.$$

Hence when the 3 kg mass hits the ground both particles are travelling at $\sqrt{(4g/5)}\,\mathrm{m\,s^{-1}}$. Since the string is no longer taut, the 2 kg mass now moves freely under gravity with a downward acceleration of $g\,\mathrm{m\,s^{-2}}$.

Using the formula $v^2 = u^2 + 2as$, where $u = \sqrt{(4g/5)}, v = 0, a = -g$:

$$0 = \frac{4}{5}g - 2gs \qquad \therefore \quad s = \frac{2}{5}.$$

Hence the 2 kg mass rises a further 0·4 m, reaching a maximum height above the ground of 4·4 m.

Exercise 5.3

1. A motor-boat of mass 1000 kg is towing a water-skier of mass 75 kg by means of a horizontal cable. The resistance to the motion of the boat is 650 N and the resistance to the motion of the skier is 125 N. If the thrust of the boat's engine is 4000 N, find (a) the acceleration, (b) the tension in the cable.

2. A car of mass 800 kg tows a trailer of mass 400 kg. The resistance to motion for both car and trailer is 0·2 N per kg. Find the tractive force exerted by the car engine and the tension in the coupling between the car and the trailer when they are travelling (a) with uniform velocity, (b) with an acceleration of $2\,\mathrm{m\,s^{-2}}$.

3. A railway engine of mass 60 tonnes is pulling two trucks each of mass 12 tonnes along a horizontal track. The resistance to motion is 80 N per tonne for the engine and 50 N per tonne for the trucks. Given that the train is travelling at constant speed, find the tractive force exerted by the engine, the tension in the coupling between the engine and the first truck and the tension in the coupling between the two trucks. If the tractive force is increased to 27 kN, find the acceleration and the new tensions in the couplings.

4. Two particles of mass 2 kg and 5 kg are connected by a light inextensible string passing over a smooth fixed pulley. If the system is moving freely with the strings taut and vertical, find the acceleration of the particles and the tension in the string.

5. Repeat question 4 for particles of mass 9 kg and 3 kg.

6. Repeat question 4 for particles of mass m and M, where $m < M$.

7. Two particles of mass $2m$ and $3m$ are connected by a light inextensible string passing over a smooth fixed pulley. If the system is moving freely with the strings vertical, find the force exerted by the string on the pulley.

8. A particle of mass 3·6 kg rests on a smooth horizontal table. It is connected by a light inextensible string passing over a smooth pulley fixed at the edge of the table to a particle of mass 1·2 kg which hangs freely. Find the acceleration of the system when it is released from rest and the tension in the string.

9. A particle of mass m rests on a smooth horizontal table. It is connected by a light inextensible string passing over a smooth pulley fixed at the edge of the table to a second particle of mass m which hangs freely. If the system is released from rest find the distance travelled in the first 0·4 seconds of the subsequent motion. Find also the force exerted by the string on the pulley.

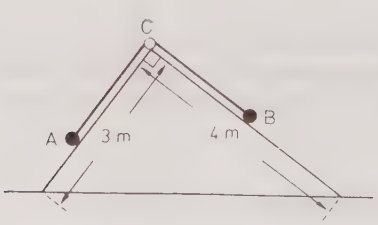
10. A particle of mass 2 kg rests on a smooth plane inclined at an angle of 30° to the horizontal. It is connected by a light inextensible string passing over a smooth pulley fixed at the top of the plane to a particle of mass 3 kg which hangs freely. Find the acceleration of the system when it is released from rest and the tension in the string. Find also the force exerted by the string on the pulley.

11. A particle of mass m rests on a smooth plane inclined at an angle α to the horizontal. It is connected by a light inextensible string passing over a smooth pulley fixed at the top of the plane to a particle of mass M which hangs freely. Find the condition that the mass m should slide down the plane when the system is released from rest.

12. Two particles of mass $3m$ and $5m$ are connected by a light inextensible string passing over a fixed smooth pulley. The system is released from rest with the strings taut and vertical. After 2 seconds the $5m$ mass hits the ground. Find the further time which elapses before the $3m$ mass reaches its greatest height.

13. Two particles of mass 0·5 kg and 0·7 kg are connected by a light inextensible string passing over a fixed smooth pulley. Initially both parts of the string are taut and vertical, and the 0·5 kg mass is moving vertically downwards with a speed of 3·5 m s^{-1}. Find the distance it travels before coming instantaneously to rest.

14.



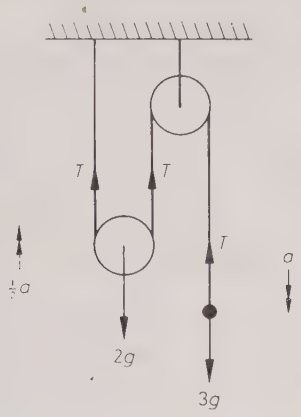
Two particles A and B, of mass 5 kg, rest on the smooth inclined faces of a fixed wedge as shown. They are connected by a light inextensible string passing over a smooth pulley C which lies in the same vertical plane as the particles. Find the acceleration of the

system when moving freely and the tension in the string. If the system is released from rest with the string taut and both particles 2 m from the pulley, find the speed of the particles when A hits the horizontal plane. Find also the distance that B has travelled when it first comes to rest.

5.4 Related accelerations

Some problems involve bodies moving with different but related accelerations. A system including both fixed and moving pulleys can be dealt with by the methods of the previous section when the relationships between the accelerations have been established.

Example 1 A light inextensible string attached to the ceiling passes under a smooth movable pulley of mass 2 kg and then over a smooth fixed pulley. A particle of mass 3 kg hangs freely from the end of the string. All parts of the string not touching the pulleys are vertical. If the system is released from rest, find the acceleration of the particle and the tension in the string.



If the particle moves a distance s m, then the movable pulley moves a distance $\frac{1}{2}s$ m in the opposite direction. Hence if the acceleration of the particle is $a\,\mathrm{m\,s^{-2}}$ downwards, then the acceleration of the movable pulley is $\frac{1}{2}a\,\mathrm{m\,s^{-2}}$ upwards.

Using Newton's 2nd law,

for the particle	$3g - T = 3a$	(1)
for the pulley	$2T - 2g = 2.\frac{1}{2}a$	(2)
$2\times(1)+(2)$	$4g = 7a$	

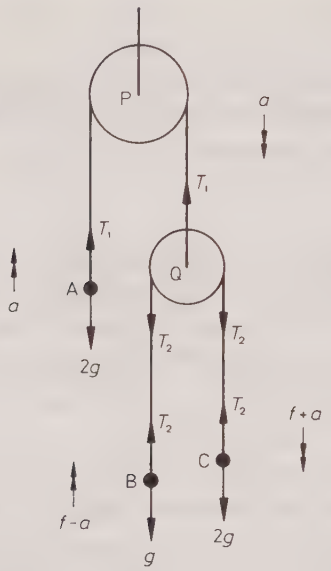
$$\therefore \quad a = \frac{4}{7}g = \frac{4}{7} \times 9{\cdot}8 = 5{\cdot}6.$$

Substituting in (1) $3g - T = 3.\frac{4}{7}g$

$$\therefore \quad T = \frac{9}{7}g = \frac{9}{7} \times 9{\cdot}8 = 12{\cdot}6.$$

Hence the acceleration of the particle is $5{\cdot}6\,\mathrm{m\,s^{-2}}$ and the tension in the string is 12·6 N.

Example 2 A light inextensible string which passes over a smooth fixed pulley P carries at one end a particle A of mass 2 kg and at the other end a smooth light pulley Q. A light inextensible string passes over pulley Q and carries at its ends a particle B of mass 1 kg and a particle C of mass 2 kg. Find the acceleration of particle A when the system is moving freely and the tensions in the two strings.



The tensions and accelerations in the different parts of the system are shown in the diagram, f being the acceleration of particles B and C relative to pulley Q.

Applying Newton's 2nd law, to pulley Q (which is assumed to have zero mass):

$$2T_2 - T_1 = 0 \qquad (1)$$

to A: $\quad T_1 - 2g = 2a \qquad (2)$

to B: $\quad T_2 - g = f - a \qquad (3)$

to C: $\quad 2g - T_2 = 2(f+a) \qquad (4)$

(1)+(2) $\quad 2T_2 - 2g = 2a \qquad \therefore \quad T_2 = a+g.$

Substituting in (3) $\quad a+g-g = f-a \qquad \therefore \quad f = 2a$

Substituting in (4) $\quad 2g-(a+g) = 2(2a+a)$

$$g - a = 6a$$

$$\therefore \quad a = \frac{1}{7}g = 1{\cdot}4.$$

Hence the acceleration of particle A is $1{\cdot}4\,\mathrm{m\,s^{-2}}$ upwards.

Thus $\quad T_2 = a+g = 1{\cdot}4+9{\cdot}8 = 11{\cdot}2$

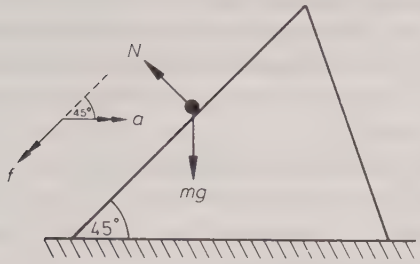
and $\quad T_1 = 2T_2 = 22{\cdot}4.$

Hence the tensions in the strings over pulleys P and Q are 22·4 N and 11·2 N respectively.

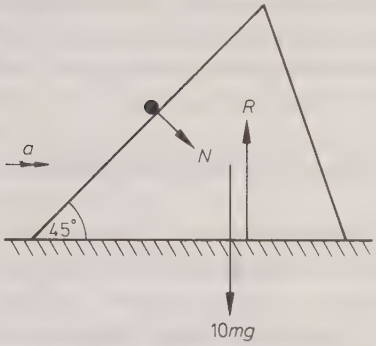
The last example concerns two bodies in contact, which are free to move with different accelerations. In problems of this kind it is helpful to consider the acceleration of one body relative to the other.

Example 3 A particle of mass m rests on a smooth face of a wedge which stands on a smooth horizontal plane. The face of the wedge is inclined at an angle of 45° to the horizontal and the mass of the wedge is $10m$. Find the acceleration of the wedge when the system is released.

Motion of the particle



Motion of the wedge



Let a be the acceleration of the wedge and let f be the acceleration of the particle relative to the wedge. [This means that the true acceleration of the particle has components of magnitudes a and f as shown in the diagram.]

Applying Newton's second law to the motion of the particle in a direction perpendicular to the wedge:

$$\searrow mg\cos 45° - N = ma\sin 45°.$$

Applying Newton's second law horizontally to the motion of the wedge:

$$\rightarrow N\sin 45° = 10ma$$

Hence
$$mg.\frac{1}{\sqrt{2}} - N = ma.\frac{1}{\sqrt{2}} \qquad (1)$$

and
$$N.\frac{1}{\sqrt{2}} = 10ma \qquad (2)$$

$\frac{1}{\sqrt{2}} \times (1)+(2)$
$$\tfrac{1}{2}mg = \tfrac{1}{2}ma + 10ma$$

$$\therefore \quad g = a + 20a$$

i.e.
$$a = \frac{g}{21} = \frac{9{\cdot}8}{21} = \frac{1{\cdot}4}{3} = \frac{7}{15}.$$

Hence the acceleration of the wedge is $7/15\ \mathrm{m\,s^{-2}}$ horizontally.

Exercise 5.4

1. A light inextensible string attached to the ceiling passes under a smooth movable pulley of mass 6 kg and then over a smooth fixed pulley. A particle of mass 1 kg hangs freely from the end of the string. All parts of the string not touching the pulleys are vertical. If the system is released from rest, find the acceleration of the particle and state its direction. Find also the tension in the string.

2. Repeat question 1 for a movable pulley of mass 5 kg and a particle of mass 4 kg.

3. A light inextensible string which passes over a smooth fixed pulley P carries at one end a particle A of mass 2 kg and at the other end a smooth light pulley Q. Particles B and C of masses 4 kg and 5 kg respectively are connected by a light inextensible string passing over pulley Q. Find the accelerations of particles A and B when the system is moving freely. Find also the tensions in the two strings.

4. Repeat question 3 for particles A, B and C of masses 4 kg, 1 kg and 2 kg.

5. Repeat question 3 for particles A, B and C of masses 5 kg, 2 kg and 1 kg. Assume in this case that pulley Q has a non-zero mass of 1 kg.

6. A particle of mass 2 kg is in contact with a smooth face of a wedge which stands on a smooth horizontal plane. The face of the wedge is inclined at 30° to the horizontal and the mass of the wedge is 4 kg. Find the acceleration of the wedge when the system is released.

7. A particle of mass m is placed on a smooth face of a wedge which stands on a smooth horizontal plane. The face of the wedge is inclined at an angle α to the horizontal and the mass of the wedge is $4m$. If the system is released from rest, find the speed of the particle relative to the wedge after 1 second.

8. A particle of mass 1 kg is placed on the smooth horizontal upper face of a wedge, which is itself standing on a smooth plane inclined at an angle α to the horizontal. The mass of the wedge is 3 kg and $\sin\alpha = \frac{1}{4}$. Find the accelerations of the particle and the wedge when the system is released.

Exercise 5.5 (miscellaneous)

1. Two forces of magnitude 10 N and 7 N act on a particle producing an acceleration of $4\,\mathrm{m\,s^{-2}}$. If the forces act at an angle of 60°, find the mass of the particle.

2. A body of mass 5 kg rests on a smooth horizontal plane. Horizontal forces of magnitudes 6, 12, 5 and F newtons act on the body in the directions 015°, 130°, 180° and 270° respectively. Given that the acceleration of the body is $3\,\mathrm{m\,s^{-2}}$, find the value of F and the direction of the acceleration.

3. A vehicle of mass 2000 kg is travelling along a straight horizontal road at 90 km/h. It is brought to rest by a constant force of F newtons in a distance of 500 m. Find F and the time taken for the car to come to rest.

4. A particle A of mass 2 kg is acted upon by a constant force of magnitude 10 N in the direction of the vector $3\mathbf{i} - 4\mathbf{j}$ and a constant force of magnitude 50 N in the direction of the vector $7\mathbf{i} + 24\mathbf{j}$. Find the acceleration of A. Initially the particle is at the origin moving with speed $39\,\mathrm{m\,s^{-1}}$ in the direction of the vector $12\mathbf{i} - 5\mathbf{j}$. Find the position vector of the particle 2 s later. (O & C)

5. A particle of mass 3 kg is being pulled across a smooth horizontal surface by a string inclined at an angle of 35° to the horizontal. If the particle started from rest and moved a distance of 5 m in the first 2 seconds of its motion, find the constant tension in the string.

6. A man of mass 70 kg is travelling in a lift. Find the force between the man and the floor of the lift when the lift is (a) ascending with an acceleration of $1{\cdot}5\,\mathrm{m\,s^{-2}}$, (b) descending with an acceleration of $0{\cdot}5\,\mathrm{m\,s^{-2}}$.

7. A load of 200 kg is lifted by means of a vertical cable through a distance of 50 m. The load accelerates uniformly from rest for 20 seconds, then continues with constant speed for 10 seconds. It then decelerates uniformly to rest in a further 10 seconds. Find the maximum speed attained and the tensions in the cable at every stage of the motion.

8. A package of mass 80 kg is suspended by a rope from a helicopter. Find the tension in the rope and its angle of inclination to the vertical when the helicopter is flying (a) with uniform velocity, (b) with a constant acceleration of $2{\cdot}5\,\mathrm{m\,s^{-2}}$ horizontally.

9. A particle of mass m slides down a smooth plane inclined at an angle α to the horizontal. Find the speed attained by the particle when it has travelled a distance x from rest.

10. A particle of mass 5 kg rests on a smooth plane inclined at an angle α to the horizontal, where $\sin\alpha = 0{\cdot}2$. If the particle is projected up the plane with an initial speed of $7\,\mathrm{m\,s^{-1}}$, find the distance up the plane that it travels.

11. A ring of mass m kg slides down a smooth wire inclined at an angle α to the horizontal. The ring is released 4·9 m vertically above a horizontal plane and hits the plane t seconds later. Find (a) the value of t if $\alpha = 30°$, (b) the value of α if $t = 1{\cdot}2$.

12. A car of mass 1200 kg is pulling a trailer of mass 500 kg up a road inclined at an angle α to the horizontal where $\sin\alpha = 0{\cdot}1$. The resistance to motion for both car and trailer is 0·15 N per kg. Find the tractive force exerted by the car engine and the tension in the coupling between the car and the trailer if they are decelerating at $0{\cdot}5\,\mathrm{m\,s^{-2}}$.

13. Two particles of mass 1·5 kg and 2 kg are connected by a light inextensible string passing over a smooth fixed pulley. If the system is released from rest with the strings taut and vertical, find the distance moved by the 1·5 kg mass relative to the 2 kg mass in the first $1\frac{1}{2}$ seconds of the subsequent motion.

14. A particle of mass 4 kg rests on a smooth horizontal platform at a distance of 2·5 m from the edge. It is connected by a light inextensible string passing over a smooth pulley at the edge of the platform to a particle of mass 0·5 kg which is

hanging freely. If the particles are released from rest, find the speed of the 4 kg mass when it reaches the edge of the platform given that initially the 0·5 kg mass is (a) 3 m, (b) 1·6 m above the ground.

15. A particle of mass 10 kg rests on a smooth plane inclined at an angle of 40° to the horizontal. It is connected by a light inextensible string passing over a smooth pulley fixed at the top of the plane to a particle of mass 4 kg which is hanging freely. Find the acceleration of the system when it is released from rest and the force exerted by the string on the pulley.

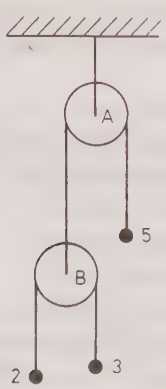
16. A particle A of mass 4 kg is connected by a light inextensible string passing over a smooth fixed pulley to a particle B of mass 3 kg. Particle B is connected by another light inextensible string to a particle C of mass 3 kg. The system is released from rest with the strings taut and vertical. Find the tensions in the strings. Given that B and C are initially 2 m and 1 m respectively above the ground, find the distance travelled by A before it first comes to rest.

17. A particle A of mass $2m$ is initially at rest on a smooth plane inclined at an angle α to the horizontal. It is supported by a light inextensible string which passes over a smooth light pulley P at the top edge of the plane. The other end of the string supports a particle B, of mass m, which hangs freely. Given that the system is in equilibrium, find α, and the magnitude and direction of the resultant force exerted by the string on the pulley.

A further particle of mass m is now attached to B and the system is released. Find, for the ensuing motion, the tension in the string, the acceleration of B and the magnitude and direction of the resultant force exerted by the string on the pulley. (L)

18. A light inextensible string has one end attached to a ceiling. The string passes under a smooth movable pulley of mass m and then over a smooth fixed pulley. A particle of mass $2m$ is attached at the free end of the string. The sections of the string not in contact with the pulleys are vertical. If the system is released from rest and moves in a vertical plane, show that the acceleration of the particle is $2g/3$ and find the tension in the string. (AEB 1977)

19.



The figure shows a light inextensible string which passes over a smooth light fixed pulley A and carries at one end a mass 5 kg and at the other end a smooth light pulley B. A light inextensible string passes over the pulley B and carries masses 3 kg and 2 kg at its ends. The system is in motion with the masses moving vertically. Find the tensions in the two moving strings stating the units. (C)

20. A crate of mass 50 kg is suspended by a light cable from a helicopter. The helicopter has a horizontal acceleration of $2\,m\,s^{-2}$. The crate is being winched aboard the helicopter with an acceleration relative to the helicopter of $0{\cdot}5\,m\,s^{-2}$. Find the tension in the cable and its angle of inclination to the vertical.

6 Friction; Hooke's law

6.1 The laws of friction

When two objects are in contact each exerts a force upon the other. As stated in Newton's third law, these forces are equal and opposite. The force exerted by a smooth surface on a body in contact with it is perpendicular to the surface. Thus there is no resistance to sliding motion across the surface. However, the force exerted by a rough surface on a body has a component perpendicular to the surface, but may also have a component which acts along or tangentially to the surface and which opposes sliding or a tendency to slide. These components are referred to as the *normal reaction* **N** and the *frictional force* **F**. Their resultant is called the *total reaction* **R**.

It can be shown experimentally that the following laws provide a fairly good approximation to the behaviour of frictional forces.

(1) When two rough surfaces are in contact friction acts to oppose the sliding of one surface relative to the other.

(2) When there is no sliding the frictional force is just sufficient to prevent the relative motion of the surfaces. However, the friction between two surfaces cannot exceed a maximum value called *limiting friction*.

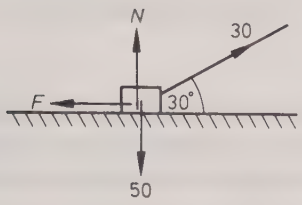
(3) The limiting value of the frictional force F is proportional to the normal reaction N. Thus the maximum value of F is μN, where μ is a constant called the *coefficient of friction*.

(4) The value of μ depends only on the nature of the two surfaces involved.

(5) When sliding occurs the frictional force takes its limiting value μN and acts in the direction that opposes the relative motion of the surfaces.

[In practice, the magnitude of the frictional force that acts during sliding is found to be slightly less than the limiting value reached just before sliding begins. Thus for any pair of surfaces the coefficient of sliding friction is slightly less than the coefficient of static friction. However, in elementary work the difference is assumed to be negligible and a single value of μ is used.]

Example 1 A block of weight 50 N rests in equilibrium on a rough horizontal plane. It is attached to a string inclined at 30° to the horizontal and the tension in the string is 30 N. Find the magnitude of the frictional force acting on the block.



Resolving horizontally

$\leftarrow F - 30\cos 30° = 0$

$\therefore \quad F = 30 \times \frac{\sqrt{3}}{2} = 15\sqrt{3}.$

Hence the magnitude of the frictional force is $15\sqrt{3}$ N.

Example 2 A block of weight 50 N rests on a horizontal plane. The coefficient of friction between the block and the plane is 0·4. A horizontal force of P N is applied to the block. Find the value of P given that the block is just about to slide.

Resolving horizontally and vertically,

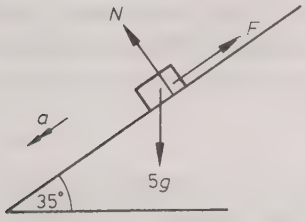
$\leftarrow F - P = 0, \quad \text{i.e.} \quad F = P$

$\uparrow N - 50 = 0, \quad \text{i.e.} \quad N = 50.$

When the block is just about to slide friction is limiting, i.e. $F = \mu N$

$\therefore \quad P = 0{\cdot}4 \times 50 = 20.$

Example 3 A block of mass 5 kg is sliding down a plane inclined at 35° to the horizontal. If the coefficient of friction between the block and the plane is 0·3, find the acceleration of the block.



Let the acceleration of the block be a m s^{-2} down the plane.

Applying Newton's second law parallel and perpendicular to the plane

$\swarrow \quad 5g\sin 35° - F = 5a \qquad (1)$

$\nwarrow \quad N - 5g\cos 35° = 0 \qquad (2)$

Since the block is sliding, $F = \mu N$

$\therefore$ using (2), $F = 0{\cdot}3 \times 5g\cos 35°.$

Substituting in (1) $5g\sin 35° - 0{\cdot}3 \times 5g\cos 35° = 5a$

$\therefore \quad a = g(\sin 35° - 0{\cdot}3\cos 35°) \approx 3{\cdot}21.$

Hence the acceleration of the block down the plane is 3·21 m s^{-2}.

Exercise 6.1

1. A block of weight 20 N rests in equilibrium on a rough horizontal plane under the action of a force of 10 N. Find the magnitude of the frictional force on the block, given that the external force acts (a) horizontally, (b) vertically downwards, (c) downwards at 30° to the vertical.

2. A block of weight 20 N rests in equilibrium on a rough plane inclined at 30° to the horizontal. Find the magnitude and direction of the frictional force on the block, given that there is a force on the block acting up the plane, which has magnitude (a) 15 N, (b) 10 N, (c) 5 N.

3. A block of weight 12 N rests on a horizontal plane. The coefficient of friction between the block and the plane is $\frac{1}{4}$. A force of P N is applied to the block, such that the block is just about to slide. Find the value of P, given that this force acts (a) horizontally, (b) upwards at 45° to the horizontal, (c) downwards at 45° to the horizontal.

4. A block of weight W rests on a rough plane inclined at 25° to the horizontal. Given that the block is just about to slide down the plane, find the coefficient of friction.

5. A particle of weight 5 N is at rest on a plane inclined at 40° to the horizontal under the action of a force of P N up the plane. The coefficient of friction is 0·2. Find the value of P if the particle is just about to slide (a) up the plane, (b) down the plane.

6. A particle of mass 2 kg is being pulled across a horizontal plane by a horizontal force of magnitude 8 N. Find the acceleration of the particle if the coefficient of friction is $\frac{1}{3}$.

7. A particle of mass 3 kg is being pulled across a rough horizontal plane by a string inclined at 35° to the horizontal. The coefficient of friction between the particle and the plane is 0·6. Find the acceleration of the particle when the tension in the string is 30 N.

8. A particle of mass 10 kg is sliding down a plane inclined at 30° to the horizontal. If the coefficient of friction is 0·2, find the acceleration of the particle.

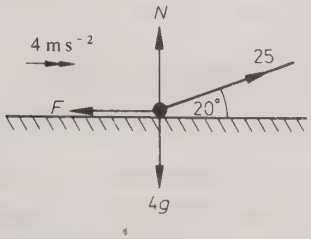
9. A particle of mass 4 kg is sliding down a plane inclined at 20° to the horizontal. If its acceleration is $2\,\mathrm{m\,s^{-2}}$, find the coefficient of friction between the particle and the plane.

10. A particle of mass 5 kg is being pulled up a plane inclined at 45° to the horizontal by a string parallel to the plane. If the coefficient of friction is $\frac{1}{4}$ and the tension in the string is 50 N, find the acceleration of the particle.

6.2 Motion on a rough surface

When a particle is in motion on a rough surface the frictional force F and the normal reaction N are connected by the equation $F = \mu N$. Many problems can be solved using this relationship together with the methods of Chapter 5.

Example 1 A particle of mass 4 kg is being pulled across a rough horizontal plane by a string inclined at an angle of 20° above the horizontal. When the tension in the string is 25 N the particle accelerates at $4\,\text{m s}^{-2}$. Find the coefficient of friction between the particle and the plane.



Applying Newton's second law horizontally and vertically

$\rightarrow 25\cos 20° - F = 4 \times 4$

$\uparrow N + 25\sin 20° - 4g = 0$

$\therefore F = 25\cos 20° - 16$

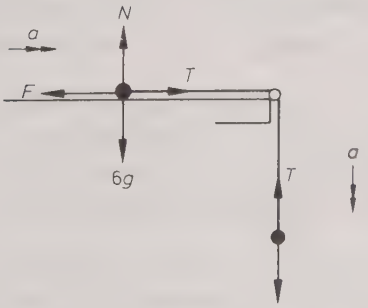
$N = 4g - 25\sin 20°.$

Since the particle is in motion, $F = \mu N$

$$\therefore \quad \mu = \frac{F}{N} = \frac{25\cos 20° - 16}{4 \times 9{\cdot}8 - 25\sin 20°} \approx 0{\cdot}244.$$

Hence the coefficient of friction is 0·244.

Example 2 A particle of mass 6 kg rests on a rough horizontal plane, the coefficient of friction between the particle and the plane being 0·4. The particle is connected by a light inextensible string passing over a smooth pulley at the edge of the plane to a second particle of mass 3 kg which hangs freely. If the system is released from rest, find the distance moved by each of the particles in the first 3 seconds of the subsequent motion.



Let the tension in the string be T N and the acceleration of the particles $a\,\text{m s}^{-2}$.

Using Newton's second law for the 6 kg mass:

$\rightarrow \quad T - F = 6a \qquad (1)$

$\uparrow N - 6g = 0 \qquad (2)$

for the 3 kg mass: $\downarrow 3g - T = 3a \qquad (3)$

When the 6 kg mass is sliding $F = \mu N$

$\therefore$ using (2) $F = 0{\cdot}4N = 0{\cdot}4 \times 6g = 2{\cdot}4g$

Substituting in (1) $T - 2{\cdot}4g = 6a$

Adding (3) $0{\cdot}6g = 9a$

$$\therefore \quad a = \frac{1}{9} \times \frac{3}{5}g = \frac{1}{15}g.$$

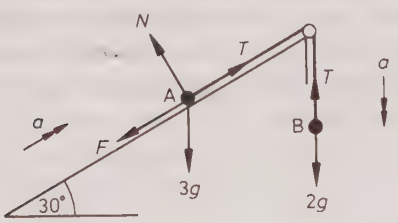
Using the formula $s = ut + \frac{1}{2}at^2$, where $u = 0$, $t = 3$, $a = \frac{g}{15}$

$$s = 0 + \frac{1}{2} \times \frac{g}{15} \times 9 = \frac{3g}{10} = 2{\cdot}94.$$

Hence the distance moved by the particles in the first 3 seconds is 2·94 m.

Example 3 A particle A of mass 3 kg rests on a rough plane inclined at 30° to the horizontal, the coefficient of friction between the particle and the plane being 1/10. It is connected by a light inextensible string passing over a smooth pulley at the top edge of the plane to a particle B of mass 2 kg which hangs freely. Find the acceleration of the system when it is released from rest and the tension in the string.

The component of the weight of particle A acting down the plane is $3g \sin 30°$, i.e. $\frac{3}{2}g$. Since this is less than the weight of particle B, A will have a tendency to move up the plane. Thus the force of friction will act down the plane.



Using Newton's second law, for particle A:

$\nearrow \quad T - F - 3g \sin 30° = 3a$ (1)

$\nwarrow \quad N - 3g \cos 30° = 0$ (2)

for particle B:

$\downarrow \quad 2g - T = 2a.$ (3)

When particle A is sliding $F = \mu N$

$\therefore$ using (2) $F = \frac{1}{10}N = \frac{3}{10}g \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{20}g.$

Substituting in (1) $T - \frac{3\sqrt{3}}{20}g - 3g \cdot \frac{1}{2} = 3a.$

Adding (3) $2g - \frac{3\sqrt{3}}{20}g - \frac{3}{2}g = 5a.$

$$\therefore \quad a = \frac{1}{5}g\left(\frac{1}{2} - \frac{3\sqrt{3}}{20}\right) = \frac{g}{100}(10 - 3\sqrt{3}) \approx 0{\cdot}47.$$

Rearranging (3) $\qquad T = 2(g - a) \approx 18{\cdot}7.$

Hence the acceleration of the particles is approximately $0{\cdot}47\,\mathrm{m\,s^{-2}}$ and the tension in the string 18·7 N.

Exercise 6.2

1. A particle of mass 4 kg is projected across a horizontal plane with a speed of $7\,\mathrm{m\,s^{-1}}$. If it comes to rest in a distance of 10 m, find the magnitude of the frictional force acting on the particle and the coefficient of friction between the particle and the plane.

2. A body of mass 5 kg is at rest on a horizontal plane, the coefficient of friction being 0·3. A horizontal force of 21 N is applied to the body. If after $3\frac{1}{2}$ seconds this force ceases to act, find the further time that elapses before the particle comes to rest.

3. A block of mass 2 kg rests on a horizontal plane. The coefficient of friction is $\frac{1}{2}$. Find the acceleration of the block when a force of 20 N acts on it (a) upwards at an angle of 20° to the horizontal, (b) downwards at an angle of 20° to the horizontal.

4. A particle slides down a plane inclined at an angle of 30° to the horizontal. The coefficient of friction is 1/7. Find the distance the particle slides in 2 seconds from rest.

5. A particle is sliding down a plane inclined at an angle α to the horizontal. Find an expression for the coefficient of friction, given that the particle moves with (a) uniform velocity, (b) constant acceleration a.

6. A block of mass 2 kg rests on a rough plane inclined at an angle α to the horizontal, where $\sin\alpha = 7/25$. The coefficient of friction is $\frac{1}{2}$. Find the acceleration of the block when a force of 50 N acts on it down the plane. Find the magnitude of the force acting parallel to the plane that would give the block an equal acceleration up the plane.

7. A body of mass 8 kg is projected down a plane inclined at an angle of 15° to the horizontal with a velocity of $10\,\mathrm{m\,s^{-1}}$. If the body comes to rest after travelling a distance of 25 m, find the magnitude of the frictional force acting on the body during its motion and the coefficient of friction.

8. A particle is placed at a point O on a plane inclined at 45° to the horizontal. The particle is projected up the plane along a line of greatest slope with velocity $15\,\mathrm{m\,s^{-1}}$. Find the condition that must be satisfied by the coefficient of friction μ if the particle is to return through 0 after it comes to rest. Given that the particle does return through 0 with a velocity of $10\,\mathrm{m\,s^{-1}}$, find the value of μ.

9. A particle of mass 3 kg rests on a rough horizontal table, the coefficient of friction between the particle and the table being $\frac{1}{3}$. The particle is connected by a light inextensible string passing over a smooth pulley at the edge of the table to a second particle of mass 4 kg which hangs freely. If the system is released from rest, find the acceleration of the particles and the tension in the string.

10. A particle A of mass 6 kg rests on a rough horizontal plane. It is connected by a light inextensible string passing over a smooth pulley at the edge of the plane to a particle B of mass 2 kg which hangs vertically. If particle B is projected vertically downwards with a velocity of $3{\cdot}5\,\mathrm{m\,s^{-1}}$, it comes to rest after travelling a distance of 2·5 m. Find the coefficient of friction between particle A and the plane. Find also the force exerted by the string on the pulley during the motion.

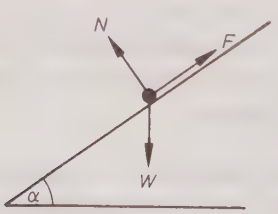
11. A particle of mass 5 kg rests on a rough plane inclined at 25° to the horizontal, the coefficient of friction between the particle and the plane being 0·2. The particle is connected by a light inextensible string passing over a smooth pulley at the top of the plane to a second particle of mass 5 kg which hangs freely. Find the acceleration of the system and the tension in the string.

12. A particle of mass 5 kg rests on a rough plane inclined at an angle α to the horizontal, the coefficient of friction between the particle and the plane being 1/8. The particle is connected by a light inextensible string passing over a smooth pulley at the top of the plane to a particle of mass 3 kg which hangs freely. Find the acceleration of the system when it is released from rest if (a) $\alpha = 10°$, (b) $\alpha = 60°$.

6.3 Equilibrium on a rough surface

When a particle is in equilibrium on a rough surface the frictional force F and the normal reaction N satisfy the inequality $F \leqslant \mu N$. When the particle is on the point of sliding $F = \mu N$. This state is sometimes called limiting equilibrium.

Example 1 A particle is placed on a rough plane inclined at an angle α to the horizontal. The coefficient of friction between the particle and the plane is μ. Find the condition that the particle should be in equilibrium.



If the particle is in equilibrium, resolving along and perpendicular to the plane,

$$\nearrow \quad F - W\sin\alpha = 0$$
$$\nwarrow \quad N - W\cos\alpha = 0$$

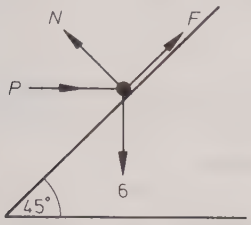
i.e. $F = W\sin\alpha$ and $N = W\cos\alpha$.

Since $F \leqslant \mu N$, $\quad W\sin\alpha \leqslant \mu W\cos\alpha \quad \therefore \quad \tan\alpha \leqslant \mu$.

Hence equilibrium will be possible if $\mu \geqslant \tan\alpha$.

In some problems it is necessary to find the least force required to maintain the equilibrium of a particle on a rough surface. As this least force just prevents motion, when it is applied the frictional force F takes its limiting value μN.

Example 2 A particle of weight 6 N is placed on a rough plane inclined at 45° to the horizontal, the coefficient of friction being $\frac{1}{2}$. Find the magnitude of the least horizontal force required to maintain equilibrium.



Consider a horizontal force of P N acting on the particle to maintain equilibrium.

Resolving along and perpendicular to the plane

$$\nearrow \quad P\cos 45° + F - 6\sin 45° = 0$$
$$\nwarrow \quad N - P\sin 45° - 6\cos 45° = 0$$

$$\therefore \quad F = 6\sin 45° - P\cos 45° = \frac{1}{\sqrt{2}}(6-P)$$
$$N = 6\cos 45° + P\sin 45° = \frac{1}{\sqrt{2}}(6+P).$$

The value of P is least when friction is limiting, i.e. when

$$F = \mu N$$

$$\therefore \quad \text{since} \quad \mu = \tfrac{1}{2}, \quad \frac{1}{\sqrt{2}}(6-P) = \frac{1}{2}\cdot\frac{1}{\sqrt{2}}(6+P)$$

i.e. $12 - 2P = 6 + P \qquad \therefore \quad P = 2$

Hence the least horizontal force required to maintain equilibrium is 2 N.

[A similar method can be used to find the least force required to move a particle across a rough surface.]

Exercise 6.3

1. A particle is at rest on a rough plane inclined at an angle α to the horizontal. The coefficient of friction is $\frac{1}{4}$. Given that the particle is on the point of sliding down the plane, find α.

2. A horizontal force of magnitude P N acts on a particle of weight 10 N at rest on a plane inclined at 40° to the horizontal. The coefficient of friction between the particle and the plane is 0·2. Find the value of P if the particle is just about to slide (a) up the plane, (b) down the plane.

3. A particle of weight 3 N rests on a rough horizontal plane. The coefficient of friction is μ. A force of 1 N acts on the particle. Find the condition that the particle should be in equilibrium if this force acts (a) horizontally, (b) upwards at 30° to the horizontal, (c) downwards at 30° to the horizontal.

4. A body of weight 4 N is placed on a rough plane inclined at an angle α to the horizontal. The coefficient of friction between the body and the plane is $\frac{1}{2}$. A force of 1·5 N acting downwards parallel to the plane is applied to the particle. Prove that the condition that the particle should be in equilibrium is $3\sec\alpha + 8\tan\alpha \leqslant 4$.

5. A body of weight 6 N is kept in equilibrium on a rough plane inclined at 30° to the horizontal by a force of P N acting up the plane. If the least value of P for which equilibrium is maintained is 1·5, find the coefficient of friction.

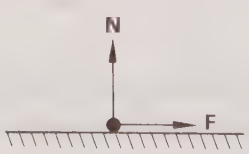
6. A block of weight 5 N is on a plane inclined at 50° to the horizontal. The coefficient of friction is 0·3. The block is kept in equilibrium by a force of P N acting up the plane. Find the range of possible values of P.

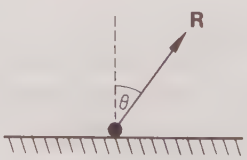
7. A particle of weight 20 N rests on a rough plane inclined at an angle α to the horizontal, where $\tan\alpha = \frac{3}{4}$. The coefficient of friction is $\frac{1}{4}$. The particle moves up the plane under the action of a force of P N. Find the least value of P, given that this force acts (a) up the plane, (b) horizontally.

8. A particle of mass M rests on a rough horizontal plane, the coefficient of friction being $\frac{1}{3}$. The particle is connected by a light inextensible string passing over a smooth pulley at the edge of the plane to a particle of mass m which hangs vertically. Find the condition that the system should remain at rest when released.

6.4 Angle of friction

The total reaction **R** of a surface on a body in contact with it is the resultant of the normal reaction **N** and the frictional force **F**.





If **R** acts at an angle θ to the normal to the surface then

$$F = R\sin\theta \quad \text{and} \quad N = R\cos\theta.$$

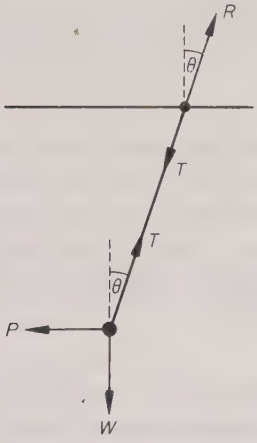
Since $F \leqslant \mu N$, $R\sin\theta \leqslant \mu R\cos\theta$, i.e. $\tan\theta \leqslant \mu$.

If λ is the acute angle such that $\tan\lambda = \mu$, then $\tan\theta \leqslant \tan\lambda$ and hence $\theta \leqslant \lambda$. The angle λ is called the *angle of friction*. It is the maximum angle which the total reaction can make with the normal to the surface. Thus the total reaction can lie anywhere on or within a 'cone of friction' with the normal to the surface as axis and λ as semi-vertical angle.

Some problems are simplified by using the total reaction **R** instead of its components **F** and **N**.

Example 1 A particle of weight **W** is suspended by a light inextensible string from a light ring. The ring can slide along a rough horizontal rod, the coefficient of friction being $\frac{1}{3}$. A horizontal force **P** is applied to the particle. Given that the system is at rest with the ring on the point of sliding along the rod, find P in terms of W.

Since the only forces acting on the ring are the tension in the string and the reaction of the rod on the ring, these forces must be equal and opposite. If the reaction makes an angle θ with the vertical, then since the ring is about to slide, $\theta = \lambda$.



Resolving the forces on the particle horizontally and vertically

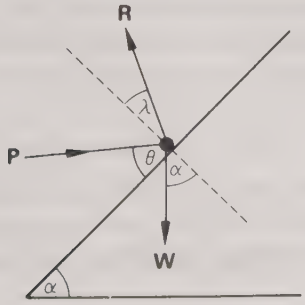
$$\leftarrow \quad P - T\sin\lambda = 0$$
$$\downarrow \quad W - T\cos\lambda = 0$$

$$\therefore \quad P = T\sin\lambda, \quad W = T\cos\lambda,$$

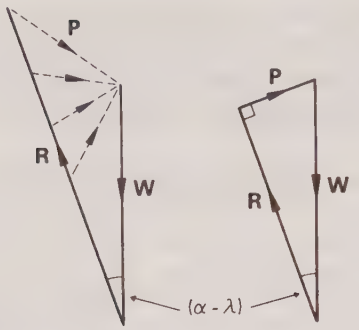
Hence $\dfrac{P}{W} = \tan\lambda = \frac{1}{3}$ so that

$$P = \tfrac{1}{3}W.$$

Example 2 A particle of weight **W** is placed on a rough plane inclined at an angle α to the horizontal. The coefficient of friction is μ, where $\mu < \tan\alpha$. Find the magnitude and direction of the least force required to prevent motion down the plane.



Suppose that sliding is prevented by a force of magnitude P acting at an angle θ to the plane. When P takes its least value, friction will be limiting and the reaction **R** will make an angle λ with the normal to the plane, where $\tan\lambda = \mu$.



Assuming that the particle is on the point of sliding, the value of P depends on the value chosen for θ. By considering the triangle of forces as θ varies, we see that the magnitude of **P** is least when its direction is perpendicular to that of **R**, i.e. when $\theta = \lambda$. Hence the least force required to prevent motion down the plane is of magnitude $W \sin(\alpha - \lambda)$ and acts at an angle λ to the plane where $\tan \lambda = \mu$.

Exercise 6.4

1. A particle of weight 12 N is suspended by a light inextensible string from a light ring. The ring can slide along a rough horizontal rod, the coefficient of friction being $\frac{1}{2}$. A force of P N is acting on the particle. The system is in equilibrium with the ring on the point of sliding along the rod. Find P given that the force is acting (a) horizontally, (b) upwards at 45° to the horizontal, (c) downwards at 30° to the horizontal.

2. One end of a light inextensible string is attached to a fixed point on a horizontal rod. The other end is fastened to a light ring which can slide on the rod. The coefficient of friction is $\frac{3}{4}$. A particle of weight 20 N is suspended from the mid-point of the string. Given that the system is in equilibrium with the ring on the point of sliding along the rod, find the tension in each part of the string and the angle of inclination to the vertical.

3. A block of weight 8 N rests on a horizontal plane. The coefficient of friction between the block and the plane is $\frac{1}{4}$. Find the magnitude and direction of the least force required to move the block.

4. A particle of weight W rests on a horizontal plane. If the angle of friction is λ, find the magnitude and direction of the least force required to move the block.

5. A particle of weight 6 N is placed on a rough plane inclined at 30° to the horizontal. The coefficient of friction is $\frac{1}{3}$. Find the magnitude and direction of the least force required (a) to prevent motion down the plane, (b) to move the particle up the plane.

6. A particle of weight W is placed on a rough plane inclined at an angle α to the horizontal. The angle of friction is λ, where $\lambda > \alpha$. Find the magnitude and direction of the least force required to move the particle (a) down the plane, (b) up the plane.

6.5 Hooke's† law

In this section we examine the behaviour of springs and elastic strings. The length of an elastic string when it is unstretched is called its *natural length*. When the string is stretched by forces exerted on its ends, its increase in length is called the *extension* of the string. It can be shown experimentally that as an elastic string is stretched the tension in the string is proportional to the extension. This result is called *Hooke's law* after the man who discovered it.

When a string is stretched beyond a certain point it ceases to obey Hooke's law. However, in elementary work with elastic strings it is assumed that conditions are such that Hooke's law can be used.

For a string of natural length l, tension T and extension x, the law is written in the form

$$T = \lambda \frac{x}{l}$$

where λ is a constant called the *modulus of elasticity*. The value of λ depends on the nature of the string in use. Since x/l is a ratio of two lengths it has a purely numerical value. Hence the modulus λ must be measured in the same units as T, i.e. in units of force. By writing $x = l$, we see that λ is equal to the tension required to double the length of an elastic string.

Example 1 An elastic string of natural length 1·5 m is stretched to 2 m. If its modulus of elasticity is 12 N, find the tension in the string.

By Hooke's law $$T = \frac{\lambda x}{l}.$$

Putting $\lambda = 12$, $l = 1{\cdot}5$, $x = 0{\cdot}5$:

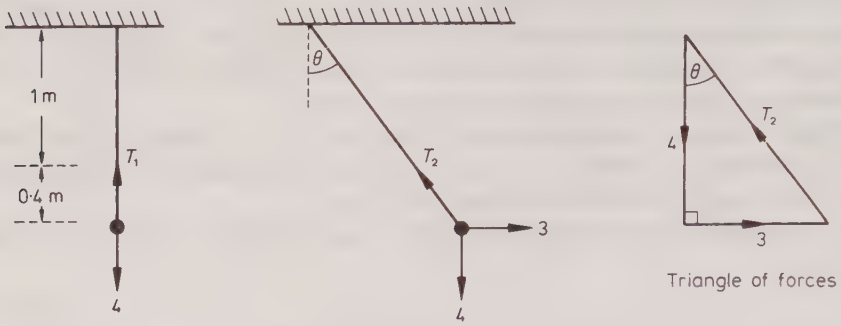
$$T = \frac{12 \times 0{\cdot}5}{1{\cdot}5} = 4$$

$\therefore$ the tension in the string is 4 N.

Example 2 A light elastic string of natural length 1 m is fixed at one end and a particle of weight 4 N is attached to the other end. When the particle hangs freely

†*Hooke, Robert* (1635–1703) English experimental physicist. He contributed to many of the important inventions of his time including the microscope. He worked on planetary motion and proposed a law of gravitation similar to Newton's. A pamphlet containing his theory of elasticity was published in 1678.

in equilibrium the length of the string is 1·4 m. The string is now held at an angle θ to the vertical by a horizontal force of 3 N acting on the particle. Find the value of θ and the new length of the string.



Triangle of forces

In the first equilibrium position $T_1 = 4$.

Using Hooke's law, $T = \dfrac{\lambda x}{l}$, where $T = 4$, $x = 0{\cdot}4$, $l = 1$,

$$4 = \frac{\lambda \times 0{\cdot}4}{1} \qquad \therefore \quad \lambda = 10.$$

Hence the modulus of elasticity for the string is 10 N.

Using the triangle of forces for the second equilibrium position

$$\tan\theta = \tfrac{3}{4} \qquad \therefore \quad \theta = 36° 52'$$

$$T_2 = \sqrt{(3^2 + 4^2)} = 5.$$

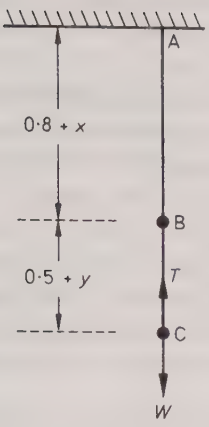
Using Hooke's law for $T = 5, \lambda = 10, l = 1$

$$5 = \frac{10x}{1} \qquad \therefore \quad x = 0{\cdot}5.$$

Hence the value of θ is 36° 52′ and the new length of the string is 1·5 m.

Hooke's law also applies to the stretching of springs.

Example 3 A spring AB of natural length 0·8 m and modulus 16 N is fixed at A. The other end is joined to a second spring BC of natural length 0·5 m and modulus 20 N. A particle of weight W N is then attached at the point C. When the system is hanging freely in equilibrium, the point C is a distance of 2·5 m vertically below A. Find the value of W.



Let the extensions in springs AB and BC be x m and y m respectively.

Applying Hooke's law to both springs

$$T = \frac{16x}{0{\cdot}8} = 20x \quad \text{and} \quad T = \frac{20y}{0{\cdot}5} = 40y$$

Since the tensions in the springs are equal,

$$20x = 40y, \quad \text{i.e.} \quad x = 2y \qquad (1)$$

Considering the two springs together, the natural length is 1·3 m and the stretched length is 2·5 m.

Hence $x + y = 2{\cdot}5 - 1{\cdot}3 = 1{\cdot}2$

Using (1) $3y = 1{\cdot}2$, i.e. $y = 0{\cdot}4$

$$\therefore \quad T = 40y = 40 \times 0{\cdot}4 = 16.$$

Since the particle is in equilibrium, $W = T = 16$.

Thus the value of W is 16.

A spring also obeys Hooke's law when compressed. We again write $T = \lambda x/l$, where in this case T represents a thrust and x a compression, i.e. a reduction in length.

Exercise 6.5

1. An elastic string of natural length 2 m is stretched to 2·5 m. If its modulus of elasticity is 20 N, find the tension in the string.

2. A spring of natural length 60 cm is stretched to a length of 1 m. If the tension in the spring is 6 N, find its modulus of elasticity.

3. A spring of natural length 10 cm is compressed to a length of 8 cm. If its modulus of elasticity is 12 N, find the thrust in the spring.

4. When the tension in an elastic string is 8 N, its length is 1·6 m. If the modulus of elasticity of the string is 24 N, find its natural length.

5. An elastic string of natural length $3a$ is fixed at one end and a particle of weight W is attached to the other end. When the particle hangs freely in equilibrium the length of the string is $5a$. If the string is held at an angle θ to the vertical by a horizontal force of magnitude $\frac{1}{2}W$, find the value of θ and the new length of the string.

6. A particle of mass 3 kg is suspended from an elastic string of natural length 0·5 m and modulus 48 N. If the particle is pulled vertically downwards and then

released when the length of the string is 1 m, find its acceleration at the instant that it is released.

7. An elastic string AB of natural length 1·2 m and modulus 10 N lies along a line of greatest slope of a smooth plane inclined at 30° to the horizontal. The end A is fixed and to the end B is attached a particle of weight 10 N. Find the length of the string when the particle at B rests in equilibrium on the plane.

8. The ends of an elastic string of natural length $4a$ are fixed to points A and B on the same horizontal level, where $AB = 3a$. A particle P of weight W is attached to the mid-point of the string and hangs in equilibrium at a depth of $2a$ below the level of AB. Find the modulus of elasticity of the string in terms of W.

9. A particle P of weight 5·2 N is hanging in equilibrium attached to two strings PA and PB, where A and B are fixed points on the same horizontal level. The string PA is inextensible, but the string PB is elastic with modulus of elasticity 6 N. Given that $AB = 1{\cdot}3$ m, $PA = 0{\cdot}5$ m and $\angle APB = 90°$, find the tension in each string and the natural length of the elastic string.

10. A spring PQ of natural length 1·5 m and modulus λ N is fixed at P. The other end is joined to a second spring QR of natural length 1 m and modulus 2λ N. A particle of weight 15 N is then attached to the end R of the second spring. When the system is hanging freely in equilibrium, the distance PR is 4 m. Find the value of λ.

11. Three light springs of natural lengths 80 cm, 100 cm and 120 cm whose moduli of elasticity are 320 N, 240 N and 180 N respectively are attached end to end and confined in a horizontal tube of length 276 cm. Calculate the thrust on the ends of the tube and the amount by which each spring is compressed.

12. A body of weight 50 N is suspended from a ceiling by a spring of natural length 1 m and modulus of elasticity 120 N. It is supported by another spring of natural length 2 m and modulus 160 N attached to the floor. The springs are vertical, and the distance from floor to ceiling is 3 m. Find the lengths of the springs.

Exercise 6.6 (miscellaneous)

1. A box of weight 400 N will just move across a rough horizontal floor when a horizontal force of 100 N is applied to it. Find the least force required to move the box if it acts (a) upwards at 60° to the horizontal, (b) downwards at 30° to the horizontal.

2. A particle of weight W rests on a horizontal plane, the coefficient of friction between the particle and the plane being $\frac{1}{3}$. When a horizontal force of magnitude P is applied to the particle its acceleration is a. When a horizontal force of magnitude $2P$ is applied, the acceleration is $4a$. Find P in terms of W.

3. A particle of mass 2 kg is placed on a rough plane inclined at an angle α to the horizontal, where $\sin\alpha = 3/5$. The coefficient of friction between the particle and the plane is 1/4. Initially the particle is at rest under the action of a force P N acting upwards parallel to the plane. Find the value of P given that the particle is about to move (a) up the plane, (b) down the plane. If the force is then removed, find the acceleration of the particle down the plane.

4. A block of weight W is placed on a rough plane inclined at an angle α to the horizontal, where $\tan\alpha = 5/12$. The block is on the point of sliding when a force of 5 N acts directly down the plane and when a force of 55 N acts directly up the plane. Find the value of W and the coefficient of friction between the block and the plane.

5. A particle of weight W rests on a rough plane inclined to the horizontal at an angle whose tangent is 2μ, where μ is the coefficient of friction between the particle and the plane. The particle is acted on by a force of magnitude P. (i) Given that the force acts horizontally in a vertical plane through a line of greatest slope, and that the particle is on the point of sliding down the plane, find the magnitude of P. (ii) Given that the force acts along a line of greatest slope and that the particle is on the point of sliding up the plane, find the magnitude of P. (C)

6. Two small light rings which are connected by a light inextensible string of length a can slide on a rough horizontal rod. A weight is attached to the mid-point of the string and the system hangs in equilibrium. If the coefficient of friction between the rings and the rod is μ, show that the greatest possible distance between the rings is $\mu a/(1+\mu^2)^{1/2}$. (O)

7. A mass of 5 kg is moved along a rough horizontal table by means of a light inextensible string which passes over a smooth light pulley at the edge of the table and is attached to a mass of 1 kg hanging over the edge of the table. The system is released from rest with the string taut, each portion being at right angles to the edge of the table. If the masses take twice the time to acquire the same velocity from rest that they would have done had the table been smooth, prove that the coefficient of friction is 1/10. After falling a distance 1/5 m, the 1 kg mass reaches the floor and comes to rest. Prove that the 5 kg mass then moves a further 1/6 m before coming to rest. [Assume throughout that the 5 kg mass does not leave the table.] (O & C)

8. Two particles A and B, of masses 5 kg and 10 kg respectively, lie with B above A on a rough slope inclined at an angle of 30° to the horizontal. The coefficient of friction between the particles and the slope is 0·2. A light inextensible string along a line of greatest slope joins A to B and a force of P N acts on B up the slope. If the system is in equilibrium with both particles just about to move up the slope, find the value of P and the tension in the string.

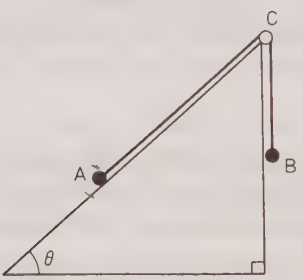
9. Three particles, A, B, C, are of masses 4, 4, 2 kg respectively. They lie at rest on a horizontal table in a straight line, with particle B attached to the mid-point of a light inextensible string. The string has particle A attached at one end and

particle C at the other, and is taut. A force of 60 N is applied to A in the direction CA produced, and a force of 15 N is applied to C in the opposite direction. Find the acceleration of the particles and the tension in each part of the string, (a) if the table is smooth, (b) if the coefficient of friction between each particle and the table is $\frac{1}{4}$. [Take g as $10\,\text{m/s}^2$.] (L)

10. A particle of mass m moves on a line of greatest slope of a plane inclined to the horizontal at an angle θ. The coefficient of friction between the particle and the plane is μ. The particle is released from rest at a point A and it slides down to B, where the distance $AB = d$. Prove that $\mu < \tan\theta$, and that V, the particle's speed at B, is given by $V^2 = 2gd(\sin\theta - \mu\cos\theta)$. The particle is projected up the plane from B with speed U, comes instantaneously to rest at A and then returns to B. If $U = 2V$, prove that (i) $\mu = \frac{3}{5}\tan\theta$, (ii) $V^2 = \frac{4}{5}gd\sin\theta$. (C)

11. A particle of mass $2m$ rests on a rough plane inclined at an angle α to the horizontal. It is connected by a light inextensible string passing over a smooth pulley at the top edge of the plane to a second particle of mass m which hangs freely. Given that when the system is released from rest it begins to move, find the condition that must be satisfied by μ, the coefficient of friction if (a) $\tan\alpha = 3/4$, (b) $\tan\alpha = 5/12$.

12.



The diagram shows a fixed wedge whose sloping face is inclined at an angle θ to the horizontal. Two particles A and B are connected by a light inextensible string which passes over a smooth pulley C at the top of the wedge.

The particle A rests on the sloping face of the wedge, the string between A and C being parallel to a line of greatest slope, and B hangs freely. Particle A has mass M, particle B has mass m and the coefficient of friction between A and the sloping plane is μ. The system is released from rest.

(i) Show that, if $\sin\theta - \mu\cos\theta \leqslant \frac{m}{M} \leqslant \sin\theta + \mu\cos\theta$, the masses remain at rest.

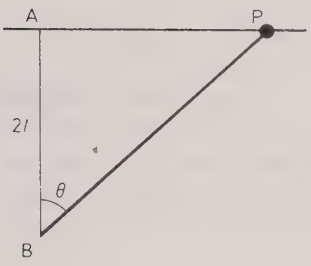
(ii) If $M = m$, $\theta = 30°$ and $\mu = \frac{1}{2}$, find expressions for the acceleration of the system and the tension in the string. (C)

13. The inclined faces of a fixed triangular wedge make angles α and β with the horizontal. Two particles of equal weight, connected by a light inextensible string which passes without friction over the vertex of the wedge, rest on the two faces of the wedge. The system rests in equilibrium with the plane containing the string perpendicular to both faces of the wedge. If both particles are in limiting equilibrium, show that $\mu = \tan\frac{1}{2}|\alpha - \beta|$, where μ is the coefficient of friction between each particle and the face of the wedge. (O)

14. A body of weight W N is being pulled across a rough horizontal plane by a string inclined at an angle θ to the horizontal. The tension in the string is $\frac{1}{2}W$N and the coefficient of friction between the body and the plane is $\frac{1}{4}$. Find the value of θ for which the acceleration of the body is greatest and the magnitude of the maximum acceleration.

15. A particle of weight 4 N is placed on a rough plane inclined at an angle of 10° to the horizontal. The coefficient of friction between the particle and the plane is 0·2. Find the magnitude and direction of the least force required to move the particle (a) down the plane, (b) up the plane.

16.

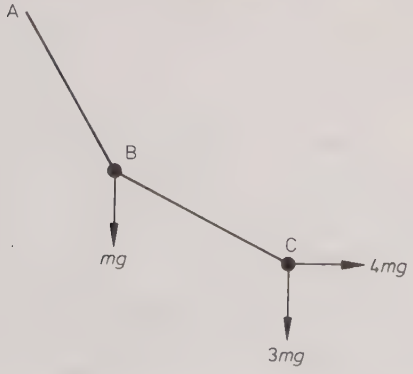


A small rough ring P of weight W is threaded on to a fixed horizontal straight wire which passes through a point A, as shown in the diagram. The ring is joined by a light elastic string of natural length l and modulus λ to a fixed point B which is at a distance $2l$ vertically below A. The ring is in equilibrium with the angle $ABP = \theta$. Find the tension in the string and the vertical and horizontal components of the force exerted by the wire on the ring. Show that

$$W \geqslant \lambda(2-\cos\theta)\left(\frac{\tan\theta}{\mu}-1\right),$$

where μ is the coefficient of friction between the ring and the wire. (JMB)

17. A particle of mass $3m$ is tied to the end C and a particle of mass m is tied at the mid-point B of a light unstretched elastic string ABC. The end A of the string is fixed and a horizontal force of magnitude $4mg$ is applied to the particle at C so that the system hangs in equilibrium as shown. Calculate (i) the tensions in BC and AB, (ii) the inclinations of BC and AB to the vertical. Given that the modulus of elasticity of the string is $6mg$, show that, for this position of equilibrium, $AB:BC = (6+4\sqrt{2}):11$. (AEB 1978)



18. A light elastic string is of natural length a, and is such that when it is stretched by a length a, the tension in it is mg.
(i) The string is fastened to two points, $2a$ apart, which are in the same

horizontal line. A particle of mass m is attached to the mid-point, and gently lowered until it comes to rest. Show that, in its equilibrium position, each of the parts of the string is inclined at θ to the horizontal, where θ is given by $2\tan\theta - \sin\theta = \frac{1}{2}$.

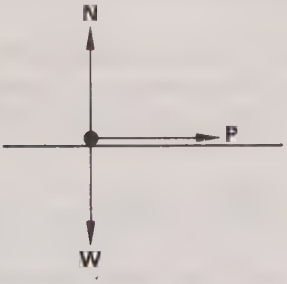
(ii) The same string is now fastened to two points, $2a$ apart, which are in the same vertical line. A particle of mass m is attached to the mid-point and gently released until it comes to rest. Show that, when the system comes to rest, the particle is $\frac{1}{4}a$ below its original position. (SU)

19. A wedge of mass M is resting on a smooth horizontal table. A particle of mass m is placed gently on a rough face of the wedge which is inclined at an angle θ to the horizontal, and slides down this face. Given that the coefficient of friction between the particle and the face is μ, where $\mu < \tan\theta$, prove that the acceleration of the wedge is $\dfrac{mg\cos\theta(\sin\theta - \mu\cos\theta)}{M + m\sin\theta(\sin\theta - \mu\cos\theta)}$. (O)

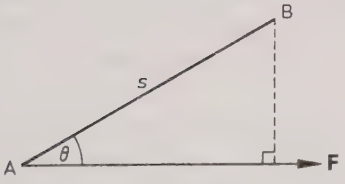
7 Work, energy and power

7.1 Work and kinetic energy

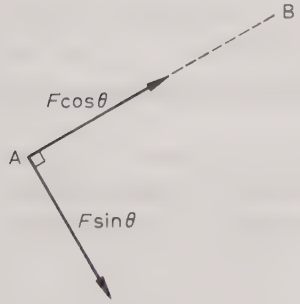
The *work done by a force* is the product of the force and the distance moved in the direction of the force by its point of application.



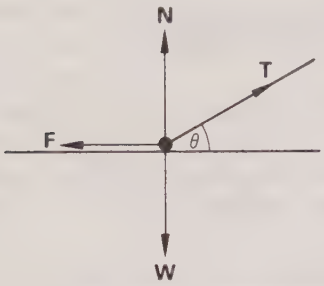
If a particle of weight **W** is pulled a distance s along a smooth horizontal plane by a horizontal force **P**, then the work done by this force is Ps. However, since the direction of motion is perpendicular to the directions of the weight **W** and the normal reaction **N**, these forces do no work.



Consider now a particle which moves a distance s from A to B under the action of a force **F**, where the angle between the displacement **AB** and the force **F** is θ.
The particle moves a distance $s\cos\theta$ in the direction of the force **F**. Hence the work done by the force is $Fs\cos\theta$.



Another way of obtaining the same result is to consider the force **F** as a sum of components along and perpendicular to **AB**. The component $F\sin\theta$ does no work but, since the particle moves a distance s in the direction of the component $F\cos\theta$, the work done is $F\cos\theta \times s$, i.e. $Fs\cos\theta$ as before.



If a particle is pulled a distance s along a rough horizontal plane by a string inclined at an angle θ to the horizontal, then the work done by the tension **T** in the string is $Ts\cos\theta$. The weight **W** and the normal reaction **N** do no work. However, the displacement of the particle in the direction of the frictional force **F** which opposes motion is $-s$. Hence the work done by the force **F** is $-Fs$. The quantity Fs is called the work done *against* friction.

The SI unit of work is the joule (J). When the point of application of a force of 1 newton moves 1 metre, the work done is 1 joule, i.e. $1\,\text{J} = 1\,\text{N m}$.

To establish a relationship between the motion of a body and the work done on it, we consider a particle of mass m moving in a straight line under the action of a constant force F. We assume that the acceleration of the particle is a and that its initial velocity is u. If when the particle has moved a distance s its velocity is v, then, as shown in §4.2, $v^2 = u^2 + 2as$. By Newton's second law,

$$F = ma$$

$$\therefore \qquad Fs = mas \quad \text{and} \quad as = \tfrac{1}{2}(v^2 - u^2)$$

Hence
$$Fs = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2.$$

The quantity $\frac{1}{2}mv^2$ is called the *kinetic energy* (K.E.) of a particle of mass m moving with velocity v and is measured in the same units as work, i.e. in joules.

Thus when a particle is moving in a straight line under the action of a constant force,

work done by force = increase in kinetic energy.

Example 1 A body of mass 4 kg is moving in a straight line under the action of a constant force. Given that its speed increases from $3\,\text{m s}^{-1}$ to $7\,\text{m s}^{-1}$, find the work done by the force.

Work done = increase in K.E. $= \tfrac{1}{2}.4.7^2 - \tfrac{1}{2}.4.3^2\,\text{J}$
$= 98 - 18\,\text{J}.$

Hence the work done by the force is 80 J.

Example 2 A car of mass 1200 kg travelling along a straight horizontal road accelerates uniformly from $10\,\text{m s}^{-1}$ to $15\,\text{m s}^{-1}$ in a distance of 200 m. If the thrust of the engine is 500 N, find the resistance to motion.

Let the resistance to motion be R N, then the total horizontal force acting on the car is $(500 - R)$ N.

Work done = increase in K.E.

$$\therefore \qquad (500-R)200 = \tfrac{1}{2}.1200.15^2 - \tfrac{1}{2}.1200.10^2 = 600(225-100)$$

$$\therefore \qquad 500 - R = 3 \times 125 = 375.$$

Hence there is resistance to motion of 125 N.

When the motion of a particle in a straight line is opposed by a constant force, it may be convenient to express the relationship between work and kinetic energy in the form:

Work done again resistance = loss of K.E.

Exercise 7.1 [Take $g = 10$.]

1. A particle of weight W is pulled a distance s across a rough horizontal plane by a horizontal force P. Find the work done by the weight W, the normal reaction N, the force P and the frictional force F.

2. A body of weight W is lifted through a distance x by a force P directed vertically upwards. Find the work done by the weight W and the force P.

3. A particle of weight W slides a distance s down a rough plane inclined at an angle α to the horizontal. Find the work done by the weight W and the normal reaction N. Find also the work done against the frictional force F.

4. A particle of weight W is pulled a distance s up a rough plane inclined at an angle α to the horizontal by a string parallel to the plane. Find the work done by the tension T and the normal reaction N. Find also the work done against the frictional force F and against gravity.

5. A body of mass 10 kg moving under the action of a constant force accelerates from $2\,\mathrm{m\,s^{-1}}$ to $5\,\mathrm{m\,s^{-1}}$. Find the work done by the force. Given that the magnitude of the force is 7 N, find the distance moved by the body.

6. A railway truck of mass 2 tonnes moving on a straight horizontal track is brought to rest from a speed of $3\,\mathrm{m\,s^{-1}}$ by a force of P N. Find the work done by this force. Given that the truck travels 20 m before coming to rest, find the value of P.

7. A bullet of mass 0·04 kg travelling at $300\,\mathrm{m\,s^{-1}}$ hits a fixed wooden block and penetrates a distance of 4 cm. Find the average resistance of the wood.

8. A car of mass 1000 kg travelling along a straight horizontal road accelerates uniformly from $15\,\mathrm{m\,s^{-1}}$ to $25\,\mathrm{m\,s^{-1}}$ in a distance of 320 m. If the resistance to motion is 145 N, find the driving force of the engine.

9. A box of mass 16 kg is pulled from rest a distance of 5 m across a smooth horizontal floor by a cable inclined at 60° to the horizontal. Find the work done by the tension in the cable given that its magnitude is 25 N.

10. A particle of mass 6 kg sliding across a rough horizontal plane comes to rest in a distance of 8 m. Given that its initial velocity was $10\,\mathrm{m\,s^{-1}}$, find the work done against friction. Find also the coefficient of friction between the particle and the plane.

11. Find the kinetic energy gained by a body of mass 2 kg falling freely from rest through a distance of 10 m. If a vertical force of P N brings the body to rest in a further distance of 8 m, find the value of P.

12. A particle of mass 5 kg slides a distance 9 m down a rough plane inclined at an angle α to the horizontal, where $\sin\alpha = \frac{3}{5}$. The coefficient of friction is $\frac{1}{2}$. Find the work done by gravity and the work done against friction. Find also the velocity attained by the particle.

7.2 Conservation of mechanical energy

The *energy* of a body can be regarded as its capacity for doing work. For instance, the kinetic energy of a body is its capacity to do work by virtue of its motion. As we saw in the previous section, a moving body does work against forces opposing motion as it loses kinetic energy. There are many other forms of energy such as heat, light, sound, chemical and electrical energy. However, our interest will be confined to forms of *mechanical energy*, i.e. energy by virtue of motion or position.

A body is said to have *potential energy* if its position is such that when released it begins to move and to gain kinetic energy. For instance, a body of mass m released at a height h above the ground will fall freely under gravity until it hits the ground with velocity v.

$$\text{Work done by gravity} = \text{gain in K.E.}$$
$$\therefore \quad mgh = \tfrac{1}{2}mv^2.$$

Thus, initially the body is said to have *gravitational potential energy* mgh by virtue of its height h above the ground.

In any given situation a *zero level* for gravitational potential energy must be decided, and then the potential energy of a body can be calculated relative to this level. Bodies above the zero level will have positive potential energy and bodies below it negative potential energy.

As stated earlier, if a particle is moving in a straight line under the action of a resultant force F then,

$$\text{work done by } F = \text{gain in K.E.}$$

If the only force doing work is gravity this result becomes,

$$\text{work done by gravity} = \text{gain in K.E.}$$

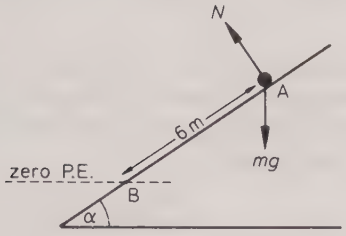
However, by definition, when a mass m falls from a height h the work done by gravity, mgh, is equal to the loss of potential energy.

Hence, $\quad$ loss of P.E. = gain in K.E.
so that $\quad$ K.E. + P.E. = constant.

This result is a simple form of the *principle of conservation of mechanical energy*.

If a particle is moving such that no external force other than gravity is doing work, then the total mechanical energy of the particle remains constant.

Example 1 A particle of mass m kg is released from rest at a point A on a smooth plane inclined at an angle α to the horizontal, where $\sin\alpha = 3/5$. Find the velocity acquired by the particle after travelling a distance of 6 m down the plane to a point B.



Let us assume that at B the particle has zero P.E. and velocity $v\,\mathrm{m\,s^{-1}}$, then
at A: K.E. $= 0$, P.E. $= mg \times 6\sin\alpha$
at B: K.E. $= \frac{1}{2}mv^2$, P.E. $= 0$.

Using the conservation of mechanical energy

$$\tfrac{1}{2}mv^2 + 0 = 0 + 6mg\sin\alpha$$

$$\therefore\quad v^2 = 2\times 6\times 9{\cdot}8\times\frac{3}{5} = \frac{6^2\times 7^2}{5^2}$$

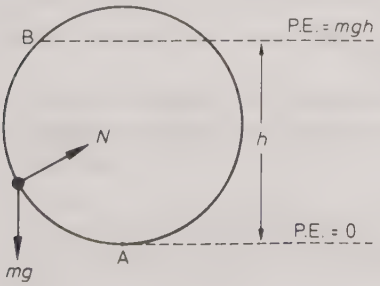
$$\therefore\quad v = \frac{6\times 7}{5} = 8{\cdot}4.$$

Hence the velocity acquired by the particle is $8{\cdot}4\,\mathrm{m\,s^{-1}}$.

[Note that although this problem could be solved using Newton's second law and one of the constant acceleration formulae, it is quicker to use an energy equation, since only velocity and distance moved are involved.]

It can be shown that the principle of conservation of mechanical energy also applies to the motion of a particle along a curved path, provided that no work is done by any external force other than gravity.

Example 2 A ring of mass m is threaded on a smooth circular wire fixed in a vertical plane. If the ring is projected from the lowest point A of the wire with velocity v and first comes to rest at a point B on the wire, express the vertical height h of B above A in terms of g and v.



Since the wire is smooth the force exerted by the wire on the ring is always perpendicular to the direction of motion and therefore does no work. Hence mechanical energy is conserved throughout the motion, i.e.

(K.E. + P.E.) at B = (K.E. + P.E.) at A

$$0 + mgh = \tfrac{1}{2}mv^2 + 0.$$

Thus $$h = v^2/2g.$$

When using energy equations to solve problems it is helpful to distinguish between *conservative* and *non-conservative* forces. When work is done against a conservative force such as gravity, there is a gain in potential energy. However, when work is done against a non-conservative force, such as friction, energy is produced in various forms, including heat and sound, but there is no gain in potential energy. Thus the principle of conservation of mechanical energy can be applied to systems of particles provided that (i) no work is done by external forces other than gravity, (ii) no work is done against non-conservative internal forces such as friction.

Example 3 A particle A of mass 2 kg is connected to a particle B of mass 3 kg by means of a light inextensible string which passes over a smooth light pulley. The system is released from rest with both parts of the string taut and vertical. Find the speed attained by each particle when A has risen 2 m.

Let the initial P.E. of the system be zero and let the final speed of each particle be $v\,\mathrm{m\,s^{-1}}$.
Thus initially the system has zero mechanical energy.
Final P.E. of $A = 2g \times 2\,\mathrm{J} = 4g\,\mathrm{J}$.
Final P.E. of $B = 3g \times (-2)\,\mathrm{J} = -6g\,\mathrm{J}$.

Final K.E. of $A = \frac{1}{2} \times 2 \times v^2\,\mathrm{J} = v^2\,\mathrm{J}$.

Final K.E. of $B = \frac{1}{2} \times 3 \times v^2\,\mathrm{J} = \frac{3}{2}v^2\,\mathrm{J}$.

Thus the final mechanical energy of the system

$$= \left(v^2 + \frac{3}{2}v^2 + 4g - 6g\right)\mathrm{J} = \left(\frac{5}{2}v^2 - 2g\right)\mathrm{J}.$$

Using conservation of mechanical energy, $\frac{5}{2}v^2 - 2g = 0$.

$$\therefore \qquad v = \sqrt{\left(\frac{4g}{5}\right)} = \sqrt{\left(\frac{4}{5} \times \frac{49}{5}\right)} = \frac{2 \times 7}{5} = 2{\cdot}8.$$

Hence each particle attains a speed of $2{\cdot}8\,\mathrm{m\,s^{-1}}$.

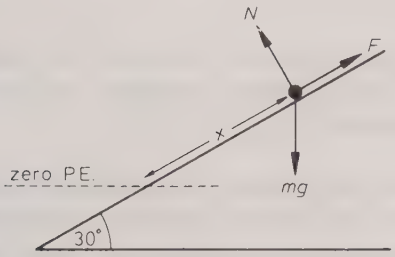
When the total mechanical energy of a particle or system of particles is changed by the action of external forces an energy equation similar to the following may be used,

initial (P.E. + K.E.) + work done = final (P.E. + K.E.).

If the mechanical energy of a system is reduced by work done against non-conservative forces a more convenient form may be,

initial (P.E. + K.E.) − work done against resistance = final (P.E. + K.E.).

Example 4 A small block of mass m slides down a rough plane inclined at an angle 30° to the horizontal. If the block acquires velocity v after travelling a distance x from rest, find an expression for the work done against friction.



Let the final P.E. of the block be zero, then the initial P.E. is $mg \times x \sin 30°$, i.e. $\frac{1}{2}mgx$.
The initial K.E. is zero and the final K.E. is $\frac{1}{2}mv^2$.

Initial (P.E. + K.E.) − work done against friction = final (P.E. + K.E.).
Work done against friction
$= \text{initial (P.E.} + \text{K.E.)} - \text{final (P.E.} + \text{K.E.)}$
$= (\frac{1}{2}mgx + 0) - (0 + \frac{1}{2}mv^2)$
$= \frac{1}{2}m(gx - v^2)$.

Exercise 7.2

1. A ball is thrown vertically downwards with a speed of $3{\cdot}5\,\text{m s}^{-1}$. Find its speed when it has travelled a distance of 5 m.

2. A particle is projected vertically upwards from a point A with a speed of $21\,\text{m s}^{-1}$. Find its position relative to A when its speed is (a) $4{\cdot}2\,\text{m s}^{-1}$, (b) $35\,\text{m s}^{-1}$.

3. A stone is thrown vertically upwards from a point A with velocity $v\,\text{m s}^{-1}$. Find an expression for the greatest height above A reached by the stone.

4. A small block is released from rest on a smooth plane inclined at 30° to the horizontal. Find the distance the particle has travelled down the plane when its speed is $6{\cdot}3\,\text{m s}^{-1}$.

5. A particle is projected with velocity $9\,\text{m s}^{-1}$ up a line of greatest slope of a smooth plane inclined at an angle α to the horizontal, where $\sin\alpha = 4/7$. Find the speed of the particle when it has travelled 5 m up the plane.

6. A tile slides from rest down a smooth roof inclined at an angle α to the horizontal, where $\sin\alpha = \frac{2}{3}$. It reaches the edge after travelling a distance of 4·5 m,

then falls to the ground 7 m below. Find the speed at which the tile is travelling as it hits the ground.

7. A particle is attached to a fixed point by means of a light inextensible string of length 0·8 m. If when the particle is released from rest the string makes an angle of 60° with the downward vertical, find the speed of the particle as it passes through its lowest point.

8. A force acts vertically upwards on a body of mass 4 kg. If the body rises vertically from rest through 5 m and acquires a speed of $3\,\mathrm{m\,s^{-1}}$, find the work done by the force.

9. Find the kinetic energy acquired by a body of mass 5 kg which falls freely from rest through a distance of 3 m. If the particle is then brought to rest by a vertical force of 70 N, find the further distance that it falls.

10. A particle of mass m is connected to a particle of mass M by means of a light inextensible string which passes over a smooth light pulley. The system is released from rest with both parts of the string taut and vertical. Given that $M > m$, find an expression for the distance moved by each particle when the velocity attained is v.

11. A particle of mass 3 kg standing on a smooth horizontal table is connected by means of a light inextensible string passing over a smooth pulley at the edge of the table to a particle of mass 2 kg hanging freely. Find the speed of the particles when they have travelled a distance of 25 cm from rest.

12. A cyclist travelling initially at $2\,\mathrm{m\,s^{-1}}$ free-wheels down a hill. At the bottom of the hill his speed is $10\,\mathrm{m\,s^{-1}}$. If the total mass of the cyclist and his machine is 90 kg and the work done against resistance is 4500 J, find the difference in level between the top and bottom of the hill.

13. In a fairground ride a car of mass 500 kg starts from rest at a point A, 30 m above the ground. After travelling 73 m along a track against a constant resistance of 600 N the car passes through a point B with a speed of $16\,\mathrm{m\,s^{-1}}$. Find the height of B above the ground.

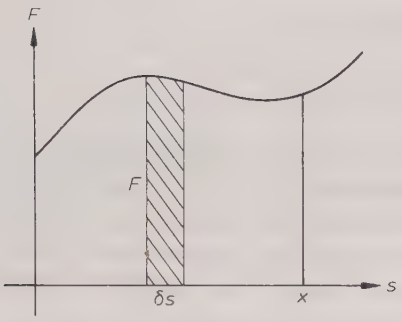
14. A particle is moving at a speed of $7\,\mathrm{m\,s^{-1}}$ when it begins to ascend a slope inclined at an angle α to the horizontal where $\sin\alpha = 5/13$. The coefficient of friction between the particle and the slope is 5/8. Find (a) the speed of the particle when it has travelled 1·95 m up the slope, (b) the total distance the particle travels up the slope before coming to rest.

15. A block of mass 5 kg is projected with velocity $5{\cdot}6\,\mathrm{m\,s^{-1}}$ up a line of greatest slope of a rough plane inclined at an angle α to the horizontal where $\sin\alpha = 0{\cdot}6$. If the block travels 2 m up the slope before coming to rest, find the work done against friction. Find also the coefficient of friction between the block and the plane.

7.3 Work done by a variable force

Consider a particle moving in a straight line under the action of a variable force **F** acting along the line. The work done by **F** in a small displacement δs is approximately equal to $F\delta s$. Thus an approximate value of the total work done in a displacement from $s = 0$ to $s = x$ is given by a sum of the form $\sum_{s=0}^{s=x} F\delta s$. Remembering that sums of this kind can be evaluated by integration, we write:

$$\text{work done} = \lim_{\delta s \to 0} \sum_{s=0}^{s=x} F ds = \int_0^x F ds.$$



Geometrically this quantity is represented by the area under the force-displacement graph for the motion.

An important application of this theory is to the stretching of a spring or elastic string. By Hooke's law, when the extension in an elastic string is x, then the tension T is given by

$$T = \frac{\lambda x}{l}$$

where λ is the modulus of elasticity and l is the natural length of the string. Hence the work done against the tension when the extension is increased from x_1 to x_2 is

$$\int_{x_1}^{x_2} T dx = \int_{x_1}^{x_2} \frac{\lambda x}{l} dx = \left[\frac{\lambda x^2}{2l}\right]_{x_1}^{x_2} = \frac{\lambda}{2l}\left(x_2^2 - x_1^2\right).$$

In particular, the work done when an elastic string is stretched from its natural length l to length $(l+x)$ is $\dfrac{\lambda x^2}{2l}$.

[A similar result can be obtained for the work done when a spring is compressed.]

Example 1 A particle of weight 10 N is suspended from a fixed point by a light elastic string of natural length 1·5 m. When the particle hangs in equilibrium the length of the string is 2 m. Find the modulus of elasticity of the string. Find also the work done in stretching the string from its natural length to a length of 2 m.

When the particle is in equilibrium the tension in the string is equal in magnitude to the weight of the particle.

Hence, using Hooke's law $T = \dfrac{\lambda x}{l}$, where $T = 10$, $l = 1{\cdot}5$ and $x = 0{\cdot}5$,

$$10 = \frac{\lambda \times 0{\cdot}5}{1{\cdot}5} \quad \therefore \quad \lambda = 30$$

$\therefore$ the modulus of elasticity of the string is 30 N.

The work done in extending a string from its natural length by an amount x is $\dfrac{\lambda x^2}{2l}$

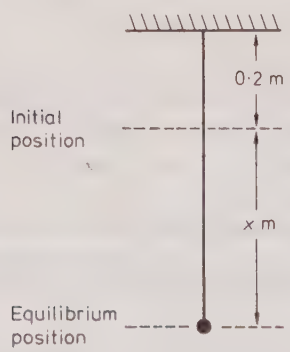
$\therefore$ the work done in extending this string by 0·5 m

$$= \frac{30 \times (0{\cdot}5)^2}{2 \times 1{\cdot}5} \text{J} = 2{\cdot}5 \text{J}.$$

Let us suppose that an elastic string is stretched and then released. As the tension acts to restore the string to its original length the work it does is equal to the work done against the tension in stretching the string. Thus, since no mechanical energy is lost, the tension in an elastic string is a conservative force. A spring or elastic string stretched from its natural length l to a length $(l+x)$ is said to have *elastic potential energy* of $\lambda x^2/2l$.

The principle of conservation of mechanical energy can be applied to systems involving springs and elastic strings, provided that no work is done by external forces (other than gravity).

Example 2 A particle of mass 2 kg is suspended from a fixed point by a spring of natural length 20 cm and modulus 10 N. Given that the extension in the spring when the particle is hanging in equilibrium is x m, find x. If the particle is released from rest with the spring vertical and unstretched, find the speed, v m s^{-1}, of the particle when it passes through its equilibrium position.



In the equilibrium position the tension in the spring is equal to the weight of the particle, i.e. $2g$ N.

Using Hooke's law, $T = \dfrac{\lambda x}{a}$

Substituting $T = 2g$, $\lambda = 10$, $a = 0{\cdot}2$

$$2g = \frac{10x}{0{\cdot}2}$$

$\therefore \quad x = 0{\cdot}04\,g = 0{\cdot}04 \times 9{\cdot}8 = 0{\cdot}392.$

Considering mechanical energy in the initial position,
let gravitational P.E. = 0,
energy stored in spring = 0,
kinetic energy = 0.

Considering mechanical energy in the equilibrium position,
gravitational P.E. $= 2g \times (-x)\,\text{J} = -0{\cdot}08\,g^2\,\text{J}$

energy stored in spring $= \dfrac{\lambda x^2}{2a}\,\text{J} = \dfrac{10(0{\cdot}04\,g)^2}{2+0{\cdot}2}\,\text{J} = 0{\cdot}04\,g^2\,\text{J}$

kinetic energy $= \frac{1}{2} \times 2 \times v^2\,\text{J} = v^2\,\text{J}$.

Using the principle of conservation of mechanical energy

$$-0{\cdot}08\,g^2 + 0{\cdot}04\,g^2 + v^2 = 0$$

$$\therefore \quad v^2 = 0{\cdot}04\,g^2, \text{ i.e. } v = 0{\cdot}2\,g = 1{\cdot}96.$$

Hence the speed with which the particle passes through its equilibrium position is $1{\cdot}96\,\text{m s}^{-1}$.

[Note that in practice it is virtually impossible to construct a system in which there is no loss of mechanical energy, since there are no truly frictionless pulleys, smooth surfaces nor perfectly elastic strings. In this chapter we are dealing with mathematical models of systems in which the work done against friction and other non-conservative forces is negligible compared with the total energy involved.]

Exercise 7.3

1. Find the work done in stretching a spring of natural length 1 m and modulus 10 N to a length of (a) 1·2 m, (b) 2 m.

2. Find the energy stored in an elastic string of natural length 4 m and modulus 2 N when its length is (a) 5 m, (b) 10 m.

3. Find the work done in stretching a spring of natural length 1·5 m and modulus 9 N from a length of 2 m to length 2·5 m.

4. The work done in stretching an elastic string of natural length 2·5 m from 3 m to 4 m in length is 6 J. Find the modulus of the string.

5. When an elastic string of natural length 1·5 m is stretched to the length 1·8 m the energy stored in it is 2·7 J. Find the energy stored in the string when its length is 2 m.

6. The work done in extending a spring from its natural length $6a$ to a length $7a$ is ka. Find the modulus of the spring in terms of k and the work done in extending the length of the spring from $7a$ to $8a$.

7. A particle of mass 8 kg is suspended from a fixed point by a spring of natural length 0·5 m and modulus 140 N. If the particle is released from rest with the spring vertical and unstretched, find the distance it falls before coming to rest instantaneously.

8. A particle of mass 5 kg on a smooth horizontal plane is attached to one end

of a string of natural length 1 m and modulus 20 N. The other end of the string is attached to a fixed point in the same plane. If the particle is released from rest with the string stretched to a length of 2 m, find its speed at the instant the string returns to its natural length.

9. A particle of mass m is attached by a light elastic string of natural length $4a$ and modulus $7mg$ to a fixed point. The particle is released from rest with the string taut and vertical, but unstretched. Find an expression for its speed when it has fallen a distance a.

10. A particle is suspended from a fixed point A by a light elastic string of natural length 2 m. When the particle hangs in equilibrium the length of the string is 2·5 m. Given that the particle is released from rest at A, find the distance it has fallen when it first comes to rest.

11. A particle of mass 7 kg is attached to one end of an elastic string of modulus $9g$ N and natural length 1·8 m. The other end of the string is fastened to a fixed point A. The point B is the position of the particle when it hangs in equilibrium. If the particle is released from rest at A, find its speed when it passes through B.

12. A particle is attached by a light elastic string of natural length 60 cm to a fixed point O. When the particle hangs in equilibrium its distance below O is 80 cm. If the particle is released from rest at a point 1 m vertically below O, find the distance of the particle from O when it next comes to rest. Find also the speed of the particle when it passes through its equilibrium position.

13. The ends of an elastic string of modulus 30 N and natural length 2 m are attached to fixed points A and B 2 m apart on the same horizontal level. A particle P is attached to the mid-point of the string then released at the mid-point of AB. If the particle falls 0·75 m before coming to rest instantaneously, find its weight.

14. A particle P of mass 2 kg is suspended from a point O by a light elastic string of length 1 m and modulus 98 N. Find the length of the string when P hangs in equilibrium. The particle is now pulled down and released from rest 1·6 m below O. Find (a) the speed of P as the string goes slack, (b) the depth below O at which P comes instantaneously to rest.

7.4 Power

Power is the rate at which a force does work. The SI unit of power is the watt (W). One watt is a rate of working of one joule per second, i.e. $1\,\text{W} = 1\,\text{J}\,\text{s}^{-1}$.
[For those who are familiar with the watt only as a unit of electrical power, it may be helpful to remember that a family saloon car has a maximum power output of approximately 40–50 kW. A sprinter may generate kinetic energy at an

average rate of about 1 kW for short periods, but most people produce mechanical energy at much lower rates.]

Example 1 Find the average power developed by a crane which lifts a load of 8 kN through a vertical distance of 15 m in 6 s.

Work done by the crane in 6 s is 8×15 kJ, i.e. 120 kJ
$\therefore$ average work done in 1 s is $120 \div 6$ kJ, i.e. 20 kJ.
Hence the average power of the crane is 20 kW.

Example 2 A pump raises water at a rate of 750 litres per minute through a vertical distance of 4 m and projects it into a reservoir at a speed of $6\,\mathrm{m\,s^{-1}}$. Find the power developed by the pump.

Since the mass of 1 litre of water is 1 kg, the mass of water raised per second $= \frac{750}{60}\text{kg} = \frac{25}{2}\text{kg}$.

Thus the P.E. given to the water each second

$$= \frac{25}{2}g \times 4\,\text{J} = 25 \times 9{\cdot}8 \times 2\,\text{J} = 490\,\text{J}.$$

The K.E. given to the water each second

$$= \frac{1}{2} \times \frac{25}{2} \times 6^2\,\text{J} = 25 \times 9\,\text{J} = 225\,\text{J}.$$

Hence the total power developed by the pump is

$(490 + 225)$ W, i.e. 715 W.

If the engine of a vehicle, such as a car or a train, produces a tractive force F newtons, then the power of the engine is the rate at which this force does work. When the vehicle is travelling at constant speed $v\,\mathrm{m\,s^{-1}}$, since the distance travelled each second is v m, the work done each second is Fv J. Hence the power of the engine is Fv watts.

A similar result is obtained for non-uniform motion by considering the work done when the vehicle moves a small distance δs in time δt. The rate of working in this interval is approximately $\frac{F\delta s}{\delta t}$, i.e. $F\frac{\delta s}{\delta t}$. However, as $\delta t \to 0$, $\frac{\delta s}{\delta t} \to \frac{ds}{dt}$. Thus the rate of working at the instant when the vehicle is moving with velocity $v\,\mathrm{m\,s^{-1}}$ must be Fv watts. Hence the power P watts, the tractive force F newtons and the velocity $v\,\mathrm{m\,s^{-1}}$ are connected by the formula

$$\boxed{P = Fv.}$$

Example 3 A car of mass 1200 kg is moving along a horizontal road against resistance to motion of 500 N. Find the power developed by the engine when the

car is travelling at $15\,\mathrm{m\,s^{-1}}$ and accelerating at $0{\cdot}25\,\mathrm{m\,s^{-2}}$. Find also the maximum speed reached by the car if the engine continues to work at the same rate.

If the pull of the engine is F newtons, then by Newton's second law,

$$F - 500 = 1200 \times 0{\cdot}25$$
$$\therefore \quad F = 300 + 500 = 800.$$

Hence the power of the engine

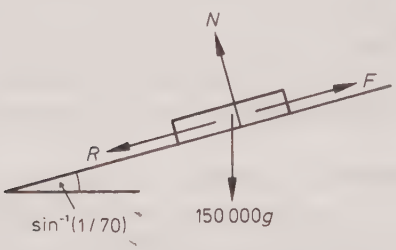
$$= 800 \times 15\,\mathrm{W} = 12\,000\,\mathrm{W} = 12\,\mathrm{kW}.$$

Let the maximum speed of the car be $v\,\mathrm{m\,s^{-1}}$. When maximum speed is reached the pull of the engine is equal in magnitude to the resistance to motion

$$\therefore \quad 12\,000 = 500\,v, \text{ i.e. } v = 24.$$

Hence the maximum speed when the engine is working at the rate of 12 kW is $24\,\mathrm{m\,s^{-1}}$.

Example 4 A train of mass 150 tonnes is ascending an incline of 1 in 70 at a constant speed of $25\,\mathrm{m\,s^{-1}}$. If the power being exerted by the engine is 750 kW, find the resistance to motion.



Let F N be the tractive force of the engine and R N the resistance to motion.
Using the formula $P = Fv$,

$$750\,000 = F \times 25$$
$$\therefore \quad F = 30\,000.$$

Applying Newton's second law up the incline

$$F - R - 150\,000\,g \times \frac{1}{70} = 0$$

$$\therefore \quad R = 30\,000 - \frac{150\,000 \times 9{\cdot}8}{70} = 30\,000 - 21\,000 = 9000.$$

Hence the resistance to motion is 9 kN.

Example 5 When a cyclist rides along a horizontal road at $v\,\mathrm{m\,s^{-1}}$ the resistance to motion is $(16 + \frac{1}{4}v^2)\,\mathrm{N}$. Find the rate at which the cyclist must work to maintain a steady speed of $6\,\mathrm{m\,s^{-1}}$.

When travelling at a steady speed the force exerted by the cyclist is equal to the resistance to motion,

$$\therefore \quad \text{his rate of work} = (16 + \tfrac{1}{4}v^2)v\,\mathrm{W}$$
$$= (16 + \tfrac{1}{4} \times 6^2)6\,\mathrm{W}$$
$$= (16 + 9)6\,\mathrm{W}$$

Hence to maintain a steady speed of $6\,\mathrm{m\,s^{-1}}$ the cyclist must work at a rate of 150 W.

Exercise 7.4

1. A man of mass 80 kg climbs a staircase of height 5 m in 10 s. Find his average rate of work against gravity.

2. If the effective power developed by a crane is 15 kW, find the time taken to lift a load of weight 50 kN through a vertical distance of 12 m.

3. Find the minimum power needed to pull an object of weight 500 N a distance of 8 m up a smooth slope inclined at 30° to the horizontal in 5 seconds.

4. A lift of mass 1500 kg is being raised by an engine working at the rate of 24·5 kW. If the lift rises 50 m in 40 s, find the load it is carrying.

5. Water from a river, flowing at $4\,m\,s^{-1}$, completely fills a pipe of cross-sectional area $3\,m^2$. If, when the water is used to drive a generator, 30% of its kinetic energy is converted into electrical energy, find the power supplied by the generator.

6. A pump raises 40 kg of water a second through a vertical distance of 10 m and delivers it in a jet with speed $12\,m\,s^{-1}$. Find the effective power developed by the pump.

7. A pump raises water at the rate of 250 litres per minute through a vertical distance of 8 m and projects it into a lake at $2\,m\,s^{-1}$. Find the power developed by the pump.

8. If the engine of a train travelling at a steady speed of $25\,m\,s^{-1}$ is working at a rate of 1800 kW, find the magnitude of the resistance to motion.

9. A motor-cyclist is riding along a horizontal road at a constant speed of $15\,m\,s^{-1}$ against a resistance of 80 N. Find the power at which the motor-cycle engine is working. Given that the total mass of the rider and his machine is 200 kg, find the initial acceleration when the power is increased to 1·5 kW.

10. A car of mass 900 kg is travelling at $20\,m\,s^{-1}$ along a horizontal road against a constant resistance of 600 N. Find the power developed by the engine if the car is (a) moving with constant speed, (b) accelerating at $0{\cdot}15\,m\,s^{-2}$, (c) decelerating at $0{\cdot}5\,m\,s^{-2}$.

11. A cable car of mass 1500 kg is pulled at a constant speed of $10\,m\,s^{-1}$ up a smooth slope inclined at an angle α to the horizontal where $\sin\alpha = 1/7$. Find the power exerted on the car.

12. A car of mass 1000 kg is travelling down an incline of 1 in 28 against a constant resistance of 750 N. Find the maximum speed that can be reached by the car when the engine is working at the rate of 12 kW.

13. A lorry of mass 2400 kg is moving at a steady speed of 60 km/h against a constant frictional resistance of 2 kN. Find the power developed by the engine if the lorry is travelling (a) along a horizontal road, (b) up a hill of inclination $\sin^{-1}(1/21)$, (c) down the same hill.

14. A train of mass 50 tonnes is ascending an incline of 1 in 60 with the engine working at the rate of 200 kW. When the train is travelling at $10\,\text{m s}^{-1}$, its acceleration is $0{\cdot}1\,\text{m s}^{-2}$. Find the frictional resistance to motion. Assuming that the resistance and the power of the engine remain constant, find the maximum speed the train can attain on this slope.

15. The constant non-gravitational resistance to the motion of a car of mass 1500 kg is 750 N. The engine of the car works at a constant rate of 20 kW. Find, in km/h, the maximum speed (a) on the level, and (b) directly up a road inclined at an angle $\arcsin(1/12)$ to the horizontal. Find also, in m/s^2, the acceleration of the car when it is travelling at 72 km/h on a level road. [Take g as $10\,\text{m/s}^2$.] (L)

16. The frictional resistance to the motion of a car of mass 1000 kg is $30v$ N where $v\,\text{m s}^{-1}$ is the speed of the car. The car ascends a hill of inclination $\arcsin\left(\frac{1}{10}\right)$, the power exerted by the engine being 12·8 kW. If the car is moving at a steady speed, prove (do not merely verify) that this steady speed is $10\,\text{m s}^{-1}$. On reaching the top of the hill, find the immediate acceleration of the car, assuming the road becomes level. (O&C)

17. The resistance to the motion of a car is $(160+cV^2)$ newtons, where c is a constant and V is the speed of the car in m/s. The mass of the car is 900 kg and, when the engine is working at the rate of 19·2 kW, the maximum speed on a level road is 20 m/s. Find the value of c. The car ascends a slope inclined to the horizontal at an angle θ, where $\sin\theta = 1/30$. If the engine continues to work at the same rate, find the acceleration of the car when its speed is 10 m/s. [Take g as $10\,\text{m/s}^2$.] (L)

18. A car of mass 1·2 tonnes is travelling along a straight, horizontal road at a constant speed of 120 km/h against a resistance of 600 newtons. Calculate, in kW, the effective power being exerted. Given that the resistance is proportional to the square of the velocity, calculate also (i) the power required to go down a hill of 1 in 30 (along the slope) at a steady speed of 120 km/h, (ii) the acceleration of the car up this hill with the engine working at 20 kW at the instant when the speed is 80 km/h. (AEB 1978)

Exercise 7.5 (miscellaneous)

1. A particle is projected at a speed of $3{\cdot}5\,\text{m s}^{-1}$ across a rough horizontal plane. Given that it comes to rest after travelling a distance of 5 m, find the coefficient of friction.

2. A particle released from rest on a smooth plane inclined at an angle α to the horizontal reaches a speed of $6\,\mathrm{m\,s^{-1}}$ after travelling 3 m down the plane. Find $\sin\alpha$.

3. A body of mass 2 kg is pulled from rest across a rough horizontal plane by a string inclined at an angle α to the horizontal where $\sin\alpha = 3/5$. The coefficient of friction is 1/3. After travelling 7·8 m the body attains a speed of $5{\cdot}2\,\mathrm{m\,s^{-1}}$. Assuming the tension in the string is constant, find its magnitude.

4. A particle is attached to a fixed point by means of a light inextensible string of length 1 m. If the particle is projected horizontally through its lowest point with a speed of $2{\cdot}8\,\mathrm{m\,s^{-1}}$, find the angle the string makes with the vertical when the particle first comes to rest.

5. A particle of mass 5 kg is projected with a speed of $2{\cdot}8\,\mathrm{m\,s^{-1}}$ down a rough plane inclined at an angle α to the horizontal, where $\sin\alpha = 5/13$. If the particle comes to rest after travelling 2·6 m, find the work done against friction. Find also the coefficient of friction between the particle and the plane.

6. The work done in stretching an elastic string from 3 m to 5 m in length is 4 J. If the modulus of the string is 3 N, find its natural length.

7. A particle of mass 2 kg on a smooth plane inclined at 30° to the horizontal is attached by means of a light inextensible string passing over a smooth pulley at the top edge of the plane to a particle of mass 3 kg which hangs freely. If the system is released from rest with both parts of the string taut, find the speed acquired by the particles when both have moved a distance of 1 m.

8. A curved chute with a smooth surface is fixed so that a point A on the chute is 2·5 m above a horizontal plane. A particle is released from A and is travelling horizontally when it leaves the chute at a point B 1·6 m above the horizontal plane. Find the speed of the particle as it hits this plane and the horizontal distance it has travelled since leaving the chute.

9. A particle P of mass m is attached to two elastic strings each of length 0·6 m and modulus kmg. The other ends of the strings are fixed to two points A and B 1·2 m apart on the same horizontal level. P is released from rest at M, the mid-point of AB and falls vertically 0·8 m before coming instantaneously to rest. Find k and also the speed of P as it passes through C, where $MC = 0{\cdot}45$ m.

10. A light elastic string obeys Hooke's law, so that when it is stretched by an amount x the tension in it is given by $T = kx$ for some constant k. Prove that the elastic energy stored in the string when its extension is x is given by $E = \frac{1}{2}kx^2$.

One end of such a string is attached to a fixed point A, and the other end is attached to a particle of mass m. The unstretched length of the string is a, and the particle hangs freely in equilibrium at a point B which is at a distance $\frac{5}{4}a$ below

A. Prove that $k = 4mg/a$. The particle is held at A, and released from rest. Find its velocity as it passes B, and the total distance it falls before coming instantaneously to rest. (C)

11. A bead of mass m slides on a smooth vertical rod and is attached to a light inextensible string which passes over a smooth peg at a distance a from the rod and has a particle of mass $2m$ fastened to its other end. The bead is held at the level of the peg and then released. Prove that it will descend $\frac{4}{3}a$ before coming to rest and find the tension in the string at this instant. (O)

12. A, B, C, D are fixed points situated at the mid-points of the sides of a square of side $2a$ with AC vertical (C above A) and BD horizontal. A weight W is held in the plane of the square by four springs, of modulus λ and natural length a, whose other ends are attached to A, B, C and D respectively. Show that in the equilibrium position the springs attached to B and D are inclined to the upward drawn vertical at an angle α where $2\cot\alpha - \cos\alpha = W/2\lambda$. Show that the amount of energy needed to raise the weight to the centre of the square must be at least $2a\lambda(\cot^2\alpha + \sin\alpha - 1)$. (W)

13. A small bead of mass m is threaded on a smooth wire which is shaped in the form of the curve $by = x^2 + b^2$ $(x \leqslant 0)$. The positive y-axis is vertically upwards and the x-axis lies in a horizontal plane. The bead is released from rest at the point where $x = -2b$ and slides down the wire to its end at the point $(0, b)$, after which the bead moves freely under gravity until it strikes the horizontal plane at a point on the x-axis. Find expressions for (i) the speed of the bead on leaving the wire, (ii) the speed of the bead on striking the x-axis, (iii) the angle that the direction of motion of the bead makes with the x-axis when it meets the x-axis, (iv) the x-coordinate of the point where the bead meets the x-axis.

Suppose instead that it is required that the bead should hit a small target which is to be placed on the x-axis at the point $(c, 0)$. Find the x-coordinate of the point on the curve at which the bead has to be released from rest if it is to achieve this. (JMB)

14. An elastic string is such that when stretched from its natural length l to a length $l+x$ the tension in the string becomes kx, where k is a positive constant. Show that the work done when the length of the string is increased from $l+x_1$ to $l+x_2$ is $\frac{1}{2}k(x_2^2 - x_1^2)$.

A particle of mass m is attached to one end of the string, the other end of which is attached to a fixed point O in a smooth inclined plane of inclination α. The particle rests in equilibrium on the plane and the extension of the string is $\frac{1}{2}l$. Show That $kl = 2mg\sin\alpha$. The particle is then pulled a further distance $b - \frac{1}{2}l$ down the line of greatest slope, and released. By considerations of energy or otherwise, determine the distance up the plane, measured from the point of release, that the particle will travel, distinguishing the cases (i) $b \leqslant l$, (ii) $l < b \leqslant 2l$. Describe briefly in words what happens if $b > 2l$. (JMB)

15. A train of mass 3×10^5 kg travels along a straight level track. The resistance to motion is $1{\cdot}5 \times 10^4$ N. Find the tractive force required to produce an acceleration of $0{\cdot}1\,\text{m s}^{-2}$, and the power in kW which is then developed by the engine when the speed of the train is $10\,\text{m s}^{-1}$. Find also the maximum speed attainable on the same track when the engine is working at a rate of 360 kW. (JMB)

16. A car of mass 1000 kg is travelling along a level road at a steady speed of 45 km/h with its engine working at $22\frac{1}{2}$ kW. Calculate in newtons the total resistance (assumed constant) due to friction, etc. The engine is disconnected and simultaneously the brakes are applied, bringing the car to rest in 30 metres. Find the force, assumed constant, exerted by the brakes. (O & C)

17. A cable car in the Alps gains height at an average rate of 900 m in $2\frac{1}{2}$ minutes. Given that the cable car has mass 1300 kg and carries 30 passengers of average mass 90 kg, determine the average power required if the efficiency of the machinery is 75%. [Take g to be $10\,\text{m s}^{-2}$.] (C)

18. A pump raises water through a vertical height of 15 m and delivers it at a speed of $10\,\text{m s}^{-1}$ through a circular pipe of internal diameter 12 cm. Find the mass of water raised per second and the effective power of the pump, giving your answers to 3 significant figures.

19. A motor lorry weighing 5 metric tonnes can develop up to 15 000 watts. It is moving up a hill with slope 1 in 20 $\left(\theta = \sin^{-1}\frac{1}{20}\right)$ against a frictional force of 300 N. This frictional force is independent of the speed. (i) Find the maximum steady speed in kilometres per hour at which the lorry can travel up this slope. (ii) Find the acceleration capable of being developed when the lorry travels up this slope at $10\,\text{km h}^{-1}$. The lorry travels on a straight horizontal road at a speed of v kilometres per hour against a braking force of $10v$ newtons and other resistances of 300 N. What is the maximum value of v? (W)

20. A locomotive of mass 15 000 kg, working at the rate of 220 kW, pulls a train of mass 35 000 kg up a straight track which rises 1 m vertically for every 50 m travelled along the track. When the speed is 10 m/s the acceleration is $0{\cdot}23\,\text{m/s}^2$. Find the frictional resistance at this speed. Given that the resistance is proportional to the speed of the train, find, in m/s, the greatest speed of the train up the slope if the rate of working is unchanged. [Take $g = 10\,\text{m/s}^2$.] (L)

21. The engine of a car, of mass M kg, works at a constant rate of H kW. The non-gravitational resistance to the motion of the car is constant. The maximum speed on level ground is V m/s. Find, in terms of M, V, H, α and g, expressions for the accelerations of the car when it is travelling at speed $\frac{1}{2}V$ m/s (a) directly up a road of inclination α, (b) directly down this road. Given that the acceleration in case (b) is twice that in case (a), find $\sin\alpha$ in terms of M, V, H and g. Find also, in terms of V alone, the greatest steady speed which the car can maintain when travelling directly up the road. (L)

22. An engine is travelling at a steady speed of 10 m/s on a level track against a constant resistance of 40 000 N. Calculate, in kW, the power output of the engine. The engine is then coupled to a carriage by a towbar. The constant resistance to the motion of the carriage is 20 000 N. If the power output of the engine is now 900 kW, calculate, in m/s, the maximum speed of the train on a level track. State the tension, in newtons, in the towbar. The train then ascends an incline of inclination arcsin (1/50) to the horizontal with the same power output, 900 kW, against the same constant frictional forces. If the total mass of the train is 340 tonne, show that the acceleration of the train when it is travelling at 5 m/s is $13/85\,\text{m/s}^2$. [Take g as $10\,\text{m/s}^2$.] (L)

23. A car of mass $2{\cdot}5 \times 10^3$ kg travels at a constant speed of $20\,\text{m s}^{-1}$ up a hill of inclination of $\sin^{-1}(1/20)$ to the horizontal, with the engine working at a power of 8×10^4 W. Given that the resistance is proportional to the cube of the speed show that the resistance at a speed of $24\,\text{m s}^{-1}$ is $4{\cdot}8 \times 10^3$ N, correct to 2 significant figures. Given that the engine is working at the same power, find the acceleration at an instant when the car is moving down a hill of inclination $\sin^{-1}(1/10)$ to the horizontal at a speed of $24\,\text{m s}^{-1}$. (C)

24. A motor-cyclist, whose total mass with his machine is 200 kg, is ascending a hill of inclination θ to the horizontal, where $\sin\theta = 1/8$. If the resistance forces total 150 N and the engine is working at 9 kW, calculate (i) the acceleration of the motor-cycle when the speed is 20 m/s, (ii) the maximum speed that can be attained by the motor-cycle up this hill. The motor-cyclist descends the same hill with a pillion rider whose mass is 80 kg. The resistive forces now total 170 N and the engine is switched off. Find the distance covered and the time taken as the speed of the motor-cycle increases from 10 m/s to 20 m/s. [Take the acceleration due to gravity to be $10\,\text{m/s}^2$.] (AEB 1977)

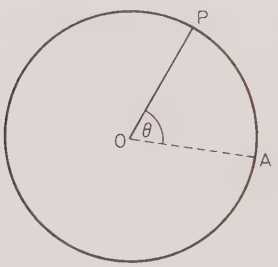
25. A car has a maximum speed of 108 km/h when moving along a horizontal road with the engine working at 36 kW. Calculate, in newtons, the total resistance to the motion of the car. The car, which is of mass 800 kg, can move down a road inclined at an angle α to the horizontal at a maximum speed of 108 km/h with the engine working at 30 kW against the same total resistance. Calculate $\sin\alpha$. Given that the total resistance varies as the square of the speed, find, in kW, the rate at which the engine is working at the instant when the car is moving along a horizontal road with an acceleration of $0{\cdot}5\,\text{m/s}^2$ at a speed of 54 km/h. [Take the acceleration due to gravity to be $10\,\text{m/s}^2$.] (AEB 1978)

26. Two points A and B are at the same level, and they are connected by a road consisting of two straight sections AC and CB, each of uniform gradient. The point C is at a height h vertically above the line AB, and the inclinations to the horizontal of AC and CB are α and β respectively. A car of mass m starts from rest at A, accelerates uniformly to reach a speed $2U$ at C, and then moves with constant retardation, attaining a speed U at B. Show that the time taken to move from A to B is $h(\operatorname{cosec}\alpha + \frac{2}{3}\operatorname{cosec}\beta)/U$. The resistance to the motion of the car is a constant force R. Find the force exerted by the engine during the climb from A

to C, the maximum power developed by the engine and the total work done by the engine. Show that the described motion from C to B can be achieved by using the brakes, with the engine switched off, only if $R \leqslant \frac{m \sin \beta}{2h}(2gh+3U^2)$. (JMB)

8 Uniform motion in a circle

8.1 Angular velocity

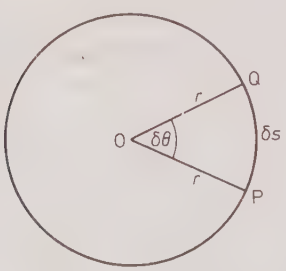


Consider a particle P moving round the circumference of a circle with centre O. If A is any fixed point on the circle and $\angle POA = \theta$, then the *angular velocity* of P is defined as the rate of change of θ and often denoted by ω.

Thus $\omega = \dot{\theta}$, i.e. $\omega = d\theta/dt$.

The value of ω may be positive or negative according to the direction of motion. Anti-clockwise rotations are usually taken to be positive. The *angular speed* of P is the magnitude of its angular velocity. The *angular acceleration* of P is the rate of change of the angular velocity, denoted by $\dot{\omega}$ or $\ddot{\theta}$.

The SI unit of angular velocity is one radian per second, i.e. 1 rad s^{-1}. Angular velocity can also be measured in revolutions per minute (rev min^{-1}).



Suppose now that a particle is moving round the circumference of a circle radius r such that when the angular velocity of the particle is ω, its speed is v. If the particle moves from P to Q in time δt, let arc $PQ = \delta s$ and $\angle POQ = \delta\theta$.

Since $\delta s = r\,\delta\theta$ we have $\dfrac{\delta s}{\delta t} = r\dfrac{\delta\theta}{\delta t}$.

As $\delta t \to 0$, $\dfrac{\delta s}{\delta t} \to \dfrac{ds}{dt}$ and $\dfrac{\delta\theta}{\delta t} \to \dfrac{d\theta}{dt}$.

Hence $$\frac{ds}{dt} = r\frac{d\theta}{dt}.$$

$\therefore$ $$\boxed{v = r\omega \quad \text{and} \quad \omega = \frac{v}{r}.}$$

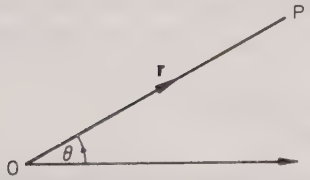
Example 1 The tip of the second hand of a large clock is 60 cm from the centre of the clock face. Find the angular speed of the second hand and the speed at which the tip is travelling.

The second hand makes 1 revolution per minute, i.e. it moves through 2π rad in 60 s

$\therefore$ its angular speed $= \frac{2\pi}{60}\,\text{rad s}^{-1} = \frac{\pi}{30}\,\text{rad s}^{-1}$.

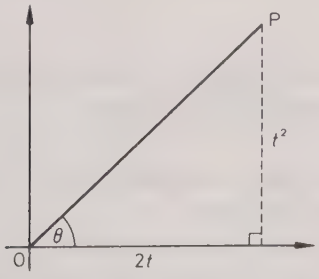
Hence the speed of the tip of the hand

$$= \frac{60}{100} \times \frac{\pi}{30}\,\text{m s}^{-1} = \frac{\pi}{50}\,\text{m s}^{-1} \approx 0{\cdot}063\,\text{m s}^{-1}.$$



The definition of angular velocity can be extended to the motion in a plane of a point P with position vector $\mathbf{r}$. If the angle between $\mathbf{r}$ and some fixed direction is θ, then the angular velocity of P about O (i.e. the angular velocity of **OP**) is $\dot{\theta}$.

Example 2 At time t, the position vector of a point P with respect to the origin O is $2t\mathbf{i} + t^2\mathbf{j}$. Find the angular velocity of **OP** at time t.



Let θ be the angle between **OP** and the direction of $\mathbf{i}$, then at time t,

$$\tan\theta = \frac{t^2}{2t} = \frac{t}{2}.$$

Differentiating with respect to t

$$\sec^2\theta\frac{d\theta}{dt} = \frac{1}{2}.$$

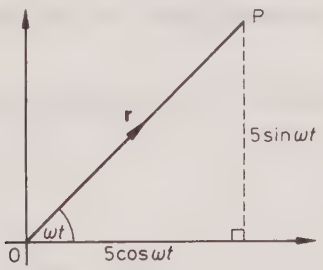
$\therefore$ $$(1+\tan^2\theta)\frac{d\theta}{dt} = \frac{1}{2}$$

$$\left(1+\frac{t^2}{4}\right)\frac{d\theta}{dt} = \frac{1}{2}$$

i.e. $$(4+t^2)\frac{d\theta}{dt} = 2.$$

Hence the angular velocity at time t is given by $\dfrac{d\theta}{dt} = \dfrac{2}{4+t^2}$.

Example 3 A particle P is moving anti-clockwise with constant angular speed ω round a circle with centre O and radius 5. Given that initially the position vector of P is $5\mathbf{i}$, find its position vector at time t. Deduce vector expressions for the velocity and acceleration of P at time t.



Since the angular speed is ω, at time t **OP** makes an angle ωt with the direction of $\mathbf{i}$. Hence at time t the position vector of P is

$$\mathbf{r} = 5(\cos \omega t\mathbf{i} + \sin \omega t\mathbf{j}).$$

Differentiating with respect to t, the velocity of P is given by

$$\mathbf{v} = \dot{\mathbf{r}} = 5(-\omega \sin \omega t\mathbf{i} + \omega \cos \omega t\mathbf{j})$$

i.e. $$\mathbf{v} = 5\omega(-\sin \omega t\mathbf{i} + \cos \omega t\mathbf{j}).$$

Similarly the acceleration of P is given by

$$\mathbf{a} = \ddot{\mathbf{r}} = 5\omega(-\omega \cos \omega t\mathbf{i} - \omega \sin \omega t\mathbf{j})$$

i.e. $$\mathbf{a} = -5\omega^2(\cos \omega t\mathbf{i} + \sin \omega t\mathbf{j})$$

[We note that $v = 5\omega\sqrt{(\sin^2 \omega t + \cos^2 \omega t)} = 5\omega$, which is consistent with the result $v = r\omega$ obtained earlier in the section. We also see that $\mathbf{a} = -\omega^2\mathbf{r}$. Thus the acceleration of the particle is directed towards O and has magnitude $5\omega^2$.]

Exercise 8.1

1. A record of diameter 18 cm is rotating at 45 rev min^{-1}. Find its angular velocity in rad s^{-1} and the speed of a point on the rim of the record.

2. A point is travelling round a circle of radius 5 m at a speed of 20 m s^{-1}. Find the angular speed of the point about the centre of the circle.

3. Find the angular speed with which the earth is rotating about its axis. Hence, taking the radius of the earth to be 6400 km, find the speed of a point on the equator in m s^{-1}.

4. Find an approximate value for the angular speed at which the earth is moving about the sun. Assuming that the earth moves round the sun in an approximately circular orbit of radius 150 million kilometres, find the speed of the earth relative to the sun.

5. Given that the tip of the minute hand of a clock is moving at a speed of $0{\cdot}001\ \mathrm{m\,s^{-1}}$, find the length of the hand to the nearest cm.

6. Find the angular velocity of a point P about the origin O at time $t > 0$, given that the position vector of P is

(a) $\mathbf{i}+t\mathbf{j}$, (b) $2t\mathbf{i}-3\mathbf{j}$, (c) $\frac{1}{t}\mathbf{i}+t\mathbf{j}$.

7. A particle P moves anti-clockwise with constant angular speed ω round a circle with centre O and radius r. Write down the position vector of P at time t given that when $t = 0$ its position vector is (a) $r\mathbf{i}$, (b) $r\mathbf{j}$.

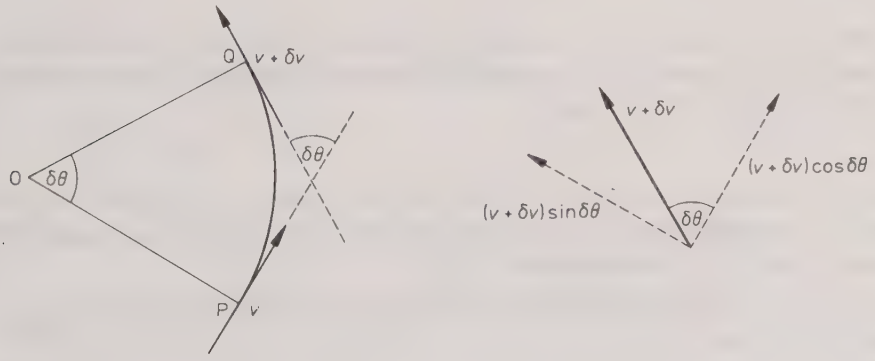
8. Use the results of question 7 to find the angular velocity of a particle P about the origin O and the speed at which the particle is travelling given that at time t its position vector is

(a) $3(\cos 2t\mathbf{i}+\sin 2t\mathbf{j})$, (b) $5(-\sin t\mathbf{i}+\cos t\mathbf{j})$,
(c) $\cos 2\pi t\mathbf{i}-\sin 2\pi t\mathbf{j}$, (d) $a(\sin kt\mathbf{i}-\cos kt\mathbf{j})$.

9. The points S and T are moving anti-clockwise round a circle with centre O (the origin) and radius a, with the same constant speed $a\omega$. When T is passing through the point with position vector $a\mathbf{j}$, S is passing through the point with position vector $a\mathbf{i}$. Find, at time t later, the position vectors of S and T. Find also, in vector form, the velocity of T relative to S. (L)

8.2 Acceleration in circular motion

Consider a particle moving with angular velocity ω round the circumference of a circle centre O, radius r. Suppose that in time δt it moves from P to Q, where $\angle POQ = \delta\theta$. Let the speed of the particle be v at P and $v+\delta v$ at Q.



From the diagrams we see that the velocity of the particle at Q has components $(v+\delta v)\sin\delta\theta$ in the direction of $\vec{PO}$ and $(v+\delta v)\cos\delta\theta$ perpendicular to $\vec{PO}$.

Thus in the direction of $\vec{PO}$, the change in velocity in time δt

$$= (v+\delta v)\sin\delta\theta$$

$$\therefore \quad \text{average acceleration} = \frac{(v+\delta v)\sin\delta\theta}{\delta t} = (v+\delta v)\frac{\sin\delta\theta}{\delta\theta}.\frac{\delta\theta}{\delta t}.$$

As $\delta t \to 0$, $\delta v \to 0$, $\dfrac{\sin\delta\theta}{\delta\theta} \to 1$ and $\dfrac{\delta\theta}{\delta t} \to \dfrac{d\theta}{dt}$

$$\therefore \quad \text{acceleration along } \vec{PO} = v\frac{d\theta}{dt} = v\omega = \frac{v^2}{r}.$$

Similarly, perpendicular to $\vec{PO}$, the change in velocity in time δt

$$= (v+\delta v)\cos\delta\theta - v$$

$$\therefore \quad \text{average acceleration} = \frac{(v+\delta v)\cos\delta\theta - v}{\delta t}$$

$$= \frac{v(\cos\delta\theta - 1)}{\delta t} + \frac{\delta v}{\delta t}\cos\delta\theta$$

$$= -\frac{v.2\sin^2\frac{1}{2}\delta\theta}{\delta t} + \frac{\delta v}{\delta t}\cos\delta\theta$$

$$= -v\frac{\delta\theta}{\delta t}\left(\frac{\sin\frac{1}{2}\delta\theta}{\frac{1}{2}\delta\theta}\right)\sin\tfrac{1}{2}\delta\theta + \frac{\delta v}{\delta t}\cos\delta\theta.$$

As $\delta t \to 0$, $\dfrac{\sin\frac{1}{2}\delta\theta}{\frac{1}{2}\delta\theta} \to 1$, $\dfrac{\delta\theta}{\delta t} \to \dfrac{d\theta}{dt}$, $\dfrac{\delta v}{\delta t} \to \dfrac{dv}{dt}$

but $\sin\frac{1}{2}\delta\theta \to 0$ and $\cos\delta\theta \to 1$

$$\therefore \quad \text{acceleration perpendicular to } \vec{PO} = \frac{dv}{dt}.$$

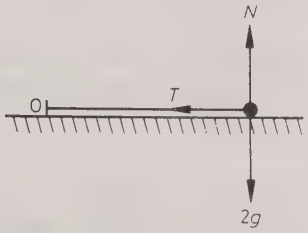
> Hence the acceleration of a particle moving in a circle with variable speed v has, at any instant, components v^2/r towards the centre of the circle and dv/dt in the direction of motion.

If v is constant, then $dv/dt = 0$. Thus a particle moving in a circle with constant speed v has acceleration v^2/r towards the centre of the circle. In terms of the angular velocity ω the acceleration is $r\omega^2$.

[This last result can also be obtained by the vector methods used in Example 3 of the previous section.]

Using Newton's second law, we find that the resultant force acting on a particle of mass m, which moves in a circle radius r with constant speed v, is of magnitude mv^2/r and is directed towards the centre of the circle.

Example 1 One end of a light inextensible string of length 0·5 m is attached to a fixed point O on a smooth horizontal table and a particle of mass 2 kg is attached to its other end. If the particle moves in a horizontal circle, centre O, with a speed of $5\,\mathrm{m\,s^{-1}}$, find the tension in the string and the reaction of the table on the particle.

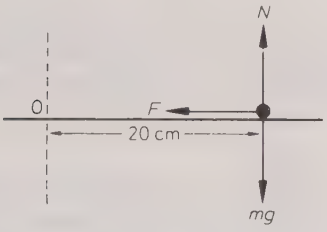


The acceleration of the particle is $5^2/0{\cdot}5\,\mathrm{m\,s^{-2}}$, i.e. $50\,\mathrm{m\,s^{-2}}$ towards O
$\therefore$ using Newton's second law horizontally and vertically,

$$\uparrow N - 2g = 0$$
$$\leftarrow \quad T = 2 \times 50 = 100.$$
$$\therefore \quad N = 2g = 19{\cdot}6$$

Hence the tension in the string is 100 N and the reaction of the table is 19·6 N.

Example 2 A particle is placed at a point 20 cm from the centre O of a rough circular disc. When the disc rotates about a vertical axis through O at an angular speed of $3{\cdot}5\,\mathrm{rad\,s^{-1}}$, the particle is on the point of slipping. Find the coefficient of friction between the particle and the disc.



The acceleration of the particle is $0{\cdot}2 \times (3{\cdot}5)^2\,\mathrm{m\,s^{-2}}$ towards O. Using Newton's second law horizontally and vertically,

$$\uparrow N - mg = 0 \quad \therefore \quad N = mg$$

$$\leftarrow \quad F = m \times 0{\cdot}2(3{\cdot}5)^2 = m \times \frac{1}{5} \times \left(\frac{7}{2}\right)^2 = \frac{49m}{20}.$$

Since the particle is on the point of slipping, $F = \mu N$

$$\therefore \quad \frac{49m}{20} = \mu mg = 9{\cdot}8\,\mu m.$$

Hence $\mu = \dfrac{49}{20 \times 9{\cdot}8} = \dfrac{1}{4}$

$\therefore$ the coefficient of friction between the particle and the disc is $\frac{1}{4}$.

Exercise 8.2

1. Find the acceleration of a particle moving uniformly along a circular path
(a) radius 4 m at a speed of $6\,\mathrm{m\,s^{-1}}$,

(b) radius 25 m at a speed of 36 km/h,
(c) radius 80 cm with angular velocity 5 rad s^{-1},
(d) radius 10 m with angular velocity 24 rev/min.

2. Find the magnitude of the resultant force acting on a particle of mass 5 kg which is moving in a circle of radius 2 m with a constant speed of 3 m s^{-1}.

3. Find the magnitude of the resultant force acting on a particle of mass 1·5 kg which is moving in a circle of radius 50 cm with a constant angular velocity of 4 rad s^{-1}.

4. One end of a light inextensible string of length 3 m is attached to a fixed point O on a smooth horizontal surface and a particle of mass 5 kg is attached to its other end. If the particle is moving in a horizontal circle, centre O, at a speed of 15 m s^{-1}, find the tension in the string and the reaction of the surface on the particle.

5. One end of a light inextensible string of length 1·6 m is attached to a fixed point O on a smooth horizontal surface and a particle of mass 4 kg is attached to the other end. If the string will break when the tension in it exceeds 90 N, find the maximum speed at which the particle can move in a horizontal circle with centre O.

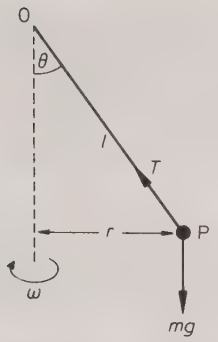
6. A particle is placed at a point 0·5 m from the centre O of a rough circular disc. When the disc rotates about a vertical axis through O at an angular speed of 2·8 rad s^{-1}, the particle is on the point of slipping. Find the coefficient of friction between the particle and the disc.

7. A small ring A is threaded on a horizontal rod PQ, the coefficient of friction between the rod and the ring being 2/7. When the rod rotates horizontally about P with a constant angular velocity of 2 rad s^{-1}, the ring does not slip along the rod. Find the maximum possible distance AP.

8. A particle A is placed on a rough horizontal platform, the coefficient of friction between the particle and the platform being 3/8. The platform rotates about a vertical axis through a fixed point O on the platform. If $OA = 1{\cdot}2$ m, find the angular velocity of the platform when the particle is on the point of slipping.

8.3 Further problems

Consider a particle P of mass m attached by means of a light inextensible string of length l to a fixed point O. The particle is set in motion so that it moves in a horizontal circle of radius r with constant angular velocity ω. Since the string traces out a cone in space, this system is called a *conical pendulum*.



Applying Newton's second law horizontally and vertically,

$\leftarrow T\sin\theta = mr\omega^2$

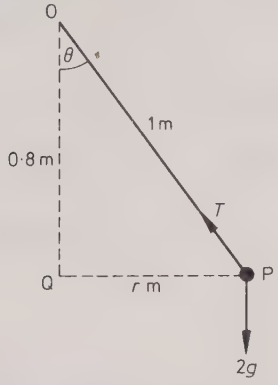
$\uparrow T\cos\theta - mg = 0.$

Substituting $r = l\sin\theta$,

$$T\sin\theta = ml\sin\theta\omega^2$$

$\therefore$ the tension in the string is $ml\omega^2$.

Example 1 A particle of mass 2 kg is attached by a light inextensible string of length 1 m to a fixed point O. The particle is made to move in a horizontal circle whose centre is 0·8 m vertically below O. Find the tension in the string and the speed of the particle.



Let the speed of the particle be v m s^{-1}, then applying Newton's second law horizontally and vertically

$$\leftarrow T\sin\theta = \frac{2v^2}{r} \qquad (1)$$

$$\uparrow T\cos\theta - 2g = 0 \qquad (2)$$

In $\triangle OPQ$, by Pythagoras' theorem

$$r = \sqrt{\{1^2 - (0{\cdot}8)^2\}} = 0{\cdot}6$$

$\therefore$ $\sin\theta = 0{\cdot}6$ and $\cos\theta = 0{\cdot}8$

Using (2) $$T = \frac{2\times 9{\cdot}8}{0{\cdot}8} = 24{\cdot}5$$

$\therefore$ the tension in the string is 24·5 N.

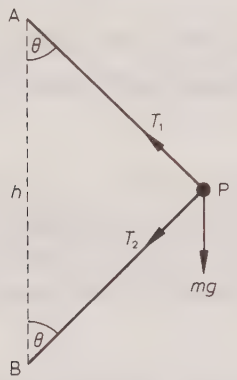
Using (1) $$v^2 = 24{\cdot}5\times 0{\cdot}6\times\frac{0{\cdot}6}{2} = 7^2\times(0{\cdot}3)^2$$

$\therefore$ $$v = 7\times 0{\cdot}3 = 2{\cdot}1.$$

Hence the speed of the particle is 2·1 m s^{-1}.

The methods used in Example 1 can also be applied to a particle sliding round the inner surface of a smooth sphere or cone in a horizontal circle. In such problems the acceleration towards the centre of the circle is produced by the horizontal component of the normal reaction of the surface on the particle.

Example 2 One end of a light inextensible string of length l is attached to a fixed point A and a particle P is attached to the other end. The ends of a second string of the same length are attached to P and to a fixed point B at a distance h $(<2l)$ vertically below A. If the particle moves in a horizontal circle with uniform angular speed ω, find the least value of ω for which both strings are taut.



Let $\angle PAB = \theta$, then $h = 2l\cos\theta$ and the radius of the circle described by the particle is $l\sin\theta$.

Hence the acceleration of the particle

$$= \omega^2 l\sin\theta.$$

Applying Newton's second law horizontally and vertically

$$\uparrow\ T_1\cos\theta - T_2\cos\theta - mg = 0$$
$$\leftarrow\ T_1\sin\theta + T_2\sin\theta = m\omega^2 l\sin\theta$$

$$\therefore\quad T_1 + T_2 = m\omega^2 l$$

$$T_1 - T_2 = \frac{mg}{\cos\theta} = \frac{2mgl}{h}.$$

Subtracting

$$2T_2 = m\omega^2 l - \frac{2mgl}{h}$$

$$\therefore\quad T_2 = \frac{ml}{h}(h\omega^2 - 2g).$$

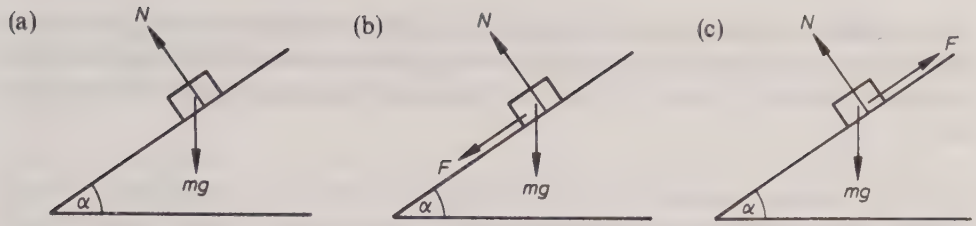
Since $T_1 > T_2$, both strings will be taut if $T_2 \geqslant 0$

i.e. if $\quad h\omega^2 - 2g \geqslant 0.$

Hence the least value of ω for which both strings are taut is $\sqrt{(2g/h)}$.

When a vehicle travels at constant speed round a bend, regarded as a circular arc, its acceleration is directed towards the centre of the arc. If the road surface is horizontal then this acceleration is produced by the frictional force between the wheels and the ground. However, if the road is banked, the horizontal components of the frictional force and of the normal reaction between the vehicle and the ground together provide the required acceleration. [We will ignore at this stage the possibility that the vehicle could overturn.]

Example 3 A car of mass m travels at constant speed v round a bend of radius r on a road banked at an angle α. The coefficient of friction between the car's tyres and the road surface is μ. Find an expression for v^2 given that (a) the car travels with no tendency to slip, (b) the car is about to slip outwards, (c) the car is about to slip inwards.



(a) In this case there is no frictional force acting.
Applying Newton's second law horizontally and vertically

$$\leftarrow N\sin\alpha = mv^2/r$$
$$\uparrow N\cos\alpha - mg = 0$$

$$\therefore\quad v^2 = \frac{rN\sin\alpha}{m} = \frac{r\sin\alpha}{m}\cdot\frac{mg}{\cos\alpha} = rg\tan\alpha.$$

Hence when there is no tendency to slip $v^2 = rg\tan\alpha$.

(b) The frictional force acts down the slope and takes its limiting value μN.
Applying Newton's second law horizontally and vertically

$$\leftarrow N\sin\alpha + \mu N\cos\alpha = mv^2/r$$
$$\uparrow N\cos\alpha - \mu N\sin\alpha - mg = 0$$

$$\therefore\quad v^2 = \frac{rN}{m}(\sin\alpha + \mu\cos\alpha) \quad \text{and} \quad N = \frac{mg}{\cos\alpha - \mu\sin\alpha}.$$

Hence when the car is about to slip outwards

$$v^2 = \frac{rg(\sin\alpha + \mu\cos\alpha)}{\cos\alpha - \mu\sin\alpha}, \quad \text{i.e.} \quad \frac{rg(\tan\alpha + \mu)}{1 - \mu\tan\alpha}.$$

(c) The frictional force acts up the slope and takes its limiting value μN.
Applying Newton's second law horizontally and vertically

$$\leftarrow N\sin\alpha - \mu N\cos\alpha = mv^2/r$$
$$\uparrow N\cos\alpha + \mu N\sin\alpha - mg = 0.$$

Hence when the car is about to slip inwards

$$v^2 = \frac{rg(\sin\alpha - \mu\cos\alpha)}{\cos\alpha + \mu\sin\alpha}, \quad \text{i.e.} \quad \frac{rg(\tan\alpha - \mu)}{1 + \mu\tan\alpha}.$$

[Note that if we substitute $\mu = \tan\lambda$, where λ is the angle of friction, the results in parts (b) and (c) may be written in the form $v^2 = rg\tan(\alpha+\lambda)$ and $v^2 = rg\tan(\alpha-\lambda)$ respectively.]

Exercise 8.3

1. A particle of mass 3 kg is attached by a light inextensible string of length 75 cm to a fixed point. The particle moves in a horizontal circle with constant speed such that the string makes an angle θ with the vertical, where $\tan\theta = 4/3$. Find the tension in the string and the speed of the particle.

2. A particle of mass 5 kg is attached by a light inextensible string of length 3·25 m to a fixed point. The particle moves in a horizontal circle with a constant angular velocity of 2·8 rad s^{-1}. Find the tension in the string and the radius of the circle.

3. A particle of mass m is moving with constant speed v in a horizontal circle of radius r around the smooth inner surface of a hollow cone of semi-vertical angle θ. Find the reaction between the particle and the cone and express v^2 in terms of r, θ and g.

4. A particle moving on the smooth inner surface of a sphere describes a horizontal circle with constant angular speed 2 rad s^{-1}. Find the depth of the circle below the centre of the sphere.

5. A particle of mass 2 kg is suspended from a fixed point by a light elastic string of natural length 1 m and modulus 19·6 N. The particle is moving in a horizontal circle with constant speed v m s^{-1}. Given that the length of the string is 2·25 m, find the value of v.

6. A light inextensible string of length $4a$ has one end fixed at a point A and the other end fixed at a point B which is at a distance $2a$ vertically below A. A smooth ring R of mass m is threaded on the string and moves in a horizontal circle with centre B such that the string is taut. Show that $AR = 5a/2$ and, assuming that the ring is free to move on the string, find its speed.

7. A light inextensible string of length $5a$ has one end fixed at a point A and the other end fixed at a point B which is vertically below A and at a distance $4a$ from it. A particle P of mass m is fastened to the midpoint of the string and moves with speed u, and with the parts AP and BP of the string both taut, in a horizontal circular path whose centre is the midpoint of AB. Find, in terms of m, u, a and g, the tensions in the two parts of the string, and show that the motion described can take place only if $8u^2 \geqslant 9ga$. (JMB)

8. A car is moving in a horizontal circle of radius 100 m on a track banked at an angle α, where $\tan\alpha = 5/16$. Find the speed of the car when there is no tendency for it to slip sideways.

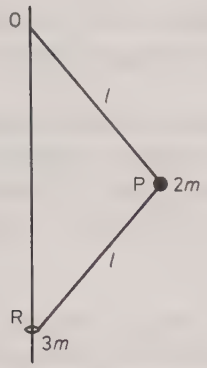
9. A car travels at 90 km/h round a bend of radius 200 m on a road banked at an angle α. Find α, given that there is no tendency to skid sideways.

10. A car travels round a bend of radius 125 m on a road banked at an angle α where $\tan\alpha = 0{\cdot}5$. The coefficient of friction between the car's tyres and the road is $\frac{1}{3}$. Find the maximum and minimum speeds at which the car can travel without slipping sideways.

11. A vehicle is approaching a bend of radius 50 m. The coefficient of friction between its wheels and the road is $\frac{1}{2}$. If the road surface is horizontal, find the

maximum speed at which the vehicle can negotiate the bend without skidding. If the vehicle is to drive round the bend at $20\,\mathrm{m\,s^{-1}}$ without skidding, find the least angle at which the road must be banked.

12.



The figure shows a particle P of mass $2m$ which is attached to a fixed point O by a light inextensible string of length l. A ring R of mass $3m$, which is attached to P by another light inextensible string of length l, is free to slide on a smooth vertical wire passing through O.

The plane OPR rotates about the wire with constant angular velocity ω, where $\omega^2 > 4g/l$. Show that $OR = 8g/\omega^2$. What happens if $\omega^2 \leqslant 4g/l$? (C)

Exercise 8.4 (miscellaneous)

1. A record is rotating at $33\frac{1}{3}$ rev/min. Find its angular speed in $\mathrm{rad\,s^{-1}}$. Find also the speed of a point on the record at a distance of 12 cm from the centre.

2. A van, whose wheels are of diameter 0·8 m, is travelling at 72 km/h. If the wheels do not slip on the road, find the number of revolutions made each second and hence the angular velocity of the wheels in $\mathrm{rad\,s^{-1}}$.

3. A point P moves such that at time t its position vector is $2t\mathbf{i} + (3-t)\mathbf{j}$. Find an expression for the angular velocity of P about the origin O at time t.

4. A particle A is moving in a horizontal circle of radius 3 m with constant speed $6\,\mathrm{m\,s^{-1}}$. A second particle B is moving in a concentric circle of radius 9 m with the same constant speed. At a given instant both particles are due east of the centre of the circles and are moving northwards. Find, in magnitude and direction, the velocity of A relative to B after a time $\frac{1}{4}\pi$ s. (O & C)

5. On an icy morning a car travelling at 42 km/h drives safely round a bend of radius 100 m but skids on a bend of radius 25 m. Assuming that the road surface is horizontal, find the range of possible values of the coefficient of friction between the car's tyres and the road.

6. One end of a light inextensible string of length 1·2 m is attached to a fixed point A on a smooth horizontal surface. A particle of mass 2 kg is attached to the other end C and a particle of mass 1 kg is attached to the mid-point B. Each

particle is moving in a horizontal circle centre A, such that the string remains straight. If the particle at C is moving at a constant speed of $6\,\mathrm{m\,s^{-1}}$, find the tensions in BC and AB.

7. Show that when a particle moves in a circle of radius r with constant speed u its acceleration is of magnitude u^2/r and is directed towards the centre of the circle.

A particle of mass m is attached to one end A of a light inelastic string of length a, the other end being attached to a fixed point O. A moves in a horizontal circle below O with constant speed $\sqrt{(3ag/2)}$. Find (i) the angle that OA makes with the vertical, (ii) the tension in the string, (iii) the time taken to complete one revolution. (SU)

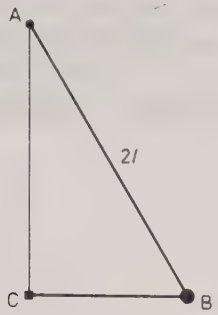
8. A particle P is attached by a light inextensible string of length l to a fixed point O. The particle is held with the string taut and OP at an acute angle α to the downward vertical, and is then projected horizontally at right angles to the string with speed u chosen so that it describes a circle in a horizontal plane. Show that $u^2 = gl\sin\alpha\tan\alpha$. The string will break when the tension exceeds twice the weight of P. Find the greatest possible value of α. (JMB)

9. A particle moves in a circle, of radius a and centre O, at constant speed u. Prove that the acceleration of the particle is u^2/a towards O. The end A of a light rigid rod AB of length 1 metre is smoothly hinged at a fixed point. A ring is threaded on AB and the coefficient of friction between the ring and the rod is $\frac{1}{3}$. The rod is made to rotate horizontally at 2 rad/s about A. Show that the ring can remain at rest relative to the rod at any position between A and C, where $AC = 5/6$ metres. The ring is placed at C and a second identical ring is threaded on the rod and placed at the mid-point of AB. The rings are connected by a taut, light inextensible string. When the rod rotates horizontally at ω rad/s, the rings are on the point of sliding on the rod. Given that the mass of each ring is 0·2 kg, calculate (i) the value of ω, (ii) the tension in the string. [Take the acceleration due to gravity to be $10\,\mathrm{m/s^2}$.] (AEB 1977)

10. One end of a light inelastic string of length a is attached to a particle P of mass m, and the other end of the string is attached to a fixed point O. P moves with constant speed in a horizontal circle whose centre C is a fixed point vertically below O. If the angular velocity at which CP rotates is ω, show that $OC = g/\omega^2$. A second string, also of length a, is fixed to P and C, and the new system rotates about OC with uniform angular velocity Ω and with both strings taut. Find expression in terms of m, a, ω and Ω for the tensions in OP and PC, and deduce that $\Omega > \omega\sqrt{2}$. (C)

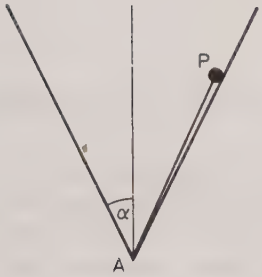
11. A particle moves in a horizontal circle on the smooth inner surface of a fixed spherical bowl of radius a. If the depth of the circle below the centre of the sphere is h, find an expression for the speed of the particle.

12.



The ends of a light inextensible string ABC of length $3l$ are attached to fixed points A and C, C being vertically below A at a distance $\sqrt{3}l$ from A. At a distance $2l$ along the string from A a particle B of mass m is attached. When both portions of the string are taut, B is given a horizontal velocity u, and then continues to move in a circle with constant speed. Find the tensions in the two portions of the string and show that the motion is possible only if $u^2 \geqslant \frac{1}{3}gl\sqrt{3}$. (JMB)

13.



A circular cone of semi-vertical angle α is fixed with its axis vertical and its vertex, A, lowest, as shown in the diagram. A particle P of mass m moves on the inner surface of the cone, which is smooth. The particle is joined to A by a light inextensible string AP of length l. The particle moves in a horizontal circle with constant speed v and with the string taut. Find the reaction exerted on P by the cone. Find the tension in the string and show that the motion is possible only if $v^2 \geqslant gl\cos\alpha$. (JMB)

14. A car moves in a horizontal circle of radius 500 m on a track which is banked at an angle α to the horizontal. When the speed is 60 m/s there is no sideways force on the wheels. Find the angle α. Find also the maximum speed round the track to avoid slipping if the coefficient of friction is 0·5. (AEB 1978)

15. A light elastic string AB of natural length a and modulus of elasticity mg is joined to a light inextensible string BC of length a. The ends A and C of the string are fastened to two fixed points with A vertically above C and $AC = a$. A particle of mass m is fixed at B and rotates with speed v in a horizontal circle. Show that if AB makes an angle $\pi/6$ with the downward vertical then $v^2 = \frac{1}{2}\sqrt{3}ag$, and find the tension in BC. (O)

16. A car moves with constant speed in a horizontal circle of radius r on a track which is banked at an angle α to the horizontal, where $\tan\alpha = \frac{3}{4}$. The coefficient of friction between the tyres and the track is $\frac{1}{2}$. Find, in terms of r and g, the range of speeds at which the car can negotiate this bend without the tyres slipping on the road surface. Show that the greatest possible speed is $\sqrt{11}$ times the least possible speed. [It may be assumed that the car will not overturn at these speeds.] (L)

17. A small ring of mass m can slide on a smooth fixed vertical rod. One end of a light inextensible string of length $2a$ is fastened to the ring, and its other end is fastened to a point of the rod. A particle of mass $4m$ attached to the mid-point of the string is moving in a horizontal circle. Show that the angular velocity of the particle in this circle is greater than $\sqrt{(3g/2a)}$. Find the speed of the particle if the tension in the lower half of the string is $4mg$. (L)

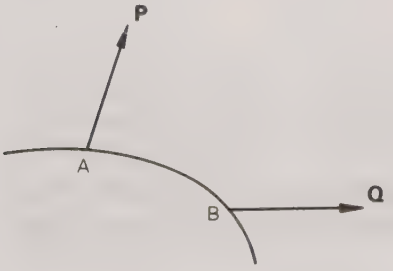
9 Coplanar forces; centre of gravity

9.1 Parallel forces, moments and couples

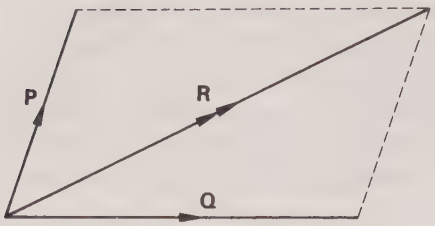
In earlier chapters we considered forces acting on bodies small enough to be treated as particles and represented by single points in space. Applying Newton's laws of motion, we found that if the forces acting on such a body have a non-zero resultant then the effect produced is an acceleration in the direction of the resultant. However, if the vector sum of the forces on the body is zero, the body either remains at rest or continues to move with unchanged velocity.

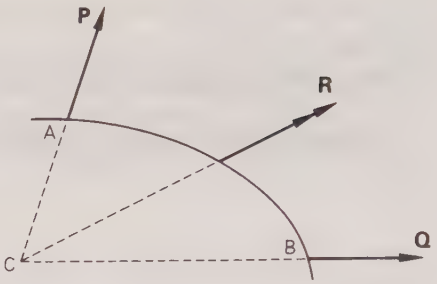
There are many situations in which this elementary approach is inadequate. From experience we know that the forces on a body may tend to produce rotation as well as the types of motion discussed in earlier work. A projectile, such as a cricket ball, may spin as it moves. A block placed on an inclined plane may topple rather than slide. Thus, when solving a problem in which the rotational effects of forces could be important, we must use a more realistic representation of the bodies involved. We will then find that the effect produced by a set of forces depends not only on the magnitude and direction of the resultant, but also on its line of action.

We now develop these ideas by considering the effect of coplanar forces on a variety of rigid bodies i.e. bodies whose shapes are not changed by the forces acting on them.

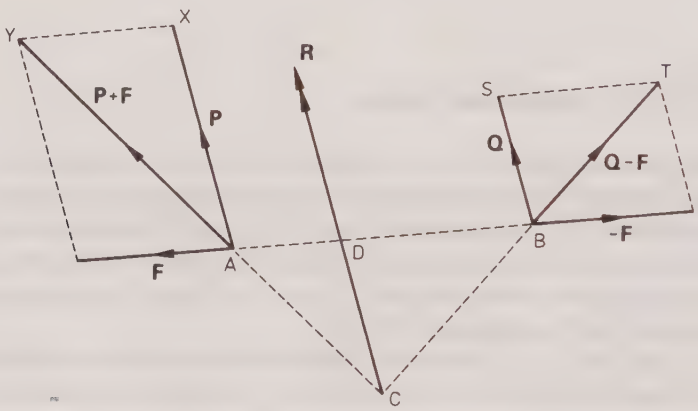


Let **P** and **Q** be *non-parallel* coplanar forces acting at points A and B of a rigid body. The magnitude and direction of their resultant **R** can be found using the parallelogram law. If the lines of action of **P** and **Q** intersect at the point C, then the line of action of **R** also passes through C.





To find the line of action of the resultant of *parallel* forces **P** and **Q** acting at points A and B of a rigid body, we introduce forces **F** and $-\mathbf{F}$ acting along AB. The application of these equal and opposite forces will have no effect on the body and leave the line of action of the resultant unaltered.



The diagram shows how the parallelogram law is used to construct a point C on the line of action of the resultant **R**, when **P** and **Q** are *like parallel forces* i.e. forces in the same sense.

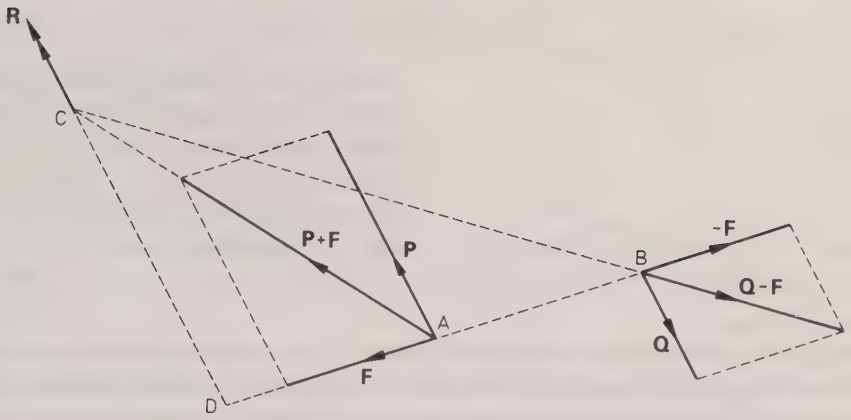
In similar triangles ADC, YXA: $\dfrac{AD}{DC} = \dfrac{YX}{XA} = \dfrac{F}{P}$

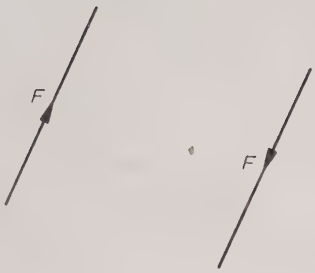
In similar triangles DBC, STB: $\dfrac{DC}{DB} = \dfrac{SB}{ST} = \dfrac{Q}{F}$

$$\therefore \quad \frac{AD}{DB} = \frac{AD}{DC} \times \frac{DC}{DB} = \frac{F}{P} \times \frac{Q}{F} = \frac{Q}{P}$$

By vector addition, the magnitude of **R** is $P + Q$.
Hence the resultant of the like parallel forces **P** and **Q** is of magnitude $P + Q$ and divides AB internally in the ratio $Q:P$.

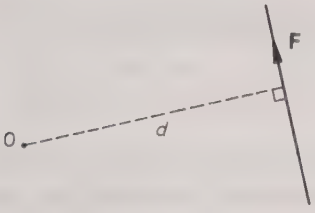
Using similar arguments it can be shown that the resultant of *unlike parallel forces* **P** and **Q**, where $P > Q$, is of magnitude $P - Q$ and divides AB externally in the ratio $Q:P$.





A pair of equal unlike parallel forces has zero resultant, but clearly does have a turning effect. Such a system is called a *couple*.

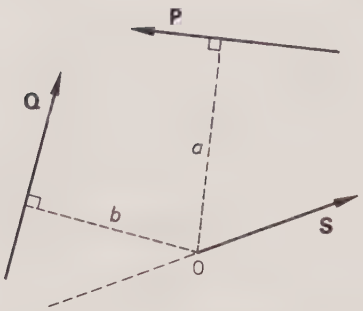
A quantity called the *moment of a force* or *torque* is used as a measure of "turning effect". The moment of a force about a given point is defined as the product of the magnitude of the force and its distance from the point. If the line of action of a force **F** is at a perpendicular distance d from a point O, then the moment of **F** about O is Fd.



[In three-dimensional work the quantity Fd is more accurately described as the moment or torque about an axis through O perpendicular to the plane containing O and the force **F**.]

The SI unit of torque is the newton metre (N m). Thus, if a force of magnitude 40 N acts along a line whose perpendicular distance from a point A is 2 m, then the moment of the force about A is 80 N m.

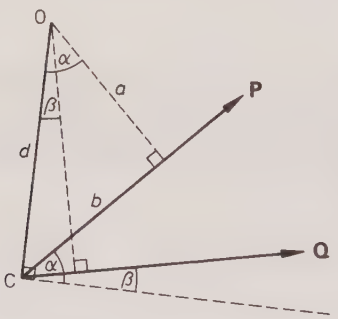
When finding the sum of the moments of a set of coplanar forces about a point, the moment of each force is taken to be positive or negative according to the sense of the rotation it would produce. The moment of a force about any point on its line of action is zero.

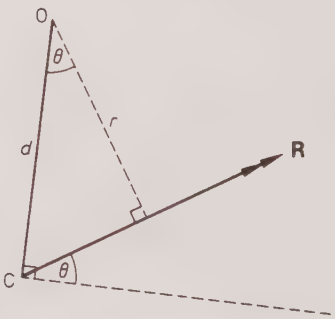


For instance, if forces **P**, **Q** and **S** act as shown in the diagram, then the sum of the anti-clockwise moments of **P**, **Q** and **S** about O is $Pa - Qb$, i.e. the combined turning effect of the system about O is $Pa - Qb$ anti-clockwise.

We now show that the sum of the moments of two coplanar forces about any point in their plane is equal to the moment of their resultant about the same point.

Suppose that **P** and **Q** are forces whose lines of action intersect at a point C. Let **R** be their resultant and let O be a point in the same plane such that $OC = d$.





Using the notation of the above diagrams, the sum of the moments of **P** and **Q** about O

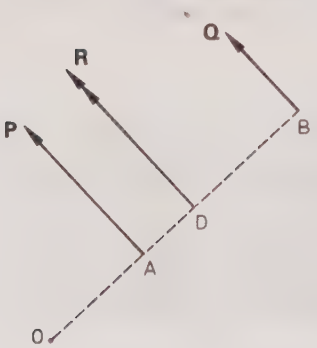
$$= Pa + Qb = Pd\cos\alpha + Qd\cos\beta = d(P\cos\alpha + Q\cos\beta)$$

The moment of **R** about O

$$= Rr = Rd\cos\theta = d(R\cos\theta).$$

Considering the components of **P**, **Q** and **R** in a direction perpendicular to OC, we have

$$P\cos\alpha + Q\cos\beta = R\cos\theta.$$



Hence the sum of the moments about O of **P** and **Q** is equal to the moment about O of **R**.

If **P** and **Q** are like parallel forces with resultant **R**, as shown in the diagram, then

$$\frac{AD}{DB} = \frac{Q}{P}$$

$$\therefore \qquad P \times AD = Q \times DB$$

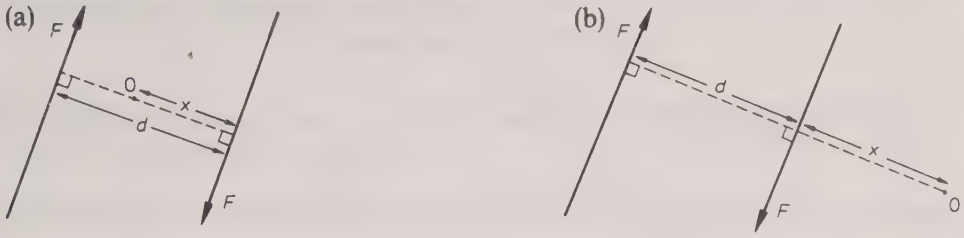
Thus the sum of the moments of **P** and **Q** about O

$= P \times OA + Q \times OB$
$= P(OD - AD) + Q(OD + DB)$
$= (P + Q)OD - P \times AD + Q \times DB$
$= R \times OD$
= the moment of **R** about O.

By extending these arguments to any set of coplanar forces with a non-zero resultant it is possible to establish the following general result called the *principle of moments.*

> The sum of the moments of a system of coplanar forces about any point in their plane is equal to the moment of their resultant about the same point

A couple has zero resultant but non-zero moment. Consider a couple consisting of a pair of equal and opposite forces of magnitude F with lines of action at a distance d apart. Suppose that O is any point in the plane of this couple.



In diagram (a) the sum of the moments about O

$= F(d - x) + Fx = Fd - Fx + Fx = Fd.$

In diagram (b) the sum of the moments about O

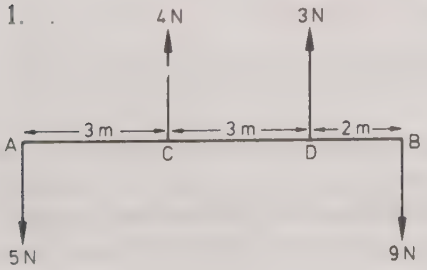
$= F(d + x) - Fx = Fd + Fx - Fx = Fd.$

Since both results are independent of x, the couple has the same moment, i.e. Fd clockwise, for all positions of O.

Hence the *moment of a couple* about any point in its plane is the product of the magnitude of one of the forces and the perpendicular distance between the forces.

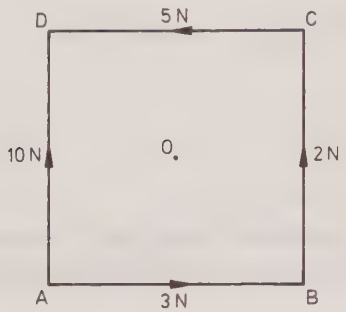
Exercise 9.1

1.



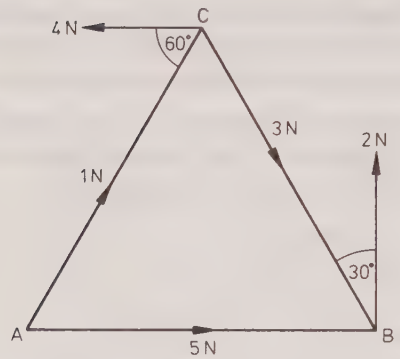
Forces act on a rod AB of length 8 m as shown in the diagram. Find the sum of the anti-clockwise moments of these forces about each of the points A, B, C and D.

2.



ABCD is a square of side 6 m with centre *O*. Forces act along the sides *AB*, *BC*, *CD*, *AD* as shown in the diagram. Find the sum of the anti-clockwise moments of these forces about each of the points *A*, *B*, *C*, *D* and *O*.

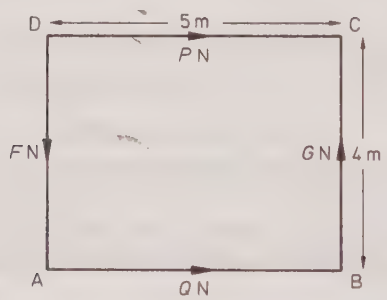
3.



ABC is an equilateral triangle of side 2 m. Find the sum of the anti-clockwise moments of the forces shown in the diagram about each of the points *A*, *B* and *C*. [You may leave your answers in surd form.]

In questions 4 to 9 forces **F**, **G**, **P** and **Q** act along the sides of the rectangle *ABCD* as shown in the diagram.

4.



The resultant of **P** and **Q** acts through a point *X* on *BC*. Find the magnitude of the resultant and the length of *BX* if (a) $P = Q = 3$, (b) $P = 6, Q = 2$.

5. The resultant of **F** and **G** acts through a point *Y* on *AB* produced. Find the magnitude of the resultant and the length of *AY* if
(a) $F = 1, G = 2$, (b) $F = 2, G = 3$.

6. Find the moment of the couple formed by forces **F** and **G** if $F = G = 10$.

7. If the forces **F** and **G** form a couple of moment 20 N m, find *F* and *G*.

8. The resultant of **F** and **P** acts through a point *S* on *AB*. Find the magnitude of the resultant and the length of *AS* if (a) $F = P = 5$, (b) $F = 12, P = 9$.

9. The resultant of **G** and **Q** acts through a point T on DA produced. Find the magnitude of the resultant and the length of AT if (a) $G = Q = 4$, (b) $G = 6, Q = 2.5$.

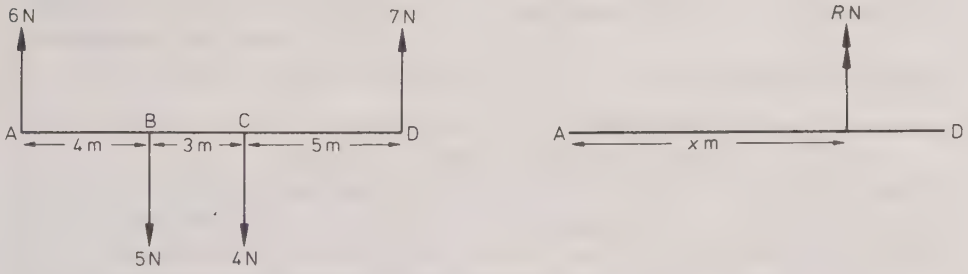
10. Four forces each of magnitude 3 N act along the sides BA, BC, CD, AD of a square $ABCD$, in the directions indicated by the order of the letters. Find the magnitude and the line of action of their resultant.

11. Three forces each of magnitude 10 N act along the sides AB, DE, AF of a regular hexagon $ABCDEF$ of side 2 m. Find the sum of the moments of these forces about each of the points A, B, C, D, E and F. Find also the magnitude and the line of action of the resultant of the three forces.

9.2 Systems of coplanar forces

We consider first a system of parallel coplanar forces with non-zero resultant. The magnitude and direction of the resultant is found by vector addition. Its line of action can be determined using the principle of moments.

Example 1 Find the magnitude and line of action of the resultant of the system of parallel forces shown in the diagram.



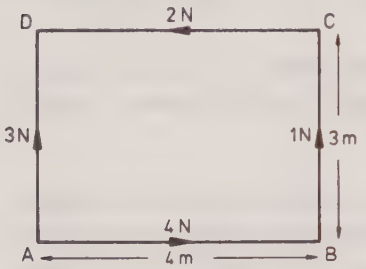
Let the resultant of the given system of forces be of magnitude R N acting at a distance x m from A.
Comparing magnitudes and moments about A)

$\uparrow \quad R = 6 - 5 - 4 + 7 = 4$
$A) \quad Rx = 0 - 5 \times 4 - 4 \times 7 + 7 \times 12 = 36$
$\therefore \quad R = 4 \quad \text{and} \quad x = 9.$

Hence the resultant has magnitude 4 N and its line of action cuts AD at a point 9 m from A.

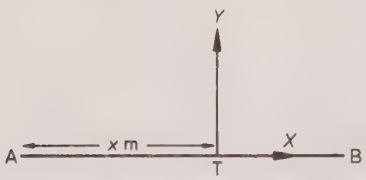
Similar methods can be applied to any set of coplanar forces with a non-zero resultant. The magnitude and direction of the resultant is found by means of a vector polygon or by resolving in two perpendicular directions. As before the line of action of the resultant is determined using the principle of moments.

Example 2



The resultant of the system of forces shown in the diagram is a force **R** acting through a point T on AB. Find the magnitude and direction of **R** and the length of AT.

Let X and Y be the components of **R** parallel and perpendicular to AB.



Resolving parallel and perpendicular to AB

$\rightarrow$ $\quad X = 4 - 2 \quad \therefore \quad X = 2$

$\uparrow$ $\quad Y = 3 + 1 \quad \therefore \quad Y = 4$

Taking moments about A

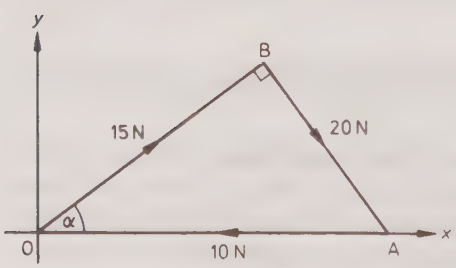
$$Yx = 1 \times 4 + 2 \times 3 = 10 \qquad \therefore \quad x = 2{\cdot}5$$

$$\therefore \quad R = \sqrt{(X^2 + Y^2)} = \sqrt{(4 + 16)} = \sqrt{20} = 2\sqrt{5} \quad \text{and}$$

$$\tan\theta = \frac{Y}{X} = \frac{4}{2} = 2, \quad \theta \approx 63{\cdot}4°.$$

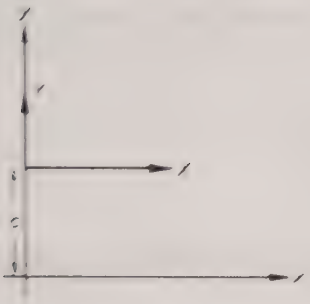
Hence the resultant has magnitude $2\sqrt{5}$ N and acts at $63{\cdot}4°$ to AB. The length of AT is 2·5 m.

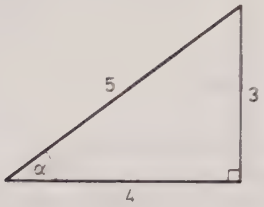
Example 3



Find the equation of the line of action of the resultant of the forces shown in the diagram, given that $\tan\alpha = \frac{3}{4}$ and that A is the point $(7, 0)$.

Let X and Y be the components of the resultant in the directions of the x- and y-axes respectively. Let the line of action cut the y-axis at the point $(0, c)$.





$\Rightarrow \quad \tan\alpha = \frac{3}{4}$
$\sin\alpha = \frac{3}{5},$
$\cos\alpha = \frac{4}{5}.$

Resolving in the directions of the x- and y- axes,

$$\rightarrow \qquad X = 15\cos\alpha + 20\sin\alpha - 10$$
$$\uparrow \qquad Y = 15\sin\alpha - 20\cos\alpha$$

$\therefore \quad X = 12 + 12 - 10 = 14, \quad Y = 9 - 16 = -7.$

Taking moments about O: $\quad Xc = 20 \times 7\cos\alpha$

$$\therefore \quad c = \frac{20 \times 7\cos\alpha}{X} = \frac{20 \times 7 \times 4}{14 \times 5} = 8$$

Hence the line of action of the resultant has gradient $Y/X = -7/14 = -1/2$ and intercept on the y-axis $c = 8$,

$\therefore$ the equation of the line of action is

$$y = -\tfrac{1}{2}x + 8 \quad \text{i.e.} \quad x + 2y = 16.$$

Two systems of forces are said to be *equivalent* if their effects are the same. In particular, two sets of coplanar forces are equivalent if

(i) the sums of the components of the forces in any given direction are equal and

(ii) the sums of the moments of the forces about any given point are equal.

Thus, the *resultant* of a system of forces is the single force equivalent to the whole system.

It also follows from conditions (i) and (ii) that any two coplanar *couples* with the same moment are equivalent. Similarly a set of coplanar couples with moments $G_1, G_2, G_3, \ldots, G_n$ is equivalent to a single couple with moment

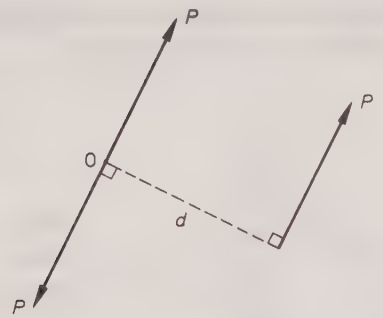
$$G_1 + G_2 + G_3 + \ldots + G_n.$$

We next consider the resultant of a force **P** acting at a point O and a couple of moment G in the same plane. The couple is equivalent to a pair of equal and opposite forces of magnitude P and perpendicular distance d apart, provided that

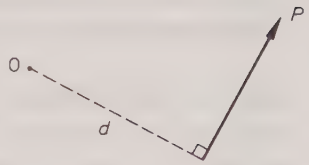
$$Pd = G.$$

Therefore the force and the couple are equivalent to the three forces shown in diagram (a) and hence to the single force shown in diagram (b).

(a)



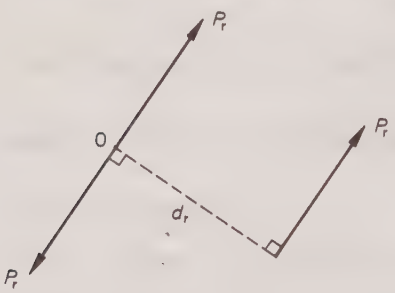
(b)



Thus the resultant of a force **P** acting at O and a couple of moment G is a force **P** acting at a distance d from O, where $d = G/P$.

Reversing the argument we may also deduce that a force **P** acting at a distance d from a point O is equivalent to a force **P** acting at O together with a couple of moment Pd.

This result can be used to find a simple equivalent system for any set of coplanar forces. Let $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$ be a system of coplanar forces acting at distances $d_1, d_2, \ldots, d_n$ respectively from a fixed point O in their plane. A typical force $\mathbf{P}_r$ acting at a distance d_r from O is equivalent to a force $\mathbf{P}_r$ acting at O together with a couple of moment $P_r d_r$. Applying this reasoning to all the forces in the system we obtain an equivalent system of forces $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$ acting at 0 together with couples of moments $P_1 d_1, P_2 d_2, \ldots, P_n d_n$.



If **R** is the vector sum of the forces and G is the sum of the moments of the original forces about O, then this new system can be replaced by a single force **R** acting at O and a couple of moment G.

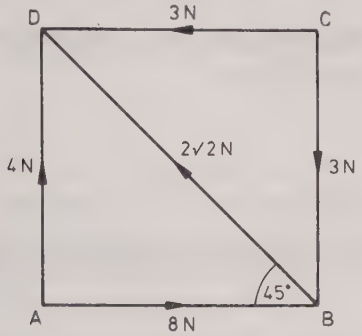
> Thus, in general, any system of coplanar forces is equivalent to a single force acting through a given point together with a couple.

We consider further four possible types of system.

(1) $R = 0$, $G = 0$. The system is in equilibrium.
(2) $R = 0$, $G \neq 0$. The system is equivalent to a couple of moment G.
(3) $R \neq 0$, $G = 0$. The system is equivalent to a single force **R** acting through O.
(4) $R \neq 0$, $G \neq 0$. The system is equivalent to a single force **R** acting at a distance G/R from O.

> Hence any system of coplanar forces which is not in equilibrium may be reduced either to a single force or to a couple.

Example 4 A square $ABCD$ of side 3 m has forces of magnitude 8, 3, 3, 4 and $2\sqrt{2}$ newtons acting along AB, CB, CD, AD and BD respectively. If the system is reduced to a force acting through A together with a couple, find the magnitude and direction of the force and the moment of the couple. If the system is reduced to a single force acting through a point E on AB at a distance d metres from A, find d.



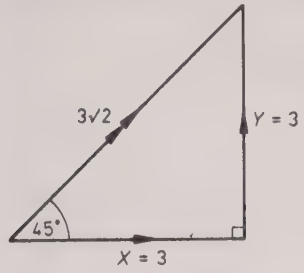
Suppose that the given system is equivalent to forces X and Y acting along AB and AD respectively, together with a couple of moment G in the sense $ABCD$.

Resolving in the directions of AB and AD:

$$\rightarrow \qquad X = 8 - 3 - 2\sqrt{2}\cos 45° = 3$$
$$\uparrow \qquad Y = 4 - 3 + 2\sqrt{2}\sin 45° = 3$$

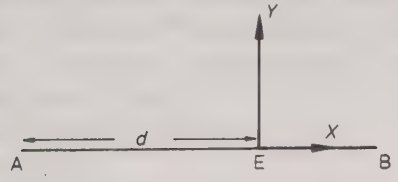
Taking moments about A↻:

$$G = 3 \times 3 - 3 \times 3 + 2\sqrt{2} \times 3\sin 45° = 6$$



Hence the system is equivalent to a force of $3\sqrt{2}$ N acting along AC together with a couple of moment 6 N m in the sense $ABCD$.

The single equivalent force through E has moment G about A and components X and Y in the directions of AB and AD respectively.

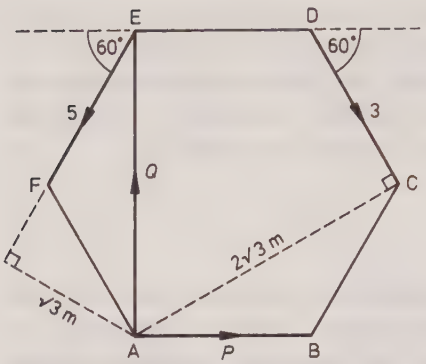


Taking moments about A↻:

$$G = Yd$$
$$\text{i.e.} \quad 6 = 3d$$
$$\therefore \quad d = 2.$$

Example 5 $ABCDEF$ is a regular hexagon of side 2 m. Forces of magnitude P, 3, 5 and Q newtons act along AB, DC, EF and AE respectively. Show that this system of forces is not in equilibrium.

(a) If the system is equivalent to a couple, find its moment and the values of P and Q.
(b) If the system is equivalent to a single force through E, find P.



The sum of the moments of the forces about A (in N m)
$= 3 \times 2\sqrt{3} - 5 \times \sqrt{3} = \sqrt{3}$.
Since this sum is not zero, the forces are not in equilibrium.

(a) If the system is equivalent to a couple, it will have the same moment about any point in the plane,
$\therefore$ the moment of the couple is $\sqrt{3}$ N m.
The sum of the components of the forces in a couple is zero in any direction,
$\therefore$ resolving in the directions of AB and AE:

$$\rightarrow \qquad P + 3\cos 60° - 5\cos 60° = 0$$
$$\uparrow \qquad Q - 3\sin 60° - 5\sin 60° = 0$$

$\therefore \quad P = 2\cos 60° = 1 \quad$ and $\quad Q = 8\sin 60° = 4\sqrt{3}$.

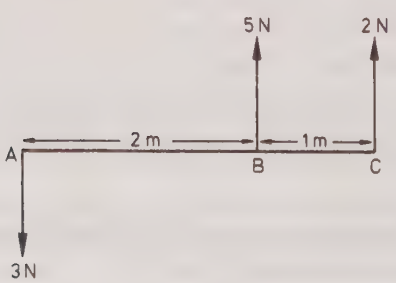
(b) If the system is equivalent to a single force through E, then the sum of the moments about E will be zero.
Taking moments about E: $P \times 2\sqrt{3} - 3 \times \sqrt{3} = 0$

$$\therefore \quad P = \frac{3\sqrt{3}}{2\sqrt{3}} = 1{\cdot}5.$$

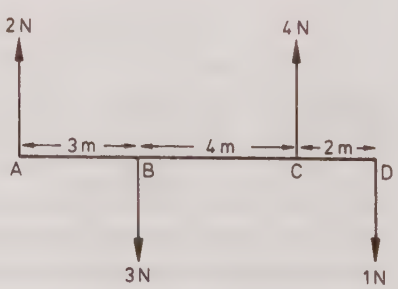
Exercise 9.2

In questions 1 to 4 the resultant of the given forces acts through a point T on AB or AB produced. In each case find the magnitude and direction of the resultant and the length of AT. [In question 4 $ABCDEF$ is a regular hexagon.]

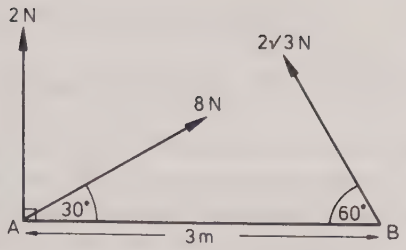
1.



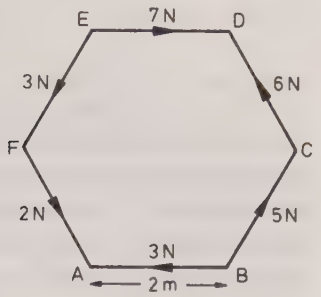
2.



3.



4.



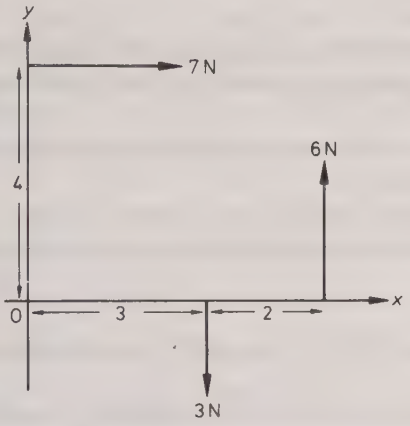
5. Forces of magnitude $8P, 7P, 5P$ and $3P$ act along the sides AB, BC, CD and DA respectively of a square $ABCD$ of side $2a$. The resultant of this system of forces cuts AB produced at E. Find

(a) the magnitude and direction of the resultant,

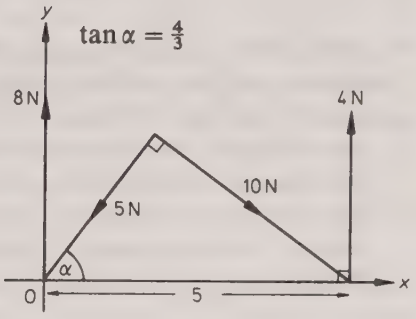
(b) the length of AE.

In questions 6 and 7 find the equation of the line of action of the resultant of the given forces.

6.



7.



In questions 8 to 19 a system of forces acts in the plane of perpendicular axes Ox and Oy. The sums of the components of the forces in the directions of Ox and Oy are X and Y respectively. The sums of the anti-clockwise moments of the forces about three non-collinear points A, B and C in the plane are G_A, G_B and G_C respectively. In each question use the given facts to decide which of the following statements could be true.

a: the system is in equilibrium,
b: the system is equivalent to a couple,
c: the system is equivalent to a single force.

8. $X = 0, Y = 0.$

9. $X = 0, Y \neq 0.$

10. $Y = 0, G_A = 0.$

11. $X = 0, G_B \neq 0.$

12. $Y \neq 0, G_C = 0.$

13. $G_A \neq 0, G_B = 0.$

14. $G_A = G_B = 0$. 15. $G_A = G_B \neq 0$. 16. $G_A = G_B \neq G_C$.

17. $G_A = G_B = G_C = 0$. 18. $G_A = G_B = G_C \neq 0$ 19. $X = 0$, $G_A = G_B = 0$.

20. In the square $OABC$ of side 2 m, O, A and B are the points $(0, 0)$, $(2, 0)$ and $(2, 2)$ respectively. Forces of magnitude 5, 8, 3 and $5\sqrt{2}$ newtons act along OA, BA, OC and AC respectively. Show that this system of forces is equivalent to a couple and find its moment. If a fifth force of magnitude $6\sqrt{2}$ newtons acts along OB, find the equation of the line of action of the resultant of the enlarged system.

21. In a rectangle $ABCD$, $AB = 12$ m, $BC = 5$ m. Forces of magnitude 6, P, 18, 10 and Q newtons act along AB, CB, CD, AD and DB respectively. Show that this system of forces is not in equilibrium. Find P and Q given that the system reduces to (a) a couple, (b) a single force acting along DC.

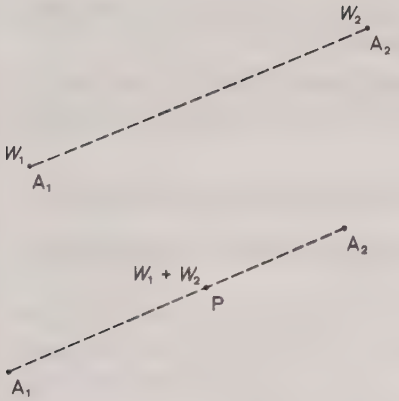
22. In a rectangle $ABCD$, $AB = 3$ m, $BC = 2$ m. Forces of magnitude 5, 8, 12, 6 and P newtons act along BA, BC, DC, AD and EA respectively, where E is a point on CD. Given that the forces are in equilibrium, find P and the distance DE. If the force along EA is replaced by an equal force acting through C, show that the system now reduces to a couple and find its moment.

23. $ABCDEF$ is a regular hexagon of side 1 m. Forces of magnitude 4, 6, 10, 8, P and 2 newtons act along AB, BC, CD, ED, FE and AF respectively, directions being indicated by the order of the letters.
(a) If the resultant passes through the centre of the hexagon find the value of P and the magnitude and direction of the resultant.
(b) The force in FE is replaced by another so that the new system reduces to a couple. Find the magnitude of the force and the moment of the couple indicating its sense. (SU)

24. The points A, B and C have coordinates $(9a, 0)$, $(0, -4a)$ and $(6a, 4a)$ respectively referred to the coordinate axes Ox and Oy.
(a) Forces $12P$, $15P$, $5P$ and P act along $\overrightarrow{OA}$, $\overrightarrow{BC}$, $\overrightarrow{CA}$ and $\overrightarrow{OB}$ respectively. Calculate the magnitude of the resultant of these forces and the equation of the line of action of this resultant.
(b) Forces $15P$, S and T act along $\overrightarrow{BC}$, $\overrightarrow{OA}$ and $\overrightarrow{CA}$ respectively.
Given that these forces reduce to a couple, calculate (i) the values of S and T, in terms of P, (ii) the magnitude of the couple, in terms of a and P, (iii) the sense of the couple. (AEB 1979)

9.3 Centre of gravity; centre of mass

A solid body may be regarded as a tightly packed collection of small particles. The *weight* of the body is the resultant of the weights of the particles. For relatively small bodies on or near the earth's surface the weights of the constituent particles are assumed to form a system of parallel forces acting vertically downwards.

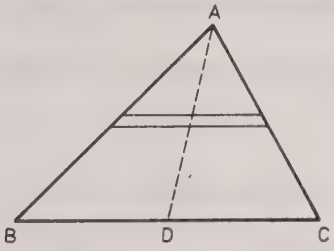


Suppose that at points A_1 and A_2 of a body there are particles of weights W_1 and W_2 respectively. As shown in §9.1, the resultant of the weights of these particles is a force of magnitude $W_1 + W_2$ acting through the point P which divides A_1A_2 in the ratio $W_2 : W_1$.

By extending this argument to include all the particles forming the body, we deduce that the resultant weight force is equal to the sum of the weights of the particles and acts through a fixed point in the body.

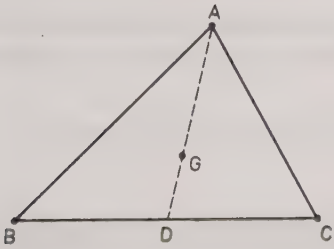
The *centre of gravity* of a body is the fixed point through which its weight acts.

The *symmetries* of a body can be used to determine its centre of gravity. If the particles in a body are evenly distributed about an axis or a plane of symmetry, then the centre of gravity of the body must lie on this axis or plane. Thus the centre of gravity of a thin uniform rod is at its mid-point and the centre of gravity of a uniform sphere is at its centre.



To find the centre of gravity of a *uniform triangular lamina ABC*, suppose that the triangle is divided into n strips parallel to the side BC. If the value of n is increased the strips become thin uniform rods with centres of gravity at their mid-points. Hence the centre of gravity of the whole lamina must lie on the line joining these mid-points i.e. on the median AD. Similarly the centre of gravity must also lie on the medians through B and C.

Thus the centre of gravity of a uniform triangular lamina is at the point of intersection of the medians.



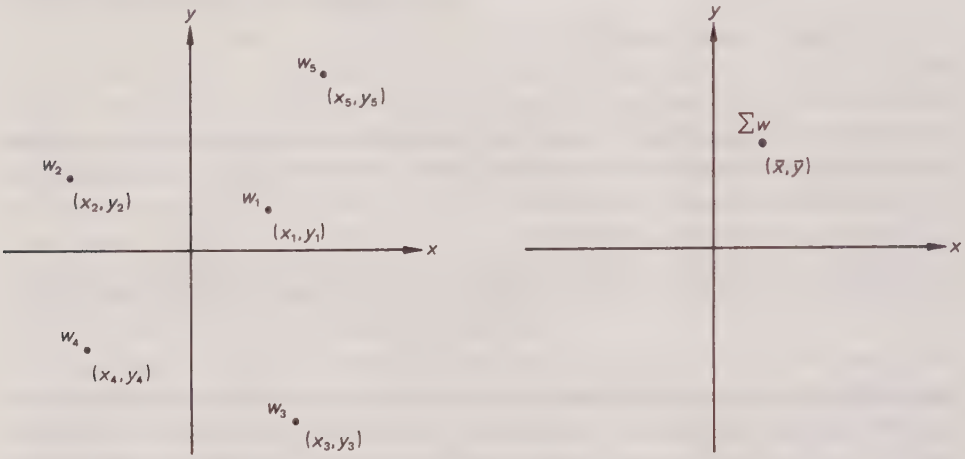
Consider now three particles each of weight W placed at the vertices of a triangle ABC. The resultant of the weights of the particles at B and C is a force of $2W$ acting at D, the mid-point of BC. Hence the resultant of the weights of all three particles is a force of $3W$ acting at the point G on AD which divides

AD in the ratio 2:1. Using similar arguments we find that G is also the point which divides the medians through B and C in the ratio 2:1.

Thus the centre of gravity of the three equal particles at A, B and C is at the point of intersection of the medians of triangle ABC, i.e. at the same point as the centre of gravity of the uniform lamina ABC.

[Note that we have also verified that this point divides each median in the ratio 2:1.]

In general the centre of gravity of a set of particles in a plane is found using the principle of moments.



Let particles with weights $w_1, w_2, \ldots, w_n$ have positions $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ relative to x- and y-axes in their plane. Suppose that the resultant weight of the set of particles acts through the point $(\bar{x}, \bar{y})$.

Assuming the x, y plane to be horizontal and applying the principle of moments we have,
for moments about the y-axis:

$$w_1x_1 + w_2x_2 + \cdots + w_nx_n = (w_1 + w_2 + \cdots + w_n)\bar{x}$$

and for moments about the x-axis:

$$w_1y_1 + w_2y_2 + \cdots + w_ny_n = (w_1 + w_2 + \cdots + w_n)\bar{y}.$$

Hence the centre of gravity of the particles is at the point $(\bar{x}, \bar{y})$,

where
$$\bar{x} = \frac{\sum wx}{\sum w}, \quad \bar{y} = \frac{\sum wy}{\sum w}.$$

This result can be expressed in vector notation by letting the position vectors of the particles be $\mathbf{r}_1 = x_1\mathbf{i} + y_1\mathbf{j}, \mathbf{r}_2 = x_2\mathbf{i} + y_2\mathbf{j}, \ldots$ and the position vector of the centre of gravity be $\bar{\mathbf{r}} = \bar{x}\mathbf{i} + \bar{y}\mathbf{j}$, so that

$$\bar{\mathbf{r}} = \frac{(w_1x_1 + w_2x_2 + \cdots)\mathbf{i}}{\sum w} + \frac{(w_1y_1 + w_2y_2 + \cdots)\mathbf{j}}{\sum w}$$

$$= \frac{w_1(x_1\mathbf{i} + y_1\mathbf{j}) + w_2(x_2\mathbf{i} + y_2\mathbf{j}) + \cdots}{\sum w}$$

Thus the position vector of the centre of gravity is $\bar{\mathbf{r}} = \dfrac{\sum w\mathbf{r}}{\sum w}$.

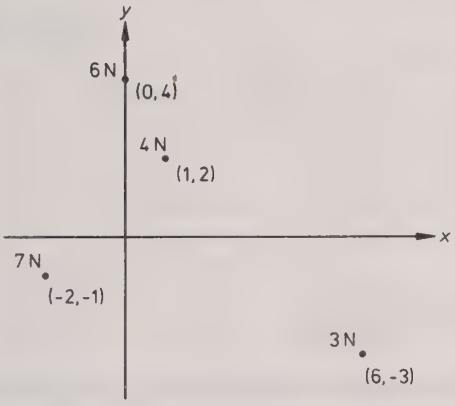
Note that in the case of three particles of weight W at the points $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ we have

$$\bar{x} = \frac{Wx_1 + Wx_2 + Wx_3}{3W} = \tfrac{1}{3}(x_1 + x_2 + x_3)$$

and similarly $\bar{y} = \tfrac{1}{3}(y_1 + y_2 + y_3)$.

Hence the centre of gravity of three equal particles at A, B, C and also of a uniform triangular lamina ABC has coordinates $(\tfrac{1}{3}\{x_1 + x_2 + x_3\}, \tfrac{1}{3}\{y_1 + y_2 + y_3\})$ and position vector $\tfrac{1}{3}\{\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3\}$.

Example 1 Find the coordinates of the centre of gravity of particles of weights 4 N, 7 N, 3 N and 6 N at the points (1, 2), (−2, −1), (6, −3) and (0, 4) respectively.



Let the centre of gravity of the particles be at the point $(\bar{x}, \bar{y})$, then the resultant weight of the particles has magnitude 20 N and acts through $(\bar{x}, \bar{y})$.

Taking moments about the y-axis,

$$20\bar{x} = 4 \times 1 + 7 \times (-2) + 3 \times 6 + 6 \times 0$$

$$\therefore \quad \bar{x} = \frac{4 - 14 + 18 + 0}{20} = \frac{8}{20} = \frac{2}{5}$$

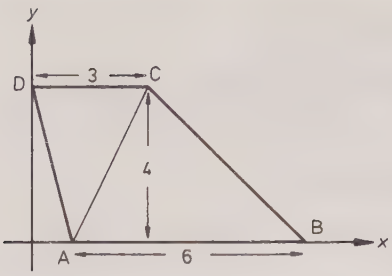
Taking moments about the x-axis,

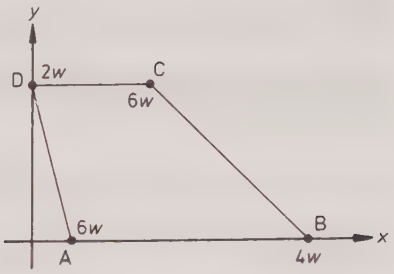
$$20\bar{y} = 4 \times 2 + 7 \times (-1) + 3 \times (-3) + 6 \times 4$$

$$\therefore \quad \bar{y} = \frac{8 - 7 - 9 + 24}{20} = \frac{16}{20} = \frac{4}{5}$$

Hence the centre of gravity has coordinates $\left(\dfrac{2}{5}, \dfrac{4}{5}\right)$.

Example 2 Find the coordinates of the centre of gravity of a uniform lamina in the form of a trapezium with vertices $A(1, 0)$, $B(7, 0)$, $C(3, 4)$ and $D(0, 4)$.





The weight of the lamina $ABCD$ is the resultant of the weights of lamina ABC and lamina ACD. Let w be the weight per unit area of the lamina, then the weight of ABC is $12w$ and is equivalent to three weights of $4w$ at points A, B and C.
Similarly the weight of ACD is $6w$ and is equivalent to three weights of $2w$ at points A, C and D.
Hence the centre of gravity $(\bar{x}, \bar{y})$ of the lamina $ABCD$ is the same as the centre of gravity of particles of weights $6w, 4w, 6w$ and $2w$ at the points A, B, C and D respectively.
Taking moments about the y-axis,

$$18w\bar{x} = 6w \times 1 + 4w \times 7 + 6w \times 3$$

Taking moments about the x-axis,

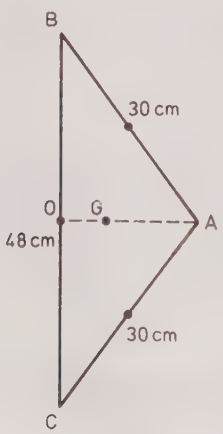
$$18w\bar{y} = 6w \times 4 + 2w \times 4$$

$$\therefore \bar{x} = \frac{52w}{18w} = \frac{26}{9}, \quad y = \frac{32w}{18w} = \frac{16}{9}.$$

Hence the centre of gravity has coordinates $\left(\frac{26}{9}, \frac{16}{9}\right)$.

The principle of moments is also used to find the centre of gravity of any body which can be divided into smaller bodies, each with known weight and centre of gravity.

Example 3 A thin uniform wire is bent to form a triangle ABC in which $AB = AC = 30$ cm and $BC = 48$ cm. Find its centre of gravity.



From symmetry the centre of gravity G lies on OA, where O is the mid-point of BC.
Each of the wires AB, AC and BC has its centre of gravity at its mid-point.
By Pythagoras' theorem, $OA = 18$ cm.

Letting w be the weight per cm of the wire, we have:

Body	Weight	Distance of centre of gravity from BC
Wire AB	$30w$	9
Wire AC	$30w$	9
Wire BC	$48w$	0
Whole body	$108w$	$\bar{x}$

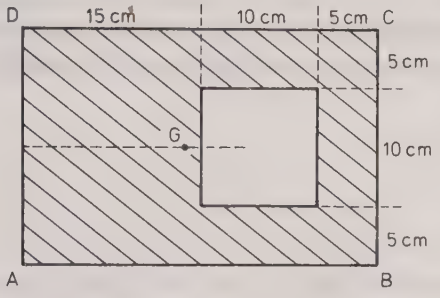
Taking moments about BC,

$$108w\bar{x} = 30w \times 9 + 30w \times 9 + 48w \times 0$$

$$\therefore \quad \bar{x} = \frac{2 \times 30w \times 9}{108w} = 5$$

$\therefore$ the centre of gravity is 5 cm from BC on the line OA.

Example 4



A square hole is cut in a uniform rectangular lamina as shown in the diagram. Find the centre of gravity of the resulting body.

From symmetry the centre of gravity G lies on the line joining the mid-points of AD and BC.

We use the positions of the centres of gravity of the complete rectangle and of the square removed to find the distance $\bar{x}$ of G from AD.
Letting w be the weight per unit area, we have:

Body	Weight	Distance of centre of gravity from AD
Rectangle	$600w$	15
Square	$100w$	20
Remainder	$500w$	$\bar{x}$

Taking moments about AD,

$$500w\bar{x} + 100w \times 20 = 600w \times 15$$

$$\therefore \quad 500w\bar{x} = 600w \times 15 - 100w \times 20$$

$$\therefore \quad \bar{x} = \frac{9000w - 2000w}{500w} = 14$$

$\therefore$ the centre of gravity is 14 cm from AD on the line joining the mid-points of AD and BC.

As shown earlier the centre of gravity of a set of particles in a plane is the point $\left(\frac{\sum wx}{\sum w}, \frac{\sum wy}{\sum w}\right)$. If $m_1, m_2, \ldots$ are the masses of the particles, their *centre of mass* is defined to be the point

$$\left(\frac{\sum mx}{\sum m}, \frac{\sum my}{\sum m}\right).$$

The weight w and the mass m of a typical particle are connected by the relationship $w = mg$, where g is the acceleration due to gravity. Thus, provided that g takes the same value for all the particles, $\frac{\sum wx}{\sum w} = \frac{\sum mgx}{\sum mg} = \frac{\sum mx}{\sum m}$ and similarly $\frac{\sum wy}{\sum w} = \frac{\sum my}{\sum m}$. It follows that, in general, the centre of gravity and the centre of mass of a body are at the same point.

[We note, however, that if a body is so large that the weights of its constituent particles are not parallel forces, then its centre of gravity cannot be found by elementary methods and does not necessarily coincide with the centre of mass.]

A *centroid* is a centre of area or volume. The centroid of a plane area coincides with the centre of mass of a uniform lamina of the same shape. The centroid of a three-dimensional figure coincides with the centre of mass of the corresponding uniform solid.

Exercise 9.3

1. Find the centre of gravity of two particles of weight 2 N and 3 N at points A and B respectively, given that $AB = 30$ cm.

2. Find the centre of gravity of two particles of weight 30 N and 70 N at points P and Q respectively, given that $PQ = 1{\cdot}5$ m.

3. A uniform triangular lamina ABC has $AB = AC = 17$ cm and $BC = 16$ cm. Find the distance of the centre of gravity from BC.

4. A uniform triangular lamina PQR has $PQ = 9$ cm, $QR = 15$ cm and $PR = 12$ cm. Find the distances of the centre of gravity from PR and PQ.

5. Find the coordinates of the centre of gravity of particles of weight 5 N, 7 N, 1 N and 3 N at the points $(1, 0)$, $(3, 1)$, $(6, 3)$ and $(0, 2)$ respectively.

6. Find the coordinates of the centre of mass of particles of mass 9 kg, 4 kg, 6 kg and 5 kg at the points $(4, 3)$, $(6, -6)$, $(-3, 0)$ and $(6, -3)$ respectively.

7. Find the position vector of the centre of mass of a uniform triangular lamina, given that the position vectors of its vertices are $3\mathbf{i} - 5\mathbf{j}$, $7\mathbf{i} + 2\mathbf{j}$ and $-\mathbf{i} + 6\mathbf{j}$.

8. Find the position vector of the centre of gravity of three particles of weight 7 N, 9 N and 4 N with position vectors $4\mathbf{i} + \mathbf{j}$, $3\mathbf{j}$ and $3\mathbf{i} + 4\mathbf{j}$.

9. A uniform lamina in the form of a quadrilateral has vertices $O(0,0)$, $A(4,-6)$, $B(7,0)$ and $C(0,6)$. If the weight per unit area of the lamina is w, show that it has the same centre of gravity as particles of weight $14w$, $7w$, $14w$ and $7w$ at O, A, B and C respectively. Hence find the centre of gravity of the lamina.

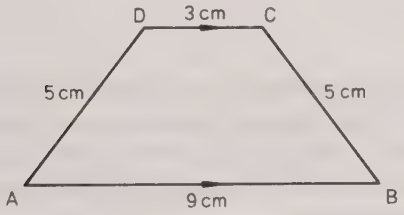
10. Use the method of the previous question to find the centre of gravity of a uniform lamina in the form of a pentagon with vertices $O(0,0)$, $A(3,0)$, $B(3,2)$, $C(0,4)$ and $D(-3,1)$.

11. A thin uniform wire is bent to form a triangle ABC in which $AB = 24\,\text{cm}$, $AC = 10\,\text{cm}$ and $\angle A = 90°$. Find the distances of the centre of gravity from AC and AB.

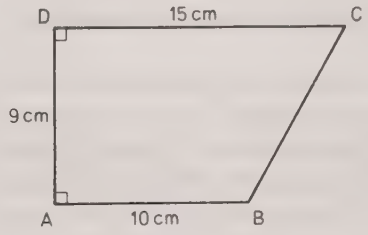
12. A thin uniform wire is bent into the shape of an isosceles trapezium $ABCD$ in which $AB = 24\,\text{cm}$, $AD = BC = 25\,\text{cm}$ and $CD = 10\,\text{cm}$. Find the distance of the centre of gravity from AB.

In questions 13 to 16 find the position of the centre of gravity of the uniform lamina shown in the given diagram.

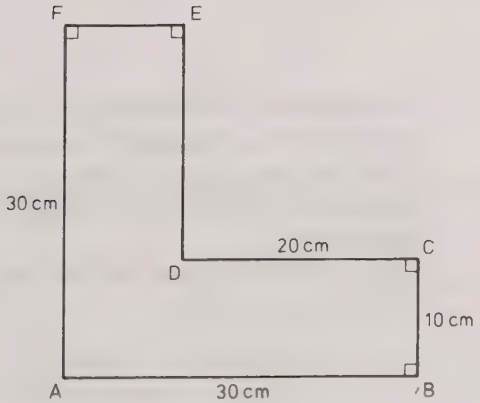
13.



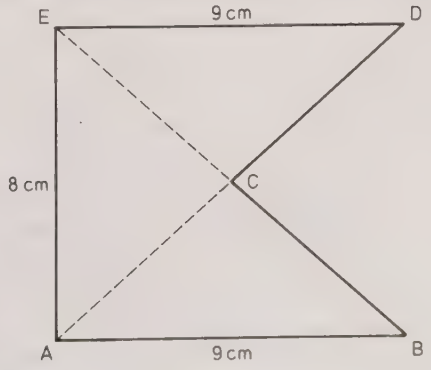
14.



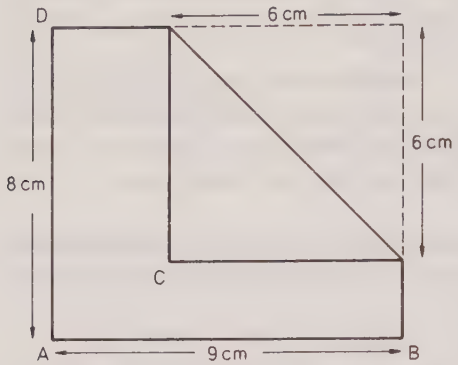
15.



16.

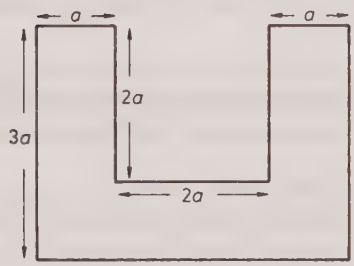


17.



Find the centre of gravity of a uniform rectangular lamina *ABCD* folded as shown in the diagram.

18.

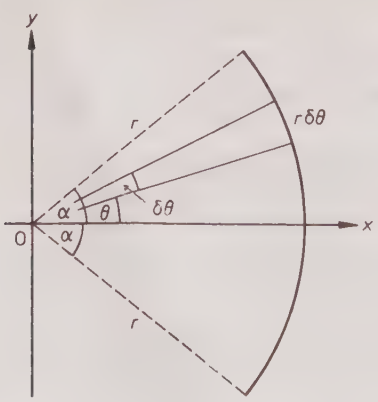


The diagram shows the central cross-section of a uniform cylindrical container weighing $2w$ per unit volume. Find its centre of gravity. If the container is now filled with a liquid weighing w per unit volume, find the new position of the centre of gravity.

9.4 Standard results by integration

As we saw in the previous section the principle of moments can be used to find the centre of gravity of any body which can be divided into smaller bodies with known weights and centres of gravity. In certain cases it is necessary to divide a body into a large number of small parts or elements. The sums of the moments of these elements about appropriate axes are then found by integration.

Example 1 Find the centre of gravity of a uniform wire in the form of an arc of a circle, radius r, subtending an angle 2α at the centre.



From symmetry the centre of gravity lies on the bisector of the angle 2α. As shown in the diagram, a typical element of arc subtends an angle $\delta\theta$ at O and is of length $r\,\delta\theta$.

Letting w be the weight per unit length of the wire, we have:

Body	Weight	Distance of centre of gravity from the y-axis
Element	$wr\,\delta\theta$	$r\cos\theta$
Whole arc	$wr\,2\alpha$	$\bar{x}$

Taking moments about the y-axis,

$$wr\,2\alpha\bar{x} \approx \sum(wr\,\delta\theta \times r\cos\theta)$$

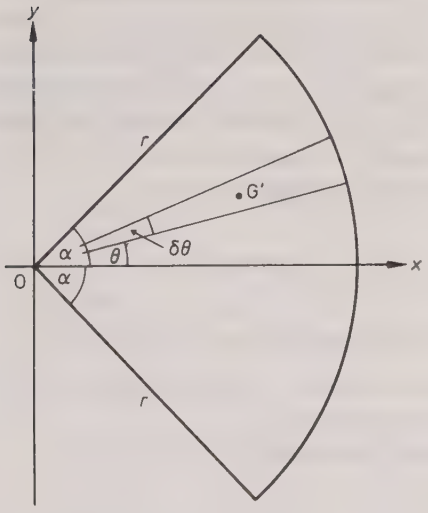
$$\therefore \quad 2wr\,\alpha\bar{x} = \int_{-\alpha}^{\alpha} wr^2\cos\theta\,d\theta$$

$$= wr^2\Big[\sin\theta\Big]_{-\alpha}^{\alpha} = 2wr^2\sin\alpha$$

$$\therefore \quad \bar{x} = \frac{r\sin\alpha}{\alpha}$$

Hence the centre of gravity of the wire is at a distance $(r\sin\alpha)/\alpha$ from the centre.

Example 2 Find the centre of gravity of a uniform lamina in the form of a sector of a circle, radius r, subtending an angle 2α at the centre.



From symmetry the centre of gravity lies on the bisector of the angle 2α. A typical element is a sector subtending an angle $\delta\theta$ at O and having area $\frac{1}{2}r^2\,\delta\theta$. This sector is approximately triangular with its centre of gravity G' at a distance $\frac{2}{3}r$ from O.

Letting w be the weight per unit area of the lamina, we have:

Body	Weight	Distance of centre of gravity from the y-axis
Element	$w\frac{1}{2}r^2\,\delta\theta$	$\frac{2}{3}r\cos\theta$
Whole sector	$w\frac{1}{2}r^2\,2\alpha$	$\bar{x}$

Taking moments about the y-axis,

$$w\tfrac{1}{2}r^2\,2\alpha\bar{x} \approx \sum(w\tfrac{1}{2}r^2\,\delta\theta \times \tfrac{2}{3}r\cos\theta)$$

$$\therefore \quad wr^2\,\alpha\bar{x} = \int_{-\alpha}^{\alpha} \tfrac{1}{3}wr^3\cos\theta\,d\theta$$

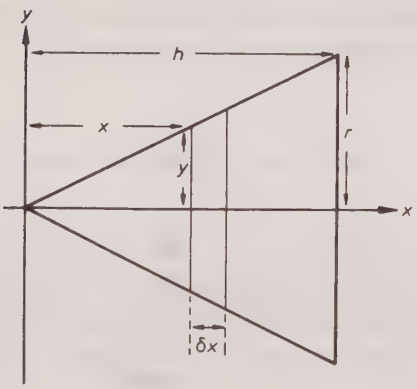
$$= \tfrac{1}{3}wr^3\Big[\sin\theta\Big]_{-\alpha}^{\alpha} = \tfrac{2}{3}wr^3\sin\alpha$$

$$\therefore \quad \bar{x} = \frac{2r\sin\alpha}{3\alpha}$$

Hence the centre of gravity of the lamina is at a distance $(\frac{2}{3}r\sin\alpha)/\alpha$ from the centre.

[This result can also be produced by showing that a uniform sector of radius r has the same centre of gravity as a uniform arc of radius $\frac{2}{3}r$.]

Example 3 Find the centre of gravity of a uniform solid right circular cone of base radius r, height h.



Let the axis of the cone lie along the x-axis, then from symmetry the centre of gravity is on the x-axis. If the cone is divided into elements by planes parallel to its base, then each element is approximately a disc, thickness δx, radius y, which has volume $\pi y^2\,\delta x$.

Letting w be the weight per unit volume of the cone, we have:

Body	Weight	Distance of centre of gravity from the y-axis
Element	$w\pi y^2\,\delta x$	x
Whole cone	$w\frac{1}{3}\pi r^2 h$	$\bar{x}$

Taking moments about the y-axis

$$w\tfrac{1}{3}\pi r^2 h\bar{x} \approx \sum(w\pi y^2\,\delta x \times x)$$

$$\therefore \quad \tfrac{1}{3}\pi r^2 hw\bar{x} = \int_0^h \pi w x y^2\,dx$$

From similar triangles, $\dfrac{y}{x} = \dfrac{r}{h}$

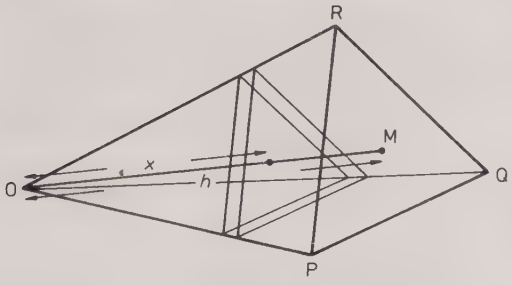
$$\therefore \quad \tfrac{1}{3}\pi r^2 hw\bar{x} = \int_0^h \frac{\pi r^2 w}{h^2} x^3 \, dx$$

$$= \frac{\pi r^2 w}{h^2}\left[\tfrac{1}{4}x^4\right]_0^h = \tfrac{1}{4}\pi r^2 h^2 w$$

$$\therefore \quad \bar{x} = \tfrac{3}{4}h$$

Hence the centre of gravity of the cone is on its axis at a distance $\frac{1}{4}h$ from its base.

Example 4 Find the centre of gravity of a uniform solid tetrahedron.



In tetrahedron $OPQR$, let M be the point of intersection of the medians of $\triangle PQR$. Let the length of OM be h and let the area of $\triangle PQR$ be A.
If the tetrahedron is divided into elements by planes parallel to the face PQR, then a typical element is a triangular lamina of thickness δx, which cuts OM at a distance x from O. Since this element is similar to $\triangle PQR$, its medians intersect on OM and its area is Ax^2/h^2.
Letting w be the weight per unit volume, we have:

Body	Weight	Distance of centre of gravity from 0
Element	$wAx^2\,\delta x/h^2$	x
Whole body	$w\frac{1}{3}Ah$	$\bar{x}$

Taking moments about an axis perpendicular to OM

$$w\tfrac{1}{3}Ah\bar{x} \approx \sum\left(\frac{wAx^2\,\delta x}{h^2} \times x\right)$$

$$\therefore \quad \tfrac{1}{3}Ahw\bar{x} = \int_0^h \frac{Aw}{h^2} x^3 \, \delta x$$

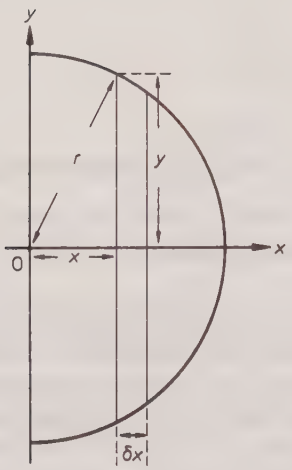
$$= \frac{Aw}{h^2}\left[\tfrac{1}{4}x^4\right]_0^h = \tfrac{1}{4}Awh^2$$

$$\therefore \quad \bar{x} = \tfrac{3}{4}h$$

Hence the centre of gravity of the tetrahedron lies one quarter of the way up a line joining the centroid of one face to the opposite vertex.

Example 5 Find the centre of gravity of a uniform solid hemisphere of radius r.

If we take x- and y-axes along and perpendicular to the axis of symmetry of the hemisphere, then its centre of gravity will be on the x-axis.



A typical element is approximately a disc of thickness δx, radius y at a distance x from the y-axis.
Letting w be the weight per unit volume of the hemisphere:

Body	Weight	Distance of centre of gravity from the y-axis
Element	$w\pi y^2\,\delta x$	x
Whole hemisphere	$w\frac{2}{3}\pi r^3$	$\bar{x}$

Taking moments about the y-axis,

$$w\tfrac{2}{3}\pi r^3\bar{x} \approx \sum(w\pi y^2\,\delta x \times x)$$

$$\therefore\quad \tfrac{2}{3}\pi r^3 w\bar{x} = \int_0^r \pi w x y^2\,dx$$

By Pythagoras' theorem, $x^2 + y^2 = r^2$,

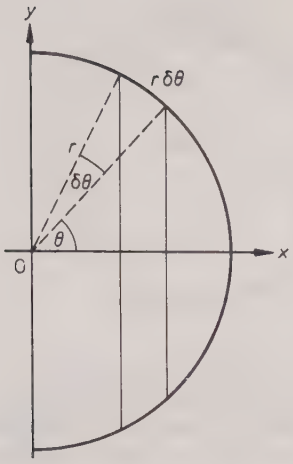
$$\therefore\quad \tfrac{2}{3}\pi r^3 w\bar{x} = \int_0^r \pi w(r^2x - x^3)\,dx$$

$$= \pi w\left[\tfrac{1}{2}r^2x^2 - \tfrac{1}{4}x^4\right]_0^r = \tfrac{1}{4}\pi r^4 w$$

$$\therefore\quad \bar{x} = 3r/8$$

Hence the centre of gravity of the hemisphere is on its axis of symmetry at a distance $3r/8$ from its plane face.

Example 6 Find the centre of gravity of a uniform hemispherical shell of radius r.



If we take x- and y-axes along and perpendicular to the axis of symmetry of the hemispherical shell, then its centre of gravity will be on the x-axis.

If the shell is divided into narrow bands by planes perpendicular to the x-axis, then a typical elemental band is approximately cylindrical with radius $r \sin\theta$ and width $r\,\delta\theta$. Letting w be the weight per unit area of the shell, we have:

Body	Weight	Distance of centre of gravity from the y-axis
Element	$w2\pi r^2 \sin\theta\,\delta\theta$	$r\cos\theta$
Whole body	$w2\pi r^2$	$\bar{x}$

Taking moments about the y-axis

$$w2\pi r^2\bar{x} \approx \sum(w2\pi r^2 \sin\theta\,\delta\theta \times r\cos\theta)$$

$$\therefore \quad 2\pi w r^2\bar{x} = \int_0^{\pi/2} 2\pi w r^3 \sin\theta\cos\theta\,d\theta$$

$$= \int_0^{\pi/2} \pi w r^3 \sin 2\theta\,d\theta$$

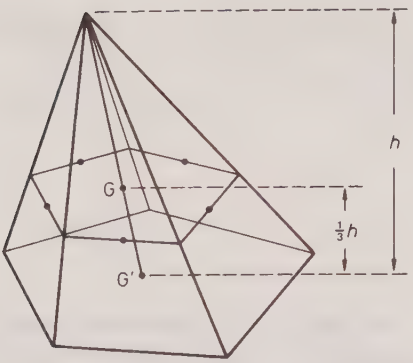
$$= \pi w r^3[-\tfrac{1}{2}\cos 2\theta]_0^{\pi/2} = \pi w r^3$$

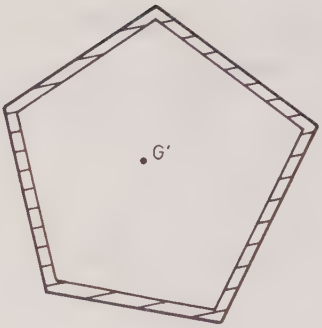
$$\therefore \quad \bar{x} = \frac{r}{2}$$

Hence the centre of gravity of the hemispherical shell is at a distance $\frac{1}{2}r$ from its base.

[For an alternative method see Exercise 9.6, question 11.]

It is not necessary to use integration to find the centres of gravity of hollow pyramids and cones.





Let us consider a thin hollow pyramid of height h, without base. Each of the faces is a triangular lamina with centre of gravity at a height $\frac{1}{3}h$ above the base. If the pyramid is divided into sections parallel to the base, we see that the centre of gravity must lie on the line joining the centres of gravity of these sections.

Hence the centre of gravity of the *hollow pyramid* lies one third of the way up a line joining the centroid of the base to the vertex.

As the number of sides of the base of the pyramid is increased indefinitely, it tends to become a cone. Hence the centre of gravity of a *hollow right circular cone*, without base, is on the axis of the cone at a distance of one third of the height of the cone from the base.

[Note that the results obtained in this section may be quoted without proof when finding the centres of gravity of composite bodies. A list of standard results will be found in the "Formulae for reference" section near the end of the book.]

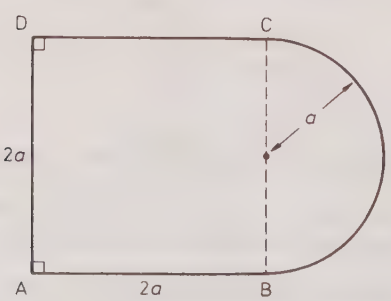
Exercise 9.4

1. Prove that the centre of gravity of a uniform wire in the form of a semi-circle of radius r is at a distance $2r/\pi$ from the centre.

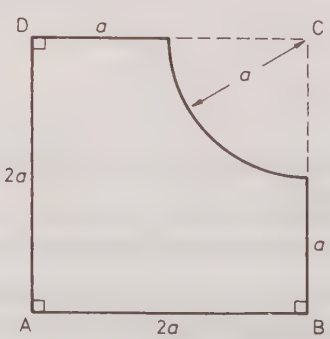
2. Prove that the centre of gravity of a uniform semi-circular lamina of radius r is at a distance $4r/3\pi$ from the centre.

In questions 3 to 6 find the distance from AD of the centre of gravity of the given uniform lamina.

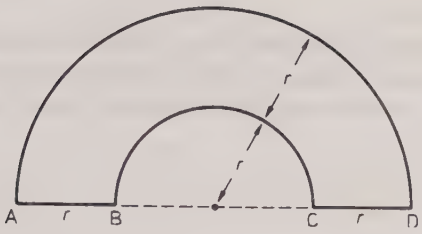
3.



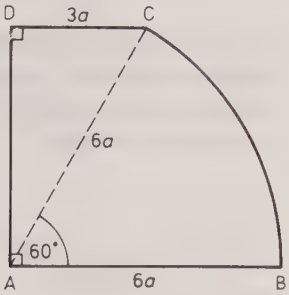
4.



5.



6.



7. A frustum is cut from a solid right circular cone of base radius r and height $2h$ by a plane parallel to the base and at a distance h from it. Find the distance of the centre of gravity of the frustum from the base.

8. A circular lamina of radius $2r$ is divided into two segments by a line at a distance r from the centre. Find the distance from the centre of the centre of gravity of (a) the larger segment, (b) the smaller segment.

9. A solid sphere of radius $3a$ is divided into two parts by a plane at a distance a from the centre. Find, by integration, the volume of the smaller part and the distance of its centre of gravity from the centre of the sphere.

10. Prove that the centre of gravity of a uniform solid pyramid with a square base lies one quarter of the height of the pyramid above the base and on the line joining the vertex to the point of intersection of the diagonals of the base.

11. Prove that the centre of gravity of a uniform solid hemisphere of radius r is a distance $3r/8$ from the centre.

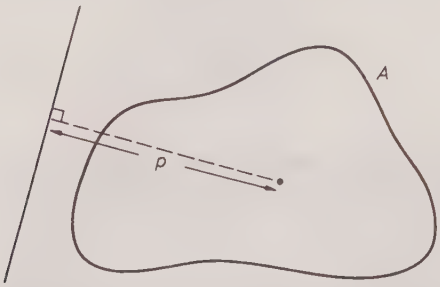
A child's toy is made up from a uniform and solid right circular cone and hemisphere. The radius of the cone is r, and its height $3r$. The radius of the hemisphere is r. The base of the cone and hemisphere are sealed together. The material from which the hemisphere is made is 3 times as heavy per unit volume as the cone material. Find the distance of the centre of gravity of the toy from the vertex of the cone. (SU)

12. Prove that the centre of gravity of a uniform thin hemispherical cup of radius r is at a distance $r/2$ from the centre.

A goblet consists of a uniform thin hemispherical cup of radius r, a circular base of the same material, thickness and radius as the cup, and an intervening stem of length r and whose weight is one-quarter of that of the cup. Show that the height of the centre of gravity above the base is $13r/14$. If the weight of the goblet is W and that of the amount of liquid that fills it is W', show that filling it raises the centre of gravity through a distance $\frac{39}{56}\left(\frac{W'}{W+W'}\right)r$. (W)

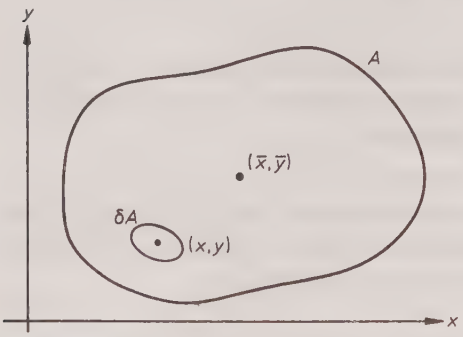
9.5 Centroids of areas and volumes

The *centroid* of a plane figure is its centre of area i.e. the point about which the area is evenly distributed. In simple cases we can use symmetry to find the centroid of a figure. However, to arrive at a more general method, we define a quantity called the *first moment of area.*



If the centroid of an area A is at a perpendicular distance p from an axis, then the first moment of A about this axis is pA.

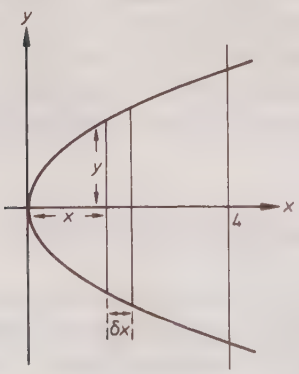
If an area is divided into a number of parts, then the sum of the first moments of the parts about any axis is equal to the first moment of the whole area about that axis.



Suppose that a region of the x, y plane has area A and centroid $(\bar{x}, \bar{y})$. If a typical element of area δA has centroid (x, y) then, comparing first moments about the coordinates axes, we have:

$$\bar{x}A = \sum x\,\delta A, \quad \bar{y}A = \sum y\,\delta A.$$

Example 1 Find the centroid of the region bounded by the curve $y^2 = x$ and the line $x = 4$.



A typical element of area is a strip of width δx, length $2y$ at a distance x from the y-axis. From symmetry the centroids of such strips and of the whole region lie on the x-axis.

Thus, we may write:

	Area	Coordinates of centroid
Element	$2y\,\delta x$	$(x, 0)$
Whole region	A	$(\bar{x}, 0)$

First moment of area about the y-axis

$$= \bar{x}A \approx \sum(x \times 2y\,\delta x)$$

$$\bar{x}A = \int_0^4 2xy\,dx$$

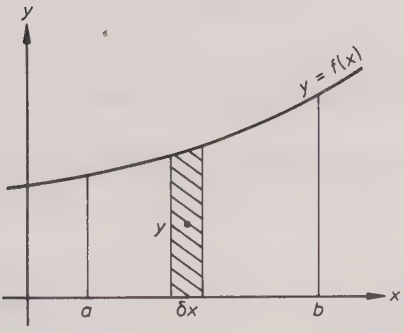
$$= \int_0^4 2x^{3/2}\,dx = 2\left[\frac{2}{5}x^{5/2}\right]_0^4 = \frac{128}{5}$$

But $$A = \int_0^4 2y\,dx$$

$$= \int_0^4 2x^{1/2}\,dx = 2\left[\frac{2}{3}x^{3/2}\right]_0^4 = \frac{32}{3}$$

$$\therefore \quad \bar{x} = \frac{128}{5} \Big/ \frac{32}{3} = \frac{128}{5} \times \frac{3}{32} = \frac{12}{5}$$

Hence the centroid of the region is the point $(12/5, 0)$.



Let us consider now the area bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$, $x = b$. A typical element of area is an approximately rectangular strip of width δx and height y.

	Area	Coordinates of centroid
Element	$y\,\delta x$	$(x, \frac{1}{2}y)$
Whole region	A	$(\bar{x}, \bar{y})$

First moment of area about the y-axis

$$= \bar{x}A \approx \sum(x \times y\,\delta x)$$

$$\therefore \quad \bar{x}A = \int_a^b xy\,dx$$

First moment of area about the x-axis

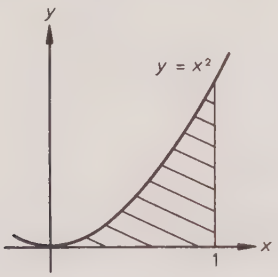
$$= \bar{y}A \approx \sum(\tfrac{1}{2}y \times y\,\delta x)$$

$$\therefore \quad \bar{y}A = \int_a^b \tfrac{1}{2}y^2\,dx$$

But $A = \int_a^b y\,dx$, so the centroid has coordinates

$$\bar{x} = \int_a^b xy\,dx \Big/ \int_a^b y\,dx \quad \bar{y} = \int_a^b \tfrac{1}{2}y^2\,dx \Big/ \int_a^b y\,dx$$

Example 2 Find the centroid of the area bounded by the curve $y = x^2$, the x-axis and the line $x = 1$.



Area of the given region

$$= \int_0^1 y\,dx = \int_0^1 x^2\,dx = \left[\frac{1}{3}x^3\right]_0^1 = \tfrac{1}{3}$$

First moment about the y-axis

$$= \int_0^1 xy\,dx = \int_0^1 x^3\,dx = \left[\frac{1}{4}x^4\right]_0^1 = \tfrac{1}{4}$$

First moment about the x-axis

$$= \int_0^1 \tfrac{1}{2}y^2\,dx = \int_0^1 \tfrac{1}{2}x^4\,dx = \left[\frac{1}{10}x^5\right]_0^1 = \frac{1}{10}$$

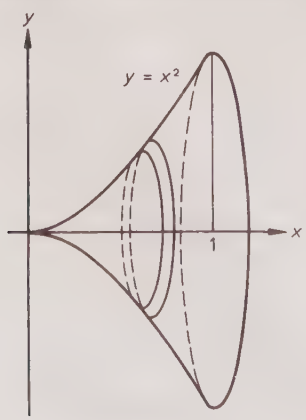
$$\therefore \quad \bar{x} = \int_0^1 xy\,dx \Big/ \int_0^1 y\,dx = \frac{1}{4} \div \frac{1}{3} = \frac{3}{4}$$

$$\therefore \quad \bar{y} = \int_0^1 \tfrac{1}{2}y^2\,dx \Big/ \int_0^1 y\,dx = \frac{1}{10} \div \frac{1}{3} = \frac{3}{10}$$

Hence the centroid of the area is the point $(\frac{3}{4}, \frac{3}{10})$.

The centroid of a solid three-dimensional figure is its centre of volume i.e. the point about which the volume is evenly distributed. The methods used to find centroids of similar to those used to find centroids of area.

Example 3 Find the centroid of the volume obtained by rotating completely about the x-axis the area bounded by the curve $y = x^2$, the x-axis and the line $x = 1$.



A typical element of volume is approximately a disc of thickness δx, radius y at a distance x from the y-axis. From symmetry the centroids of such discs and of the whole solid lie on the x-axis.

Thus we may write:

	Volume	Coordinates of centroid
Element	$\pi y^2\,\delta x$	$(x, 0)$
Whole solid	V	$(\bar{x}, 0)$

First moment of volume about the y-axis

$$= \bar{x}V \approx \sum(x \times \pi y^2\,\delta x)$$

$$\therefore \quad \bar{x}V = \int_0^1 \pi x y^2\,dx$$

$$= \pi \int_0^1 x^5\,dx = \pi\left[\frac{1}{6}x^6\right]_0^1 = \frac{\pi}{6}$$

But $$V = \int_0^1 \pi y^2\,dx$$

$$= \pi \int_0^1 x^4\,dx = \pi\left[\frac{1}{5}x^5\right]_0^1 = \frac{\pi}{5}$$

$$\therefore \quad \bar{x} = \frac{\pi}{6}\Big/\frac{\pi}{5} = \frac{\pi}{6} \times \frac{5}{\pi} = \frac{5}{6}$$

Hence the centroid of the solid is at the point $(5/6, 0)$.

Exercise 9.5

Find the first moments about the y-axis and the x-axis of the areas bounded by the following curves and straight lines. In each case use your results to find the coordinates of the centroid of the area.

1. $y = x^2 + 1$, $y = 0$, $x = 0$ and $x = 2$.

2. $y = x(x + 2)$ for $x \geqslant 0$, $y = 0$ and $x = 3$.

3. $y = \dfrac{1}{x^2}$, $y = 0$, $x = 1$ and $x = 2$.

4. $y = x^2$ for $x \geqslant 0$, $x = 0$ and $y = 1$.

Find the centroids of the areas bounded by the following curves and straight lines.

5. $y^2 = x^3$ and $x = 4$.

6. $y^2 = 4x$ and $x = 1$.

7. $y = \cos x$ for $-\frac{1}{2}\pi \leqslant x \leqslant \frac{1}{2}\pi$ and $y = 0$.

8. $x^2y = 4$, $y = 1$ and $y = 4$.

9. $y = e^x$, $y = 0$, $x = 0$ and $x = 1$.

10. $y = \sin x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, $y = 0$ and $x = \frac{1}{2}\pi$.

Find the centroids of the solids formed by rotating completely about the x-axis the regions defined by the following inequalities.

11. $y \geqslant x(x - 4)$, $y \leqslant 0$.

12. $y^2 \leqslant 9x$, $y \geqslant 0$, $x \leqslant 1$.

13. $xy \leqslant 4$, $y \geqslant 0$, $1 \leqslant x \leqslant 2$.

14. $0 \leqslant y \leqslant e^{-x}$, $0 \leqslant x \leqslant 1$.

15. Sketch the curve $y^2 = x(3 - x)^2$. Find the centroid of the area enclosed by the loop.

16. Find the centroid of the solid formed by rotating through π radians about the y-axis the region bounded by the curve $y = 4x^2$ and the straight lines $y = 1$ and $y = 4$.

17. Use integration by parts to evaluate

$$\int_1^2 \ln x \, dx, \quad \int_1^2 x \ln x \, dx \quad \text{and} \quad \int_1^2 (\ln x)^2 \, dx.$$

Hence find, correct to 2 decimal places, the coordinates of the centroid of the area bounded by the curve $y = \ln x$, the x-axis and the line $x = 2$.

18. Find the area of the region R bounded by the curve $y = x^2$ and the straight line $y = x$. Show that the first moments of R about the x- and y-axes are given by

$$\int_0^1 \tfrac{1}{2}(x^2 - x^4)\, dx \quad \text{and} \quad \int_0^1 (x^2 - x^3)\, dx \text{ respectively.}$$

Hence find the centroid of the region R.

Exercise 9.6 (miscellaneous)

1. All forces in this question act in the plane of a triangle ABC in which $AB = 4a$, $AC = 3a$ and the angle $A = 90°$. Forces of magnitude $17P$, $15P$, $3P$ act along AB, BC, AC respectively in the directions indicated by the order of the letters. Calculate
(a) the magnitude of the resultant of these three forces and the tangent of the angle made by its line of action with AB,
(b) the distance from A of the point where the line of action of the resultant cuts AB. A couple G is now added to the system and the resultant of this enlarged system acts through the point B. Calculate the magnitude and sense of G. (L)

2. A rigid square lamina $ABCD$ of side a is subject to forces of magnitude $1, 2, 3, 1, 3\sqrt{2}$ and $\lambda\sqrt{2}$ units acting along AB, BC, CD, AD, AC and DB respectively

in the directions indicated by the order of the letters. Given that the direction of the resultant force is parallel to AC, find λ. With this value of λ, find the total moment about A of the forces acting on the lamina. Hence, or otherwise, find AE in terms of a, where E is the intersection of AB with the line of action of the resultant force.

(JMB)

3. (a) State conditions which will ensure that a system of forces in a plane will reduce to a couple.
   (b) State conditions which will ensure that a system of forces in a plane is in equilibrium.
   (c) $ABCD$ is a square of side a. Forces of size 1, 3, 3 and 7 act respectively along AB, BC, CD and DA in the directions indicated by the letters. (i) Find the sizes of the forces which must act along AC and BD so that the six forces will be equivalent to a couple, and find the moment of the couple. (ii) If the forces in DA, AC and BD are to be replaced so that the system is in equilibrium, how will this have to be done? (SU)

4. An equilateral triangle ABC has side $2a$ and D, E are the mid-points of BC, CA respectively. Fixed forces of magnitude $P, 2P, 4P$ act along AB, BC, CA respectively and variable forces of magnitude x, y act along AD, BE respectively, the direction of each force being indicated by the order of the letters. Find (in terms of P) the values of x and y if

(a) the system is equivalent to a force through B parallel to AD, stating the magnitude of this force;
(b) the system is equivalent to a couple, stating the moment of the couple. (O&C)

5. Forces 2, 4, 6, $2p$, $2q$ and 18 newtons act along the sides AB, BC, CD, ED, EF and AF respectively of a regular hexagon $ABCDEF$, the directions of the forces being indicated by the order of the letters. If the system is in equilibrium, find, by resolving parallel and perpendicular to AB, the values of p and q. Check your results by finding the moment of the forces about O, the centre of the hexagon.

The forces along ED, EF and AF are now replaced by a coplanar force through O and a coplanar couple. If the resulting system is in equilibrium and if the length of each side of the hexagon is 2 metres, calculate

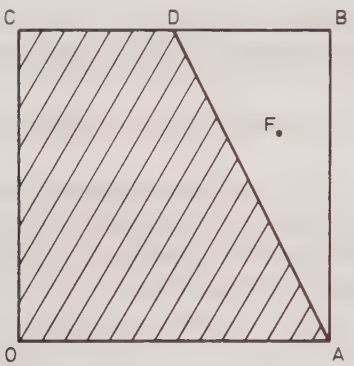
(a) the magnitude of this force through O,
(b) the magnitude of the couple. (L)

6. The centre of a regular hexagon $ABCDEF$ of side a is O. Forces of magnitude $P, 2P, 3P, 4P, mP$ and nP act along $\vec{AB}, \vec{BC}, \vec{CD}, \vec{DE}, \vec{EF}$, and $\vec{FA}$ respectively. Given that the resultant of these six forces is of magnitude $3P$ acting in a direction parallel to $\vec{EF}$,

(i) determine the values of m and n,
(ii) show that the sum of the moments of the forces about O is $9Pa\sqrt{3}$.
The mid-point of EF is M.
(iii) Find the equation of the line of action of the resultant referred to OM as x-axis and OA as y-axis.

The forces mP and nP acting along $\vec{EF}$ and $\vec{FA}$ are removed from the system. The remaining four forces and an additonal force Q, which acts through O, reduce to a couple. Calculate the magnitude of Q and the moment of the couple. (AEB 1979)

7.



The diagram shows a square $OABC$ of side a. The mid-point of BC is D. Show that, with respect to OA and OC as axes, the coordinates of the centroid F of the triangular region ABD are $(5a/6, 2a/3)$. Find the coordinates of the centre of mass of a uniform lamina in the form of the figure $OADC$. (JMB)

8. A uniform thin sheet of paper has the shape of a rectangle $OABC$, where $OA = 20\,\text{cm}$ and $OC = 12\,\text{cm}$. The side AB is folded down so as to lie entirely along OA. Find the distances of the centre of gravity of the folded sheet from OA and OC. (O&C)

9. A uniform solid hemisphere of radius a is cut into two parts by a plane parallel to the plane face of the hemisphere and at a distance $\frac{1}{2}a$ from it. Find the volume of each of the two parts and the position of the centre of mass of each of the two parts. (AEB 1980)

10. Prove that the centre of gravity of a uniform circular arc subtending an angle 2θ at the centre of a circle of radius a is $a \sin\theta/\theta$ from the centre. Deduce that the centre of gravity of a uniform sector bounded by that arc and the radii to its extremities is $\frac{2}{3}a \sin\theta/\theta$ from the centre.

Show also that the centre of gravity of a segment of a circular lamina cut off by a chord subtending a right angle at the centre of the circle is $\frac{2}{3}\sqrt{2}a/(\pi - 2)$ from the centre. (W)

11. Find by integration the distance of the centroid of a uniform solid hemisphere of radius a from its plane face.

A hollow sphere of external radius R and internal radius r is cut in half by a plane through its centre. Show that the distance of the centroid of each half from the centre of the sphere is

$$\frac{3(R + r)(R^2 + r^2)}{8(R^2 + Rr + r^2)}.$$

Deduce that the centroid of a hemispherical shell of radius R is at a distance $R/2$ from its centre. (L)

12. Find the area A of the loop of the curve $y^2 = x^2(2 - x)$. Find also the first moment about the y-axis of the volume generated when the area A is rotated through π radians about the x-axis. (AEB 1975)

13. Find the area of the region in the first quadrant enclosed between the curve $y = x^2 + 4$, the line $y = 8$ and the y-axis. The region is rotated through four right-angles about Oy to form a uniform solid. Find the volume and the coordinates of the centre of mass of the solid. (C)

14. Sketch on one diagram the graphs of $x^3y = 1$ and $y = \sqrt{x}$, for positive values of x. Show that the area of the finite region bounded by the straight lines $y = 0$ and $x = 2$ and an arc of each of the above curves is 25/24. Find also the coordinates of the centroid of the region. (C)

15. Find the area of the finite region between the curve $y = x^2 + 1$ and the line $y = 2x + 1$. The region is rotated through four right angles about Ox, to produce a uniform solid of revolution. Find the volume and the coordinates of the centre of gravity of this solid. (C)

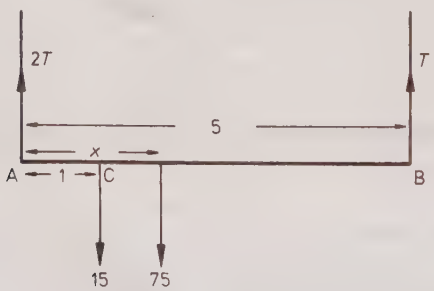
16. Sketch the arc of the curve given parametrically by $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, for which $0 \leqslant \theta \leqslant 2\pi$. Find the area of the finite region enclosed by this arc and the x-axis. Find also the coordinates of the centroid of the region. (L)

10 Equilibrium of rigid bodies

10.1 Parallel forces in equilibrium

If a set of coplanar forces is in equilibrium then
(i) the sum of the components of the forces in any given direction is zero and
(ii) the sum of the moments of the forces about any given point is zero.
These facts are used to find unknown quantities in many statics problems.

Example 1 A non-uniform rod AB of length 5 m and weight 75 N is supported in a horizontal position by two vertical strings attached to its ends. A weight of 15 N is attached to the rod at the point C, where $AC = 1$ m. If the tensions in the strings at A and B are $2T$ N and T N respectively, find the value of T. Find also the distance, x m, of the centre of gravity of the rod from A.



Considering the vertical forces on the rod,

$\uparrow \qquad 2T - 15 - 75 + T = 0$

$\therefore \qquad 3T = 90$

Hence $\qquad T = 30.$

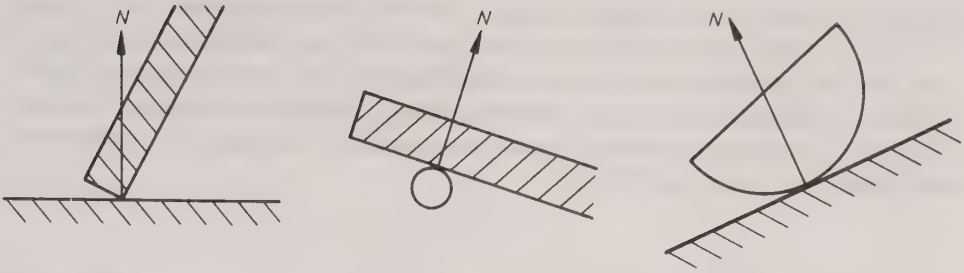
Taking moments clockwise about A,

$A\circlearrowright \qquad 15 \times 1 + 75 \times x - T \times 5 = 0$

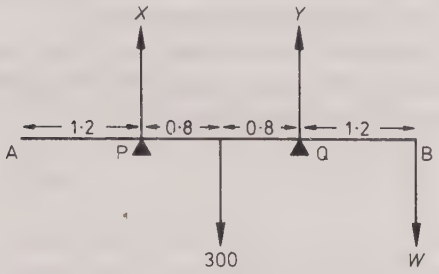
$\therefore \qquad 75x = 5T - 15 = 135$

Hence $\qquad x = 1{\cdot}8.$

When a rigid body rests on an object described as *smooth*, (e.g. a smooth support or a smooth table,) then the reaction on the body at any point of contact is assumed to be perpendicular to the surfaces in contact.



Example 2 A uniform plank AB of length 4 m and weight 300 N rests horizontally on smooth supports at P and Q, where $AP = QB = 1{\cdot}2$ m. Find the maximum weight that can be placed at B without tilting the plank.



Let us assume that the plank is in equilibrium with reactions of X N and Y N at P and Q respectively and a weight of W N at B.

Taking moments about Q,

$$X \times 1{\cdot}6 + W \times 1{\cdot}2 - 300 \times 0{\cdot}8 = 0$$

$\therefore \quad 1{\cdot}2W = 240 - 1{\cdot}6X$

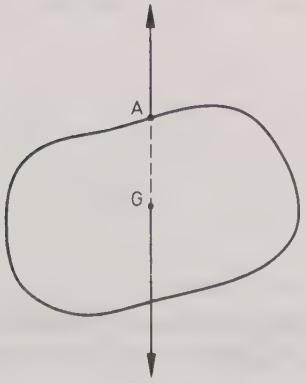
Since the reaction at P acts vertically upwards, $X \geqslant 0$,

$\therefore \quad 1{\cdot}2W \leqslant 240$ i.e. $W \leqslant 200$.

Hence the maximum weight that can be placed at B without tilting the plank is 200 N.

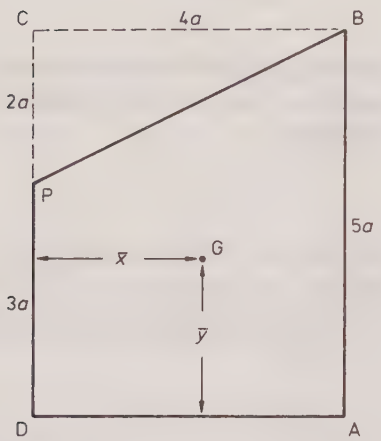
[Note that when a weight of 200 N is placed at B, the reaction at P is zero and the plank just fails to rotate about Q.]

When a body is in equilibrium under the action of *two forces*, these forces must be equal in magnitude and act in opposite directions along the same line.



In particular, when a body freely suspended from a point A is in equilibrium, the two forces acting on it, namely its weight and the force at A, must act along the same vertical line. We deduce that the body hangs in equilibrium with its centre of gravity G vertically below A.

Example 3 A uniform rectangular lamina $ABCD$ of weight $5W$ has $AB = 5a$ and $BC = 4a$. The point P lies on the edge DC and $DP = 3a$. The triangle BCP is cut away and the remaining trapezium is suspended from P. Find the angle that DP makes with the vertical when the trapezium hangs freely in equilibrium. Find also the moment of the couple which would be required to maintain the trapezium in equilibrium with D vertically below P.



We first find the position of G, the centre of gravity of the lamina $ABPD$. Rectangle $ABCD$ has area $20a^2$ and weight $5W$. Since triangle BCP has area $4a^2$, its weight must be W.

Body	Weight	Distance of centre of gravity from CD	Distance of centre of gravity from DA
Rectangle $ABCD$	$5W$	$2a$	$5a/2$
Triangle BCP	W	$4a/3$	$13a/3$
Remainder	$4W$	$\bar{x}$	$\bar{y}$

Taking moments about CD,

$$4W \times \bar{x} + W \times \frac{4a}{3} = 5W \times 2a$$

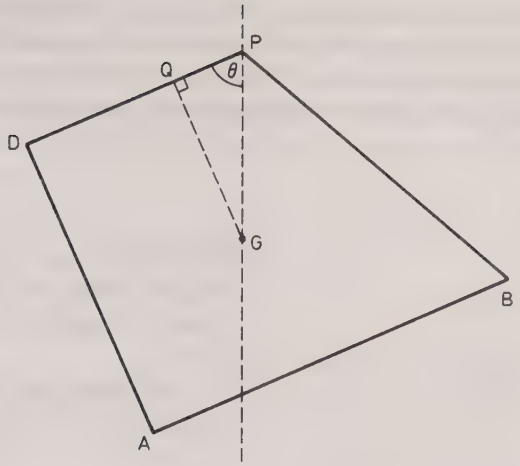
$$\therefore \quad \bar{x} = \frac{1}{4}\left(10a - \frac{4a}{3}\right) = \frac{13a}{6}$$

Taking moments about DA,

$$4W \times \bar{y} + W \times \frac{13a}{3} = 5W \times \frac{5a}{2}$$

$$\therefore \quad \bar{y} = \frac{1}{4}\left(\frac{25a}{2} - \frac{13a}{3}\right) = \frac{49a}{24}$$

When the trapezium hangs in equilibrium from P, the line PG must be vertical. Let θ be the angle DP then makes with the vertical and let Q be the foot of the perpendicular from G to DP.



$$GQ = \bar{x} = \frac{13a}{6}, \quad DQ = \bar{y} = \frac{49a}{24}, \qquad \therefore \quad QP = \frac{23a}{24}.$$

Thus $\tan\theta = \dfrac{GQ}{QP} = \dfrac{13}{6} \times \dfrac{24}{23} = \dfrac{52}{23}$ and $\theta \approx 66{\cdot}1°$.

Hence the angle made by DP with the vertical is $66{\cdot}1°$.

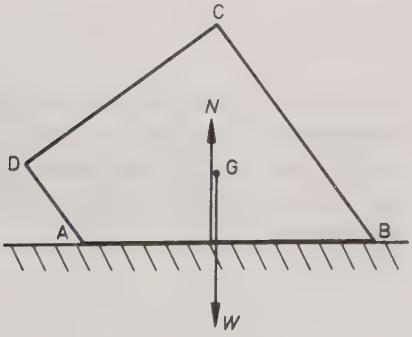
Suppose now that the trapezium is in equilibrium with D vertically below P and that a couple of moment M in the sense DAB is required to maintain this position. Taking moments anti-clockwise about P,

$$M - 4W \times \frac{13a}{6} = 0 \qquad \therefore \quad M = \frac{26Wa}{3}.$$

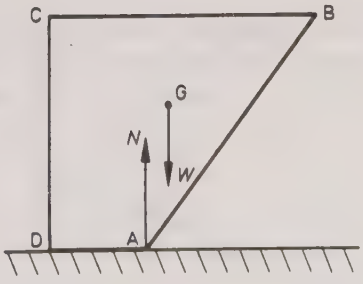
Hence the moment of the required couple is $26Wa/3$.

When a body rests in equilibrium on a horizontal plane, the weight of the body and the force exerted by the plane on the body must act along the same line, namely the vertical line through the centre of gravity.

In the case of a lamina with an edge AB resting on a smooth horizontal plane, the force N exerted by the plane on the lamina is the resultant of normal reactions at the points of contact along AB. Thus N must act through some point on AB. It follows that, if the vertical line through the centre of gravity G passes through a point on the edge AB, then the force N can act to maintain equilibrium. Otherwise equilibrium is not possible and the lamina will topple.

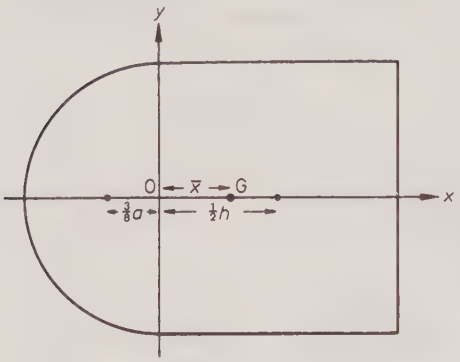


Lamina in equilibrium



Lamina will topple

Example 4 A uniform solid is formed by joining a hemisphere of radius a to a circular cylinder of radius a and height h, so that the plane face of the hemisphere coincides with a plane face of the cylinder. Show that this solid can rest in equilibrium on a horizontal plane with the curved surface of the cylinder touching the plane if $h^2 \geqslant \frac{1}{2}a^2$.



If we take x- and y-axes as shown in the diagram, then the centre of gravity G of the combined solid must lie on the x-axis.
Let w be the weight per unit volume of the solid.

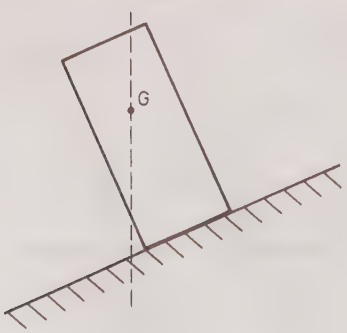
Body	Weight	x-coordinate of centre of gravity
Cylinder	$\pi a^2 hw$	$\frac{1}{2}h$
Hemisphere	$\frac{2}{3}\pi a^3 w$	$-\frac{3}{8}a$
Whole body	$(h + \frac{2}{3}a)\pi a^2 w$	$\bar{x}$

Taking moments about the y-axis,

$$(h + \tfrac{2}{3}a)\pi a^2 w\bar{x} = \pi a^2 hw \times \tfrac{1}{2}h - \tfrac{2}{3}\pi a^3 w \times \tfrac{3}{8}a$$
$$\therefore \quad (h + \tfrac{2}{3}a)\bar{x} = \tfrac{1}{2}h^2 - \tfrac{1}{4}a^2$$

When this solid is placed with the curved surface of the cylinder touching a horizontal plane, it will rest in equilibrium if the vertical line through G passes through a point of contact between the plane and the solid.
This condition will be satisfied if $\bar{x} \geqslant 0$ i.e. if $h^2 \geqslant \frac{1}{2}a^2$.

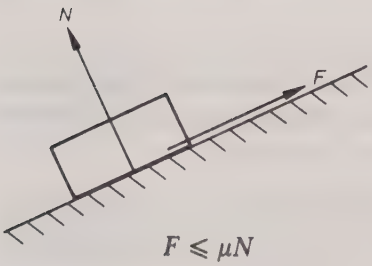
For a body to rest in equilibrium on a rough inclined plane the force exerted by the plane on the body must act along a vertical line through the centre of gravity of the body.



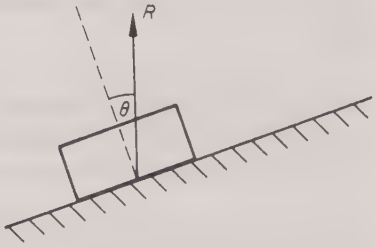
For the block shown in this diagram equilibrium is not possible. Since the vertical line through the centre of gravity G falls outside the base of the block, it will topple.

The reaction R between a body and a rough surface is the resultant of two components, the normal reaction N and a friction force F. According to the laws of friction, if μ is the coefficient of friction between the body and the surface, then $F \leqslant \mu N$. We can also express this condition in terms of λ, the angle of friction, and θ, the angle made by R with the normal to the surface, since $\tan \lambda = \mu$ and $\tan \theta = F/N$.

$$F \leqslant \mu N \Leftrightarrow \tan \theta \leqslant \mu \Leftrightarrow \theta \leqslant \lambda.$$



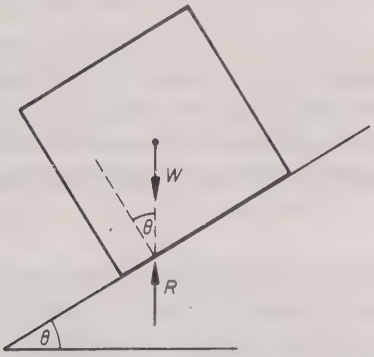
$F \leqslant \mu N$



$\tan \theta \leqslant \mu, \quad \theta \leqslant \lambda$

Example 5 A uniform solid cube is in equilibrium with one face resting on a rough inclined plane and one pair of opposite faces vertical. The coefficient of friction between the cube and the plane is μ. The angle of inclination of the plane is gradually increased. Show that equilibrium will be broken by sliding rather than toppling if $\mu < 1$.

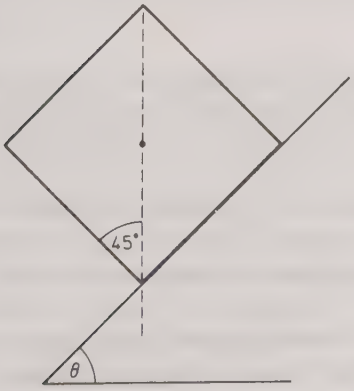
(1)



Let θ be the angle of inclination of the plane to the horizontal. When the cube is in equilibrium, as shown in diagram (1), the total reaction R acts vertically. The angle between its line of action and the normal to the plane is θ. When the cube is on the point of sliding, friction is limiting,

$$\therefore \qquad \tan \theta = \mu.$$

(2)



In diagram (2) the cube is on the point of toppling, since the vertical line through its centre of gravity passes through the lower edge of the base. Thus, when the cube is about to topple,

$$\tan \theta = \tan 45° = 1.$$

Hence equilibrium will be broken by sliding if $\tan\theta$ reaches the value μ before it reaches the value 1, i.e. if $\mu < 1$.

Exercise 10.1

1. A beam AB of length 4·5 m is supported in a horizontal position by two vertical cables attached to its ends. The tensions in the cables at A and B are 160 N and 200 N respectively. Find the weight of the beam and the distance of its centre of gravity from A.

2. A uniform rod AB of length 3 m and weight 50 N rests horizontally on smooth supports at A and B. A load of 24 N is attached to the rod at a point C, where $AC = 1$ m. Find the forces exerted on the rod by the supports.

3. A uniform bar of length 2·8 m and weight 80 N has loads of 20 N and 40 N attached to its ends A and B respectively. If the bar balances in a horizontal position when smoothly supported at C, find the distance of C from A.

4. A uniform rod $APQB$ of length 2 m rests horizontally on smooth supports at P and Q, where $PQ = 1{\cdot}2$ m. If the reaction at P is twice the reaction at Q, find the distance AP. Given that when a weight of 5 N is attached to the rod at B the reactions at P and Q are equal, find the weight of the rod.

5. A uniform plank ABC of weight 400 N is held in a horizontal position by a smooth support at B and a vertical rope attached at C, where $AB = 1$ m and $AC = 5$ m. If the rope is likely to break when the tension in it exceeds 750 N, find
(a) the maximum weight that can be placed on the plank at the mid-point of BC,
(b) the maximum distance that a man of weight 800 N can safely walk from B towards C.

6. A non-uniform beam $AXYB$ of length 4 m and weight 400 N is maintained in a horizontal position by smooth supports at X and Y, where $AX = YB = 1$ m. A boy of weight 500 N can just walk to the end A of the beam without overturning it. If the boy walks along the beam towards B, find his distance from B when the beam is just about to tip.

7. A uniform right circular solid cone of weight W is suspended by two vertical strings attached to the ends A and B of a diameter of its base. If the cone hangs in equilibrium with its vertex vertically below A, find the tensions in the strings.

8. $ABCD$ is a uniform rectangular lamina of weight 10 N. A particle of weight W N is attached to the lamina at B. When the system is freely suspended from A it rests in equilibrium with the mid-point of BC vertically below A. Find the value of W.

9. A uniform wire is bent into the form of a trapezium $ABCD$ in which $\angle ABC = \angle BCD = 90°$, $AB = 6a$, $BC = 4a$ and $CD = 3a$. Find the distances of

the centre of gravity from AB and BC. If the trapezium is freely suspended from A, find the angle which AB makes with the vertical.

10. A uniform triangular lamina ABC has $AB = 4a, BC = 5a, CA = 3a$. The lamina is folded through 180° along LN, where $L$and N are the mid-points of BC and AB respectively, so that the vertex B is at A. Find the coordinates of the centre of mass of the folded lamina relative to AN and AC as x-axis and y-axis respectively. The folded lamina is suspended from C and hangs in equilibrium under gravity. Find the tangent of the angle that CA makes with the vertical. (C)

11. A uniform wire is bent to form a semi-circular arc together with the diameter AB joining the ends of the arc. If the wire hangs in equilibrium from a smooth pivot at A, find the tangent of the angle that AB makes with the vertical.

12. A uniform wire AB of weight $2W$is bent into a semi-circular arc. A particle of weight W is attached to the end B and the system is freely suspended from A. Find the tangent of the angle that AB makes with the vertical.

13. Three uniform rods AB, BC, CD, each of length $2a$ and weight W are rigidly joined together to form three sides of a square $ABCD$. If the system hangs freely from a smooth pivot at A, find the angle that AB makes with the vertical. Find also, in terms of W and a, the moment of the couple required to maintain the system in equilibrium with BC horizontal.

14. $ABCD$ is a uniform square lamina of side $6a$ and weight W. P is a point on the diagonal AC, such that $AP = 2PC$. The square having diagonal PC is cut away and the remaining body is freely suspended from B. Find the angle which BA makes with the vertical. Find also the moment of the couple required to maintain the body in equilibrium with BA horizontal.

15. $ABCD$ is a uniform rectangular lamina in which $BC = 2a$. P is the mid-point of BC and R is a point on AB such that $AR = a$ and $RB = ka$. The rectangle $BPQR$ is cut away and the remaining body is placed in a vertical plane with AR resting on a horizontal plane. Find the maximum value of k given that the lamina does not topple about R.

16. A uniform lamina of weight $3W$ in the form of a trapezium $ABCD$ has $\angle ABC = \angle BCD = 90°, AB = BC = a$ and $CD = 2a$. The lamina is placed in a vertical plane with AB resting on a horizontal plane. Find, in terms of W, the greatest weight that can be attached to the lamina at D without causing it to topple about A.

17. A solid hemisphere of radius a and a solid right circular cone of height h and base radius a are made from the same uniform material and are joined together with their plane faces completely in contact. O is the centre of the common base. Find the position of the centre of gravity of the whole body.

(a) The body is suspended freely from a point A on the edge of the common base and hangs in equilibrium under gravity. If $h = 3a$, find, correct to the nearest degree, the acute angle made by AO with the vertical.
(b) The body is found to rest in equilibrium when it is placed on a horizontal table with *any* point of its hemispherical surface in contact with the table. Find h in terms of a. (O&C)

18. A uniform thin hemispherical bowl rests in equilibrium with its axis of symmetry horizontal and its curved surface on a rough inclined plane. Find the angle between the plane and the horizontal. Find also the least possible value of the coefficient of friction between the bowl and the plane.

19. A uniform solid hemisphere rests in equilibrium with its curved surface on a rough plane inclined at an angle α to the horizontal, where $\sin\alpha = 0{\cdot}3$. Find the angle between the plane face of the hemisphere and the vertical.

20. A uniform right circular solid cone of radius a and height $3a$ rests in equilibrium with its base in contact with a rough inclined plane. The angle of inclination of the plane is then increased steadily.
(a) Assuming that the cone does not slide down the plane, find the angle of inclination of the plane to the horizontal when the cone is about to topple.
(b) If the coefficient of friction between the cone and the plane is $\frac{1}{2}$, find the angle of inclination of the plane to the horizontal when the cone is about to slide.

21. A uniform solid cylinder of radius a and height h is placed with one plane face resting on a rough inclined plane. The angle of inclination of the plane is then gradually increased. Show that the cylinder will slide before it topples if the coefficient of friction between the cylinder and the plane is less than $\frac{2a}{h}$.

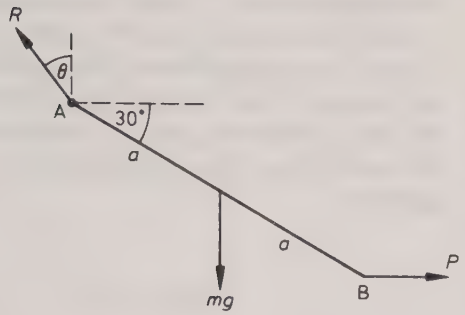
10.2 Three force problems

When a body is in equilibrium under the action of *three forces*, the vector sum of these forces must be zero. Hence three non-parallel forces in equilibrium can be represented in magnitude and direction by the sides of a triangle. We also know that the sum of the moments of coplanar forces in equilibrium about any point in their plane is zero.

Thus one approach to three force problems is to
(i) take moments about any suitable point, and
(ii) use a triangle of forces.

Example 1 A uniform rod AB of mass m, which is smoothly hinged at A, is maintained in equilibrium by a horizontal force P acting at B. Given that the rod is

inclined at 30° to the horizontal with B below A, find (a) an expression for P, (b) the magnitude and direction of the reaction at the hinge.



[Note that the reaction at a *smooth hinge* is a single force which may act in any direction. The system of reactions in a hinge which is *not smooth* may reduce to a single force together with a couple.]

Let the length of the rod be $2a$ and let the reaction at the hinge be of magnitude R acting at an angle θ to the vertical.

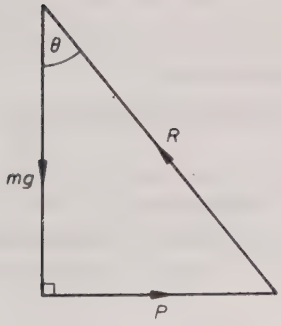
(a) The weight of the rod is mg and acts through the mid-point of the rod. Thus taking moments anti-clockwise about A, we have

$$P \times 2a \sin 30° - mg \times a \cos 30° = 0$$

$$\therefore \qquad Pa - \frac{\sqrt{3}}{2} mga = 0$$

Hence $P = \frac{\sqrt{3}}{2} mg.$

(b)



Using the triangle of forces,

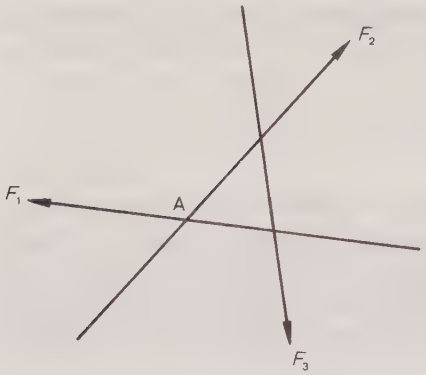
$$R = \sqrt{\{(mg)^2 + P^2\}} = mg\sqrt{(1 + \tfrac{3}{4})}$$

$$= \frac{\sqrt{7}}{2} mg$$

$$\tan\theta = \frac{P}{mg} = \frac{\sqrt{3}}{2} \qquad \therefore \quad \theta \approx 40{\cdot}9°$$

Hence the reaction at the hinge has magnitude $\frac{\sqrt{7}}{2} mg$ and makes an angle $\tan^{-1}\frac{\sqrt{3}}{2} \approx 40{\cdot}9°$ with the vertical.

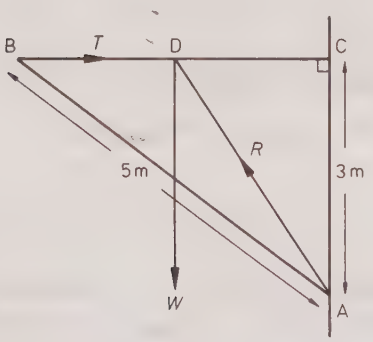
We now establish a property of three force systems which is a useful aid to problem solving. Suppose that a rigid body is in equilibrium under the action of three coplanar forces F_1, F_2 and F_3. Then either all three forces are parallel or the lines of action of two of them meet.



If F_1 and F_2 act along lines which intersect at a point A, then the moments of F_1 and F_2 about A are zero. Since F_1, F_2 and F_3 are in equilibrium, the moment of F_3 about A must also be zero. Hence the line of action of F_3 passes through A. We may deduce the following general principle.

> If a rigid body is in equilibrium under the action of three coplanar forces, then the lines of action of the forces are either parallel or concurrent.

Example 2 A uniform pole AB of length 5 m is smoothly hinged at A and supported at B by a light elastic string BC of natural length 3 m and modulus 60 N. The end C of the string is fixed 3 m vertically above A. Given that the system is in equilibrium with BC horizontal, find in newtons the tension T in the string, the weight W of the pole and magnitude R of the reaction at A.



The lines of action of T and W meet at D, the mid-point of BC,

$\therefore$ the third force R must also act through D.

By Pythagoras' theorem in $\triangle ABC$,

$$BC = 4\,\text{m},$$

$\therefore$ the extension in the elastic string

$$= (4 - 3)\,\text{m} = 1\,\text{m}.$$

By Hooke's law, $T = \dfrac{\lambda x}{l}$

where $\lambda = 60$, $l = 3$ and $x = 1$,

$$\therefore \quad T = \frac{60 \times 1}{3} = 20.$$

Hence the tension in the elastic string is 20 N.

Triangle of forces

Comparing the diagrams we see that the triangle of forces is similar to $\triangle ADC$. By Pythagoras' theorem in $\triangle ADC$,

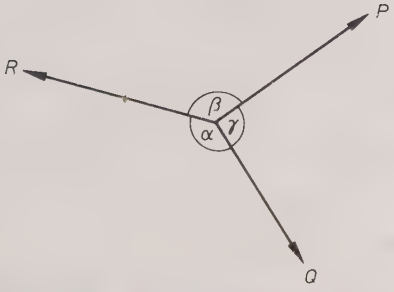
$$AD = \sqrt{(AC^2 + DC^2)} = \sqrt{(3^2 + 2^2)}\,\text{m} = \sqrt{13}\,\text{m},$$

$$\therefore \quad \frac{T}{2} = \frac{W}{3} = \frac{R}{\sqrt{13}}$$

$$\therefore \quad W = \frac{3}{2}T = 30$$

and $$R = \frac{\sqrt{13}}{2}T = 10\sqrt{13} \approx 36{\cdot}1.$$

Hence the weight of the pole is 30 N and the magnitude of the reaction at A is 36·1 N.



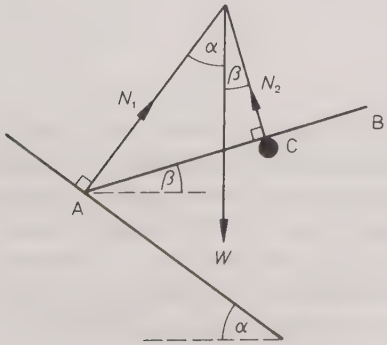
When solving three force problems in which angles rather than lengths are known, *Lami's theorem* may be useful. It states that for forces P, Q and R in equilibrium, as shown in the diagram.

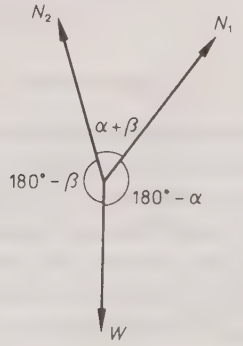
$$\frac{P}{\sin\alpha} = \frac{Q}{\sin\beta} = \frac{R}{\sin\gamma}$$

Example 3 A uniform rod AB of length $5a$ and weight W rests with the end A on a smooth plane inclined at an angle α to the horizontal, where $\sin\alpha = 3/5$. It is maintained in equilibrium by a smooth support at C. If the rod makes an angle β with the horizontal, where $\sin\beta = 7/25$, find

(a) the magnitudes of the reactions at A and C,
(b) the distance AC.

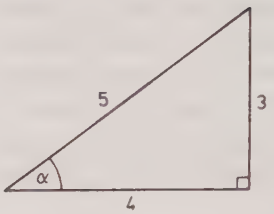
The reaction at A is perpendicular to the inclined plane and the reaction at C is perpendicular to the rod. Let their magnitudes be N_1 and N_2 respectively. Since the forces N_1, N_2 and W are in equilibrium, their lines of action are concurrent.





(a) By Lami's theorem, $\dfrac{N_1}{\sin(180^\circ - \beta)} = \dfrac{N_2}{\sin(180^\circ - \alpha)} = \dfrac{W}{\sin(\alpha + \beta)}$

$$\therefore \quad \frac{N_1}{\sin\beta} = \frac{N_2}{\sin\alpha} = \frac{W}{\sin(\alpha + \beta)} \qquad (1)$$



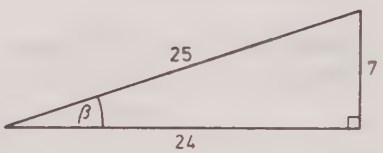
$$\sin\alpha = \frac{3}{5} \Rightarrow \cos\alpha = \frac{4}{5}$$

$$\sin\beta = \frac{7}{25} \Rightarrow \cos\beta = \frac{24}{25}$$

$$\therefore \quad \sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$= \frac{3}{5} \times \frac{24}{25} + \frac{4}{5} \times \frac{7}{25}$$

$$= \frac{72 + 28}{125} = \frac{100}{125} = \frac{4}{5}$$



Substituting these values in equation (1),

$$\frac{25}{7}N_1 = \frac{5}{3}N_2 = \frac{5}{4}W$$

$$\therefore \quad N_1 = \frac{7}{25} \times \frac{5}{4}W = \frac{7}{20}W \quad \text{and} \quad N_2 = \frac{3}{5} \times \frac{5}{4}W = \frac{3}{4}W$$

Hence the magnitudes of the reactions at A and C are $7W/20$ and $3W/4$ respectively.

(b) The distance of the centre of gravity of the rod from A is $5a/2$. Let the distance AC be x.

Taking moments anti-clockwise about A,

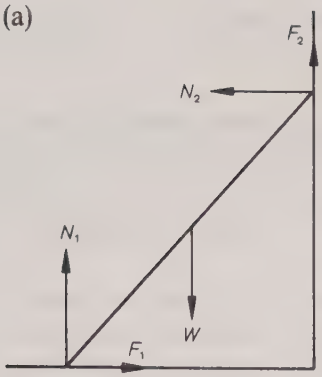
$$N_1 \times 0 + N_2 \times x - W \times \frac{5}{2}a\cos\beta = 0$$

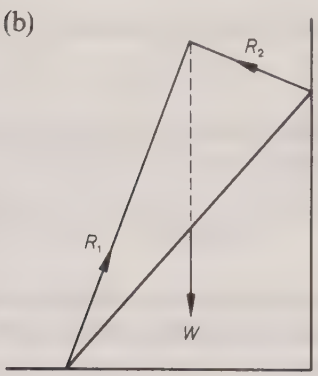
$$\therefore \quad \frac{3}{4}Wx - \frac{5}{2} \times \frac{24}{25}Wa = 0$$

$$\therefore \quad x = \frac{4}{3} \times \frac{5}{2} \times \frac{24}{25}a = \frac{16}{5}a$$

Hence the distance AC is $16a/5$.

In problems involving contact between rough surfaces the reaction at any point of contact can be treated as the resultant of the normal reaction and a frictional force or as a single force. For instance, a problem concerning a ladder resting on a rough horizontal floor and against a rough vertical wall may be solved using diagram (a) and the general methods to be described in §10.3. Alternatively, as indicated in diagram (b), the set of forces on the ladder can be reduced to a three force system by using the total reactions at the points of contact with the floor and the wall.

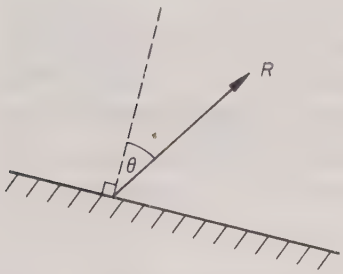




When using this latter approach, it is helpful to remember that if θ is the angle between the total reaction and the normal to the surfaces in contact, then

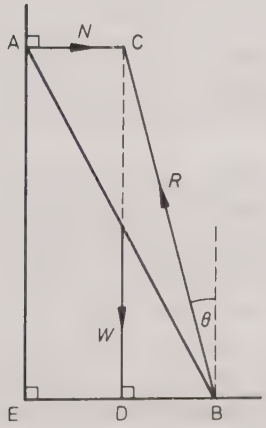
$$\tan\theta \leqslant \mu \quad \text{and} \quad \theta \leqslant \lambda,$$

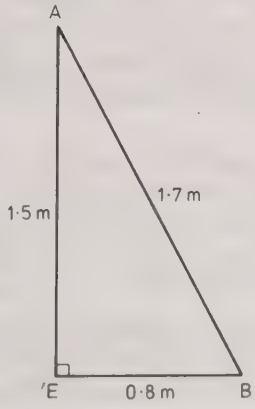
where μ is the coefficient of friction and λ is the angle of friction.

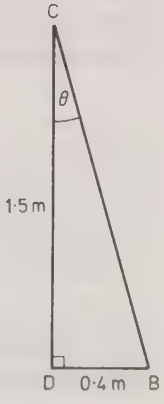


Example 4 A uniform rod AB of length 1·7 m is placed with the end A against a smooth vertical wall and the end B on rough horizontal ground at a distance of 0·8 m from the wall. If the coefficient of friction between the rod and the ground is μ, find the least value of μ for which equilibrium is possible.

If the rod is in equilibrium under the action of its weight W, the normal reaction N at A and the reaction R at B, then the lines of action of these three forces must be concurrent. Let C be their point of intersection and let θ be the angle the reaction R makes with the vertical.







By Pythagoras' theorem in $\triangle ABE$, $AE^2 + EB^2 = AB^2$

$\therefore \quad AE = \sqrt{\{(1{\cdot}7)^2 - (0.8)^2\}}\,\text{m} = 1{\cdot}5\,\text{m}.$

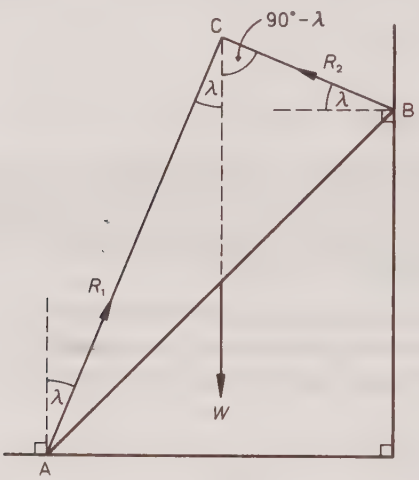
Since the weight of the rod acts vertically through its mid-point, D must be the mid-point of EB,

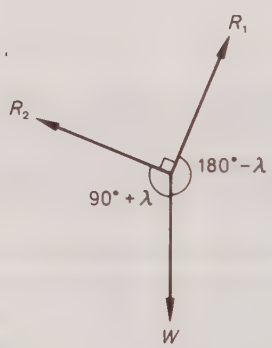
$$\therefore \quad \text{using } \triangle BCD, \quad \tan\theta = \frac{0{\cdot}4}{1{\cdot}5} = \frac{4}{15}.$$

Equilibrium is possible provided that $\tan\theta \leqslant \mu$.
Hence the least possible value of μ is 4/15.

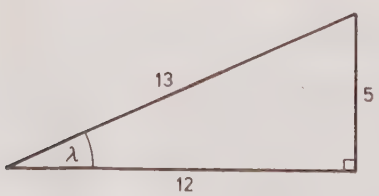
Example 5 A uniform ladder AB of weight W rests in limiting equilibrium with the end A on rough horizontal ground and the end B against a rough vertical wall. If the coefficient of friction at each end of the ladder is 5/12, find the magnitudes of the reactions at A and B.

Since the ladder is in limiting equilibrium, the reaction R_1 at A acts at an angle λ to the vertical and the reaction R_2 at B acts at an angle λ to the horizontal, where $\tan\lambda = 5/12$. If the lines of action of R_1 and R_2 meets at C, then the weight of the ladder also acts through C and $\angle ACB = \lambda + (90° - \lambda) = 90°$.





By Lami's theorem, $\dfrac{R_1}{\sin(90° + \lambda)} = \dfrac{R_2}{\sin(180° - \lambda)} = \dfrac{W}{\sin 90°}$



$$\therefore \quad \frac{R_1}{\cos\lambda} = \frac{R_2}{\sin\lambda} = W$$

Since $\tan\lambda = \frac{5}{12}$,

$\cos\lambda = \frac{12}{13}$ and $\sin\lambda = \frac{5}{13}$,

$\therefore \quad R_1 = W\cos\lambda = \frac{12}{13}W$ and $R_2 = W\sin\lambda = \frac{5}{13}W$.

Hence the reactions at A and B are $12W/13$ and $5W/13$ respectively.

Exercise 10.2

[Take g as $10\,\text{m s}^{-2}$.]

1. A uniform rod AB of weight 20 N is smoothly hinged at B and maintained in equilibrium by a horizontal force of P N acting at A. If the rod is inclined at 45° to the horizontal with A above B, find (a) the value of P, (b) the magnitude and direction of the reaction at the hinge.

2. A non-uniform bar AB of length $2a$ and mass m is maintained in a horizontal position by two strings AC and BC attached to a fixed point C above the bar. If $\angle CAB = 60°$ and $\angle ABC = 30°$, find expressions for the tensions in the strings. Find also the distance of the centre of gravity of the bar from A.

3. A uniform rod AB of length 40 cm and weight 8 N rests on a rough peg at C, where $AC = 15$ cm, and against a smooth vertical wall at A. If the rod is in equilibrium at an angle α to the horizontal, where $\sin\alpha = 4/5$, find the magnitudes of the reactions at A and at C.

4. A uniform bar AB of length 2 m and mass 3 kg rests on a smooth peg at X, where $AX = 0{\cdot}5$ m and against a rough vertical wall at A. If the bar is in equilibrium at 60° to the horizontal, find the magnitude and direction of the force exerted by the wall on the bar.

5. A sphere of radius 40 cm and weight 30 N rests in equilibrium on a smooth plane inclined at an angle α to the horizontal, where $\tan\alpha = 8/15$. A horizontal string joins a point on the sphere to a fixed point on the plane. Find the length of the string and the tension in the string.

6. A rod AB of length 40 cm, mass 6 kg and centre of mass G, is suspended by a light inextensible string AXB of length 80 cm, which passes over a smooth peg at X. Find the tension in the string and the angle the rod makes with the vertical when it hangs in equilibrium if (a) $AG = GB$, (b) $AG:GB = 3:5$.

7. A uniform rod of weight W is in equilibrium with one end resting against a smooth vertical plane and the other end against a smooth plane inclined at an angle α to the horizontal, where $\tan\alpha = 5/12$. The line of intersection of the planes is horizontal. If the rod makes an angle θ with the vertical, find $\tan\theta$. Find also the magnitudes of the reactions at the vertical and inclined planes.

8. A non-uniform rod AB of weight $3W$ rests with the end A on a smooth plane inclined at 30° to the horizontal. It is maintained in equilibrium at 30° to the horizontal by a smooth peg at C, where $AC = 3a$. Find (a) the magnitudes of the reactions at A and C, (b) the distance of the centre of gravity of the rod from A.

9. A uniform rod AB of length 1 m is smoothly hinged at A and the end B is attached by means of a light elastic string of natural length 1 m, modulus 20 N to a point C which is 1 m vertically above A. Given that the tension in the string is 12 N,

find (a) the weight of the rod, (b) the magnitude and direction of the reaction at the hinge.

10. A uniform rod AB of weight $4W$ rests in equilibrium with the end A against a smooth vertical wall. A light elastic string XY of natural length 24 cm, modulus $15W$ is attached to a point X vertically above A and a point Y on the rod. Given that $XY = 32$ cm and $\angle XYB = 90°$, find the tension in the string and the length of the rod.

11. A smooth hemispherical bowl of radius $5\sqrt{5}$ cm is fixed with its rim horizontal. A uniform rod AB is in equilibrium with the end A resting inside the bowl and a point C of the rod against the rim, where $AC = 20$ cm. If the rod is inclined at an angle α to the horizontal, find $\tan \alpha$. Find also the length of the rod.

12. A uniform ladder of length 6·5 m rests with one end against a smooth vertical wall and the other end on rough horizontal ground at a distance of 2·5 m from the wall. If the foot of the ladder is about to slip, find the coefficient of friction between the ladder and the ground.

13. A uniform pole of length $2a$ and mass m rests in equilibrium at an angle α to the horizontal, with one end against a smooth vertical wall and the other end on rough horizontal ground. Find expressions for (a) the force exerted on the pole by the wall, (b) the least possible value of the coefficient of friction between the pole and the ground.

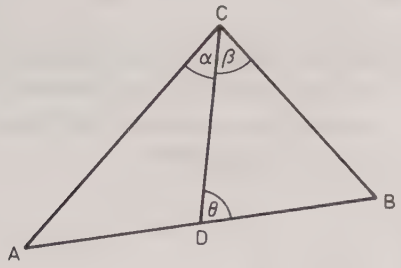
14. A uniform ladder of weight W and length $2l$ rests with its upper end against a smooth vertical wall. Its lower end stands on rough horizontal ground, and the coefficient of friction between the ladder and the ground is $\frac{1}{2}$. The ladder is in limiting equilibrium. Find the angle the ladder makes with the horizontal. Find also the reaction at the wall. A man of weight W climbs the ladder. Find how far up it he can go before the ladder will slip. Find also how far up the ladder he can go when a load of weight W is placed on the foot of the ladder. (L)

15. A uniform rod AB of length a and weight W has a light ring attached to the end A. The ring is free to slide along a straight wire inclined at 30° to the horizontal. The rod is maintained in equilibrium by a light inextensible string of length a attached to the end B. The other end of the string is attached to the wire at a fixed point C above A. (a) If the wire is smooth, find the angle θ which the string makes with the vertical. (b) If the string is vertical, find the least value of the coefficient of friction between the ring and the wire.

16. A heavy uniform rod of length $2a$ rests in equilibrium against a small smooth horizontal peg with one end of the rod on a rough horizontal floor. If the height of the peg above the floor is b and friction is limiting when the rod makes an angle $\frac{\pi}{6}$ with the floor, show that the coefficient of friction between the floor and the rod is $\frac{\sqrt{3}a}{(8b - 3a)}$. (O)

17. A uniform rod AB of length $10a$ rests in limiting equilibrium with the end A on horizontal ground and the end B against a vertical wall. The coefficient of friction at each end of the rod is μ. Given that A is at a distance $6a$ from the wall, find the value of μ.

18. (a)



In the given diagram D is the mid-point of AB. Using the sine rule or otherwise prove that

$$2\cot\theta = \cot\alpha - \cot\beta.$$

(b) A uniform ladder rests at an angle θ to the vertical, with one end on horizontal ground and the other end against a vertical wall. The coefficients of friction between the ladder and the ground and between the ladder and the wall are μ_1 and μ_2 respectively. Given that the ladder is on the point of slipping, use part (a) to prove that $\tan\theta = 2\mu_1/(1 - \mu_1\mu_2)$.

10.3 General conditions for equilibrium

We have already established that if a body is in equilibrium under the action of a set of coplanar forces, then the sum of their components in any direction is zero and the sum of their moments about any point is zero. In this section we consider sets of conditions that are sufficient to ensure equilibrium.

Suppose that a system of coplanar forces is such that

(i) the sum of the components in a direction d_1 is zero,
(ii) the sum of the components in a direction d_2, not parallel to d_1, is zero, and
(iii) the sum of the moments about a point A in the plane is zero.

This system of forces is not equivalent to a single force, because no force can have zero components in two directions. The system does not reduce to a couple, because a couple has the same non-zero moment about every point in its plane. Thus, if a set of forces acting on a rigid body satisfies conditions (i), (ii) and (iii), it follows that the body is in equilibrium.

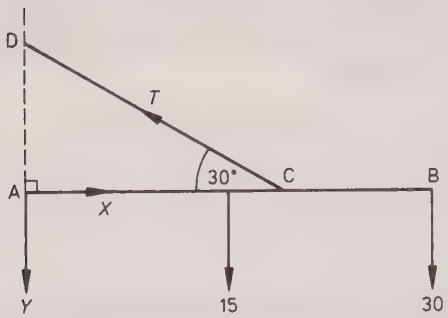
We can use this set of equilibrium conditions to solve problems involving forces in a *vertical plane* by

(i) resolving horizontally,
(ii) resolving vertically, and
(iii) taking moments about a suitable point.

In this way we obtain up to three equations which we then solve to find any unknown quantities.

Example 1 A uniform rod AB of weight 15 N and length 4 m is smoothly hinged at A and has a particle of weight 30 N attached to it at B. A light inextensible string,

attached to the rod at a point C, where $AC = 2{\cdot}5$ m, and to a point D vertically above A, keeps the rod in a horizontal position. If the angle between the rod and the string is 30°, find (a) the tension in the string, (b) the magnitude and direction of the reaction at the hinge.



Let the tension in the string be T N and let the horizontal and vertical components of the reaction at the hinge be X N and Y N respectively.

(a) Taking moments anti-clockwise about A,

$$T \times 2{\cdot}5 \sin 30° - 15 \times 2 - 30 \times 4 = 0$$

i.e. $$T \times \tfrac{5}{2} \times \tfrac{1}{2} - 30 - 120 = 0$$

$$\therefore \quad T = 150 \times \tfrac{4}{5} = 120.$$

Hence the tension in the string is 120 N

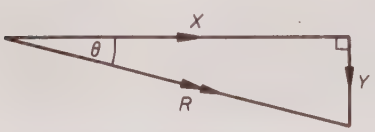
(b) Resolving horizontally and vertically,

$$\rightarrow \qquad X - T\cos 30° = 0 \qquad (1)$$

$$\downarrow \qquad Y + 15 + 30 - T\sin 30° = 0 \qquad (2)$$

From (1), $X = 120 \times \dfrac{\sqrt{3}}{2} = 60\sqrt{3}$

From (2), $Y = 120 \times \tfrac{1}{2} - 45 = 15$



If the resultant reaction R acts at an angle θ to the horizontal, then

$$R^2 = X^2 + Y^2, \tan\theta = \frac{Y}{X}.$$

$$\therefore \quad R = \sqrt{\{(60\sqrt{3})^2 + 15^2\}} = 15\sqrt{\{(4\sqrt{3})^2 + 1^2\}} = 15\sqrt{(48 + 1)} = 105,$$

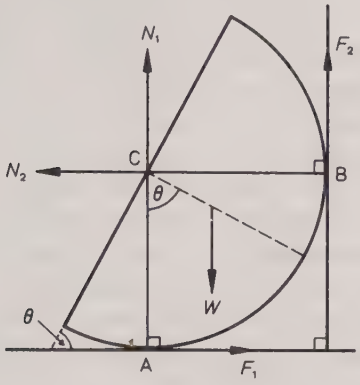
$$\tan\theta = \frac{15}{60\sqrt{3}} = \frac{1}{4\sqrt{3}} \quad \text{i.e.} \quad \theta \approx 8{\cdot}2°.$$

Hence the reaction at the hinge has magnitude 105 N and acts at 8·2° to the horizontal.

When a rigid body is kept in equilibrium by frictional forces, then at each point of contact with a rough surface the frictional force F acts along the surface, opposing any tendency to slide. It also satisfies the condition $F \leqslant \mu N$, where N is the normal

reaction and μ is the coefficient of friction. If the body is just about to slip, then friction is limiting at *every* point of contact and in each case $F = \mu N$.

Example 2 A uniform solid hemisphere rests in limiting equilibrium with its curved surface in contact with a horizontal floor and a vertical wall. The coefficient of friction at the floor is $\frac{1}{4}$ and at the wall is $\frac{1}{2}$. If θ is the angle of inclination of the plane face of the hemisphere to the horizontal, find $\sin\theta$.



Let the hemisphere have centre C and radius a. Let the forces on it be as shown in the diagram. Since the hemisphere is in limiting equilibrium, friction must be limiting at both A and B,

$\therefore \quad F_1 = \frac{1}{4}N_1, \quad F_2 = \frac{1}{2}N_2.$

Resolving horizontally and vertically,

$$\rightarrow \qquad F_1 - N_2 = 0 \qquad (1)$$

$$\uparrow \qquad N_1 + F_2 - W = 0 \qquad (2)$$

Substituting for F_1 in (1):

$$\tfrac{1}{4}N_1 - N_2 = 0 \qquad \therefore \quad N_1 = 4N_2$$

Substituting for N_1 and F_2 in (2):

$$4N_2 + \tfrac{1}{2}N_2 - W = 0 \qquad \therefore \quad N_2 = \tfrac{2}{9}W$$

Hence $F_1 = N_2 = \frac{2}{9}W$ and $F_2 = \frac{1}{2}N_2 = \frac{1}{9}W$.

The centre of gravity of the hemisphere is on its axis of symmetry at a distance $\frac{3}{8}a$ from C.

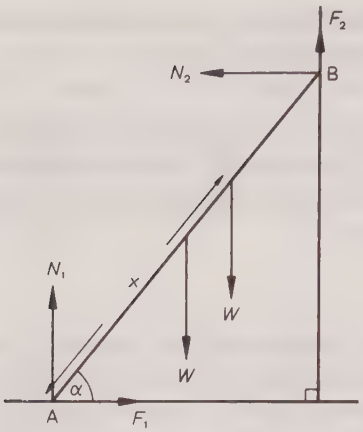
Thus, taking moments clockwise about C,

$$W \times \tfrac{3}{8}a\sin\theta - F_1 \times a - F_2 \times a = 0$$

$$\therefore \qquad \tfrac{3}{8}aW\sin\theta - \tfrac{2}{9}aW - \tfrac{1}{9}aW = 0$$

Hence $\frac{3}{8}\sin\theta - \frac{3}{9} = 0$ i.e. $\sin\theta = \frac{8}{9}$.

Example 3 A uniform ladder AB of length $2a$ and weight W rests with the end A on horizontal ground and the end B against a vertical wall. The coefficients of friction at A and B are $\frac{2}{5}$ and $\frac{1}{2}$ respectively. The ladder makes an angle α with the horizontal, where $\tan\alpha = \frac{5}{4}$. If a man of weight W starts to climb the ladder, find his distance from A when the ladder begins to slip.



Let the reactions at A and B when the ladder is about to slip be as shown in the diagram. Since friction must be limiting at both A and B,

$$F_1 = \tfrac{2}{5}N_1, \quad F_2 = \tfrac{1}{2}N_2.$$

Resolving horizontally and vertically,

$$\rightarrow \qquad F_1 - N_2 = 0 \qquad (1)$$
$$\uparrow \qquad N_1 + F_2 - 2W = 0 \qquad (2)$$

Substituting for F_1 in (1),

$$\tfrac{2}{5}N_1 - N_2 = 0 \qquad \therefore \quad N_1 = \tfrac{5}{2}N_2$$

Substituting for N_1 and F_2 in (2),

$$\tfrac{5}{2}N_2 + \tfrac{1}{2}N_2 - 2W = 0 \qquad \therefore \quad N_2 = \tfrac{2}{3}W$$

Hence $F_2 = \tfrac{1}{2}N_2 = \tfrac{1}{3}W$

Let the man be a distance x from A when the ladder begins to slip.
Taking moments clockwise about A,

$$W \times x\cos\alpha + W \times a\cos\alpha - F_2 \times 2a\cos\alpha - N_2 \times 2a\sin\alpha = 0$$

Dividing by $\cos\alpha$,

$$Wx + Wa - 2F_2a - 2N_2a\tan\alpha = 0$$
$$\therefore \quad Wx + Wa - \tfrac{2}{3}Wa - \tfrac{4}{3}Wa \times \tfrac{5}{4} = 0$$
$$\therefore \quad x = \tfrac{5}{3}a + \tfrac{2}{3}a - a = \tfrac{4}{3}a.$$

Hence the man is at a distance of $4a/3$ from A when the ladder begins to slip.

The set of conditions discussed earlier, for the equilibrium of a rigid body under the action of coplanar forces, involved the sums of components in two directions and the sum of moments about one point. We now consider whether any other combinations of similar conditions will also ensure equilibrium.

To be certain that a set of forces is not equivalent to a couple, we need to know that the system has zero moment about at least one point. A system with zero moments about two points A and B may have a resultant whose line of action passes through A and B. We can rule out this possibility if we find that the components of the forces in some direction, not perpendicular to AB, have zero sum or that the system has zero moment about a third point not on AB.

We have then three sets of equilibrium conditions. A system of coplanar forces must be in equilibrium if

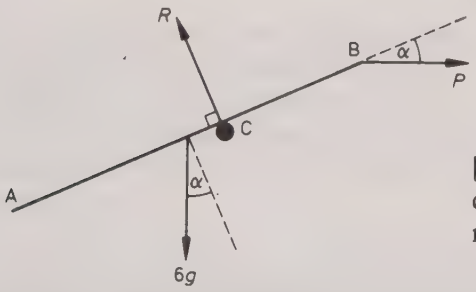
(a) the sums of the components in two non-parallel directions are each zero, and the sum of the moments about one point is zero;
or (b) the sums of the moments about two points A and B are each zero, and the sum of the components in a direction, not perpendicular to AB, is zero;
or (c) the sums of the moments about three points, not in a straight line, are each zero.

Thus when tackling problems we may
(a) resolve in two directions and take moments about one point;
or (b) take moments about two points A and B then resolve in a direction not perpendicular to AB;
or (c) take moments about three points which are not collinear.

Since each method involves putting three different restrictions on the system of forces concerned, we will obtain three *independent* equations connecting any unknown quantities i.e. three equations each giving information which could not be obtained from the other two. [Note that since these three equations ensure the equilibrium of the system, no new information can be gained from further equations formed by resolving or taking moments.]

The following examples show that, by careful choice of the points about which we take moments and of the directions in which we resolve, we may
(1) avoid equations involving quantities that we do not need to find,
(2) form equations containing as few unknowns as possible.

Example 4 A uniform rod AB of mass 6 kg rests on a smooth peg at a point C. The rod is kept in equilibrium at an angle α to the horizontal, where $\tan\alpha = 5/12$, by a horizontal force of P newtons acting at the upper end B of the rod. Find the value of P.
[Take g as $10\,\mathrm{m\,s^{-2}}$.]



[We try to avoid equations involving distances or the reaction at C, as we do not need to find these quantities.]

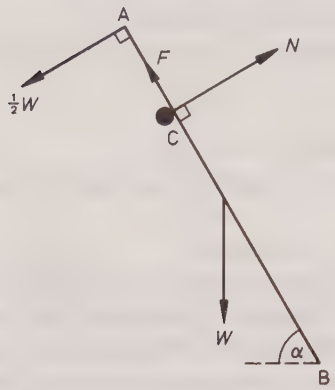
Resolving along the rod in the direction AB,

$$P\cos\alpha - 6g\sin\alpha = 0$$

$$\therefore \quad P = 6g\tan\alpha = 6 \times 10 \times \tfrac{5}{12} = 25.$$

Example 5 A uniform bar AB of weight W and length $4a$ rests on a rough peg at a point C, where $AC = a$. The bar is maintained in equilibrium, with A above B, by a force of magnitude $\frac{1}{2}W$ acting at A in a direction perpendicular to the bar. Find the

angle α, which the bar makes with the horizontal. Given that the bar is about to slip, find the coefficient of friction μ between the bar and the peg.



[In this example we try to form equations which simplify the working as much as possible.]

Taking moments clockwise about C,

$$W \times a\cos\alpha - \tfrac{1}{2}W \times a = 0 \qquad \therefore \cos\alpha = \tfrac{1}{2}$$

Hence $\alpha = 60°$.

Let F be the frictional force at C and let N be the normal reaction.
Taking moments clockwise about the mid-point of AB,

$$N \times a - \tfrac{1}{2}W \times 2a = 0 \qquad \therefore \quad N = W$$

Resolving along the bar in the direction BA,

$$F - W\sin\alpha = 0 \qquad \therefore \quad F = \frac{\sqrt{3}}{2}W$$

Since the bar is about to slip, $F = \mu N$.

Hence $\mu = \dfrac{F}{N} = \dfrac{\sqrt{3}}{2}$.

Exercise 10.3

[Take g as $10\,\mathrm{m\,s^{-2}}$.]

1. A uniform rod AB of weight 20 N and length 2 m is freely hinged at A. The rod is held in equilibrium at 30° to the horizontal, with B higher than A, by a light inextensible string of length 1·25 m. The string joins a point C on the rod to a fixed point D at a distance of 1·25 m vertically above A. If a particle of weight 5 N is suspended from B, find (a) the tension in the string, (b) the horizontal and vertical components of the reaction at A.

2. A uniform rod AB of mass 6 kg and length 3 m is freely hinged at A. A load of mass 2 kg is attached to the rod at B. The rod is kept in equilibrium by a light inextensible string CD of length 1·5 m, which is attached to a point C on the rod and

to a fixed point D on the same horizontal level as A. Given that $AC = 2\,\text{m}$ and $AD = 2{\cdot}5\,\text{m}$, find (a) the tension in the string, (b) the magnitude and direction of the reaction at the hinge.

3. A uniform square lamina $ABCD$ of mass 6 kg is hinged at A so that it is free to move in a vertical plane. It is maintained in equilibrium, with B vertically below A, by a horizontal force acting at C and a vertical force acting at D each of magnitude P newtons. Find (a) the value of P, (b) the magnitude and direction of the force exerted by the hinge on the lamina.

4. A uniform rod AB of weight 16 N is freely hinged at A. It is kept in equilibrium at an angle of 30° to the horizontal, with B lower than A, by a force acting at B. Find the magnitude and direction of the least force required at B to maintain equilibrium. Find also the corresponding horizontal and vertical components of the reaction at A.

5. A uniform rod AB of length 3 m and weight 40 N rests in equilibrium with the lower end A on a smooth plane inclined at an angle α to the horizontal, where $\tan\alpha = 3/4$, and a point C of the rod against a smooth peg. If the rod makes an angle θ with the horizontal, where $\tan\theta = 7/24$, find (a) the magnitudes of the reactions at A and C, (b) the distance AC.

6. A uniform rod AB of mass 13 kg is in limiting equilibrium with its lower end A on a rough horizontal plane and its upper end B attached to a light inextensible string inclined at an angle θ to the upward vertical, where $\tan\theta = 12/5$. The string passes over a smooth fixed pulley at a point vertically above A and supports at its other end a mass of 6·5 kg. Find (a) the coefficient of friction between the rod and the plane, (b) the angle the rod makes with the horizontal.

7. A smooth hemisphere of radius 60 cm is fixed with its plane face on a rough horizontal floor. A uniform rod AB of length 1 m and weight 10 N rests in equilibrium in a vertical plane with the end A on the floor and a point C of the rod in contact with the hemisphere, where $AC = 80\,\text{cm}$. Given that the rod is on the point of slipping, find the reaction at C and the coefficient of friction between the rod and the floor.

8. A uniform ladder AB of length 5 m and mass 12 kg rests with the end A on rough horizontal ground and the end B in contact with a smooth vertical wall. The foot of the ladder is at a distance of 1·4 m from the wall. Find the magnitudes of the reactions at A and B. If the ladder is about to slip, find the coefficient of friction between the ladder and the ground.

9. A uniform beam AB of length $2a$ and weight W rests with its end A against a smooth vertical wall and B on a rough horizontal plane. The vertical plane through AB is perpendicular to the wall. When it makes an angle of α with the wall the beam is on the point of slipping. Find the coefficient of friction between the beam and the horizontal plane. Find, also, what horizontal force applied to the mid-point of the

beam (in the vertical plane through AB) will just cause the beam to slip at B towards the wall. (SU)

10. A thin uniform hemispherical bowl, of weight W and radius a, rests in limiting equilibrium with its plane face at an angle θ to the horizontal. Its curved surface is in contact with a rough horizontal floor and a rough vertical wall. If the coefficient of friction at each point of contact is $\frac{1}{3}$, find the magnitudes of the frictional forces at the floor and the wall. Find also the value of $\sin\theta$.

11. A uniform solid hemisphere rests in limiting equilibrium with its plane face at an angle α to the horizontal, where $\sin\alpha = 8/13$. Its curved surface is in contact with rough horizontal ground and a rough vertical wall. If the coefficient of friction at both points of contact is μ, find the value of μ.

12. A uniform rod AB of weight W and length $4a$ rests with its end A on a rough horizontal plane and a point C of the rod, where $AC = 3a$, in contact with a smooth peg. Given that the rod makes an angle θ with the horizontal, where $\sin\theta = 3/5$, find the magnitude of the reaction at C and the horizontal and vertical components of the reaction at A. If μ is the coefficient of friction between the rod and the horizontal plane, show that $\mu \geqslant 24/43$.

13. A uniform ladder of weight W rests at 60° to the horizontal with its foot on rough horizontal ground and the other end against a smooth vertical wall. If the ladder is just about to slip when a man of weight $2W$ stands at the top, find the coefficient of friction between the ladder and the ground. Find the frictional force between the ladder and the ground when the man is standing at the mid-point of the ladder.

14. A uniform ladder of length $2a$ and weight W rests against a smooth vertical wall and has its foot on rough horizontal ground. The coefficient of friction between the ladder and the ground is $\frac{1}{2}$. If the ladder is placed at an angle α to the horizontal, where $\tan\alpha = \frac{3}{2}$, find the distance that a man of weight $3W$ can climb up the ladder before it slips.

15. A uniform pole AB of length 8 m and mass 60 kg rests in limiting equilibrium with the end A on horizontal ground and the end B against a vertical wall. If the coefficient of friction at each end of the pole is $\frac{1}{3}$, find the angle the pole makes with the horizontal. The pole is now placed at an angle α to the horizontal, where $\tan\alpha = 2$, and a body of mass m kg is attached to the pole at B without causing the pole to slip. Find the maximum value of m and the magnitude of the corresponding normal reaction at A.

16. A uniform circular hoop of weight 15 N hangs over a rough horizontal peg A. A horizontal force P N, acting in the plane of the hoop, is applied at the other end B of the diameter through A. If the hoop is on the point of slipping when AB is inclined at

an angle α to the downward vertical, where $\tan\alpha = 4/3$, find the value of P and the coefficient of friction between the hoop and the peg.

17. A uniform rod AB, of weight W and length $6a$, rests with its lower end A on a smooth horizontal plane and a point C of the rod against a smooth peg, where $AC = 4a$. The rod is kept in equilibrium at an angle α to the horizontal by a force of magnitude $\frac{1}{4}W$ acting along the rod. Find the magnitude of the reaction at A and the value of $\sin\alpha$.

18. A uniform ladder, of weight W and length l, rests in a vertical plane with one end against a smooth vertical wall, the wall being perpendicular to the vertical plane through the ladder. The other end of the ladder rests on horizontal ground, the coefficient of friction between the ladder and the ground being 1/4. The ladder is inclined at an angle θ to the horizontal, where $\tan\theta = 24/7$. A man of weight $10W$ climbs up the ladder. Show that the man can reach a height of $6l/7$ above the ground before the ladder begins to slip and calculate the force exerted by the ladder on the wall at this instant. (L)

19. A uniform ladder of length 7 m rests against a vertical wall with which it makes an angle of 45°, the coefficients of friction between the ladder and the wall and the ladder and the (horizontal) ground being $\frac{1}{3}$ and $\frac{1}{2}$ respectively. A boy whose weight is one half that of the ladder slowly ascends the ladder. How far along the ladder will he be when the ladder slips? [Assume the ladder is a uniform rod which is perpendicular to the line of intersection of the wall and ground.] (O&C)

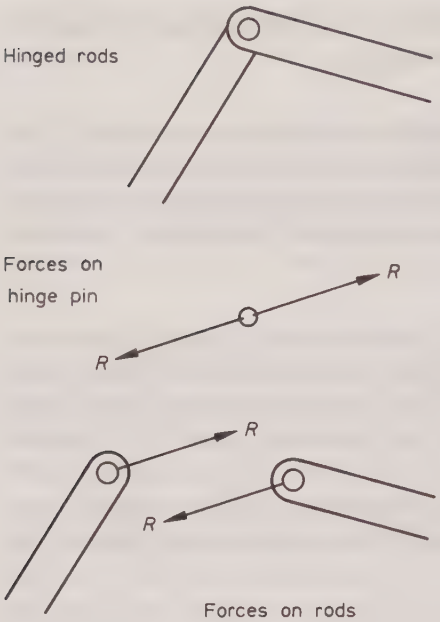
20. A uniform circular disc, of radius a and weight w, stands in a vertical plane upon a rough horizontal floor with the point P of its circumference in contact with an equally rough vertical wall and its plane at right angles to that of the wall. A particle of equal weight w is attached to a point Q of the disc between its centre O and P so that the disc is on the point of slipping. Draw a good diagram indicating clearly the normal reactions and the friction forces acting on the disc. Show that

$$OQ = \frac{2a\mu(1 + \mu)}{1 + \mu^2},$$

where μ is the coefficient of friction. (W)

10.4 Rigid body systems

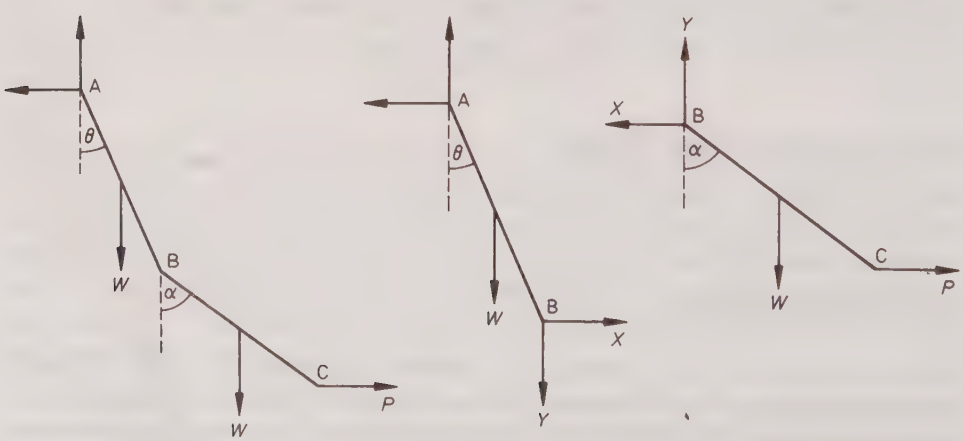
When two or more rigid bodies are in equilibrium under the action of a set of coplanar forces, then the whole system can be treated as a single body in equilibrium under the action of the external forces. Each separate body in the system is in equilibrium under the action of a set of forces which includes both the external forces on the body and the forces exerted on it by the other bodies in the system.



We first consider two rods connected by a smooth light hinge. Each rod exerts a force on the hinge and, according to Newton's third law, the hinge exerts an equal and opposite force on the rod. Thus, provided no external force acts on the hinge, the effect of the hinge is to set up equal and opposite forces on the two rods.

Example 1 Two equal uniform rods AB and BC, each of length $2a$ and weight W, are freely hinged together at B and hang freely from the point A. Equilibrium is maintained by a horizontal force P acting at C. Given that BC makes an angle α with the downward vertical, where $\tan\alpha = 4/3$, find the value of P and the horizontal and vertical components of the force exerted by the rod AB on the rod BC. Find also the angle made by AB with the vertical.

Let the horizontal and vertical components of the reaction at B be X and Y respectively and let θ be the angle AB makes with the vertical.



External forces Forces on AB Forces on BC

Consider first the forces on the rod BC.
Taking moments about B,

$$\circlearrowright \qquad P \times 2a\cos\alpha - W \times a\sin\alpha = 0$$
$$\therefore \qquad P = \tfrac{1}{2}W\tan\alpha = \tfrac{2}{3}W.$$

Resolving horizontally and vertically,

$$\leftarrow \qquad X - P = 0 \qquad \therefore \quad X = P = \tfrac{2}{3}W$$
$$\uparrow \qquad Y - W = 0 \qquad \therefore \quad Y = W$$

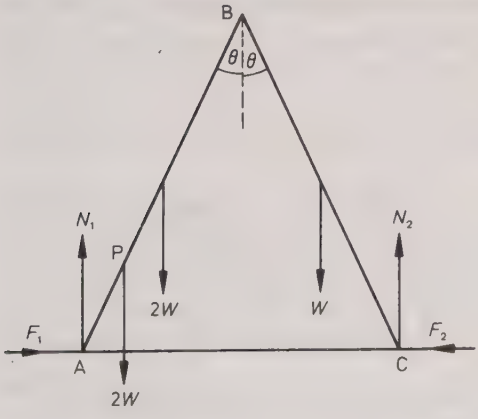
Hence the horizontal and vertical components of the force exerted by the rod AB on the rod BC are $\frac{2}{3}W$ and W respectively.
Consider now the forces on the rod AB.
Taking moments about A,

$$\circlearrowleft \quad W \times a\sin\theta + Y \times 2a\sin\theta - X \times 2a\cos\theta = 0$$
$$\therefore \qquad Wa\sin\theta + 2Wa\sin\theta - \tfrac{4}{3}Wa\cos\theta = 0$$

$$\therefore \quad 3Wa\sin\theta = \tfrac{4}{3}Wa\cos\theta$$
$$\therefore \quad \tan\theta = \tfrac{4}{9} \quad \text{and} \quad \theta \approx 24°.$$

Hence AB makes an angle of 24° with the vertical.

Example 2 Two uniform rods AB and BC, each of length $2a$, have weights $2W$ and W respectively. The rods are freely jointed at B and rest in a vertical plane with A and C on a rough horizontal plane, the coefficient of friction at both points of contact being $\frac{1}{2}$. A particle of weight $2W$ is attached to a point P of the rod AB, where $AP = \frac{1}{2}a$. Given that the rods are inclined at an angle θ to the vertical, find the maximum value of θ for which equilibrium is possible.



Let the reactions at A and C be as shown in the diagram.

Consider first the forces on the whole system.
Resolving horizontally and vertically,

$$\rightarrow \qquad F_1 - F_2 = 0 \qquad (1)$$
$$\uparrow \qquad N_1 + N_2 - 5W = 0 \qquad (2)$$

Taking moments about A,

$$\circlearrowright \quad N_2 \times 4a\sin\theta - W \times 3a\sin\theta - 2W \times a\sin\theta - 2W \times \tfrac{1}{2}a\sin\theta = 0$$
$$\therefore \quad 4N_2 - 3W - 2W - W = 0 \quad \text{giving} \quad N_2 = \tfrac{3}{2}W.$$
From (2), $\qquad N_1 = 5W - N_2 = \tfrac{7}{2}W.$

Using the laws of friction, $F_1 \leqslant \mu N_1$ and $F_2 \leqslant \mu N_2$
Since $\mu = \frac{1}{2}$, $F_1 \leqslant \frac{7}{4}W$ and $F_2 \leqslant \frac{3}{4}W$
However, using (1), we may write $F_1 = F_2 = F$
$\therefore$ in any equilibrium position, $F \leqslant \frac{3}{4}W$ (3)
Taking moments about B for the forces on rod BC,

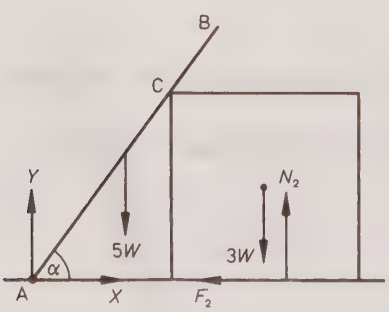
$$\circlearrowleft \quad N_2 \times 2a\sin\theta - W \times a\sin\theta - F_2 \times 2a\cos\theta = 0$$
$$\therefore \quad 3Wa\sin\theta - Wa\sin\theta - 2Fa\cos\theta = 0$$
$$\therefore \quad 2Wa\sin\theta = 2Fa\cos\theta$$
$$\text{Thus} \quad \tan\theta = F/W$$
$$\therefore \text{ using (3)}, \quad \tan\theta \leqslant 3/4$$

Hence the maximum value of θ for which equilibrium is possible is $\tan^{-1}(3/4) \approx 36{\cdot}9°$.

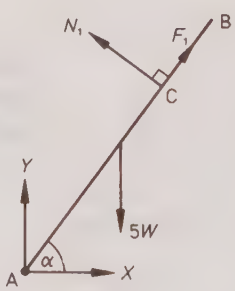
[Note that when θ takes this maximum value, slipping is about to occur at C.]

We now consider problems concerning two bodies in contact. By Newton's third law, the forces exerted by each body on the other are equal and opposite. Thus, as in the case of jointed bodies, unknown quantities can be found by resolving and taking moments, considering either the external forces acting on the system as a whole, or both internal and external forces acting on each body separately. In any given problem we try to simplify the working by avoiding equations involving quantities that we do not need to find.

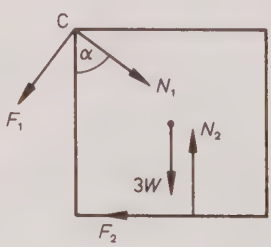
Example 3 A uniform cube of side $12a$ and weight $3W$ rests on a rough horizontal table. A uniform rod AB of length $20a$ and weight $5W$ is freely hinged at A to a fixed point on the table. A point C of the rod rests against the mid-point of one edge of the cube, this edge being perpendicular to the vertical plane containing the rod. When the rod is at an angle α to the horizontal, where $\sin\alpha = 4/5$, the cube is on the point of sliding across the table without toppling. If the coefficient of friction at C is $\frac{1}{2}$, find the coefficient of friction between the cube and the table.



Forces on whole system



Forces on rod



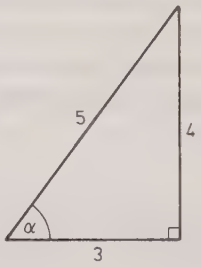
Forces on cube

Let F_1, N_1 and F_2, N_2 be the frictional forces and normal reactions at C and the base of the cube respectively. Consider first the forces on the rod.

Taking moments about A,

$$\circlearrowleft \qquad N_1 \times 12a \operatorname{cosec}\alpha - 5W \times 10a\cos\alpha = 0$$

Using a right-angled triangle,



if $\sin\alpha = \frac{4}{5}$, then $\operatorname{cosec}\alpha = \frac{5}{4}$, $\cos\alpha = \frac{3}{5}$

$\therefore \quad N_1 \times 12a \times \frac{5}{4} - 5W \times 10a \times \frac{3}{5} = 0$

Hence $N_1 = 2W$.
Since the cube is about to slide, $F_1 = \frac{1}{2}N_1 = W$.
Consider now the forces on the cube.
Resolving horizontally and vertically,

$$\leftarrow \qquad F_2 + F_1\cos\alpha - N_1\sin\alpha = 0 \qquad (1)$$

$$\uparrow \qquad N_2 - 3W - F_1\sin\alpha - N_1\cos\alpha = 0 \qquad (2)$$

From (1), $\quad F_2 + \frac{3}{5}W - \frac{8}{5}W = 0 \quad \therefore \quad F_2 = W$

From (2), $\quad N_2 - 3W - \frac{4}{5}W - \frac{6}{5}W = 0 \quad \therefore \quad N_2 = 5W$

Since the cube is on the point of sliding, the coefficient of friction between the cube and the table is given by F_2/N_2 and is equal to 1/5.

Exercise 10.4

1. Two uniform rods AB and BC, of weights 5 N and 10 N respectively, are freely hinged at B and hang freely from a fixed hinge at A. The system is kept in equilibrium by a horizontal force of magnitude 5 N acting at C. Find (a) the magnitudes of the reactions at the hinges A and B, (b) the angles the rods AB and BC make with the vertical.

2. Two equal uniform rods, each of weight W, are freely jointed at B and hang freely from a hinge at A. The end C is attached by means of a light inextensible string to a point D on the same horizontal level as A. When the system is in equilibrium the rod AB is inclined at 45° to the vertical and the rod BC is horizontal. Find the tension in the string and the angle it makes with the vertical.

3. Two uniform rods AB and BC, each of length $2a$, have weights W and $3W$ respectively. The rods are smoothly jointed at B. The ends A and C of the rods are smoothly hinged to fixed points at the same horizontal level and a distance $2a$ apart. If the system is in equilibrium with B below AC, find the horizontal and vertical components of the reaction at B.

4. Two equal uniform rods, AB and AC, each of length $2a$ and weight W, are freely jointed at A. The end C is freely hinged to a point on a rough horizontal plane, the end B rests on the plane and is about to slip, and both rods are in a vertical plane. If the coefficient of friction between B and the plane is μ and the angle ABC is θ, (i) prove that $\mu = \frac{1}{2}\cot\theta$, (ii) find, in terms of μ, the magnitude and direction of the reaction at the hinge at C, (iii) find the magnitude of the reaction at the joint A.
(SU)

5. Two uniform rods AB and BC, freely jointed together at B, are of equal length, but have weights $4W$ and W respectively. The end A is freely hinged to a fixed point on a rough vertical wall and the end C rests against the wall at a point vertically below A. The system rests in limiting equilibrium with each rod inclined at an angle α to the vertical, where $\tan\alpha = 2$. Find (a) the coefficient of friction between the rod BC and the wall, (b) the horizontal and vertical components of the reaction of AB on BC.

6. Two uniform rods AB and AC, of length $2a$, but of weights $2W$ and W respectively, are smoothly jointed at A. The rods rest in a vertical plane with B and C on a rough horizontal plane, the coefficient of friction between the rods and the plane being 0·4. Given that the system is in limiting equilibrium, determine whether slipping is about to occur at B or at C and find angle BAC.

7. Two equal uniform rods, AB and BC, each of mass $4m$, are smoothly hinged together at B and stand in a vertical plane with A and C on a rough horizontal floor. The angle ABC is 2α where $\tan\alpha = \frac{1}{2}$, and the coefficient of friction at both A and C is μ. A mass m is fixed to the mid-point of AB. Show that equilibrium is possible provided that $\mu \geqslant 9/34$. What happens if $\mu = \frac{1}{4}$?

8. Two smooth uniform spheres of weights W and $2W$ are each suspended from a fixed point O by a light inextensible string and rest against each other. The angle between the strings is 90° and the string supporting the heavier sphere makes an angle α with the vertical, where $\tan\alpha = \frac{3}{4}$. Find (a) the tensions in the strings, (b) the angle of inclination to the horizontal of the line joining the centres of the spheres.

9. A uniform rod AB of length $4a$ and weight W is freely hinged at A. A second uniform rod CD of length $5a$ and weight $2W$ is freely hinged at C, which lies at a distance $4a\sqrt{3}$ from A in the same horizontal plane. The rods rest in equilibrium in a vertical plane with the rod CD in contact with the end B of the rod AB. If the rods are on the point of slipping when $BC = 4a$, find the coefficient of friction between the rods.

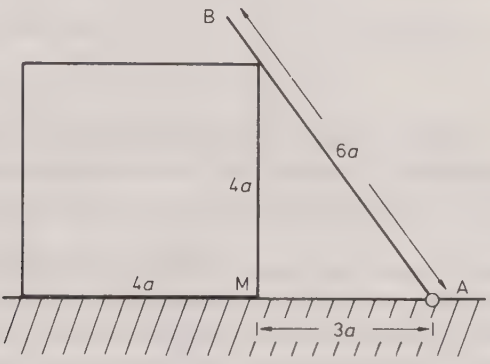
10. Two uniform circular cylinders rest in contact on a horizontal table. A third cylinder is supported by the first two. The cylinders are identical and their axes are horizontal and parallel. The coefficient of friction between any pair of cylinders and between the lower cylinders and the table is μ. Find the least value of μ for which equilibrium is possible.

11. A uniform rod OA of weight W and length $2a$ is pivoted smoothly to a fixed point at the end O. A light inextensible string, also of length $2a$, has one end attached to the end A of the rod, its other end being attached to a small heavy ring of weight $\frac{1}{2}W$ which is threaded on to a fixed rough straight horizontal wire passing through O. The system is in equilibrium with the string taut and the rod making an angle α with the downward vertical at O. By taking moments about O for the rod and ring

together, or otherwise, find the normal component R of the reaction exerted by the wire on the ring. Find also the frictional force F on the ring. The equilibrium is limiting when $\alpha = 60°$. Determine the coefficient of friction between the wire and the ring. (O&C)

12. Two uniform rods AB, AC, each of weight W and length $2a$, are smoothly jointed at A. The rods rest on a smooth circular cylinder of radius a with their plane perpendicular to the axis of the cylinder which is horizontal. When the rods are in equilibrium in a vertical plane with A vertically above the axis of the cylinder, angle $BAC = 2\theta$. Show that the force exerted by the cylinder on each rod is of magnitude W cosec θ. Show also that $\sin^3\theta = \cos\theta$. Hence show that there is only one such position of equilibrium. (O&C)

13.



A uniform cube, of edge $4a$ and weight W, rests on a smooth horizontal surface (see diagram). A uniform smooth rod AB, of length $6a$ and weight w, has one end freely hinged to a point A of the surface. The rod rests against the mid-point of a horizontal edge of the top face of the cube. The rod is in a vertical plane through the centre of the cube. A light inelastic string of length $3a$ joins A to the mid-point M of the horizontal edge of the cube nearest to A. The system is in equilibrium. (i) Find the magnitude of the reaction between the rod and the cube. (ii) Find the magnitude of the reaction between the cube and the horizontal surface. (iii) Find the tension in the string AM. (iv) Show that $18w \leqslant 125$W. (C)

10.5 Light frameworks

The expression *light framework* is used to describe a system of rods whose weights are negligible compared with the other forces involved. It is also assumed that the rods are smoothly jointed together at their ends.

Let us consider the two resultant forces acting at the ends of a typical light rod AB in such a framework. If the rod is in equilibrium these forces must be equal and opposite. By taking moments about A and B we also find that both forces act along AB.

Forces on a rod in tension

Forces exerted by a rod in tension

Forces on a rod in thrust

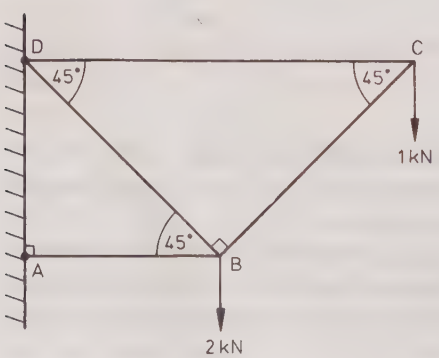
Forces exerted by a rod in thrust

As shown in the diagrams above, a rod may either be in *tension* or in *thrust* i.e. under *compression*. It follows from Newton's third law that at each end of the rod, the force exerted by the rod on the joint is equal and opposite to the force exerted by the joint on the rod.

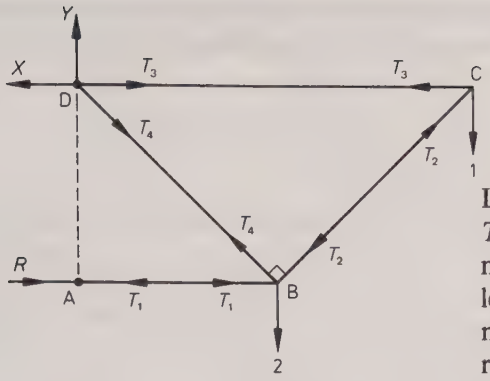
Elementary problems about light frameworks can be solved by considering the equilibrium of the external forces on the framework and the equilibrium of the forces at each joint. In general, unknown forces at joints are found by resolving in suitable directions. However, when dealing with three forces at a joint it is sometimes quicker to use a triangle of forces.

Whatever method is chosen, it is important to draw a large clear diagram showing the forces acting at each joint. Sometimes it is difficult to decide whether a particular rod is in tension or in thrust. In this case we may choose to indicate one type of stress in the diagram, but if calculation leads to a negative value for this stress, we deduce that it is of the other type.

Example 1



The diagram shows a framework of four smoothly jointed light rods. The framework is freely hinged at A and D to a vertical wall. Loads of 2 kN and 1 kN are hung from B and C respectively. Find the magnitudes of the reactions at A and B. Find also the stresses in the rods, stating whether they are tensions or thrusts.



Let the stresses in the rods be thrusts T_1, T_2 and tensions T_3, T_4 as shown. Let the magnitude of the reaction at A be R and let the horizontal and vertical components of the reaction at D be X and Y respectively.

Since there are only two forces acting at A, the reaction R must be a horizontal force such that

$$R - T_1 = 0 \tag{1}$$

Consider now the forces on the whole framework.
Let $AB = AD = a$, then taking moments about D,

↻ $\quad R \times a - 2 \times a - 1 \times 2a = 0 \quad \therefore \quad R = 4$

Hence the magnitude of the reaction at A is 4 kN.
Resolving horizontally and vertically,

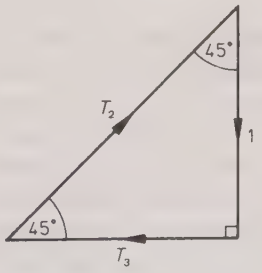
$\leftarrow \quad X - R = 0 \quad \therefore \quad X = R = 4$
$\uparrow \quad Y - 2 - 1 = 0 \quad \therefore \quad Y = 3$

Hence the magnitude of the reaction at D

$$= \sqrt{(X^2 + Y^2)}\,\text{kN} = \sqrt{(4^2 + 3^2)}\,\text{kN} = 5\,\text{kN}$$

From (1), $T_1 = R = 4$.
Using the triangle of forces for the joint C,

$$T_2 = \sqrt{2}, \quad T_3 = 1.$$



Resolving the forces at B in the direction BD,

↖ $\quad T_4 - 2\cos 45° - T_1 \cos 45° = 0$

$$\therefore \quad T_4 = 2 \times \frac{1}{\sqrt{2}} + 4 \times \frac{1}{\sqrt{2}} = 3\sqrt{2}$$

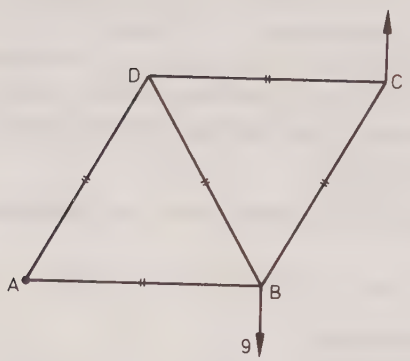
Hence the stresses in AB and BC are thrusts of magnitude 4 kN and $\sqrt{2}$ kN respectively. The stresses in BD and CD are tensions of magnitude $3\sqrt{2}$ kN and 1 kN respectively.

Exercise 10.5

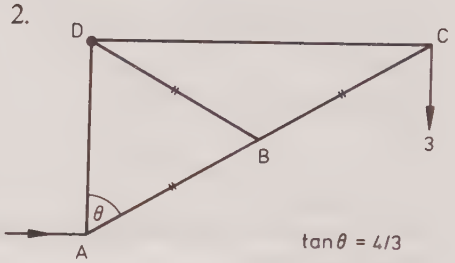
In questions 1 to 8 each diagram shows a light framework consisting of smooth jointed rods. The large dots represent points at which the framework is smoothly

hinged to a fixed point. External forces acting at other points are indicated by arrows and given in kilonewtons. Find the stresses in the rods, stating whether they are tensions or thrusts.

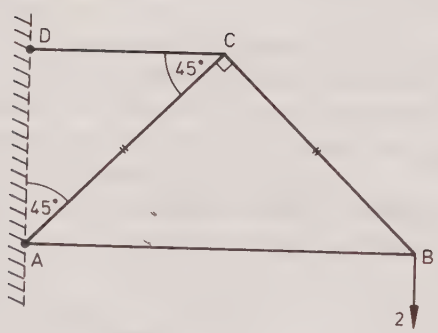
1.



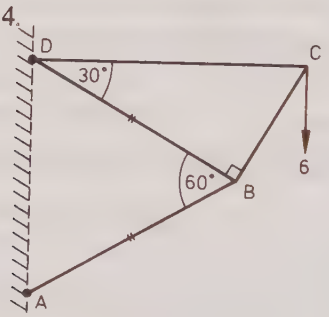
2.



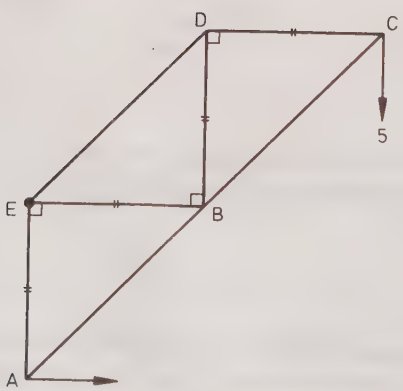
3.



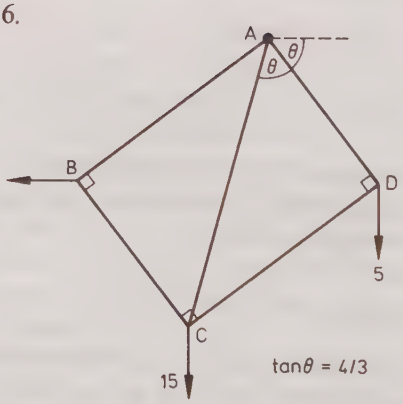
4.



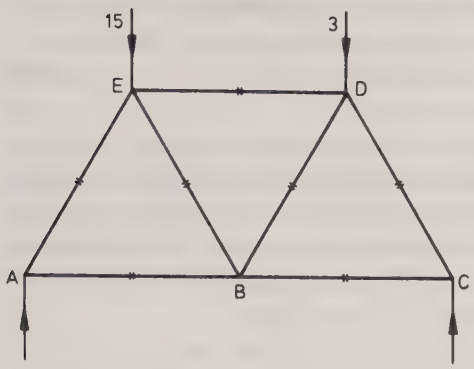
5.



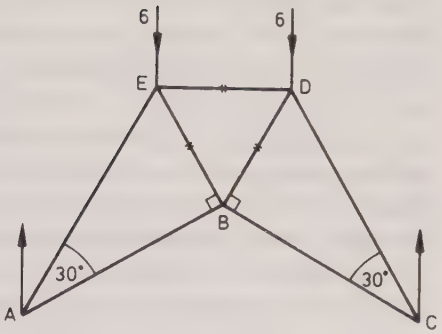
6.



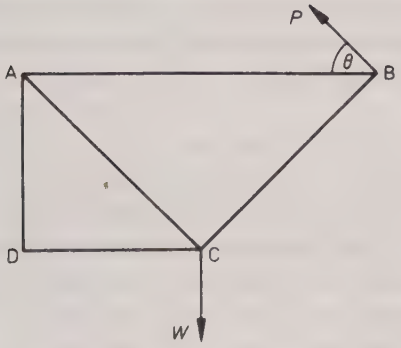
7.



8.



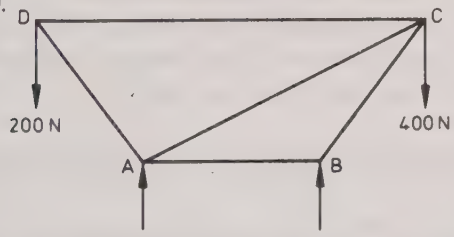
9.



The diagram represents a framework of five smoothly jointed light rods with $AB = \sqrt{2}BC = \sqrt{2}CA = 2CD = 2DA$. The framework is freely hinged to a support at D. A weight W is hung from C and the framework is supported with CD horizontal by a force P at B making an angle θ with the horizontal.

If the magnitude of the stress in CA is twice the magnitude of the stress in BC, find P and the stresses in the rods, specifying whether they are tensions or thrusts. (O)

10.



The diagram shows a smoothly jointed framework consisting of five light rigid rods with $AB = BC = AD = 5$ metres and $DC = 11$ metres. The framework is simply supported at A and B and is in equilibrium in a vertical plane with AB and DC horizontal and with loads of 400 N and 200 N at C and D respectively.
(i) Show that the magnitude of the force supporting the framework at B is 520 N.
(ii) Find the magnitude of the force acting in each member of the framework.
(iii) State which members of the framework are in tension. (AEB 1978)

Exercise 10.6 (miscellaneous)

[Take g as $10\,\mathrm{m\,s^{-2}}$.]

1. A uniform plank $ABCD$ of length $6a$ and weight W rests on two supports B and C; $AB = BC = CD = 2a$. A man of weight $4W$ wishes to stand on the plank at A, for

which purpose he places a counter-balancing weight Y at a point P between C and D at a distance x from D. Show that the least value of Y required is $7Wa/(4a - x)$. Show that, if the board is also to be in equilibrium when he is not standing on it, then the greatest value of Y that may be used is $Wa/(2a - x)$. (JMB)

2. A uniform lamina $ABCD$ is in the form of a trapezium in which $DC = AD = a$, $AB = 2a$ and $\angle BAD = \angle CDA = 90°$. When the lamina is freely suspended from a point E in BC the edge AB is vertical. Show that $BE:EC = 4:5$. Find the tangent of the angle which AB makes with the vertical when the lamina is freely suspended from B. The lamina is suspended by two vertical strings attached at C and D. Find the ratio of the tensions in the strings if CD is horizontal. (L)

3. Three equal uniform rods AB, BC, CA, each of weight w, are joined together to form a triangle ABC. A particle of weight W is attached to B and the system is suspended freely from the point A. In equilibrium the rod AB makes an angle θ with the vertical. Prove that $\tan\theta = \sqrt{3}w/(2W + 3w)$. (O)

4. A uniform thin sheet of cardboard of weight W is in the form of a square $ABCD$ of side $6a$. A cut is made along DO, where O is the point of intersection of the diagonals, and the triangular portion AOD is folded over along AO and stuck to the portion AOB with which it now coincides. Find the distances of the centre of gravity of the resulting object from AB and from BC and show that the centre of gravity lies on BD. If this object hangs freely from A, find the angle which BO makes with the horizontal. A particle of weight w is now attached to the object at D so that, in equilibrium, BO is horizontal. Find w. (L)

5. A uniform right circular solid cone of radius r and height h rests with its base on a rough plane which is initially horizontal. The inclination of the plane to the horizontal is gradually increased. Show that the cone will slide before it topples if the coefficient of friction between the plane and the cone is less than $4r/h$. (O)

6. If three non-parallel coplanar forces are in equilibrium, show that they must be concurrent.

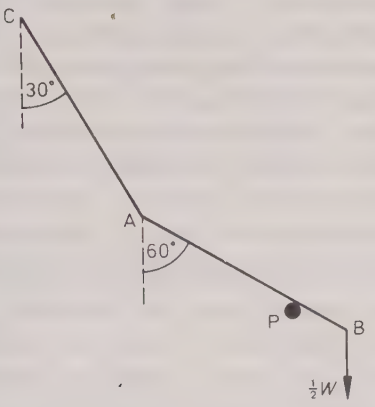
A uniform rod AB of length $2a$ and weight W has one end A in contact with a vertical wall and to the other end B is attached one end of a light string. The other end of the string is attached to the wall at C, the point at a height $2a$ vertically above A. The rod makes an angle α ($0° < \alpha < 90°$) with the downward vertical. Show that equilibrium is impossible if the wall is smooth. If $\alpha = 60°$ and the rod is in equilibrium, show that the least possible value of the coefficient of friction between the rod and the wall is $1/\sqrt{3}$. (L)

7. A uniform rectangular lamina $ABCD$ of mass 1 kg has $AB = 0{\cdot}8$ m and $AD = 0{\cdot}6$ m. The lamina is smoothly pivoted at A. It is kept in equilibrium in a vertical plane with the diagonal AC horizontal and with B as the lowest point by a force P acting at right angles to CD. Find the magnitude of P and the magnitude and direction of the force exerted on the lamina by the pivot if (a) the force P acts through

the mid-point of CD, (b) the force P acts at C. Find also the magnitude and the line of action of P when the force exerted by the pivot acts in the direction BA. (L)

8. A uniform rod AB, of length $2l$ and weight W, is smoothly hinged to a vertical wall at A. It is held at rest in a horizontal position by a light inextensible string attached at B and to a point C which is vertically above and at a distance l from A. Calculate the tension in the string and show that the reaction at A is equal in magnitude to the tension. When a load of weight $4W$ is attached to the rod at a point D the horizontal and vertical components of the reaction at A are equal in magnitude. Find AD. (L)

9. A thin uniform rod AB of length 3 m and mass 5 kg is freely pivoted to a fixed point A. A light elastic string BC of modulus 30 N has one end attached to B and its other end C fixed to a point at the same level as A, where $AC = 5$ m. When the system is in equilibrium $BC = 4$ m. Calculate (a) the tension in the string. (b) the natural length of the string. (c) the resultant force, in magnitude and direction, exerted on the rod at A. (L)

10. 

The diagram shows a uniform rod AB, of length $4c$ and weight W, resting on a rough fixed peg P, where $AP = 3c$. A particle of weight $\frac{1}{2}W$ hangs from B. One end of a light string is tied at A and the other end is tied at a fixed point C. The rod and the string are in the same vertical plane and are inclined at angles of 60° and 30° respectively to the vertical. (i) Show that the tension in the string is $\frac{1}{6}W\sqrt{3}$. (ii) Given that equilibrium is limiting, calculate the coefficient of friction between the peg and the rod. (iii) Given that $AC = 4c$ and that the string is elastic with modulus of elasticity $W\sqrt{3}$, calculate the natural length of the string. (AEB 1979)

11. A uniform square lamina $ABCD$ of weight W and side $2a$ rests in a vertical plane with the vertex A in contact with a rough horizontal plane. The lamina is kept in equilibrium by a force P acting at the vertex C in the direction of DC produced. Given that the height of the vertex D above the horizontal plane is $6a/5$, show that $P = W/10$. Find the coefficient of friction between the lamina and the plane if the lamina is in limiting equilibrium. (L)

12. A uniform rod AB, of weight W and length $2l$, rests in equilibrium with the end A on rough horizontal ground and with the end B in contact with a smooth vertical wall, which is perpendicular to the vertical plane containing the rod. If AB makes an angle α with the horizontal, where $\tan\alpha = 4/3$, find the least possible value of μ, the coefficient of friction between the rod and the ground, for equilibrium to be

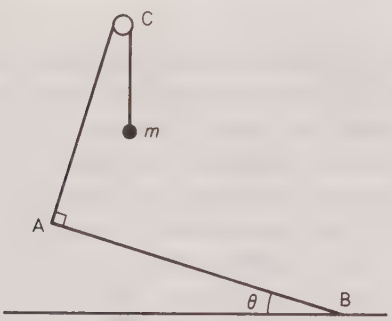
preserved. If $\mu = \frac{1}{2}$, find the distance from A to the highest point of the rod at which a load of weight W can be attached without equilibrium being disturbed. (L)

13. Prove that the position of the centre of gravity of a uniform solid hemisphere of radius a is distant $\frac{3}{8}a$ from the centre of its plane face.

A uniform solid hemisphere rests with its curved surface on a rough horizontal surface and also against a rough vertical surface. The coefficient of friction at both points of contact is $\frac{1}{4}$. If the hemisphere is just about to slip, show that its top surface is inclined at an angle of just under 52° to the horizontal. (SU)

14. A uniform bar, of length $3a$ and mass m, rests in limiting equilibrium at an angle α to the horizontal with one end A on rough horizontal ground and the other end B against a smooth vertical wall. A horizontal force P is now applied to the bar at a point C, where $AC = a$. Find in terms of m, α and g, the value of P which will just cause the bar to slip towards the wall and the corresponding reaction between the bar and the wall.

15.



AB is a uniform rod of mass M. The end B rests on a rough horizontal table (coefficient of friction μ), and a light inextensible string is attached to the end A, passes over a smooth pulley at C and carries a mass m at its other end. The system rests in equilibrium with the string at right angles to AB. Find an expression for θ in terms of M and m and show that equilibrium is not possible unless $2m < M$. Show also that $\mu \geqslant \dfrac{m\sqrt{(M^2 - 4m^2)}}{M^2 - 2m^2}$. (SU)

16. A hemisphere of radius a, whose curved surface is smooth, is fixed with its flat base on a rough horizontal table. A uniform rod AB of length $4a$ and weight W rests in equilibrium against the hemisphere, its point of contact with the hemisphere lying between A and B. The end A of the rod is in contact with the table and AB makes an acute angle θ with the horizontal. Prove that the frictional force F at A has magnitude $2W\sin^2\theta$ and find the normal reaction R at A. If the coefficient of friction between the end A and the table is $\frac{1}{2}$, prove that $3\tan^2\theta + 2\tan\theta - 1 \leqslant 0$ and deduce the maximum value of θ consistent with equilibrium. (O&C)

17. $ABCD$ is a rectangular lamina, freely pivoted at a point X on AB. The length of AB is $2a$ and of BC is a. Forces $3W$, $5W$, $2W$ and $7W$ act along the sides AB, BC, DC and AD respectively in the directions indicated by the order of the letters. Find the position of X so that the lamina is in equilibrium and, in this case, determine the reaction of the pivot on the lamina in magnitude and direction. If, instead, the lamina is pivoted at a point Y on CD and is in equilibrium, deduce from your results the position of Y. (O&C)

18. Explain, with the aid of a diagram, the meaning of the expression coefficient of friction.

A straight uniform rod AB of weight W rests in limiting equilibrium with the end A on horizontal ground and the end B against a vertical wall. The vertical plane containing AB is perpendicular to the wall. The coefficient of friction between the rod and the ground is 4/5. The coefficient of friction between the rod and the wall is 3/5. Given that the inclination of AB to the horizontal is α, calculate (i) the normal reactions at A and B in terms of W, (ii) the numerical value of $\tan\alpha$. (AEB 1980)

19. A uniform right circular heavy cylinder, of height h and radius a, rests with one plane face upon a rough horizontal plane. A gradually increasing horizontal force is applied to a point on the edge of the upper face, and the force passes through the axis of the cylinder. Show that the cylinder will slide before it topples if $\mu h < a$, where μ is the coefficient of friction between the cylinder and the horizontal plane. (O)

20. A uniform solid hemisphere of radius a has mass M. A concentric hemisphere of mass $M/8$ is removed. Show that the centre of mass of the remainder is at a distance $45a/112$ from the centre O of the plane base. A particle of mass M is attached to a point X on the outer edge of the plane base of the remainder. Find, relative to OX and the axis of symmetry of the hemisphere as axes, the coordinates of the centre of mass G of the combined solid (i.e. remainder together with attached mass). The combined solid is placed with its curved surface on a horizontal plane and rests in equilibrium. Find the tangent of the angle between the plane base of the solid and the horizontal. (C)

21. Prove by integration that the centre of mass of a uniform solid right circular cone of height h and base radius r is at a distance $\frac{3}{4}h$ from the vertex. Such a cone is joined to a uniform solid right circular cylinder, of the same material, with base radius r and height l, so that the plane base of the cone coincides with a plane face of the cylinder. Find the centre of mass of the solid thus formed. Show that, if $6l^2 \geqslant h^2$, this solid can rest in equilibrium on a horizontal plane, with the curved surface of the cylinder touching the plane. Given that $l = h$, show that the solid can rest in equilibrium with its conical surface touching the plane provided that $r \geqslant \frac{1}{4}h\sqrt{5}$. (JMB)

22. A toy consists of a solid hemisphere of radius a to which is glued a solid circular cylinder of radius a and height $2a$ so that the plane end of the hemisphere is in complete contact with a plane end of the cylinder. The cylinder is made of uniform material of density ρ, and the hemisphere is made of uniform material of density $k\rho$. The toy is designed so that if placed on a horizontal table with the hemisphere downwards and then tilted to one side, it will return to the vertical position. Show that $k > 8$. The toy is placed on a desk of slope α where $\sin\alpha = 1/8$, sufficiently rough to prevent slipping. It rests in equilibrium with the hemisphere in contact with the desk. Find an expression giving the (acute) angle β made by its axis of symmetry with the vertical. Hence deduce that $k \geqslant 13\frac{1}{2}$. (O&C)

23. Using a diagram, explain the meaning of the term angle of friction. A uniform straight pole AB of length 2·5 metres stands in limiting equilibrium with the end A on horizontal ground and the end B against a vertical wall. The vertical plane containing AB is perpendicular to the wall and A is at a distance 0·7 metres from the wall. The angles of friction between the pole and the ground and between the pole and the wall are equal. Show that the line of action of the resultant force exerted on the pole by the ground at A is perpendicular to the line of action of the resultant force exerted on the pole by the wall at B. Hence, or otherwise, show that the coefficient of friction at A and B is 1/7. Given that the pole weighs 50 N, find the magnitude of the resultant reactions (i) between the pole and the ground, (ii) between the pole and the wall. (AEB 1978)

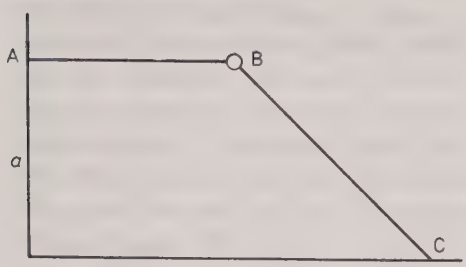
24. A uniform rod of weight W rests in contact with two parallel, horizontal, smooth pegs, passing over the higher peg and under the lower one. The lower end of the rod rests on a smooth horizontal plane. A force of magnitude P acts on the upper end of the rod in the direction of the rod and tending to move the rod upwards. The length of the rod is l, and its inclination to the horizontal is θ. If the upper and lower pegs are at distances a and b respectively from the lower end of the rod, prove that, so long as the rod remains in equilibrium, the reaction of the upper peg on the rod is $\dfrac{Wl\cos\theta - 2Pb\cot\theta}{2(a-b)}$. Given that $l > 2a$, determine which of the three forces exerted on the rod by the two pegs and the ground will first become zero as P is gradually increased from zero. (C)

25. A uniform rod AB, of length l and weight W, is smoothly hinged at A. The point C is vertically above A and is such that $AC = l$. An elastic string, of unstretched length $\frac{1}{2}l$, obeys Hooke's law and has modulus W. The ends of the string are attached to B and C so that the rod rests in equilibrium in an inclined position. Show that the tension in the string is $W/3$. Find the magnitude of the couple which when applied to the rod will maintain the system in equilibrium with $AB = BC$. (L)

26. A uniform rod AB of weight W and length $2a$ has the end A freely hinged to a fixed point. The other end B is freely hinged to a uniform rod BC of weight W and length $6a$. The ends A and C are joined by a light inelastic string whose length is such that AB is perpendicular to BC, and the system hangs in equilibrium. Show that AB is inclined at 45° to the vertical. Find the tension in AC and the horizontal and vertical components of the reaction on BC at B. (C)

27. Two uniform rods AB and BC of the same thickness and material, and of length 4 metres and 3 metres respectively, are freely hinged together and rest in a vertical plane with the ends A and C on a rough horizontal plane. The system is in limiting equilibrium when the angle ABC is 90°. Determine how equilibrium will be broken when the angle ABC is slightly increased beyond 90° and show that the coefficient of friction between the rods and the ground is 84/163. (W)

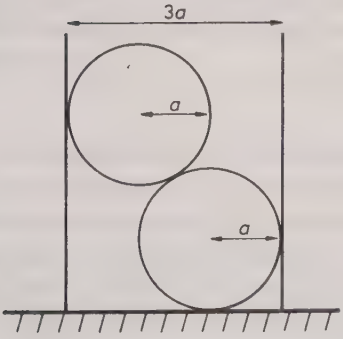
28.



A uniform rod AB of length a and weight W is smoothly jointed at B to a uniform rod BC of length $a\sqrt{2}$ and also of weight W. The system is in equilibrium with C resting on a rough horizontal floor at a distance $2a$ from a rough vertical wall, A resting against the wall, and AB horizontal at a height a above the floor. The plane of the rods is perpendicular to the floor and the wall. (i) Find the frictional force exerted by the wall at A. (ii) Find the vertical and horizontal components of the force exerted on BC at C. (iii) Show that equilibrium is possible only if the coefficient of friction at C is at least $\frac{2}{3}$. (iv) Find the least value of the coefficient of friction at A for equilibrium to be possible. (JMB)

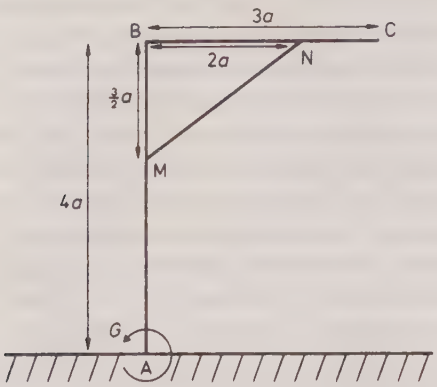
29. Three uniform straight rods, each of weight W, are smoothly jointed together to form an equilateral triangle ABC. The joint B is hinged to a fixed support and the system is free to rotate about B in a vertical plane. The system is held in equilibrium with BC horizontal and A uppermost by a horizontal light string which connects A with a fixed point. Show that the tension in the string is $W\sqrt{3}$ and find the magnitude and direction of the force exerted on the triangle by the support at B. Find the horizontal and vertical components of all the forces which are exerted on the rod AC, and indicate them clearly on your diagram. (JMB)

30. The diagram shows two parallel vertical walls at a distance $3a$ apart, with two uniform smooth cylinders, each of radius a and weight W. The axes of the cylinders are horizontal and parallel to the walls. Each cylinder is in contact with the other and with one wall; the lower cylinder rests on horizontal ground. Find the magnitudes of the reactions between the two cylinders and between each cylinder and the corresponding wall. (C)



31. A uniform sphere of radius a and weight $W/\sqrt{3}$ rests on a rough horizontal table. A uniform rod AB of weight $2W$ and length $2a$ is freely hinged at A to a fixed point on the table and leans against the sphere so that the centre of the sphere and the rod lie in a vertical plane. The rod makes an angle of $60°$ with the horizontal. Show that the frictional force between the rod and the sphere is $\frac{1}{3}W$. The coefficient of friction at each point of contact is μ. What is the smallest value of μ which makes equilibrium possible? (O&C)

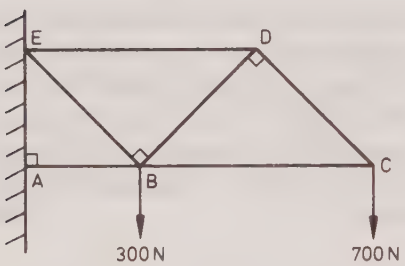
32.



The diagram shows a uniform post AB, of length $4a$ and weight W, smoothly hinged to the ground at A. A horizontal uniform beam BC, of length $3a$ and weight W, is smoothly hinged to AB at B. The beam is supported by a light inelastic rod MN smoothly jointed to AB at M and to BC at N, where $BM = \frac{1}{2}a$ and $BN = 2a$. A weight $3W$ hangs from C, and the system is maintained in equilibrium, with AB vertical, by an external couple of magnitude G applied to AB, as shown.

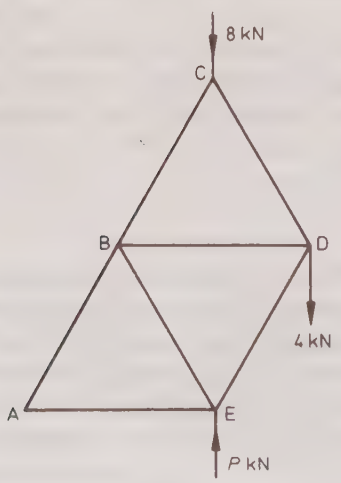
Find (i) the thrust in the rod MN, (ii) the horizontal and vertical components of the forces on the post AB at A and at B, (iii) the value of G. (C)

33.



The diagram shows a framework $ABCDE$, consisting of 6 light rigid rods which are smoothly jointed. The framework is smoothly hinged to a vertical wall at A and E and the triangles ABE, BED and BCD are all right-angled and isosceles. When loads of 300 N and 700 N are hung from B and C respectively, the framework is in equilibrium in a vertical plane. (i) Explain why the reaction on the hinge at A is horizontal and find its magnitude. (ii) Find the magnitude and the direction of the reaction on the hinge at E. (iii) Find the magnitude of the force acting in each member of the framework and state which rods are in compression. (AEB 1978)

34.

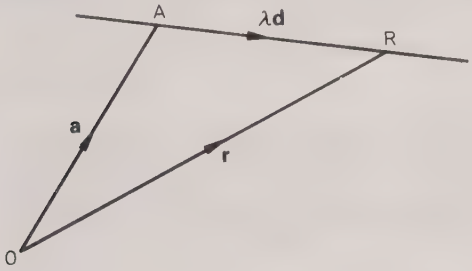


The diagram shows a framework of 7 light equal rods, smoothly jointed at their ends. It is freely hinged to a vertical wall at A. Forces of 8 kN and 4 kN are applied at C and D as shown. A force of size P kN is applied vertically upwards at E in order to keep the rod AE in a horizontal position. Calculate (i) the magnitude of P; (ii) the stresses in the rods, stating if they are in tension or under compression; (iii) the magnitude and direction of the reaction of the support at A on the frame. (SU)

11 Vector methods

11.1 Forces and their lines of action

In this chapter we consider the application of vector methods to systems of forces given in vector form, often in terms of unit vectors **i**, **j** and **k**.



The diagram shows a straight line which passes through a fixed point A with position vector **a** and which is parallel to a given vector **d**. Let **r** be the position vector of a point R on the line.

Since **AR** is parallel to **d**, then, for some scalar λ,

$$\mathbf{AR} = \lambda\mathbf{d}$$

$$\therefore \quad \mathbf{r} - \mathbf{a} = \lambda\mathbf{d}$$

$$\text{i.e.} \quad \mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$$

Thus every point on the line has a position vector of the form $\mathbf{a} + \lambda\mathbf{d}$. Moreover, each real value of the parameter λ corresponds to some point on the line.

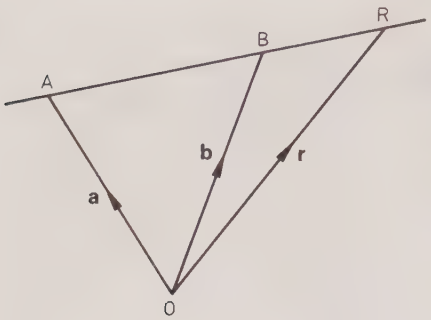
The equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$ is called the *vector equation* of the line through A parallel to **d**. The vector **d** is a *direction vector* of the line.

Example 1 Find the vector equation of the line which passes through the point with position vector $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and which is parallel to the vector $2\mathbf{j} - \mathbf{k}$.

The vector equation of the given line is

$$\mathbf{r} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k} + \lambda(2\mathbf{j} - \mathbf{k})$$

$$\text{i.e.} \quad \mathbf{r} = 2\mathbf{i} + (2\lambda - 1)\mathbf{j} + (3 - \lambda)\mathbf{k}.$$



Consider now the line through two points A and B with position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. If a point R, with position vector $\mathbf{r}$, lies on this line, then for some scalar λ,

$$\mathbf{AR} = \lambda\mathbf{AB}$$
$$\therefore \quad \mathbf{r} - \mathbf{a} = \lambda(\mathbf{b} - \mathbf{a})$$
$$\therefore \quad \mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}).$$

Hence the vector equation of the line through A and B is $\mathbf{r} = (1 - \lambda)\mathbf{a} + \lambda\mathbf{b}$.

Example 2 Find the vector equation of the straight line which passes through the points $A(1, 0, -2)$ and $B(2, 3, -1)$.

The position vectors of A and B are $\mathbf{i} - 2\mathbf{k}$ and $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ respectively.
Hence the vector equation of the line through A and B is

$$\mathbf{r} = (1 - \lambda)(\mathbf{i} - 2\mathbf{k}) + \lambda(2\mathbf{i} + 3\mathbf{j} - \mathbf{k})$$
i.e. $$\mathbf{r} = (\lambda + 1)\mathbf{i} + 3\lambda\mathbf{j} + (\lambda - 2)\mathbf{k}.$$

Example 3 A force $\mathbf{P}$ of magnitude 28 N acts at the point A with position vector $2\mathbf{j} + \mathbf{k}$ in the direction of the vector $3\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$. A second force $\mathbf{Q}$ of magnitude 27 N also acts at A, but in the direction of the vector $\mathbf{i} - 4\mathbf{j} - 8\mathbf{k}$. Find the magnitude of the resultant of the forces $\mathbf{P}$ and $\mathbf{Q}$. Obtain a vector equation for its line of action.

$$|3\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}| = \sqrt{\{3^2 + (-2)^2 + 6^2\}} = \sqrt{\{9 + 4 + 36\}} = 7$$
$$|\mathbf{i} - 4\mathbf{j} - 8\mathbf{k}| = \sqrt{\{1^2 + (-4)^2 + (-8)^2\}} = \sqrt{\{1 + 16 + 64\}} = 9$$

$\therefore$ the unit vectors in the directions of $\mathbf{P}$ and $\mathbf{Q}$ are $\frac{1}{7}(3\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})$ and $\frac{1}{9}(\mathbf{i} - 4\mathbf{j} - 8\mathbf{k})$ respectively.

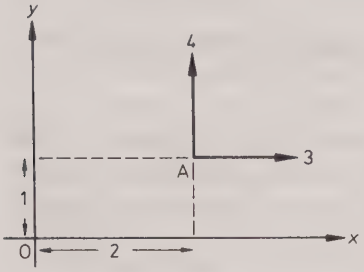
Thus $$\mathbf{P} = 28 \times \tfrac{1}{7}(3\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}) = 12\mathbf{i} - 8\mathbf{j} + 24\mathbf{k}$$
and $$\mathbf{Q} = 27 \times \tfrac{1}{9}(\mathbf{i} - 4\mathbf{j} - 8\mathbf{k}) = 3\mathbf{i} - 12\mathbf{j} - 24\mathbf{k}$$
$$\therefore \quad \mathbf{P} + \mathbf{Q} = 15\mathbf{i} - 20\mathbf{j} = 5(3\mathbf{i} - 4\mathbf{j})$$
$$\therefore \quad |\mathbf{P} + \mathbf{Q}| = 5\sqrt{\{3^2 + (-4)^2\}} = 5\sqrt{\{9 + 16\}} = 25$$

Hence the resultant of $\mathbf{P}$ and $\mathbf{Q}$ acts in the direction of the vector $3\mathbf{i} - 4\mathbf{j}$ and has magnitude 25 N.
Since $\mathbf{P}$ and $\mathbf{Q}$ act at A, their resultant must also act at A. Thus a vector equation for its line of action is

$$\mathbf{r} = 2\mathbf{j} + \mathbf{k} + \lambda(3\mathbf{i} - 4\mathbf{j})$$
i.e. $$\mathbf{r} = 3\lambda\mathbf{i} + (2 - 4\lambda)\mathbf{j} + \mathbf{k}.$$

The approach of Example 3 can also be used to find the magnitude and direction of the resultant of a set of coplanar forces with different points of application. However, to find the line of action of the resultant we must use the principle of moments introduction in §9.1.

[Note that unless otherwise stated we will assume that vector quantities are measured in the appropriate SI units.]



To illustrate how the moments of individual forces are calculated, we suppose that a force $\mathbf{F} = 3\mathbf{i} + 4\mathbf{j}$ acts at the point A with position vector $2\mathbf{i} + \mathbf{j}$.

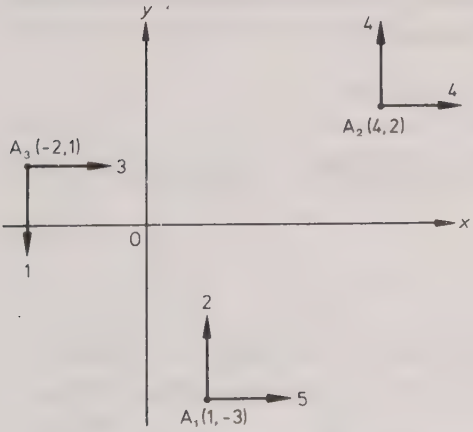
Taking anti-clockwise moments as positive, the moment of $\mathbf{F}$ about O

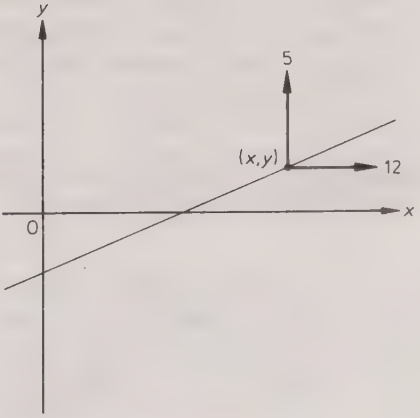
= (moment about O of the force $3\mathbf{i}$) + (moment about O of the force $4\mathbf{j}$)
$= -(3 \times 1) + (4 \times 2)\,\mathrm{N\,m} = 5\,\mathrm{N\,m}.$

Example 4 Coplanar forces $\mathbf{F}_1 = 5\mathbf{i} + 2\mathbf{j}$, $\mathbf{F}_2 = 4\mathbf{i} + 4\mathbf{j}$ and $\mathbf{F}_3 = 3\mathbf{i} - \mathbf{j}$ act at points A_1, A_2 and A_3 with position vectors $\mathbf{i} - 3\mathbf{j}, 4\mathbf{i} + 2\mathbf{j}$ and $-2\mathbf{i} + \mathbf{j}$ respectively. Find the magnitude of the resultant of these forces and the Cartesian equation of its line of action.

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = (5 + 4 + 3)\mathbf{i} + (2 + 4 - 1)\mathbf{j} = 12\mathbf{i} + 5\mathbf{j}$$
$$\therefore\ |\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3| = \sqrt{(12^2 + 5^2)} = \sqrt{(144 + 25)} = 13.$$

Hence the magnitude of the resultant of the three forces is 13 N.





Let the point with position vector $x\mathbf{i} + y\mathbf{j}$ be some point on the line of action of the resultant, then the anti-clockwise moment of the resultant about O is $5x - 12y$.

Taking moments anti-clockwise about O for the original forces,

$$\text{moment of } \mathbf{F}_1 = 5 \times 3 + 2 \times 1 = 17$$
$$\text{moment of } \mathbf{F}_2 = -4 \times 2 + 4 \times 4 = 8$$
$$\text{moment of } \mathbf{F}_3 = -3 \times 1 + 1 \times 2 = -1$$

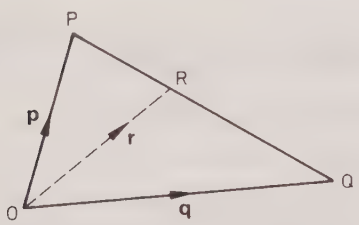
$\therefore$ using the principle of moments,

$$5x - 12y = 17 + 8 - 1$$

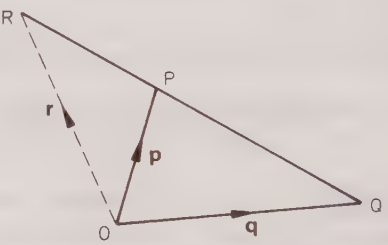
Hence the Cartesian equation of the line of action of the resultant is $5x - 12y = 24$.

Let us consider three points P, Q and R such that R divides PQ in the ratio $\mu:\lambda$. We will observe the convention that (i) if λ and μ have the same sign then R lies between P and Q, and (ii) if λ and μ have opposite signs then R does not lie between P and Q.

(i)



(ii)



In both cases $\quad \lambda\mathbf{PR} = \mu\mathbf{RQ}$

$\therefore \quad \lambda(\mathbf{OR} - \mathbf{OP}) = \mu(\mathbf{OQ} - \mathbf{OR})$

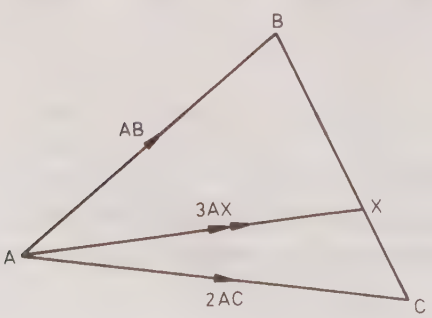
$\therefore \quad \lambda\mathbf{OR} - \lambda\mathbf{OP} = \mu\mathbf{OQ} - \mu\mathbf{OR}$

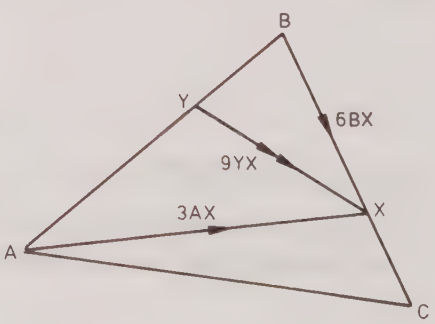
Hence $\quad (\lambda + \mu)\mathbf{OR} = \lambda\mathbf{OP} + \mu\mathbf{OQ}$

This result is called the *ratio theorem* and may also be stated as follows:

> If two forces acting at a point O are represented in magnitude and direction by $\lambda\mathbf{OP}$ and $\mu\mathbf{OQ}$, then their resultant is represented by $(\lambda + \mu)\mathbf{OR}$, where R is the point which divides PQ in the ratio $\mu:\lambda$.

Example 5 Forces represented by **AB**, 2**AC** and 4**BC** act along the sides of a triangle ABC. Find their resultant.





By the ratio theorem, $\mathbf{AB} + 2\mathbf{AC} = 3\mathbf{AX}$,
where X is the point which divides BC in the ratio 2:1.

Since $BX:XC = 2:1$, we have $BC = \frac{3}{2}BX$,

$\therefore \quad \mathbf{AB} + 2\mathbf{AC} + 4\mathbf{BC} = 3\mathbf{AX} + 6\mathbf{BX}$.

By the ratio theorem, $3\mathbf{AX} + 6\mathbf{BX} = 9\mathbf{YX}$,

where Y is the point which divides AB in the ratio 6:3 i.e. 2:1. Hence the resultant of the three forces acts through X and is represented by $9\mathbf{YX}$, where X divides BC in the ratio 2:1 and Y divides AB in the ratio 2:1.

Exercise 11.1

1. Forces of magnitude 10 N and 5 N act in the directions of the vectors $3\mathbf{i} - 4\mathbf{j}$ and $4\mathbf{i} + 3\mathbf{j}$ respectively. Express each force as a vector in terms of $\mathbf{i}$ and $\mathbf{j}$. Hence find the resultant of the forces as a vector.

2. A force $\mathbf{R}$ of magnitude $4\sqrt{2}$ N acting in the direction of the vector $\mathbf{i} - \mathbf{j}$ is the resultant of two forces $\mathbf{P}$ and $\mathbf{Q}$ which act in the directions of the vectors $\mathbf{i} + \mathbf{j}$ and $\mathbf{i} - 7\mathbf{j}$ respectively. Find the magnitudes of $\mathbf{P}$ and $\mathbf{Q}$.

3. In a parallelogram $OABC$ the position vectors of A and B relative to O are $\mathbf{a} = 4\mathbf{i} + 3\mathbf{j}$ and $\mathbf{b} = 3\mathbf{j} + 3\mathbf{k}$ respectively. Forces of magnitude 20 N, $9\sqrt{2}$ N and 5 N act along OA, OB and OC respectively. Find the magnitude of their resultant.

4. In each of the following cases obtain a vector equation for the line of action of the resultant of forces $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$ acting at a point with position vector $\mathbf{a}$.
(a) $\mathbf{F}_1 = 2\mathbf{i} - 3\mathbf{j}$, $\mathbf{F}_2 = \mathbf{i} + 5\mathbf{j}$, $\mathbf{F}_3 = -\mathbf{i} + 2\mathbf{j}$; $\mathbf{a} = \mathbf{i} + \mathbf{j}$.
(b) $\mathbf{F}_1 = -3\mathbf{i} + 2\mathbf{k}$, $\mathbf{F}_2 = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{F}_3 = 4\mathbf{j}$; $\mathbf{a} = 2\mathbf{j} - \mathbf{k}$.
(c) $\mathbf{F}_1 = \mathbf{i} - 5\mathbf{j} + 8\mathbf{k}$, $\mathbf{F}_2 = 2\mathbf{i} - \mathbf{j}$, $\mathbf{F}_3 = -3\mathbf{i} + \mathbf{k}$; $\mathbf{a} = 3\mathbf{i}$.

5. Forces $\mathbf{P} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{Q} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$ act through points with position vectors $2\mathbf{i} - \mathbf{j}$ and $9\mathbf{i} - \mathbf{k}$ respectively. Write down vector equations for the lines of action of $\mathbf{P}$ and $\mathbf{Q}$. Hence find the position vector of the point through which both forces act. Find also a vector equation for the line of action of the resultant of $\mathbf{P}$ and $\mathbf{Q}$.

6. Forces $\mathbf{F}_1 = \mathbf{i} + 7\mathbf{j} + 4\mathbf{k}$ and $\mathbf{F}_2 = 5\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$ act through the origin O. A third force $\mathbf{F}_3 = \mathbf{i} - 4\mathbf{j} + 6\mathbf{k}$ acts through the point A with position vector $\mathbf{a} = 3\mathbf{i} + 5\mathbf{j}$. Show that the resultant of $\mathbf{F}_1$ and $\mathbf{F}_2$ also acts through A. Find the magnitude of the resultant of all three forces and obtain a vector equation for its line of action.

7. Coplanar forces $\mathbf{F}_1 = \mathbf{i} - 2\mathbf{j}$, $\mathbf{F}_2 = 3\mathbf{i} + 4\mathbf{j}$, $\mathbf{F}_3 = -5\mathbf{i} + \mathbf{j}$ act at points A_1, A_2, A_3 with position vectors $4\mathbf{i} + 5\mathbf{j}$, $3\mathbf{i} - 2\mathbf{j}$, $\mathbf{i} + 4\mathbf{j}$ respectively. Find the Cartesian equation of the line of action of the resultant of these forces.

8. Coplanar forces $\mathbf{F}_1 = 6\mathbf{i} + 2\mathbf{j}, \mathbf{F}_2 = -\mathbf{i} + 7\mathbf{j}, \mathbf{F}_3 = 4\mathbf{i} - 3\mathbf{j}$ act at points A_1, A_2, A_3 with position vectors $\mathbf{i} - \mathbf{j}, 4\mathbf{j}, 2\mathbf{i} + 3\mathbf{j}$ respectively. Find the magnitude of the resultant and the tangent of the angle it makes with the vector $\mathbf{i}$. Find also the Cartesian equation of its line of action.

9. The following sets of forces act along the edges of a triangle ABC. In each case find the resultant force, using a clear diagram to illustrate your answer.

(a) $2\mathbf{AB}, 3\mathbf{AC}, 6\mathbf{BC}$,
(b) $\mathbf{CB}, 3\mathbf{AB}, 5\mathbf{CA}$,
(c) $2\mathbf{BC}, \mathbf{BA}, 2\mathbf{AC}$,
(d) $6\mathbf{AB}, 4\mathbf{CA}, 3\mathbf{CB}$.

10. Three forces acting at a point P are represented by $\mathbf{PA}, \mathbf{PB}$ and $\mathbf{PC}$. Show that their resultant is given by $3\mathbf{PG}$, where G is the centroid of triangle ABC.

11. Forces represented by $3\mathbf{AB}, 2\mathbf{CB}, 4\mathbf{CD}$ and $6\mathbf{AD}$ act along the sides of a square $ABCD$. Show that their resultant is $15\mathbf{XY}$, where X divides AC in the ratio 2:3 and Y divides BD in the ratio 2:1.

12. Three forces represented by $\mathbf{PQ}, \mathbf{QR}$ and $\mathbf{RP}$ act along the sides of a triangle PQR. Show that these forces are equivalent to a couple whose moment is represented by twice the area of the triangle.

11.2 Use of vectors in kinematics and dynamics

Suppose that a particle P moving with constant velocity $\mathbf{v}$ has position vector $\mathbf{r}_0$ initially and position vector $\mathbf{r}$ at time t.

It follows that $$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}.$$

This equation gives the position vector of P at any instant and can be taken as the vector equation of the straight line path of the particle.

The next example shows the use of such equations in a problem concerning the *relative motion* of two ships.

Example 1 At noon ships A and B have position vectors $3\mathbf{i} + 2\mathbf{j}$ and $\mathbf{i} - 4\mathbf{j}$ respectively and constant velocities $4\mathbf{i} + 3\mathbf{j}$ and $3\mathbf{i} + 5\mathbf{j}$ respectively, where distances are measured in kilometres and times in hours. Find vectors to represent (a) the velocity of A relative to B, (b) the displacement of A relative to B, t hours after noon. Find also the least distance between the ships.

(a) The velocity of A relative to B

$$= (\text{velocity of } A) - (\text{velocity of } B)$$
$$= (4\mathbf{i} + 3\mathbf{j}) - (3\mathbf{i} + 5\mathbf{j}) = \mathbf{i} - 2\mathbf{j}.$$

(b) The position vector of A after t hours is given by

$$\mathbf{r}_A = 3\mathbf{i} + 2\mathbf{j} + t(4\mathbf{i} + 3\mathbf{j})$$

The position vector of B after t hours is given by

$$\mathbf{r}_B = \mathbf{i} - 4\mathbf{j} + t(3\mathbf{i} + 5\mathbf{j})$$

$\therefore$ the displacement of A relative to B after t hours is

$$\mathbf{r}_A - \mathbf{r}_B = 2\mathbf{i} + 6\mathbf{j} + t(\mathbf{i} - 2\mathbf{j})$$

i.e. $\mathbf{r}_A - \mathbf{r}_B = (2 + t)\mathbf{i} + (6 - 2t)\mathbf{j}$.

The distance between the ships

$$\begin{aligned} = |\mathbf{r}_A - \mathbf{r}_B| &= \sqrt{\{(2 + t)^2 + (6 - 2t)^2\}} \\ &= \sqrt{\{4 + 4t + t^2 + 36 - 24t + 4t^2\}} \\ &= \sqrt{\{5t^2 - 20t + 40\}} \\ &= \sqrt{\{5(t - 2)^2 + 20\}} \end{aligned}$$

Since $(t - 2)^2$ is never negative, this expression takes its least value when $t = 2$. Hence the least distance between the ships is $\sqrt{20}$ km i.e. $2\sqrt{5}$ km.

[Note that the least distance can also be found by writing

$$f(t) = 5t^2 - 20t + 40$$

Differentiating, $f'(t) = 10t - 20, \quad f''(t) = 10$

$\therefore \quad f'(t) = 0 \Leftrightarrow 10t - 20 = 0 \Leftrightarrow t = 2.$

Since $f(2) = 20$ and $f''(2) > 0$, the function $f(t)$ has a minimum value of 20.]

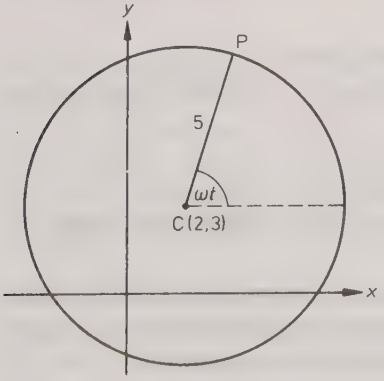
In general, if a moving particle P has position vector $\mathbf{r}$ at time t, then its velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ are given by

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} \quad \text{and} \quad \mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}$$

or, using a dot to indicate differentiation with respect to time,

$$\mathbf{v} = \dot{\mathbf{r}} \quad \text{and} \quad \mathbf{a} = \ddot{\mathbf{r}}.$$

Example 2 A particle P has position vector $\mathbf{r}$ at time t and moves in a circle with constant angular velocity ω. The centre of the circle is the point C with position vector $\mathbf{c} = 2\mathbf{i} + 3\mathbf{j}$, and when $t = 0$, $\mathbf{CP} = 5\mathbf{i}$. Find an expression for $\mathbf{r}$ and hence for the velocity $\mathbf{v}$ of the particle at time t. Find also the time at which P first moves parallel to the vector $\sqrt{3}\mathbf{i} - \mathbf{j}$.



Since P moves on a circle with centre C, the vector $\mathbf{CP}$ has constant magnitude and rotates with constant angular velocity ω. Initially $\mathbf{CP} = 5\mathbf{i}$ and $|\mathbf{CP}| = |5\mathbf{i}| = 5$. Thus, at time t, the

vector **CP** has magnitude 5 units and makes an angle ωt with the direction of **i**. It follows that

$$\mathbf{CP} = 5\cos\omega t\mathbf{i} + 5\sin\omega t\mathbf{j}$$
$$\therefore \quad \mathbf{r} - \mathbf{c} = 5\cos\omega t\mathbf{i} + 5\sin\omega t\mathbf{j}$$
$$\therefore \quad \mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + 5\cos\omega t\mathbf{i} + 5\sin\omega t\mathbf{j}$$

Differentiating with respect to t,

$$\mathbf{v} = \dot{\mathbf{r}} = -5\omega\sin\omega t\mathbf{i} + 5\omega\cos\omega t\mathbf{j}.$$

Thus when P is moving parallel to the vector $\sqrt{3}\mathbf{i} - \mathbf{j}$ there is a scalar λ such that

$$-5\omega\sin\omega t\mathbf{i} + 5\omega\cos\omega t\mathbf{j} = \lambda(\sqrt{3}\mathbf{i} - \mathbf{j})$$

i.e. $-5\omega\sin\omega t = \lambda\sqrt{3}$ and $5\omega\cos\omega t = -\lambda$

$$\therefore \quad \tan\omega t = \frac{\sin\omega t}{\cos\omega t} = \sqrt{3}$$

Hence P first moves parallel to the vector $\sqrt{3}\mathbf{i} - \mathbf{j}$ when $\omega t = \pi/3$ i.e. when $t = \pi/3\omega$.

[Note that at this instant P is not moving in the direction of the vector $\sqrt{3}\mathbf{i} - \mathbf{j}$, but in the opposite direction.]

In problems concerning the forces on moving objects *Newton's second law* is useful. It states that if a particle of mass m moves under the action of a force **F** with acceleration **a**, then

$$\mathbf{F} = m\mathbf{a}.$$

Example 3 A particle of mass 3 kg moves so that its position vector after t seconds is $\mathbf{r} = 8t\mathbf{i} - 5\mathbf{j} + 2t^3\mathbf{k}$. Find (a) the speed of the particle when $t = 1$, (b) the force acting on the particle when $t = 3$.

Let **v** and **a** be the velocity and acceleration of the particle after t seconds, then

$$\mathbf{v} = \dot{\mathbf{r}} = 8\mathbf{i} + 6t^2\mathbf{k}$$
and $\mathbf{a} = \ddot{\mathbf{r}} = 12t\mathbf{k}$.

(a) When $t = 1$, $\mathbf{v} = 8\mathbf{i} + 6\mathbf{k}$

$$\therefore \quad v = |8\mathbf{i} + 6\mathbf{k}| = \sqrt{(8^2 + 6^2)} = 10.$$

Hence, when $t = 1$, the speed of the particle is $10\,\mathrm{m\,s^{-1}}$.

(b) Let **F** be the force on the particle after t seconds, then by Newton's second law,

$$\mathbf{F} = 3\mathbf{a}.$$

When $t = 3$, $\mathbf{a} = 36\mathbf{k}$ $\quad\therefore\quad$ $\mathbf{F} = 108\mathbf{k}$.

Hence, when $t = 3$, the force acting on the particle is $108\mathbf{k}$ (in newtons).

Example 4 A particle of unit mass is acted upon at time t by a force **F**, where $\mathbf{F} = 4\mathbf{i} - 6t\mathbf{j}$. At time $t = 0$ the particle is at the point with position vector $\mathbf{i} + \mathbf{j}$ and its velocity is $5\mathbf{i}$. Find the position vector of the particle at time $t = 2$.

Let $\mathbf{r}$, $\mathbf{v}$ and $\mathbf{a}$ be the position vector, velocity and acceleration of the particle at time t.
By Newton's second law, $\mathbf{F} = m\mathbf{a}$.
Since the particle has unit mass, $\mathbf{F} = \mathbf{a}$.

$$\therefore \qquad \mathbf{a} = \frac{d\mathbf{v}}{dt} = 4\mathbf{i} - 6t\mathbf{j}$$

Integrating, $\qquad \mathbf{v} = 4t\mathbf{i} - 3t^2\mathbf{j} + \mathbf{c},$

where $\mathbf{c}$ is a constant vector.
When $t = 0$, $\mathbf{v} = 5\mathbf{i}$, $\qquad \therefore \quad \mathbf{c} = 5\mathbf{i}$

$$\therefore \qquad \mathbf{v} = \frac{d\mathbf{r}}{dt} = (4t + 5)\mathbf{i} - 3t^2\mathbf{j}$$

Integrating, $\qquad \mathbf{r} = (2t^2 + 5t)\mathbf{i} - t^3\mathbf{j} + \mathbf{d},$

where $\mathbf{d}$ is a constant vector.
When $t = 0$, $\mathbf{r} = \mathbf{i} + \mathbf{j}$, $\qquad \therefore \quad \mathbf{d} = \mathbf{i} + \mathbf{j}$
$\therefore \quad \mathbf{r} = (2t^2 + 5t + 1)\mathbf{i} + (1 - t^3)\mathbf{j}$

Hence, when $t = 2$, the position vector of the particle is

$$\mathbf{r} = (8 + 10 + 1)\mathbf{i} + (1 - 8)\mathbf{j} = 19\mathbf{i} - 7\mathbf{j}.$$

Exercise 11.2

1. Particles A and B are moving with constant velocities $5\mathbf{i} + 2\mathbf{j}$ and $2\mathbf{i} + \mathbf{j}$ respectively. Initially the particle A has position vector $3\mathbf{i} + 4\mathbf{j}$ and the particle B has position vector $5\mathbf{i} + 8\mathbf{j}$. Write down expressions for $\mathbf{r}_A$ and $\mathbf{r}_B$, the position vectors of the particles at time t. Find the vector $\mathbf{AB}$ at time t. Hence find the minimum distance between the particles.

2. Particles P and Q are moving with constant velocities $3\mathbf{i} - 2\mathbf{j}$ and $-\mathbf{i} + 4\mathbf{j}$ respectively. Initially the particle P has position vector $\mathbf{i} + 5\mathbf{j}$ and the particle Q has position vector $3\mathbf{i} + 2\mathbf{j}$. Find the velocity of P relative to Q and the displacement of P relative to Q at time t. Show that the particles collide and state the value of t when the collision occurs.

3. At noon ships A and B have position vectors $10\mathbf{i} + 3\mathbf{j}$ and $12\mathbf{i} - 15\mathbf{j}$ respectively. Both ships move with constant velocity, the velocity of A being $3\mathbf{i} + 8\mathbf{j}$ and the velocity of B relative to A being $-5\mathbf{i} + 4\mathbf{j}$. The unit of distance is the kilometre and the unit of time is the hour. Find an expression for the vector $\mathbf{AB}$, t hours after noon. Hence find the shortest distance between the ships. Find also the position vectors of the ships A and B when they are closest together.

4. A particle P is moving with constant angular velocity ω on a circle of radius 8 units, with centre at the origin. Given that initially P has position vector $8\mathbf{i}$, find its position vector $\mathbf{r}$ at time t. Hence show that $\ddot{\mathbf{r}} = -\omega^2\mathbf{r}$.

5. A particle P moves so that its position vector at time t is given by $\mathbf{r} = \mathbf{i} - 2\mathbf{j} + 3(\mathbf{i}\sin 2t + \mathbf{j}\cos 2t)$. Show that P is moving on a circle, stating the position vector of the centre of the circle and its radius. Find also the speed at which P is moving.

6. Two particles A and B are moving such that, at time t, their position vectors are $\mathbf{r}_A = 3(1 + \cos t)\mathbf{i} + (2\sin t)\mathbf{j}$ and $\mathbf{r}_B = (5 - 6t)\mathbf{i} + (3 + 4t)\mathbf{j}$. Find the value of t when the particles are first moving (a) in the same direction, (b) in opposite directions.

7. The position vector of a particle P at time t is $\mathbf{r} = (1 - t^2)\mathbf{i} + (3t - 5t^2)\mathbf{j}$. Find the time at which P is moving (a) directly towards the origin, (b) directly away from the origin.

8. A particle of mass m moves so that its position vector at time t is $\mathbf{r}$. Find its velocity $\mathbf{v}$ and the force $\mathbf{F}$ acting on it, given that

(a) $\mathbf{r} = 4t^2\mathbf{i} + (3t^3 - 2t)\mathbf{j}$,
(b) $\mathbf{r} = 5\mathbf{i} - 2t^3\mathbf{j} + (t^2 - 1)\mathbf{k}$,
(c) $\mathbf{r} = (2t - t^2)\mathbf{i} + (3\sin 2t)\mathbf{j}$,
(d) $\mathbf{r} = (te^{-t})\mathbf{i} - 2t\mathbf{j}$.

9. A particle of mass 5 kg moves so that its position vector after t seconds is $\mathbf{r} = (\cos 2t)\mathbf{i} + (4\sin 2t + 3)\mathbf{j}$. Find (a) the speed of the particle when $t = \pi/3$, (b) the force acting on the particle when $t = \pi/2$.

10. A particle of mass 2 kg starts from rest at the origin and moves under the action of the force $\mathbf{F}$, where $\mathbf{F} = 8\mathbf{i} + 20\mathbf{j}$. Find the velocity and the position vector of the particle after 3 seconds.

11. A particle of unit mass starts from rest at the point with position vector $2\mathbf{i} - 3\mathbf{j}$. At time t the force acting on the particle is $12t\mathbf{i} - 16\mathbf{j}$. Find the speed of the particle and its position vector at time $t = 2$.

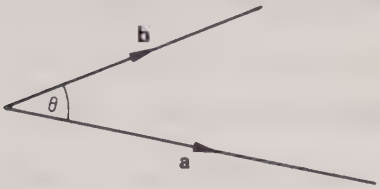
12. A particle of mass 3 kg is acted upon at time t by a force $\mathbf{F}$, where $\mathbf{F} = 6\mathbf{i} - 36t^2\mathbf{j} + 54t\mathbf{k}$. At time $t = 0$ the particle is at the point with position vector $\mathbf{i} - 5\mathbf{j} - \mathbf{k}$ and its velocity is $3(\mathbf{i} + \mathbf{j})$. Find the position vector of the particle at time $t = 1$.

13. A particle of mass 2 kg moves from rest at the origin under the action of two forces. One force $\mathbf{P}$ has magnitude 22 N and acts in the direction of the vector $2\mathbf{i} - 6\mathbf{j} + 9\mathbf{k}$. The other force $\mathbf{Q}$ has magnitude 30 N and acts in the direction of the vector $4\mathbf{j} - 3\mathbf{k}$. Find the acceleration of the particle and the distance it travels in the first 3 seconds of its motion.

14. A particle P is projected vertically upwards from a point O with a speed of $16\,\mathrm{m\,s^{-1}}$. At the same instant a second particle Q is projected horizontally from a point A, 25 m vertically above O, with speed $12\,\mathrm{m\,s^{-1}}$. Using $\mathbf{i}$ and $\mathbf{j}$ as unit vectors in the horizontal and upward vertical directions respectively, find expressions for the position vectors of P and Q with respect to O at time t after projection. Hence find the least distance between the particles.

[Take g as $10\,\mathrm{m\,s^{-2}}$.]

11.3 Scalar product and its applications



The *scalar product* (or dot product) of two vectors **a** and **b** is defined as the scalar $ab\cos\theta$, where θ is the angle between **a** and **b**.

The scalar product is denoted by **a . b**, which is read "**a** dot **b**".

Thus $$\mathbf{a}.\mathbf{b} = ab\cos\theta$$

The scalar product **a . a** is sometimes written as $\mathbf{a}^2$.

[Note that although a product of three real numbers a, b, c can be evaluated by writing $abc = (ab)c$, it is not possible to define a scalar product of three vectors in a similar way. Since **a . b** is a scalar, the expressions (**a . b**) **. c** and **a . b . c** are meaningless. Similarly, since **a . a** is a scalar, the expression $\mathbf{a}^3$ cannot be defined as (**a . a**) **. a**.]

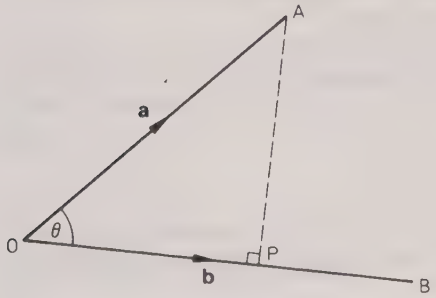
The definition of the scalar product gives rise to various useful properties.

(4) For any vector **a**, $$\mathbf{a}.\mathbf{a} = a^2\cos 0° = a^2.$$

(5) If non-zero vectors **a** and **b** are perpendicular, then

$$\mathbf{a}.\mathbf{b} = ab\cos 90° = 0.$$

Conversely, if non-zero vectors **a** and **b** are such that **a . b** $= 0$, then **a** is perpendicular to **b**.

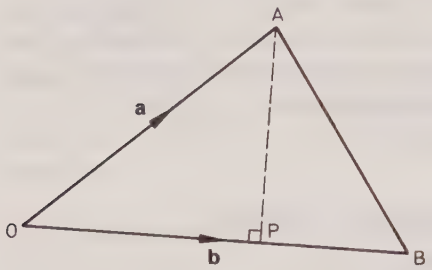


The diagram shows two vectors **a** and **b** represented by line segments $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively. If the vector **a** is resolved along and perpendicular to **b**, then its component in the direction of **b** is $a\cos\theta$.

Thus the component of **a** in the direction of **b** is $\dfrac{\mathbf{a}.\mathbf{b}}{b}$ or $\mathbf{a}.\hat{\mathbf{b}}$, where $\hat{\mathbf{b}}$ is the unit vector in the direction of **b**.

In geometrical work the quantity $\mathbf{a}.\hat{\mathbf{b}}$ is called the *projection* of **a** on **b**. In the diagram the length of this projection is represented by the distance OP.

Example 1 The points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ with respect to an origin O. Show that the area of triangle OAB is given by $\frac{1}{2}\sqrt{\{a^2b^2 - (\mathbf{a}.\mathbf{b})^2\}}$.



The projection of $\mathbf{a}$ on $\mathbf{b}$ is

$$(\mathbf{a}.\mathbf{b})/b,$$

$\therefore$ if P is the foot of the perpendicular from A to OB,

$$OP^2 = (\mathbf{a}.\mathbf{b})^2/b^2$$

Using Pythagoras' theorem in $\triangle OAP$, $OP^2 + AP^2 = OA^2$

$$\therefore \quad AP^2 = OA^2 - OP^2 = a^2 - \frac{(\mathbf{a}.\mathbf{b})^2}{b^2}$$

Hence the area of $\triangle OAB = \frac{1}{2} \times OB \times AP$

$$= \tfrac{1}{2}\sqrt{\left\{b^2 \times \left(a^2 - \frac{(\mathbf{a}.\mathbf{b})^2}{b^2}\right)\right\}} = \tfrac{1}{2}\sqrt{\{a^2b^2 - (\mathbf{a}.\mathbf{b})^2\}}.$$

When considering the unit vectors $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ the definition of scalar product leads to the results

$$\mathbf{i}.\mathbf{i} = \mathbf{j}.\mathbf{j} = \mathbf{k}.\mathbf{k} = 1, \quad \mathbf{i}.\mathbf{j} = \mathbf{j}.\mathbf{k} = \mathbf{k}.\mathbf{i} = 0$$

Thus for vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$,

$$\begin{aligned}\mathbf{a}.\mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}).(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\ &= a_1b_1\mathbf{i}.\mathbf{i} + a_1b_2\mathbf{i}.\mathbf{j} + a_1b_3\mathbf{i}.\mathbf{k} \\ &\quad + a_2b_1\mathbf{j}.\mathbf{i} + a_2b_2\mathbf{j}.\mathbf{j} + a_2b_3\mathbf{j}.\mathbf{k} \\ &\quad + a_3b_1\mathbf{k}.\mathbf{i} + a_3b_2\mathbf{k}.\mathbf{j} + a_3b_3\mathbf{k}.\mathbf{k}\end{aligned}$$

$\therefore$

$$\boxed{\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3}$$

Example 2 Find the component of the force $\mathbf{F} = 2\mathbf{i} + 13\mathbf{j} - 8\mathbf{k}$ in the direction of the vector $\mathbf{d} = 2\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}$. Hence resolve $\mathbf{F}$ into component forces parallel and perpendicular to $\mathbf{d}$.

Let $\hat{\mathbf{d}}$ be the unit vector in the direction of $\mathbf{d}$.

$$d = |2\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}| = \sqrt{\{2^2 + (-6)^2 + 3^2\}} = \sqrt{\{4 + 36 + 9\}} = 7$$

$$\therefore \quad \hat{\mathbf{d}} = \frac{\mathbf{d}}{d} = \tfrac{2}{7}\mathbf{i} - \tfrac{6}{7}\mathbf{j} + \tfrac{3}{7}\mathbf{k}$$

Hence the component of $\mathbf{F}$ in the direction of $\mathbf{d}$

$$= \mathbf{F}.\hat{\mathbf{d}} = (2\mathbf{i} + 13\mathbf{j} - 8\mathbf{k}).(\tfrac{2}{7}\mathbf{i} - \tfrac{6}{7}\mathbf{j} + \tfrac{3}{7}\mathbf{k})$$

$$= \tfrac{4}{7} - \tfrac{78}{7} - \tfrac{24}{7} = -\tfrac{98}{7} = -14.$$

Resolving **F** parallel and perpendicular to **d**, the component force parallel to **d**

$= (\mathbf{F}.\hat{\mathbf{d}})\hat{\mathbf{d}} = -14(\frac{2}{7}\mathbf{i} - \frac{6}{7}\mathbf{j} + \frac{3}{7}\mathbf{k}) = -4\mathbf{i} + 12\mathbf{j} - 6\mathbf{k}.$

Thus the component force perpendicular to **d**

$= (2\mathbf{i} + 13\mathbf{j} - 8\mathbf{k}) - (-4\mathbf{i} + 12\mathbf{j} - 6\mathbf{k}) = 6\mathbf{i} + \mathbf{j} - 2\mathbf{k}.$

If the angle between **a** and **b** is θ, then $\mathbf{a}.\mathbf{b} = ab\cos\theta$.

Thus
$$\boxed{\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{ab} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{ab}}$$

It follows that non-zero vectors **a** and **b** are perpendicular if and only if

$$a_1b_1 + a_2b_2 + a_3b_3 = 0.$$

Example 3 Find the angle θ between the vectors $\mathbf{a} = 4\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} - 5\mathbf{j} - 4\mathbf{k}$.

$\mathbf{a}.\mathbf{b} = 4 \times 3 + 5 \times (-5) + 3 \times (-4) = 12 - 25 - 12 = -25$

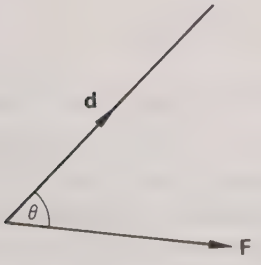
$a = \sqrt{\{4^2 + 5^2 + 3^2\}} = \sqrt{\{16 + 25 + 9\}} = \sqrt{50}$

$b = \sqrt{\{3^2 + (-5)^2 + (-4)^2\}} = \sqrt{\{9 + 25 + 16\}} = \sqrt{50}$

$$\therefore \quad \cos\theta = \frac{\mathbf{a}.\mathbf{b}}{ab} = \frac{-25}{\sqrt{50} \times \sqrt{50}} = -\frac{25}{50} = -\frac{1}{2}$$

Hence the angle between the vectors is 120°.

We next show that the *work done by a force* can be expressed as a scalar product. We recall that if the point of application of a constant force **F** moves a distance s in a direction which makes an angle θ with the direction of **F**, then the work done by the force is $Fs\cos\theta$.



If we now suppose that the point of application of a constant force **F** is given a displacement **d**, where the angle between the vectors **F** and **d** is θ, then,

work done by $\mathbf{F} = Fd\cos\theta = \mathbf{F}.\mathbf{d}$.

[We will assume that forces, displacements and work are measured in newtons, metres and joules respectively.]

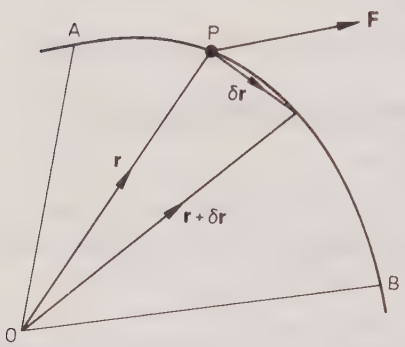
Example 4 A constant force $\mathbf{F} = 3\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}$ acts on a particle. Find the work done by $\mathbf{F}$ when the particle moves from the point with position vector $\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ to the point with position vector $4\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

The displacement of the particle

$$= (4\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) - (\mathbf{i} + 3\mathbf{j} - \mathbf{k}) = 3\mathbf{i} - \mathbf{j} + 4\mathbf{k}$$

$\therefore$ the work done by $\mathbf{F}$

$$= (3\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}).(3\mathbf{i} - \mathbf{j} + 4\mathbf{k}) = 9 + 4 - 8 = 5.$$



Consider now a variable force $\mathbf{F}$ acting on a particle P as it moves along a curve from a point A to a point B. If the path from A to B is regarded as a series of small displacements, then the work done by $\mathbf{F}$ in a typical displacement $\delta\mathbf{r}$ is approximately equal to $\mathbf{F}.\delta\mathbf{r}$.

$\therefore$ total work done by $\mathbf{F}$

$$\approx \sum_{A}^{B} \mathbf{F}.\delta\mathbf{r}$$

Letting $\delta\mathbf{r} \to 0$, we deduce that as the particle moves from A to B,

$$\boxed{\text{the work done by } \mathbf{F} \text{ is } \int_A^B \mathbf{F}.d\mathbf{r}} \tag{1}$$

If $\mathbf{F}$ and $\mathbf{r}$ are given as functions of the time t, then we write

$$\int_A^B \mathbf{F}.d\mathbf{r} = \int_{t_1}^{t_2} \mathbf{F}.\frac{d\mathbf{r}}{dt}dt = \int_{t_1}^{t_2} \mathbf{F}.\mathbf{v}\,dt \tag{2}$$

where t_1 and t_2 are the times at which the particle passes through A and B respectively.

Example 5 A particle moves along a curve so that its position vector at time t is given by $\mathbf{r} = t^3\mathbf{i} + 2t\mathbf{j}$. If one of the forces acting on the particle is a force $\mathbf{F}$, where $\mathbf{F} = 2\mathbf{i} - t\mathbf{j}$ at time t, find the work done by $\mathbf{F}$ in the interval from $t = 0$ to $t = 3$.

$$\mathbf{r} = t^3\mathbf{i} + 2t\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = 3t^2\mathbf{i} + 2\mathbf{j}$$

$\therefore$ the work done by $\mathbf{F}$ from $t = 0$ to $t = 3$

$$= \int_0^3 \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt$$

$$= \int_0^3 (2\mathbf{i} - t\mathbf{j}) \cdot (3t^2\mathbf{i} + 2\mathbf{j})\, dt$$

$$= \int_0^3 (6t^2 - 2t)\, dt$$

$$= \left[2t^3 - t^2\right]_0^3 = 54 - 9 = 45.$$

If a particle of mass m is moving with velocity $\mathbf{v}$ at time t, then its *kinetic energy* at that instant is $\frac{1}{2}mv^2$. We can establish the relationship between work and kinetic energy by considering the motion of this particle along a curved path AB. We will assume that the particle passes through A at time t_1 with velocity $\mathbf{v}_1$ and through B at time t_2 with velocity $\mathbf{v}_2$. If $\mathbf{F}$ is the resultant force acting on the particle at time t and W is the work done by $\mathbf{F}$ in the interval $t_1 \leqslant t \leqslant t_2$, then using our earlier results (1) and (2),

$$W = \int_A^B \mathbf{F} \cdot d\mathbf{r} = \int_{t_1}^{t_2} \mathbf{F} \cdot \mathbf{v}\, dt.$$

Using Newton's second law, $\mathbf{F} = m\mathbf{a} = m\dfrac{d\mathbf{v}}{dt}$

$\therefore$ $$W = \int_{t_1}^{t_2} m\frac{d\mathbf{v}}{dt} \cdot \mathbf{v}\, dt$$

But, using the product rule proved in Exercise 7.5 question 25,

$$\frac{d}{dt}(v^2) = \frac{d}{dt}(\mathbf{v} \cdot \mathbf{v}) = \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} + \frac{d\mathbf{v}}{dt} \cdot \mathbf{v} = 2\frac{d\mathbf{v}}{dt} \cdot \mathbf{v}$$

$\therefore$ $$\int \frac{d\mathbf{v}}{dt} \cdot \mathbf{v}\, dt = \tfrac{1}{2}v^2 + \text{constant}.$$

Hence $$W = \left[\tfrac{1}{2}mv^2\right]_{t_1}^{t_2} = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2$$

Thus

work done by $\mathbf{F}$ = K.E. at B − K.E. at A

Example 6 A particle of mass 10 kg moves under the action of a force $\mathbf{F}$, so that its position vector after t seconds is given by $\mathbf{r} = 3t\mathbf{i} - \mathbf{j} + t^2\mathbf{k}$. Find the work done by $\mathbf{F}$ in the interval $1 \leqslant t \leqslant 5$.

$$\mathbf{r} = 3t\mathbf{i} - \mathbf{j} + t^2\mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = 3\mathbf{i} + 2t\mathbf{k}$$

$$\Rightarrow v^2 = 3^2 + (2t)^2 = 9 + 4t^2$$

$\therefore$ when $t = 1$, $v^2 = 13$ and when $t = 5$, $v^2 = 109$.

Work done by $\mathbf{F}$ = (K.E. when $t = 5$) − (K.E. when $t = 1$)
$= \frac{1}{2} \times 10 \times 109 - \frac{1}{2} \times 10 \times 13$ J
$= 480$ J.

The *power* of a force is the rate at which it does work. Let us suppose that at time t a force $\mathbf{F}$ acts at a point with position vector $\mathbf{r}$. If the point of application of $\mathbf{F}$ is given a displacement $\delta\mathbf{r}$ in a small interval of time δt, then the work done is approximately $\mathbf{F}.\delta\mathbf{r}$ and the average rate of working is $(\mathbf{F}.\delta\mathbf{r})/\delta t$.
$\therefore$ the power of the force $\mathbf{F}$ at time t

$$= \lim_{\delta t \to 0} \frac{\mathbf{F}.\delta\mathbf{r}}{\delta t} = \lim_{\delta t \to 0} \mathbf{F}.\frac{\delta\mathbf{r}}{\delta t} = \mathbf{F}.\frac{d\mathbf{r}}{dt}.$$

Hence if the point of application of a force $\mathbf{F}$ is moving with velocity $\mathbf{v}$, then

the power of $\mathbf{F}$ is $\mathbf{F}.\mathbf{v}$.

Example 7 At time t a particle of unit mass is moving with velocity $\mathbf{v}$ under the action of a force $\mathbf{F} = 4t\mathbf{i} + 3\mathbf{k}$. Given that when $t = 0$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$, find an expression for the velocity $\mathbf{v}$ at time t. Hence find an expression for the power of $\mathbf{F}$ at time t.

By Newton's second law, $\mathbf{F} = m\mathbf{a}$.
Since $\mathbf{F} = 4t\mathbf{i} + 3\mathbf{k}$, $m = 1$ and $\mathbf{a} = \frac{d\mathbf{v}}{dt}$,

we have $\frac{d\mathbf{v}}{dt} = 4t\mathbf{i} + 3\mathbf{k}$,

Integrating, $\mathbf{v} = 2t^2\mathbf{i} + 3t\mathbf{k} + \mathbf{c}$,

where $\mathbf{c}$ is a constant vector.

But when $t = 0$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$ $\quad\therefore\quad \mathbf{c} = \mathbf{i} + 2\mathbf{j}$.
Hence $\mathbf{v} = (2t^2 + 1)\mathbf{i} + 2\mathbf{j} + 3t\mathbf{k}$.
The power of $\mathbf{F}$ at time $t = \mathbf{F}.\mathbf{v}$
$= (4t\mathbf{i} + 3\mathbf{k}).(\{2t^2 + 1\}\mathbf{i} + 2\mathbf{j} + 3t\mathbf{k})$
$= 4t(2t^2 + 1) + 3 \times 3t$
$= 8t^3 + 4t + 9t = 8t^3 + 13t$.

Exercise 11.3

1. Two particles A and B are moving with constant velocities $8\mathbf{i} + \mathbf{j} - 4\mathbf{k}$ and $4\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$ respectively. Find the angle between the paths of the particles. Find also the speed of A relative to B.

2. Two forces $\mathbf{F}_1 = 4\mathbf{j} - 5\mathbf{k}$ and $\mathbf{F}_2 = 2\mathbf{i} - 5\mathbf{j} - \mathbf{k}$ act on a particle. Find the magnitude of the resultant force acting on the particle and the angle the resultant makes with the force $\mathbf{F}_1$.

3. Find the component of the force $\mathbf{F}$ in the direction of the vector $\mathbf{d}$ if
(a) $\mathbf{F} = 3\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$, $\mathbf{d} = 6\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$,
(b) $\mathbf{F} = \mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$, $\mathbf{d} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$,
(c) $\mathbf{F} = 7\mathbf{i} - \mathbf{j} - 4\mathbf{k}$, $\mathbf{d} = 3\mathbf{i} - 5\mathbf{j} + 4\mathbf{k}$,
(d) $\mathbf{F} = 3\mathbf{i} + 5\mathbf{j} - \mathbf{k}$, $\mathbf{d} = 4\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}$.

4. Resolve the force $\mathbf{F}$ into component forces parallel and perpendicular to the vector $\mathbf{d}$ if
(a) $\mathbf{F} = 11\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$, $\mathbf{d} = 4\mathbf{i} - 3\mathbf{k}$,
(b) $\mathbf{F} = 5\mathbf{i} - 22\mathbf{j} + 19\mathbf{k}$, $\mathbf{d} = -5\mathbf{j} + 12\mathbf{k}$,
(c) $\mathbf{F} = 18\mathbf{i} + 35\mathbf{j} - 21\mathbf{k}$, $\mathbf{d} = 2\mathbf{i} - 6\mathbf{j} + 9\mathbf{k}$,
(d) $\mathbf{F} = 3\mathbf{i} - 4\mathbf{j} + 10\mathbf{k}$, $\mathbf{d} = 4\mathbf{i} - 8\mathbf{j} + \mathbf{k}$.

5. Find the work done when the point of application of a force $\mathbf{F}$ moves through a displacement $\mathbf{d}$, given that
(a) $\mathbf{F} = 3\mathbf{i} - 7\mathbf{j} + 6\mathbf{k}$, $\mathbf{d} = 5\mathbf{i} + \mathbf{j} - \mathbf{k}$,
(b) $\mathbf{F} = -2\mathbf{i} + 3\mathbf{j} - 8\mathbf{k}$, $\mathbf{d} = 6\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}$,
(c) $\mathbf{F} = 15\mathbf{i} + 7\mathbf{k}$, $\mathbf{d} = 10\mathbf{j} + 3\mathbf{k}$,
(d) $\mathbf{F} = 3\mathbf{i} - 9\mathbf{j} - 4\mathbf{k}$, $\mathbf{d} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

6. At time t a particle of mass m is moving with position vector $\mathbf{r}$ and velocity $\mathbf{v}$ under the action of a force $\mathbf{F}$. In each of the following cases find an expression for $\mathbf{v}$ and hence determine the work done by $\mathbf{F}$ in the given interval of time.
(a) $\mathbf{r} = 3t\mathbf{i} + t^2\mathbf{j}$, $0 \leqslant t \leqslant 2$,
(b) $\mathbf{r} = 5\mathbf{i} - t^2\mathbf{j} + 3t\mathbf{k}$, $2 \leqslant t \leqslant 4$,
(c) $\mathbf{r} = t^3\mathbf{i} - 4t\mathbf{j} + 3\mathbf{k}$, $1 \leqslant t \leqslant 3$,
(d) $\mathbf{r} = t\mathbf{i} - \mathbf{j} + (t^2 - 7t)\mathbf{k}$, $2 \leqslant t \leqslant 5$.

7. At time t a particle of mass m is moving with position vector $\mathbf{r}$ and velocity $\mathbf{v}$ under the action of a force $\mathbf{F}$. Find expressions for $\mathbf{F}$, $\mathbf{v}$ and the power of $\mathbf{F}$ at time t, given that
(a) $\mathbf{r} = 2t^2\mathbf{i} - 5t\mathbf{k}$,
(b) $\mathbf{r} = \mathbf{i} + t^3\mathbf{j} - 3t^2\mathbf{k}$,
(c) $\mathbf{r} = t\mathbf{i} - \mathrm{e}^{-t}\mathbf{j}$,
(d) $\mathbf{r} = \sin t\mathbf{j} + 3\cos t\mathbf{k}$.

8. A particle of unit mass is moving under the action of a constant force $\mathbf{F} = \mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$. Find the work done by $\mathbf{F}$ when the particle moves from the point A with position vector $5\mathbf{j} + 2\mathbf{k}$ to the point $\mathbf{B}$ with position vector $3\mathbf{i} + \mathbf{k}$. If the speed of the particle at A is $6\,\mathrm{m\,s^{-1}}$, find the speed of the particle at B.

9. A particle of mass 3 kg moves under the action of a constant force $\mathbf{F}$ with acceleration $\mathbf{a} = 3\mathbf{i} + 2\mathbf{k}$. Find the work done by $\mathbf{F}$ as the particle moves from the point A with position vector $4\mathbf{i} - 9\mathbf{j} + 21\mathbf{k}$ to the point B with position vector $10\mathbf{i} + 15\mathbf{j} - 7\mathbf{k}$. If the particle passes through A with velocity $12\mathbf{j} - 16\mathbf{k}$ and through

B with velocity $\lambda(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$, find the value of λ and the power of $\mathbf{F}$ at the points A and B.

10. A particle of unit mass is acted upon at time t by a force $\mathbf{F} = 2t\mathbf{i} - 3\mathbf{j}$. Given that the initial velocity of the particle is $\mathbf{j} + \mathbf{k}$, find an expression for its velocity $\mathbf{v}$ at time t. Hence find the work done by $\mathbf{F}$ in the interval $0 \leqslant t \leqslant 2$ and the power of $\mathbf{F}$ when $t = 2$.

11. A particle of mass 2 kg is acted upon at time t by a force $\mathbf{F} = 8\mathbf{i} - 4\cos t\mathbf{j} + 2t\mathbf{k}$. When $t = 0$ the velocity of the particle is $6\mathbf{i}$. Find expressions for the velocity of the particle and the power of $\mathbf{F}$ at time t.

12. A particle moves along a curve so that its position vector at time t is $\mathbf{r} = (3t - 2)\mathbf{i} + t^4\mathbf{j}$. If one of the forces acting on the particle is $\mathbf{F} = 5t^2\mathbf{i} - \mathbf{j}$, find by integration the work done by $\mathbf{F}$ in the interval $0 \leqslant t \leqslant 2$.

13. A particle moves along a curve with velocity $\mathbf{v} = \mathbf{i} + 10t^2\mathbf{j} - 3t\mathbf{k}$ at time t. If one of the forces acting on the particle is $\mathbf{F} = 2t\mathbf{i} + t^2\mathbf{j} - 2t\mathbf{k}$, find the power of $\mathbf{F}$ in terms of t. Hence find the work done by $\mathbf{F}$ in the interval from $t = 1$ to $t = 3$.

14. A particle moves on a curve with vector equation $\mathbf{r} = (\lambda - 12)\mathbf{i} - \lambda^2\mathbf{j} + \mathbf{k}$, where λ is a parameter. One of the forces acting on the particle is a variable force $\mathbf{F}$. Write down, in the form of a definite integral, an expression for the work done by $\mathbf{F}$ as the particle moves from the point where $\lambda = 1$ to the point where $\lambda = 2$. Evaluate this integral given that $\mathbf{F} = -4\mathbf{r}$.

Exercise 11.4 (miscellaneous)

1. (i) The force $\mathbf{F}_1$, of magnitude 26 N, acts at the origin in the direction of the vector $(3\mathbf{i} + 4\mathbf{j} + 12\mathbf{k})$. A second force $\mathbf{F}_2$, of magnitude 30 N, also acts at the origin, but in the direction of the vector $(6\mathbf{j} - 8\mathbf{k})$. Calculate the magnitude and the direction of the resultant of the forces $\mathbf{F}_1$ and $\mathbf{F}_2$. Calculate also the cosine of the angle between $\mathbf{F}_1$ and $\mathbf{F}_2$.
   (ii) A particle of mass 2 kg starts from rest at the origin and the force $\mathbf{F}_3$, where $\mathbf{F}_3 = (6\mathbf{i} + 26\mathbf{j})$ N, acts on it as it moves. Find the position vector of the particle 2 seconds later. (L)

2. The force acting at time t $(0 \leqslant t < 2)$ on a particle of unit mass is $24t^2\mathbf{i} + 6\mathbf{j}$. At time $t = 0$ the particle is at rest at the point with position vector $-2\mathbf{i} + 3\mathbf{j}$. Find the position vector of the particle at time $t = T$ $(0 \leqslant T < 2)$. For time $t \geqslant 2$ the force acting on the particle is $6\mathbf{j}$. Find the position vector of the particle at time $t = 3$. (L)

3. State the centre and radius of the circle which has vector equation $\mathbf{r} = 6\mathbf{i} + 8\mathbf{j} + 6(\mathbf{i}\cos\theta + \mathbf{j}\sin\theta)$. A particle P of mass 2 kg moves on this circle with

constant angular velocity $\pi/12$ radians per second. Write down the position vector of P at time t given that, at $t = 0$, P is at the point corresponding to $\theta = 0$. Calculate the components parallel to **i** and **j** of the resultant vector force on the particle when $t = 4$. Find the position vector of the particle when $t = 8$ given that the forces on the particle cease to act when $t = 4$. (L)

4. The position vector, relative to the origin O, of an object of mass m moving in a horizontal plane in which **i** and **j** are perpendicular unit vectors is given by

$$\mathbf{r} = \frac{\cos\theta}{1 + \cos\theta}\mathbf{i} + \frac{\sin\theta}{1 + \cos\theta}\mathbf{j}, \quad -\pi < \theta < \pi.$$

Obtain an expression for r, the magnitude of **r**. Given that $r^2\dfrac{d\theta}{dt} = h$ where h is constant, show that the velocity of the object is $-h\sin\theta\mathbf{i} + h(1 + \cos\theta)\mathbf{j}$. Show that the force acting on the object is of magnitude mh^2/r^2, and find the direction of this force. (JMB)

5. A particle is projected from the origin O with initial velocity $u\mathbf{i} + v\mathbf{j}$, where **i** and **j** are unit vectors in the horizontal and upward vertical directions respectively, and moves freely under gravity. Show that its velocity and position vector with respect to O, at time t after projection, are given by $\dot{\mathbf{r}} = u\mathbf{i} + (v - gt)\mathbf{j}$ and $\mathbf{r} = ut\mathbf{i} + (vt - \frac{1}{2}gt^2)\mathbf{j}$ respectively.

For the remainder of this question components of velocities are measured in $\mathrm{m\,s^{-1}}$ and components of displacements in m. Take g as $10\,\mathrm{m\,s^{-2}}$.

Two particles, A and B, are projected simultaneously from the origin with velocities $-30\mathbf{i} + 90\mathbf{j}$ and $30\mathbf{i} + 110\mathbf{j}$ respectively. Both particles impinge on a plane which contains the origin, meeting the plane on a line of greatest slope which has the equation $\mathbf{r} = \lambda(3\mathbf{i} + \mathbf{j})$, where λ is a scalar. Show that the times of flight of A and B are equal and that the distance between the points of impact is $400\sqrt{10}\,\mathrm{m}$. Find the cosine of the acute angle between the direction of motion of A at the instant of impact and the line of greatest slope of the plane. (JMB)

6. At noon two ships A and B have the following position and velocity vectors:

	Position vector	*Velocity vector*
Ship A	$10\mathbf{i} + 5\mathbf{j}$	$-2\mathbf{i} + 4\mathbf{j}$
Ship B	$2\mathbf{i} - \mathbf{j}$	$2\mathbf{i} + 7\mathbf{j}$

where **i** and **j** are unit vectors in the directions East and North respectively, and where the speeds are measured in kilometres per hour and the distances in kilometres. If they continue on their respective courses, (i) find the position vector of A after a time t hours; (ii) show that they will collide, and give the time of the collision; (iii) determine how far ship A will have travelled between noon and the collision. (SU)

7. A ship A is travelling on a course of 060° at a speed of $30\sqrt{3}$ km/h and a ship B is travelling on a course of 030° at 20 km/h. At noon B is 260 km due east of A.

Using unit base vectors **i** and **j** pointing east and north respectively find $\overrightarrow{AB}$ in terms of **i**, **j** and t at time t hours in the afternoon. Hence, or otherwise, calculate the least distance between A and B, to the nearest kilometre, and the time at which they are nearest to one another. (L)

8. An aeroplane A moves in space with a variable velocity **v** and is attacked by a guided missile B which moves with constant speed u, where $u > |v|$. The guidance system of B is such that, if from any instant the velocity of A stayed constant, B would hit A without having to change direction. Show that the velocity of A relative to B is

$$\left[\frac{\mathbf{r}.\mathbf{v} - \sqrt{\{(\mathbf{r}.\mathbf{v})^2 + (u^2 - \mathbf{v}^2)\mathbf{r}^2\}}}{\mathbf{r}^2}\right]\mathbf{r},$$

where **r** is the position vector of A relative to B. (O)

9. At time t a particle is in motion with velocity **v** and is being acted upon by a variable force **F**. Write down expressions for (i) the power at time t, (ii) the work done by **F** during the time interval $0 \leqslant t \leqslant T$.

The particle, of mass m, moves in a plane where **i** and **j** are perpendicular unit vectors so that its position vector at time t is given by $\mathbf{r} = 2a\cos 2t\mathbf{i} + a\sin 2t\mathbf{j}$, where a is a positive constant. Derive expressions for the velocity **v** and the force **F** at time t. Obtain an expression in terms of t for the power at time t and show that the work done by **F** during the interval $0 \leqslant t \leqslant T$ is $3ma^2(1 - \cos 4T)$. If T varies, find the maximum value of the work done by **F** and determine also the smallest value of T for which this maximum value is reached. (JMB)

10. A smooth fixed wire is in the shape of the curve given by

$$\mathbf{r} = a(\theta - \sin\theta)\mathbf{i} + a(1 + \cos\theta)\mathbf{j},\ 0 \leqslant \theta \leqslant 2\pi,$$

where **i**, **j** are unit vectors in the horizontal and upward vertical directions respectively. A particle threaded on the wire is released from rest at the point A ($\theta = 0$). Write down, in terms of θ and its derivative, an equation expressing the conservation of energy during the motion. Show that the time taken for the particle to reach B ($\theta = 2\pi$) is $2\pi\sqrt{(a/g)}$. (JMB)

12 Impulse and momentum

12.1 The impulse of a force

When a constant force $\mathbf{F}$ acts for a time t, then the *impulse* of the force is defined as the vector quantity $\mathbf{F}t$.

The SI unit of impulse is the newton second (N s).

Example 1 A force $\mathbf{F} = 4\mathbf{i} - 3\mathbf{j}$ acts on a particle for 3 seconds. Find the magnitude of the impulse of the force.

If the impulse of the force is $\mathbf{I}$, then

$$\mathbf{I} = (4\mathbf{i} - 3\mathbf{j}) \times 3$$
$$\therefore \quad |\mathbf{I}| = 3|4\mathbf{i} - 3\mathbf{j}| = 3\sqrt{(4^2 + 3^2)} = 15.$$

Hence the magnitude of the impulse is 15 N s.

When a variable force $\mathbf{F}$ acts from time t_1 to time t_2, then the impulse of the force is defined as $\int_{t_1}^{t_2} \mathbf{F}\, dt$.

Example 2 A particle is moving under the action of a force $\mathbf{F} = 2\mathbf{i} - 3t^2\mathbf{j} + 4t\mathbf{k}$ at the time t. Find the impulse given to the particle in the interval $1 \leqslant t \leqslant 2$.

$$\text{Impulse} = \int_1^2 (2\mathbf{i} - 3t^2\mathbf{j} + 4t\mathbf{k})\, dt$$
$$= \Big[2t\mathbf{i} - t^3\mathbf{j} + 2t^2\mathbf{k}\Big]_1^2$$
$$= (4\mathbf{i} - 8\mathbf{j} + 8\mathbf{k}) - (2\mathbf{i} - \mathbf{j} + 2\mathbf{k})$$
$$= 2\mathbf{i} - 7\mathbf{j} + 6\mathbf{k}.$$

In some circumstances it is difficult to determine the exact nature of the forces involved. For example, in collisions between moving objects comparatively large

unknown forces act for very short intervals of time. We will now show that the impulse of such a force can be found by considering the effect it produces.

Suppose that at time t a particle of mass m is moving with velocity $\mathbf{v}$ under the action of a force $\mathbf{F}$. Let $\mathbf{I}$ be the impulse given to the particle by $\mathbf{F}$ in the interval $t_1 \leqslant t \leqslant t_2$. Whether $\mathbf{F}$ is constant or variable,

$$\mathbf{I} = \int_{t_1}^{t_2} \mathbf{F}\, dt$$

Using Newton's second law in its original form,

$$\mathbf{F} = \frac{d(m\mathbf{v})}{dt},$$

where $m\mathbf{v}$ is the *momentum* of the particle.

$$\therefore \quad \mathbf{I} = \int_{t_1}^{t_2} \frac{d(m\mathbf{v})}{dt} dt$$

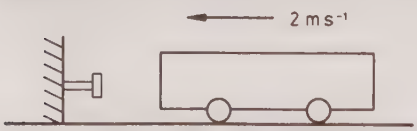
$$\mathbf{I} = \Big[m\mathbf{v} \Big]_{t_1}^{t_2}$$

$$\mathbf{I} = (m\mathbf{v})_{at\, t_2} - (m\mathbf{v})_{at\, t_1}$$

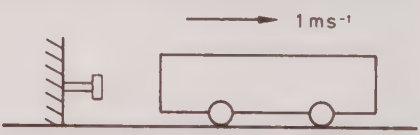
i.e. impulse = change in momentum

[Because of the relationship between impulse and momentum, both quantities are usually measured in newton seconds. An alternative unit of momentum is the kilogramme metre per second. However, since $1\,\mathrm{N} = 1\,\mathrm{kg\,m\,s^{-2}}$, it follows that $1\,\mathrm{N\,s} = 1\,\mathrm{kg\,m\,s^{-1}}$.]

Example 3 A truck of mass 1400 kg moving along a straight horizontal track at $2\,\mathrm{m\,s^{-1}}$ runs into fixed buffers and rebounds at a speed of $1\,\mathrm{m\,s^{-1}}$. If the truck and the buffers are in contact for 1·2 s, find the average force exerted by the buffers on the truck.



Before impact



After impact

Let the direction away from the buffers be positive and let F N be the average force exerted by the buffers.

Impulse = change in momentum

$$\therefore \quad F \times 1{\cdot}2 = 1400 \times 1 - 1400 \times (-2)$$

$$\therefore \quad F = \frac{1400 \times 3}{1{\cdot}2} = 3500$$

Hence the average force exerted by the buffers is 3500 N.

Example 4 A particle of mass 5 kg is moving with velocity $3\mathbf{i} - 7\mathbf{j}$ when it is given an impulse $\mathbf{I} = -7\mathbf{i} + 16\mathbf{j}$. If the velocity of the particle after the impulse is $\mathbf{v}$, find $\mathbf{v}$. Find also the change in kinetic energy.

Impulse = change in momentum

$\therefore \quad -7\mathbf{i} + 16\mathbf{j} = 5\mathbf{v} - 5(3\mathbf{i} - 7\mathbf{j})$

$\therefore \quad 5\mathbf{v} = -7\mathbf{i} + 16\mathbf{j} + 15\mathbf{i} - 35\mathbf{j}$

i.e. $\mathbf{v} = \frac{1}{5}(8\mathbf{i} - 19\mathbf{j})$

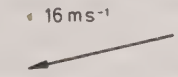
Initial speed $= \sqrt{(3^2 + 7^2)} = \sqrt{(9 + 49)} = \sqrt{58}$

New speed $= \frac{1}{5}\sqrt{(8^2 + 19^2)} = \frac{1}{5}\sqrt{(64 + 361)} = \frac{1}{5}\sqrt{425} = \sqrt{17}$.

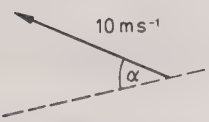
$\therefore$ there is a loss in kinetic energy of

$$(\tfrac{1}{2} \times 5 \times 58 - \tfrac{1}{2} \times 5 \times 17)\,\mathrm{J} \quad \text{i.e.} \quad 102{\cdot}5\,\mathrm{J}.$$

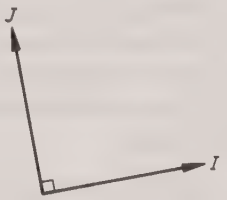
Example 5 A ball of mass 0·25 kg strikes a wall at a speed of $16\,\mathrm{m\,s^{-1}}$ and rebounds with a speed of $10\,\mathrm{m\,s^{-1}}$. If the angle between the velocities of the ball before and after impact is α, where $\tan\alpha = \frac{3}{4}$, find the magnitude of the impulse exerted by the wall on the ball.



Before impact



After impact



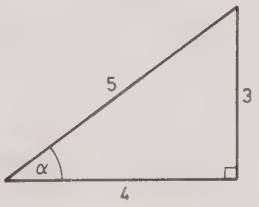
Components of the impulse

Resolving parallel and perpendicular to the initial velocity of the ball,

$\nearrow \qquad I = 0{\cdot}25(-10\cos\alpha) - 0{\cdot}25(-16)$

$\nwarrow \qquad J = 0{\cdot}25(10\sin\alpha)$

Using a right-angled triangle,



$$\sin\alpha = \tfrac{3}{5}, \quad \cos\alpha = \tfrac{4}{5}.$$

$\therefore \quad I = 0{\cdot}25(-8) - 0{\cdot}25(-16) = 2$

$J = 0{\cdot}25 \times 6 = 1{\cdot}5$

Hence the magnitude of the impulse exerted by the wall

$$= \sqrt{(2^2 + 1{\cdot}5^2)}\,\mathrm{N\,s} = 2{\cdot}5\,\mathrm{N\,s}.$$

When a jet of water is directed at a wall, the reaction of the wall on the water produces a *continuous change in momentum*. Provided that the water is being delivered at a steady rate and with constant velocity, the force exerted by the wall can be determined by considering the momentum lost by the water each second.

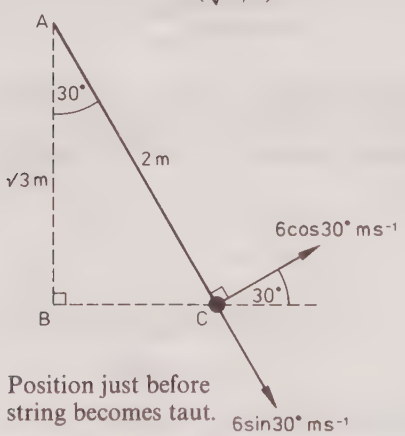
Example 6 A hose delivers water horizontally at a rate of 20 kg per second with a speed of $25\,\mathrm{m\,s^{-1}}$. The water strikes a vertical wall and does not rebound. Find the force exerted on the wall.

Let F N be the force exerted on the wall, then the force exerted by the wall is also F N,
$\therefore$ impulse exerted in 1 second $= F \times 1\,\mathrm{N\,s} = F\,\mathrm{N\,s}$.
Mass of water delivered in 1 second $= 20\,\mathrm{kg}$
$\therefore$ loss of momentum in 1 second $= 20 \times 25\,\mathrm{N\,s} = 500\,\mathrm{N\,s}$
However, impulse = change in momentum,
$\therefore \qquad F = 500$
Hence the force exerted on the wall is 500 N.

An instantaneous impulse acting along a string is called an *impulsive tension*. It may produce changes in momentum in the direction of the string, but has no effect perpendicular to the string. In particular, if a stationary particle is jerked into motion by the sudden tightening of a string attached to it, then the initial velocity of the particle will be in the direction of the string.

Example 7 One end of a light inextensible string of length 2 m is fixed at a point A on a smooth horizontal plane. The other end is attached to a particle of mass 2 kg. The particle is placed at a point B on the plane, where $AB = \sqrt{3}\,\mathrm{m}$, then projected horizontally with speed $6\,\mathrm{m\,s^{-1}}$ in a direction perpendicular to AB. Find the speed of the particle immediately after the string has become taut. Find also the impulsive tension in the string.

If C is the point on the path of the particle at which the string first becomes taut, $\angle CAB = \cos^{-1}(\sqrt{3}/2) = 30°$.



Position just before string becomes taut.

As shown in the diagram, just before the string tightens the velocity of the particle has components $6\sin 30°\,\mathrm{m\,s^{-1}}$ along AC and $6\cos 30°\,\mathrm{m\,s^{-1}}$ perpendicular to AC. When the string becomes taut the particle loses its momentum in the direction of AC, but its momentum perpendicular to AC is unchanged.

Hence just after the string has become taut, the speed of the particle is $6\cos 30°\,\mathrm{m\,s^{-1}}$ i.e. $3\sqrt{3}\,\mathrm{m\,s^{-1}}$.
The loss of momentum along AC is produced by the impulsive tension in the string,

$\therefore \qquad$ impulsive tension $= 2 \times 6\sin 30°\,\mathrm{N\,s} = 6\,\mathrm{N\,s}$.

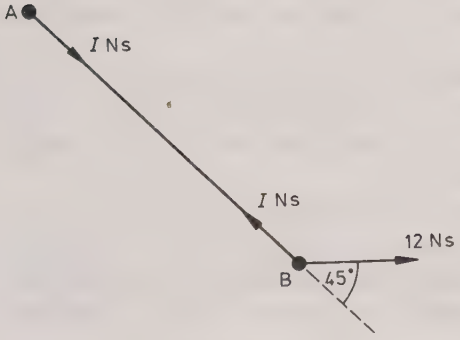
In problems which involve systems of two or more particles the impulse-momentum

relationship can be used both for the system as a whole and for particles taken separately. When considering the impulse given to the whole system, we will see that internal impulses occur in equal and opposite pairs and cancel each other out. Thus we may write:

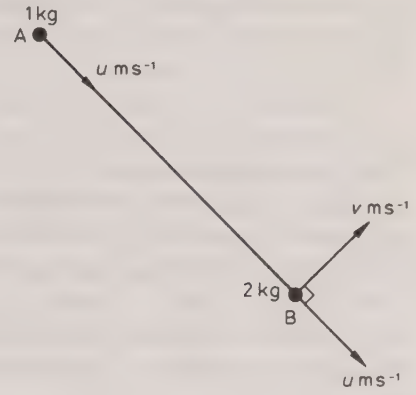
sum of external impulses = total change in momentum.

The next example concerns two particles connected by a light inextensible string. In this case the impulsive tensions acting on the particles are equal and opposite, so they can be disregarded when dealing with the system as a whole.

Example 8 Particles A and B, of masses 1 kg and 2 kg respectively, are connected by a light inextensible string. The particles are at rest on a smooth horizontal plane with the string taut. A horizontal impulse of magnitude 12 N s is applied to particle B in a direction inclined at 45° to the direction of AB. Find the velocities with which A and B start to move. Find also the impulsive tension in the string.



Impulse diagram



Velocity diagram

Let the initial velocity of B have components u and v along and perpendicular to AB respectively. Since the string is taut, A will start to move with speed u along AB.

For A and B together, impulse = change in momentum,

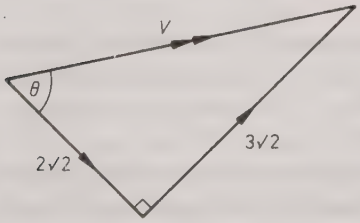
$\therefore$ resolving along and perpendicular to AB,

$$\searrow \qquad 12\cos 45° = 1 \times u + 2 \times u$$

$$\nearrow \qquad 12\sin 45° = 2 \times v$$

$$\therefore \qquad u = 2\sqrt{2} \quad \text{and} \quad v = 3\sqrt{2}$$

Thus the initial velocity of A is $2\sqrt{2}\,\text{m s}^{-1}$ along AB.



If the initial velocity of B is $V\,\text{m s}^{-1}$ at an angle θ to AB, then

$$V = \sqrt{\{(2\sqrt{2})^2 + (3\sqrt{2})^2\}} = \sqrt{26}$$

$$\theta = \tan^{-1}(3/2) \approx 56{\cdot}3°$$

Thus the initial velocity of B is $\sqrt{26}\,\text{m s}^{-1}$ at an angle of 56·3° to AB.

Let I N s be the impulsive tension in the string, then considering the motion of A,

$$I = 1 \times 2\sqrt{2}.$$

Hence the impulsive tension in the string is $2\sqrt{2}$ N s.

Exercise 12.1

1. When a constant force $\mathbf{F}$ acts for time t, the impulse of the force is $\mathbf{I}$.
(a) If $\mathbf{F} = 2\mathbf{i} - 7\mathbf{j}$ and $t = 4$, find $\mathbf{I}$.
(b) If $\mathbf{I} = 6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and $t = 3$, find $\mathbf{F}$.
(c) If $\mathbf{F} = -12\mathbf{i} + 5\mathbf{j}$ and $t = 5$, find $|\mathbf{I}|$.
(d) If $\mathbf{I} = 6\mathbf{i} + 12\mathbf{j} - 4\mathbf{k}$ and $t = 3{\cdot}5$, find $|\mathbf{F}|$.

2. Find the impulse of the variable force $\mathbf{F}$ in the stated interval of time.
(a) $\mathbf{F} = 5t\mathbf{i} + 3\mathbf{j}$, $0 \leqslant t \leqslant 2$.
(b) $\mathbf{F} = \mathbf{i} - 6t^2\mathbf{j} + 2t\mathbf{k}$, $1 \leqslant t \leqslant 3$.
(c) $\mathbf{F} = \cos 2t\mathbf{i} + 3\sin 2t\mathbf{j}$, $0 \leqslant t \leqslant \frac{1}{2}\pi$.

3. Find the momentum of
(a) a man of mass 80 kg running at $6\,\mathrm{m\,s^{-1}}$,
(b) a car of mass 1200 kg travelling at $25\,\mathrm{m\,s^{-1}}$,
(c) a bullet of mass 15 g moving at $800\,\mathrm{m\,s^{-1}}$.

4. A particle of mass m moves so that its position vector at time t is $\mathbf{r}$. Find, as a vector, the momentum of the particle at time T if
(a) $\mathbf{r} = 3\mathbf{i} + t^2\mathbf{j}$, $m = 2$, $T = 3$;
(b) $\mathbf{r} = -4t\mathbf{i} + t^3\mathbf{k}$, $m = 0{\cdot}5$, $T = 2$;
(c) $\mathbf{r} = 5\cos 3t\mathbf{i} + 2\mathbf{j} + 5\sin 3t\mathbf{k}$, $m = 4$, $T = \pi$.

5. A stone of mass 3 kg is falling at a speed of $18\,\mathrm{m\,s^{-1}}$ when it hits the ground. If the stone is brought to rest by the impact, find the impulse exerted by the ground.

6. A ball of mass 0·2 kg strikes a wall when moving horizontally at $12\,\mathrm{m\,s^{-1}}$. If the ball rebounds horizontally at $8\,\mathrm{m\,s^{-1}}$, find the impulse exerted by the wall.

7. A bullet of mass 0·03 kg is fired into a fixed block of wood at a speed of $400\,\mathrm{m\,s^{-1}}$. If the bullet is brought to rest in 0·02 s, find the average resistance exerted by the wood.

8. A particle of mass 5 kg is moving under the action of a constant force of magnitude 8 N. Initially the speed of the particle is $12\,\mathrm{m\,s^{-1}}$. Find its speed after 3 seconds if the force (a) acts in the direction of motion, (b) directly opposes motion.

9. A particle of mass 2 kg is projected across a rough horizontal plane with a speed of $21\,\mathrm{m\,s^{-1}}$. Find the time in which the particle comes to rest, given that the coefficient of friction is 2/7.

10. A particle of mass 0·5 kg is moving with velocity $\mathbf{u} = 6\mathbf{i} - 7\mathbf{j}$ when it is given an impulse $\mathbf{I} = 3\mathbf{i} + 11\mathbf{j}$. Find the new velocity of the particle and the average force required to bring it to rest in a further 3 seconds.

11. A particle of mass 0·8 kg is moving with velocity $4\mathbf{i} + 7\mathbf{j}$ when it strikes a fixed obstacle and rebounds with velocity $-6\mathbf{i} + 2\mathbf{j}$. Find the impulse exerted by the particle on the obstacle. Find also the kinetic energy lost by the particle in the collision.

12. A particle of mass 3 kg is moving with speed $6\,\mathrm{m\,s^{-1}}$ when it receives an impulse of magnitude 24 N s. Find the speed of the particle immediately after the impulse and the change in the kinetic energy of the particle if the direction of the impulse and the original direction of motion are (a) the same, (b) opposite, (c) perpendicular.

13. A ball of mass 0·2 kg strikes a wall at a speed of $12\,\mathrm{m\,s^{-1}}$ and rebounds with a speed of $3{\cdot}5\,\mathrm{m\,s^{-1}}$. If its path is diverted through 90° by the impact, find the magnitude of the impulse exerted on the wall.

14. An object of mass 4 kg strikes a barrier at a speed of $10\,\mathrm{m\,s^{-1}}$ and rebounds at a speed of $5\,\mathrm{m\,s^{-1}}$. If the angle between the velocities of the object before and after impact is 60°, find the magnitude of the impulse exerted on the barrier.

15. A hose is discharging water at the rate of 50 litres per second. The jet of water is travelling horizontally with a speed of $12\,\mathrm{m\,s^{-1}}$ when it strikes a vertical wall. Assuming that the water does not rebound, find the force exerted on the wall. [Take the mass of 1 litre of water to be 1 kg.]

16. A stream of water falling vertically from a pipe at the rate of 75 kg per second exerts a force of 900 N on the ground below. Assuming that the ground destroys the momentum of the water, find the speed at which the water hits the ground.

17. In a factory small screws are falling vertically onto a conveyor belt without bouncing at a rate of 48 kg per minute. Just before impact the screws are travelling at $2\,\mathrm{m\,s^{-1}}$. If the conveyor belt moves horizontally at $1{\cdot}5\,\mathrm{m\,s^{-1}}$, find the magnitude and direction of the average force it exerts on the screws.

18. One end of a light inextensible string of length $\sqrt{3}$ m is fixed at a point A on a smooth horizontal plane. The other end is attached to a particle of mass 3 kg. The particle is placed at a point B on the plane, where $AB = 1$ m, then projected horizontally with speed $4\,\mathrm{m\,s^{-1}}$. Find the speed of the particle immediately after the string has become taut and the impulsive tension in the string, given that the angle between the initial velocity of the particle and the direction BA is (a) 60°, (b) 90°.

19. Particles A and B, of masses 3 kg and 2 kg respectively, are connected by a light inextensible string. The particles are at rest on a smooth horizontal plane with the string taut. A horizontal impulse of magnitude 12·5 N s is applied to particle B in a direction inclined at an angle α to the direction of AB, where $\tan\alpha = 4/3$. Find the

velocities with which A and B start to move. Find also the impulsive tension in the string.

20. Particles A and B, both of mass m, are connected by a light inextensible string. A particle C of mass $2m$ is attached to the mid-point of the string. The particles are at rest on a smooth horizontal plane with the string taut and with AC at right angles to BC. A horizontal impulse of magnitude $12m$ is applied to C in a direction inclined at an angle α to AC, such that both parts of the string remain taut. Find the speeds with which the particles begin to move if (a) $\alpha = 45°$, (b) $\alpha = 60°$.

12.2 Conservation of momentum

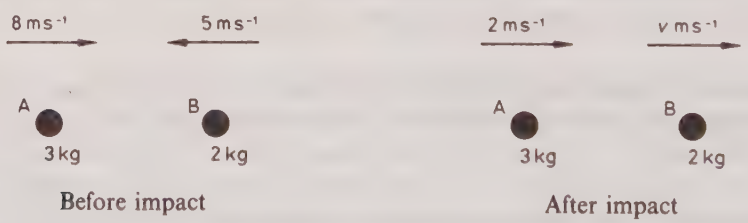
When two particles collide they exert equal and opposite impulses on each other. Therefore the changes in the momentum of the particles are also equal and opposite. It follows that the total momentum of the two particles together remains unchanged. A similar argument can be used in any situation in which equal and opposite impulses act on two bodies, e.g. when a bullet is fired from a gun or when a light inextensible string connecting two particles suddenly becomes taut.

These examples are special cases of a more general result called the *principle of conservation of linear momentum.*

> If no external force acts on a system of particles in a particular direction, then there can be no change in the total momentum of the system in that direction.

Example 1 Two particles A and B, of masses 3 kg and 2 kg respectively, are travelling in opposite directions along the same straight line when they collide. Just before the impact A is moving with speed $8\,\mathrm{m\,s^{-1}}$ and B with speed $5\,\mathrm{m\,s^{-1}}$. After the impact the speeds of A and B are $2\,\mathrm{m\,s^{-1}}$ and $v\,\mathrm{m\,s^{-1}}$ respectively. Find the value of v given that after the impact A and B are travelling in (a) the same direction, (b) opposite directions.

(a)



Momentum before impact = momentum after impact

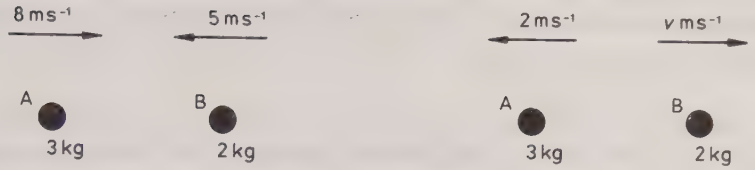
$\therefore$ taking the direction of motion of A as positive,

$$3 \times 8 + 2 \times (-5) = 3 \times 2 + 2 \times v$$

i.e. $$14 = 6 + 2v$$

Thus $$v = 4$$

(b)



Before impact After impact

Momentum before impact = momentum after impact

$\therefore \quad 3 \times 8 + 2 \times (-5) = 3 \times (-2) + 2 \times v$

i.e. $14 = -6 + 2v$

Thus $v = 10.$

Example 2 A bullet of mass 0·02 kg is fired from a gun of mass 6 kg with a speed of 450 m s^{-1}. Find the speed of the gun's recoil. Find also the constant force required to bring the gun to rest in 0·5 seconds.

Let v m s^{-1} be the speed of the gun and let F N be the force required to bring the gun to rest in 0·5 seconds.

Momentum after firing = momentum before firing

$\therefore \quad 6 \times v + 0{\cdot}02 \times (-450) = 0$

i.e. $6v - 9 = 0$

$v = 1{\cdot}5$

Hence the speed of the gun's recoil is 1·5 m s^{-1}.
As the gun is brought to rest,

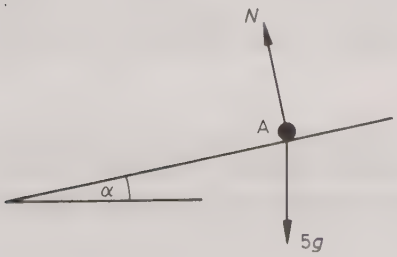
impulse = change in momentum

$\therefore \quad F \times 0{\cdot}5 = 6 \times 1{\cdot}5$

i.e. $F = 18$

Hence the constant force required is 18 N.

Example 3 A particle A of mass 5 kg slides from rest down a smooth plane inclined at an angle α to the horizontal, where $\sin\alpha = 1/6$. After 3 s it collides with a particle B of mass 1 kg which is sliding down the plane at 2·5 m s^{-1}. If the two particles coalesce, find their common speed immediately after impact.



The change in the momentum of A is equal to the impulse received.
$\therefore$ just before the collision the momentum of A down the plane

$= 5g \sin\alpha \times 3$ N s

$= 5 \times 9{\cdot}8 \times \frac{1}{6} \times 3$ N s

$= 24{\cdot}5$ N s

Momentum is conserved in the collision between A and B. Thus if $v\,\mathrm{m\,s^{-1}}$ is their common speed after impact,

$$24{\cdot}5 + 1 \times 2{\cdot}5 = (5 + 1)v$$
$$\therefore \qquad v = 4{\cdot}5$$

Hence the common speed of the particles after impact is $4{\cdot}5\,\mathrm{m\,s^{-1}}$.

Exercise 12.2

1. A railway truck of mass 12 000 kg is moving along a straight level track at $7\,\mathrm{m\,s^{-1}}$ when it collides with another truck of mass 8000 kg moving in the same direction at $2\,\mathrm{m\,s^{-1}}$. After the collision the trucks move on together. Find the common speed of the trucks after impact.

2. A bullet of mass 0·04 kg is fired horizontally into a block of wood of mass 1·4 kg which rests on a smooth horizontal surface. The bullet becomes embedded in the block and after impact they move on together at a speed of $20\,\mathrm{m\,s^{-1}}$. Find the speed at which the bullet is fired.

3. Two particles, A of mass 5 kg and B of mass 2 kg, are moving horizontally along the same straight line when they collide. Both come to rest instantaneously. If the velocity of A was $3\,\mathrm{m\,s^{-1}}$ just before the impact, find the velocity of B and the kinetic energy lost in the impact.

4. A gun of mass 4 kg fires a bullet of mass 20 g at a speed of $500\,\mathrm{m\,s^{-1}}$. Find the speed of the gun's recoil and the energy of the explosion in the gun.

5. Two particles A and B, of mass 3 kg and 5 kg respectively, collide. Just before impact A is moving with speed $8\,\mathrm{m\,s^{-1}}$ and B is moving with speed $4\,\mathrm{m\,s^{-1}}$ in the opposite direction. Immediately after the impact A rebounds with speed $7\,\mathrm{m\,s^{-1}}$. Find the speed of B immediately after impact.

6. Two particles X and Y, of mass 4 kg and 3 kg respectively, collide. Just before impact X is moving with speed $13\,\mathrm{m\,s^{-1}}$ and Y is moving with speed $5\,\mathrm{m\,s^{-1}}$ in the opposite direction. Immediately after the impact X moves on, with no change in direction, at $4\,\mathrm{m\,s^{-1}}$. Find the speed of Y immediately after impact.

7. A particle of mass m and velocity $4\mathbf{i} + \mathbf{j}$ collides with a particle of mass $2m$ and velocity $\mathbf{i} - 8\mathbf{j}$. If the particles coalesce on impact, find their common velocity just after the collision.

8. Two particles, A of mass $3m$ and B of mass $2m$, are moving with velocities $4\mathbf{i} - 6\mathbf{j}$ and $2\mathbf{i} + 7\mathbf{j}$ respectively when they collide. Immediately after the impact A moves on in the same direction as before, but with half its original speed. Find the velocity of B just after the collision.

9. A particle of mass 2 kg moving with speed $10\,\mathrm{m\,s^{-1}}$ collides with a stationary particle of mass 7 kg. Immediately after impact the particles move with the same speed but in opposite directions. Find the kinetic energy lost in the collision.

10. Two particles, of masses $3m$ and $2m$ moving in opposite directions with speeds u and $4u$ respectively, collide. Given that no kinetic energy is lost in the collision, find the speeds of the particles immediately after impact.

11. Three trucks A, B and C, each of mass 2 tonnes, are moving on a straight horizontal track in the same direction with speeds of 10, 5 and $3\,\mathrm{m\,s^{-1}}$ respectively. When any two trucks collide, they become coupled. Thus after two collisions the three trucks move on with a common speed. Find this speed and find also the total loss of kinetic energy in the collisions.

12. A truck A of mass 3000 kg moving at $4\,\mathrm{m\,s^{-1}}$ collides with a truck B of mass 2000 kg moving at $2\,\mathrm{m\,s^{-1}}$ in the same direction. If after impact truck A continues to move in the same direction but at a speed of $3\,\mathrm{m\,s^{-1}}$, find the new speed of truck B. If the trucks are in contact for 1·5 s, find the average force each exerts on the other.

13. A van of mass 1800 kg begins to tow a stationary car of mass 1200 kg. Just before the tow-rope tightens, the van is moving at a speed of $1\,\mathrm{m\,s^{-1}}$. Find the common speed of the vehicles just after the rope tightens. Find also the impulsive tension in the rope.

14. A bullet of mass 0·03 kg is fired from a gun with a horizontal velocity of $600\,\mathrm{m\,s^{-1}}$. If the gun is then brought to rest in 1·2 s by a horizontal force which rises uniformly from zero to P N and then falls uniformly to zero, find the value of P.

15. A particle of mass 2 kg slides from rest down a smooth plane inclined at an angle of 30° to the horizontal. After 5 seconds it collides with a particle of mass 3 kg which is sliding down the plane at a speed of $7\,\mathrm{m\,s^{-1}}$. If the two particles coalesce on impact find their common velocity (a) just after the collision, (b) 2 seconds after the collision.

16. Two small beads A and B, each of mass m, are threaded on a smooth straight wire inclined at an angle α to the horizontal. Bead A is released from rest and begins to slide down the wire. At same instant bead B is projected up the wire towards A with speed u. After a time t the two beads collide and are brought to rest by the impact. Find an expression for t in terms of u, g and α.

12.3 Work, energy and momentum

When faced with a problem in particle dynamics, we will now have to decide whether to consider work-energy relationships, impulse-momentum relationships or a

combination of both. The two fundamental equations are:

Work done = change in kinetic energy
Impulse = change in momentum

Example 1 A bullet of mass m is fired horizontally with velocity V into a fixed block of wood and penetrates a distance d into it. If the block exerts a constant resistance R, which stops the bullet in a time T, find expressions for R and T in terms of m, V and d.

Work done against resistance = loss of kinetic energy

$$\therefore \quad Rd = \tfrac{1}{2}mV^2$$

Hence
$$R = mV^2/2d$$

Impulse = change in momentum

$$\therefore \quad RT = mV$$

Hence
$$T = \frac{mV}{R} = mV \times \frac{2d}{mV^2} = \frac{2d}{V}.$$

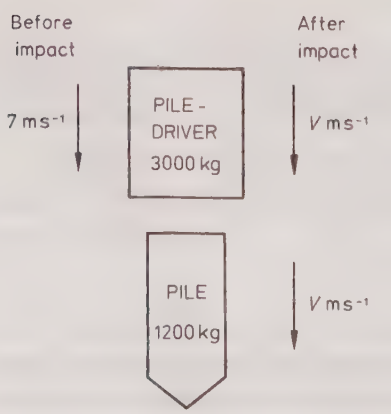
[Note that as the resistance is constant this problem could also be solved using Newton's second law and the constant acceleration formulae $v^2 = u^2 + 2as$, $v = u + at$.]

Example 2 A pile-driver of mass 3000 kg falls freely from a height of 2·5 m and strikes a pile of mass 1200 kg. After the blow the pile and the driver move on together. If the pile is driven a distance of 0·5 m into the ground, find the speed at which the pile starts to move into the ground and the average resistance of the ground to penetration.

Let $v\,\mathrm{m\,s^{-1}}$ be the downward velocity of the pile-driver just before impact. Using the formula $v^2 = u^2 + 2as$, where $u = 0, a = 9{\cdot}8, s = 2{\cdot}5$, we have

$$v^2 = 0 + 2 \times 9{\cdot}8 \times 2{\cdot}5 = 49$$

$$\therefore \quad v = 7$$



Let $V\,\mathrm{m\,s^{-1}}$ be the common velocity of the pile and the driver after impact. Using the principle of conservation of momentum,

$$7 \times 3000 + 0 \times 1200 = V(3000 + 1200)$$

$$\therefore \quad V = \frac{7 \times 3000}{4200} = 5$$

Hence the pile starts to move into the ground at $5\,\mathrm{m\,s^{-1}}$.

Let R N be the average resistance of the ground, then the resultant force acting on pile and driver together is $(R - 4200g)$ N.

$$\text{Work done} = \text{change in kinetic energy}$$
$$\therefore \quad (R - 4200g) \times 0{\cdot}5 = \tfrac{1}{2} \times 4200 \times 5^2$$
$$\therefore \quad R - 4200g = 4200 \times 25$$
$$\therefore \quad R = 4200(25 + 9{\cdot}8) = 146\,160$$

Hence the average resistance of the ground is 146 160 N.

When a particle moves in a vertical plane there are changes in its gravitational potential energy. Provided that no force other than gravity is doing work, we may then apply the principle of conservation of mechanical energy in the form:

$$\boxed{\text{K.E.} + \text{P.E.} = \text{constant}}$$

Example 3 A particle A, of mass m, is attached to a fixed point by means of a light inextensible string and hangs in equilibrium. A particle B, of mass $2m$, is travelling horizontally with speed $5u$ when it collides with particle A. Immediately after the impact B continues to move in the same direction but with speed $3u$. If A swings to a height h above its original position before coming instantaneously to rest, find an expression for h in terms of u and g.

Let v be the speed of particle A after impact.

Just before impact Just after impact

Using the principle of conservation of momentum,

$$m \times 0 + 2m \times 5u = m \times v + 2m \times 3u$$
$$10mu = mv + 6mu$$
$$\therefore \quad v = 4u$$

Hence the speed of A immediately after impact is $4u$.

When A is swinging freely its direction of motion is always at right angles to the string. Thus the tension in the string does no work and mechanical energy is conserved.

P.E. after impact $= 0$, K.E. after impact $= \frac{1}{2}m(4u)^2 = 8mu^2$.
P.E. at height $h = mgh$, K.E. at height $h = 0$.

$$\therefore \quad 0 + 8mu^2 = mgh + 0$$
$$\text{Hence} \quad h = 8u^2/g.$$

Example 4 Particles A and B, of masses 1 kg and 2 kg respectively, are attached to the ends of a light inextensible string passing over a smooth light pulley. The system is released from rest with both parts of the string taut and vertical. After A has travelled 1·2 m it hits and coalesces with a stationary particle C, of mass 2 kg. Find the new speed of the system and the impulsive tension in the string at the instant when C is picked up.

Taking the initial P.E. of each particle as zero, after the system has travelled 1·2 m the P.E. of A is $1{\cdot}2g$ J and the P.E. of B is $-1{\cdot}2 \times 2g$ J i.e. $-2{\cdot}4g$ J.

The initial K.E. of the system is zero. If v m s^{-1} is the speed of the particles after they have travelled 1·2 m, the K.E. acquired is $\frac{1}{2}(1 + 2)v^2$ J i.e. $1{\cdot}5v^2$ J.
Since no external force is doing work, the mechanical energy of the system remains constant.

$$\therefore \quad 1{\cdot}5v^2 + (1{\cdot}2g - 2{\cdot}4g) = 0$$
$$\text{i.e.} \quad 1{\cdot}5v^2 = 1{\cdot}2 \times 9{\cdot}8$$
$$\therefore \quad v = 2{\cdot}8$$

Thus, just before C is picked up the system is travelling at 2·8 m s^{-1}.
Let V m s^{-1} be the new speed of the system and J N s the impulsive tension in the string when C is picked up.

Impulse = change in momentum

Thus for B: $J = 2 \times 2{\cdot}8 - 2 \times V$

i.e. $J = 5{\cdot}6 - 2V$ (1)

For A and C: $J = (1 \times V + 2 \times V) - (1 \times 2{\cdot}8 + 2 \times 0)$

i.e. $J = 3V - 2{\cdot}8$ (2)

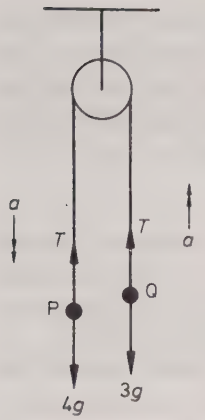
Subtracting (2) from (1): $0 = 8{\cdot}4 - 5V$

$\therefore \quad V = 1{\cdot}68$

Substituting in (1): $J = 5{\cdot}6 - 2 \times 1{\cdot}68 = 2{\cdot}24$.

Hence the new speed of the system is 1·68 m s^{-1} and the impulsive tension in the string is 2·24 N s.

Example 5 A particle P of mass 4 kg is connected to a particle Q of mass 3 kg by means of a light inextensible string which passes over a smooth light pulley. The system is released from rest with both parts of the string taut and vertical. After 3·5 s particle P hits the ground without rebounding. Assuming that particle Q does not hit the pulley, find the further time which elapses before the string is again taut. Find also the speed with which P leaves the ground.



Initial position

Let $a\,\mathrm{m\,s^{-2}}$ be the acceleration of the particles and let $T\,\mathrm{N}$ be the tension in the string.
Using Newton's second law,

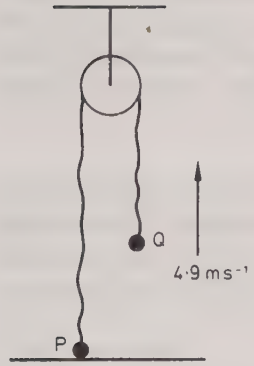
for P: $\quad 4g - T = 4a \qquad (1)$
for Q: $\quad T - 3g = 3a \qquad (2)$
(1) + (2): $\quad g = 7a$

$$\therefore \qquad a = \frac{g}{7} = 1{\cdot}4$$

Using the formula $v = u + at$, where $u = 0$, $a = 1{\cdot}4$, $t = 3{\cdot}5$,

$$v = 0 + 1{\cdot}4 \times 3{\cdot}5 = 4{\cdot}9$$

Hence when P hits the ground, Q has an upward velocity of $4{\cdot}9\,\mathrm{m\,s^{-1}}$.

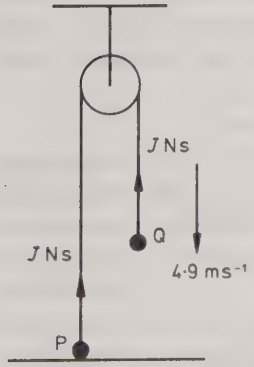


When P hits the ground

While P is on the ground Q moves freely under gravity. Hence the time which elapses before Q returns to the position where the string slackened is given by the formula $s = ut + \frac{1}{2}at^2$, where $s = 0$, $u = 4{\cdot}9$ and $a = -9{\cdot}8$.

$$\therefore \quad 0 = 4{\cdot}9t - 4{\cdot}9t^2$$
$$\therefore \quad t = 0 \quad \text{or} \quad t = 1.$$

Hence the string is again taut after a further time of 1 s.
Since there is no loss of energy in this time, it follows that as the string tightens Q has a downward velocity of $4.9\,\mathrm{m\,s^{-1}}$.



As the string becomes taut again

Let $V\,\mathrm{m\,s^{-1}}$ be the speed with which P is jerked into motion and let $J\,\mathrm{N\,s}$ be the impulsive tension in the string.

Impulse = change in momentum

Thus for P: $\quad J = 4V$
and for Q: $\quad J = 3 \times 4{\cdot}9 - 3V$
Subtracting: $\quad 0 = 7V - 3 \times 4{\cdot}9$
$\therefore$ $\quad V = 2{\cdot}1$

Hence P leaves the ground at a speed of $2{\cdot}1\,\mathrm{m\,s^{-1}}$.

If we look again at the worked examples in this chapter we will see that a collision or the sudden tightening of a string often results in a loss of kinetic energy. This happens because some of the kinetic energy of the bodies involved is converted into other forms such as heat, sound or light. It is therefore important to stress that the principle of conservation of mechanical energy cannot usually be applied to a system when impulsive forces are acting.

Exercise 12.3

1. A car of mass 1200 kg travelling in a straight line at a speed of $20\,\mathrm{m\,s^{-1}}$ is brought to rest in a distance of 50 m by a constant braking force. Find the magnitude of this force and the time taken to stop the car.

2. A bullet of mass 25 g is fired horizontally with velocity $600\,\mathrm{m\,s^{-1}}$ into a fixed block of wood and penetrates a distance of 50 cm into it. Find the constant resistance exerted by the wood and the time taken to bring the bullet to rest.

3. A railway truck of mass 3000 kg moving at $4\,\mathrm{m\,s^{-1}}$ collides with a truck of mass 2000 kg moving in the opposite direction at $1\,\mathrm{m\,s^{-1}}$. If the trucks move on together after the collision, find their common speed just after impact. If a constant retarding force of 400 N acts on the trucks after impact, find the distance they travel before coming to rest.

4. A particle is projected at a speed of $7\,\mathrm{m\,s^{-1}}$ across a rough horizontal plane. After travelling a distance of 2 m it collides with a stationary particle of equal mass. The two particles coalesce and travel a further distance d m before coming to rest. If the coefficient of friction between the particles and the plane is 5/12, find the value of d.

5. A stone of mass 5 kg is dropped from a point 10 m above horizontal ground. Find the impulse exerted by the ground on the stone if (a) it comes to rest without rebounding, (b) it rises to a maximum height of 10 cm after impact.

6. A particle of mass 2 kg is projected vertically upwards from a point A at a speed of $21\,\mathrm{m\,s^{-1}}$. At the same instant a particle of mass 1 kg is dropped from a point 30 m vertically above A. Find the time which elapses before the two particles collide. If the particles coalesce on impact, find the speed at which they pass through A.

7. A pile-driver of mass 1500 kg falls freely from a height of 1·6 m and strikes a pile of mass 500 kg. After the blow the pile and the driver move on together. If the pile is driven a distance of 0·3 m into the ground, find the speed at which the pile starts to move into the ground and the average resistance of the ground to penetration.

8. A pile-driver of mass 1200 kg falls freely from a height of 3·6 m and strikes without rebounding a pile of mass 800 kg. The blow drives the pile a distance of 36 cm into the ground. Find the resistance of the ground, assumed to be constant. Find also the time for which the pile is in motion.

9. A particle P is released from rest at a point A on a smooth plane inclined at 30° to the horizontal. When it has reached a point B on the plane, P collides with a particle Q, of equal mass, moving up the plane at $7\,\mathrm{m\,s^{-1}}$. After the collision P moves up the plane coming to rest instantaneously at A. Given that Q is brought to rest by the impact, find the distance AB.

10. A bullet of mass 30 g is fired into a stationary block of wood of mass 5 kg and becomes embedded in it. The block is freely suspended from a fixed point and after impact swings to a height of 40 cm above its initial position. Find, correct to 3 significant figures, the velocity of the bullet just before impact.

11. Two beads, A of mass m and B of mass $2m$, are threaded on a smooth circular wire fixed in a vertical plane. Initially A is at rest at the lowest point O of the wire and B is released from rest at a height h above O. Just before B collides with A it is moving with velocity v and immediately after impact A starts to move with velocity v. Find, in terms of h, the height above O at which each bead first comes to rest.

12. A particle P is attached to one end of a light inextensible string of length 1·25 m, the other end being attached to a fixed point A. The string is taut and inclined at an angle of 60° to the downward vertical when P is released from rest. As P reaches its lowest point B, it strikes a stationary particle Q, of equal mass, which is free to move horizontally. If the speed of Q just after impact is $2{\cdot}1\,\mathrm{m\,s^{-1}}$, find the maximum height above B to which P rises after the collision.

13. Particles A and B, of masses 1 kg and 3 kg respectively are attached to the ends of a light inextensible string passing over a smooth light pulley. The system is released from rest with both parts of the string taut and vertical. After A has travelled 1·8 m it hits and coalesces with a stationary particle C, of mass 4 kg. Find the new velocity of the system and the impulsive tension in the string at the instant when C is picked up.

14. Particles A and B, of masses 1 kg and 2 kg respectively, are connected by a light inextensible string which passes over a smooth light pulley. The system is released from rest with the string taut and the hanging parts vertical. After 4·5 s A picks up a stationary particle of mass 3 kg. Find the velocity of the system immediately after impact and the further time which elapses before the system first comes instantaneously to rest.

15. A particle A of mass 5 kg is connected to a particle B of mass 2 kg by means of a light inextensible string which passes over a smooth light pulley. The system is released from rest with both parts of the string taut and vertical. After 3·5 s particle A hits the ground without rebounding. Assuming that particle B does not hit the pulley, find the further time which elapses before the string is again taut. Find also the speed with which A leaves the ground.

16. Particles P and Q, of masses m and $2m$ respectively, are connected by a light inextensible string which passes over a smooth light pulley. The system is released from rest with the string taut and both particles at a height h above the ground.

Assuming that Q hits the ground without rebounding and that P does not hit the pulley, find expressions for (a) the greatest height above the ground reached by P, (b) the impulsive tension in the string when Q leaves the ground again.

17. A particle A, of mass 4 kg, rests on a smooth plane inclined at an angle of 30° to the horizontal. It is connected by a light inextensible string passing over a smooth pulley fixed at the top of the plane to a particle B, of mass 3 kg, which hangs freely. The system is released from rest with both parts of the string taut. When B has fallen a distance of 6·3 m A picks up a particle C, of mass 3·5 kg, which was previously at rest. Find the velocity of the system immediately after C has been picked up. Find also the loss in kinetic energy as C is picked up, giving your answer correct to 3 significant figures.

18. A pile of mass M is being driven into the ground by a pile-driver of mass m. The pile-driver is released from rest at a height h above the top of the pile and falls freely under gravity on to the top of the pile. The pile-driver rebounds vertically from the top of the pile and, after the first impact, it comes first to rest at a point $\frac{3}{4}h$ below the original point of release. (i) Find the kinetic energy lost by the pile-driver in the first impact. (ii) Find the kinetic energy gained by the pile in the first impact, assuming that the momentum of the system is conserved in the impact. (iii) Given that the pile is driven a distance $\frac{1}{16}h$ into the ground by the first impact, show that the frictional resistance, assumed constant, between the pile and the ground is $(36m^2 + M^2)g/M$. (C)

19. Two particles, A and B, of masses $2m$ and $3m$ respectively, are attached to the ends of a light inextensible string of length c and are placed close together on a horizontal table. The particle A is projected vertically upwards with speed $\sqrt{(6gc)}$. (i) Show that, at the instant immediately after the string tightens, B is moving with velocity $\frac{4}{5}\sqrt{(gc)}$. (ii) State the impulse of the tension in the string. (iii) Find the height to which A rises above the table before it comes to instantaneous rest. (iv) Calculate the loss in kinetic energy due to the tightening of the string. (AEB 1979)

20. A cube of mass 100 g and side 14 cm is at rest on a smooth horizontal table. A bullet of mass 20 g is fired with velocity 400 m s^{-1} into the centre of a vertical face of the cube. The bullet travels through the cube in a direction perpendicular to this face and emerges undeflected with velocity 200 m s^{-1}. Find the velocity of the cube as the bullet emerges. Find the kinetic energy gained by the cube and the kinetic energy lost by the bullet. If the cube moves a distance of d cm before the bullet emerges, write down the distance moved by the bullet as it passes through the cube. By considering the work done on the cube and on the bullet, determine the value of d and the magnitude of the constant force exerted by the cube on the bullet.

12.4 Newton's law of restitution

In a collision between two elastic bodies the impulses exerted by each body on the other produce changes in momentum and the bodies bounce away from each other.

If both bodies are moving along the line of action of these impulses at the instant the collision occurs, then the impact is said to be *direct*. Otherwise it is described as *indirect* or *oblique*. Both types of elastic impact were investigated by Sir Isaac Newton and as a result of his experimental work he was able to formulate a law of impact which is known as *Newton's law of restitution* or as *Newton's experimental law*. In its general form the law concerns the indirect impact of two moving bodies. However, we first consider its application to the direct impact of one body with a fixed plane.

Our everyday experience tells us that if several objects made of different materials are dropped from a given height on to horizontal ground, then the various objects will probably rebound to different heights after impact. A lump of plasticine will become flattened and show little tendency to rebound, whereas a "superball" will regain most of its original height. The behaviour of most objects will come somewhere between these two extremes.

Further observation shows that when a body strikes a fixed plane there is a period of time during which the body is compressed and its speed is reduced to zero. This is followed by a period of restitution in which the body regains its original shape and starts to rebound. While the body is being compressed some of its original kinetic energy is stored as elastic potential energy and this is then released as the body rebounds. The remaining kinetic energy is converted into other forms such as heat energy. Thus, the behaviour of a particular body during impact depends on its elasticity.

Experimental evidence suggests that in a series of direct collisions between a given elastic body a fixed plane the ratio of the speed of the body after impact to its speed before impact is approximately constant. Thus, in its simplest form, Newton's law of restitution states that if the speed of the body before impact is u and its speed after impact is v, then

$$v = eu,$$

where e is constant for that particular body. The constant e is known as the *coefficient of restitution* between the body and the plane and its value always lies between 0 and 1. If $e = 0$, collisions between the body and the plane are said to be *inelastic*, but if $e = 1$ they are described as *perfectly elastic*.

Example 1 A ball dropped from a height of 9 m above horizontal ground rebounds to a height of 4 m. Find the coefficient of restitution between the ball and the ground.

Substituting $u = 0, a = g, s = 9$ in the formula $v^2 = u^2 + 2as$,
we have: $v^2 = 0 + 2g \times 9$
$\therefore \quad v = 3\sqrt{(2g)}$

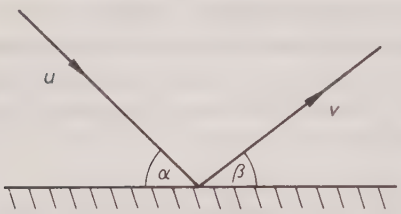
Hence the speed of the ball just before impact is $3\sqrt{(2g)}\,\mathrm{m\,s^{-1}}$.
Substituting $v = 0, a = -g, s = 4$ in the formula $v^2 = u^2 + 2as$,
we have: $0 = u^2 - 2g \times 4$
$\therefore \quad u = 2\sqrt{(2g)}$

Hence the speed of the ball just after impact is $2\sqrt{(2g)}\,\mathrm{m\,s^{-1}}$.

Thus, if e is the coefficient of restitution,

$$2\sqrt{(2g)} = e \times 3\sqrt{(2g)}$$
$$\therefore \quad e = \tfrac{2}{3}.$$

When considering indirect impacts between an elastic body and a smooth fixed plane, it is found that Newton's law of restitution can be applied to motion perpendicular to the plane.



Let us suppose that a body strikes a smooth fixed plane while travelling with speed u in a direction inclined at an angle α to the plane, then rebounds with speed v at an angle β to the plane.

Since the plane is smooth no force acts along the plane and therefore there is no change in the component of velocity along the plane

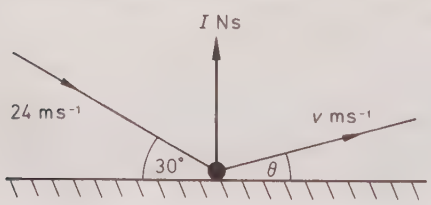
i.e. $$v\cos\beta = u\cos\alpha.$$

However, the impulse of the normal reaction of the plane changes the component of velocity perpendicular to the plane in accordance with Newton's law,

i.e. $$v\sin\beta = eu\sin\alpha,$$

where e is the coefficient of restitution.

Example 2 A particle of mass 2 kg strikes a fixed smooth plane while moving at a speed of $24\,\mathrm{m\,s^{-1}}$ in a direction making an angle of 30° with the plane. If the coefficient of restitution between the particle and the plane is $\frac{1}{4}$, find the speed at which the particle rebounds. Find also the impulse exerted by the plane on the particle.



Let the velocity of the particle after impact be $v\,\mathrm{m\,s^{-1}}$ at an angle θ to the plane and let the impulse exerted by the plane be I N s.

There is no change in velocity parallel to the plane

$$\therefore \quad v\cos\theta = 24\cos 30° = 12\sqrt{3}.$$

By Newton's law of restitution perpendicular to the plane,

$$v\sin\theta = \tfrac{1}{4} \times 24\sin 30° = 3$$

Squaring and adding: $v^2\cos^2\theta + v^2\sin^2\theta = (12\sqrt{3})^2 + 3^2$

$$\therefore \quad v = \sqrt{(432 + 9)} = 21.$$

Hence the speed of the particle after impact is $21\,\mathrm{m\,s^{-1}}$.

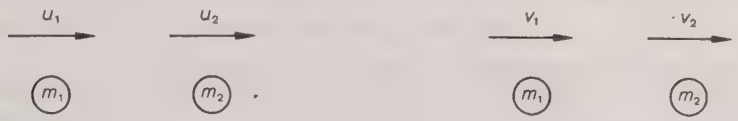
Considering the motion of the particle perpendicular to the plane,

impulse = change in momentum

$\therefore \quad I = 2 \times v \sin\theta + 2 \times 24 \sin 30° = 30.$

Hence the impulse exerted by the plane is 30 N s.

When applied to the direct impact of two moving bodies, Newton's law of restitution states that the ratio of the relative speed of separation after impact to the relative speed of approach before impact is constant.



Before impact ($u_1 > u_2$) After impact ($v_1 < v_2$)

Let us suppose that two spheres of masses m_1 and m_2 collide when moving along the same straight line with velocities u_1 and u_2 respectively. Let the velocities of the spheres after impact be v_1 and v_2 respectively. Provided that all velocities are measured in the same direction, as shown in the diagram, the relative speed of approach is $(u_1 - u_2)$ and the relative speed of separation is $(v_2 - v_1)$.

By Newton's law of restitution

$$v_1 - v_2 = -e(u_1 - u_2)$$

where e is the coefficient of restitution.

Assuming that no external impulse acts during the impact, we can also apply the principle of conservation of momentum to give:

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

Example 3 A sphere A, of mass $2m$ and moving with speed $6u$, collides directly with a sphere B of mass $3m$ moving in the same direction with speed $3u$. After the impact the speed of B is $5u$. Find the coefficient of restitution between the spheres.

Let the speed of A after impact be v, as shown in the diagram.

Using the principle of conservation of momentum,

$$2m \times 6u + 3m \times 3u = 2m \times v + 3m \times 5u$$
$$\therefore \quad 12mu + 9mu = 2mv + 15mu$$
$$\therefore \quad v = 3u.$$

Using Newton's law of restitution,

$$v - 5u = -e(6u - 3u)$$
$$\therefore \quad 3u - 5u = -3eu$$
$$\therefore \quad e = \tfrac{2}{3}$$

Hence the coefficient of restitution between the spheres is $\frac{2}{3}$.

Example 4 A sphere of mass 5 kg, moving at $8\,\mathrm{m\,s^{-1}}$, collides directly with a sphere of mass 3 kg moving in the opposite direction at $7\,\mathrm{m\,s^{-1}}$. If the coefficient of restitution between the spheres is 0·6, find their speeds after impact. Find also the magnitude of the impulse exerted on each sphere during the collision.

Let the velocities of the spheres after impact be $v_1\,\mathrm{m\,s^{-1}}$ and $v_2\,\mathrm{m\,s^{-1}}$, as shown in the diagram.
Using conservation of momentum,

$$5 \times 8 - 3 \times 7 = 5v_1 + 3v_2$$
$$\therefore \quad 5v_1 + 3v_2 = 19 \qquad (1)$$

Using Newton's law of restitution,

$$v_1 - v_2 = -0{\cdot}6\{8 - (-7)\}$$
$$\therefore \quad v_1 - v_2 = -9 \qquad (2)$$

Adding 3 × (2) to (1): $8v_1 = -8$
Hence $v_1 = -1$ and $v_2 = 8$.

Thus after impact the speed of the 5 kg sphere is $1\,\mathrm{m\,s^{-1}}$ and the speed of the 3 kg sphere is $8\,\mathrm{m\,s^{-1}}$. The directions of both spheres are reversed in the collision.

Considering the motion of the 3 kg sphere,

$$\text{impulse} = \text{change in momentum}$$
$$= 3 \times 8 - 3 \times (-7)\,\mathrm{N\,s} = 45\,\mathrm{N\,s}$$

Hence the magnitude of the impulse exerted on each sphere is 45 N s.

Exercise 12.4

1. A ball of mass 0·2 kg is dropped from a point 10 m above horizontal ground. If the coefficient of restitution between the ball and the ground is $\frac{1}{2}$, find (a) the height above the ground to which the ball rebounds, (b) the loss of kinetic energy in the impact, (c) the impulse exerted by the ground.

2. A ball dropped from a height of 8 m above horizontal ground rebounds to a height of 4·5 m. Find the coefficient of restitution between the ball and the ground. If the ball is then dropped from a height of 12·8 m and allowed to bounce twice, find the maximum height it reaches after the second bounce.

3. A particle is projected vertically upwards with speed $7\,\mathrm{m\,s^{-1}}$ from a point A, which is 2 m above a horizontal plane. After striking this plane the particle first comes instantaneously to rest at A. Find the coefficient of restitution between the particle and the plane.

4. A particle of mass 3 kg strikes a fixed smooth plane while moving at a speed of $20\,\mathrm{m\,s^{-1}}$ in a direction making an angle α with the plane, where $\sin\alpha = 3/5$. If the coefficient of restitution between the particle and the plane is 2/3, find the speed at which the particle rebounds. Find also the impulse exerted by the plane on the particle.

5. A particle strikes a fixed smooth plane while travelling in a direction inclined at 45° to the plane. After impact its direction makes an angle of 30° with the plane. Find the coefficient of restitution between the particle and the plane. If on another occasion the particle is moving in a direction inclined at 60° to the plane, find its direction after impact with the plane.

6. A particle strikes a smooth fixed plane and rebounds in a direction at right angles to its direction just before impact. If the coefficient of restitution is $\frac{1}{3}$, find the acute angle between the initial direction of motion of the particle and the plane.

7. A particle is projected with speed $6{\cdot}5\,\mathrm{m\,s^{-1}}$ at an angle θ to the horizontal, where $\tan\theta = 12/5$, from a point A on a smooth horizontal plane. The particle strikes the plane again at B, then rebounds and strikes the plane for a second time at C. The coefficient of restitution between the particle and the plane is 5/9. Find the speed and direction with which the particle rebounds from the plane at B. Show that the distance AC is 100/21 m.

8. A sphere of mass $4m$ and moving with speed $5u$, collides directly with a sphere of mass $3m$, which is at rest. If the coefficient of restitution between the spheres is 0·4, find their speeds after impact. Find also the loss in kinetic energy due to the impact.

9. A sphere A, of mass 1 kg and moving with speed $12\,\mathrm{m\,s^{-1}}$, collides directly with a sphere B, of mass 4 kg, moving in the same direction with speed $2\,\mathrm{m\,s^{-1}}$. Given that sphere A is brought to rest by the collision, find the coefficient of restitution between the spheres.

10. A sphere moving with speed $7\,\mathrm{m\,s^{-1}}$ collides directly with an identical sphere moving in the opposite direction with speed $2\,\mathrm{m\,s^{-1}}$. Find the speeds of the spheres after impact if the coefficient of restitution is (a) $\frac{1}{3}$, (b) $\frac{2}{3}$. In each case state whether the spheres are travelling in the same direction or in opposite directions.

11. A sphere of mass m moving with speed $9u$ collides directly with a sphere of mass M moving in the same direction with speed $5u$. After impact the spheres move in the same direction with speeds $6u$ and $7u$. Find the coefficient of restitution between the spheres and the ratio $m:M$.

12. Three perfectly elastic particles A, B, C, with masses 4 kg, 2 kg, 3 kg respectively, lie at rest in a straight line on a smooth horizontal table. Particle A is projected towards B with speed $15\,\mathrm{m\,s^{-1}}$ and after A has collided with B, B collides with C. Find the velocities of the particles after the second collision and state whether there will be a third collision.

13. Three identical spheres A, B, C, lie at rest in a straight line on a smooth horizontal table. Particle A is projected towards B with speed u. After the collision between A and B, B moves on to hit C. The coefficient of restitution between any two of the spheres is $\frac{1}{2}$. Find the speeds of the spheres after the second collision and state whether there will be a third collision.

14. A sphere A of mass $2m$, moving on a smooth horizontal floor with speed $5u$, collides directly with a sphere B of mass m, which is at rest. If the coefficient of restitution between the spheres is 4/5, find the speed of B after impact. Sphere B goes on to strike a vertical wall, which is at right angles to the direction of motion of the spheres. A is then brought to rest by a second collision with B. Find the coefficient of restitution between B and the wall.

15. A small smooth sphere moves on a horizontal table and strikes an identical sphere lying at rest on the table at a distance d from a vertical wall, the impact being along the line of centres and perpendicular to the wall. Prove that the next impact between the spheres will take place at a distance $2de^2/(1 + e^2)$ from the wall, where e is the coefficient of restitution for all impacts involved. (L)

16. The coefficient of restitution between two small, smooth spheres A and B is e. The mass of A is m and of B is em. Show that, if A is at rest and B is projected with velocity u to collide directly with A, B is reduced to rest by the collision. If, instead, B is at rest and A is projected with the same speed to hit B directly, find the velocities of A and B after the collision. Show that the loss of kinetic energy is the same in each collision and is greatest when $e = \frac{1}{2}$. (SU)

17. Two small spheres of masses m and $2m$ are connected by a light inextensible string of length $2a$. When the string is taut and horizontal, its mid-point is fixed and the spheres are released from rest. The coefficient of restitution between the spheres is $\frac{1}{2}$. Show that the first impact brings the heavier sphere to rest, and the second impact brings the lighter sphere to rest. Find the velocity of each sphere immediately after the third impact. (L)

18. Three small smooth spheres A, B, C, of equal radius but of mass $m, 2m$ and m respectively, lie at rest and separated from one another on a smooth horizontal table in the order ABC with their centres in a straight line. Sphere A is projected with speed V directly towards sphere B. The coefficient of restitution at each collision is e, where $0 < e < 1$. Find, in terms of V and e, the speeds of the three spheres (a) after the first collision (between A and B), (b) after the second collision (between B and C). Hence show that there will be at least three collisions if $e^2 - 7e + 1 > 0$. (O&C)

Exercise 12.5 (miscellaneous)

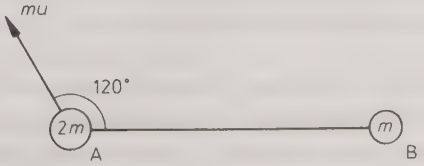
1. Two particles of masses $2m$ and $3m$ are moving towards each other with speeds $4u$ and $2u$ respectively. The direction of motion of the heavier particle is reversed by the impact and its speed after impact is u. Find (i) the magnitude of the impulse, (ii) the loss of energy. (C)

2. A particle of mass 4 kg moves under the action of a force $\mathbf{F}$ in the interval of time from $t = 0$ to $t = 2$. When $t = 0$ the velocity of the particle is $2\mathbf{i} - 3\mathbf{j}$. Given that $\mathbf{F} = 6t\mathbf{i} + 2\mathbf{j}$, find (a) the impulse of $\mathbf{F}$, (b) the velocity of the particle when $t = 2$, (c) the gain in kinetic energy.

3. A particle of mass m moves in a horizontal plane Oxy with speed v along the x-axis in the positive direction. It is subjected to a horizontal impulse $\mathbf{I}$ which turns its direction of motion through 30° in an anticlockwise sense and reduces its speed to $v/\sqrt{3}$. Find the vector $\mathbf{I}$. At the same instant an impulse $-\mathbf{I}$ is applied to a particle of mass $3m$ which is at rest. Find the magnitude and direction of the resultant velocity of this particle. (JMB)

4. A pump raises water from an underground reservoir through a height of 10 m and delivers it at a speed of $9\,\mathrm{m\,s^{-1}}$ through a circular pipe of internal diameter 20 cm. Taking 1 litre of water to have a mass of 1 kg and g to have the value $9{\cdot}8\,\mathrm{m\,s^{-2}}$ find (i) the mass of water raised per second, correct to 3 sig. fig.; (ii) the kinetic energy imparted to the water each second, correct to 3 sig. fig.; (iii) the effective power of the pump correct to 2 sig. fig.; (iv) the actual power of the pump if it is 70% efficient. If the water hits a vertical wall horizontally and does not rebound calculate the magnitude of the force exerted on the wall, correct to 2 sig. fig. (SU)

5. A light inextensible string AB has a particle of mass $2m$ attached at A and a particle of mass m attached at B. The particles are placed on a smooth horizontal table with the string taut. A horizontal impulse is applied to the particle at A of magnitude mu in a direction which makes an angle of 120° with AB. (i) Show that B



starts to move along BA with speed $u/6$. (ii) Find the components of the initial velocity of the particle at A along BA and perpendicular to BA and state the magnitude of the impulse of the tension in the string. (AEB 1977)

6. A composite particle of mass $m_1 + m_2$ is formed as a result of a head-on collision between two particles of masses m_1, m_2 travelling with speeds u_1, u_2 respectively. Find the speed of the composite particle in terms of m_1, m_2, u_1, u_2. Show that the collision leads to a loss of kinetic energy of amount

$$\frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right)(u_1 + u_2)^2. \qquad \text{(W)}$$

7. A gun of mass M fires a shell of mass m horizontally. The recoil of the gun is opposed by a constant force F which brings the gun to rest in t seconds. Show that the energy of the explosion is $\dfrac{F^2t^2(M + m)}{2Mm}$. (O)

8. A particle of mass m is dropped from rest from a point A which is at a height h above a horizontal surface. After hitting the surface the particle rebounds vertically and reaches a maximum height of $\frac{1}{2}h$ above the surface at the point B. Find expressions for the loss of energy and the impulse on the particle at the impact. At the instant of the particle's impact with the surface, a second particle, of mass $4m$, is dropped from rest at A. Show that the two particles collide at B. The two particles coalesce when they collide at B. When the combined particle hits the surface, one-half of its kinetic energy is lost in the impact. Show that, after the impact, the combined particle reaches a greatest height of $41h/100$ above the surface. (C)

9. Two beads, one of mass 3 g and the other of mass 2 g, are threaded on a smooth circular wire which is fixed with its plane vertical. The heavier bead is initially at rest at the lowest point L of the wire. The lighter bead is released from rest and is moving with speed $2\ \text{m s}^{-1}$ when it strikes the other bead. There is no loss of energy in the impact. Find the speed of each bead immediately after impact. In the subsequent motion each bead comes to rest instantaneously before reaching the highest point of the wire. Find, to the nearest millimetre, the height above L at which each bead comes first to rest. (C)

10. A pile-driver of mass 2000 kg is raised 1·6 m above a pile of mass 1500 kg and then dropped on to it. After impact the driver and the pile move together and the pile is driven 0·2 m into the ground. Find (i) the velocity of the pile-driver just before it hits the pile, (ii) the common velocity of pile and driver immediately after impact, (iii) the loss of energy on impact, stating the units in which it is measured, (iv) the mean resistance of the ground to penetration. (AEB 1980)

11. A particle A of mass $3m$ lies on a smooth horizontal rectangular table at a distance $5a$ from a point C at the edge of the table. The particle is attached to one end of a light inelastic string of length $6a$ which passes over a small smooth pulley at C.

To the other end of the string is attached a particle B of mass $2m$. The horizontal floor is at a depth $4a$ below C. The particle B is released from rest at C and motion takes place in a vertical plane perpendicular to the edge of the table. Show that the speed of the particles immediately after the string becomes taut is $\frac{2}{5}\sqrt{(2ga)}$. Find (i) the magnitude of the impulse which acts on the particle A, (ii) the loss of kinetic energy of the system due to the string becoming taut, (iii) the speed of the particle B just before it hits the floor. (C)

12. Two masses, one 3 kg and the other 2 kg, are connected by a light inextensible string which passes over a fixed smooth pulley and the system is released from rest. Find the acceleration of the system and show that the tension in the string is 23·52 N. After falling 2 metres the heavier mass strikes a fixed horizontal plane and is brought to rest. If the lighter mass does not reach the pulley, show that after 4/7 seconds the 3 kg mass is again in motion. Find the speed with which it begins to ascend and the impulsive tension in the string. (W)

13. Two particles A and B, each of mass m and connected by a light inextensible string of length $4a$, are resting on a smooth horizontal table with $AB = 2a$. A particle C of mass $3m$ is attached to the midpoint of the string and rests on the table so that $AC = BC$. The particles A and B are then simultaneously projected along the table with equal velocities perpendicular to AB and in the sense away from C. Show that, when the string tightens, the initial velocities of the particles A and B are each inclined at angle $\frac{1}{3}\pi$ with the line AB. (O)

14. A particle P of mass $2m$ is connected to a particle Q of mass $3m$ by means of a light inextensible string which passes over a small smooth pulley. The particles are released from rest with the string taut and its hanging parts vertical. (i) Calculate the acceleration of P and the tension in the string. At the instant when P is moving with velocity v it collides and coalesces with a stationary particle of mass $3m$ to form a particle P_1. (ii) Prove that the velocity of P_1 immediately after the collision is $5v/8$. (iii) Calculate the magnitude of the impulse of the tension in the string. (iv) Calculate the loss in kinetic energy of the whole system due to the collision.

The string breaks and particle Q hits a horizontal floor when moving with speed u and bounces to a height $u^2/(6g)$ above the floor. (v) Calculate the coefficient of restitution between Q and the floor. (AEB 1979)

15. A bullet of mass m travelling horizontally with speed u strikes perpendicularly a wooden block of mass M which is free to move horizontally in the line of motion of the bullet. The resistance to penetration is assumed uniform and the bullet comes to rest after penetrating a distance a into the block. Show that the ultimate common velocity of the system is $mu/(m + M)$ and that the total loss of kinetic energy during the penetration is $mMu^2/2(m + M)$. Deduce the value of the resistance and show that (i) the block moves a distance $am/(m + M)$ before the bullet comes to rest relative to the block, (ii) the penetration by the bullet ceases after a time $2a/u$. (W)

16. A particle of mass m is attached to one end of a light inextensible string of length a, the other end being attached to a fixed point O. Initially the string is taut and

makes an angle α ($\leqslant 90°$) with the downward vertical at O. The particle is released and, when it is at its lowest point A, coalesces with a stationary particle of mass m. The combined body next comes to rest with the string making an angle β with the downward vertical at O. Show that $1 - \cos\beta = \frac{1}{4}(1 - \cos\alpha)$.

The motion continues so that each time the body passes through A it coalesces with a stationary particle of mass m. If $\alpha = 90°$, (a) prove that the speed of the body just before passing through A for the fourth time is $\frac{1}{4}\sqrt{(2ag)}$, (b) find the angle (correct to the nearest degree) made by the string with the downward vertical at O when the body next comes to instantaneous rest after passing through A for the fourth time, (c) find the ratio of the tensions in the string just before and just after the body passes through A for the fourth time. (O&C)

17. Two equal particles are projected at the same instant from points A and B on horizontal ground, the first from A with speed u at an angle of elevation α and the second from B with speed v at an angle of elevation β. They collide directly when they are moving horizontally in opposite directions. Find v in terms of u, α and β, and show that $AB = \dfrac{u^2 \sin\alpha \sin(\alpha + \beta)}{g \sin\beta}$. The particles coalesce on impact. Find, in terms of u, α and β, the speed of the combined particle immediately after the collision. In the case when $\alpha = 30°$ and $\beta = 60°$ find AC, where C is the point where the combined particle returns to the ground. (JMB)

18. A ball is thrown vertically upwards from the floor with velocity V. It rebounds from the ceiling, which is at height h above the floor, and then rebounds from the floor. After this second rebound, it just reaches the ceiling again. If the coefficient of restitution between the ball and ceiling is e, and between the ball and floor is f, prove that

$$V^2 = \frac{2gh(1 - f^2 + e^2 f^2)}{e^2 f^2}. \qquad \text{(SU)}$$

19. A particle of mass m strikes a fixed smooth plane at speed u while moving in a direction inclined at an angle α to the plane, then rebounds at an angle β to the plane. Find expressions for (a) the coefficient of restitution between the particle and the plane, (b) the speed of the particle after impact, (c) the impulse exerted on the plane.

20. A particle of mass m, projected from a point A with speed u at an angle of elevation α, strikes a smooth vertical wall at a point B and after rebounding returns to A. Show that the total time taken to move from A to B and then back to A is $(2u \sin\alpha)/g$. Given that e is the coefficient of restitution between the particle and the wall, find expressions for (a) the loss of kinetic energy on impact, (b) the distance of A from the wall.

21. Two elastic spheres of masses m and M collide directly. Their velocities before impact are 12 m/s and 6 m/s respectively in the same direction, and after impact they are 5 m/s and 8 m/s respectively also in the same direction. Find the coefficient of restitution and also the ratio $m:M$. Find also the velocities of the spheres after

impact if they collide directly when travelling in opposite directions, the velocity of the heavier sphere being 4 m/s and that of the lighter sphere 14 m/s. (L)

22. Three small spheres A, B and C, of equal radii and masses m, $2m$ and $3m$ respectively, are placed with their centres in a straight line on a smooth horizontal table with B between A and C. The sphere A is given a velocity $5u$ in the direction $\overrightarrow{AB}$ and, as a result of the ensuing collision between A and B, B moves towards C with speed $3u$. Calculate (i) the magnitude and the sense of the velocity of A after the collision, (ii) the coefficient of restitution between A and B, (iii) the loss in kinetic energy due to the collision between A and B.

The sphere B strikes C and as a result C receives an impulse of $4mu$. Calculate (iv) the velocities of B and C after their collision, (v) the coefficient of restitution between B and C. (AEB 1978)

23. A small sphere A, of mass m moving with velocity $2u$ on a smooth horizontal plane, impinges directly on a small sphere B, of equal radius and mass $2m$ moving with velocity u in the same direction on the plane. Given that after impact B moves with velocity $3u/2$, calculate (i) the coefficient of restitution between A and B, (ii) the loss in kinetic energy due to the impact. The sphere B continues to move with velocity $3u/2$ until it hits a vertical wall from which it rebounds and is then brought to rest by a second impact with the approaching sphere A. Calculate (iii) the coefficient of restitution between B and the wall, (iv) the final speed of the sphere. (AEB 1980)

24. Particles A and B of mass m and $2m$ respectively lie at rest on a smooth horizontal plane. They are projected towards each other both with speed u. Show that, if the coefficient of restitution between the particles is $\frac{3}{4}$, their directions of motion are reversed on impact and find their speeds after impact. Find, also, the loss of kinetic energy during impact. If A subsequently hits a smooth vertical plane normally, show that A and B will collide again only if the coefficient of restitution between A and the vertical plane is greater than $1/8$. (SU)

25. Two particles A, B of masses M, m respectively are attached to a fixed point O by light inextensible strings of equal length. The particle A is held so that its string is taut and horizontal. It is then released so that it strikes the particle B which was hanging in equilibrium. Prove that if OB just reaches the horizontal then $Me = m$, where e is the coefficient of restitution between the particles. (O)

26. Two particles, A of mass $2m$ and B of mass m, moving on a smooth horizontal table in opposite directions with speeds $5u$ and $3u$ respectively, collide directly. Find their velocities after the collision in terms of u and the coefficient of restitution e. Show that the magnitude of the impulse exerted by B on A is $16mu(1 + e)/3$. Find the value of e for which the speed of B after the collision is $3u$. Moving at this speed B subsequently collides with a stationary particle C of mass km, and thereafter remains attached to C. Find the velocity of the combined particle and find the range of values of k for which a third collision will occur. (JMB)

27. Identical beads P and Q are threaded on a smooth circular hoop fixed in a horizontal plane. Initially Q is at rest at a point A of the hoop when P collides with it. Denoting the coefficient of restitution between P and Q by e, find the values of e for which the next collision takes place at A. (JMB)

13 Further particle dynamics

13.1 Acceleration as a function of displacement

Let us consider a particle moving along a straight line so that at time t its displacement from a fixed point on the line is s. The velocity v and the acceleration a of the particle at time t are given by:

$$v = \frac{ds}{dt} \quad \text{and} \quad a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

Suppose now that the acceleration a is regarded as a function of the displacement s, then, using the chain rule,

$$a = \frac{dv}{dt} = \frac{dv}{ds} \times \frac{ds}{dt} = \frac{dv}{ds} \times v = v\frac{dv}{ds}$$

$\therefore$ if $a = f(s)$, then $$v\frac{dv}{ds} = f(s),$$

giving $$\int v\,dv = \int f(s)\,ds$$

$\therefore$ $$\tfrac{1}{2}v^2 = \int f(s)\,ds$$

i.e. $$v^2 = 2\int f(s)\,ds.$$

Thus if the acceleration a is a function of the displacement s, the relationship $a = v\dfrac{dv}{ds}$ may be used to obtain an expression for v^2 in terms of s.

Example 1 A particle of mass m moves in a straight line under the action of a force directed towards a fixed point O of the line. The magnitude of the force is $3ms^2$, where

s is the displacement of the particle from O at time t. Given that the particle starts from rest at a distance $2a$ from O, find the speed of the particle as it reaches O.

If v is the velocity of the particle at time t, then its acceleration is $v\dfrac{dv}{ds}$ in the direction of the displacement s. Since the force of magnitude $3ms^2$ acts in the opposite direction,

by Newton's second law, $\quad mv\dfrac{dv}{ds} = -3ms^2$

$$\therefore \qquad v\frac{dv}{ds} = -3s^2$$

$$\therefore \qquad \int v\,dv = -\int 3s^2\,ds$$

$$\tfrac{1}{2}v^2 = -s^3 + c$$

But $v = 0$ when $s = 2a$,

$$\therefore \qquad 0 = -8a^3 + c \quad \text{i.e.} \quad c = 8a^3$$

Thus $\qquad v^2 = 2(8a^3 - s^3)$,

and when $s = 0$, $\qquad v^2 = 16a^3$.

Hence when the particle reaches O its speed is $4a^{3/2}$.

Example 2 Assuming that the earth is a sphere of radius R, the acceleration due to gravity of an object at a distance x from the centre of the earth is inversely proportional to x^2 when above the earth's surface and equal to g on the surface. Given that an object projected vertically upwards from the earth's surface with speed u rises to a maximum height R above the surface, find u in terms of g and R.

The acceleration of the object is inversely proportional to the square of the displacement x and in the opposite direction,

$$\therefore \qquad v\frac{dv}{dx} = -\frac{k}{x^2},$$

where v is the velocity and k is constant.

Since the acceleration is $-g$ when $x = R$,

$$-g = -\frac{k}{R^2} \quad \text{giving} \quad k = gR^2.$$

$$\therefore \qquad v\frac{dv}{dx} = -\frac{gR^2}{x^2}$$

$$\therefore \qquad \int v\,dv = -\int \frac{gR^2}{x^2}\,dx$$

$$\tfrac{1}{2}v^2 = \frac{gR^2}{x} + c$$

Given that $v = 0$ when $x = 2R$,

$$0 = \tfrac{1}{2}gR + c \quad \text{i.e.} \quad c = -\tfrac{1}{2}gR$$

$$\therefore \quad v^2 = \frac{2gR^2}{x} - gR$$

Since $v = u$ when $x = R$, $u^2 = \dfrac{2gR^2}{R} - gR = gR$

Hence $u = \sqrt{(gR)}$.

Exercise 13.1

In questions 1 to 4 a particle moves in a straight line so that after t seconds its displacement from a fixed point on the line is s metres and its velocity is $v\,\mathrm{m\,s^{-1}}$.

1. Given that the acceleration of the particle is $1/(1 + t)\,\mathrm{m\,s^{-2}}$ and that the particle starts from rest, find its speed after 2 seconds.

2. Given that the particle is subject to a retardation of $5e^{-t}\,\mathrm{m\,s^{-2}}$ and that its initial speed is $4\,\mathrm{m\,s^{-1}}$, find the time at which the particle first comes to rest.

3. Given that the acceleration of the particle is $2s\,\mathrm{m\,s^{-2}}$ and that $v = 2$ when $s = 0$, find the speed of the particle when $s = 4$.

4. Given that the particle is subject to a retardation of $2\sin^2 s\,\mathrm{m\,s^{-2}}$ and that $v = 1$ when $s = \pi/4$, find, correct to 3 significant figures, the speed of the particle when $s = \pi/3$.

5. A particle of mass m moving along a straight line is subject to a force $4mx$ acting towards a fixed point O on the line, where x is the displacement of the particle from O. Given that the particle starts from rest at a distance a from O, find the speed of the particle as it passes through O.

6. The acceleration due to gravity of an object at a distance x from the centre of the earth is inversely proportional to x^2 when above the earth's surface and equal to g on the surface. The earth is assumed to be a sphere of radius R. If an object falls from rest at a height $2R$ above the earth's surface, find its speed when it reaches a height R above the surface.

7. A particle of mass m is moving in a straight line under the action of a force $4m/x^3$ directed towards a fixed point O of the line, x being the displacement of the particle from O. Given that initially the particle is at a distance h from O, moving away from O at a speed of $2/h$, find expressions for (a) dx/dt in terms of x, and (b) x in terms of t and h.

8. A particle of mass m is moving in a straight line under the action of a force of magnitude $\frac{1}{4}e^{2x}$, where x is the displacement of the particle from a fixed

point O on the line. Initially the particle is at a point A travelling towards O at speed u. Given that the force on the particle acts in the direction of $\vec{OA}$ and that $OA = \ln(2u)$, find an expression for dx/dt in terms of x. Hence show that $x = \ln\left(\frac{2u}{1+ut}\right)$.

13.2 Acceleration as a function of velocity

We consider again a particle moving in a straight line so that at time t its displacement from a fixed point on the line is s and its velocity is v. If the acceleration of the particle is a function of the velocity, $f(v)$ say, then we may write:

$$(1)\ \frac{dv}{dt} = f(v), \quad \text{or} \quad (2)\ v\frac{dv}{ds} = f(v).$$

Equation (1) is used when a relationship between velocity and time is required. However, equation (2) may be used to obtain a relationship between velocity and displacement.

Example 1 A particle of mass m is projected vertically upwards from a point O with speed u and after time t its velocity is v. Given that the particle moves under gravity in a medium which exerts a resistance kv per unit mass, where k is a positive constant, find the time taken by the particle to reach its greatest height.

As the particle rises to its highest point the forces acting on it are its weight mg and the resistance mkv, both directed vertically downwards.

Thus, by Newton's second law, $\quad -mg - mkv = m\frac{dv}{dt}$

$$\therefore \qquad -(g + kv) = \frac{dv}{dt}$$

$$1 = -\frac{1}{(g + kv)}\frac{dv}{dt}$$

$$\therefore \qquad \int dt = -\int \frac{1}{g + kv}\,dv$$

$$t = -\frac{1}{k}\ln(g + kv) + c$$

Since $v = u$ when $t = 0$, $\quad c = \frac{1}{k}\ln(g + ku)$

$$\therefore \qquad t = -\frac{1}{k}\ln(g + kv) + \frac{1}{k}\ln(g + ku) = \frac{1}{k}\ln\left(\frac{g + ku}{g + kv}\right)$$

$$\therefore \quad \text{when } v = 0,\ t = \frac{1}{k}\ln\left(\frac{g + ku}{g}\right)$$

Hence the time taken by the particle to reach its greatest height is $\frac{1}{k}\ln\left(1 + \frac{ku}{g}\right)$.

Example 2 A particle of mass m moves in a straight line under the action of a retarding force $mv^4/2k^4$, where k is a positive constant and v is the velocity of the particle. Initially the particle is at a distance k^2 from a fixed point A, travelling away from A at speed k. Find the speed of the particle when its displacement from A is x. Find also the time which elapses before the particle reaches the point B, where $AB = 4k^2$.

Writing the acceleration of the particle in the form $v\,dv/dx$ and using Newton's second law we have:

$$-\frac{mv^4}{2k^4} = mv\frac{dv}{dx}$$

$$\therefore \quad \frac{1}{k^4} = -\frac{2}{v^3}\frac{dv}{dx}$$

$$\therefore \quad \int \frac{1}{k^4}\,dx = \int\left(-\frac{2}{v^3}\right)dv$$

Thus
$$\frac{x}{k^4} = \frac{1}{v^2} + c$$

Since $v = k$ when $x = k^2$, $\quad \frac{1}{k^2} = \frac{1}{k^2} + c \quad$ i.e. $\quad c = 0$

$$\therefore \quad \frac{x}{k^4} = \frac{1}{v^2}$$

$$v^2 = \frac{k^4}{x}$$

Hence when the displacement of the particle from A is x, its speed is $k^2/\sqrt{x}$.
$\therefore$ if x is the displacement from A at time t,

$$\frac{k^2}{\sqrt{x}} = \frac{dx}{dt}$$

$$\therefore \quad \int k^2\,dt = \int x^{1/2}\,dx$$

$$k^2 t = \tfrac{2}{3}x^{3/2} + C$$

Since $x = k^2$ when $t = 0$, $\quad C = -\tfrac{2}{3}k^3$

$$\therefore \quad k^2 t = \tfrac{2}{3}(x^{3/2} - k^3)$$

Thus when $x = 4k^2$, $\quad k^2 t = \tfrac{2}{3}(8k^3 - k^3) = 14k^3/3$

Hence the time which elapses before the particle reaches B is $14k/3$.

Example 3 A car of mass m moving on a straight horizontal road is subject to a constant resistance R. The engine is working at a constant rate kR. Given that in time t the car accelerates to a speed v from rest, where $v < k$, find an expression for t in terms of m, R, k and v. Find also an expression for the distance x travelled by the car in this time.

If the tractive force exerted by the engine is F when the car is travelling at speed v, then its rate of working $= kR = Fv$,

$$\therefore \qquad F = kR/v$$

Using Newton's second law, the equation of motion is

$$\frac{kR}{v} - R = m\frac{dv}{dt}$$

$$\therefore \qquad R\left(\frac{k-v}{v}\right) = m\frac{dv}{dt} \tag{1}$$

$$\therefore \qquad \int R\,dt = \int m\left(\frac{v}{k-v}\right)dv$$

$$= \int m\left(\frac{k}{k-v} - 1\right)dv$$

$$\therefore \qquad Rt = m\{-k\ln(k-v) - v\} + c$$

Since $v = 0$ when $t = 0$, $\quad c = mk\ln k$

$$\therefore \qquad Rt = m\{-k\ln(k-v) - v + k\ln k\}$$

Hence $$t = \frac{m}{R}\left\{k\ln\left(\frac{k}{k-v}\right) - v\right\}$$

Substituting $\dfrac{dv}{dt} = v\dfrac{dv}{dx}$ in equation (1),

$$R\left(\frac{k-v}{v}\right) = mv\frac{dv}{dx}$$

$$\therefore \qquad \int R\,dx = \int m\left(\frac{v^2}{k-v}\right)dv$$

$$\begin{array}{r} -v-k \\ -v+k\,\overline{)\,v^2} \\ \underline{v^2 - kv} \\ kv \\ \underline{kv - k^2} \\ k^2 \end{array}$$

By long division

$$\frac{v^2}{k-v} = -v - k + \frac{k^2}{k-v}$$

$$\therefore \qquad \int R\,dx = \int m\left\{\frac{k^2}{k-v} - v - k\right\}dv$$

$$\therefore \qquad Rx = m\{-k^2\ln(k-v) - \tfrac{1}{2}v^2 - kv\} + C$$

Since $v = 0$ when $x = 0$, $\quad C = mk^2\ln k$

$$\therefore \qquad Rx = m\{-k^2\ln(k-v) - \tfrac{1}{2}v^2 - kv + k^2\ln k\}$$

Hence $$x = \frac{m}{R}\left\{k^2\ln\left(\frac{k}{k-v}\right) - \tfrac{1}{2}v^2 - kv\right\}.$$

In certain types of problem it is quicker to use definite integrals rather then indefinite integrals. For instance, if in Example 3 we had been asked to find the time which elapses as the car accelerates from speed $\frac{1}{3}k$ to speed $\frac{2}{3}k$, we could have deduced from equation (1) that:

$$\int_{t_1}^{t_2} R\,dt = \int_{k/3}^{2k/3} m\left(\frac{v}{k-v}\right)dv$$

where $t = t_1$ when $v = \frac{1}{3}k$ and $t = t_2$ when $v = \frac{2}{3}k$

$$\therefore \quad R\int_{t_1}^{t_2} dt = m\int_{k/3}^{2k/3}\left(\frac{k}{k-v} - 1\right)dv$$

$$\therefore \quad R\Big[t\Big]_{t_1}^{t_2} = m\Big[-k\ln(k-v) - v\Big]_{k/3}^{2k/3}$$

$$\begin{aligned}\therefore \quad R(t_2 - t_1) &= m(-k\ln\tfrac{1}{3}k - \tfrac{2}{3}k) - m(-k\ln\tfrac{2}{3}k - \tfrac{1}{3}k)\\ &= mk(\ln\tfrac{2}{3}k - \ln\tfrac{1}{3}k + \tfrac{1}{3} - \tfrac{2}{3})\\ &= mk(\ln 2 - \tfrac{1}{3})\end{aligned}$$

Hence the time taken is $\dfrac{mk}{R}(\ln 2 - \frac{1}{3})$.

Exercise 13.2

1. A particle of unit mass, moving in a straight line with speed v at time t, is subject to a retarding force of magnitude $(3 + 2v)$. Given that initially $v = 66$, find the value of t when $v = 6$.

2. A particle travelling in a straight line is subject to a retardation of $(1 + v^2)$, where v is its speed. If the initial speed of the particle is u, find an expression for the distance it travels before coming to rest.

3. A particle is moving in a straight line so that when the displacement of the particle from a fixed point on the line is x, its velocity is v. Given that the acceleration of the particle is $k(1 - v)$, where k is constant and that $x = 0$ when $v = 0$, find an expression for x in terms of v.

4. A particle is projected horizontally from a point O on a horizontal plane with an initial velocity $2a/b$ where a and b are positive constants. The particle is subject to a retardation $a + bv$ where v is the velocity at time t from the commencement of the motion. Show that $bv = a(3e^{-bt} - 1)$ and, using this result or otherwise, find the distance gone by the particle when it comes to instantaneous rest. (O&C)

5. The retardation of a particle moving in a straight line with speed v is proportional to v^3. The initial speed of the particle is u and s is the displacement of the particle from its initial position at time t. Given that $v = \frac{1}{2}u$ when $s = 2/u$, find an expression for v in terms of u and s. Hence find an expression for t in terms of u and s.

6. Show that the acceleration of an object moving along a straight line may be written as $v\,dv/ds$.

A vehicle of mass 2500 kg moving on a straight course is subject to a single resisting force in the line of motion of magnitude kv newtons, where v metres per second is the velocity and k is constant. At 100 km/h this force is 2000 N. The vehicle is slowed down from 100 km/h to 50 km/h. Find (i) the distance travelled, (ii) the time taken. (AEB 1980)

7. A particle falls from rest and, at any moment, it experiences a resistance of size kmv, where m is the mass of the particle, v is its velocity and k is a constant.

(i) Show that, from the equation of motion we can deduce

$$\int \frac{dv}{g - kv} = \int dt.$$

(ii) Using this equation, show that the velocity at any time is

$$\frac{g}{k}(1 - e^{-kt}).$$

(iii) Show that, no matter for how long the particle falls, its velocity never exceeds a certain value. If this value is V, show that k is g/V. (SU)

8. A particle of mass m moving under gravity is subject to a resistance of magnitude mkv^2, where v is its speed and k is constant. Initially the particle is travelling vertically downwards at speed u, where $u^2 < g/k$. Find the speed of the particle when it has fallen a distance s.

9. A train, when braking, is subject to a retardation of $\left(1 + \frac{v}{100}\right)\text{m s}^{-2}$, where v is its speed at any time. The brakes are applied when it is travelling at $20\,\text{m s}^{-1}$.

(i) Show that it takes just over 18 seconds to come to rest.

(ii) Show that it will travel approximately 177 m after the brakes are applied.

$$\left[\text{It may prove useful to know that } \frac{v}{v + 100} \text{ can be written as } 1 - \frac{100}{v + 100}\right].$$

(iii) If it then accelerates with acceleration $\left(1 + \frac{v^2}{100}\right)\text{m s}^{-2}$, find, correct to 3 significant figures, how long it takes to attain a speed of $20\,\text{m s}^{-1}$. (SU)

10. A particle of mass m moves, in the direction of x increasing, along the positive x-axis under the action of two forces. These are (i) a force toward the origin O of magnitude kv, and (ii) a force away from O of magnitude kU^2/v, where k and U are positive constants and v is the speed of the particle at time t. Write down the equation of motion of the particle in the form of a differential equation in v and t, and show that the time taken for the particle to accelerate from speed $\frac{1}{4}U$ to speed $\frac{1}{2}U$ is $\frac{m}{2k}\ln\frac{5}{4}$.

Write the equation of motion in the form of a differential equation in v and x, and hence show that the distance travelled by the particle while it accelerates from speed $\frac{1}{4}U$ to speed $\frac{1}{2}U$ is given by

$$\frac{m}{k}\int_{U/4}^{U/2}\left(\frac{U^2}{U^2-v^2}-1\right)dv.$$

Show that this distance is $\dfrac{mU}{4k}\left(2\ln\dfrac{9}{5}-1\right)$. (C)

11. A particle of mass m moves in a straight line under the action of a retarding force of magnitude $k(4+v^2)$, where v is the speed of the particle at time t and k is constant. If the initial speed of the particle is u, find an expression of the time taken by the particle to come to rest. Find also an expression, in terms of m, k and u, for the distance travelled in this time.

12. An engine of mass m kilograms, driven by a constant tractive force of P newtons, experiences a resistance which is k times its speed in metres/second. Show that, when moving horizontally from rest, its speed t seconds after the start is $\dfrac{P}{k}(1-e^{-kt/m})$ metres/second. Hence deduce that it will have travelled mP/ek^2 metres in m/k seconds. What power is being developed by the engine when the time is m/k seconds after the start? (W)

13. A train is pulled along a level track by an engine which exerts a constant pull P newtons at all speeds and the total resistance to motion varies as the square of the speed $v\,\mathrm{m\,s^{-1}}$. The mass of the engine and train combined is 161 700 kg, the maximum speed attained is $120\,\mathrm{km\,h^{-1}}$ and the power then developed is 1078 kW. Show that (a) $P = 32\,340$, (b) $5\times10^4 v\dfrac{dv}{ds} = 10^4 - 9v^2$, where s metres is the distance travelled. Hence show that the distance travelled from rest while attaining a speed of $20\,\mathrm{m\,s^{-1}}$ is about 1240 metres. (C)

14. A train of mass M moves on a straight horizontal track. At speeds less than V the resultant force on the train is constant and equal to P; at speeds not less than V the rate of working of the force is constant and equal to PV. Show that a speed $v(>V)$ is attained from rest in a time $M(V^2+v^2)/2PV$ and that the distance travelled in this time is $M(V^3+2v^3)/6PV$. (W)

15. A car of mass m moving on a level road is subject to a constant resistance R. The engine is working at a constant rate H. Find the time taken for the speed of the car to increase from u to $2u$, where $2Ru < H$. Find also, in terms of m, R, H and u, the distance travelled in this time.

16. A car of mass m moving on a level road is subject to a resisting force kv, where v is the speed of the car at time t and k is constant. When the engine of the car is working at full power S, the car can travel at a steady speed U. Find the

time taken by the car to accelerate under full power from rest to speed V, where $V < U$. Find also the distance travelled in this time. Give both answers in terms of m, S, U and V.

13.3 Simple harmonic motion

Any motion which satisfies a differential equation of the form

$$\ddot{x} = -\omega^2 x$$

where ω is constant, is described as *simple harmonic motion* or S.H.M.

For a particle moving in a straight line, the variable x is the displacement of the particle from a fixed point on the line. However, x may also be a displacement measured along a curved path. In the case of circular motion or the rotation of a rigid body x could represent an angle.

The diagram shows a particle P executing simple harmonic motion along a straight line. If x is the displacement of P from a fixed point O and $\ddot{x} = -\omega^2 x$, then the magnitude of the acceleration of P is a constant multiple of the distance OP. Since when $x > 0, \ddot{x} < 0$ and when $x < 0, \ddot{x} > 0$, the acceleration is always directed towards O. This leads to an alternative definition of simple harmonic motion.

When a particle moves in a straight line with acceleration always directed towards a fixed point of the line and proportional to its distance from that point, the particle is said to be executing simple harmonic motion.

The differential equation $\ddot{x} = -\omega^2 x$, i.e. $\dfrac{d^2x}{dt^2} = -\omega^2 x$, can be solved by expressing the acceleration of the particle in the form $v\,dv/dx$, where v is the velocity at time t, to give:

$$v\frac{dv}{dx} = -\omega^2 x$$

$$\therefore \quad \int 2v\,dv = -\int 2\omega^2 x\,dx$$

$$v^2 = -\omega^2 x^2 + c$$

If $v = 0$ when $x = a$, we find that

$$v^2 = \omega^2(a^2 - x^2) \tag{1}$$

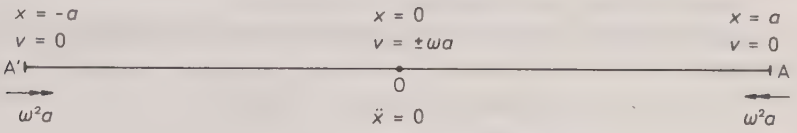
It can be shown that this leads to the general solution:

$$x = a\sin(\omega t + \varepsilon) \tag{2}$$

or the equivalent form:

$$x = A\cos\omega t + B\sin\omega t \tag{3}$$

From equations (1) and (2) we deduce that the particle is oscillating between points A and A', which lie on opposite sides of O at a distance a from O. The point O is called the *centre* or *central point* of the oscillation and a is the *amplitude*.



From equation (1) we see that the particle is momentarily at rest when it reaches A or A', and achieves its maximum speed ωa when passing through O.

Since $x = A\cos\omega t + B\sin\omega t$

and $v = -A\omega\sin\omega t + B\omega\cos\omega t$,

when ωt is increased to $\omega t + 2\pi$ the same values of x and v are obtained. This means that at times t and $t + 2\pi/\omega$ the particle passes through the same point with the same velocity. Hence $2\pi/\omega$ is the time taken to perform a complete oscillation and is called the *period* of the motion. In particular, $2\pi/\omega$ is the time taken by the particle to move from A to A' and back again to A.

The *frequency*, i.e. the number of complete oscillations per unit time, is $\omega/2\pi$. Summarising the properties of S.H.M. we have:

$\ddot{x} = -\omega^2 x$,
$v^2 = \omega^2(a^2 - x^2)$, where a is the amplitude,
$x = a\sin(\omega t + \varepsilon)$, or
$x = A\cos\omega t + B\sin\omega t$,
$T = 2\pi/\omega$, where T is the period of oscillations.

When solving problems involving S.H.M. standard formulae may be quoted without proof. It is usually best to begin be determining a and ω.

Example 1 A particle is performing simple harmonic oscillations between points A and A' which are 6 m apart. When the particle is at a distance $\sqrt{5}$ m from the mid-point O of AA' its speed is 10 m s^{-1}. Find (a) the period of the motion, (b) the maximum speed of the particle, (c) its maximum acceleration.

Let the equation of motion of the particle be

$$\ddot{x} = -\omega^2 x,$$

then

$$v^2 = \omega^2(a^2 - x^2),$$

where x m is the displacement of the particle from O, v m s^{-1} is its speed and a m is the amplitude of the motion.
But $AA' = 6\,\text{m} = 2a\,\text{m}$, which gives $a = 3$

$$\therefore \quad v^2 = \omega^2(9 - x^2)$$

Since $v = 10$ when $x = \sqrt{5}$, $100 = \omega^2(9 - 5)$ $\quad \therefore \quad \omega = 5$

(a) The period of the motion $= \dfrac{2\pi}{\omega}$ seconds $= \dfrac{2\pi}{5}$ seconds.

(b) The speed of the particle is greatest when $x = 0$
$\therefore$ the maximum speed $= \omega a\,\text{m s}^{-1} = 15\,\text{m s}^{-1}$.

(c) The acceleration of the particle is greatest when the magnitude of $\omega^2 x$ is greatest,
$\therefore$ the maximum acceleration $= \omega^2 a\,\text{m s}^{-2} = 75\,\text{m s}^{-2}$.

In S.H.M. problems involving time we need to find suitable expressions for x and v in terms of t. The forms of these expressions depend on the starting point chosen for the motion. One way of approaching such problems is to write:

$$x = A\cos\omega t + B\sin\omega t, \tag{3}$$
$$v = -A\omega\sin\omega t + B\omega\cos\omega t, \tag{4}$$

then to use any given conditions to determine A and B. We first apply this method to two important special cases.

Case I: $x = a$ when $t = 0$.

Substituting in (3): $a = A$
Since $v = 0$ when $x = a$, from (4): $B = 0$

$\therefore$

$$\boxed{x = a \quad \text{when} \quad t = 0 \;\Rightarrow\; x = a\cos\omega t.}$$

Case II: $x = 0, v > 0$ when $t = 0$.

Substituting in (3): $0 = A$
$\therefore \quad x = B\sin\omega t$
Since the amplitude of the motion is a and $v > 0$ when $t = 0$, we must have $B = a$.

$\therefore$

$$\boxed{x = 0, v > 0 \quad \text{when} \quad t = 0 \;\Rightarrow\; x = a\sin\omega t.}$$

Example 2 A particle moving in a straight line performs simple harmonic oscillations about a point O with amplitude 2 m and maximum speed 1 m s^{-1}. If P

and Q are the two points on the line which lie at a distance $\sqrt{2}$ m from O, find the time taken by the particle to move directly from P to Q.

Let the equation of motion be $\ddot{x} = -\omega^2 x$, then, assuming that $x = a$ when $t = 0$, $x = a\cos\omega t$.
Since the amplitude of the motion is 2 m, $a = 2$.
The maximum speed of the particle $= \omega a \,\mathrm{m\,s^{-1}} = 1\,\mathrm{m\,s^{-1}}$

$\therefore \quad \omega = \frac{1}{2}$
Thus $x = 2\cos\frac{1}{2}t$.

Let P and Q be the points at which $x = \sqrt{2}$ and $x = -\sqrt{2}$ respectively.

For $x = \sqrt{2}$: $\sqrt{2} = 2\cos\frac{1}{2}t$
i.e. $\cos\frac{1}{2}t = 1/\sqrt{2}$

$\therefore$ the particle first passes through P when $\frac{1}{2}t = \frac{1}{4}\pi$ i.e. when $t = \frac{1}{2}\pi$.

For $x = -\sqrt{2}$: $-\sqrt{2} = 2\cos\frac{1}{2}t$
i.e. $\cos\frac{1}{2}t = -1/\sqrt{2}$

$\therefore$ the particle first passes through Q when $\frac{1}{2}t = \frac{3}{4}\pi$ i.e. when $t = \frac{3}{2}\pi$.
Hence the particle moves directly from P to Q in a time of π seconds.

Example 3 A particle is performing simple harmonic motion about a point O with amplitude 5 m and period $\frac{1}{2}\pi$ seconds. If P is the point at which the speed of the particle is $10\,\mathrm{m\,s^{-1}}$, find the time taken by the particle to move directly from O to P.

Let the equation of motion of the particle be $\ddot{x} = -\omega^2 x$, and let $x = a\sin\omega t$, so that the particle is at O when $t = 0$.
Since the amplitude of the motion is 5 m, $a = 5$.
The period of oscillation $= \dfrac{2\pi}{\omega}\,\mathrm{s} = \dfrac{\pi}{2}\,\mathrm{s}, \qquad \therefore \quad \omega = 4.$
Thus $x = 5\sin 4t$ and $v = 20\cos 4t$.

Substituting $v = 10$: $10 = 20\cos 4t$
$\therefore$ $\cos 4t = \frac{1}{2}$

$\therefore$ the particle first passes through P when $4t = \pi/3$. Hence the particle moves directly from O to P in a time of $\pi/12$ seconds.

Example 4 A particle moves on a straight line through a fixed point O so that at time t seconds its displacement from O is x metres and its equation of motion is $\ddot{x} = -9x$. Given that $x = 5$ and $\dot{x} = -6$ when $t = \pi/6$, find the position and speed of the particle when $t = 2\pi/3$.

The general solution of the equation of motion is

$$x = A\cos 3t + B\sin 3t$$
$$\therefore \quad \dot{x} = -3A\sin 3t + 3B\cos 3t$$

Since $x = 5$ and $\dot{x} = -6$ when $t = \pi/6$,

$$5 = A\cos\tfrac{1}{2}\pi + B\sin\tfrac{1}{2}\pi$$

i.e. $5 = B$

and $-6 = -3A\sin\tfrac{1}{2}\pi + 3B\cos\tfrac{1}{2}\pi$

i.e. $-6 = -3A$

$\therefore$ $A = 2$ and $B = 5$

Thus $x = 2\cos 3t + 5\sin 3t$,

$\dot{x} = -6\sin 3t + 15\cos 3t$.

$\therefore$ when $t = 2\pi/3$, $x = 2\cos 2\pi + 5\sin 2\pi = 2$

and $\dot{x} = -6\sin 2\pi + 15\cos 2\pi = 15$

i.e. the particle is 2 m from O moving at $15\,\text{m s}^{-1}$.

Exercise 13.3

[Standard formulae may be quoted without proof.]

1. A particle moves in a straight line with simple harmonic motion of amplitude 2·5 m. If the period of oscillation is π seconds, find the maximum speed of the particle and its maximum acceleration.

2. A particle moves in a straight line with simple harmonic motion about the point O as centre. The maximum speed of the particle is $6\,\text{m s}^{-1}$ and its maximum acceleration is $18\,\text{m s}^{-2}$. Find (a) the amplitude of the motion, (b) the period of the motion, (c) the speed of the particle when it is 1 m from O.

3. A particle of mass 3 kg moves in a straight line with simple harmonic motion between two points A and A' which are 15 m apart. Given that when the particle is at a distance of 3 m from A its speed is $2\,\text{m s}^{-1}$, find the period of the motion. Find also the greatest force exerted on the particle during the motion.

4. A mass of 10 kg moves with simple harmonic motion. When it is 2 m from the centre of the oscillation, the velocity and acceleration of the body are $12\,\text{m s}^{-1}$ and $162\,\text{m s}^{-2}$ respectively. Calculate (i) the number of oscillations per minute; (ii) the amplitude of the motion; (iii) the force being applied to the body when it is at the extremities of its motion. (SU)

5. A particle P moves in a straight line so that its acceleration is always directed towards a point O in its path and is of magnitude proportional to the distance OP. When P is at the point A, where $OA = 1$ m, its speed is $3\sqrt{3}$ m/s and when P is at the point B, where $OB = \sqrt{3}$ m, its speed is 3 m/s. Calculate the maximum speed attained by P and the maximum value of OP. Show that P takes $\pi/18$ seconds to move directly from A to B. Find, in m/s correct to 2 significant figures, the speed of P one second after it passes O. (L)

6. A particle, A, is performing simple harmonic oscillations about a point O with amplitude 2 m and period 12π s. Find the least time from the instant when A passes through O until the instant when (i) its displacement is 1 m, (ii) its velocity is half that at O, (iii) its kinetic energy is half that at O. (AEB 1978)

7. A particle P is describing simple harmonic motion in the horizontal line $ADCB$, where $AD = DC = \frac{1}{2}CB$. The speed of P as it passes through C is 5 m/s and P is instantaneously at rest at A and B. Given that P performs 3 complete oscillations per second, calculate (i) the distance AB, (ii) the speed of P as it passes through D, (iii) the distance of P from C at an instant when the acceleration of P is 18π m/s^2, (iv) the time taken by P to go directly from D to A. (AEB 1978)

8. A particle P moves in a straight line with simple harmonic motion of period 2 seconds and maximum speed 4 m/s. Find the speed of P when it is at the point A which is $2/\pi$ metres from O, the centre of the path. Find also the time taken by P to move directly from O to A. When P is passing through O it strikes and adheres to a stationary particle which is free to move. If each particle is of mass 2 kg, find the kinetic energy lost in the collision. (L)

9. A particle moves in a straight line so that at time t its displacement from a fixed point on the line is x and its equation of motion is $\ddot{x} = -4x$. Given that $x = 3$ and $\dot{x} = -6$ when $t = \pi/4$, find (a) x in terms of t, (b) the values of x and $\dot{x}$ when $t = 3\pi/4$, (c) the least positive value of t for which $x = 0$.

10. A particle moves in a straight line so that at time t its displacement from a fixed point on the line is x and its equation of motion is $\ddot{x} = -16x$. Given that $x = 3$ and $\dot{x} = 4\sqrt{3}$ when $t = 0$, find (a) x in terms of t, (b) the least positive value of t for which $\dot{x} = 0$, (c) the amplitude of the oscillation.

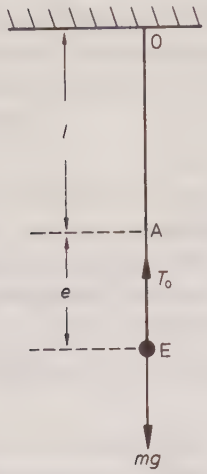
11. A particle P moves round a circle, diameter AB, with constant angular speed ω. A second particle Q moves along AB so that PQ is always perpendicular to AB, i.e. Q is the projection of P on AB. Show that the motion of Q is simple harmonic.

13.4 Forces producing simple harmonic motion

To show that a given set of forces acting on a particle produce simple harmonic motion we must prove that the equation of motion of the particle can be expressed in the form $\ddot{x} = -\omega^2 x$.

Since the tension in an elastic string is proportional to the extension in the string, it may be possible for a particle attached to an elastic string or a spring to perform simple harmonic motion.

Let us suppose that a light elastic string of natural length l and modulus of elasticity λ has one end fastened to a fixed point O. A particle of mass m is



attached to the other end. Let e be the extension in the string and T_0 the tension when the particle hangs in equilibrium at the point E.

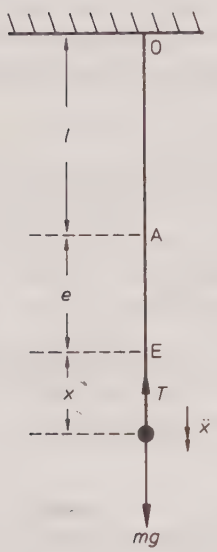
By Hooke's law, $T_0 = \dfrac{\lambda e}{l}$

Since the particle is in equilibrium,

$$T_0 - mg = 0$$

$$\therefore \quad \frac{\lambda e}{l} = mg$$

Consider now the forces on the particle when its displacement from E is x vertically downwards.



By Hooke's law, $T = \dfrac{\lambda(x+e)}{l}$

where T is the tension in the string. Applying Newton's second law vertically downwards:

$$mg - T = m\ddot{x}$$

$$\therefore \quad \frac{\lambda e}{l} - \frac{\lambda(x+e)}{l} = m\ddot{x}$$

$$\therefore \quad -\frac{\lambda x}{l} = m\ddot{x}$$

Hence the equation of motion of the particle is $\ddot{x} = -\dfrac{\lambda}{lm}x$. Since this equation is of the form $\ddot{x} = -\omega^2 x$, where $\omega = \sqrt{(\lambda/lm)}$, the particle can perform simple harmonic motion with centre E and period $2\pi\sqrt{(lm/\lambda)}$. The amplitude a of the motion is the maximum distance below E reached by the particle.

Provided that $a < e$ the string will remain taut throughout the motion and complete simple harmonic oscillations will be possible. However, if $a > e$ the particle will perform simple harmonic motion when it is below A, but move freely under gravity when the string becomes slack.

[Note that in the case of a particle suspended from a spring of natural length l, complete S.H.M. involving both extension and compression of the spring is theoretically possible when $a > e$.]

Example 1 A light elastic string of natural length l and modulus mg has one end fastened to a fixed point O. The other end of the string is attached to a particle of

mass m which hangs in equilibrium at a point E. Find the distance OE. The particle is now pulled down to a point A at a distance $5l/2$ vertically below O. If at time $t = 0$ the particle is released from rest at A, show that the subsequent motion is simple harmonic and find the speed of the particle as it passes through E. Find also the time at which the particle first passes through the point B which lies at a distance $7l/4$ vertically below O.

When the particle is at E let the tension in the string be T_0 and the extension e.

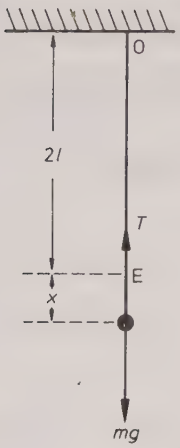
By Hooke's law: $T_0 = \dfrac{mg \times e}{l}$

But since the particle is in equilibrium, $T_0 = mg$

$$\therefore \quad \frac{mg \times e}{l} = mg$$

$$e = l$$

Hence $OE = l + e = 2l$.



Let T be the tension in the string when the displacement of the particle from E is x vertically downwards.

By Hooke's law: $T = \dfrac{mg(x + l)}{l}$

Applying Newton's second law vertically downwards:

$$mg - T = m\ddot{x}$$

$$\therefore \quad mg - \frac{mg(x + l)}{l} = m\ddot{x}$$

$$\therefore \quad \ddot{x} = -\frac{g}{l}x$$

Since this equation is of the form $\ddot{x} = -\omega^2 x$, where $\omega = \sqrt{(g/l)}$, the particle performs simple harmonic motion with centre E.
The speed v of the particle is given by the formula

$$v^2 = \omega^2(a^2 - x^2)$$

When the particle is at A, $x = \frac{1}{2}l$ and $v = 0$ $\quad \therefore \quad a = \frac{1}{2}l$

Hence $$v^2 = \frac{g}{l}(\tfrac{1}{4}l^2 - x^2).$$

$$\therefore \quad \text{when } x = 0, v^2 = \frac{g}{l} \times \frac{l^2}{4} = \frac{gl}{4}$$

Hence as the particle passes through E its speed is $\frac{1}{2}\sqrt{(gl)}$.

Since $x = \frac{1}{2}l$ and $v = 0$ when $t = 0$,

$$x = \tfrac{1}{2}l\cos\omega t, \quad \text{where} \quad \omega = \sqrt{(g/l)}.$$

When $x = -\frac{1}{4}l$, $\quad -\frac{1}{4}l = \frac{1}{2}l\cos\omega t$

i.e. $\quad \cos\omega t = -\frac{1}{2}$

$\therefore$ the particle first passes through B when $\omega t = 2\pi/3$

i.e. when $t = \dfrac{2\pi}{3}\sqrt{\left(\dfrac{l}{g}\right)}$.

We include the following example as a reminder that many problems involving particles attached to elastic strings can be solved using energy considerations rather than the theory of simple harmonic motion.

Example 2 An elastic string of natural length l and modulus $4mg$ has one end fastened to a fixed point O. The other end of the string is attached to a particle of mass m. If the particle is released from rest at O, find the distance it falls before coming instantaneously to rest at a point A.

Let x be the extension in the string when the particle reaches A. The kinetic energy of the particle is zero when it is released from O and when it comes to rest at A.
Since $OA = l + x$, the loss in gravitational potential energy is $mg(l + x)$.
The energy stored in the elastic string as it is stretched to the length $(l + x)$ is $4mgx^2/2l$ i.e. $2mgx^2/l$.
Thus, using the principle of conservation of mechanical energy,

$$\frac{2mgx^2}{l} = mg(l + x)$$

$\therefore \quad 2x^2 = l(l + x)$

$$2x^2 - lx - l^2 = 0$$

$$(x - l)(2x + l) = 0$$

Since $x > 0$, we must have $x = l$.
Hence the particle falls a distance $2l$ before coming to rest.

In the next example we consider oscillations in a horizontal plane.

Example 3 A particle P of mass m lies on a smooth horizontal table and is attached to two fixed points A, B on the table by two elastic strings each of natural length l. The strings AP, PB have moduli $2mg$ and mg respectively and $AB = 5l$. If E is the equilibrium position of the particle, find AE. Given that the particle is released from rest at the point C on AB such that $AC = l$, show that it performs simple harmonic motion, stating the period and the amplitude of the oscillations.

Let T be the tension in both strings when the particle is in equilibrium at E.
If $AE = l + e$, then the extensions in the strings AP and PB are e and $3l - e$ respectively.

By Hooke's law for AP: $T = \dfrac{2mge}{l}$

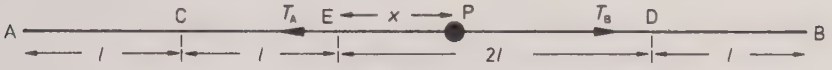
By Hooke's law for BP: $T = \dfrac{mg(3l - e)}{l}$

$$\therefore \quad \frac{2mge}{l} = \frac{mg(3l - e)}{l}$$

$$2e = 3l - e$$

$$\therefore \quad e = l$$

Hence $AE = l + e = 2l$.



Suppose now that the displacement of the particle P from E is x, as shown in the diagram.
Assuming that both strings are taut, i.e. that P lies between the points C and D, the extension in AP is $(l + x)$ and the extension in PB is $(2l - x)$.
Let T_A and T_B be the tensions in the strings AP and PB respectively, then applying Hooke's law,

$$T_A = \frac{2mg(l + x)}{l}, \quad T_B = \frac{mg(2l - x)}{l}$$

Using Newton's second law in the direction AB,

$$T_B - T_A = m\ddot{x}$$

$$\therefore \quad \frac{mg(2l - x)}{l} - \frac{2mg(l + x)}{l} = m\ddot{x}$$

$$-\frac{3mgx}{l} = m\ddot{x}$$

$$\therefore \quad \ddot{x} = -\frac{3g}{l}x$$

Thus the equation of motion of the particle is of the form $\ddot{x} = -\omega^2 x$, where $\omega = \sqrt{(3g/l)}$. Hence when both strings are taut the particle performs simple harmonic motion with centre E and period $2\pi\sqrt{(l/3g)}$.
Since the particle starts from rest at C, where $CE = l$, the amplitude of the simple harmonic motion is l.
The diagram shows that complete oscillations of this amplitude are possible with both strings taut. Thus when the particle is released at C it performs simple harmonic motion of period $2\pi\sqrt{(l/3g)}$ and amplitude l.

These examples illustrate the fact that in simple harmonic motion the centre of oscillation is an equilibrium position. If a displacement x from this equilibrium position is considered, then the equation of motion is obtained in the standard form

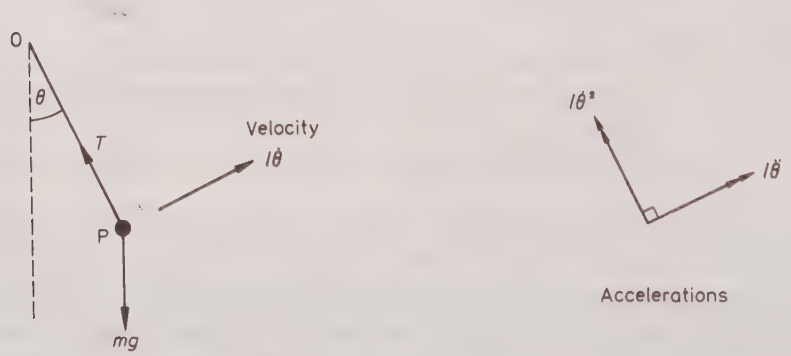
$$\ddot{x} = -\omega^2 x.$$

However, if a displacement from a point which is not an equilibrium position is used, then the simple harmonic motion equation takes the more general form

$$\ddot{x} + \omega^2 x = \text{constant}.$$

Finally, we show that the motion of a *simple pendulum* is approximately simple harmonic. A simple pendulum consists of a particle, known as the "bob", suspended from a fixed point by a light inextensible string. When the particle swings in a vertical plane with the string taut, the bob moves along a circular arc.

It was shown in Chapter 8 that when a particle moves in a circle with variable speed v, then the acceleration of the particle has components v^2/r towards the centre of the circle and dv/dt in the direction of motion. These results can now be used to find the equation of motion of a simple pendulum.



Let the bob of the pendulum be a particle P of mass m and let the length of the string be l. At the instant when the string makes an angle θ with the downward vertical, P is moving perpendicular to the string with velocity $l\dot{\theta}$. Hence the acceleration of P has components $l\dot{\theta}^2$ and $l\ddot{\theta}$ along and perpendicular to the string.

Applying Newton's second law perpendicular to the string,

$$mg \sin\theta = -ml\ddot{\theta}$$

Hence
$$\ddot{\theta} = -\frac{g}{l}\sin\theta$$

However, if θ is small, $\sin\theta \approx \theta$

$\therefore$
$$\ddot{\theta} \approx -\frac{g}{l}\theta$$

Thus small oscillations of a simple pendulum are approximately simple harmonic. The period of complete oscillations is $2\pi\sqrt{\frac{l}{g}}$.

A pendulum which beats seconds is called a *seconds pendulum*. Such a pendulum swings through its equilibrium position once every second and therefore each complete oscillation takes 2 seconds. Hence the length of a seconds pendulum is found by writing:

$$2\pi\sqrt{\frac{l}{g}} = 2.$$

Exercise 13.4

1. A particle of mass m is attached to one end of a light elastic string of length a and modulus $\frac{3}{2}mg$. The other end of the string is attached to a fixed point O and the particle hangs in equilibrium under gravity at the point E. Find the distance OE. If the particle is a *further* distance x below E show that the resultant force acting on the particle is proportional to x. The particle is pulled down to the point at a distance a below E and released from rest. Show that, in the subsequent motion and while the string is taut, the particle executes Simple Harmonic Motion and that its distance below E at time t after being released is

$$a\cos\left\{\left(\frac{3g}{2a}\right)^{1/2}t\right\}. \qquad \text{(O\&C)}$$

2. A light elastic string of natural length l has one end fastened to a fixed point O. The other end of the string is attached to a particle of mass m. When the particle hangs in equilibrium the length of the string is $\frac{7}{4}l$. The particle is displaced from equilibrium so that it moves vertically with the string taut. Show that the motion is simple harmonic with period $\pi\sqrt{(3l/g)}$.

At time $t = 0$ the particle is released from rest at a point A at a distance $\frac{3}{2}l$ vertically below O. Find (i) the depth below O of the lowest point L of the motion, (ii) the time taken to move from A to L, (iii) the depth below O of the particle at time $t = \frac{1}{3}\pi\sqrt{\frac{3l}{g}}$. (C)

3. A particle of mass m is attached to one end of a light elastic string of natural length l and modulus $2mg$. The other end of the string is fixed at a point A. The

particle rests on a support B vertically below A, with $AB = 5l/4$. Find the tension in the string and the reaction exerted on the particle by the support B. The support B is suddenly removed. Show that the particle will execute simple harmonic motion and find (i) the depth below A of the centre of oscillation, (ii) the period of the motion. (JMB)

4. A light elastic spring AB of natural length l has the end A attached to a fixed point and hangs vertically under gravity with a particle of mass m attached at B. The system is set in motion and performs small vertical oscillations. Show that the motion is simple harmonic. The particle of mass m at B is replaced by a particle of mass αm, and it is found that the period of oscillation is doubled. Find the value of α. (C)

5. One end of an elastic string of modulus mg and natural length a is attached to a fixed point O. To the other end A are attached two particles P and Q, P having mass $2m$ and Q having mass m. The particles hang down in equilibrium under gravity. If Q falls off, show that P subsequently performs simple harmonic motion and state the period and amplitude of this motion. If on the other hand P falls off, find the distance from O of the highest point reached by Q. (O&C)

6. An elastic string of natural length $2a$ and modulus λ has its ends attached to two points A, B on a smooth horizontal table. The distance AB is $4a$ and C is the mid-point of AB. A particle of mass m is attached to the mid-point of the string. The particle is released from rest at D, the mid-point of CB. Denoting by x the displacement of the particle from C, show that the equation of motion of the particle is

$$\frac{d^2x}{dt^2} + \frac{2\lambda}{ma}x = 0.$$

Find the maximum speed of the particle and show that the time taken for the particle to move from D directly to the mid-point of CD is $\frac{\pi}{3}\left(\frac{ma}{2\lambda}\right)^{1/2}$ (JMB)

7. A particle P of mass m is attached by two light elastic strings to two fixed points A and B on a smooth horizontal table. Each string is of natural length l and $AB = 4l$. If the strings AP and PB have moduli of elasticity λ and 3λ respectively, find the length of AP when the particle is in equilibrium. If the particle is given a small displacement from the equilibrium position along the line AB, show that while both strings are taut the particle executes simple harmonic motion. Find the period of the oscillations. Given that the string PB just becomes slack in the ensuing motion, find the maximum speed of the particle.

8. Two fixed points A and B on a smooth horizontal table are at a distance $5l$ apart. A particle of mass m lies between A and B. It is attached to A by means of a light elastic string of modulus mg and natural length l and to B by means of a light elastic string of modulus $2mg$ and natural length $2l$. If O is the point at which the particle would rest in equilibrium, find the distance OA. The particle is projected towards O

from the mid-point of AB with speed u. Show that while both strings remain taut the particle performs simple harmonic motion. Find the period and the amplitude of this motion. Show that the particle will perform complete simple harmonic oscillations if $2u^2 \leqslant 3gl$.

9. A particle of mass m moving on a smooth horizontal plane is attached to a point O of the plane by an elastic string of natural length l and modulus λ. Initially the particle is at the point A which lies at a distance l from O and is moving away from O at speed u. If the particle first comes to rest at the point B, use energy considerations to find the distance AB. Show that when the particle is moving in the line AB its equation of motion is of the form $\ddot{x} = -\omega^2 x$, where x is its displacement from A. Find the time taken by the particle to move directly from A to B. Find also the further time which elapses before the particle next comes to rest.

10. A particle of mass m is attached to two elastic strings each of natural length a. The first string has modulus $3mg$ and its other end is attached to a fixed point A. The second string has modulus $6mg$ and its other end is attached to a fixed point B which is at a distance $3a$ vertically below A. Show that the particle can rest in equilibrium at a point O between A and B with both strings taut. Find the distance y of O from A. The particle is in motion in the line AB with both strings taut. At time t, the particle is at P where $OP = x$. Prove that $\dfrac{d^2x}{dt^2} = -\dfrac{9g}{a}x$.

The particle is projected vertically upwards from the point at a distance $2a$ below A with speed u. Find the maximum value of u for which the upper string does not become slack in the subsequent motion. For this maximum value of u find the time taken for the particle to travel from O to its highest point. (O&C)

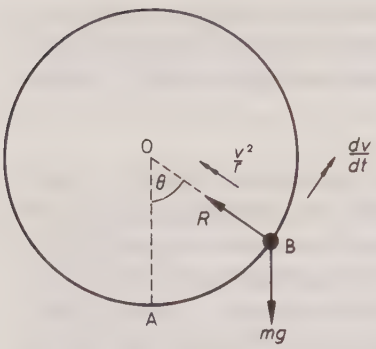
11. Find, to the nearest millimetre, the length of a seconds pendulum, (a) at Greenwich, where $g \approx 9{\cdot}81$, (b) at the equator, where $g \approx 9{\cdot}78$, (c) at the North Pole, where $g \approx 9{\cdot}83$, (d) on the surface of the moon, where $g \approx 1{\cdot}62$.

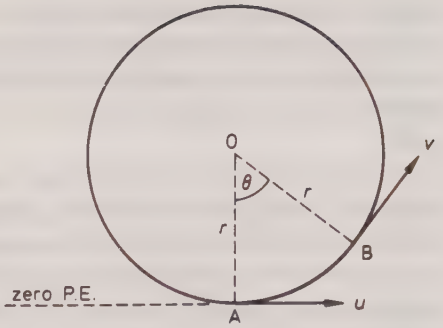
12. A musician wishes to use a simple pendulum as a makeshift metronome. Regarding a complete oscillation as two "beats", find, to the nearest centimetre, the length of pendulum required to produce (a) 50 beats per minute, (b) 80 beats per minute, (c) 100 beats per minute, (d) 160 beats per minute. [Take g as $9{\cdot}8\ \mathrm{m\,s^{-2}}$.]

13.5 Motion in a vertical circle

It was established in Chapter 8 that the acceleration of a particle moving in a circle, radius r, with variable speed v has components v^2/r towards the centre of the circle and dv/dt in the direction of motion.

When dealing with motion in a vertical circle we can obtain equations of motion by applying Newton's second law in these two directions. However, it may sometimes be more convenient to use an energy equation.





We consider first a small bead of mass m sliding on a smooth circular wire, of centre O and radius r, fixed in a vertical plane. Let u be the speed of the bead as it passes through the lowest point A of the wire and let v be the speed of the bead when it is at a point B such that $\angle AOB = \theta$. Since the wire is smooth the reaction R of the wire on the bead is perpendicular to the wire. Using Newton's second law radially and tangentially,

$$R - mg\cos\theta = m\frac{v^2}{r}$$

$$mg\sin\theta = -m\frac{dv}{dt}$$

The reaction R is perpendicular to the direction of motion and therefore does no work. This means that we can apply the principle of conservation of mechanical energy:

$$\text{(K.E. + P.E.) at } A = \text{(K.E. + P.E.) at } B$$

$$\therefore \quad \tfrac{1}{2}mu^2 = \tfrac{1}{2}mv^2 + mgr(1 - \cos\theta) \qquad (1)$$

As θ increases the speed of the bead decreases. If the bead comes to rest before it reaches the highest point of the wire it will slide back down the wire and perform oscillations about A. However, if the speed of the bead is greater than zero for all values of θ, it will execute complete circles.

From (1): $\quad v^2 = u^2 - 2gr(1 - \cos\theta)$

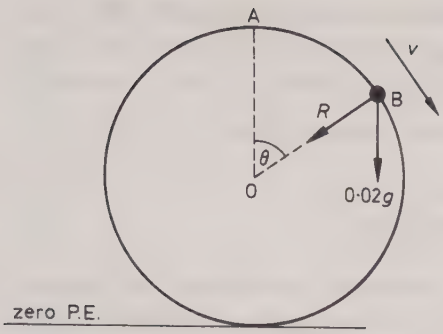
$\therefore$ if $\quad v^2 > 0$ for all θ

then $\quad u^2 > 2gr(1 - \cos\theta)$ for all θ

Since the greatest value taken by $(1 - \cos\theta)$ is 2, the condition for performing complete circles is

$$u^2 > 4gr.$$

Example 1 A small ring of mass 0·02 kg is threaded on a smooth circular wire of centre O fixed in a vertical plane. The ring is slightly displaced from rest at the highest point A of the wire. Find the magnitude and direction of the reaction of the wire on the ring when the radius to the ring makes an angle of 45° with (a) the upward vertical, (b) the downward vertical.



Let v m s^{-1} be the speed of the ring when it is at a point B such that $\angle AOB = \theta$. Let the reaction of the wire on the ring be R N towards O and let the radius of the wire be r m.

Using the principle of conservation of mechanical energy,

(K.E. + P.E.) at A = (K.E. + P.E.) at B

$\therefore \quad 0 + 0{\cdot}02g \times 2r = \frac{1}{2} \times 0{\cdot}02 \times v^2 + 0{\cdot}02g \times r(1 + \cos\theta)$

$\therefore \quad 4gr = v^2 + 2gr(1 + \cos\theta)$

Thus $\quad v^2 = 2gr(1 - \cos\theta).$

Applying Newton's second law towards O,

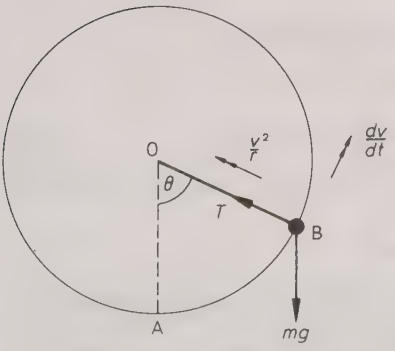
$$R + 0{\cdot}02g\cos\theta = 0{\cdot}02 \times \frac{v^2}{r}$$

$\therefore \quad R + 0{\cdot}02g\cos\theta = 0{\cdot}02 \times 2g(1 - \cos\theta)$

Thus $\quad R = 0{\cdot}02g(2 - 3\cos\theta).$

(a) When $\theta = 45°$, $R = 0{\cdot}02 \times 9{\cdot}8 \times (2 - 3\cos 45°) \approx -0{\cdot}024$.
Hence when the radius to the ring makes an angle of 45° with the upward vertical the reaction of the wire is 0·024 N *away from O*.

(b) When $\theta = 135°$, $R = 0{\cdot}02 \times 9{\cdot}8 \times (2 - 3\cos 135°) \approx 0{\cdot}808$.
Hence when the radius to the ring makes an angle of 45° with the downward vertical the reaction of the wire is 0·808 N *towards O*.



Suppose now that a particle of mass m, suspended from a fixed point O by a light inextensible string of length r, moves in a vertical circle. Let u be the speed of the particle as it passes through its lowest point A and let v be the speed of the particle when it is at a point B such that $\angle AOB = \theta$.

Applying Newton's second law towards O,

$$T - mg\cos\theta = m\frac{v^2}{r} \qquad (1)$$

As in the case of a bead sliding on a circular wire, it can also be shown that

$$v^2 = u^2 - 2gr(1 - \cos\theta) \tag{2}$$

$$\therefore \quad T = \frac{m}{r}\{u^2 - 2gr(1 - \cos\theta)\} + mg\cos\theta$$

i.e.
$$T = \frac{mu^2}{r} - 2mg + 3mg\cos\theta \tag{3}$$

Thus when the particle passes through A its speed v and the tension T in the string take their maximum values. As θ increases, both v and T decrease, but the exact nature of the subsequent motion depends on the magnitude of u. We now consider in more detail the following possibilities:

(i) v reaches zero before T,
(ii) T reaches zero before v,
(iii) both v and T are positive for all values of θ.

Case (i): The particle comes to rest with the string taut.

Substituting $v = 0$ in (1) and (2), then rearranging:

$$T = mg\cos\theta \tag{4}$$
$$2gr - u^2 = 2gr\cos\theta \tag{5}$$

Using (4): $T > 0$ when $v = 0$
$\Leftrightarrow$ $\cos\theta > 0$ when $v = 0$

Hence the particle comes to rest at a point in the lower half of the circle.

Using (5): $\cos\theta > 0$ when $v = 0$
$\Leftrightarrow$ $2gr - u^2 > 0$
$\Leftrightarrow$ $2gr > u^2$

Hence the particle performs oscillations below the level of O with the string taut if and only if $u^2 < 2gr$.

Case (ii): The string becomes slack while the particle is moving.

Substituting $T = 0$ in (1) and (3), then rearranging:

$$v^2 = -gr\cos\theta \tag{6}$$
$$2gr - u^2 = 3gr\cos\theta \tag{7}$$

Using (6): $v^2 > 0$ when $T = 0$
$\Leftrightarrow$ $-1 < \cos\theta < 0$ when $T = 0$

Hence the string can become slack only at a point in the upper half of the circle.

Using (7): $-1 < \cos\theta < 0$ when $T = 0$
$\Leftrightarrow$ $-3gr < 3gr\cos\theta < 0$ when $T = 0$
$\Leftrightarrow$ $-3gr < 2gr - u^2 < 0$
$\Leftrightarrow$ $0 < u^2 - 2gr < 3gr$
$\Leftrightarrow$ $2gr < u^2 < 5gr$

Hence the string becomes slack at a point above the level of O if and only if $2gr < u^2 < 5gr$. The particle will then leave its circular path and move freely under gravity until the string is again taut.

Case (iii): *The particle performs complete circles*

Using (2): $v > 0$ for all θ

$\Leftrightarrow$ $u^2 > gr(2 - 2\cos\theta)$ for all θ

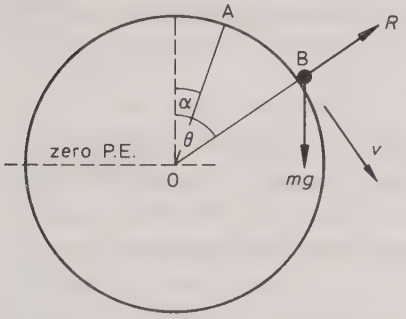
Using (3): $T > 0$ for all θ

$\Leftrightarrow$ $u^2 > gr(2 - 3\cos\theta)$ for all θ

Since $\cos\theta$ takes values from -1 to $+1$, both conditions are satisfied if and only if $u^2 > 5gr$. Hence the particle performs complete circles if and only if $u^2 > 5gr$.

Lastly we illustrate, by means of a worked example, a further way in which motion in a vertical circle may arise.

Example 2 A particle of mass m is released from rest at a point A on the outer surface of a smooth fixed sphere of centre O and radius r. Given that OA makes an angle α with the upward vertical, find an expression for the speed at which the particle is travelling when it leaves the surface of the sphere.



Let v be the speed of the particle when it reaches a point B such that OB makes an angle θ with the upward vertical. Since the sphere is smooth the reaction R of the sphere on the particle acts in the direction OB.

Since the reaction does no work mechanical energy is conserved during the motion,

$\therefore$ (K.E. + P.E.) at A = (K.E. + P.E.) at B

$\therefore$ $0 + mg \times r\cos\alpha = \frac{1}{2}mv^2 + mg \times r\cos\theta.$

i.e. $$2gr\cos\alpha = v^2 + 2gr\cos\theta \qquad (1)$$

Applying Newton's second law towards O,

$$mg\cos\theta - R = m\frac{v^2}{r} \qquad (2)$$

The particle will leave the surface of the sphere when R vanishes.
Putting $R = 0$ in equation (2) gives $gr\cos\theta = v^2$

$\therefore$ from (1): $2gr\cos\alpha = v^2 + 2v^2$

i.e. $$v^2 = \tfrac{2}{3}gr\cos\alpha$$

Hence the speed at which the particle leaves the surface is $\sqrt{(\frac{2}{3}gr\cos\alpha)}$.

Exercise 13.5

1. A particle P of mass 2 kg is suspended from a fixed point O by a light inextensible string of length 2·5 m and hangs in equilibrium at a point A. The particle is projected from A with horizontal speed 5 m s^{-1}. Given that when $\angle AOP = \theta$, the speed of the particle is v m s^{-1} and the tension in the string is T N, find expressions for v^2 and T in terms of θ. Hence find the values of v and T when $\theta = 40°$. Find also to the nearest degree, the greatest value attained by θ in the subsequent motion.

2. A small bead of mass 0·5 kg moves on a smooth circular wire of radius 2 m which is fixed in a vertical plane. When the radius to the bead makes an angle of 60° with the downward vertical the reaction between the bead and the wire is of magnitude 12 N. Find the speed of the bead at this instant and the magnitude of the reaction when the bead passes through the lowest point of the wire.

3. A particle of mass m is suspended from a fixed point O by a light inextensible string of length a. When the particle is in its lowest position it is given a horizontal speed of $\sqrt{(3ag)}$. If v is the speed of the particle when the string makes an angle θ with the upward vertical, find $\cos\theta$ and an expression for v^2 at the instant when the string becomes slack.

4. A particle of mass m is placed inside a smooth hollow sphere of radius a. The particle is set in motion at the lowest point of the sphere with speed $\sqrt{(\lambda ag)}$. Show that the particle will perform complete circles if $\lambda > 5$.

5. A small ring P of mass m is threaded on a smooth circular wire, of centre O and radius r, fixed in a vertical plane. Initially the ring is slightly displaced from the highest point of the wire. Find the magnitude of the force exerted by the ring on the wire when OP makes an angle θ with the upward vertical. Find the range of values of $\cos\theta$ for which this force acts towards O.

6. A particle of mass m slides down the smooth outside surface of a fixed sphere of radius a. At the top of the sphere its velocity is horizontal and of magnitude u. If θ is the angle that the radius to the particle makes with the upward vertical, show that the reaction between the particle and the sphere is

$$mg(3\cos\theta - 2) - mu^2/a.$$

Show that the particle, when just displaced from rest at the top, leaves the surface when $\theta = \cos^{-1}\frac{2}{3}$, and find its speed at that instant. (W)

7. A light rod OP of length a is free to rotate in a vertical plane about a fixed point O, and a mass m is attached to it at P. Denote by θ the angle between OP and the downward vertical. When $\theta = 0$, P is moving horizontally with speed u. Find the tension in the rod as a function of θ, and show that the value of the function is positive throughout the motion if $u^2 > 5ag$. Find the least value of the tension when $u^2 = 6ag$. Show that, if $u^2 < 2ag$, P remains below the level of O and deduce that, in this case also, the value obtained for the tension is positive throughout the motion. (JMB)

8. A particle P is projected horizontally with speed u from the lowest point A of the smooth inside surface of a fixed hollow sphere of internal radius a.

(i) In the case when $u^2 = ga$ show that P does not leave the surface of the sphere. Show also that, when P has moved halfway along its path from A towards the point at which it first comes to rest, its speed is $\sqrt{\{ga(\sqrt{3} - 1)\}}$.

(ii) Find u^2 in terms of ga in the case when P leaves the surface at a height $3a/2$ above A, and find, in terms of a and g, the speed of P as it leaves the surface. (JMB)

9. A particle of mass m, attached to a fixed point O by an inelastic string of length l, is allowed to fall from a point in the horizontal through O at a distance $l\cos\alpha$ from O. Find the speed of the particle when it first begins to move in a circle. In the subsequent motion, show that, when the string first makes an angle θ ($< \frac{1}{2}\pi - \alpha$) with the downward vertical, the speed v of the particle is $\sqrt{\{2gl(\cos\theta - \sin^3\alpha)\}}$ and the tension in the string is $3mv^2/2l + mg\sin^3\alpha$. Will the tension vanish during the subsequent motion? (W)

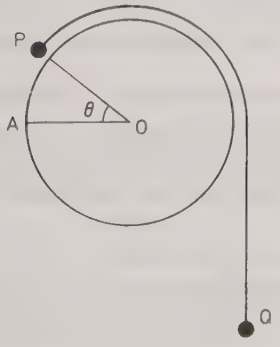
10. One end of a light inextensible string of length a is attached to a particle of mass m. The other end is attached to a fixed point O which is at a height $5a/2$ above the horizontal ground. Initially the string is taut and horizontal. The particle is then projected vertically downward with velocity $(2ag)^{\frac{1}{2}}$. When the string has turned through an angle θ ($< \pi$) find the velocity v of the particle and show that the tension in the string is $mg(2 + 3\sin\theta)$.

(a) If the string can withstand a tension of at least $5mg$, prove that the string will not break.

(b) If the string can withstand a tension of at most $7mg/2$, find the values of θ and v when the string breaks. In this case find the time to reach the vertical through O. Hence show that the particle strikes the ground at the point vertically below O. (O&C)

11. Two small smooth pegs O and C are fixed at the same level and $OC = a$. A light inextensible string of length $3a$ has one end attached to O and a particle P of mass m hanging at its other end vertically below O. The particle is projected with speed u parallel to OC. Find the speed of P when OP first makes an acute angle θ with the downward vertical and show that the tension in the string is then $mg(3\cos\theta - 2) + mu^2/3a$. Show that the string will reach a horizontal position if $u^2 \geqslant 6ga$. Given that $u^2 > 6ga$, find the tension when, after the string has struck the peg C, the moving portion CP makes an angle ϕ above the horizontal, and show that P will complete a semicircle with centre C if $u^2 \geqslant 12ga$. (JMB)

12.



A particle P of mass m is at rest at a point A on the smooth outer surface of a fixed sphere of centre O and radius a, OA being horizontal. The particle is attached to one end of a light inextensible string which is taut and passes over the sphere, its other end carrying a particle Q of mass $2m$ which hangs freely. The string lies in the vertical plane containing OA. The system is released from rest,

and after time t the angle POA is θ, as shown in the diagram. Show that, while P remains in contact with the sphere,

$$3a\left(\frac{d\theta}{dt}\right)^2 = 2g(2\theta - \sin\theta).$$

Find the tension in the string and the reaction of the sphere on P in terms of m, g and θ. (JMB)

Exercise 13.6 (miscellaneous)

1. A particle is in motion along a straight line. When at a distance x from a fixed point of the line its velocity is v. Show that its acceleration is $v\,dv/dx$.

The gravitational force per unit mass at a distance $x(> a)$ from the earth's centre is ga^2/x^2, where a is the radius of the earth, and g is constant. A particle is projected vertically upwards from the earth's surface at A with speed u. Show that if $u^2 < 2ag$, the particle rises to a maximum height $au^2/(2ag - u^2)$ above A. What happens if $u^2 > 2ag$? (O&C)

2. A man pushes a loaded cart of total mass m along a straight horizontal path against a constant resisting force λm. The cart starts from rest, and the man exerts a force, in the direction of the motion, of magnitude $Ke^{-\alpha x}$, where x is the distance moved, α is a positive constant and K is a constant greater than λm. Write down a differential equation, involving x and the speed v, for the motion of the cart (while it continues to move), and hence obtain v in terms of x and the given constants. (JMB)

3. A particle moves in a straight line in such a manner that when the particle is at the point P, at a distance x from some fixed point O on the line, the only force acting on the particle is in the direction **OP** and has magnitude $m\mu\left\{\frac{a}{x^3} - \frac{1}{x^2}\right\}$, where m is the mass of the particle and μ a positive constant.

If the particle is projected from the point $x = a$ with speed V in the direction **OP** find the set of values to which V must belong if the particle reaches the point $x = 2a$ but does not reach the point $x = 4a$. (C)

4. A particle moves along the x-axis so that its displacement x from the origin O at time t satisfies the differential equation $\frac{d^2x}{dt^2} + 9x - 18 = 0$. Its speed when $t = \pi/12$ is $21/\sqrt{2}$ in the direction of the positive x-axis and its speed when $t = \pi/6$ is 15 in the *opposite* direction. Find x at any time t. What is the greatest distance from O reached by the particle and what is its greatest speed? If the motion starts at $t = 0$, find the time at which the particle first passes through O. [The general solution of the differential equation may be quoted.] (O&C)

5. A particle is projected vertically upwards under (constant) gravity g from a point A on the earth's surface with an initial speed k. At time t after projection, the

particle is at height x above A and its speed is v; the air resistance is gv/k per unit mass. Show that, before the particle reaches its highest point, $\frac{dv}{dt} = -\frac{g}{k}(v + k)$, and $v = k(2e^{-gt/k} - 1)$. Hence find the corresponding value of x as a function of t. (O&C)

6. A bullet of mass 40 g is fired horizontally with velocity 800 m/s into a fixed block of material which exerts a constant resistance, R newtons, which stops it in 0·5 m. Show that $R = 2{\cdot}56 \times 10^4$ and find the time taken. An identical bullet is fired at the same velocity into a second fixed block which offers a resistance which varies as the square root of the speed of the bullet. The initial value of this resistance is $2{\cdot}56 \times 10^4$ N. Write down the equation of motion for the bullet and find the time taken to stop it. Explain why the bullet could never be stopped by a material which offered a resistance proportional to the velocity. (AEB 1979)

7. With the usual notation, prove that $\frac{dv}{dt} = v\frac{dv}{ds}$. A particle P of mass m moves in a straight line and starts from a point O with velocity u. When $OP = x$, where $x \geqslant 0$, the velocity v of P is given by $v = u + \frac{x}{T}$, where T is a positive constant. Show that, at any instance, the force acting on P is proportional to v. Given that the velocity of P at the point A is $3u$, calculate (i) the distance OA in terms of u and T, (ii) the time, in terms of T, taken by P to move from O to A, (iii) the work done, in terms of m and u, in moving P from O to A. (AEB 1980)

8. A point P is travelling in the positive direction on the x-axis with acceleration proportional to the square of its speed v. At time $t = 0$ it passes through the origin with speed gT and with acceleration g. Show that $dv/dx = v/(gT^2)$, and hence obtain an expression for v in terms of x, g and T. Prove that, at time t,

$$x = gT^2 \ln\left(\frac{T}{T-t}\right).$$

Sketch the graph of x against t for $0 \leqslant t < T$. (L)

9. A particle is projected vertically upwards with speed u from a point O. The air resistance is proportional to the fourth power of the velocity and is initially equal to the weight of the particle. If v is the speed of the particle when it is at a height z above O, show that $\frac{d}{dz}(\frac{1}{2}v^2) + g\left(1 + \frac{v^4}{u^4}\right) = 0$.

Solve this equation by writing w for v^2 and so prove that the particle is instantaneously at rest when its height above O is $\pi u^2/8g$. (W)

10. A body of mass m falls from rest under gravity through a resisting medium. The resistance is kv^2 per unit mass where k is constant and v is the velocity when the body has fallen a distance x.

(i) Given that u is the limiting value of the velocity, i.e. the velocity for which there would be zero acceleration, establish the differential equation $v\dfrac{dv}{dx} = k(u^2 - v^2)$.

(ii) By solving this differential equation obtain an expression for v in terms of x. (JMB)

11. A locomotive of mass M works at a constant rate H. Its motion is resisted by a force Mkv, where v is the speed at time t and k is constant. If $c^2 = H/(Mk)$, show that the equation of motion of the locomotive can be written $v\dfrac{dv}{dt} = k(c^2 - v^2)$.

Prove that the locomotive will increase its speed from $\frac{1}{3}c$ to $\frac{1}{2}c$ in time $\{1/(2k)\}\ln(32/27)$, and in this interval of time will travel a distance $\dfrac{c}{2k}\ln(1{\cdot}5) - \dfrac{c}{6k}$. (O&C)

12. A car of mass M is driven by an engine working at constant power and the resistance to motion is proportional to the square of the speed. If R is the resistance at full speed V on a horizontal road, prove that the distance covered as the speed increases from v_1 to v_2 is $\dfrac{MV^2}{3R}\log_e\left(\dfrac{V^3 - v_1^3}{V^3 - v_2^3}\right)$. (O)

13. (a) A particle travelling in a straight line is subject to a retardation of $(v^2 + 25)\,\mathrm{m\,s^{-2}}$, where $v\,\mathrm{m\,s^{-1}}$ is its speed at t seconds. If its initial speed is $12\,\mathrm{m\,s^{-1}}$, show that it travels a distance of $\log_e(13/5)$ metres before coming to rest.

(b) A particle moving with SHM oscillates between points A and B which are 10 m apart. When it is 1 m from A, its speed is $16\,\mathrm{m\,s^{-1}}$. Find (i) the periodic time; (ii) the maximum speed it attains; (iii) the maximum acceleration; (iv) the shortest time it takes to travel from A to C, where AC is $\frac{1}{4}AB$. (SU)

14. A particle moves along a straight line with simple harmonic motion of amplitude 0·5 m and of period 4 seconds. Find the maximum speed in m/s and the maximum acceleration in $\mathrm{m/s^2}$ of the particle. Find also in m/s the speed of the particle when it is 0·25 m from its central position. (Answers may be left in terms of π.) Find also the least time which elapses between two instants when the speed of the particle is half its maximum speed. (L)

15. A particle of mass m moves in simple harmonic motion about O in a straight line, under the action of a restoring force of magnitude proportional to the distance from O. At time $t = 0$ the speed of the particle is zero. After 1 second the speed of the particle is $2\,\mathrm{m\,s^{-1}}$, after a further second the speed is $2\sqrt{3}\,\mathrm{m\,s^{-1}}$ and subsequently the particle passes through O for the first time. Show that the speed of particle when it passes through O is $4\,\mathrm{m\,s^{-1}}$ and find (i) the period and amplitude of the motion, (ii) the time at which the speed of the particle is equal to $2\,\mathrm{m\,s^{-1}}$ after it first passes through O.

A stationary particle of mass m is placed at O and the moving particle collides and coalesces with this particle. The combined particle moves under the action of the

same restoring force as before. Find the period of the subsequent motion and the speed when the displacement is $12/\pi$ m from O. (C)

16. At time t the displacement from the origin of a particle moving along the x-axis under the action of a single force is given by $x = a\cos(\omega t + \alpha)$ where a, ω and α are constants. Find the acceleration of the particle in terms of its displacement from the origin and describe the force acting on the particle. What is the periodic time of this motion?

Two particles, each moving in simple harmonic motion, pass through their centres of oscillation at time $t = 0$. They are next at their greatest distances from their centres at times $t = 2$ s and $t = 3$ s respectively, having been at the same distance from their respective centres at time $t = 1$ s. Show that the amplitudes of their motions are in the ratio $1:\sqrt{2}$. When will the particles next be at their centres, simultaneously? When will the initial conditions next be repeated? (W)

17. A particle moves on a straight line through a fixed point O so that at time t its displacement from O is x and its equation of motion is $\ddot{x} = -\omega^2 x$. If when $t = 0$, $x = d$ and $\dot{x} = u$, show that $x = d\cos\omega t + (u/\omega)\sin\omega t$. Given that the first three times at which the particle passes through the point $x = d$ are $t = 0, t = 2$ and $t = 8$, find d in terms of u.

18. A rough horizontal platform moves horizontally in simple harmonic motion. A particle P of mass 2 kg rests on the platform without slipping and is at O, the centre of oscillations when the motion starts; the initial speed of P is 4 m s^{-1}. The period of the S.H.M. is 8 s. Find (i) the amplitude of the motion correct to 3 sig. fig., (ii) the distance OP, 1 s after the motion starts, correct to 3 sig. fig., (iii) an expression for the time taken for the particle to be first displaced a distance 1 m from O. Do not evaluate the expression.

Draw a diagram showing the forces acting on P with its acceleration and obtain the maximum frictional force acting on P. Hence give the range of values of the coefficient of friction in order that P should not slip. (SU)

19. The end A of a light elastic string AB, obeying Hooke's law and of natural length 0·5 m, is fixed. When a particle of mass 2 kg is attached to the string at B and hangs freely under gravity, the extension of the string in the equilibrium position is 0·075 m. Calculate, in newtons, the modulus of elasticity of the string. The particle is now pulled down vertically a further 0·1 m and released. Show that, until the string becomes slack, the motion of the particle is simple harmonic. Show that the time that elapses before the particle first passes through the equilibrium position is $(\pi\sqrt{3})/40$ s and find the speed of the particle when it is in this position. [Take g as 10 m/s^2]. (L)

20. One end of a light elastic string of natural length l and modulus λ is attached to a fixed point A, and a small mass m, attached to the other end, hangs in equilibrium at B. A small mass $2m$ is made to adhere to the under side of the first mass, and the combined mass is released from rest at B. Show that the combined mass oscillates

about a point C which is at a depth $l + \dfrac{3mgl}{\lambda}$ below A, and that the equation of motion is

$$\frac{d^2x}{dt^2} = -\frac{\lambda}{3ml}x,$$

where x is the depth below C of the combined mass at time t. Show that the speed of the combined mass when $x = 0$ is $2g\sqrt{(ml/3\lambda)}$. Find the force of adhesion between the masses in terms of m, g, λ and x. (JMB)

21. A particle P of mass m lies on a smooth horizontal table and is attached to two fixed points A, B on the table by two light elastic strings, each obeying Hooke's law and of natural lengths $2l, 3l$ and of moduli $2mg, mg$, respectively. If $AB = 7l$, show that, when P is in equilibrium, $AP = 5l/2$. The particle P is held at rest at the point C in the line AB, where $AC = 3l$ and C lies between A and B, and is then released. Show that the motion of P is simple harmonic of period $\pi\sqrt{(3l/g)}$. Find the maximum speed of P during the motion. (O&C)

22. One end of a light elastic string, of natural length $2a$ and modulus $4mg$, is attached to a fixed point A of a smooth horizontal table. A particle of mass m is attached to the other end of the string. The particle is released from rest from a point C of the table, where $AC = 3a$. Show that the particle reaches C again after time $(4 + \pi)\sqrt{(2a/g)}$. (L)

23. A scale pan of mass m is suspended from a fixed point by an elastic spring, of modulus $2mg$ and natural length l. When the scale pan is hanging at rest, a particle of mass $\frac{1}{2}m$ falls vertically on to the scale pan, striking it with speed $\sqrt{(lg)}$. Find the amplitude of the oscillations which the scale pan starts to make if the particle rebounds, the coefficient of restitution between the particle and the scale pan being $\frac{1}{4}$. (L)

24. A particle rests on the *outside* of a fixed smooth vertical hoop, of radius a, at its highest point A, and a second particle rests on the *inside* of the hoop at its lowest point B. The particle at A is just disturbed from rest and the particle at B is given a horizontal velocity v such that both particles leave the hoop at the same point P. Show that (i) the radius OP is inclined to the vertical at an angle $\cos^{-1}\frac{2}{3}$, (ii) $v = 2\sqrt{(ag)}$. (W)

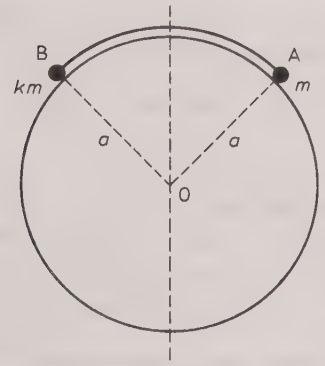
25. A smooth wire in the form of a circle of centre O and radius a is fixed in a vertical plane. Two beads, P of mass $2m$ and Q of mass m, are threaded on to the wire. Initially Q is at rest at A, the lowest point of the wire, and P is held at a point B on the wire, where the angle AOB is $\frac{1}{3}\pi$.

The particle P is then released. Find the speed of P when the angle AOP is θ and P has not yet reached A, and find the reaction then exerted on P by the wire. Show that, when P reaches A, its speed is $\sqrt{(ga)}$. Show that, when the particles collide, the impulse of the reaction between them is of magnitude $\frac{2}{3}m(1 + e)\sqrt{(ga)}$, where e is the

coefficient of restitution. In the case when $e = 1/5$ find the height above A which Q subsequently reaches before coming instantaneously to rest. (JMB)

26. A particle is at rest at the lowest point A inside a smooth hollow sphere of radius a. It is projected horizontally with velocity $\sqrt{(7ga/2)}$. Show that it loses contact with the sphere at a height $3a/2$ above A and will next strike the sphere at A. (C)

27.



Two particles A and B of masses m and km respectively, where $k < 1$, are joined by a light inextensible string of length $\frac{1}{2}\pi a$. The string is taut and the particles are held, at the same horizontal level, in contact with a fixed smooth cylinder of radius a. The axis of the cylinder is horizontal and the string is in a vertical plane perpendicular to the axis of the cylinder. The system is released from rest. If θ is the angle made with the vertical by the radius OA of the cylinder when the particle A is about to lose contact with the cylinder, show that $(3 + k)\cos\theta + 2k\sin\theta = (1 + k)\sqrt{2}$. (C)

28. Two small equal beads, threaded on a vertical smooth circular wire of radius a, are connected by a light inextensible string of length a. One bead is held at the highest point of the wire with the string taut and is then released. Find the speed of the beads at the instant the string first becomes slack. (O)

29. A particle of mass m is projected horizontally with velocity $\mathbf{u}$ from a point O in a medium producing a resisting force $-mk\dot{\mathbf{r}}$ where k is a positive constant and $\mathbf{r}$ is the position vector of the particle relative to O. The acceleration due to gravity is $\mathbf{g}$. Write down the equation of motion (involving $\ddot{\mathbf{r}}$) and show that at time t

(i) $\dfrac{d}{dt}(\mathbf{r}e^{kt}) = (\mathbf{u} + \mathbf{g}t)e^{kt}$,

(ii) $\mathbf{r} = \dfrac{1}{k}(1 - e^{-kt})\mathbf{u} + \dfrac{1}{k^2}(e^{-kt} + kt - 1)\mathbf{g}$.

If k is so small that k^2 may be neglected compared with k, show that approximately $\mathbf{r} = \mathbf{u}t(1 - \frac{1}{2}kt) + \mathbf{g}t^2(\frac{1}{2} - \frac{1}{6}kt)$. (JMB)

14 Probability

14.1 Permutations and combinations

A *permutation* is an arrangement of objects chosen from a given set. In this section we discuss ways of finding the total number of permutations of a set of objects under various conditions. For example, consider the number of ways in which we could arrange four guests in a row of four numbered chairs at a concert.

Any one of the 4 people may sit in the first chair. When the first person is seated, there are 3 people to be considered for the second place. This means that for each of the 4 ways of choosing the first person, there are 3 ways of choosing the second person to be seated. Similarly, in each of these $4 \times 3 = 12$ cases, we are left with 2 ways of choosing the occupant of the third chair. When 3 people are seated, the fourth chair must be filled by the 1 remaining person. Hence the total number of ways of seating the 4 people is $4.3.2.1$, i.e. 24.

Using the letters A, B, C, D to represent the 4 people, these 24 arrangements are as follows:

A B C D	B A C D	C A B D	D A B C
A B D C	B A D C	C A D B	D A C B
A C B D	B C A D	C B A D	D B A C
A C D B	B C D A	C B D A	D B C A
A D B C	B D A C	C D A B	D C A B
A D C B	B D C A	C D B A	D C B A

Since products of the type $4.3.2.1$ frequently arise in this type of work, the *factorial* notation is used. We write $4.3.2.1 = 4!$ which is read 'four factorial' or 'factorial four'. Thus for any positive integer n, $n! = n(n-1)(n-2)\ldots3.2.1$.

The result of the above seating problem can now be generalised.

> The number of permutations or arrangements of n different objects is $n!$.

Example 1 In how many different ways can 4 people chosen from a set of 6 be seated in a row of four chairs?

There are 6 possible choices for the first chair, then 5 choices for the second, 4 for the third and 3 for the fourth. Hence the total number of ways of seating 4 people out of a set of six is 6.5.4.3, i.e. 360.
[In factorial notation this number is 6!/2!]

In general, the number of permutations or arrangements of r different objects chosen from a set of n objects, written ${}_nP_r$, is given by ${}_nP_r = \dfrac{n!}{(n-r)!}$.

When $r = n$ this expression becomes ${}_nP_n = n!/0!$, but as shown above ${}_nP_n = n!$. Hence, it is convenient to define 0! to be 1.

In some problems the order in which objects are arranged is not important. The number of *combinations* of r different objects out of a set of n is the number of different selections, irrespective of order, and is denoted by ${}_nC_r$, or sometimes $\binom{n}{r}$.

Let us suppose that a set of 3 cards is to be dealt from a pack of 52 playing cards. The number of permutations of 3 cards chosen from $52 = 52!/49! = 52.51.50$. However, since any set of 3 cards can be arranged in 3! different ways, each set of 3 cards will appear 6 times in the list of all possible arrangements. For instance, the set containing cards X, Y and Z will appear as XYZ, XZY, YXZ, YZX, ZXY and ZYX. Hence the total number of ways of selecting 3 cards $= \dfrac{52!}{3!49!} = \dfrac{52.51.50}{3.2.1} = 22\,100.$

In general, the number of ways of selecting r objects from n unlike objects $= {}_nC_r = \dfrac{n!}{r!(n-r)!}$.

Example 2 A committee of 5 is to be formed from 12 men and 8 women. In how many ways can the committee be chosen so that there are 3 men and 2 women on it?

Number of ways of choosing 3 men from 12

$$= {}_{12}C_3 = \frac{12!}{3!9!} = \frac{12.11.10}{3.2.1} = 220.$$

Number of ways of choosing 2 women from 8

$$= {}_8C_2 = \frac{8!}{2!6!} = \frac{8.7}{2.1} = 28.$$

$\therefore$ the total number of ways of forming the committee

$$= {}_{12}C_3 \times {}_8C_2 = 220 \times 28 = 6160.$$

Exercise 14.1

1. Find the number of ways of arranging 6 different books on a shelf.

2. Find the number of arrangements of 8 items on a shopping list.

3. A lady has 8 house plants. In how many ways can she arrange 6 of them in a line on a window sill?

4. In a competition 6 household products chosen from 10 are to be listed in order of preference. In how many ways can this be done?

5. There are 25 entrants in a gymnastics competition. In how many different ways can the gold, silver and bronze medals be awarded?

6. In how many different ways can 8 books be arranged on a shelf if 2 particular books must be placed next to each other?

7. Three girls and 4 boys are to sit in a row of 7 chairs. If the girls wish to sit in adjacent chairs, how many different arrangements are possible?

8. How many numbers less than 3500 can be formed using one or more of the digits 1, 3, 5, 7, if (a) no digit can be repeated, (b) repetitions are allowed?

9. Verify that ${}_nC_{n-r} = {}_nC_r$ for the cases
(a) $n = 8, r = 5$; (b) $n = 10, r = 2$.

10. In how many ways can
(a) 4 photographs be chosen from 10 proofs,
(b) 3 representatives be chosen from 20 students,
(c) a hand of 5 cards be dealt from a set of 13,
(d) 11 players be selected from 12 cricketers?

11. Nine people are to go on a journey in cars which can take 2, 3 and 4 passengers respectively. In how many different ways can the party travel, assuming that the seating arrangements inside the cars are not important?

12. In how many ways can a set of 12 unlike objects be divided into (a) 2 sets of 6, (b) 3 sets of 4, (c) 6 sets of 2?

13. A team of 5 students, including a captain and a reserve, is to be selected for a general knowledge contest. In how many ways can the team be chosen from a short list of 12?

14. Find the number of different selections of two letters which can be made from the letters of the word PROBABILITY. How many of these selections do not contain a vowel?

15. Find the number of ways in which 8 books can be distributed to 2 boys, if each boy is to receive at least 2 books.

16. A committee of 6 is to be formed from 13 men and 7 women. In how many ways can the committee be selected.given that (a) it must consist of 4 men and 2 women, (b) it must have at least one member of each sex?

14.2 The language of sets

A *set* is a collection of objects. These objects are the *elements* of the set. A set may be defined either by listing the elements or by describing their properties, e.g. $A = \{2,4,7\}$, B is the set of all triangles, $C = \{\ldots, -5, -3, -1, 1, 3, \ldots\}$ or {odd integers}. Using the colon : to mean 'such that', we may also write $P = \{p : p \text{ is a prime number}\}$, which is read '$P$ is the set of all elements p such that p is a prime number'. Whatever form is chosen it must always be possible to decide from the definition whether or not an object belongs to the set.

The statement $x \in S$ means 'x is an element of S' or 'x belongs to S'. Similarly $x \notin S$ means 'x is not an element of the set S'. Thus, for the sets defined above, $2 \in A$ but $2 \notin C$ and $9 \in C$ but $9 \notin P$.

Two sets are *equal* if they contain exactly the same elements, e.g. $\{16, 25, 36, 49, 64, 81\} = \{36, 81, 16, 25, 64, 49\} = \{\text{two digit squares}\}$. The way in which the elements are listed or described is not important.

Sometimes all the elements of one set are contained in another. If every element of a set A also belongs to a set B, then A is a *subset* of B, written $A \subseteq B$. If A is a subset of B and there is at least one element of B which is not an element of A, then A is a *proper subset* of B and we write $A \subset B$. For instance, if $X = \{a,c\}$, $Y = \{a,c,e\}$ and $Z = \{a, b, c, d\}$, then $X \subset Y$ and $X \subset Z$ but $Y \not\subset Z$. We may also write $Y \supset X$ and $Z \supset X$ using the symbol $\supset$ to mean 'contains as a proper subset'.

Three important properties of the set inclusion relation follow from the definition.

(1) Any set A is a subset of itself, i.e. $A \subseteq A$.

(2) $A \subseteq B$ and $B \subseteq C \Rightarrow A \subseteq C$.

(3) $A \subseteq B$ and $B \subseteq A \Leftrightarrow A = B$.

[Note that if p and q represent statements, then $p \Rightarrow q$ means 'if p then q' or 'p implies q'. The symbol $\Leftarrow$ means 'is implied by'. We write $p \Leftrightarrow q$ meaning 'p implies and is implied by q' when the statements p and q are equivalent.]

All the objects under consideration in any mathematical discussion form a set called the *universal set* (or *universe of discourse*), often denoted by $\mathscr{E}$. For instance, in a geometrical problem the universal set may be a set of points, a set of lines or a set of plane figures. Once a universal set $\mathscr{E}$ has been defined any other set considered must be one of its subsets. Thus for any set A, $A \subseteq \mathscr{E}$.

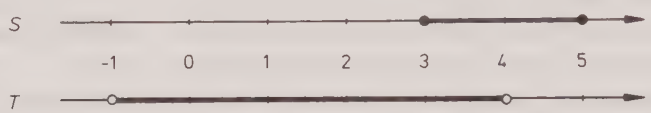
The set with no elements is called the *null* or *empty set*, written $\varnothing$ or sometimes $\{\ \}$. The empty set $\varnothing$ is considered to be a subset of any set A, i.e. $\varnothing \subseteq A$.

The *complement* of a set A is the set containing all the elements of $\mathscr{E}$ which are not elements of A and is written A'. For instance, if $\mathscr{E} = \{1,2,3,4,5\}$ and $A = \{1,3,5\}$ then $A' = \{2,4\}$. It follows from the definition that for any set A, $(A')' = A$. We also find that $\varnothing' = \mathscr{E}$ and $\mathscr{E}' = \varnothing$.

The *union* of sets A and B, written $A \cup B$, is the set containing all elements which belong to A or B (or both). For instance, if $A = \{1,2,3,4\}$ and $B = \{1,3,5\}$, then $A \cup B = \{1,2,3,4,5\}$.

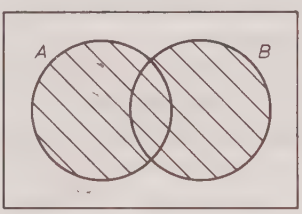
The *intersection* of sets A and B, written $A \cap B$, is the set containing all elements which belong to both A and B. For instance, if $A = \{2,4,6,8\}$ and $B = \{1,2,3,4\}$, then $A \cap B = \{2,4\}$.

Example 1 If $S = \{x \in \mathbb{R} : 3 \leqslant x \leqslant 5\}$ and $T = \{x \in \mathbb{R} : -1 < x < 4\}$, find $S \cup T$ and $S \cap T$.

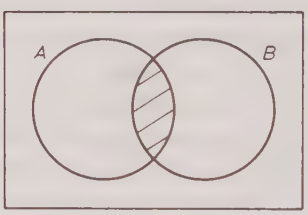


Representing S and T on the real number line we find that
$S \cup T = \{x \in \mathbb{R} : -1 < x \leqslant 5\}$ and $S \cap T = \{x \in \mathbb{R} : 3 \leqslant x < 4\}$.

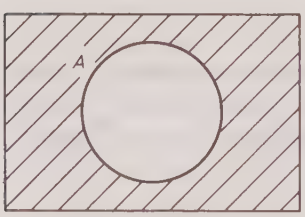
Venn† diagrams are used to represent relationships between sets. The universal set is shown as a rectangle and other sets as regions, often circles, within this rectangle.



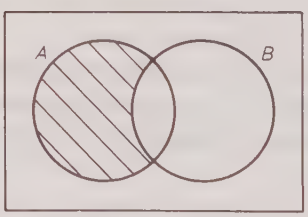
$A \cup B$



$A \cap B$



A'



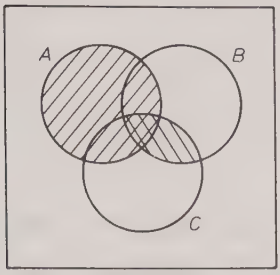
$A \cap B'$

Venn diagrams can be used to demonstrate properties of union and intersection, but such verifications are not regarded as rigorous proofs of the results.

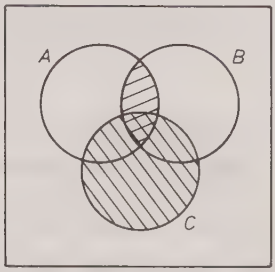
†*Venn, John* (1834–1923) English logician. He developed the theories of probability and mathematical logic. In his *Symbolic Logic* (1881) he improved the diagrammatic methods of earlier writers.

Example 2 Verify using Venn diagrams, the associative law

$$A \cap (B \cap C) = (A \cap B) \cap C.$$

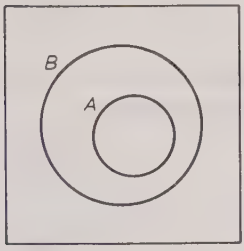


(1) $A \cap (B \cap C)$

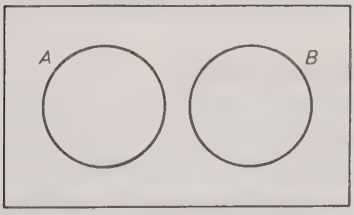


(2) $(A \cap B) \cap C$

In (1) the sets A and $(B \cap C)$ are shaded in different ways, so that the region shaded twice represents $A \cap (B \cap C)$. Similarly in (2) the region shaded twice represents $(A \cap B) \cap C$. Hence, in these diagrams, $A \cap (B \cap C) = (A \cap B) \cap C$.



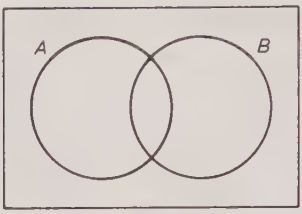
This diagram illustrates the fact that A is a subset of B, i.e. $A \subset B$. We see that in this case $A \cup B = B$ and $A \cap B = A$.



Here the sets A and B have no common elements, i.e. $A \cap B = \varnothing$. Such sets are said to be *disjoint*.

A', the complement of A, contains no elements of A,
$\therefore$ A and A' are disjoint and $A \cap A' = \varnothing$.
Since every element of the universal set $\mathscr{E}$ belongs to either A or A', for any set A, $A \cup A' = \mathscr{E}$.

Venn diagrams can also be used in problems concerning the numbers of elements in sets. The number of elements in a finite set A is called the *order* or *cardinal number* of the set and is denoted by $n(A)$.



By considering the diagram we see that in the sum $n(A)+n(B)$, the elements of $A \cap B$ have been counted twice.

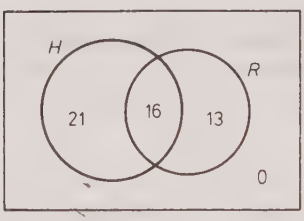
Hence $\boxed{n(A \cup B) = n(A)+n(B)-n(A \cap B).}$

Example 3 On a shelf of 50 books, there are 37 historical novels and 29 romances, but no other types of book. What conclusion may be drawn from this?

Let H and R be the sets of historical novels and romances respectively, then $n(H) = 37$, $n(R) = 29$ and $n(H \cup R) = 50$.

$$n(H \cup R) = n(H)+n(R)-n(H \cap R)$$
$$50 = 37+29-n(H \cap R)$$
$$\therefore \quad n(H \cap R) = 16.$$

It follows that there are 16 historical romances on the shelf.



The diagram shows one way of illustrating this result.

Example 4 In a survey of 100 housewives it is found that 65 use brand X and 55 use brand Y washing powder. What can be said about the numbers of housewives using both X and Y.

Let X and Y be the sets of housewives using those brands, then

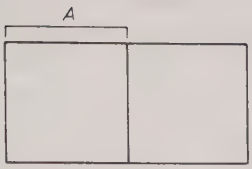
$$n(X \cup Y) = n(X)+n(Y)-n(X \cap Y)$$
$$= 65+55-n(X \cap Y)$$
$$= 120-n(X \cap Y).$$

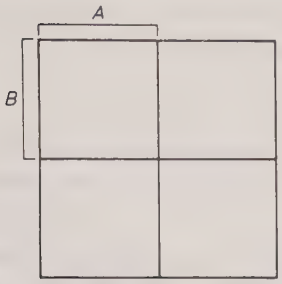
As there may be housewives who use neither X nor Y, we may assume only that $n(X \cup Y) \leqslant 100$,

$\therefore \quad n(X \cap Y) \geqslant 20$, i.e. at least 20 housewives use both X and Y.

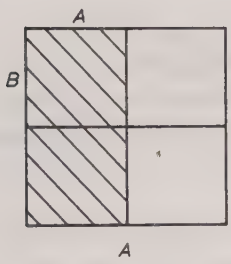
Karnaugh† *maps* provide a useful alternative to Venn diagrams when dealing with intersecting sets. Karnaugh maps for one set A and for two sets A and B are given below.

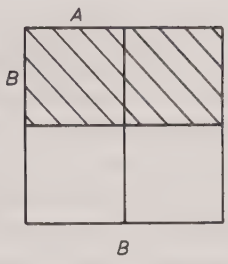
† Karnaugh maps representing sets are developments of diagrams used to simplify logic circuits in computers and electronic control systems, as described by *M. Karnaugh* in his article *The Map Method for Synthesis of Combinational Logic Circuits* (1953).

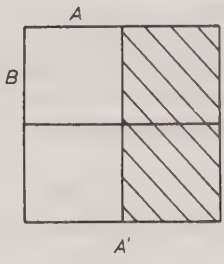


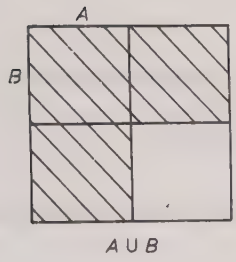


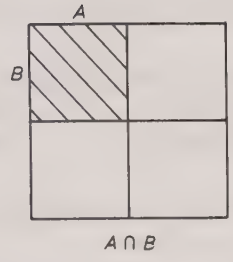
Any set can be represented on a Karnaugh map by shading the appropriate region.

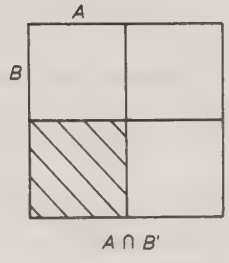


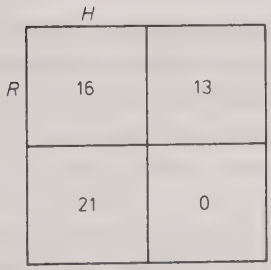






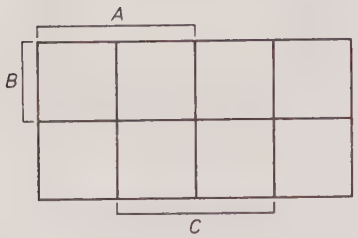


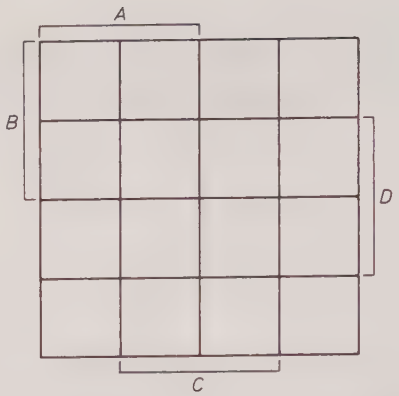




A Karnaugh map could have been used to illustrate the result of Example 3.

The Karnaugh system can be extended to larger numbers of sets. The maps for 3 sets and 4 sets are given here.





Exercise 14.2

1. If $A = \{x, y, z\}$, $B = \{x, p, q, r\}$, $C = \{x, y, p\}$, list the following sets:
(a) $A \cap B$, (b) $B \cup C$, (c) $(A \cup B) \cup C$,
(d) $(A \cup C) \cap B$, (e) $(A \cap C) \cap B$, (f) $B \cup (A \cap C)$.

2. Find $P \cup Q$ and $P \cap Q$ in each of the following cases
(a) $P = \{x \in \mathbb{R} : -2 < x < 4\}$, $Q = \{x \in \mathbb{R} : 0 < x < 7\}$,
(b) $P = \{x \in \mathbb{R} : 2 \leqslant x \leqslant 9\}$, $Q = \{x \in \mathbb{R} : x \leqslant 5\}$,
(c) $P = \{x \in \mathbb{R} : |x| < 10\}$, $Q = \{x \in \mathbb{R} : -4 < x < 3\}$,
(d) $P = \{x \in \mathbb{R} : -3 < x < 1\}$, $Q = \{x \in \mathbb{R} : 1 \leqslant x \leqslant 6\}$.

3. If $\mathscr{E} = \{1, 2, 3, 4, 5\}$, $A = \{1, 2, 3\}$, $B = \{5\}$, $C = \{3, 4\}$, list the following sets:
(a) $A \cap B$, (b) $C \cap A'$, (c) $(B \cup C) \cap \mathscr{E}$,
(d) $B' \cap C'$, (e) $A \cup \varnothing$, (f) $A' \cap (B \cup C)$.

4. If $\mathscr{E} = \{a, b, c, d\}$, $X = \{a, b\}$, $Y = \{b, c\}$, find expressions for the following sets using the symbols X, Y, X', Y', $\cup$ and $\cap$.
(a) $\{c, d\}$, (b) $\{b\}$, (c) $\{d\}$,
(d) $\{a, b, d\}$, (e) $\{a, c, d\}$, (f) $\{a, c\}$.

5. In each of the following cases draw a Venn diagram showing the universal set $\mathscr{E}$ and three sets A, B and C, then shade the region representing the given set.
(a) $A \cap B \cap C$, (b) $A' \cap B$, (c) $B' \cup C$,
(d) $(A \cup B) \cap C'$, (e) $A \cup (B \cap C')$, (f) $A' \cap B' \cap C'$.

6. Use Venn diagrams to verify the following:
(a) $A \cap (A \cup B) = A$, (b) $(A \cap B)' = A' \cup B'$,
(c) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

7. Draw a Venn diagram to show the relationships between the following sets: $\mathscr{E} = \{\text{all triangles}\}$, $P = \{\text{isosceles triangles}\}$, $Q = \{\text{equilateral triangles}\}$, $R = \{\text{right-angled triangles}\}$, $S = \{\text{acute angled triangles}\}$.

8. Draw a Venn diagram to show the following sets and their non-empty intersections: $\mathscr{E} = \{\text{positive integers}\}$, $A = \{\text{even numbers}\}$, $B = \{\text{multiples of 3}\}$, $C = \{\text{multiples of 9}\}$, $D = \{\text{multiples of 12}\}$.

9. Simplify the following expressions, using Venn diagrams if necessary.
(a) $A \cup (A \cap B)$, (b) $A \cap (A' \cap B)$,
(c) $(A \cap B) \cup (B' \cap A)$, (d) $(A \cup B') \cap [A \cup (B \cap C)]$.

10. In a survey of 100 children, it is found that 47 have at least one brother, 58 have at least one sister and that 32 children have both. How many of the children have no brothers and sisters?

11. Find an expression for $n(A \cup B \cup C)$ in terms of the numbers of elements in the sets A, B, C and their intersections.

12. Out of 50 people who enrol for evening classes, 30 enrol for Mathematics, 15 for Further Mathematics, 22 for Physics and 8 enrol for none of these subjects. Denoting the sets of students who enrol for Mathematics, Further Mathematics and Physics by M, F and P respectively, what conclusions may be drawn about the values of $n(M \cap F)$, $n(M \cap P)$ and $n(M \cap F \cap P)$?

13. In each of the following cases draw a Karnaugh map for three sets A, B and C, then shade the region representing the given set.
(a) $A \cap B \cap C$, (b) $A \cap C'$, (c) $B' \cup C$,
(d) $A' \cap B \cap C'$, (e) $A \cup (B' \cap C)$, (f) $(A' \cup B') \cap C'$.

14. Using Karnaugh maps or otherwise, simplify the following expressions.
(a) $A \cap (A \cup B')$, (b) $(B \cap C) \cup (B \cap C')$,
(c) $(A \cap B \cap C) \cup (A \cap B' \cap C)$, (d) $(A \cup B \cup C) \cap (A' \cup B \cup C)$.

14.3 Elementary theory of probability

In mathematics probability is the numerical value assigned to the likelihood that a particular event will take place. For instance, if we throw an unbiased die, we have equal chances of scoring any of the numbers 1, 2, 3, 4, 5 and 6. Since there is one chance in six of throwing a 3 the probability of the event occurring is said to be 1/6. Similarly when tossing a coin the probability that it lands heads is considered to be 1/2. This does not, of course, mean that in any two tosses we expect the coin to fall heads once, only that in a long series of tosses the number of heads will be approximately half the total number of throws.

To arrive at a formal definition, consider the set S of all possible *outcomes* of an experiment or *trial*. A set of this kind is sometimes called a *possibility space* or *sample space*. Any *event* A can be represented by the subset A of S, which contains all the outcomes in which the event occurs. If S contains a finite number

of *equally likely* outcomes, then the probability $P(A)$ that the event A will occur is given by

$$P(A) = \frac{n(A)}{n(S)}, \quad \text{i.e.} \quad \frac{\text{no. of favourable outcomes}}{\text{no. of possible outcomes}}.$$

If the event A is impossible, then $A = \varnothing$ and $P(A) = 0$. However, if the event A is certain to occur, then $A = S$ and $P(A) = 1$. Otherwise $P(A)$ will take some value between 0 and 1.

[When probabilities are calculated from experimental data, a definition such as the following may be used.

$$\text{Estimated probability} = \frac{\text{no. of successes}}{\text{no. of trials}}.$$

However, this value for the probability could be unreliable unless large numbers of observations are used.]

Example 1 What is the probability of drawing an ace at random from a pack of cards?

Since there are 4 aces in a pack of 52 cards, the probability of drawing an ace is 4/52, i.e. 1/13.

Example 2 If a letter is chosen at random from the word FACETIOUS, what is the probability that it is a vowel?

Since there are 5 vowels out of a total of 9 letters, the probability of choosing a vowel is 5/9.

Returning to the problem of throwing an unbiased die, the probability of throwing a 3 is 1/6, but the probability of not throwing a 3 is 5/6. We notice that the sum of these probabilities is 1. More generally, in a set S of equally likely possible outcomes, if A' denotes the subset of outcomes in which the event A does not occur, then A and A' are called *complementary* events and

$$P(A') = \frac{n(A')}{n(S)} = \frac{n(S) - n(A)}{n(S)} = 1 - \frac{n(A)}{n(S)}$$

$$\therefore \quad \boxed{P(A') = 1 - P(A).}$$

Hence, it seems reasonable to assume that in any sample space, if p is the probability that an event occurs and q is the probability that the event does not occur, then $p + q = 1$.

[Note that the event A' is sometimes referred to as 'not A' and may also be written $\bar{A}$.]

For experiments in which a trial is a throw of two or more dice, we can construct the sample space from the set of outcomes for a single die, $S = \{1,2,3,4,5,6\}$. For instance, in the two-dice case, the possible outcomes can be expressed as ordered pairs such as (1,3), (4,6) and (6,4) formed from the elements of S.

A *sample space diagram* can be used to find probabilities as shown in the next example.

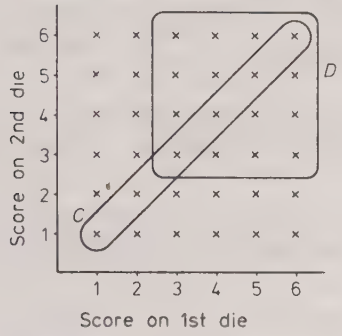
Example 3 If two dice are thrown together, what is the probability of the following events?

A: scoring a total of 2

B: scoring a total of 3

C: the same score on both dice

D: 3 or more on each die.



The diagram shows that when two dice are thrown, there are 36 possible outcomes. A score of 2 is obtained only when the score on each die is 1

$$\therefore \quad P(A) = \frac{1}{36}.$$

There are two outcomes (1,2) and (2,1) which produce a score of 3

$$\therefore \quad P(B) = \frac{2}{36} = \frac{1}{18}.$$

The set of outcomes in which both dice show the same score has 6 elements, as indicated in the diagram

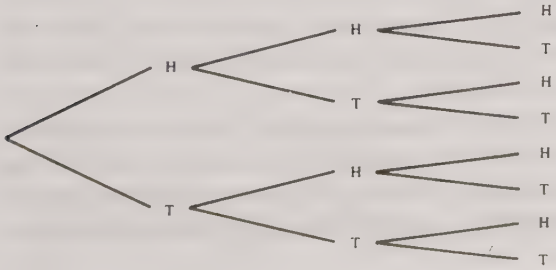
$$\therefore \quad P(C) = \frac{6}{36} = \frac{1}{6}.$$

The diagram also shows that there are 16 outcomes in which both dice show 3 or more

$$\therefore \quad P(D) = \frac{16}{36} = \frac{4}{9}.$$

Example 4 If a coin is spun three times, what is the probability that it lands heads once and tails twice?

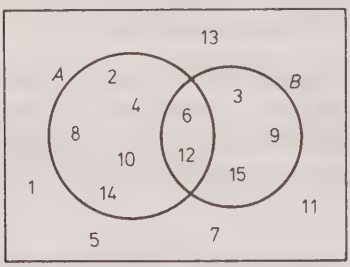
[In this case the elements of the sample space may be represented by the paths along a simple *tree diagram*.]



Since the given event occurs in 3 out of 8 equally likely outcomes, i.e. HTT, THT, TTH, the required probability is 3/8.

Let us now consider two different events A and B which may occur in an experiment. The set containing the outcomes in which A or B (or both) occur is the set $A \cup B$, so the probability of event A or event B (or both) is written $P(A \cup B)$. Similarly, the probability that both A and B occur is $P(A \cap B)$.

Example 5 An integer is chosen at random from the set $\{1, 2, 3, \ldots, 14, 15\}$. If A is the event of choosing an even number and B is the event of choosing a multiple of 3, find $P(A \cap B)$, $P(A \cup B)$ and $P(A' \cap B)$.



Using the possibility space diagram,

$$P(A \cap B) = \frac{2}{15}$$

$$P(A \cup B) = \frac{10}{15} = \frac{2}{3}$$

$$P(A' \cap B) = \frac{3}{15} = \frac{1}{5}$$

Some problems can be solved using the theory of permutations and combinations.

Example 6 If 4 cards are selected at random from a pack of 52, what is the probability that exactly 3 of them are diamonds?

The number of ways of selecting 3 diamonds from $13 = {}_{13}C_3$.
The number of ways of selecting 1 card from the 39 which are not diamonds $= 39$.
$\therefore$ the number of ways of selecting 4 cards, 3 of which are diamonds $= 39 \times {}_{13}C_3$.
The number of ways of selecting any 4 cards from $52 = {}_{52}C_4$.
Hence the probability that exactly 3 cards out of 4 are diamonds

$$= 39 \times {}_{13}C_3 \div {}_{52}C_4 = 39 \times \frac{13!}{3!10!} \times \frac{4!48!}{52!} \approx 0{\cdot}0412.$$

For a trial in which the possible outcomes are not equally likely, it is more difficult to arrive at a precise definition of the probability of an event. However, the subsets of a sample space S are regarded as events whatever the exact nature of the set S. In all cases $P(A)$, the probability of an event A, is a number or a weighting associated with the subset A. We can consider P to be a *probability mapping* which maps any subset A onto a real number p, where $0 \leqslant p \leqslant 1$. In any experiment the set of events (i.e. the set of subsets of the sample space) to be considered together with the appropriate probability mapping may be referred to as the *probability space*.

Exercise 14.3

In questions 1 to 4 find the probability of events A, B and C.

1. A die is thrown. A: scoring a four, B: scoring an odd number, C: scoring four or less.

2. A card is drawn from a pack. A: drawing a red card, B: drawing a seven, C: drawing a king, a queen or a jack.

3. Three coins are tossed. A: three heads, B: at least two tails, C: at least one of each.

4. It rained on exactly two days last week. A: it rained on Monday and Tuesday, B: it rained on two consecutive days, C: it rained on neither Monday nor Tuesday.

5. For the events in question 1, find
(a) $P(A \cup B)$, (b) $P(A \cap B)$, (c) $P(B \cup C)$, (d) $P(B \cap C')$.

6. For the events in question 2, find
(a) $P(A \cup B)$, (b) $P(A' \cap B)$, (c) $P(B \cap C')$, (d) $P(A \cap C')$.

7. An integer is chosen at random from the first 200 positive integers. Find the probability that it is (a) not divisible by 5, (b) a perfect square, (c) divisible by both 5 and 2, (d) divisible by neither 2 nor 7.

8. Two unbiased dice are thrown. Find the probability that the product of the scores is (a) odd, (b) a multiple of 3, (c) a multiple of 12.

9. A domino is drawn from a standard set of 28. Find the probability that the sum of its spots is (a) 2, (b) 8, (c) odd, (d) even.

10. In a cafeteria 80% of the customers order chips and 60% order peas. If 20% of those ordering peas do not want chips, find the probability that a customer chosen at random orders chips but not peas.

11. The points A, B, C, D and E are the vertices of a regular pentagon. All possible lines joining pairs of these points are drawn. If two of these lines are chosen at random, that is the probability that their point of intersection is (a) inside the pentagon, (b) one of the points A, B, C, D, E.

12. A box contains 4 discs numbered 1, 2, 2, 3. A disc is drawn from the box then replaced and a second disc is drawn. With the aid of a sample space diagram or otherwise, find the probability that (a) the total score is 6, (b) the total score is 4, (c) the numbers drawn are different, (d) the difference between the two numbers drawn is less than two.

13. A shopper with a keen eye for a bargain has bought a carton of assorted tins without labels. Assuming that the tins came from a large warehouse containing equal numbers of tins of soup, peaches and rice pudding, find the probability that (a) the first tin opened contains soup, (b) the first tin does not contain soup but the second does, (c) none of the first three tins contains soup. [You may find tree diagrams helpful.]

14. Two cards are selected at random without replacement from a set of five numbered 2, 3, 4, 4, 5. With the aid of a diagram or otherwise, find the probability that the numbers on the cards (a) are both even, (b) have a difference of 2, (c) have a sum of 7, (d) have a sum of 8.

15. A bag contains 3 blue beads and 2 red beads. A second bag contains 1 blue bead and 3 red beads. If one bead is drawn from each bag, find the probability that the beads are (a) both blue, (b) one blue and one red.

16. Write down in factorial form the number of different arrangements of the letters of the word EQUILIBRIUM. One of these arrangements is chosen at random. Find the probability that (a) the first two letters of the arrangement are consonants, (b) all the vowels are together.

17. There are 8 dusters in a drawer, 5 orange and 3 pink. If two of the dusters are chosen at random simultaneously, what is the probability that there will be one of each colour.

18. Four balls, two red, one blue and one white, are placed in a bag. The balls are then drawn at random from the bag one at a time and not replaced. Find the probability that (a) the first ball drawn is red, (b) the last ball drawn is white, (c) the red balls are drawn consecutively.

19. A hand of three cards is dealt from a well shuffled pack of 52. Find the probability that the hand contains (a) exactly one ace, (b) three cards of the same suit, (c) no two cards of the same suit.

20. On a plate of 12 assorted cakes, 3 are doughnuts. If 3 cakes are selected at random, find the probabilities that 0, 1, 2, 3 doughnuts are chosen.

21. A bag contains 25 clothes pegs, 15 plastic pegs and 10 wooden pegs. If 4 pegs are taken from the bag at random, find the probability that (a) all 4 are plastic, (b) 2 are plastic and 2 are wooden.

22. Four books are selected at random from a shelf containing 3 cookery books, 5 novels and 2 biographies. Find the probability that the four books (a) are all novels, (b) are 2 novels and 2 cookery books, (c) include at least one biography, (d) include at least one of each type.

14.4 Sum and product laws

If A and B are two different events which may occur in a sample space S, the set theory result

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

discussed in §14.2 can be used to find the probability that A or B (or both) will occur.

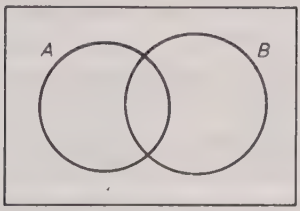
$$P(A \cup B) = \frac{n(A \cup B)}{n(S)} = \frac{n(A) + n(B) - n(A \cap B)}{n(S)}$$

$$= \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} - \frac{n(A \cap B)}{n(S)}$$

$$\therefore \quad \boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B).}$$

This is the *addition law* of probability.

Example 1 A card is drawn at random from a pack. A is the event of drawing an ace and B is the event of drawing a diamond. Find $P(A)$, $P(B)$, $P(A \cap B)$ and deduce $P(A \cup B)$.



$$P(A) = \frac{4}{52} = \frac{1}{13}, \; P(B) = \frac{13}{52} = \frac{1}{4}.$$

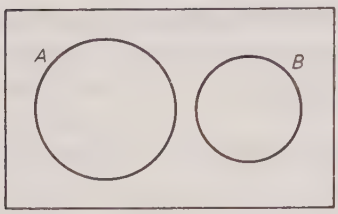
$P(A \cap B)$ is the probability of drawing the ace of diamonds, which is 1/52.

Hence using the addition law,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}.$$

Events A and B are said to be *mutually exclusive* if they cannot occur at the same time, i.e. if the sets A and B are disjoint. This means that $A \cap B = \varnothing$ and $P(A \cap B) = 0$, so that for mutally exclusive events the addition law reduces to $P(A \cup B) = P(A) + P(B)$. In particular, if $B = A'$, then $P(A \cup B) = P(A \cup A') = P(S) = 1$, giving the result $1 = P(A) + P(A')$, i.e. $P(A') = 1 - P(A)$, which was obtained in the previous section.

Example 2 A disc is drawn at random from a box containing 12 discs of which 6 are red and 2 are blue. If event A is the choice of a red disc and event B is the choice of a blue disc, find $P(A)$, $P(B)$ and deduce $P(A \cup B)$.



$$P(A) = \frac{6}{12} = \frac{1}{2},\ P(B) = \frac{2}{12} = \frac{1}{6}$$

Since a disc cannot be both red and blue, the events A and B are mutually exclusive,

$$\therefore\quad P(A \cup B) = P(A) + P(B)$$

$$= \frac{6}{12} + \frac{2}{12} = \frac{8}{12} = \frac{2}{3}.$$

A set of events is said to be *exhaustive* if at least one of them is certain to occur in any trial, e.g. if the events A and B are exhaustive, then $P(A \cup B) = 1$ and the addition law becomes $P(A) + P(B) - P(A \cap B) = 1$. This will reduce to $P(A) + P(B) = 1$, if A and B are also mutually exclusive.

In general, for events $A_1, A_2, \ldots, A_n$ which are both mutually exclusive and exhaustive

$$P(A_1) + P(A_2) + \ldots + P(A_n) = 1.$$

Two events A and B are said to be *independent* if the occurrence of event A has no effect on the probability of event B. However, if the probability of event B changes when A is known to have occurred then A and B are *dependent* upon one another.

Let us consider a playing card drawn at random from a pack and events A, B and C defined as follows: A: drawing a king, B: drawing a club, C: drawing a black king. The probability of drawing a club from the complete pack is 1/4. If the card drawn is known to be a king, i.e. if event A has occurred, then the probability that the card is a club remains 1/4. Hence events A and B are independent. Since $P(A) = 1/13$, $P(B) = 1/4$ and $P(A \cap B) = 1/52$, we find that $P(A \cap B) = P(A).P(B)$. However, if the card drawn is known to be a black king, i.e. if event C has occurred, then the probability that the card is a club is now 1/2 rather than 1/4. Hence events B and C are not independent. Since $P(B) = 1/4$, $P(C) = 1/26$ and $P(B \cap C) = 1/52$, in this case $P(B \cap C) \neq P(B).P(C)$.

This example illustrates the fact that

> two events A and B are independent if and only if
>
> $$P(A \cap B) = P(A).P(B)$$

This is the simplest form of the *multiplication* or *product law*. [The extension to events which are not independent is dealt with in the next section.]

When two dice are thrown, the scores obtained are independent of each other, so the product law may be applied. This provides an alternative approach to Example 3, §14.3.

The probability of throwing 1 on a single die is $\frac{1}{6}$, so the probability of throwing two 1's, i.e. of scoring a total of 2, is $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$.

The probabilities of the outcomes (1,2) and (2,1) are both equal to $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$. Hence the probability of scoring a total of 3 is $\frac{1}{36} + \frac{1}{36} = \frac{2}{36} = \frac{1}{18}$.

For any score on the 1st die, the probability of the 2nd score being the same is $\frac{1}{6}$. Thus the probability of throwing the same score on both dice is $\frac{1}{6}$.

Since the probability of scoring 3 or more on a single die is $\frac{2}{3}$, the probability of throwing 3 or more on each die is $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$.

Example 3 A boy must throw a 6 on a single die to start a game. Find the probability that he succeeds at his third attempt.

The probability of not throwing a 6 in his first turn $= \frac{5}{6}$.

The probability of not throwing a 6 in his second turn $= \frac{5}{6}$.

The probability of throwing a 6 in his third turn $= \frac{1}{6}$.

Hence the probability that he succeeds at his third attempt

$$= \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} = \frac{25}{216}.$$

Example 4 A card is drawn from a pack and then replaced, the suit being noted. Find the probability that in a sequence of four such trials exactly two diamonds are drawn.

The number of orders in which two diamonds and two other cards may be drawn in the four trials is $\frac{4!}{2!2!}$, i.e. 6.

In each of these 6 cases, the probability of drawing two diamonds and two other cards in the appropriate order is $\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^2$.

Hence the probability that two diamonds are drawn

$$= 6 \times \frac{3.3}{4.4.4.4} = \frac{27}{128}.$$

Exercise 14.4

1. If A and B are random events such that $P(A) = 5/12$, $P(B) = 1/3$ and $P(A \cap B) = 1/4$, find $P(A \cup B)$.

2. In a certain school the probability that a pupil takes part in the annual musical evening is 0·25. The probability that a pupil helps with the school play is 0·1. If the probability of doing both is 0·02, find the probability that a pupil is involved in at least one of these activities.

3. A butcher observes that the probability that a certain customer buys beef is $\frac{1}{2}$ and the probability that she buys pork is $\frac{1}{3}$. If the probability that this customer buys neither pork nor beef is $\frac{1}{4}$, find the probability that she buys both beef and pork.

In questions 4 and 5 for the given events A, B and C
(i) write down any pairs of events which are (a) mutually exclusive, (b) exhaustive;
(ii) find (a) $P(A \cap B)$, (b) $P(A \cup C)$, (c) $P(B \cap C)$,
(d) $P(A' \cup C)$, (e) $P(B \cap C')$, (f) $P(A' \cup B')$.

4. A card is drawn at random from a pack of 52. A: drawing a black card, B: drawing a red card, C: drawing a heart.

5. Two unbiased dice are thrown and the total score recorded. A: a score less than 8, B: a score divisible by 5, C: a score greater than 4.

6. The events A and B are mutually exclusive. The events A, B and C are exhaustive. If $P(B) = 0{\cdot}6$ and $P(C) = 0{\cdot}3$, find the range of possible values of $P(A')$.

In questions 7 and 8, for the given events A, B and C, find $P(A)$, $P(B)$, $P(C)$, $P(A \cap B)$, $P(B \cap C)$, $P(A \cap C)$. Write down any pairs of events which are independent.

7. A card is drawn at random from a pack of 52. A: drawing a black card, B: drawing the jack of hearts, C: drawing a court card (i.e. jack, queen or king).

8. A disc is drawn at random from a bag containing discs numbered 1 to 48. A: drawing a multiple of 3, B: drawing a multiple of 4, C: drawing a multiple of 6.

9. If A and B are two independent events such that $P(A) = 1/4$ and $P(B) = 3/5$, find
(a) $P(A \cap B)$, (b) $P(A \cup B)$, (c) $P(A \cap B')$, (d) $P(A' \cup B')$.

10. A and B are two independent events, such that $P(A \cap B) = 1/3$ and $P(A \cup B) = 9/10$. Given that $P(A) > P(B)$, find $P(A)$ and $P(B)$.

11. If A, B and C are independent random events such that $P(A) = 2/5$, $P(B) = 1/3$, $P(C) = 3/4$, calculate $P(A \cup B \cup C)$ and $P(A' \cup B' \cup C')$.

12. A box contains 5 red balls and 3 blue balls. A second box contains 2 red balls and 3 blue balls. Find the probability that (a) if a ball is drawn at random from each box both will be blue, (b) if two balls are drawn at random from each box all four will be red.

13. A die is biased so that the probability of throwing a six is $\frac{1}{4}$. If the die is thrown twice find the probability of (a) two sixes, (b) at least one six.

14. A man estimates that the probability that he will be early for work is 1/3, the probability that he will be on time is 1/2 and the probability that he will be late is 1/6. Find the probability that in a particular 3-day period he will (a) arrive early every day, (b) never arrive on time, (c) be late at least once.

15. Three unbiased dice are thrown. Find the probability that (a) the score on each die is the same, (b) the total score is 5, (c) the total score is even.

16. Alan and Bob are playing a game which ends when either player is two points ahead of his opponent. If the probability that Alan wins any particular point is $\frac{2}{3}$, find the probabilities that the game ends when the number of points played is (a) 2, (b) 3, (c) 4.

17. Tom and Peter throw a single die in turn. The first player to throw a six wins. If Tom throws first, find the probability that Peter wins on his first, second or third turn. [Give your answer to 3 significant figures.]

18. Cards are drawn at random, with replacement, from a pack of 52. Find the probability that (a) the first two cards drawn are spades, (b) two out of the first three cards drawn are spades, (c) the third card drawn is a club, (d) the fourth card is the first black card to be drawn.

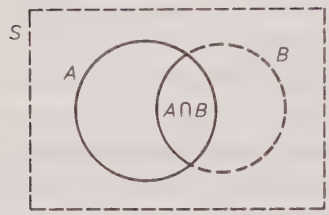
19. A certain committee has 6 members. The probability that any member will attend a particular meeting is 4/5. Find the probability that the chairman and at least 4 of the 5 other members will attend.

20. An unbiased coin is tossed 5 times. Find the probabilities of (a) 4 heads, (b) 3 heads, (c) at least 2 heads.

21. A bag contains 3 red discs, 4 blue discs and 5 green discs. A trial consists of selecting a disc at random, noting its colour then replacing it. In three such trials, what is the probability of selecting (a) three red discs, (b) two blue discs, (c) one green and two blue discs?

14.5 Conditional probability

The probability of an event B, given that an event A has occurred, is called the *conditional probability* of B upon A, or the probability of B given A, and is written $P(B|A)$. This means that $P(B|A)$ is the probability that B occurs, considering A as sample space.



Since the subset of A in which B occurs is $A \cap B$,

$$P(B|A) = \frac{n(A \cap B)}{n(A)} = \frac{n(A \cap B)}{n(S)} \Big/ \frac{n(A)}{n(S)}$$

$$\therefore \quad \boxed{P(B|A) = \frac{P(A \cap B)}{P(A)}.}$$

The general statement of the product law is obtained by rearranging this result:

$$P(A \cap B) = P(A).P(B|A).$$

Similarly

$$P(A \cap B) = P(B).P(A|B).$$

In the case of independent events the probability of B is unaffected by the occurrence of A, so $P(B|A) = P(B)$

$$\therefore \quad P(A \cap B) = P(A).P(B).$$

Example 1 Two cards are drawn in succession from a pack. What is the probability that they are both diamonds if (a) the first card is replaced before the second is drawn, (b) the first card is not replaced?

(a) Since both cards are drawn from a full pack, in each case the probability of drawing a diamond is $\frac{13}{52}$, i.e. $\frac{1}{4}$

$\therefore$ with replacement the probability of two diamonds is

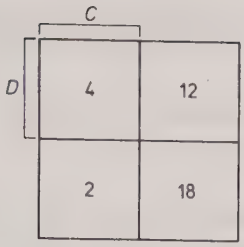
$$\frac{1}{4} \times \frac{1}{4} = \frac{1}{16}.$$

(b) The probability that the second card is a diamond given that a diamond has already been drawn is $\frac{12}{51} = \frac{4}{17}$,

$\therefore$ without replacement, the probability of two diamonds is

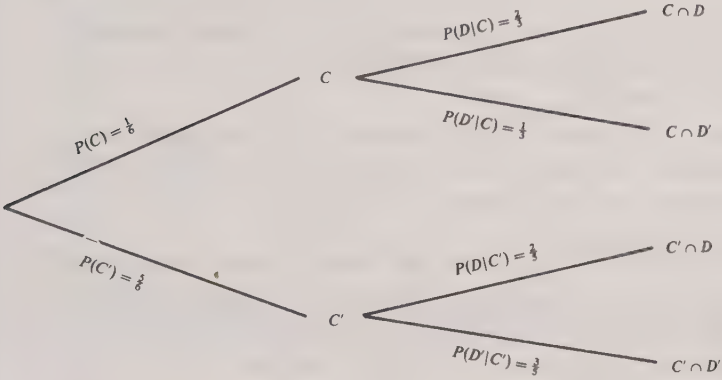
$$\frac{1}{4} \times \frac{4}{17} = \frac{1}{17}.$$

Consider now a trial in which two dice are thrown, where event C is throwing the same score on both dice and event D is throwing 3 or more on each of the dice.



The 36 possible outcomes in the sample space for this trial can be represented in a Karnaugh map for events C and D.

Relationships between probabilities associated with such a trial are often displayed in a *probability tree diagram.*



$$P(C \cap D) = \frac{1}{6}.\frac{2}{3} = \frac{1}{9}$$

$$P(C \cap D') = \frac{1}{6}.\frac{1}{3} = \frac{1}{18}$$

$$P(C' \cap D) = \frac{5}{6}.\frac{2}{5} = \frac{1}{3}$$

$$P(C' \cap D') = \frac{5}{6}.\frac{3}{5} = \frac{1}{2}$$

Probabilities can also be recorded in a *contingency table* as shown below. The diagram on the left shows which probabilities should be inserted in each part of the table. The completed table is on the right.

	D	D'	
C	$P(C \cap D)$	$P(C \cap D')$	$P(C)$
C'	$P(C' \cap D)$	$P(C' \cap D')$	$P(C')$
	$P(D)$	$P(D')$	

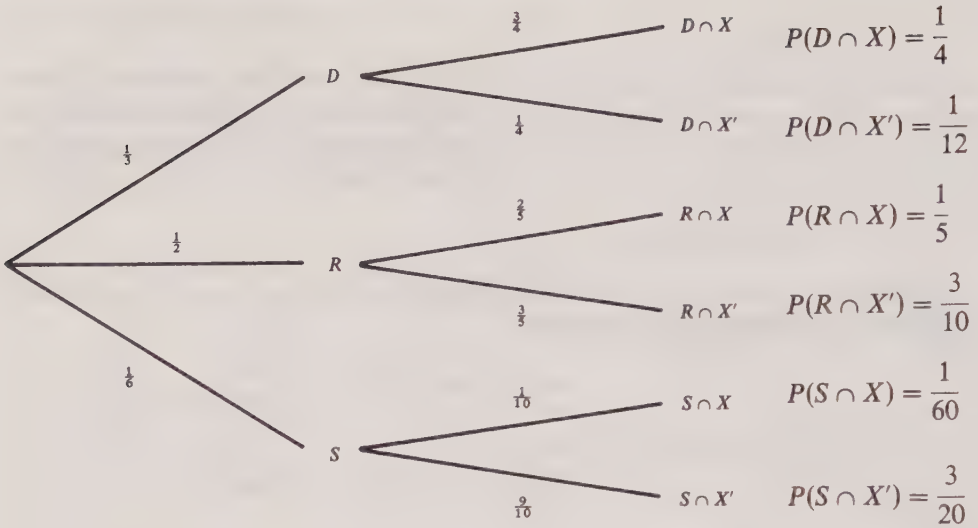
	D	D'	
C	$\frac{1}{9}$	$\frac{1}{18}$	$\frac{1}{6}$
C'	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{5}{6}$
	$\frac{4}{9}$	$\frac{5}{9}$	1

Both the tree diagram and the contingency table demonstrate that

$$P(C \cap D)+P(C \cap D')+P(C' \cap D)+P(C' \cap D') = P(C)+P(C') = 1.$$

Example 2 A motorist plans a journey and event X is arrival at his destination in less than three hours. He estimates the probabilities of dry weather D, rain R or snow S to be $\frac{1}{3}$, $\frac{1}{2}$ and $\frac{1}{6}$ respectively. The probabilities of event X in these conditions are $\frac{3}{4}$, $\frac{2}{5}$ and $\frac{1}{10}$ respectively. What is the probability that (a) the motorist completes his journey in under 3 hours, (b) if he fails to arrive in less than 3 hours, there was a fall of snow?

(a) On the assumption that events D, R and S are mutually exclusive, the following tree diagram can be constructed.



$$P(X) = P(D \cap X) + P(R \cap X) + P(S \cap X) = \frac{1}{4} + \frac{1}{5} + \frac{1}{60} = \frac{7}{15}$$

$\therefore$ the probability that the journey is completed in less than 3 hours is 7/15.

(b) $P(S|X') = \dfrac{P(S \cap X')}{P(X')} = \dfrac{P(S \cap X')}{1 - P(X)} = \dfrac{\frac{3}{20}}{1 - \frac{7}{15}} = \dfrac{3}{20} \cdot \dfrac{15}{8} = \dfrac{9}{32}.$

$\therefore$ given that the motorist failed to arrive in less than three hours, the probability that it snowed is 9/32.

	D	R	S
X	1/4	1/5	1/60
X′	1/12	3/10	3/20

It is sometimes helpful to display probabilities on a Karnaugh map. We give here the probability map for Example 2. This shows clearly that

$$P(X) = \frac{1}{4} + \frac{1}{5} + \frac{1}{60} = \frac{7}{15}.$$

When finding $P(S|X')$ we consider X' as sample space. Thus, using the lower half of the diagram only,

$$P(S|X') = \frac{3}{20} \Big/ \left(\frac{1}{12} + \frac{3}{10} + \frac{3}{20}\right) = \frac{3}{20} \Big/ \frac{32}{60} = \frac{9}{32}.$$

Exercise 14.5

Questions 1 to 5 concern a trial in which a domino is drawn from a complete set of 28. Event A: it has at least one six on it. Event B: it is a double, i.e. has two identical numbers on it. Event C: the sum of the numbers is six or less.

1. Find $P(A)$, $P(B)$, $P(C)$.

2. Find $P(B|A)$, $P(C|A)$, $P(A \cap B)$, $P(A \cap C)$.

3. Find $P(B|C)$, $P(C|B)$, $P(B|C')$, $P(C|B')$.

4. Find $P(B \cap C)$, $P(B \cap C')$, $P(B' \cap C)$.

5. Find $P(A \cap C')$, $P(A' \cap B')$, $P(A' \cap C')$.

Questions 6 to 10 Repeat questions 1 to 5 for a trial in which a card is drawn from a normal pack. Event A: drawing a king or a queen. Event B: drawing a black card. Event C: drawing a heart.

11. A bag contains 4 black discs and 2 white ones. If discs are drawn from the bag one at a time, find the probability that 2 black discs followed by 2 white discs are drawn (a) with replacement, (b) without replacement.

12. A bag contains 10 cards bearing the letters of the word STATISTICS. Find the probability that the first four letters drawn at random from the bag are S, I, T, S in that order (a) with replacement, (b) without replacement.

13. The probability that it will be foggy on a November morning is $\frac{1}{3}$. The probability that Mr. Jones will be late for work when it is foggy is $\frac{1}{2}$. The probability that he will be late if it is not foggy is $\frac{1}{8}$. If on a particular November morning Mr. Jones is late, find the probability that it is foggy.

14. On a day on the summer holidays the probability that Jane will go swimming is 0·6. The probability that both Jane and Karen will go swimming is 0·45. If Jane is seen in the swimming pool what is the probability that Karen also swims that day?

15. The probability that a light bulb has a life of more than 1000 hours is $\frac{1}{2}$ and the probability of a life of more than 2000 hours is $\frac{1}{10}$. What is the probability that a bulb that has already been in use for 1000 hours will last a further 1000 hours?

16. A football star is having injury problems. When he is playing the probability that his team will win is $\frac{3}{4}$, but otherwise it is only $\frac{1}{2}$. The probability that the player will be fit this Saturday is $\frac{1}{3}$. Find the probability that his team will win the match.

17. The probability that it will be sunny tomorrow is 0·25. The probability that Mrs. Brown will go shopping tomorrow is 0·2. The probability that both these events occur is 0·15. Find the probability that (a) neither event occurs, (b) Mrs. Brown goes shopping, given that it is sunny, (c) it is sunny given that Mrs. Brown goes shopping.

18. A bag contains 5 black beads and 3 white beads. A second bag contains 3 black beads and 5 white beads. A bead is drawn at random from the first bag and placed in the second bag. A bead is then drawn at random from the second bag

and placed in the first. Find the probability that each bag now contains (a) 4 black and 4 white beads, (b) the same numbers of each colour as it did initially.

19. A box A contains 3 red balls and 4 black balls. A box B contains 3 red balls and 2 black balls. One box is selected at random and then from that box one ball is selected at random. Find (a) the probability that the ball is red, (b) the probability that the ball came from box A, given that it is red.

20. A member of a slimming club claims that the probability that she will have a chocolate biscuit with her morning coffee is $\frac{1}{6}$ and that on the days that she resists the temptation, the probability that she exceeds her daily calorie allowance is $\frac{1}{5}$. If the probability that she consumes too many calories on any day is $\frac{1}{4}$, find the probability that she does so on a day when she has had a chocolate biscuit. Given that yesterday she kept within her calorie allowance, find the probability that she ate a chocolate biscuit.

Exercise 14.6 (miscellaneous)

1. How many even five-digit numbers can be formed using the digits 0, 1, 2, 3, 4, 5, 6 (excluding numbers which start with 0) if (a) no digit may be repeated, (b) repetition is allowed?

2. Find the number of different ways in which the letters of the word ISOSCELES can be arranged. How many of these arrangements (a) begin and end with E, (b) begin with S, (c) end with a vowel?

3. A man has found 6 records that he likes in a shop, but cannot decide which of them to buy. Assuming that he buys at least one, find the number of different selections he could make.

4. Six lines are drawn, no two of which are parallel. If no more than two of the lines pass through any one point, find the number of triangles formed.

5. In how many ways can a group of 9 people attending a conference be split into three sets of 3 people for discussion? Later the same people form new sets of 3. In how many ways can this be done if no two people remain together?

6. Find the number of ways in which 5 books can be distributed between three people A, B and C if the books are (a) all different, (b) indistinguishable.

7. Draw a Venn diagram to show the following sets and their non-empty intersections: $\mathscr{E}$ = {all quadrilaterals}, T = {trapeziums}, R = {rectangles}, P = {parallelograms}, K = {kites, i.e. two pairs of adjacent sides equal}. Describe the elements of the sets $K \cap R$ and $K \cap P$.

8. Decide whether the following statements are true or false for sets A, B and C, giving brief reasons for your answers.

(a) $A \cap B = A \cup B \Leftrightarrow A = B$,
(b) $A \cup B = A \cup C \Leftrightarrow B = C$,
(c) $A \cap B = A \cap C \Leftrightarrow A = B \cap C$,
(d) $A \cap (B' \cup C') = \varnothing \Leftrightarrow A \subset (B \cup C)$.

9. In a certain school half the pupils under 16 years of age are girls, two thirds of the girls in the school are under 16 and a quarter of the pupils are boys of 16 and over. If there are 540 girls in the school, find the total number of pupils.

10. A drawer contains 4 different pairs of socks. Find the probability that (a) if 2 socks are selected at random they will form a pair, (b) if 4 socks are selected at random they will form two pairs.

11. (a) Two dice are thrown together, and the scores added. What is the probability that (i) the total score exceeds 8, (ii) the total score is 9, or the individual scores differ by 1, or both?
(b) A bag contains 3 red balls and 4 black ones. 3 balls are picked out, one at a time and not replaced. What is the probability that there will be 2 red and 1 black in the sample?
(c) A committee of 4 is to be chosen from 6 men and 5 women. One particular man and one particular woman refuse to serve if the other person is on the committee. How many different committees may be formed? (SU)

12. An analysis of the other subjects taken by A-level Mathematics candidates in a certain year showed that 20% of them took Further Mathematics, 50% took Physics and 5% took both Further Mathematics and Physics. A candidate is chosen at random from those who took A-level Mathematics. (i) Calculate the probability that the chosen candidate took neither Further Mathematics nor Physics. (ii) Given that the chosen candidate took at least one of Further Mathematics and Physics, calculate the probability that the candidate took Further Mathematics. (JMB)

13. Two cards are drawn simultaneously from a pack of 52. What is the probability that both are spades if one card is known to be (a) black, (b) red, (c) a spade, (d) a king?

14. An experiment consists of tossing two fair coins one after the other. Let X be the set of events in which the first coin is a head, Y be the set of events in which the second coin is a head and Z be the set of events in which both coins are heads or both coins are tails. Calculate the following probabilities: $p(X)$, $p(Y)$, $p(Z)$, $p(X \cap Z)$, $p(X \cap Y \cap Z)$. (AEB 1977)

15. Three cards are drawn at random without replacement from a pack of ten cards which are numbered from 1 to 10, respectively. Calculate (i) the probability that the numbers drawn consist of two even numbers and one odd number, (ii) the probability that at least one of the numbers drawn is a perfect square greater than 1, (iii) the probability that the smallest number drawn is the 5. (JMB)

16. A disc of diameter 5 cm is tossed at random on to a large sheet of paper ruled with parallel lines 8 cm apart. Assuming the lines to be of negligible thickness, find the probability that the disc lands on a line. If the paper is now ruled with 8 cm squares, find the probability that (a) the disc lands on a line, (b) the disc lands on the intersection of two lines.

17. A pack of 52 cards contains 4 suits each of 13 cards. If 13 cards are taken at random from the pack what is the probability that exactly 10 of them are spades? [You may take $\binom{52}{13} = 6{\cdot}35 \times 10^{11}$.] (O&C)

18. A bag contains 5 red, 4 orange and 3 yellow sweets. One after another, three children select and eat one sweet each. When the bag contains n sweets, the probability of any one child choosing any particular sweet is $1/n$. What are the probabilities that (a) they all choose red sweets, (b) at least one orange sweet is chosen, (c) each chooses a different colour, (d) all choose the same colour? [Answers to this question may be left as fractions in their lowest terms.] (O&C)

19. (a) Two independent events are such that there is a probability of $\frac{1}{6}$ that they will both occur and a probability of $\frac{1}{3}$ that neither will occur. Calculate their individual probabilities of occurring.
(b) A fair cubical die has three of its faces coloured red, two coloured blue and one coloured white. If the die is thrown six times calculate the probabilities that (i) a red face will be uppermost at least once, (ii) a red face will be uppermost exactly three times, (iii) each colour will be uppermost exactly twice. (W)

20. Four cards are drawn at random from a pack, one at a time with replacement. Find the probability that (a) no heart is drawn, (b) four hearts are drawn, (c) two hearts and two diamonds are drawn (in any order), (d) one card from each suit is drawn. (L)

21. Two events A and B are such that $P(A) = 0{\cdot}4$ and $P(A \cup B) = 0{\cdot}7$. (i) Find the value of $P(A' \cap B)$. (ii) Find the value of $P(B)$ if A and B are mutually exclusive. (iii) Find the value of $P(B)$ if A and B are independent. (JMB)

22. The following are three of the classical problems in probability.
(a) Compare the probability of a total of 9 with the probability of a total of 10 when three fair dice are tossed once (Galileo and Duke of Tuscany).
(b) Compare the probability of at least one six in 4 tosses of a fair die with the probability of at least one double-six in 24 tosses of two fair dice (Chevalier de Méré).
(c) Compare the probability of at least one six when 6 dice are rolled with the probability of at least two sixes when 12 dice are rolled (Pepys to Newton).
Solve each of these problems. (AEB 1978)

23. (a) From an ordinary pack of 52 cards two are dealt face downwards on a table. What is the probability that (i) the first card dealt is a heart, (ii) the second card dealt is a heart, (iii) both cards are hearts, (iv) at least one card is a heart?

(b) Bag A contains 3 white counters and 2 black counters whilst bag B contains 2 white and 3 black. One counter is removed from bag A and placed in bag B without its colour being seen. What is the probability that a counter removed from bag B will be white?
(c) A box of 24 eggs is known to contain 4 old and 20 new. If 3 eggs are picked at random determine the probability that (i) 2 are new and the other old, (ii) they are all new. (SU)

24. A hand of 5 cards is drawn from a pack of 52 playing cards. Find the probability of drawing (a) 5 cards of the same suit, (b) a pair plus a triple (e.g. 2 sevens and 3 aces). If a hand consists of 2 aces, a king, a jack and a 4 and a player keeps the pair of aces but discards the other 3 cards and draws 3 more cards from the remaining 47, find the probability of obtaining a pair plus a triple. (L)

25. If A, B and C are exhaustive events, such that A cannot occur at the same time as either B or C, show that
(a) $P(A)+P(B)+P(C) = 1+P(B \cap C)$, (b) $P(A') = P(B \cup C)$.

26. Two cards are drawn without replacement from a pack of playing cards. Using a tree diagram, or otherwise, calculate the probability (a) that both cards are aces, (b) that one (and only one) card is an ace, (c) that the two cards are of different suits. Given that at least one ace is drawn, find the probability that the two cards are of different suits. (L)

27. (a) Two events A and B are such that $P(A) = \frac{1}{3}$ and $P(B) = \frac{1}{2}$. If A' denotes the complement of A, calculate $P(A' \cap B)$ in each of the cases when (i) $P(A \cap B) = \frac{1}{8}$, (ii) A and B are mutually exclusive, (iii) A is a subset of B.
(b) A Scottish court may give any one of the three verdicts: 'guilty', 'not guilty' and 'not proven'. Of all the cases tried by the court, 70% of the verdicts are 'guilty', 20% are 'not guilty' and 10% are 'not proven'. Suppose that when the court's verdict is 'guilty', 'not guilty' and 'not proven', the probabilities that the accused is really innocent are 0·05, 0·95 and 0·25, respectively. Calculate the probability that an innocent person will be found 'guilty' by the court. (W)

28. Suppose that letters sent by first and second class post have probabilities of being delivered a given number of days after posting according to the following table (weekends are ignored).

Days to delivery	1	2	3
1st class	0·9	0·1	0
2nd class	0·1	0·6	0·3

The secretary of a committee posts a letter to a committee member who replies immediately using the same class of post. What is the probability that four or more days are taken from the secretary posting the letter to receiving the reply if (a) first class, (b) second class post is used? The secretary sends out four letters and each member replies immediately by the same class of post. Assuming the letters move independently, what is the probability that the secretary receives (a)

all the replies within three days using first class post, (b) at least two replies within three days using second class post? (O)

29. In a game, three cubical dice are thrown by a player who attempts to throw the same number on all three. What is the chance of the player (a) throwing the same number on all three, (b) throwing the same number on just two? If the first throw results in just two dice showing the same number, then the third is thrown again. If no two dice show the same number, then all are thrown again. The player then comes to the end of his turn. What is the chance of the player succeeding in throwing three identical numbers in a complete turn? What is the chance that all the numbers are different at the end of a turn? (O&C)

30. An experiment is performed with a die and two packs of cards. The die is thrown, and if it shows 1, 2, 3 or 4 a card is drawn at random from the first pack, which contains the usual 52 cards; if the score on the die is 5 or 6 a card is drawn from the second pack, which contains only 39 cards, all the clubs having been removed. X denotes the event 'The first pack with 52 cards is used', and Y denotes the event 'The card drawn is a diamond'. Calculate the probabilities (i) $P(X)$, (ii) $P(X \cap Y)$, (iii) $P(Y)$, (iv) $P(Y|X)$, (v) $P(X|Y)$. (C)

31. Each of two boxes contains ten discs. In one box four of the discs are red, two are white and four are blue; in the other box two are red, three are white and five are blue. One of these two boxes is chosen at random and three discs are drawn at random from it without replacement. Calculate (i) the probability that one disc of each colour will be drawn, (ii) the probability that no white disc will be drawn, (iii) the most probable number of white discs that will be drawn. Given that three blue discs were drawn, calculate the conditional probability that they came from the box that contained four blue discs. (W)

32. Write down an equation, in terms of probabilities, corresponding to each of the statements (i) the events A and B are independent, (ii) the events A and C are mutually exclusive. The events A, B and C are such that A and B are independent and A and C are mutually exclusive. Given that $P(A) = 0{\cdot}4$, $P(B) = 0{\cdot}2$, $P(C) = 0{\cdot}3$, $P(B \cap C) = 0{\cdot}1$, calculate $P(A \cup B)$, $P(C|B)$, $P(B|A \cup C)$. Also calculate the probability that one and only one of the events B, C will occur. (W)

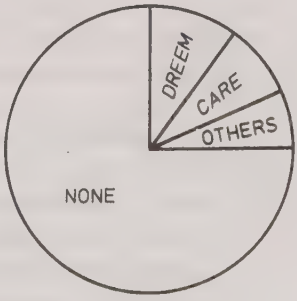
33. A pack of sixteen playing cards consists of the ace, king, queen and jack of each of the four suits, spades, hearts, diamonds and clubs. A man is dealt four cards at random from the pack. (i) Calculate the probability that he has been dealt exactly two aces. (ii) The man is asked whether he has been dealt any aces. He truthfully replies that he has. Calculate the probability that he has been dealt exactly two aces. (iii) By accident the man displays one of his four cards, which is seen to be the ace of spades. Calculate the probability that he has been dealt exactly two aces. Give your answers correct to two significant figures. (C)

15 Elementary statistics

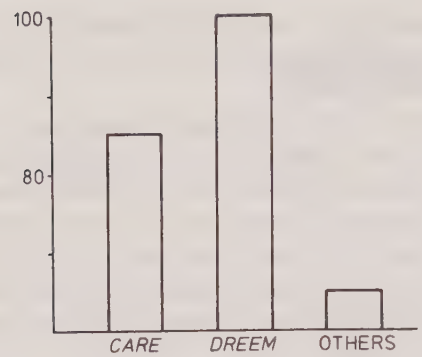
15.1 Graphical representation of data

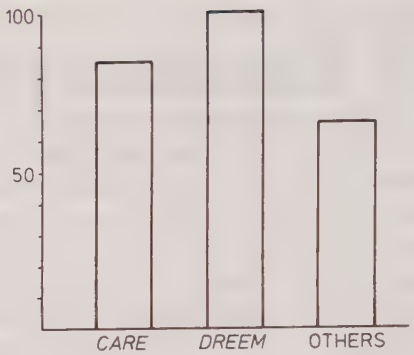
Statistics may be described as a subject concerned with the analysis of data obtained from such sources as experiments and surveys. In elementary work this will involve making tables, charts and graphs to present the data as clearly as possible. In more advanced work various numerical measures of the properties of collections of data are studied. These may be used to compare sets of observations, to draw conclusions from information gathered and to make predictions about future events. An important aspect of the subject is measuring the significance of such conclusions and predictions.

We first consider three common ways of presenting information in graphical form. Below are two *pie charts* showing the results of an investigation into which brands of fabric softener housewives use. The charts both show the relative proportions of the different brands used, but the first is quite misleading as it gives no indication that the majority of housewives use no softener at all.

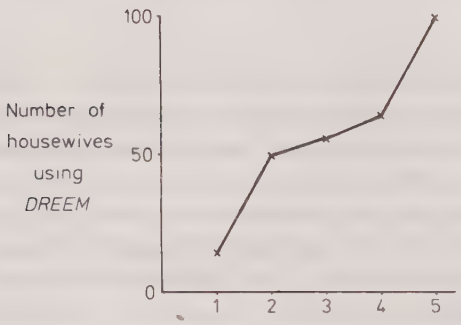


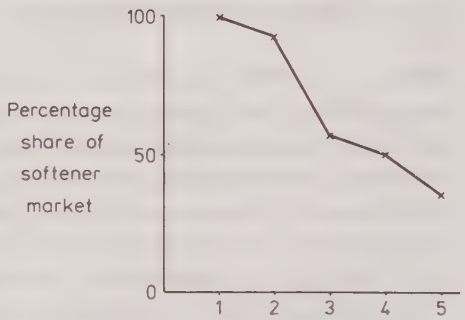
The same results can also be illustrated in the form of a *bar chart* or *block graph*. In both the given charts the vertical scale represents the number of housewives using the products. However, the first diagram exaggerates the lead of 'DREEM' in the market by starting the vertical axis at 60 instead of 0.





The following *line graphs* both represent the results of 5 successive surveys. This type of graph is used to illustrate a trend or pattern in results. However, in this case either graph on its own could be misleading because although sales of 'DREEM' are rising, its share of the market in fabric softeners is falling.





It is clear from these examples that care must be taken when making or interpreting graphs and charts, but it is also important to question the validity of the original data. In this survey a *sample* of 1000 housewives were questioned, but this is only a small proportion of the total *population* of housewives. The results could have been distorted because most of the people chosen belonged to the same income group or lived in the same area. The most reliable results are usually obtained by taking a *random sample*, in which each member of the population has an equal chance of being selected.

Another factor which may affect this type of investigation is the framing of the questions asked. For example, one person might give a different answer to each of the following four questions:

(1) Which product gives best value for money?
(2) Which product do you buy regularly?
(3) Which product did you buy this week?
(4) Which product do you prefer?

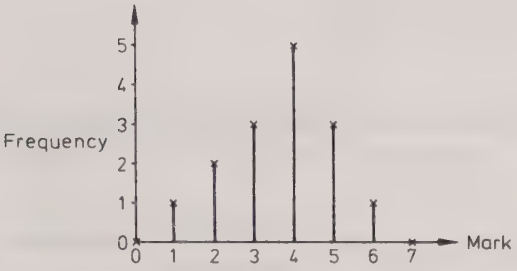
A survey based on only one of these questions could give misleading results.

Having considered some of the ways in which statistical information can be misunderstood or even misused, we now examine some more accurate ways of analysing and presenting results.

Example A In a short test given to a set of 15 students the marks were as follows:

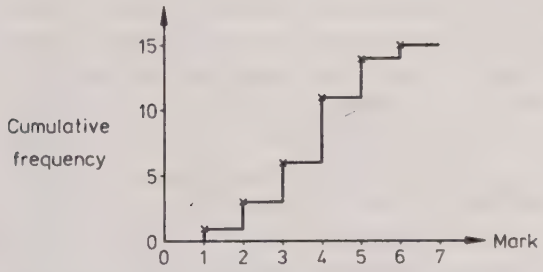
2 4 1 3 5 4 4 3 5 4 2 6 3 5 4

Since the test mark, x, of any particular student can take only certain values in the range 1 to 6, it is called a *discrete variable*. For a variable of this type it is helpful to make a table called a *frequency distribution* showing the frequency with which each value occurs. The corresponding diagram is called a *frequency chart*.



Mark	Frequency
1	1
2	2
3	3
4	5
5	3
6	1

When it is necessary to examine sums of frequencies, such as the number of students who scored 4 marks or less, a *cumulative frequency distribution* may be used. For a discrete variable x the cumulative frequency is the frequency of values *less than or equal to* x. As shown in the *cumulative frequency chart*, for a discrete variable, the cumulative frequency is a *step function*.



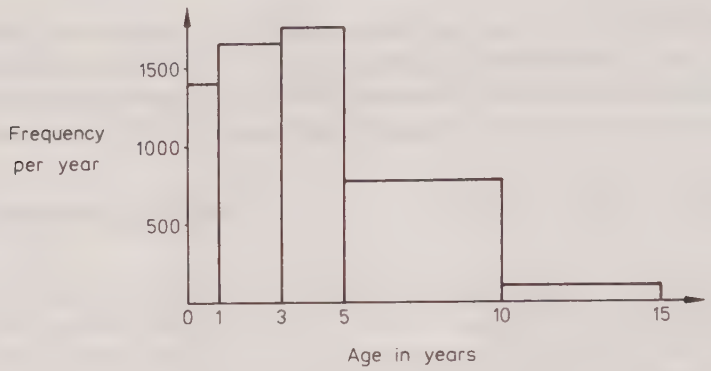
Mark	Cumulative frequency
1	1
2	3
3	6
4	11
5	14
6	15

Example B The table below shows the numbers of children known to have contracted whooping-cough in 1965.

Age	No. of cases (to the nearest 100)
Under 1 year	1400
1–2 years	3300
3–4 years	3500
5–9 years	3900
10–14 years	500

Since the age of a child can take any real value between 0 and 15 years, it is called a *continuous variable*. Because the variable is continuous and because the table gives frequencies for age groups of different sizes, a simple frequency chart

similar to the one used in Example A could be misleading. To allow for the various age ranges we draw a *histogram*, in which the *areas* (not the heights) of the columns are proportional to the frequencies.



This means that the height of each column represents $\frac{\text{frequency}}{\text{column width}}$ (sometimes called *frequency density*). In this case the vertical scale may be taken to represent the number of cases of whooping-cough per one year age group.

The *frequency polygon* shown here is obtained from the histogram by joining the mid-point of the tops of the columns. As several frequency polygons can be drawn in the same diagram, they are often used for comparing distributions.

For a continuous variable x, it is convenient to regard the cumulative frequency as the frequency of values *less than* x. In this example the cumulative frequency does not move in a series of jumps, but increases gradually throughout the range 0 to 15 years. Thus the values in the cumulative frequency table are used to draw a curve called the *cumulative frequency graph* or *ogive*.

Age	Cumulative frequency
Under 1 year	1400
Under 3 years	4700
Under 5 years	8200
Under 10 years	12 100
Under 15 years	12 600

Example C The table below shows the number of staff meals issued per day in a college refectory over a period of 40 days.

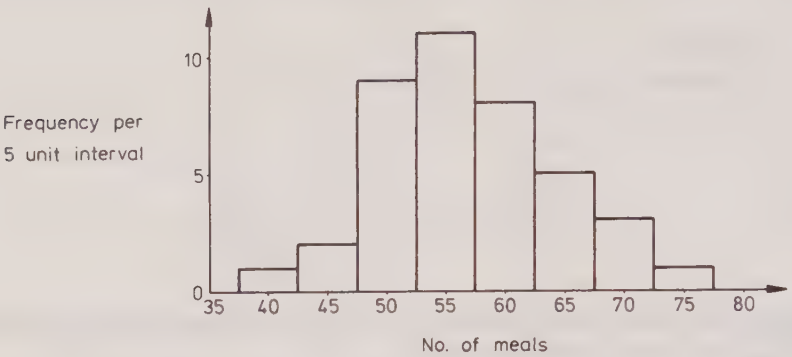
56	50	63	56	58	53	52	71	59	52
45	55	42	62	62	60	57	48	63	68
55	65	50	59	54	54	46	51	73	64
53	50	61	51	58	52	57	67	53	69

In this example the variable is again discrete. However, a frequency table of the type used in Example A would have little meaning as there is a range of 31 from the lowest value 42 to the highest 73. Any pattern in the distribution of observations will be shown more clearly by grouping the numbers. The range could be divided into intervals 40–50, 50–60, 60–70 and 70–80. One difficulty about this choice arises when we come to observations such as 50 which appear in two classes. This could be avoided by choosing intervals such as 40–49, 50–59, ... [In certain circumstances, it may be better to regard 50 as $\frac{1}{2}$ in the 40–50 class and $\frac{1}{2}$ in the 50–60 class.] The other main difficulty is that this grouping produces only 4 classes and will not illustrate the distribution in any detail. From this point of view, the intervals 40–44, 45–49, ... would be better. However, the mid-points of intervals are often used in calculations, so it is convenient if these are round numbers. Thus a reasonable choice in this example is the grouping 38–42, 43–47, ... into intervals of length 5 units with mid-points 40, 45, ... The *class limits* are the lowest and highest observations which could occur in each class, namely 38 and 42, 43 and 47, ... However, the *class boundaries* or *end-points* are taken, in this case, to be 37·5, 42·5, ...

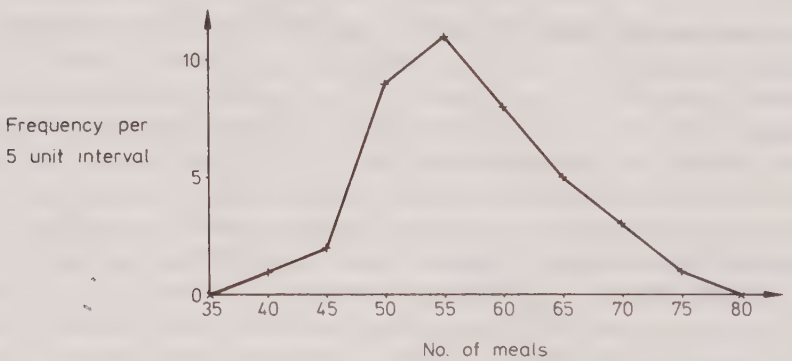
We may now draw up the frequency distribution for the grouped data using *tally marks* as shown below.

No. of meals		Frequency
38–42	I	1
43–47	II	2
48–52	𝍸 I	9
53–57	𝍸 𝍸 I	11
58–62	𝍸 III	8
63–67	𝍸	5
68–72	III	3
73–77	I	1

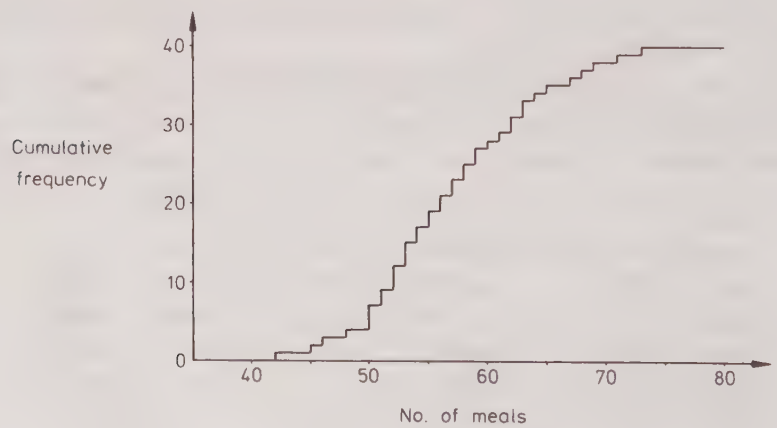
In the histogram of this distribution, since the columns are of equal width, their heights as well as their areas will be proportional to the frequencies.



When constructing a frequency polygon it is possible to plot zero frequencies at each end of the diagram to produce a closed polygon.

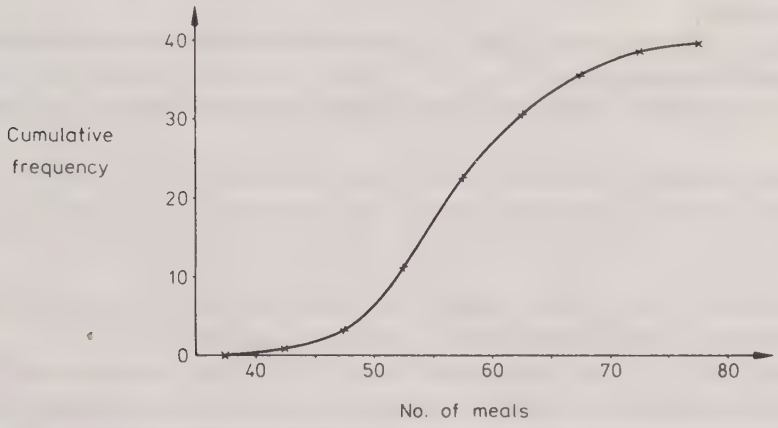


As in Example A the cumulative frequency chart constructed from the original data is a step diagram.

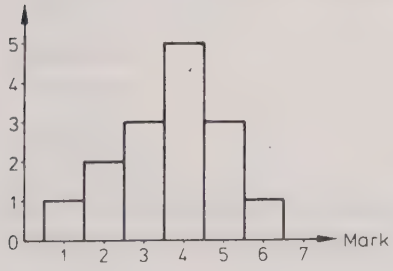


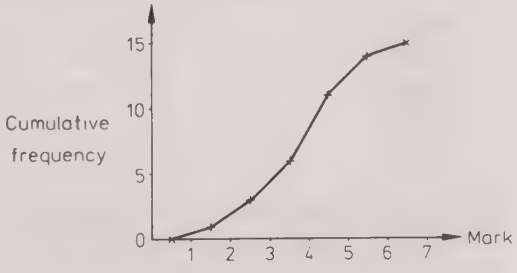
However, a cumulative frequency graph drawn using the grouped frequency distribution will provide a sufficiently good approximation for many purposes. To construct this graph we treat the number of meals as a continuous variable plotting the cumulative frequencies against the corresponding upper class boundaries.

No. of meals less than:	37·5	42·5	47·5	52·5	57·5	62·5	67·5	72·5	77·5
Cumulative frequency	0	1	3	12	23	31	36	39	40



In a similar way we could produce a histogram and a cumulative frequency polygon for the distribution in Example A by regarding it as a grouped distribution with class boundaries 0·5, 1·5, 2·5, ... However, such artificially produced diagrams have very little true meaning.





Exercise 15.1

[The data in questions 3 to 8 will be referred to in later exercises.]

1. Conduct a survey on a suitable topic of your own choice, such as the reading or television viewing habits of a group of students. Consider carefully (a) the questions to be used, (b) the way in which the data is to be recorded, (c) the graphical representation of the data collected. How could such a survey be conducted and the data presented in order to produce distorted or misleading results?

2. The table below shows the sales in two successive years of a particular type of car.

	2-door super	4-door super	4-door de-luxe	estate
1st year	8100	10 800	18 000	3600
2nd year	15 000	21 000	25 000	11 000

(a) Draw two pie charts whose areas allow the total sales for the two years to be compared.
(b) Draw a comparative bar chart, i.e. one in which there are two bars for each model of the car.
(c) Comment briefly on the advantages of the two ways of representing the data.

3. In an experiment the following 20 values of a variable x were recorded.

5	6	2	4	4	5	7	5	5	6
4	7	6	5	3	5	6	4	6	5

(a) Given that x is a discrete variable which can take only integral values, draw a frequency distribution chart and a cumulative frequency chart.
(b) Given that x is a continuous variable and that the above values are rounded to the nearest integer, draw a histogram and a cumulative frequency polygon.
(c) Given that x is a continuous variable and that the integral value n is recorded when $n \leqslant x < n+1$, draw a histogram and a cumulative frequency polygon.

4. The table below gives the number of dresses sold each day in a small shop over a 5-week period.

	Monday	Tuesday	Wednesday	Friday	Saturday
Week 1	2	2	4	4	9
Week 2	0	4	7	3	7
Week 3	1	0	3	1	6
Week 4	3	3	2	2	8
Week 5	1	3	5	3	8

Draw a frequency distribution chart and a cumulative frequency chart for the distribution. Comment briefly on other ways of representing the data graphically in order to show any patterns in the distribution more clearly.

5. In an investigation the following 50 values of a variable x were recorded:

6	11	12	8	9	8	9	4	2	7
9	7	4	11	7	5	8	7	6	4
9	6	3	4	6	7	5	6	6	9
4	9	7	1	4	7	7	4	10	9
5	8	10	7	4	9	5	7	9	11

Decide whether it would be more appropriate to represent the cumulative frequency distribution by a step diagram or an ogive given that (a) x is a discrete

variable which takes only integral values, (b) x is a continuous variable with values rounded to the nearest integer. Draw the appropriate graph in each case.

6. The length to the nearest minute of a man's journey to work over a period of 40 days are as follows:

57	56	33	56	49	56	44	56	45	58
62	57	42	68	49	44	59	63	50	58
53	58	50	56	60	59	45	51	65	53
50	38	65	43	51	63	56	61	67	61

Construct a suitable grouped frequency table, stating the class boundaries that you have used. Draw the corresponding histogram and cumulative frequency curve. Estimate from your graph the percentage of days on which the journey takes at least 45 minutes.

7. In a particular year 80 children enter a certain large infants' school. The ages of the children on the day they first attend the school are given in the table below.

Age in years	$4\frac{1}{4}$–	$4\frac{1}{2}$–	$4\frac{3}{4}$–	5–	$5\frac{1}{4}$–	$5\frac{1}{2}$–	6–	7–	$7\frac{3}{4}$–
Frequency	0	10	35	18	7	3	5	2	0

Draw a histogram, a frequency polygon and a cumulative frequency polygon to represent this distribution. Can you account for the entries in the table for children of $5\frac{1}{2}$ years and over? Estimate the probability that a child in the catchment area of this school is under 5 years old when he first attends school, stating any assumptions you have made.

8. The grouped frequency distribution of the test marks obtained by 150 students is given below.

Class limits	1–5	6–10	11–15	16–20	21–25
Frequency	0	4	12	22	30
Class limits	26–30	31–35	36–40	41–45	Over 45
Frequency	31	24	14	8	5

Draw a histogram to represent the data given that (a) the test was marked out of 50, (b) the test was marked out of 60, (c) the top mark was 54. Construct a cumulative frequency polygon in case (b). Give two reasons why this diagram represents an approximation to the true cumulative frequency distribution. Explain why a cumulative frequency curve may give a better approximation.

15.2 Mode, median and mean

The mode, median and mean of a distribution are all known as measures of *location* (or *central tendency*). For a collection of numerical data, they all indicate the general level or location of the observations by giving an 'average' or central value.

The *mode* is the observation which occurs most frequently. In some cases there are two or more modes.

The *median* is the middle observation when the observations are listed in order of magnitude. If the number of items is even, the average of the middle two is used.

The *mean* of a set of observations is the average or arithmetic mean. For a set of n observations $x_1, x_2, \ldots, x_n$, the mean $\bar{x} = \frac{1}{n}(x_1 + x_2 + \ldots + x_n)$. Using the Σ notation, this becomes $\bar{x} = \frac{1}{n}\sum_{r=1}^{n} x_r$ or simply $\bar{x} = \frac{1}{n}\sum x$.

Of these measures of location, the mean is the one used most frequently, although it can be distorted by extreme values at the ends of a distribution. The median has the advantage that it is not affected in this way.

Example 1 Find the mode, median and mean of the following sets of observations (a) 6, 7, 7, 3, 8, 5, 3, 9; (b) 18, 13, 14, 13, 11, 12, 13, 12, 11.

(a) The distribution has two modes, 3 and 7.

Arranged in order of magnitude the observations are 3, 3, 5, 6, 7, 7, 8, 9. Thus the median is $\frac{1}{2}(6+7) = 6{\cdot}5$. The mean is $\frac{1}{8}(6+7+7+3+8+5+3+9) = 6$.

(b) The mode of this distribution is 13.

Arranged in order of magnitude the observations are 11, 11, 12, 12, 13, 13, 13, 14, 18. Thus the median is 13.

The mean is $\frac{1}{9}(2.11 + 2.12 + 3.13 + 14 + 18) = 13$.

[Note that arranging a set of observations in order of magnitude is called forming an *array*.]

We now return to Examples A, B and C discussed in the previous section.

Example A From both the frequency distribution and the frequency chart, it is clear that the mode is 4.

Since there are 15 marks, the median is the 8th mark in order of magnitude. Hence, using the cumulative frequency distribution or the corresponding chart, we find that the median is 4.

When calculating the mean, the total of the marks can be found using the frequency distribution.

Mark (x)	Frequency (f)	fx
1	1	1
2	2	4
3	3	9
4	5	20
5	3	15
6	1	6
	15	55

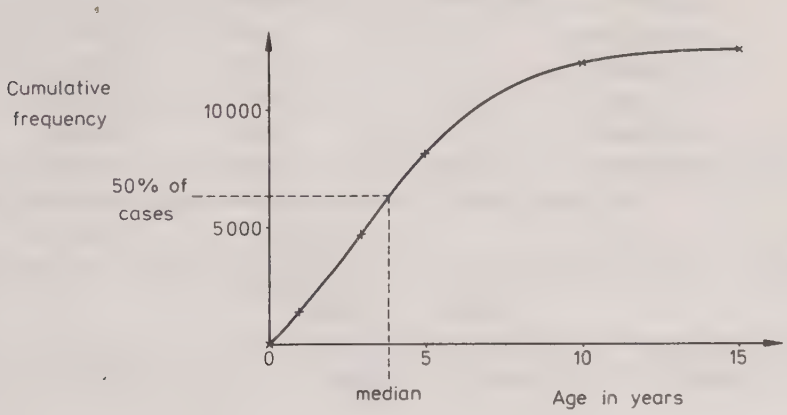
The formula for the mean can now be expressed as

$$\bar{x} = \frac{1}{n}\sum fx \quad \text{where} \quad n = \sum f.$$

Hence the mean mark $= \frac{55}{15} \approx 3{\cdot}67.$

Example B For grouped data it is not possible to find a true mode, as the values of individual observations are not known. Instead the term *modal class* may be used to describe the interval corresponding to the highest column in the histogram. In this case the modal class is the 3–4 year age group. [Note that this is not the group with the highest number of cases of whooping-cough.]

The median is a value of the variable which has equal numbers of observations above and below it. For grouped data an approximate value of the median can be found either graphically or by calculation. In this example the median is the age below which 6300 whooping-cough cases lie. Thus the median can be estimated from the ogive by reading off the age corresponding to a cumulative frequency of 6300 cases, which is approximately 4 years.

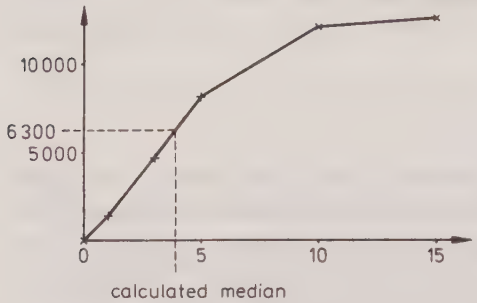


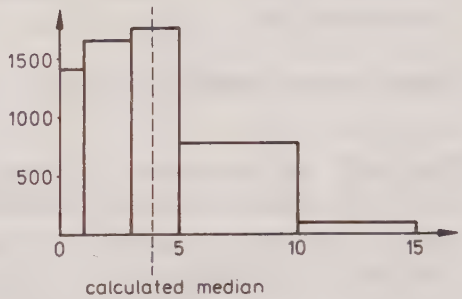
To estimate the median of a grouped frequency distribution by calculation, we assume that the observations in any class are evenly spaced, i.e. that the known points on the ogive can be joined with straight lines. This way of estimating intermediate values of a function is called *linear interpolation*. In this example there are 4700 cases under 3 years old and 8200 under 5 years old, so the median age lies in the 3–4 year class which contains 3500 cases. Since there are a total of 6300 cases below the median, 1600 of the cases in the 3–4 year class must lie below the median and 1900 above it. Thus we assume that the median divides this 2 year interval in the ratio 1600 : 1900

$$\therefore \quad \text{the median} = 3 + \left(\frac{1600}{3500} \times 2\right) \text{years} \approx 3{\cdot}9 \text{ years}.$$

Finding the median by calculation is equivalent to using a cumulative frequency polygon rather than a curve. This calculated value can also be regarded as the

value of the variable at which a vertical line divides the histogram into two equal areas.





To find an approximate value for the mean of a grouped frequency distribution, we again assume that observations are evenly spaced. It then follows that the mean of the observations in any one class lies at the mid-point of the class interval. Hence if there are f observations in a class with mid-value x, then the sum of these observations is taken to be fx.

Class	Mid-value (x)	Frequency (f)	fx
Under 1 year	0·5	1400	700
1–2 years	2	3300	6600
3–4 years	4	3500	14 000
5–9 years	7·5	3900	29 250
10–14 years	12·5	500	6250
		$n = 12\,600$	56 800

$$\bar{x} = \frac{1}{n}\sum fx = \frac{56\,800}{12\,600} \approx 4{\cdot}5$$

Hence the mean age is approximately 4·5 years.

Example C From the original data this distribution is found to have modes 50, 52 and 53. As these values occur only three times each, this information is probably of little value. The fact that the 53–57 group is the modal class may be more useful.

There are 40 observations, so the median is the average of the 20th and 21st, which is 56. An estimate of the median value can be obtained from the grouped frequency distribution, either graphically or by calculation, using the methods given in Example B. To estimate the median graphically we again use the ogive and obtain an approximate value of 56. To estimate the median by calculation we note, from the cumulative frequency table in §15.1, that 12 values of the variable are less than 52·5 and 23 values are less than 57·5. Since the median lies between the 20th and 21st value, it must be in the 5 unit interval from 52·5 to 57·5, with 8 values below it and 3 above. Assuming that these 11 observations are evenly spaced, the approximate value obtained for the median is $52{\cdot}5 + \left(\frac{8}{11} \times 5\right) \approx 56{\cdot}1$.

If it is necessary to find the mean from the original data without the aid of a calculator, it helps to use the deviations of the observations from an arbitrary origin. Taking 55 as origin the deviations are:

1	−5	8	1	3	−2	−3	16	4	−3
−10	0	−13	7	7	5	2	−7	8	13
0	10	−5	4	−1	−1	−9	−4	18	9
−2	−5	6	−4	3	−3	2	12	−2	14

Cancelling equal and opposite numbers, the sum of the deviations reduces to

$$16+8+18+6+12+14 = 74$$

$$\therefore \quad \text{the mean} = 55 + \frac{74}{40} \approx 56{\cdot}9.$$

In this case a good approximation to the value of the mean can be obtained from the grouped data. This estimated value is found to be $56\frac{7}{8}$.

Exercise 15.2

1. Find the mode, median and mean of the following sets of integers
(a) 3, 4, 2, 3, 1, 3, 5, 7, 3, 4,
(b) 24, 38, 8, 13, 25, 3, 29,
(c) 21, 19, 20, 20, 22, 20, 21, 22, 21.

2. If an additional integer 20 is included in each of the lists given in question 1, find the new values of the mode, median and mean in each case.

In questions 3 to 8 use the data given in the corresponding questions in Exercise 15.1.

3. Find the mode, median and mean of the distribution, distinguishing where necessary between cases (a), (b) and (c).

4. Find the mode, median and mean of the distribution. Comment briefly on the usefulness of these measures in this context.

5. Use a frequency distribution table to find the mean of this distribution. Comment on the significance of your answer in cases (a) and (b).

6. (i) State the modal class of the grouped frequency distribution obtained in Exercise 15.1 and use this distribution to estimate the mean of the original data. Use the cumulative frequency curve to obtain a value for the median.
(ii) Form an array of the 40 items of data and write down the mode and the median of the distribution. Use a working origin of 50 to find the mean.
(iii) Comment on your answers to (i) and (ii).

7. (i) Calculate the approximate values of the median and mean of this distribution.

(ii) Assuming that children under $5\frac{1}{2}$ years of age enter a reception class, calculate the median age and the mean age of such children.
(ii) Discuss the significance of your answers.

8. Use linear interpolation to find the median of this distribution. Estimate the value of the mean in each of the cases (a), (b) and (c).

9. Construct a flow diagram for a procedure to find the mean of a set of numbers $x_1, x_2, \ldots, x_r$ which occur with frequencies $f_1, f_2, \ldots, f_r$ using a pocket calculator without built-in statistics functions. [Note that unless it is possible to calculate $\sum f$ and $\sum fx$ simultaneously, it will be necessary to input the numbers $f_1, f_2, \ldots, f_r$ twice.]

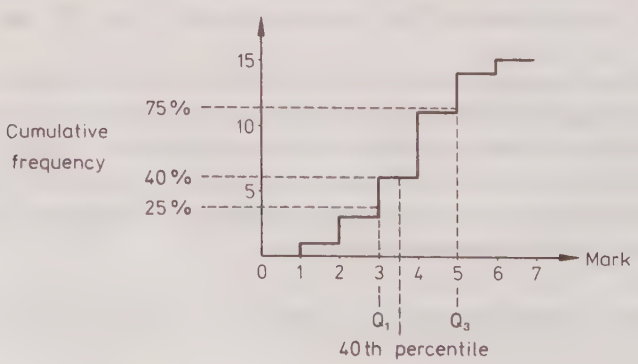
15.3 Measures of spread

Consider two sets of numbers (i) 1, 13, 15, 20, 21 and (ii) 12, 13, 14, 14, 17. Both have mean 14, but the first is more widely spread than the second. Thus, in addition to measures of location it is useful to have measures of *spread*, sometimes called *dispersion* or *variation*. The simplest of these is the *range*, which is the difference between the highest and the lowest items of data. In the above examples the ranges are (i) 20, (ii) 5. The main disadvantage of the range is that it is often determined by two freak values at the ends of a distribution.

Other measures of spread are obtained using quantities called *percentiles*. Theoretically, the percentiles divide a distribution into 100 equal parts. For instance, the 40th percentile is a value which has 40% of the observations in a distribution on or below it and 60% on or above it. Clearly the 50% percentile is the median. Other percentiles are found by similar methods to those used to find the median in the previous section. The interval between any two percentiles can be used as a measure of spread. Particularly useful in this way are the 25th percentile, referred to as the *first* or *lower quartile* Q_1, and the 75th percentile, the *third* or *upper quartile* Q_3. The *semi-interquartile range* is half the interval between the lower and upper quartiles, i.e. $\frac{1}{2}(Q_3 - Q_1)$. This is a more useful measure than the range because it is not affected by a few untypical values.

We return again to the examples discussed earlier in the chapter.

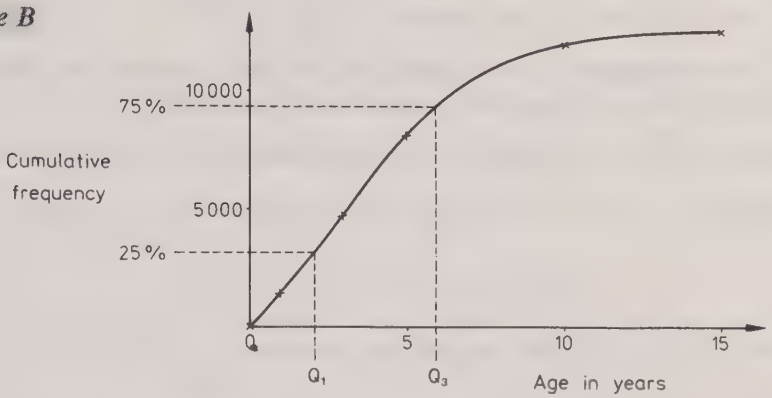
Example A



Using the cumulative frequency chart we find that the 40th percentile lies between 3 and 4. We take its value to be 3·5. The lower quartile is 3 and the upper quartile is 5

$\therefore$ the semi-interquartile range is $\frac{1}{2}(5-3)$, i.e. 1.

Example B



There are a total of 12600 observations. Hence the lower quartile is the age below which 3150 of the cases occur. From the graph this is approximately 2·1 years. Similarly the upper quartile corresponds to a cumulative frequency of 9450 and is approximately 5·9 years. Hence the semi-interquartile range is $\frac{1}{2}(5{\cdot}9-2{\cdot}1)$ years, i.e. 1·9 years.

By calculation the lower quartile is estimated, using the cumulative frequency table, to be

$$1+\left(\frac{\{3150-1400\}}{3300}\times 2\right)\text{years} \approx 2{\cdot}1 \text{ years.}$$

The upper quartile is $5+\left(\frac{\{9450-8200\}}{3900}\times 5\right)$ years $\approx 6{\cdot}6$ years.

Hence the calculated estimate of the semi-interquartile range is $\frac{1}{2}(6{\cdot}6-2{\cdot}1)$ years, i.e. 2·25 years.

[It appears likely that the whooping-cough cases are not spread evenly throughout the 5–9 year age group and that more cases occur in the lower half of the class. Thus it is probable that the estimate of 1·9 years obtained graphically is more accurate than the calculated value of 2·25 years.]

Further measures of spread are obtained by considering the set of *deviations* of the values of a variable x from the mean $\bar{x}$. For a set of n observations the sum of these deviations

$$=\sum(x-\bar{x})=\sum x-n\bar{x}=\sum x-n\left(\frac{1}{n}\sum x\right)=0.$$

Thus the mean deviation is always zero and cannot be used as a measure of spread. However, the *mean absolute deviation*, i.e. $\frac{1}{n}\sum|x-\bar{x}|$, is sometimes useful.

The main disadvantage when using this quantity is that moduli are difficult to manipulate algebraically.

The most frequently used measures of dispersion are the *variance*, which is the mean squared deviation, $\text{var}(x) = \frac{1}{n}\sum(x-\bar{x})^2$ and the *standard deviation*, which is the square root of the variance $\sqrt{\left\{\frac{1}{n}\sum(x-\bar{x})^2\right\}}$.

[Note that the variance is measured in square units, whereas the units of the standard deviation are the same as those of the original data.]

For examples (i) and (ii) considered at the beginning of the section, we have

(i) $\text{variance} = \frac{1}{5}\{(1-14)^2+(13-14)^2+(15-14)^2+(20-14)^2+(21-14)^2\}$

$$= \frac{1}{5}\{169+1+1+36+49\} = 51{\cdot}2$$

$\therefore$ the variance is 51·2 and the standard deviation is $\sqrt{(51{\cdot}2)} \approx 7{\cdot}2$.

(ii) $\text{variance} = \frac{1}{5}\{(-2)^2+(-1)^2+0^2+0^2+3^2\} = 2{\cdot}8$

$\therefore$ the variance is 2·8 and the standard deviation is $\sqrt{(2{\cdot}8)} \approx 1{\cdot}7$.

To avoid cumbersome calculations with a mean which is not a whole number, we now rearrange the formula for the variance.

$$\text{var}(x) = \frac{1}{n}\sum(x-\bar{x})^2 = \frac{1}{n}\sum(x^2-2x\bar{x}+\bar{x}^2)$$

$$= \frac{1}{n}\sum x^2 - \frac{1}{n}\sum 2x\bar{x} + \frac{1}{n}\sum \bar{x}^2$$

$$= \frac{1}{n}\sum x^2 - 2\bar{x}.\frac{1}{n}\sum x + \frac{1}{n}.n\bar{x}^2$$

$$= \frac{1}{n}\sum x^2 - 2\bar{x}^2 + \bar{x}^2$$

Hence the variance,

$$\boxed{\text{var}(x) = \frac{1}{n}\sum x^2 - \bar{x}^2.}$$

When working from a frequency table in which a value x has frequency f we may write:

$$\boxed{\text{var}(x) = \frac{1}{n}\sum f(x-\bar{x})^2 = \frac{1}{n}\sum fx^2 - \bar{x}^2.}$$

In Example A, assuming that a calculator with built-in statistical data programs is not available, the variance and standard deviation can be obtained in the following way.

Mark (x)	Frequency (f)	fx	fx^2
1	1	1	1
2	2	4	8
3	3	9	27
4	5	20	80
5	3	15	75
6	1	6	36
	15	55	227

$$n = \sum f = 15, \qquad \bar{x} = \frac{1}{n}\sum fx = \frac{55}{15}$$

$$\therefore \text{ var}(x) = \frac{1}{n}\sum fx^2 - \bar{x}^2 = \frac{1}{15} \times 227 - \left(\frac{55}{15}\right)^2.$$

Hence the variance is 1·69 and the standard deviation is 1·30 (both to 3 s.f.).

The variance of a grouped frequency distribution can be estimated using the formula $\text{var}(x) = \frac{1}{n}\sum fx^2 - \bar{x}^2$ by taking the values of x to be the mid-values in each class.

In Example B the class intervals are quite long, so the estimated variance is likely to differ noticeably from the true value. However, in Example C the class intervals are shorter, so the grouped frequency distribution should give a better value for the variance.

From the original data the variance is found to be approximately 50·7 (measured in 'square meals') and the standard deviation is 7·12 meals. Using the grouped data the estimated variance is 57·1 and the standard deviation 7·56.

Exercise 15.3

1. Find the median, the quartiles and the semi-interquartile range for the following sets of integers.
(a) 8, 3, 5, 7, 11, 9, 7, 6, 7, 5, 8, 11,
(b) 28, 18, 32, 27, 42, 23, 49, 28, 33, 30,
(c) 2, −4, 10, 0, −5, 11, 6, −4, 11.

2. Find the mean, the variance and the standard deviation of the sets of integers given in question 1.

In questions 3 to 8 use the data given in the corresponding questions in Exercise 15.1.

3. Use the appropriate cumulative frequency diagram to find the following in cases (a), (b) and (c):
(i) the 40th and the 70th percentile,
(ii) the semi-interquartile range.

4. Find the variance and standard deviation of the distribution of sales.

5. (i) Use the appropriate cumulative frequency graph to estimate the lower quartile, median and upper quartile of this distribution in cases (a) and (b).
(ii) Use the frequency distribution table to find the standard deviation of the distribution.

6. Find the lower and upper quartiles of this distribution using (i) the original data, (ii) the grouped data and linear interpolation, (iii) an ogive based on the grouped data. Comment on any differences between your answers.

7. (i) Calculate approximate values, to the nearest month, of the 30th and the 65th percentile.
(ii) Estimate the variance of the distribution, stating the units in which your answer is measured.

8. The 150 test marks are to be graded as follows: the top 15% of students, grade I; the next 30%, grade II; the next 35%, grade III; the bottom 20%, grade IV. Use a cumulative frequency polygon to suggest suitable class limits for each grade.

9. Construct a flow diagram for a procedure to find the standard deviation of a set of numbers $x_1, x_2, \ldots, x_r$ which occur with frequencies $f_1, f_2, \ldots, f_r$ using a pocket calculator without built-in statistical functions. [Assume that $n = \sum f$ and $m = \frac{1}{n}\sum fx$ have already been calculated and may be read as data.]

15.4 Change of variable

Suppose we have a distribution of n values of a variable x with mean $\bar{x}$ and standard deviation σ_x. Consider a new variable y, such that $y = ax + b$, where a and b are constants. Let the distribution of the n values of y have mean $\bar{y}$ and standard deviation σ_y, then

$$\bar{y} = \frac{1}{n}\sum y = \frac{1}{n}\sum (ax+b) = \frac{1}{n}\sum ax + \frac{1}{n}\sum b$$

$$= a.\frac{1}{n}\sum x + \frac{1}{n}.nb = a\bar{x} + b$$

$$\text{var}(y) = \frac{1}{n}\sum(y-\bar{y})^2 = \frac{1}{n}\sum\{(ax+b)-(a\bar{x}+b)\}^2$$

$$= \frac{1}{n}\sum(ax-a\bar{x})^2 = a^2 . \frac{1}{n}\sum(x-\bar{x})^2 = a^2\,\text{var}(x).$$

Hence, if $y = ax+b$, then $\bar{y} = a\bar{x}+b$ and $\text{var}(y) = a^2\,\text{var}(x)$.

Example 1 During an industrial dispute a trade union negotiated a pay increase for its members of £8 a week plus 10%, so that a man earning £x per week received an increase of £$(8+x/10)$ per week. For the original wage distribution the mean was £120 per week and the standard deviation £20 per week. Find the new mean and standard deviation.

Let £y be the new weekly wage of a man originally earning £x per week, then

$$y = x+8+\frac{x}{10} = \frac{11}{10}x+8$$

$$\therefore \quad \bar{y} = \frac{11}{10}\bar{x}+8 = \left(\frac{11}{10}\times 120\right)+8 = 140.$$

$$\text{New s.d.} = \frac{11}{10}\times \text{original s.d.} = \frac{11}{10}\times 20 = 22.$$

Hence the new mean weekly wage is £140 and the standard deviation is £22.

A change of variable is sometimes used to simplify the calculation of the mean and standard deviation of a distribution. For instance, if a set of values of a variable x appear to have a mean approximately equal to x_0, then we can take this *assumed mean* as a working origin. If we let $y = x-x_0$, then $\bar{x} = x_0+\bar{y}$ and var (x) = var (y).

Example 2 The weights in kilogrammes of the players selected for a certain Rugby Union football team are 76, 79, 85, 86, 76, 75, 86, 105, 88, 104, 102, 103, 89, 92, 95. Find the mean and standard deviation of the distribution.

If x kg is the weight of a member of the team, let $y = x-90$, then the 15 values of y for the team are -14, -11, -5, -4, -14, -15, -4, 15, -2, 14, 12, 13, -1, 2, 5.

Cancelling equal and opposite values of y,

$$\bar{y} = \frac{1}{15}(-5-4) = -\frac{9}{15} = -0{\cdot}6.$$

$$\sum y^2 = (-14)^2+(-11)^2+\ldots+2^2+5^2 = 1563$$

$$\therefore \quad \text{var}(y) = \frac{1}{15}\sum y^2-\bar{y}^2 = \frac{1563}{15}-(0{\cdot}6)^2 = 103{\cdot}84$$

$$\therefore \quad \bar{x} = 90+\bar{y} = 89{\cdot}4 \quad \text{and} \quad \text{var}(x) = \text{var}(y) = 103{\cdot}84$$

Hence the mean weight is 89·4 kg and the standard deviation is 10·2 kg.

When using a calculator a change of variable may be used to improve accuracy. For instance, if the values of a variable are given to 5 digits or more, squares of at least 9 digits will be produced in an accurate calculation of the variance. A change to a variable whose values have fewer digits would reduce the truncation and rounding errors generated by a calculator of limited capacity.

In certain situations a change of variable of the form $y = \frac{1}{c}(x - x_0)$ is useful.

If $y = \frac{1}{c}(x - x_0)$ then $x = x_0 + cy$

$\therefore \quad \bar{x} = x_0 + c\bar{y}$ and $\text{var}(x) = c^2 \text{var}(y)$.

Applying a transformation of this type to a distribution may be referred to as '*coding*' the data. We illustrate the method using the grouped frequency distribution obtained in Example C, §15.1. We take an assumed mean of 55 as the value of x_0 and the class width 5 as the value of c, so that $y = \frac{1}{5}(x - 55)$.

Class	Mid-value (x)	y	Frequency (f)	fy	fy^2
38–42	40	−3	1	−3	9
43–47	45	−2	2	−4	8
48–52	50	−1	9	−9	9
53–57	55	0	11	0	0
58–62	60	1	8	8	8
63–67	65	2	5	10	20
68–72	70	3	3	9	27
73–77	75	4	1	4	16
			40	15	97

$$n = \sum f = 40, \quad \sum fy = 15, \quad \sum fy^2 = 97$$

$$\bar{y} = \frac{1}{n}\sum fy = \frac{15}{40} = \frac{3}{8}$$

$$\therefore \quad \bar{x} = 55 + 5.\frac{3}{8} \approx 56{\cdot}9$$

$$\text{var}(y) = \frac{1}{n}\sum fy^2 - \bar{y}^2 = \frac{97}{40} - \left(\frac{3}{8}\right)^2$$

$$\therefore \quad \text{s.d. of } x \text{ values} = 5\sigma_y = 5\sqrt{\left(\frac{97}{40} - \frac{9}{64}\right)} \approx 7{\cdot}56.$$

Hence the mean of the distribution is 56·9 and the standard deviation is 7·56.

Exercise 15.4

1. Find the mean, variance and standard deviation of the numbers

1, 3, 3, 4, 6, 6, 7, 8, 8, 9.

Deduce the mean, variance and standard deviation of the following sets of numbers:
(a) 4, 6, 6, 7, 9, 9, 10, 11, 11, 12,
(b) 4, 12, 12, 16, 24, 24, 28, 32, 32, 36,
(c) 4·1, 4·3, 4·3, 4·4, 4·6, 4·6, 4·7, 4·8, 4·8, 4·9.

2. The mean and standard deviation of a set of mid-day temperatures for the month of June are 23°C and 5°C respectively. Find the mean and standard deviation for the same distribution measured in degrees Fahrenheit.

3. Use a frequency table and an assumed mean of 5 to calculate the mean, variance and standard deviation of the distribution of the discrete variable x given in question 3. Exercise 15.1. Deduce the mean, variance and standard deviation of the variable in case (c).

4. (i) Use an appropriate change of variable to estimate the standard deviation of the data given in question 6, Exercise 15.1 from the grouped frequency table.
(ii) With the aid of a calculator, find the standard deviation of the original distribution.
(iii) Comment on the difference between your answers.

5. Use an appropriate change of variable to find the mean and standard deviation of the distribution given in question 8, Exercise 15.1 assuming that the test is marked out of 60. The marks are to be scaled using the formula $z = ax + b$, where x is the original mark and z the new mark. If the mean and standard deviation of the scaled marks are to be approximately 50 and 15 respectively, find suitable values of a and b correct to 2 significant figures.

6. The variable x_1 represents the mark of a student in an examination. The distribution of marks for a certain group of students has mean $\bar{x}_1$ and standard deviation σ_1. Given that the marks are to be scaled so that the values of the new mark x_2 have mean $\bar{x}_2$ and standard deviation σ_2, prove that

$$x_2 = \bar{x}_2 + \frac{\sigma_2}{\sigma_1}(x_1 - \bar{x}_1).$$

Use this formula to standardise the marks of students A, B and C as given in the table below, taking $\bar{x}_2 = 50$ and $\sigma_2 = 15$. Hence find a combined mark for each student.

Subject	Mean	Standard deviation	A	B	C
Mathematics	60	20	90	96	70
English	45	'10	52	46	70

Exercise 15.5 (miscellaneous)

1. The following table gives an analysis by numbers of employees of the size of UK factories of less than 1000 employees manufacturing clothing and footwear.

Number of employees	11–19	20–24	25–99	100–199	200–499	500–999	Total
Number of factories	1500	800	2300	700	400	100	5800

Calculate as accurately as the data allow the mean and median of this distribution, showing your working. If 90% of the factories have less than N employees, estimate N. (O & C)

2. An inspection of 34 aircraft assemblies revealed a number of missing rivets as shown in the following table.

Number of rivets missing	0–2	3–5	6–8	9–11	12–14	15–17	18–20	21–23
Frequency	4	9	11	6	2	1	0	1

Draw a cumulative frequency curve. Use this curve to estimate the median and the quartiles of the distribution. (O & C)

3. The distribution of the times taken when a certain task was performed by each of a large number of people was such that its twentieth percentile was 25 minutes, its fortieth percentile was 50 minutes, its sixtieth percentile was 64 minutes and its eightieth percentile was 74 minutes. Use linear interpolation to estimate (i) the median of the distribution, (ii) the upper quartile of the distribution, (iii) the percentage of persons who performed the task in 40 minutes or less. (JMB)

4. The weights of a number of school children were measured and the mean weight was found to be 54·1 kg, with standard deviation 3·2 kg. Subsequently it was discovered the weighing machine was reading 0·9 kg under the true weight. How should the mean and standard deviation be revised? Give a clear indication of your reasoning. (O & C)

5. A frequency distribution has values $x_1, x_2, \ldots, x_n$ with frequencies $f_1, f_2, \ldots, f_n$ respectively. It has mean $\bar{x}$ and standard deviation σ. Find the mean and standard deviation of the distribution which has values $d_1, d_2, \ldots, d_n$ with frequencies $f_1, f_2, \ldots, f_n$ respectively, where $d_r = \dfrac{x_r - \bar{x}}{\sigma}$. (AEB 1977)

6. 100 pupils were tested to determine their intelligence quotient (I.Q.), and the results were as follows:

I.Q.	45–	55–	65–	75–	85–	95–	105–	115–	125–134
No. of pupils	1	1	2	6	21	29	24	12	4

All I.Q.s are given to the nearest integer.
(i) Calculate the mean, and the standard deviation.
(ii) Draw a cumulative frequency graph, and estimate how many pupils have I.Q.s within 1 s.d. on either side of the mean. (SU)

7. The haemoglobin levels were measured in a sample of 50 people and the results were as follows, each being correct to 1 place of decimals.

13·5	15·6	16·3	12·3	13·1	14·2	12·4	11·3	14·0	14·6
13·6	14·8	12·7	10·9	11·0	11·4	15·0	10·1	15·4	11·3
10·7	14·6	13·5	15·1	12·1	12·0	14·2	11·4	15·0	13·3
13·2	9·1	16·9	14·2	15·0	13·6	14·8	11·4	14·8	15·7
13·5	13·5	12·9	13·8	13·7	16·2	11·6	13·8	14·2	10·7

(a) Group the data into eight classes, 9·0–9·9, 10·0–10·9, ..., 16·0–16·9.
(b) What are the smallest and largest possible measurements which could be included in the class 9·0–9·9?
(c) Draw a histogram of the grouped data and use it to estimate the median value of the sample, showing your working.
(d) Find the true median of the sample. (SU)

8. A small firm has a total labour force of 267 of whom 250 are craftsmen and the rest managers. The distribution of the weekly pay of all employees is as follows, where, for example, 22·5– means that employees in this group earn at least £22.50, but less than £25.00.

Craftsmen Wages in £	No. of employees	Managers Wages in £	No. of employees
20.0–	29	50.0–	3
22.5–	33	52.5–	9
25.0–	35	55.0–	5
27.5–	40		
30.0–	51		
32.5–	37		
35.0–	25		

Display this information on a block diagram. Calculate (using a working zero of £31.25, or otherwise) the mean pay of (i) the craftsmen, (ii) the managers, (iii) all 267 employees. Comment on these means as they describe the situation in the firm and discuss other measures, such as the median or mode which might be used instead or as well. (SU)

9. The following table gives the maximum temperature in 60 cities on a certain day. Readings are to the nearest degree.

72	61	43	54	54	48	59	55	61	50	55	48
30	66	41	55	48	57	61	48	46	61	30	50
66	73	54	48	66	61	45	57	48	70	68	43
52	50	46	64	46	50	50	50	48	37	45	53
64	50	39	32	66	68	41	70	48	73	39	43

Gather these measurements into classes at intervals of 5°, starting at 29·5°. From the information in this form construct a cumulative frequency table and draw a graph to display the cumulative frequency. Use this to find the median and semi-interquartile range. For the grouped distribution calculate the mean and standard deviation, explaining what effect the grouping is likely to have had on each.

(AEB 1978)

10. On 1 January 1975 the heights of ten boys, measured in centimetres, were 172·1, 168·6, 174·1, 174·6, 176·1, 167·6, 174·1, 170·1, 172·6, 171·1. Find the mean and standard deviation of these heights. If on 1 January 1976 each boy was 2·5 cm taller, calculate the mean height and variance on that date. If the heights had been measured in inches, find the variance of the heights of the ten boys stating the units in which your answer is given. [Take 2·54 cm = 1 inch.] (L)

11. For a finite population of size N, the mean of a measurement X is μ_X and the variance σ_X^2 is given by $\sigma_X^2 = \frac{1}{N}\sum (X-\mu_X)^2$. If $Y = aX + b$, where a and b are fixed constants, show that $(aN\sigma_X)^2 = N\sum Y^2 - (\sum Y)^2$.

For the finite population given in the table below, calculate the average earnings and the standard deviation of the earnings.

Earnings to nearest £	18–22	23–27	28–32	33–37	38–42
Number of workers	2	2	2	4	5
Earnings to nearest £	**43–47**	**48–52**	**53–57**	**58–62**	**63–67**
Number of workers	7	4	7	9	8

(L)

12. The set of positive integers $x_1, x_2, \ldots, x_9$ has a mean of 5 and $\sum_{i=1}^{9} (x_i - 5)^2 = 36$. A tenth integer (x_{10}) is added to the set, and the mean and variance of the set of ten integers are denoted by $\bar{x}$ and Var (x) respectively.

(a) Calculate the values of $\bar{x}$ and Var (x) if $x_{10} = 5$.
(b) Calculate the values of $\bar{x}$ and Var (x) if $x_{10} = 9$.
(c) If $0 < x_{10} < 5$ state two inequalities satisfied by $\bar{x}$.
(d) Without doing long calculations, suggest which values for x_{10} would result in Var (x) being greater than 4.

[In this question you should take Var $(x) = \frac{1}{10}\sum_{i=1}^{10} (x_i - \bar{x})^2$.] (L)

13. (i) A distribution, A, contains N_1 observations and has mean and standard deviation μ_1 and σ_1 respectively. A second distribution, B, of N_2 observations has mean and standard deviation μ_2 and σ_2 respectively. The standard deviation of the combined distribution of N_1+N_2 observations is σ. Show that

$$(N_1+N_2)\sigma^2 = N_1\sigma_1^2+N_2\sigma_2^2+\frac{N_1N_2}{N_1+N_2}(\mu_1-\mu_2)^2.$$

(ii) Two teachers A and B each marked 100 exam papers with the following results

Mark	35	36	37	38	39	40	41	42	43	44	45
No. marked by A	4	10	14	18	12	12	10	8	4	5	3
No. marked by B	8	15	16	20	18	5	3	5	2	2	6

The mean mark for A is 39·23 with variance 6·4371. The mean mark for B is 38·51. Calculate the variance of B's marks. It is decided to add 0·72 to each of B's marks to make the mean for his papers the same as for A's. Calculate the mean and variance of the combined set of 200 papers (with B's marks adjusted).

(AEB 1975)

14. The following table gives the frequency distribution of 100 observed values of a random variable x.

x =	1	2	3	4	5	6
Frequency =	12	13	30	20	18	7

(a) Suppose x is the score obtained on a throw of a certain die.
(i) Calculate the mean and the variance of the observed scores.
(ii) Use the data to estimate the probability of throwing an even score with this die. Hence estimate the probability that six throws of the die will result in three even and three odd scores.

(b) Suppose, instead, that x is a continuous random variable and that in compiling the above frequency table the observed values were recorded to the nearest integer. Use linear interpolation to estimate the median of the observed values and the proportion of the observed values which exceeded 4·8. (JMB)

16 Probability distributions

16.1 Discrete probability distributions

The theory of probability is used to set up mathematical models of random experiments or trials. In its simplest form a *probability model* of an experiment consists of a description of the sample space (i.e. the set of possible outcomes) together with estimates of the probabilities associated with the outcomes. Let us consider, for instance, a trial in which a single unbiased die is thrown and the score noted. This can be described as a trial with a sample space containing 6 outcomes each with probability 1/6. We can use this model to make predictions about the behaviour of dice in experiments involving large numbers of such trials. The theory can be tested by comparing these predictions with the observed results of the experiments.

Since the score obtained in any particular throw of a single die cannot be predicted in advance, the trial is said to be *random* and the score obtained is called a *random variable*. In general, the values of a random variable are real numbers assigned to the elements of a sample space. This leads to the following formal definition.

> A random variable X is a function which assigns a real number x to each element s of a sample space S.

A random variable such as the height of students in a class, which may take any real value within a given interval, is said to be *continuous*. A random variable such as the score on a die, which takes only certain specified values, is described as *discrete*.

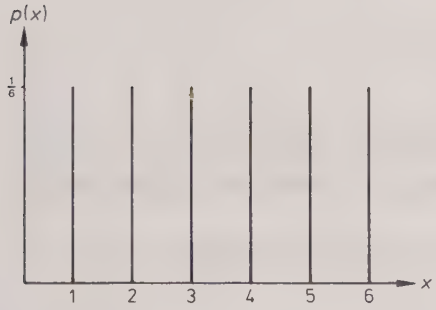
As we saw in Chapter 14, any subset A of a sample space S is called an event and the probability that event A occurs is denoted by $P(A)$. Thus the probability that the random variable X takes the value x may be written $P(X = x)$. We can now define a probability function on the set of values taken by X.

The probability function p of the discrete random variable X is defined by $p(x) = P(X = x)$, where $0 \leqslant p(x) \leqslant 1$ for all values of x.

The set of values $x_1, x_2, x_3, \ldots$ of a discrete random variable X, together with the corresponding values $p_1, p_2, p_3, \ldots$ of the probability function p, form a *discrete probability distribution.*

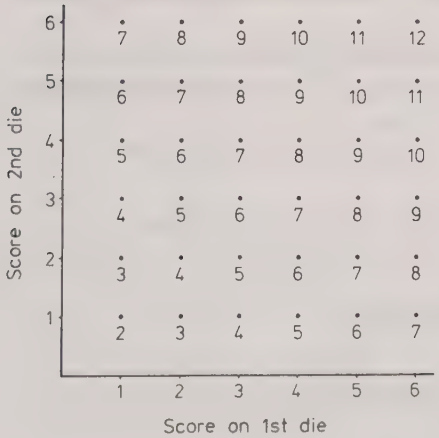
Since the sum of the probabilities of all possible outcomes of any experiment must be 1, all discrete probability distributions have the following property.

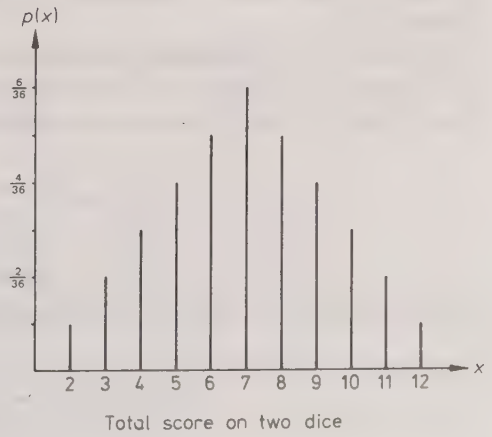
If a discrete random variable X takes the n values $x_1, x_2, \ldots x_n$ with probabilities $p_1, p_2, \ldots, p_n$ respectively, then $\sum_{r=1}^{n} p_r = 1$.



If X represents the score obtained when a single die is thrown, then $p(x) = 1/6$ for $x = 1, 2, 3, 4, 5$ and 6. A distribution such as this, in which all values of the random variable are equally likely, is said to be *rectangular* or *uniform.*

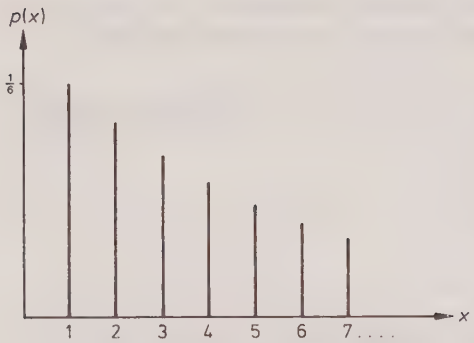
For a trial in which two dice are thrown and the total score noted the sample space S can be regarded as a set of 36 ordered pairs such as (1, 3), (4, 6) and (6, 4). The total score obtained in any throw is a discrete random variable defined on S and takes the values 2, 3, 4, ..., 12. The following diagrams show the sample space and corresponding probability distribution for this two dice trial.





A further type of discrete probability distribution is obtained when a die is thrown repeatedly and the random variable X is the number of throws needed to obtain a six. We then find that

$$p(1) = \frac{1}{6}, p(2) = \frac{5}{6} \times \frac{1}{6}, p(3) = \left(\frac{5}{6}\right)^2 \times \frac{1}{6}, p(4) = \left(\frac{5}{6}\right)^3 \times \frac{1}{6}, \ldots$$



Clearly X may take any one of the infinite set of values $1, 2, 3, 4, \ldots$ Since the corresponding probabilities form a geometric progression, this distribution is described as *geometric*.

In order to discuss the properties of probability distributions in more detail we need to define quantities similar to the measures of location and spread used to describe frequency distributions.

For a frequency distribution in which a set of n observations $x_1, x_2, \ldots, x_n$ occur with frequencies $f_1, f_2, \ldots, f_n$,

$$\text{the mean } \bar{x} = \frac{1}{n}\sum f_r x_r, \text{ where } n = \sum f_r,$$

$$\text{the variance} = \frac{1}{n}\sum f_r(x_r - \bar{x})^2 = \frac{1}{n}\sum f_r x_r^2 - \bar{x}^2$$

If n is fairly large the relative frequency f_r/n is approximately equal to the probability $p(x_r)$. Thus we obtain suitable definitions of the mean and variance of a probability distribution by replacing f_r/n by $p(x_r)$ in the formulae used for frequency distributions.

The mean of a probability distribution is called the *expectation* or *expected value* of the random variable X and is denoted by $E(X)$ or μ. The *variance* is denoted by $\mathrm{Var}(X)$ or σ^2. These quantities are defined as follows:

$$E(X) = \mu = \sum x_r p(x_r)$$
$$\mathrm{Var}(X) = \sigma^2 = \sum (x_r - \mu)^2 p(x_r)$$

The expression for $\mathrm{Var}(X)$ can be rearranged to give:

$$\mathrm{Var}(X) = \sigma^2 = \sum x_r^2 p(x_r) - \mu^2$$

For the two dice trial discussed earlier:

$$E(X) = 2.\tfrac{1}{36} + 3.\tfrac{2}{36} + \ldots + 12.\tfrac{1}{36} = 7$$
$$\text{Var}(X) = 2^2.\tfrac{1}{36} + 3^2.\tfrac{2}{36} + \ldots + 12^2.\tfrac{1}{36} - 7^2 = \tfrac{35}{6} \approx 5{\cdot}83.$$

Example 1 A box contains 10 coloured discs of which 2 are red. A man pays 10 pence to play a game in which discs are drawn one at a time from the box, without replacement, until a red disc is drawn. He will receive 25 pence if the first disc drawn is red, 20 pence if the second disc is red, 5 pence if the third disc is red, but nothing otherwise. Find the man's expected profit or loss in any game.

If the random variable X is the amount in pence received in any game, then X takes the values 25, 20, 5 and 0.

$p(25) = \frac{2}{10} = \frac{1}{5}$, $p(20) = \frac{8}{10} \times \frac{2}{9} = \frac{8}{45}$ and $p(5) = \frac{8}{10} \times \frac{7}{9} \times \frac{2}{8} = \frac{7}{45}$.

$\therefore$ the expected value of X

$$= 25 \times p(25) + 20 \times p(20) + 5 \times p(5) + 0 \times p(0)$$
$$= 25 \times \tfrac{1}{5} + 20 \times \tfrac{8}{45} + 5 \times \tfrac{7}{45} + 0 = 9\tfrac{1}{3}$$

Since the man pays 10 pence per game, his expected loss in pence is $\frac{2}{3}$.

In the next example we make use of the following results.

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1), \quad \sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1), \quad \sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2.$$

Example 2 The probability distribution of a random variable X is given by

$$P(X = r) = k(n - r) \quad \text{for} \quad r = 1, 2, 3, \ldots, n.$$

Find k in terms of n. Find also the mean and variance of X.

Since X is a discrete random variable, $\sum_1^n p(r) = 1$.

But $\sum_1^n p(r) = \sum_1^n k(n-r) = k\sum_1^n n - k\sum_1^n r$

$$= kn^2 - k.\tfrac{1}{2}n(n+1) = \tfrac{1}{2}kn(n-1)$$

Hence $k = 2/n(n-1)$.

$$E(X) = \sum_1^n rp(r) = \sum_1^n kr(n-r)$$

$$= kn\sum_1^n r - k\sum_1^n r^2$$

$$= kn.\frac{1}{2}n(n+1) - k.\frac{1}{6}n(n+1)(2n+1)$$

$$= \frac{2}{n(n-1)}.\frac{n(n+1)}{6}\{3n - (2n+1)\} = \frac{1}{3}(n+1)$$

$$\text{Var}(X) = \sum_1^n r^2 p(r) - \mu^2 = \sum_1^n kr^2(n-r) - \frac{1}{9}(n+1)^2$$

$$= kn\sum_1^n r^2 - k\sum_1^n r^3 - \frac{1}{9}(n+1)^2$$

$$= kn.\frac{1}{6}n(n+1)(2n+1) - k.\frac{1}{4}n^2(n+1)^2 - \frac{1}{9}(n+1)^2$$

$$= \frac{2}{n(n-1)}.\frac{n^2(n+1)}{12}\{2(2n+1) - 3(n+1)\} - \frac{1}{9}(n+1)^2$$

$$= \frac{1}{6}n(n+1) - \frac{1}{9}(n+1)^2$$

$$= \frac{1}{18}(n+1)\{3n - 2(n+1)\} = \frac{1}{18}(n+1)(n-2)$$

Thus the mean and variance of X are $\frac{1}{3}(n+1)$ and $\frac{1}{18}(n+1)(n-2)$ respectively.

Exercise 16.1

1. A student has a week-end job in a warehouse. The probability that he will be given 6 hours work is 1/6, the probability of 9 hours work is 1/3 and the probability of 12 hours work is 1/2. Find the expected number of hours the student will work.

2. A man pays x pence to play a game in which two unbiased dice are rolled. If the total score on the dice is an even number n, then the man receives n pence. Otherwise he receives nothing. Calculate the value of x which makes the game fair.

3. A bag contains one 50p coin, three 20p coins, seven 10p coins and several 5p coins. Given that when one coin is selected at random the expectation is 10p, find the number of 5p coins. Find also the expectation when two coins are selected at random without replacement.

4. Three discs are drawn at random, without replacement from a bag containing three black discs and five red discs. Find (a) the most probable number of black discs drawn, (b) the expected number of black discs drawn.

5. A bag X contains 3 red balls and 6 white balls and a second bag Y contains 5 red balls and 4 white balls. A ball is chosen at random from bag X and placed in bag Y. A ball is then chosen at random from bag Y and placed in bag X. If R is the number of red balls in bag X after these operations, find the probability distribution of R. Hence find the expected value of R.

6. A simple 'fruit machine' has three wheels on which there are pictures of fruit. When 10p is paid into the machine, the wheels spin and then come to rest independently, each showing one fruit. The probabilities that the three wheels show

a lemon are 0·4, 0·2 and 0·1 respectively. The machine pays out 10p for 1 lemon, 30p for 2 lemons and £1·60 for 3 lemons. Find the expected profit or loss per play.

7. In a certain gambling game a player nominates an integer x from 1 to 6 inclusive and he then throws three fair cubical dice. Calculate the probabilities that the number of x's thrown will be 0, 1, 2 and 3. The player pays 5 pence per play of the game and he receives 48 pence if the number of x's thrown is three, 15 pence if the number of x's thrown is two, 5 pence if only one x is thrown and nothing otherwise. Calculate the player's expected gain or loss per play of the game. (JMB)

8. In a certain snooker match the first man to win 3 frames wins the match. The winner of each frame receives £20 and the winner of the match receives an additional £40. Find the expected cost of the prizes in a match between players A and B if the probability that A wins a frame against B is (a) 0·5, (b) 0·2.

9. A player throws a die whose faces are numbered 1 to 6 inclusive. If the player obtains a six he throws the die a second time, and in this case his score is the sum of 6 and the second number; otherwise his score is the number obtained. The player has no more than two throws. Let X be the random variable denoting the player's score. Write down the probability distribution of X, and determine the mean of X. Show that the probability that the sum of two successive scores is 8 or more is 17/36. Determine the probability that the first of two successive scores is 7 or more, given that their sum is 8 or more. (C)

10. At a horse show there is a jumping competition in which each horse attempts to clear 4 successive fences, each fence being more difficult than the previous one. A particular horse has probability $(5 - i)/5$ of clearing the ith fence $(i = 1, 2, 3, 4)$, independent of its performance over the other fences. (i) Find the probability that the horse clears all four fences. (ii) Find the probability that the horse does not clear any fence. (iii) Determine the expected value of the number of fences that the horse clears. (C)

11. A disc is drawn from a bag containing 10 discs numbered $0, 1, 2, \ldots 9$. The random variable X is defined as the square of the number drawn. Find $E(X)$ and $\mathrm{Var}(X)$.

12. A box contains 6 cards all of different colours. A card is drawn at random from the box and its colour noted. If three cards are drawn, with replacement, find the mean and variance of the number of different colours obtained.

13. Let X denote the number of heads obtained when a fair coin is tossed three times. Find the mean and the variance of X. A pack of six cards consists of three cards labelled H and the other three labelled T. Three cards are drawn at random without replacement from the pack. Let Y denote the number of H's obtained. Calculate the probabilities that Y takes the values 0, 1, 2 and 3 respectively. Show that the mean of Y is equal to the mean of X and express the variance of Y as a percentage of the variance of X. (JMB)

14. In a certain field, each puffball which is growing in one year gives rise to a number, X, of new puffballs in the following year. None of the original puffballs is present in the following year. The probability distribution of the random variable X is as follows:

$$P(X = 0) = P(X = 2) = 0{\cdot}3, P(X = 1) = 0{\cdot}4.$$

Find the probability distribution of Y, the number of puffballs resulting from there being two puffballs in the previous year, and show that the variance of Y is 1·2. Hence, or otherwise, determine the probability distribution of the number, Z, of puffballs present in year 3, given that there was a single puffball present in year 1. Find also the mean and variance of Z. (C)

15. A cubical die is biased in such a way that the probability of scoring $n (n = 1 \text{ to } 6)$ is proportional to n. Determine the mean value and variance of the score obtained in a single throw. What would be the mean and variance if the score showing were doubled? (O&C)

16. A random variable R takes values $1, 2, \ldots, n$ with equal probabilities. Determine the expectation μ of R and show that the variance σ^2 is given by $12\sigma^2 = n^2 - 1$. Find $P(|R - \mu| > \sigma)$ in the case $n = 100$. (C)

17. The probability distribution of a random variable R is defined by $P(R = r) = kr$ for $r = 1, 2, \ldots, n$. Find the value of k. Hence find the mean and variance of the distribution.

18. Ten identically shaped discs are in a bag; two of them are black, the rest white. Discs are drawn at random from the bag in turn and NOT replaced. Let X be the number of discs drawn up to AND INCLUDING the first black one. List the values of X and the associated theoretical probabilities. Calculate the mean value of X and its standard deviation. What is the most likely value of X? If instead each disc is replaced before the next is drawn, construct a similar list of values and probabilities and point out the chief differences between the two lists. (SU)

19. (a) Expand $(1 - x)^{-2}$ in a series of ascending powers of x, giving the first three terms and the term in x^{r-1}.
    (b) A boy is throwing stones at a tin can. The probability of hitting the can with any particular stone is p. Find the probability that the boy first hits the can with the rth stone. If the random variable R is the number of throws needed to hit the can, find the expected value of R in terms of p.

20. A faulty gas-lighter works successfully, on average, once in three attempts. When using it one goes on trying until it does work and the gas lights. What is the probability that the gas lights (i) first time, (ii) second time, (iii) at the fifth attempt? What is the probability that it will not have worked after five attempts? The sellers of the lighter promise that if one is returned, they will pay 25p for each time it fails to work before the gas is lit. What is one's expectation in taking up this offer? What is the probability of getting one's money back (or more) if the original price was £1? If

four people take up this offer, what is the probability that at least two get their money back (or more)? (AEB 1978)

21. An ordinary unbiased cubical die is to be thrown repeatedly. Show that the probability that a score of 6 will occur for the first time on the nth throw with the first $(n-1)$ throws resulting in one odd score and $(n-2)$ even scores (other than 6) is equal to $(n-1)(\frac{1}{4})(\frac{1}{3})^{n-1}$ for $n \geqslant 2$. If the die is to be thrown as many times as it is necessary for a score of 6 to occur, find, using the above result, or otherwise, the probability that only one of the throws will result in an odd score.

[Hint: For $|x| < 1$, $\sum_{r=0}^{\infty} (r+1)x^r = (1-x)^{-2}$] (JMB)

16.2 The binomial distribution

A *Bernouilli*† *trial* is one in which there are two possible outcomes, usually referred to as success and failure. If the probability of success is p and the probability of failure is q, then $p + q = 1$.

If a random variable X represents the number of successes in a sequence of n independent repetitions of a Bernouilli trial, then the resulting probability distribution is called a *binomial distribution.*

Consider, for example, a trial in which an unbiased die is rolled once and success is defined as obtaining a six. In one trial the probability of success is 1/6 and the probability of failure is 5/6. Suppose now that the random variable X is the number of sixes obtained in a sequence of 5 rolls of the die.

Clearly $$P(X = 0) = (\tfrac{5}{6})^5$$

The number of arrangements of a sequence of 4 failures and 1 success = 5.
The probability of each such arrangement = $(\frac{5}{6})^4(\frac{1}{6})$

$$\therefore \quad P(X = 1) = 5(\tfrac{5}{6})^4(\tfrac{1}{6})$$

The number of arrangements of a sequence of 3 failures and 2 successes = $\frac{5!}{3!2!} = 10$.
The probability of each such arrangement = $(\frac{5}{6})^3(\frac{1}{6})^2$

$$\therefore \quad P(X = 2) = 10(\tfrac{5}{6})^3(\tfrac{1}{6})^2$$

Using similar arguments it follows that

$$P(X = 3) = 10(\tfrac{5}{6})^2(\tfrac{1}{6})^3, P(X = 4) = 5(\tfrac{5}{6})(\tfrac{1}{6})^4, P(X = 5) = (\tfrac{1}{6})^5.$$

We see that the probabilities of 0, 1, 2, 3, 4, 5 sixes in a sequence of 5 rolls of a die are given by the terms in the binomial expansion of $(\frac{5}{6} + \frac{1}{6})^5$.

To obtain a more general result let us assume that the probability of success in a certain trial is p and that the probability of failure is q, where $q = 1 - p$.

†*Bernouilli, Jacques* (1654–1705) Swiss mathematician: one of the members of a large family who were all gifted mathematically. His *Ars Conjectandi* of 1713 contains a considerable quantity of work on probability.

The number of arrangements of a sequence of $(n - r)$ failures and r successes is $\dfrac{n!}{(n-r)!r!}$, written $\binom{n}{r}$ or ${}_nC_r$.
The probability of each such arrangement $= q^{n-r}p^r$.
Hence the probability of r successes in n independent trials is $\binom{n}{r}q^{n-r}p^r$.
Thus the probabilities that $r = 0, 1, 2, \ldots, n$ are the terms in the binomial expansion of $(q + p)^n$. This distribution is described as the binomial distribution with *parameters* n and p.

A random variable X is said to have a binomial distribution $B(n, p)$, if

$$P(X = r) = \binom{n}{r}q^{n-r}p^r \quad \text{for} \quad r = 0, 1, 2, \ldots, n,$$

where $0 < p < 1$ and $q = 1 - p$.

This binomial distribution arises when X represents the number of successes in n *independent* trials, each trial having the *constant* probability p of success.

Example 1 A doctor estimates that his treatment of a particular illness is successful in $66\frac{2}{3}\%$ of cases. Find the probability that he will treat successfully exactly 4 out of the 6 patients currently in his care. Find also the probability of success in at least 2 out of the 6 cases.

If X is the number of successes, then X has a binomial distribution with parameters $n = 6$ and $p = \frac{2}{3}$.

$$P(X = 4) = \binom{6}{4}\left(\tfrac{1}{3}\right)^2\left(\tfrac{2}{3}\right)^4 = \frac{6!}{2!4!} \times \frac{2^4}{3^6} = \tfrac{80}{243} \approx 0{\cdot}329$$

$\therefore$ the probability of success in exactly 4 out of 6 cases is 0·329.

$P(X = 0) = \left(\tfrac{1}{3}\right)^6 = \tfrac{1}{729}$, $P(X = 1) = \binom{6}{1}\left(\tfrac{1}{3}\right)^5\left(\tfrac{2}{3}\right) = \tfrac{4}{243}$,

$\therefore$ the probability of at least 2 successes

$= 1 - \tfrac{1}{729} - \tfrac{4}{243} = \tfrac{716}{729} \approx 0{\cdot}982.$

If a random variable X has the binomial distribution $B(n, p)$, then

$$\sum_{r=0}^{n} P(X = r) = \sum_{r=0}^{n}\binom{n}{r}q^{n-r}p^r = (q + p)^n = 1.$$

Thus we have verified that for the distribution $B(n, p)$,

$$\sum_{r=0}^{n} p_r = 1, \quad \text{where} \quad p_r = P(X = r).$$

We now obtain the mean and variance of X.

$$E(X) = \sum_{r=0}^{n} r\binom{n}{r} q^{n-r}p^r$$

$$= \sum_{r=0}^{n} \frac{r(n!)}{(n-r)!r!} q^{n-r}p^r$$

$$= \sum_{r=1}^{n} \frac{n!}{(n-r)!(r-1)!} q^{n-r}p^r$$

$$= np \sum_{r=1}^{n} \frac{(n-1)!}{(n-r)!(r-1)!} q^{n-r}p^{r-1} = np(q+p)^{n-1} = np.$$

$$\text{Var}(X) = \sum_{r=0}^{n} r^2\binom{n}{r} q^{n-r}p^r - \{E(X)\}^2$$

Writing $r^2 = r(r-1) + r$, we obtain:

$$\text{Var}(X) = \sum_{r=0}^{n} \frac{r(r-1)n!}{(n-r)!r!} q^{n-r}p^r + \sum_{r=0}^{n} \frac{r(n!)}{(n-r)!r!} q^{n-r}p^r - n^2p^2$$

$$= \sum_{r=2}^{n} \frac{n!}{(n-r)!(r-2)!} q^{n-r}p^r + np - n^2p^2$$

$$= n(n-1)p^2 \sum_{r=2}^{n} \frac{(n-2)!}{(n-r)!(r-2)!} q^{n-r}p^{r-2} + np - n^2p^2$$

$$= n(n-1)p^2(q+p)^{n-2} + np - n^2p^2$$
$$= n(n-1)p^2 + np - n^2p^2$$
$$= np - np^2 = np(1-p) = npq$$

$\therefore$ the mean and variance of the binomial distribution $B(n, p)$ are given by

$$\mu = np, \quad \sigma^2 = npq = np(1-p).$$

Example 2 The components produced by a particular machine are tested by taking samples containing 5 components and noting the number of rejects in each. The given table shows the results for 100 samples

Number of rejects (x)	0	1	2	3	4	5
Frequency (f)	10	27	31	20	9	3

(a) Calculate the mean and variance of the number of rejects.
(b) Estimate the probability p that a component selected at random is a reject.

(c) By assuming that the number of rejects in a sample of 5 components has a binomial distribution with parameters $n = 5$ and p, write down the theoretical mean and variance.

(d) Obtain the expected frequency distribution for 100 samples.

(a) $\sum f = 10 + 27 + 31 + 20 + 9 + 3 = 100$

$\sum fx = 0 \times 10 + 1 \times 27 + 2 \times 31 + 3 \times 20 + 4 \times 9 + 5 \times 3 = 200$

$\sum fx^2 = 0^2 \times 10 + 1^2 \times 27 + 2^2 \times 31 + 3^2 \times 20 + 4^2 \times 9 + 5^2 \times 3 = 550$

$$\therefore \quad \text{the mean } \bar{x} = \frac{\sum fx}{\sum f} = \frac{200}{100} = 2,$$

$$\text{the variance} = \frac{\sum fx^2}{\sum f} - \bar{x}^2 = \frac{550}{100} - 2^2 = 1{\cdot}5.$$

(b) Total number of rejects $= \sum fx = 200$

Total number of components $= 5 \times \sum f = 500$

$\therefore$ the probability p that any component is a reject $= \frac{200}{500} = \frac{2}{5}$.

(c) The mean and variance of the binomial distribution with parameters $n = 5$ and $p = 2/5$ are respectively:

$$\mu = np = 2, \quad \sigma^2 = np(1 - p) = 1{\cdot}2.$$

(d) $$P(\text{no rejects}) = \left(\frac{3}{5}\right)^5 \approx 0{\cdot}08,$$

$$P(1 \text{ reject}) = \binom{5}{1}\left(\frac{3}{5}\right)^4\left(\frac{2}{5}\right) \approx 0{\cdot}26,$$

$$P(2 \text{ rejects}) = \binom{5}{2}\left(\frac{3}{5}\right)^3\left(\frac{2}{5}\right)^2 \approx 0{\cdot}35,$$

$$P(3 \text{ rejects}) = \binom{5}{3}\left(\frac{3}{5}\right)^2\left(\frac{2}{5}\right)^3 \approx 0{\cdot}23,$$

$$P(4 \text{ rejects}) = \binom{5}{4}\left(\frac{3}{5}\right)\left(\frac{2}{5}\right)^4 \approx 0{\cdot}08,$$

$$P(5 \text{ rejects}) = \left(\frac{2}{5}\right)^5 \approx 0{\cdot}01.$$

Multiplying these probabilities by 100 gives the expected frequencies in 100 samples (correct to the nearest integer).

Number of rejects	0	1	2	3	4	5
Expected frequency	8	26	35	23	8	1

[Note that because of rounding errors the sum of these expected frequencies is 101. This does not invalidate the results.]

We now consider an alternative way of finding the mean and variance of certain discrete probability distributions including the binomial distribution.

The *probability generating function* (p.g.f.) for the random variable X, which takes the values $0, 1, 2, \ldots, r, \ldots$ with probabilities $p_0, p_1, p_2, \ldots, p_r, \ldots$ is given by

$$G(t) = p_0 + p_1 t + p_2 t^2 + \ldots + p_r t^r + \ldots$$

Thus by using a "dummy" variable t, we are able to write down a single expression $G(t)$ representing a complete probability distribution. We note that since $\sum p_r = 1$, it follows that $G(1) = 1$.

For the binomial distribution with parameters n and p,

$$G(t) = q^n + \binom{n}{1} q^{n-1} pt + \binom{n}{2} q^{n-2} p^2 t^2 + \ldots + \binom{n}{r} q^{n-r} p^r t^r + \ldots$$

$\therefore$ the p.g.f. of the binomial distribution $B(n, p)$ is

$$G(t) = (q + pt)^n, \quad \text{where} \quad q = 1 - p.$$

If $G(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3 + \ldots + p_r t^r + \ldots$ then, differentiating with respect to t,

$$\begin{aligned} G'(t) &= p_1 + 2p_2 t + 3p_3 t^2 + \ldots + rp_r t^{r-1} + \ldots \\ G''(t) &= 2p_2 + 3.2p_3 t + \ldots + r(r-1)p_r t^{r-2} + \ldots \end{aligned}$$

$$\begin{aligned} \therefore \quad G'(1) &= p_1 + 2p_2 + 3p_3 + \ldots + rp_r + \ldots = \sum rp_r \\ G''(1) &= 2p_2 + 3.2p_3 + \ldots + r(r-1)p_r + \ldots \\ &= \sum r(r-1)p_r = \sum r^2 p_r - \sum rp_r \end{aligned}$$

Hence $E(X) = \sum rp_r = G'(1)$

and $\text{Var}(X) = \sum r^2 p_r - [E(X)]^2 = G''(1) + G'(1) - [G'(1)]^2$

i.e. for a distribution with p.g.f. $G(t)$

$$\mu = G'(1), \quad \sigma^2 = G''(1) + G'(1) - [G'(1)]^2.$$

For the binomial distribution $B(n, p)$:

$$\begin{aligned} G(t) &= (q + pt)^n \\ G'(t) &= np(q + pt)^{n-1} \qquad \therefore \quad G'(1) = np \\ G''(t) &= n(n-1)p^2(q + pt)^{n-2} \qquad \therefore \quad G''(1) = n(n-1)p^2 \end{aligned}$$

Hence $\mu = np$, $\sigma^2 = n(n-1)p^2 + np - n^2p^2 = np(1-p)$.

These results are the same as those obtained earlier by elementary methods.

Exercise 16.2

1. An unbiased die is thrown 9 times. Find the probability of (a) 2 sixes, (b) at least 2 sixes. Find also the mean and variance of the number of sixes thrown.

2. In a large batch of plastic mouldings 25% have faults. If a random sample of 10 mouldings is inspected, find the probability that the sample contains

(a) 3 faulty mouldings, (b) no more than 2 faulty mouldings. Find also the mean and variance of the number of faulty mouldings.

3. The probability that a certain darts player hits the "treble 20" with one dart is 0·3. Find the probability that the player scores at least one treble 20 with 3 darts. If the probability of scoring at least one treble 20 with n darts is greater than 0·9, find the least possible value of n.

4. One in eight of the torches assembled by a machine are imperfect. Eight torches are chosen at random. Find the probability that (a) one is imperfect, (b) at least 6 are imperfect. The probability that a box of n torches contains none that are imperfect is less than 0·1. Find the range of possible values of n.

5. Given that the binomial distribution $B(n, p)$ has mean 9·6 and standard deviation 2·4, find n and p. Hence obtain the probability generating function of the distribution.

6. Find the probability generating function of a binomial distribution whose mean is 1·8 and whose variance is 0·99. If the random variable X has this binomial distribution, (i) state the greatest possible value that X may take (with non-zero probability), (ii) write down a numerical expression for the probability that a randomly observed value of X will be equal to 3. (JMB)

7. A lake contains 1000 fish of species A, 2000 fish of species B and no other fish. Ten fish are taken from the lake at random. Explain why the probabilities of obtaining various numbers of fish of species A in the ten can be very well approximated by a binomial probability distribution, and hence estimate to three significant figures the probability of obtaining 4 fish of species A and six of species B. Estimate also to three significant figures the probability that in a catch of 5 fish altogether there are more fish of species A than there are of species B. (C)

8. A manufacturing company sends out invoices to its customers allowing 5% discount to those who settle their accounts within 15 days. In the past, 40% of the customers took advantage of the discount terms. On a particular day the company sent out 10 invoices. Examine critically the appropriateness of the binomial distribution model for this situation, discussing (a) the requirement of constant probability from trial to trial, (b) the independence of trials. Assuming the model to be appropriate, calculate the probability that, of the 10 invoices, less than 2 are settled within 15 days. For days when 10 invoices are sent out, calculate the mean and variance of the distribution of the number of invoices settled within 15 days. (L)

9. The numbers of males in a sample of 810 litters of 4 cats are recorded in the table below:

No. of males	0	1	2	3	4
No. of litters	156	326	242	74	12

Calculate the mean and variance of the number of males in this sample of litters. Obtain a corresponding table of the above form for a binomial distribution with $p = \frac{1}{3}$.

10. Six dice were thrown together 1000 times. At each throw, the number of ones obtained was noted and the frequency is shown in the table below:

No. of 1's in a single throw	0	1	2	3	4	5	6
No. of throws	370	380	150	90	9	1	0

Calculate the mean and variance of the number of ones obtained in the experiment. Produce a corresponding table of the expected frequencies, to the nearest integer in each case, based on the assumption that the six dice are unbiased and that the 1000 throws are independent. (L)

11. (a) Berg and Korner have a long rivalry in tennis: in the last 25 games, Berg has won 15 times. They start a new series of 8 games. Assuming that the binomial model may be applied, what is the probability that Berg will win at least 6 of these?

(b) Prove that the mean of the binomial distribution is np. A set of 100 pods, each containing 4 peas, was examined to see how many of the peas were good. The following were the results:

No. of good peas in pod	0	1	2	3	4
No. of pods	7	20	35	30	8

(i) What was the probability of getting a good pea? (ii) Calculate the theoretical frequencies of 0, 1, 2, 3, 4 good peas, using the associated theoretical binomial distribution. (SU)

12.

X	frequency
0	36
1	37
2	20
3	5
4	2

The number of wrong numbers (X) obtained in making a set of ten telephone calls is recorded in the given table which records the results obtained in making 100 sets of 10 telephone calls. Assuming that wrong numbers occur randomly, suggest a suitable statistical model to correspond to the making of 10 telephone calls and recording the number of wrong numbers obtained. Using this model estimate the percentage of sets of ten telephone calls in which 2 or more wrong numbers are obtained. What will this percentage be if the chance of getting a wrong number when making a call is doubled? (L)

13. (i) Find the probability of obtaining exactly two sixes in six throws of a single die [You may assume that $(5/6)^5 = 0{\cdot}402$.]

(ii) If two dice are to be thrown together and the sum of their scores is recorded, find the probability that (a) the sum in one throw will be eleven, (b) a score of

eleven will occur exactly twice in six throws [assume that $17^4/18^5 = 0{\cdot}044$], (c) in one throw the score will be less than eight, given that exactly one die shows a four. (L)

14. A crossword puzzle is published in *The Times* each day of the week, except Sunday. A man is able to complete, on average, 8 out of 10 of the crossword puzzles. (i) Find the expected value and the standard deviation of the number of completed crosswords in a given week. (ii) Show that the probability that he will complete at least 5 in a given week is 0·655 (to 3 significant figures). (iii) Given that he completes the puzzle on Monday, find, to three significant figures, the probability that he will complete at least 4 in the rest of the week. (iv) Find, to three significant figures, the probability that, in a period of four weeks, he completes 4 or less in only one of the four weeks. (C)

15. 4 girls and 3 boys plan to meet together on the following Saturday. The probability that each boy will be present is $\frac{2}{3}$ independently of the other boys. Find the probabilities that (a) 0, (b) 1, (c) 2, (d) 3 boys will be present. The probability that each girl will be present is $\frac{1}{2}$ independently of the other girls and of the boys. (e) Find the probability that the number of girls present will equal the number of boys. (f) Find the probability that both sexes will be present. (g) Afterwards it was reported that the gathering had included at least one boy and at least one girl. What is the probability that there were equal numbers of boys and girls in the light of this additional information? (O&C)

16. Each trial of a binomial experiment has a constant probability p of yielding a success. (i) Given that 7 successes occurred in 15 independent trials show that the conditional probability that the first and last trials were successes is 1/5. (ii) Find the value of p given that in 20 independent trials the probability of exactly 4 successes is twice that of exactly 6 successes. (JMB)

17. The random variable X has the binomial distribution defined by

$$P(X = r) = \binom{n}{r}(1-p)^{n-r}p^r, \quad r = 0, 1, 2, \ldots, n.$$

Find an expression for $P(X = r)/P(X = r - 1)$. Hence deduce the most likely values of X when (a) $n = 20$, $p = 0{\cdot}4$, (b) $n = 14$, $p = 0{\cdot}2$.

18. Cards are drawn at random and with replacement from an ordinary pack of playing cards until three spades have been drawn. Find the probability that the number of draws required is (a) exactly six, (b) at least six. Find an expression for P_n, the probability that exactly n cards have to be drawn to get three spades ($n \geqslant 3$). Hence, or otherwise, find the most likely value (or values) of n. (L)

19. A random variable X has p.g.f. $G(t) = 1/(2-t)^2$. Find the expressions for (a) $P(X = r)$, (b) $E(X)$, (c) $\mathrm{Var}(X)$.

20. A random variable X is defined as the number of throws of an unbiased die needed to obtain a six. Find the p.g.f. for X and hence determine $E(X)$ and $\mathrm{Var}(X)$.

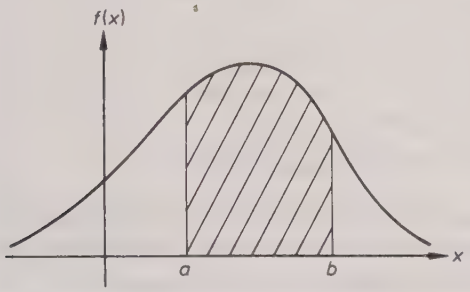
16.3 Continuous probability distributions

If X is a continuous random variable which may take any real value within some given interval, $0 \leqslant x \leqslant 10$ say, then it is difficult to give a meaning to probabilities such as $P(X = 2)$, $P(X = 7{\cdot}32054)$, $P(X = 17/39)$. If $P(X = x)$ were non-zero for all values of x, the sum of all such probabilities would be infinite. This means that we must regard the value of $P(X = x)$ as zero. Thus, in order to set up a suitable probability model we clearly need to define a new type of function.

The *probability density function* (p.d.f.) of the continuous random variable X is a function $f(x)$ such that

$$P(a \leqslant X \leqslant b) = \int_a^b f(x)\,dx.$$

[Since $P(X = a) = P(X = b) = 0$, the values of $P(a < X < b)$, $P(a \leqslant X < b)$ and $P(a < X \leqslant b)$ are also given by this integral.]



It follows that we can now think of probabilities as areas under the graph of $f(x)$. As probabilities are never negative, $f(x) \geqslant 0$ for all values of x. Since the sum of the probabilities of all possible outcomes of an experiment is 1, the total area under the graph must be 1.

i.e. $$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

For a discrete random variable X, $E(X) = \mu = \sum xp(x)$ and $\mathrm{Var}(X) = \sigma^2 = \sum(x - \mu)^2 p(x) = \sum x^2 p(x) - \mu^2$. To obtain similar formulae for the mean and variance of a continuous random variable X, let us consider the probability that the value of X lies between x and $x + \delta x$. Provided that δx is small, this probability is given approximately by $f(x)\delta x$. Thus we let

$$E(X) = \lim_{\delta x \to 0} \sum xf(x)\delta x = \int_{-\infty}^{\infty} xf(x)\,dx.$$

Applying similar arguments to the variance, we have:

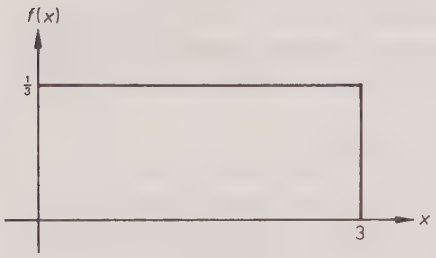
For a continuous random variable X,

$$E(X) = \mu = \int_{-\infty}^{\infty} xf(x)\,dx$$

$$\mathrm{Var}(X) = \sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)\,dx = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2.$$

One of the simplest types of continuous probability distribution is the *rectangular* or *uniform* distribution. A continuous random variable X is said to be *uniformly distributed* over the interval a to b if $f(x)$ has a constant non-zero value for $a \leqslant x \leqslant b$ and is zero otherwise.

Example 1 Find the mean and variance of the continuous random variable X which is uniformly distributed over the interval from 0 to 3.



Since the total area under the graph of the probability density function $f(x)$ must be 1,

$f(x) = \frac{1}{3}$ for $0 \leqslant x \leqslant 3$

$f(x) = 0$ otherwise.

$$\therefore \quad E(X) = \int_0^3 x \times \frac{1}{3}\,dx = \left[\frac{1}{6}x^2\right]_0^3 = \frac{3}{2}$$

$$\text{Var}(X) = \int_0^3 x^2 \times \frac{1}{3}\,dx - \left(\frac{3}{2}\right)^2 = \left[\frac{1}{9}x^3\right]_0^3 - \frac{9}{4} = 3 - \frac{9}{4} = \frac{3}{4}$$

Hence the mean of the distribution is $1\frac{1}{2}$ and the variance is $\frac{3}{4}$.

A *mode* or *modal value* of a continuous random variable X with probability density function $f(x)$ is a value of x for which $f(x)$ takes a maximum value. Thus the modes of a distribution are the x-coordinates of the maximum points on the graph of $f(x)$.

The *median* of a continuous random variable X with probability density function $f(x)$ is the value m such that $\int_{-\infty}^{m} f(x)\,dx = 0{\cdot}5$.

Example 2 A continuous random variable X has probability density function $f(x) = kx(9 - x^2)$ for $0 \leqslant x \leqslant 3$ and $f(x) = 0$ otherwise. Find (a) the value of k, (b) the mean, (c) the mode, (d) the median.

(a) $$\int_0^3 f(x)\,dx = k\int_0^3 (9x - x^3)\,dx = k\left[\frac{9x^2}{2} - \frac{x^4}{4}\right]_0^3 = \frac{81k}{4}$$

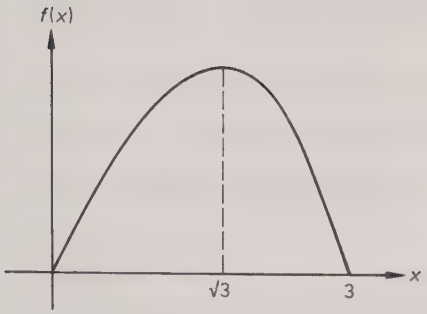
Since the total area under the graph of $f(x)$ must be 1, $k = 4/81$.

(b) $$E(X) = \int_0^3 xf(x)\,dx = \frac{4}{81}\int_0^3 (9x^2 - x^4)\,dx$$

$$= \frac{4}{81}\left[3x^3 - \frac{x^5}{5}\right]_0^3 = 4\left(1 - \frac{3}{5}\right) = \frac{8}{5}$$

Hence the mean of the distribution is 1·6.

(c)



$f(x) = \frac{4}{81}(9x - x^3)$,

$f'(x) = \frac{4}{81}(9 - 3x^2) = \frac{4}{27}(3 - x^2)$

$\therefore$ in the interval $0 \leqslant x \leqslant 3$, $f'(x) = 0$ when $x = \sqrt{3}$.
The sketch shows that this gives a maximum value of $f(x)$.

Hence the mode of the distribution is 1·73 (to 3 s.f.)

(d) If m is the median, $\int_0^m f(x)\,dx = \frac{1}{2}$

$$\therefore \quad \frac{4}{81}\int_0^m (9x - x^3)\,dx = \frac{4}{81}\left[\frac{9x^2}{2} - \frac{x^4}{4}\right]_0^m = \frac{1}{81}(18m^2 - m^4) = \frac{1}{2}$$

$$36m^2 - 2m^4 = 81$$
$$2m^4 - 36m^2 + 81 = 0$$

$$\therefore \quad m^2 = \frac{36 \pm \sqrt{(36^2 - 8.81)}}{4} = \frac{36 \pm 18\sqrt{(4 - 2)}}{4} = 9 \pm \frac{9}{2}\sqrt{2}$$

Since $0 < m < 3$, we must have $m^2 = 9 - \frac{9}{2}\sqrt{2}$

$$\therefore \quad m = \sqrt{\left\{9 - \frac{9}{2}\sqrt{2}\right\}} \approx 1{\cdot}62$$

Hence the median of the distribution is 1·62 (to 3 s.f.).

Another function used when modelling continuous probability distributions is defined as follows:

The *cumulative distribution function* (c.d.f.) of the continuous random variable X is the function $F(x)$ such that

$$F(x) = P(X \leqslant x).$$

If X has probability density function $f(x)$, then

$$F(t) = P(X \leqslant t) = \int_{-\infty}^{t} f(x)\,dx.$$

It follows that: $F'(x) = f(x)$.

Example 3 A random variable X has cumulative distribution function

$$F(x) = \begin{cases} 0 & \text{for } x \leqslant 0 \\ \frac{1}{6}x^2 & \text{for } 0 \leqslant x \leqslant 2 \\ -2 + 2x - \frac{1}{3}x^2 & \text{for } 2 \leqslant x \leqslant 3 \\ 1 & \text{for } x \geqslant 3. \end{cases}$$

Find (a) $P(X \leqslant 1{\cdot}5)$, (b) the median, (c) the probability density function $f(x)$.

(a) $P(X \leqslant 1{\cdot}5) = F(1{\cdot}5) = \frac{1}{6}(1{\cdot}5)^2 = 0.375$.

(b) The median is the value m such that $F(m) = \frac{1}{2}$.

Since $F(0) = 0$ and $F(2) = \frac{2}{3}$, m lies in the interval from 0 to 2 and thus $F(m) = \frac{1}{6}m^2$.

$\therefore$ $\frac{1}{6}m^2 = \frac{1}{2}$ which gives $m = \sqrt{3}$.

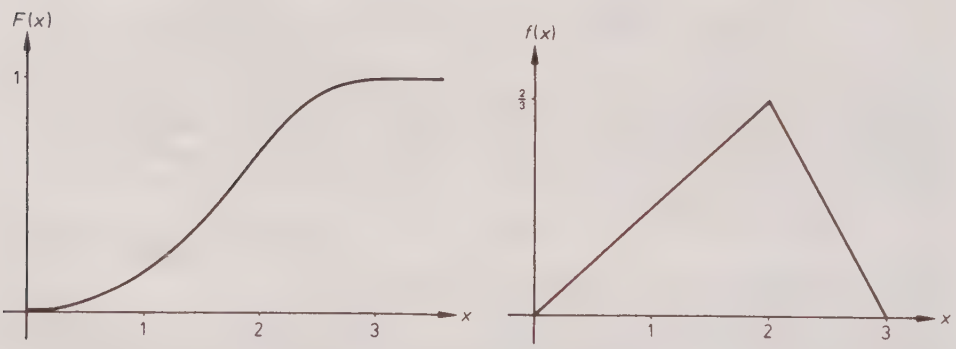
Hence the median of the distribution is 1·73 (to 3 s.f.).

(c) The probability density function $f(x) = F'(x)$.

$\therefore$ for $0 \leqslant x \leqslant 2$, $f(x) = \dfrac{d}{dx}(\frac{1}{6}x^2) = \frac{1}{3}x$;

for $2 \leqslant x \leqslant 3$, $f(x) = \dfrac{d}{dx}(-2 + 2x - \frac{1}{3}x^2) = 2 - \frac{2}{3}x$;

otherwise, $f(x) = 0$.



Example 4 A random variable X has probability density function $f(x) = \frac{1}{2}x$ for $0 \leqslant x \leqslant 2$ and $f(x) = 0$ otherwise. Find the cumulative distribution function $F(x)$ of the variable X. Hence find the cumulative distribution function $G(y)$ of the variable $Y = X^2$. Deduce the probability density function $g(y)$.

For $0 \leqslant t \leqslant 2$, $F(t) = \displaystyle\int_0^t \tfrac{1}{2}x\,dx = \left[\tfrac{1}{4}x^2\right]_0^t = \tfrac{1}{4}t^2$.

$\therefore$ the cumulative distribution function of X is given by

$F(x) = 0 (x \leqslant 0)$, $F(x) = \frac{1}{4}x^2 (0 \leqslant x \leqslant 2)$, $F(x) = 1 (x \geqslant 2)$.

$G(y) = P(Y \leqslant y) = P(X^2 \leqslant y) = P(X \leqslant \sqrt{y}) = F(\sqrt{y})$

$\therefore$ the cumulative distribution function of Y is given by

$G'(y) = 0 (y \leqslant 0)$, $G(y) = \frac{1}{4}y (0 \leqslant y \leqslant 4)$, $G(y) = 1 (y \geqslant 4)$.

Since $g(y) = G'(y)$, we find that $g(y) = \frac{1}{4} (0 \leqslant y \leqslant 4)$ and $g(y) = 0$ otherwise.

[We see that Y is uniformly distributed over the interval from 0 to 4.]

Exercise 16.3

1. Find the mean and variance of the continuous random variable X which is uniformly distributed over the interval
(a) from 0 to 1, (b) from 1 to 5,
(c) from 0 to k, (d) from a to b, where $b > a$.

2. A continuous random variable X has probability density function

$$f(x) = k(3x^2 - x^3) \quad \text{for} \quad 0 \leqslant x \leqslant 3 \quad \text{and} \quad f(x) = 0 \text{ otherwise.}$$

Find the value of k. Find also the mean, mode and standard deviation of X.

3. (a) A continuous variable x is distributed at random between the values 2 and 3 and has a probability density function of $6/x^2$. Find the median value of x.
(b) A continuous random variable X takes values between 0 and 1, with a probability density function of $Ax(1 - x)^3$. Find the value of A, and the mean and standard deviation of X. (SU)

4. A continuous variable X is distributed at random between two values, $x = 0$ and $x = 2$, and has a probability density function of $ax^2 + bx$. The mean is 1·25. (i) Show that $b = \frac{3}{4}$, and find the value of a. (ii) Find the variance of X. (iii) Verify that the median value of X is approximately 1·3. (iv) Find the mode. (SU)

5. The random variable X has a probability density function given by

$$P(x) = \begin{cases} kx(1 - x^2) & (0 \leqslant x \leqslant 1), \\ 0 & \text{elsewhere,} \end{cases}$$

k being a constant. Find the value of k and find also the mean and variance of this distribution. Find the median of the distribution. (O&C)

6. The continuous random variable X has probability density function given by

$$f(x) = \begin{cases} k(1 + x^2) & \text{for} \quad -1 \leqslant x \leqslant 1, \\ 0 & \text{otherwise,} \end{cases}$$

where k is a constant. Find the value of k, and determine $E(X)$ and $\text{Var}(X)$. A is the event $X > \frac{1}{2}$; B is the event $X > \frac{3}{4}$. Find (i) $P(B)$, (ii) $P(B|A)$. (C)

7. Petrol is delivered to a garage every Monday morning. At this garage the weekly demand for petrol, in thousands of units, is a continuous random variable X distributed with a probability density function of the form

$$f(x) = \begin{cases} ax^2(b - x) & 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise.} \end{cases}$$

(i) Given that the mean weekly demand is 600 units, determine the values of a and b. (ii) If the storage tanks at this garage are filled to their total capacity of 900 units every Monday morning, what is the probability that in any given week the garage will be unable to meet the demand for petrol? (AEB 1978)

8. A continuous random variable X has the probability density function f defined by

$$\begin{aligned} f(x) &= \tfrac{1}{3}cx, & 0 \leqslant x < 3, \\ f(x) &= c, & 3 \leqslant x \leqslant 4, \\ f(x) &= 0, & \text{otherwise,} \end{aligned}$$

where c is a positive constant. Find (i) the value of c, (ii) the mean of X, (iii) the value, a, for there to be a probability of 0·85 that a randomly observed value of X will exceed a. (JMB)

9. A continuous random variable X takes values between 0 and 1 only and its probability density function f is given by $f(x) = kx^2(1 - x)$. Prove that $k = 12$, and find $E(X)$. The cumulative distribution function F is such that $F(x) = P(X \leqslant x)$. Express $F(x)$ in terms of x, and sketch the graph of $y = F(x)$. Verify by calculation that the median of X lies between 0·60 and 0·62. (C)

10. The random variable X has the probability density function

$$f(x) = ae^{-ax} (0 < x < \infty).$$

Show that the cumulative distribution function of X is $F(x) = 1 - e^{-ax}$. The above distribution, with parameter $a = 0{\cdot}8$, is proposed as a model for the length of life, in years, of a species of bird. Find the expected frequencies, of a total of 50 birds, that would fall in the class intervals (years) 0–, 1–, 2–, 3–, 4–, 5 and over. (O)

11. A random variable X takes values in the interval $0 < x < 3$ and has probability density function $f(x)$, where

$$f(x) = \begin{cases} ax & (0 < x \leqslant 1), \\ \frac{1}{2}a(3 - x) & (1 < x < 3). \end{cases}$$

(i) Find the value of a. (ii) Find the expected value and the standard deviation of X. (iii) Find the (cumulative) distribution function $F(x)$. (iv) Find $P(|X - 1| < \frac{1}{2})$. (C)

12. The continuous random variable X has probability density function $f(x)$ defined by

$$f(x) = \begin{cases} c/x^4 & (x < -1) \\ c(2 - x^2) & (-1 \leqslant x \leqslant 1) \\ c/x^4 & (x > 1). \end{cases}$$

(i) Show that $c = \frac{1}{4}$. (ii) Sketch the graph of $f(x)$. (iii) Determine the cumulative distribution function $F(x)$. (iv) Determine the expected value of X and the variance of X. (C)

13. The continuous random variable X has (cumulative) distribution function given by

$$F(x) = \begin{cases} (1 + x)/8 & (-1 \leqslant x \leqslant 0), \\ (1 + 3x)/8 & (0 \leqslant x \leqslant 2), \\ (5 + x)/8 & (2 \leqslant x \leqslant 3), \end{cases}$$

with $F(x) = 0$ for $x < -1$, and $F(x) = 1$ for $x > 3$.
(i) Sketch the graph of the probability density function $f(x)$.
(ii) Determine the expectation of X and the variance of X.
(iii) Determine $P(3 \leqslant 2X \leqslant 5)$. (C)

14. The length X of an offcut of wooden planking is a random variable which can take any value up to 0·5 m. It is known that the probability of the length being not more than x metres $(0 \leqslant x \leqslant 0{\cdot}5)$ is equal to kx. Determine (i) the value of k, (ii) the probability density function of X, (iii) the expected value of X, (iv) the standard deviation of X (correct to 3 s.f.). (C)

15. The (cumulative) distribution function for a continuous random variable X is given by

$$F(x) = \begin{cases} 0 & \text{for} \quad x < -2 \\ k(4x - \frac{1}{3}x^3 + \frac{16}{3}) & \text{for} \quad -2 \leqslant x \leqslant 2, \\ 1 & \text{for} \quad x > 2. \end{cases}$$

Find (i) the value of k, (ii) $P(-1 < X < 1)$, (iii) the probability density function for X, (iv) the mean and variance of X. (C)

16. A random variable X has probability density function $f(x) = \frac{2}{3}x$ for $1 \leqslant x \leqslant 2$ and $f(x) = 0$ otherwise. Find the cumulative distribution function $F(x)$ of the variable X. Hence find the cumulative distribution function $G(y)$ of the variable $Y = X^{1/2}$. Deduce the probability density function $g(y)$.

17. X is a continuous random variable which is uniformly distributed in the interval $-1 \leqslant x \leqslant 1$. A second random variable Y is given by $Y = X^2$. Show that, if $0 < y < 1$, then $P(Y \leqslant y) = \sqrt{y}$, and deduce the probability density function of Y. Two independent observations of Y are made, and Z denotes the larger of the two observations. By considering the probability that both values of Y are

less than or equal to z, where $0 \leqslant z \leqslant 1$, or otherwise, prove that z is uniformly distributed. Determine the probability density function of W, the smaller of the two values of Y. (C)

18. Write down the probability density function and the cumulative distribution function of a random variable which is uniformly distributed over the interval $[a, b]$.

A person drives to work, a distance of 12 miles. The time he takes, in minutes, to cover the journey is uniformly distributed over the interval [20, 30]. Let V denote the car's average speed in miles per hour on such a journey. Find an expression for $P(V \leqslant v)$, where v is an arbitrary value between 24 and 36. Hence, or otherwise, find (i) the probability density function of V, (ii) the probability that on such a journey the car's average speed will be in excess of 30 m.p.h., (iii) the median average speed for the journey. (W)

19. A continuous random variable X has probability density function defined by

$$f(x) = 2x/a^2 \qquad 0 \leqslant x \leqslant a$$
$$f(x) = 0, \qquad \text{otherwise.}$$

(a) Find the variance of X. (b) Find the cumulative distribution function of X. (c) If $Y = 1 - (X/a)$, find (i) $P(Y > X)$, (ii) the probability density function of Y. (W)

20. A random variable X has probability density function $f(x) = |x|$ for $-1 \leqslant x \leqslant 1$ and $f(x) = 0$ otherwise. Find the probability density function $g(y)$ of the random variable $Y = X^2$.

16.4 The normal distribution

The *normal distribution* is the most important continuous probability distribution in theoretical statistics. It is found to provide a good probability model for the distributions of many variables such as the heights of people and other measurements of biological characteristics. One of the early uses of the normal distribution was as a model for the distribution of errors in experimental observations.

A continuous random variable X is said to be *normally distributed* with mean μ and variance σ^2 if it has probability density function.

$$f(x) = \frac{1}{\sigma\sqrt{(2\pi)}} e^{-(x-\mu)^2/2\sigma^2} \quad (\sigma > 0)$$

This distribution is denoted by $N(\mu, \sigma^2)$.

We cannot prove by elementary methods that this is a valid definition of a probability distribution. Clearly $f(x) \geqslant 0$ for all values of x, but for now we must

assume that the total area under the graph of $f(x)$ is 1 and that the distribution has mean μ and variance σ^2.

The graph of $f(x)$ is a bell-shaped curve called the *normal curve*. We see that
(1) the curve is symmetrical about the line $x = \mu$;
(2) the x-axis is an asymptote to the curve;
(3) when $x = \mu$, $f(x)$ takes its maximum value $1/\sigma\sqrt{(2\pi)}$ (i.e. $0{\cdot}4/\sigma$ approximately).
As described in the previous section, the probabilities associated with a continuous random variable X are represented by areas under the graph of the p.d.f. $f(x)$. Thus for the distribution $N(\mu, \sigma^2)$,

$$P(X \leqslant a) = \int_{-\infty}^{a} f(x)\,dx = \int_{-\infty}^{a} \frac{1}{\sigma\sqrt{(2\pi)}} e^{-(x-\mu)^2/2\sigma^2}\,dx.$$

This integral can be simplified by substituting

$$z = \frac{x-\mu}{\sigma} \quad \text{and} \quad dz = \frac{1}{\sigma}dx.$$

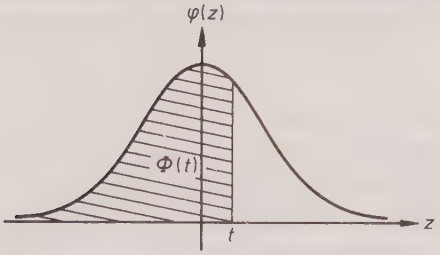
Writing $c = \dfrac{a-\mu}{\sigma}$, we have

$$P(X \leqslant a) = P(Z \leqslant c) = \int_{-\infty}^{c} \frac{1}{\sqrt{(2\pi)}} e^{-z^2/2}\,dz.$$

Hence the new random variable $Z = (X - \mu)/\sigma$ has probability density function $\phi(z) = \dfrac{1}{\sqrt{(2\pi)}} e^{-z^2/2}$. We deduce that Z is normally distributed with mean 0 and variance 1. We therefore call Z a *standardised* or *standard* normal variable. To summarise:

> If a normal variable X has the distribution $N(\mu, \sigma^2)$ then the random variable $Z = (X - \mu)/\sigma$ has the *standard normal distribution* $N(0, 1)$.

Using this change of variable any problem involving a normal distribution can be restated in terms of the standard distribution $N(0, 1)$.



The (cumulative) *distribution function* for the standard normal variable Z is denoted by $\Phi(z)$,
where $\Phi(z) = P(Z \leqslant z)$,

$$\therefore \quad \Phi(t) = \int_{-\infty}^{t} \phi(z)\,dz,$$

where $\phi(z) = \dfrac{1}{\sqrt{(2\pi)}} e^{-z^2/2}$.

Since these integrals are difficult to evaluate, tables of values have been compiled. Table 1, which appears towards the end of this book, gives values of $\Phi(z)$ for $0 \leqslant z \leqslant 4$. For example, $P(Z \leqslant 2{\cdot}18) = \Phi(2{\cdot}18) = 0{\cdot}98537$. If intermediate values of $\Phi(z)$ are required linear interpolation can be used.

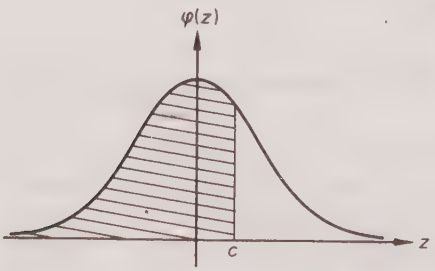
Example 1 Use Table 1 to estimate the value of $\Phi(0{\cdot}6354)$.

$\Phi(0{\cdot}64) - \Phi(0{\cdot}63) = 0{\cdot}7389 - 0{\cdot}7357 = 0{\cdot}0032$

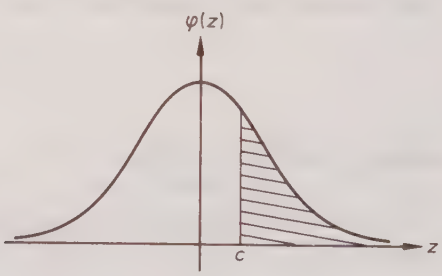
$\therefore$ by linear interpolation,

$\Phi(0{\cdot}6354) = 0{\cdot}7357 + \frac{54}{100} \times 0{\cdot}0032 \approx 0{\cdot}7374.$

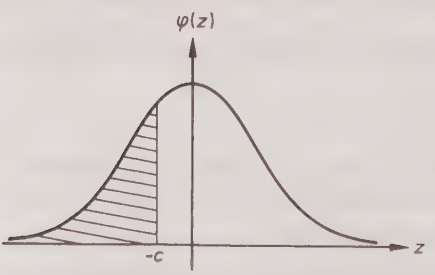
The following diagrams show how the symmetry of the normal distribution can be used to obtain $\Phi(z)$ for negative values of z.



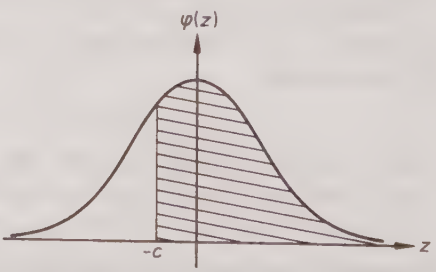
$P(Z \leqslant c) = \Phi(c)$



$P(Z \geqslant c) = 1 - \Phi(c)$



$P(Z \leqslant -c) = 1 - \Phi(c)$



$P(Z \geqslant -c) = \Phi(c)$

We may apply these results as follows:

$$P(Z \leqslant -0.72) = 1 - \Phi(0{\cdot}72) = 1 - 0{\cdot}7642 = 0.2358$$
$$P(-1{\cdot}5 \leqslant Z \leqslant -0{\cdot}5) = P(Z \geqslant -1{\cdot}5) - P(Z \geqslant -0.5)$$
$$= \Phi(1{\cdot}5) - \Phi(0{\cdot}5) = 0{\cdot}9332 - 0{\cdot}6915 = 0{\cdot}2417$$

In the next example we use our knowledge of the standard normal distribution to find probabilities associated with a different normal distribution.

Example 2 Given that a continuous random variable X is normally distributed with mean 20 and variance 16, find $P(15 \leqslant X \leqslant 24)$.

X has mean $\mu = 20$ and standard deviation $\sigma = 4$,
$\therefore$ the corresponding standard normal variable is

$$Z = (X - \mu)/\sigma = (X - 20)/4$$

$$P(X \leqslant 24) = P\left(Z \leqslant \frac{24 - 20}{4}\right) = P(Z \leqslant 1) = \Phi(1) = 0{\cdot}8413$$

$$P(X \leqslant 15) = P\left(Z \leqslant \frac{15 - 20}{4}\right) = P(Z \leqslant -1{\cdot}25) = 1 - \Phi(1{\cdot}25) = 0{\cdot}1056$$

$\therefore$ $P(15 \leqslant X \leqslant 24) = 0{\cdot}8413 - 0{\cdot}1056 = 0{\cdot}7357.$

We now demonstrate a different way of using values of the normal distribution function.

Example 3 Given that $\Phi(t) = 0{\cdot}81$, use Table 1 to estimate the value of t.

$\Phi(0{\cdot}87) = 0{\cdot}8078$, $\Phi(0{\cdot}88) = 0{\cdot}8106$,
$\therefore$ if $\Phi(t) = 0{\cdot}81$, then $0{\cdot}87 < t < 0{\cdot}88$.
$\Phi(t) - \Phi(0{\cdot}87) = 0{\cdot}0022$, $\Phi(0{\cdot}88) - \Phi(0{\cdot}87) = 0{\cdot}0028$,
$\therefore$ by linear interpolation, $t = 0{\cdot}87 + \frac{22}{28} \times 0.01 \approx 0.878$.

For certain values of $\Phi(z)$, the values of z can be obtained directly from Table 2, which gives *percentage points* of the normal distribution $N(0, 1)$. For instance, given that $\Phi(t) = 0{\cdot}8$, we know that 20% of the distribution lies "to the right" of the value $z = t$. Thus, using Table 2, $t = 0{\cdot}842$.

Example 4 A random variable X is normally distributed with mean μ and variance σ^2. Given that $P(X < 8) = 0{\cdot}05$ and $P(X > 15) = 0{\cdot}1$, find μ and σ^2.

Letting $Z = (X - \mu)/\sigma$ and using tables,
$0{\cdot}05 = P(Z > 1{\cdot}645) = P(Z < -1{\cdot}645)$
and $0{\cdot}1 = P(Z > 1{\cdot}282)$.

$$\therefore \quad -1{\cdot}645 = \frac{8 - \mu}{\sigma} \quad \text{i.e.} \quad 1{\cdot}645\sigma = \mu - 8 \tag{1}$$

$$1{\cdot}282 = \frac{15 - \mu}{\sigma} \quad \text{i.e.} \quad 1{\cdot}282\sigma = 15 - \mu \tag{2}$$

Adding (1) and (2): $2{\cdot}927\sigma = 7$

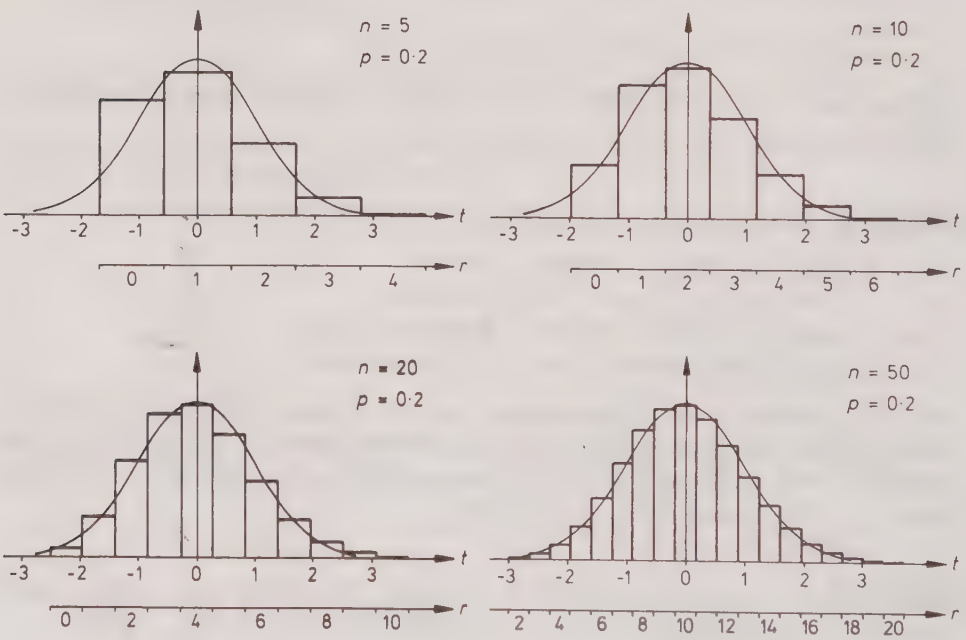
$$\therefore \quad \sigma = \frac{7}{2{\cdot}927} \quad \text{and} \quad \sigma^2 \approx 5{\cdot}72$$

Substituting in (1): $\mu = 8 + \dfrac{1{\cdot}645 \times 7}{2{\cdot}927} \approx 11{\cdot}9$

The normal distribution arises theoretically as a limit of the binomial distribution, as described in the following theorem:

> If R is a random variable with distribution $B(n, p)$, then the distribution of the variable $T = (R - \mu)/\sigma$, where $\mu = np$ and $\sigma = \sqrt{(npq)}$, tends to $N(0, 1)$ as n tends to infinity.

The proof of this statement is rather difficult, but we can demonstrate the limiting process involved by means of an example. The given set of histograms shows the distribution of the standardised binomial variable T as the value of n increases, but p remains fixed.



Transforming back to variables with mean $\mu = np$ and variance $\sigma^2 = npq$ we draw the following important conclusion.

> If R is a random variable with distribution $B(n, p)$, then for large n, the distribution of R can be approximated by the normal distribution $N(np, npq)$, where $q = 1 - p$.

This approximation will be close enough for most practical purposes if both np and nq are greater than 5. However, more accurate results will be obtained if both np and nq are greater than 20.

Before considering a numerical example we note that when using the distribution of a continuous variable X as an approximation to the distribution of a discrete variable R, it is necessary to make a *continuity correction*. For instance, if R takes only the integral values $0, 1, 2, \ldots$, then these are assumed to correspond to the intervals $X \leqslant 0{\cdot}5$, $0{\cdot}5 \leqslant X \leqslant 1{\cdot}5$, $1{\cdot}5 \leqslant X \leqslant 2{\cdot}5, \ldots$.

Let us now suppose that R has a binomial distribution with $n = 50$, $p = 0{\cdot}2$, then R will have mean $\mu = np = 10$ and variance $\sigma^2 = npq = 8$. Thus the corresponding normal variable X also has mean 10 and variance 8. Using $N(10, 8)$ as an approximation to $B(50, 0{\cdot}2)$:

$$\begin{aligned} P(12 \leqslant R \leqslant 15) &= P(11{\cdot}5 \leqslant X \leqslant 15{\cdot}5) \\ &= P\left(\frac{11{\cdot}5 - 10}{\sqrt{8}} \leqslant \frac{X - 10}{\sqrt{8}} \leqslant \frac{15{\cdot}5 - 10}{\sqrt{8}}\right) \\ &= P(0{\cdot}5303 \leqslant Z \leqslant 1{\cdot}9445) \\ &= \Phi(1{\cdot}9445) - \Phi(0{\cdot}5303) \\ &= 0{\cdot}9741 - 0{\cdot}7020 = 0{\cdot}2721 \end{aligned}$$

$$\therefore \quad P(12 \leqslant R \leqslant 15) \approx 0{\cdot}27.$$

This turns out to be a fairly good approximation to the true value, which is 0·26 (to 2 d.p.). Even better results are obtained when the value of p is closer to 0·5.

Exercise 16.4

1. Given that X is normally distributed with mean 0 and variance 1, use Table 1 to find

(a) $P(X \leqslant 0{\cdot}3)$, (b) $P(X \geqslant 1{\cdot}2)$,
(c) $P(X > -2)$, (d) $P(X \leqslant -2{\cdot}23)$,
(e) $P(0{\cdot}8 \leqslant X \leqslant 1{\cdot}8)$, (f) $P(-1{\cdot}1 < X < 0{\cdot}5)$.

2. Given that Z is a standardised normal variable, use linear interpolation to find

(a) $P(Z \leqslant 1{\cdot}537)$, (b) $P(Z < -0.862)$,
(c) $P(Z > -1{\cdot}9035)$, (d) $P(Z \geqslant 2{\cdot}0067)$,
(e) $P(|Z| \geqslant 1{\cdot}263)$, (f) $P(|Z + 1| < 0{\cdot}616)$.

3. Given that X is normally distributed with mean 5 and standard deviation 1, find

(a) $P(X < 6{\cdot}7)$, (b) $P(X \leqslant 4{\cdot}2)$, (c) $P(3{\cdot}7 < X < 5{\cdot}7)$.

4. Given that X is normally distributed with mean 0·2 and standard deviation 0·01, find

(a) $P(X \geqslant 0{\cdot}205)$, (b) $P(X > 0{\cdot}198)$, (c) $P(0{\cdot}199 \leqslant X \leqslant 0{\cdot}201)$.

5. Given that X is normally distributed with mean 12 and variance 1·5, find
(a) $P(X \leqslant 14)$, (b) $P(X > 9{\cdot}6)$, (c) $P(9 \leqslant X \leqslant 10)$.

6. Use Table 1 to estimate the value of t given that
(a) $\Phi(t) = 0{\cdot}67$, (b) $\Phi(t) = 0{\cdot}72$, (c) $\Phi(t) = 0{\cdot}19$.

7. Use Table 2 to find the value of t such that
(a) $\Phi(t) = 0{\cdot}7$, (b) $\Phi(t) = 0{\cdot}99$, (c) $\Phi(t) = 0{\cdot}9995$.

8. The weights of a large number of cats are normally distributed with mean 1·55 kg and standard deviation 0·2 kg. Find the percentage of cats with weights (a) less than or equal to 1 kg, (b) between 1·2 and 1·3 kg, (c) between 1·5 and 1·75 kg, (d) greater than or equal to 2 kg. (L)

9. The weights of oranges are distributed normally with a mean of 96 g and standard deviation 25 g. For selling purposes, oranges are divided into three grades, I, II and III. Grade I oranges weigh more than 126 g, grade III oranges weigh less than 60 g. All remaining oranges are classed grade II. Calculate the percentage of oranges in each grade. Grade I, II and III oranges sell respectively for 12p, 8p and 4p each. Calculate the expected receipts from the sale of 80 000 oranges. (L)

10. Tests made on 2 types of electric light bulb show the following: Type A, lifetime distributed normally with an average life of 1150 hours and a standard deviation of 30 hours. Type B, long-life bulb, average lifetime of 1900 hours, with standard deviation of 50 hours. (i) What percentage of bulbs of type A could be expected to have a life of more than 1200 hours? (ii) What percentage of type B would you expect to last longer than 1800 hours? (iii) What lifetime limit would you estimate would contain the central 80% of the production of type A? (SU)

11. Batteries for a transistor radio have a mean life under normal usage of 160 hours, with a standard deviation of 30 hours. Assuming the battery life follows a normal distribution, (a) calculate the percentage of batteries which have a life between 150 hours and 180 hours; (b) calculate the range, symmetrical about the mean, within which 75% of the battery lives lie; (c) if a radio takes four of these batteries and requires all of them to be working, calculate the probability that the radio will run for at least 135 hours. (O&C)

12. Adult men in a certain country have heights which are normally distributed with a mean of 1·81 m and a standard deviation of 0.05 m.
(a) Estimate the number of men in a random sample of 100 men whose heights, when measured to the nearest 0·01 m, exceed 1·90 m.
(b) Find the probability that a man selected at random will have a height, when measured to the nearest 0·01 m, less than 1·72 m.
(c) Find the height which is exceeded by 90% of the men. (L)

13. Find the mean and variance of the normally distributed variable X in each of the following cases:
(a) $P(X > 3) = 0{\cdot}5$, $P(3 < X < 5) = 0{\cdot}1$.
(b) $P(X > 20) = P(X < 10) = 0{\cdot}05$.
(c) $P(X < 60) = 0{\cdot}01$, $P(X > 90) = 0{\cdot}3$.
(d) $P(X > 10) = 0{\cdot}1$, $P(9 < X < 10) = 0{\cdot}2$.

14. A machine is producing curtain rails whose lengths are normally distributed with mean 2·4 m. If 5% of the rails are longer than 2·42 m, find the standard deviation of the distribution. If the machine is adjusted so that 95% of the rails have lengths between 2·38 m and 2·42 m, find the new standard deviation of the distribution.

15. In an examination 30% of the candidates fail and 10% achieve distinction. Last year the pass mark (out of 200) was 84 and the minimum mark required for a distinction was 154. Assuming that the marks of the candidates were normally distributed, estimate the mean mark and the standard deviation. (O&C)

16. A random variable R has the binomial distribution $B(n, p)$. In each of the following cases find both the probability of r successes and the normal approximation to it.
(a) $n = 10$, $p = \frac{1}{2}$, $r = 6$,
(b) $n = 20$, $p = \frac{1}{2}$, $r = 12$,
(c) $n = 50$, $p = \frac{1}{2}$, $r = 30$,
(d) $n = 20$, $p = \frac{1}{4}$, $r = 4$,
(e) $n = 50$, $p = \frac{1}{4}$, $r = 10$,
(f) $n = 100$, $p = \frac{1}{4}$, $r = 20$.

17. Estimate the probability of obtaining more than 60 heads in 100 tosses of an unbiased coin.

18. In a large consignment of peaches 20% of the fruit are bruised. Estimate the probability of fewer than 10 bruised peaches in a box of 60.

19. Tutorial groups, each of 9 students, are picked at random from a large population of which two-thirds are men. (i) What is the probability that a group contains men only? (ii) Calculate the probability that the number of women in a group is either 1, 2, 3, 4 or 5. (iii) Compare this with the probability that a normal variable with the same mean and variance as the distribution appropriate to (ii) should lie between 0·5 and 5·5. (AEB 1979)

20. In a particular community the probability of a man having brown eyes is 2/5. If a random sample of 6 men is to be taken, find the probability that (a) exactly 3 men, (b) at least 3 men will have brown eyes. A second random sample of 100 men is to be taken. Use the normal distribution as an approximation to the binomial distribution, to estimate to 3 decimal places the probabilities that (a) at least half the men have brown eyes and (b) the number of men with brown eyes is between 30 and 45 inclusive. (L)

21. It is known that 30% of an apple crop has been attacked by insects. A random sample of 150 is selected from the crop. Assuming that the distribution of the

number of damaged apples in random samples of 150 may be approximated to by a normal distribution, estimate the probability that (a) more than half the sample is damaged, (b) less than 10% is damaged, (c) the number of damaged apples in the sample lies between 35 and 50, inclusive. (L)

22. In Urbania selection for the Royal Flying Corps (RFC) is by means of an aptitude test based on a week's intensive military training. It is known that the scores of potential recruits on this test follow a normal distribution with mean 45 and standard deviation 10. (i) What is the probability that a randomly chosen recruit will score between 40 and 60? (ii) What percentage of the recruits is expected to score more than 30? (iii) In a particular year 100 recruits take the test. Assuming that the pass mark is 50, calculate the probability that less than 35 recruits qualify for the RFC. (AEB 1978)

16.5 The Poisson distribution

The *Poisson*† *distribution* was originally derived as a limit of the binomial distribution. It is found to provide a useful model for distributions in which the variable is the number of occurrences of a particular event in a given interval of time or space. Some examples of variables which may have a Poisson distribution are: the number of customers entering a shop in an hour, the number of claims received by an insurance company in a day, the number of flaws in a given length of copper wire, the number of earthworms in a given area of land.

To demonstrate the way in which the Poisson distribution arises as a limit of the binomial distribution, we consider the number of telephone calls received in an hour by a small business. Assuming that the average number of calls received per hour is 4, we can construct a simple binomial model by dividing each hour into 12 five-minute "trials" and taking the probability of a call in any five-minute interval to be 1/3. However, since the firm may receive more than one call in a five-minute period, this model is not entirely satisfactory. It would be better to regard an hour as 60 one-minute trials, taking the probability of a call in any one-minute interval to be 1/15. Remembering the speed with which a modern switchboard can process calls, we should perhaps reduce the time interval to 10 seconds or even to 2 seconds. The given table shows that we are producing a sequence of binomial distributions with increasing numbers of trials but constant mean.

Interval	n	p	Mean	Variance
5 minutes	12	$\frac{1}{3}$	4	$4 \times \frac{2}{3} \approx 2{\cdot}67$
1 minute	60	$\frac{1}{15}$	4	$4 \times \frac{14}{15} \approx 3{\cdot}73$
10 seconds	360	$\frac{1}{90}$	4	$4 \times \frac{89}{90} \approx 3{\cdot}96$
2 seconds	1800	$\frac{1}{450}$	4	$4 \times \frac{449}{450} \approx 3{\cdot}99$

The limit of this sequence is the Poisson distribution with mean and variance 4.

†*Poisson, Siméon Denis* (1781–1840) French mathematician. His main contributions were to the theory of electricity and magnetism, but in 1837 he published *Recherches sur la probabilité des jugements*, dealing with probability distributions.

We obtain the general form of the Poisson distribution by considering the limit of the binomial distribution $B(n, p)$ as n tends to infinity in such a way that the mean $\mu = np$ remains constant.

$$P(X = r) = \binom{n}{r} p^r (1-p)^{n-r}$$

$$= \frac{n(n-1)\dots(n-r+1)}{r!}\left(\frac{\mu}{n}\right)^r\left(1-\frac{\mu}{n}\right)^{n-r}$$

$$= \frac{n}{n}\left(\frac{n-1}{n}\right)\dots\left(\frac{n-r+1}{n}\right)\frac{\mu^r}{r!}\left(1-\frac{\mu}{n}\right)^n\left(1-\frac{\mu}{n}\right)^{-r}$$

$$= \left(1-\frac{1}{n}\right)\dots\left(1-\frac{r-1}{n}\right)\frac{\mu^r}{r!}\left(1-\frac{\mu}{n}\right)^n \Big/ \left(1-\frac{\mu}{n}\right)^r$$

$$\therefore \quad \lim_{n\to\infty} P(X = r) = \frac{\mu^r}{r!}\lim_{n\to\infty}\left(1-\frac{\mu}{n}\right)^n$$

It can be shown that

$$\lim_{n\to\infty}\left(1-\frac{\mu}{n}\right)^n = e^{-\mu}$$

Thus, as $n \to \infty$, $P(X = r) \to \dfrac{\mu^r}{r!}e^{-\mu}$.

This result leads to the following definition:

> A discrete random variable X is said to have a Poisson distribution with parameter μ, where $\mu > 0$, if
>
> $$P(X = r) = \frac{\mu^r}{r!}e^{-\mu} \quad \text{for} \quad r = 0, 1, 2, \dots$$

Considering again the limit of the distribution $B(n, p)$ as $n \to \infty$, we see that if the mean $\mu = np$ remains constant, then $p \to 0$ and the variance $\sigma^2 = np(1-p)$ tends to μ. This suggests that the Poisson distribution has mean μ and variance μ. These results can proved either directly or by using the appropriate probability generating function. For instance:

$$E(X) = \sum_{r=0}^{\infty} r\left(\frac{\mu^r}{r!}e^{-\mu}\right) = \mu e^{-\mu}\sum_{r=1}^{\infty}\frac{\mu^{r-1}}{(r-1)!}$$

$$= \mu e^{-\mu}\left(1+\mu+\frac{\mu^2}{2!}+\frac{\mu^3}{3!}+\dots\right) = \mu e^{-\mu}\times e^{\mu} = \mu.$$

[For other proofs see Exercise 16.5 questions 12 and 13.]

It follows from this discussion that the Poisson distribution is a suitable model for the distribution of the number of events occurring in a specified interval of continuous time or space, provided that

(a) the events occur at random and independently, but may not occur simultaneously;
(b) the expected number of events in any interval is proportional to the size of the interval.

Example 1 A small business receives an average of 4 telephone calls per hour. Use the appropriate Poisson distribution to find the probability that (a) 7 calls are received in a two hour period, (b) up to 3 calls are received in an interval of 30 minutes.

(a) The expected number of calls in 2 hours is 8.
Thus if X is the number of calls received in 2 hours, X has a Poisson distribution with parameter 8

$$\therefore \quad P(X = 7) = \frac{8^7}{7!}e^{-8} \approx 0{\cdot}140.$$

Hence the probability that 7 calls are received in a two hour period is 0·140.

(b) The expected number of calls in 30 minutes is 2.
Thus if Y is the number of calls received in 30 minutes, Y has a Poisson distribution with parameter 2

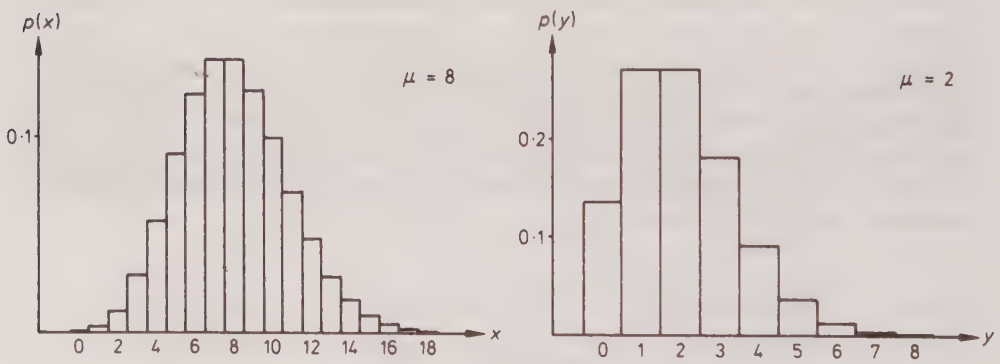
$$\therefore \quad P(Y \leqslant 3) = P(Y = 0) + P(Y = 1) + P(Y = 2) + P(Y = 3)$$

$$= e^{-2} + 2e^{-2} + \frac{2^2}{2!}e^{-2} + \frac{2^3}{3!}e^{-2}$$

$$= (1 + 2 + 2 + \tfrac{4}{3})e^{-2} \approx 0{\cdot}857.$$

Hence the probability that up to 3 calls are received in 30 minutes is 0·857.

The following diagrams show the Poisson distributions used in Example 1.



We have seen that the Poisson distribution with parameter $\mu = np$ is a limit of the binomial distribution $B(n, p)$. This means that under certain conditions a Poisson distribution can be used as an approximation to a binomial distribution with the same mean. In practice a Poisson approximation gives fairly accurate results when $n > 50$, $p < 0{\cdot}1$ and $np \leqslant 5$. It may also produce acceptable approximations to binomial distributions which do not satisfy these conditions. However, for a

binomial distribution such that both np and $n(1-p)$ are greater than 5, a normal approximation is usually more appropriate.

It can be shown that as μ tends to infinity, the distribution of the standardised Poisson variable $(X-\mu)/\sqrt{\mu}$ approaches the standard normal distribution $N(0, 1)$. It follows that for large μ, the normal distribution $N(\mu, \mu)$ can be used as an approximation to the Poisson distribution with parameter μ. Fairly good approximations are obtained for $\mu > 5$, but for accurate results it is necessary to have $\mu > 20$.

Exercise 16.5

1. In a school laboratory an average of 8 test-tubes are broken in 20 working days. Find the probability that (a) 5 test-tubes will be broken in the next 10 days; (b) fewer than 2 test-tubes will be broken in the next 3 days.

2. The number of heavy vehicles driving past a certain historic building in an hour is assumed to have a Poisson distribution with mean 3. Find the probability that (a) in the next hour more than 3 heavy vehicles will drive by; (b) in the next two hours fewer than 5 heavy vehicles will drive by.

3. A textile manufacturer estimates that there is an average of 2 faults in every 100 metres of fabric he produces. Find the probability that there is at least 1 fault in a 10-metre length of fabric. Write down an expression for the probability that there is no fault in an n-metre length of fabric. What is the greatest integer n such that this probability is greater than 0·9?

4. The number of plants of a certain species found in one square metre of woodland is assumed to have a Poisson distribution with mean 0·003. Find the probability of observing at least 2 specimens in an area of 400 m^2. A botanist is more than 99% certain that he will find at least one specimen in an area of n m^2. Find the least possible value of the integer n.

In questions 5 to 8 draw up a table showing the values, correct to 3 d.p., of the probabilities $p(0), p(1), p(2), \ldots$ for (a) the binomial distribution $B(n, p)$; (b) the Poisson distribution with parameter $\mu = np$; (c) the normal approximation to (a); (d) the normal approximation to (b). Comment briefly on your results.

5. $n = 20, p = 0{\cdot}5$.

6. $n = 20, p = 0{\cdot}1$.

7. $n = 50, p = 0{\cdot}1$.

8. $n = 100, p = 0{\cdot}08$.

9. It is found that on average 0·5% of the watches produced by a certain factory are faulty. Estimate the probability that (a) in a batch of 200 watches there are no faulty watches; (b) in a batch of 500 watches more than 2 are faulty.

10. On average 1 out of 50 patients treated with a certain drug suffer side effects. Estimate the probability that (a) if 80 patients take the drug then exactly 2 will have side effects, (b) if 200 patients take the drug then at least 4 will have side effects.

11. In a certain experiment the probability of success in any trial is 0·03. Use appropriate approximations to find the probabilities of (a) at least 2 successes in 50 trials, (b) at least 4 successes in 100 trials, (c) at least 20 successes in 500 trials.

12. Use the identity $r^2 \equiv r(r-1) + r$ to show that

$$\sum_{r=0}^{\infty} \frac{r^2 x^r}{r!} = x^2 e^x + x e^x.$$

Hence show that the Poisson distribution with parameter μ has variance μ.

13. Show that the probability generating function for the Poisson distribution with parameter μ is given by $G(t) = e^{\mu(t-1)}$. Hence show that the distribution has mean μ and variance μ.

14. The number of road accidents at a certain traffic roundabout has been found to follow a Poisson distribution with a mean of 0·8 accidents per week. Calculate, correct to 2 significant figures, the probabilities that (i) there will be at least 2 accidents in a particular week, (ii) there will be exactly 3 accidents in a particular three-week period. After road works have been carried out, the traffic is observed for a period of 8 weeks and in that time only 1 accident occurs. Calculate the probability of at most 1 accident in 8 weeks if the mean is still 0·8 accidents per week. Comment briefly on this result. (C)

15. (a) I recorded the number of phone calls I received over a period of 150 days:

No. of calls	0	1	2	3	4
No. of days	51	54	36	6	3

(i) Find the average number of calls per day. (ii) Calculate the frequencies of the comparable Poisson distribution.

(b) A firm selling electrical components packs them in boxes of 60. On average 2% of the components are faulty. What is the chance of getting more than 2 defective components in a box? (Use the Poisson distribution.) (SU)

16. An aircraft has 116 seats. The airline has found, from long experience, that on average 2·5% of people with tickets for a particular flight do not arrive for that flight. If the airline sells 120 seats for a particular flight, determine, using a suitable approximation, the probability that more than 116 people arrive for that flight. Determine also the probability that there are empty seats on the flight. (C)

17. A number of different types of fungi are distributed at random in a field. Eighty per cent of these fungi are mushrooms, and the remainder are toadstools. Five per cent of the toadstools are poisonous. A man who cannot distinguish between mushrooms and toadstools, wanders across the field and picks a total of 100 fungi. Determine, correct to 2 significant figures, using approximations, the probability that the man has picked (i) at least 20 toadstools, (ii) exactly two poisonous toadstools. (C)

18. A new telephone directory is to be published, and before publication takes place the entries have to be checked for misprints and any necessary corrections made. Previous experience suggests that, on average, 0·1% of the entries will require correction, and that entries requiring correction will be randomly distributed. The directory will contain 240 000 entries altogether, printed on 800 pages with 300 entries per page. There are two methods of making corrections: for method A the costs are £C per page containing one entry requiring correction and £3C per page containing two or more entries requiring correction; for method B the costs are £2C per page containing one or more entries requiring correction. Calculate the expected costs of correction for each method. (C)

19. In a certain office the number of reams of typing paper used per month has a Poisson distribution with mean 25. Estimate the probability that more than 25 reams of paper will be used in the next month. Given that it is more than 95% certain that there is enough paper in the stationery cupboard to last a month, find the minimum number of reams of paper in the cupboard.

20. Show that the variance of a Poisson distribution is equal to the mean. The number of bacteria in 1 ml of inoculum has a Poisson distribution with mean 2·0. If at least 3 bacteria are needed for a dose to be infective, find the probability that a dose of 1 ml will cause infection. Find approximate limits, symmetrical about the mean, between which lies 95% of the distribution of the number of bacteria in 100 ml of inoculum. (O&C)

16.6 The algebra of expectation and variance

The expectation $E(X)$ and the variance $\mathrm{Var}(X)$ of a random variable X were defined earlier in the chapter.

$$E(X) = \mu = \begin{cases} \sum x_r p(x_r), & \text{for discrete } X \\ \displaystyle\int_{-\infty}^{\infty} x f(x)\,dx, & \text{for continuous } X \end{cases}$$

$$\mathrm{Var}(X) = \sigma^2 = \begin{cases} \sum (x_r - \mu)^2 p(x_r), & \text{for discrete } X \\ \displaystyle\int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx, & \text{for continuous } X \end{cases}$$

If a new random variable $g(X)$ is defined in terms of X by a formula such as $g(X) = 2X + 3$ or $g(X) = X^2$, then the expectation of $g(X)$ is given by:

$$E[g(X)] = \begin{cases} \sum g(x_r) p(x_r), & \text{for discrete } X \\ \displaystyle\int_{-\infty}^{\infty} g(x) f(x)\,dx & \text{for continuous } X \end{cases}$$

If X is a discrete variable which takes the values $x_1, x_2, \ldots$ then we can show that this formula for $E[g(X)]$ is consistent with the definition of $E(X)$ by letting $Y = g(X)$.

(i) If Y takes distinct values $y_1 = g(x_1)$, $y_2 = g(x_2), \ldots$, then

$$P(Y = y_r) = P\big(g(X) = g(x_r)\big) = P(X = x_r) = p(x_r),$$

$$\therefore \quad E[g(X)] = E(Y) = \sum_{\text{all } r} y_r P(Y = y_r) = \sum_{\text{all } r} g(x_r)p(x_r).$$

(ii) If the values $g(x_1), g(x_2), \ldots$ are not all distinct then we have relationships such as $y_k = g(x_i) = g(x_j) = \ldots$. In these cases

$$P(Y = y_k) = P(X = x_i) + P(X = x_j) + \ldots = p(x_i) + p(x_j) + \ldots$$

$$\therefore \quad y_k P(Y = y_k) = y_k p(x_i) + y_k p(x_j) + \ldots$$
$$= g(x_i)p(x_i) + g(x_j)p(x_j) + \ldots$$

$$\therefore \quad E[g(X)] = E(Y) = \sum_{\text{all } k} y_k P(Y = y_k) = \sum_{\text{all } r} g(x_r)p(x_r).$$

Thus the formula for $E[g(X)]$ is valid for discrete X. We will assume that it is also valid for continuous X.

Example 1 A discrete random variable X takes the values 0, 1, 2, 3 and $p(0) = 0{\cdot}1$, $p(1) = p(3) = 0{\cdot}2$, $p(2) = 0{\cdot}5$. Find $E(X^2 - X)$.

$$E(X^2 - X) = \sum (x^2 - x)p(x)$$
$$= (0^2 - 0)0{\cdot}1 + (1^2 - 1)0{\cdot}2 + (2^2 - 2)0{\cdot}5 + (3^2 - 3)0{\cdot}2$$
$$= 0 + 0 + 2 \times 0{\cdot}5 + 6 \times 0{\cdot}2 = 2{\cdot}2.$$

Example 2 Find $E(X^3)$ for the continuous random variable X which has probability density function $f(x) = 2(1 - x)$ for $0 \leqslant x \leqslant 1$ and $f(x) = 0$ otherwise.

$$E(X^3) = \int_0^1 x^3 f(x)\,dx = \int_0^1 x^3 . 2(1 - x)\,dx$$

$$= 2\int_0^1 (x^3 - x^4)\,dx = 2\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1 = 2\left(\frac{1}{4} - \frac{1}{5}\right) = 0{\cdot}1.$$

Two useful properties of expectation follow from the definition of $E[g(X)]$.

> If X is a random variable, discrete or continuous,
> (i) $E(aX + b) = aE(X) + b$, for constants a and b.
> (ii) $E[g(X) + h(X)] = E[g(X)] + E[h(X)]$.

We give here the proof of (i) for discrete X. The remaining proofs are left as an exercise.

$$E(aX + b) = \sum (ax_r + b)p(x_r)$$
$$= \sum \{ax_r p(x_r) + bp(x_r)\} = a\sum x_r p(x_r) + b\sum p(x_r)$$

But $\quad \sum x_r p(x_r) = E(X) \quad$ and $\quad \sum p(x_r) = 1$

Hence $\quad E(aX + b) = aE(X) + b.$

It is now possible to express the variance of a random variable in terms of expected values:

$$\text{Var}(X) = E[(X - \mu)^2], \quad \text{where} \quad \mu = E(X)$$

Thus, using results (i) and (ii):

$$\text{Var}(X) = E(X^2 - 2\mu X + \mu^2) = E(X^2) - 2\mu E(X) + \mu^2 = E(X^2) - \mu^2$$

i.e.
$$\boxed{\text{Var}(X) = E(X^2) - [E(X)]^2.}$$

Hence if a and b are constants,

$$\begin{aligned}\text{Var}(aX + b) &= E[(aX + b)^2] - [E(aX + b)]^2 \\ &= E(a^2X^2 + 2abX + b^2) - [aE(X) + b]^2 \\ &= a^2E(X^2) + 2abE(X) + b^2 - a^2[E(X)]^2 - 2abE(X) - b^2 \\ &= a^2E(X^2) - a^2[E(X)]^2 = a^2\{E(X^2) - [E(X)]^2\}\end{aligned}$$

i.e.
$$\boxed{\text{Var}(aX + b) = a^2\,\text{Var}(X)}$$

To develop the theory of expectation further we must now consider the distribution of two discrete random variables X and Y with probability functions $p_X(x)$ and $p_Y(y)$ respectively. Let X take the values $x_1, x_2, \ldots, x_i, \ldots$ and let Y take the values $y_1, y_2, \ldots, y_j, \ldots$

The *joint probability distribution* of X and Y is the set of all probabilities $p(x_i, y_j)$, where $p(x_i, y_j) = P(X = x_i, Y = y_j)$.

For example if the possible values of X are 0, 1, 2 and the values of Y are 0, 1, 2, 3, then the joint distribution will be a set of 12 probabilities, which can be displayed in a two-way table as follows:

X \ Y	0	1	2	3	Total
0	$p(0,0)$	$p(0,1)$	$p(0,2)$	$p(0,3)$	$p_X(0)$
1	$p(1,0)$	$p(1,1)$	$p(1,2)$	$p(1,3)$	$p_X(1)$
2	$p(2,0)$	$p(2,1)$	$p(2,2)$	$p(2,3)$	$p_X(2)$
Total	$p_Y(0)$	$p_Y(1)$	$p_Y(2)$	$p_Y(3)$	1

As indicated in the table, the row totals give the values of $p_X(x)$. For instance

$$p(0,0) + p(0,1) + p(0,2) + p(0,3) = p_X(0).$$

Similarly the column totals are the values of $p_Y(y)$. Since these probabilities appear

in the margins, they are called the *marginal distributions* of X and Y. Using this table we could obtain the probability distribution of a random variable such as $X + Y$. We see that $P(X + Y = 0) = p(0, 0)$, $P(X + Y = 1) = p(1, 0) + p(0, 1)$, and so on. However, it is possible to obtain general formulae for the expected values of $X + Y$ and other functions of X and Y.

Taking into account all ordered pairs (x_i, y_j):

$$\begin{aligned} E(X + Y) &= \sum_i \sum_j (x_i + y_j) p(x_i, y_j) \\ &= \sum_i \left[\sum_j x_i p(x_i, y_j) \right] + \sum_j \left[\sum_i y_j p(x_i, y_j) \right] \\ &= \sum_i \left[x_i \sum_j p(x_i, y_j) \right] + \sum_j \left[y_j \sum_i p(x_i, y_j) \right] \\ &= \sum_i x_i p_X(x_i) + \sum_j y_j p_Y(y_j) = E(X) + E(Y). \end{aligned}$$

Similar arguments can be used to extend this result to any *linear combination* of random variables.

> If X and Y are random variables and a and b are constants, then
>
> $$E(aX + bY) = aE(X) + bE(Y).$$
>
> If $X_1, X_2, \ldots, X_n$ are random variables, then
>
> $$E(X_1 + X_2 + \ldots + X_n) = E(X_1) + E(X_2) + \ldots + E(X_n).$$

[These results are valid for both discrete and continuous variables, but in the continuous case the proofs require integration methods not covered in this course.]

If random variables X and Y are *independent*, then

$$P(X = x_i, Y = y_j) = P(X = x_i) \times P(Y = y_j)$$

i.e.

$$p(x_i, y_j) = p_X(x_i) p_Y(y_j)$$

$$\begin{aligned} E(XY) &= \sum_i \sum_j x_i y_j p(x_i, y_j) \\ &= \sum_i \left[\sum_j x_i y_j p_X(x_i) p_Y(y_j) \right] \\ &= \sum_i \left[x_i p_X(x_i) \sum_j y_j p_Y(y_j) \right] \\ &= \left[\sum_i x_i p_X(x_i) \right] \left[\sum_j y_j p_Y(y_j) \right] = E(X) E(Y). \end{aligned}$$

$\therefore$

> If X and Y are independent random variables, then
>
> $$E(XY) = E(X)E(Y).$$

We are now ready to consider the variance of a linear combination of random variables. In particular:

$$\begin{aligned}\mathrm{Var}(X+Y) &= E[(X+Y)^2] - [E(X+Y)]^2 \\ &= E(X^2 + 2XY + Y^2) - [E(X) + E(Y)]^2 \\ &= E(X^2) + 2E(XY) + E(Y^2) - [E(X)]^2 - 2E(X)E(Y) - [E(Y)]^2 \\ &= \{E(X^2) - [E(X)]^2\} + 2\{E(XY) - E(X)E(Y)\} \\ &\quad + \{E(Y^2) - [E(Y)]^2\}\end{aligned}$$

The quantity $E(XY) - E(X)E(Y)$ is called the *covariance* of X and Y, and is denoted by $\mathrm{Cov}(X, Y)$.
Thus, in general:

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + 2\,\mathrm{Cov}(X, Y) + \mathrm{Var}(Y).$$

However, if the variables X and Y are independent,

$$\mathrm{Cov}(X, Y) = E(XY) - E(X)E(Y) = 0$$

$$\therefore \quad \mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y).$$

As before, this basic result can be extended:

If X and Y are independent random variables and a and b are constants, then

$$\mathrm{Var}(aX + bY) = a^2\,\mathrm{Var}(X) + b^2\,\mathrm{Var}(Y).$$

Two important special cases are given by:

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X-Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$$

If $X_1, X_2, \ldots, X_n$ are independent random variables,

$$\mathrm{Var}(X_1 + X_2 + \ldots + X_n) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \ldots + \mathrm{Var}(X_n).$$

Example 3 In the ice-cream dessert served in a motorway cafeteria, the masses of the portions of ice-cream are distributed about a mean of 80 g with standard deviation 6 g. The fruit topping has mean mass 20 g and standard deviation 4 g. If the cost of 100 g of ice-cream is 5 p and the cost of 100 g of topping is 10 p, find the mean and standard deviation of the cost of a dessert.

Let the random variables X and Y represent the masses in grammes of portions of ice-cream and topping respectively. If Z is the total cost in pence of the dessert, then

$$Z = \tfrac{5}{100}X + \tfrac{10}{100}Y = \tfrac{1}{20}X + \tfrac{1}{10}Y$$

$$\therefore \quad E(Z) = \tfrac{1}{20}E(X) + \tfrac{1}{10}E(Y) = \tfrac{80}{20} + \tfrac{20}{10} = 6,$$

$$\text{Var}(Z) = (\tfrac{1}{20})^2 \text{Var}(X) + (\tfrac{1}{10})^2 \text{Var}(Y)$$
$$= \frac{6^2}{20^2} + \frac{4^2}{10^2} = \frac{9+16}{100} = \frac{1}{4}$$

Hence the cost of a dessert has mean 6 p and standard deviation $\frac{1}{2}$ p.

We now state without proof two important theorems concerning linear combinations of normal random variables.

If X and Y are independent normal random variables with distributions $N(\mu_x, \sigma_x^2)$ and $N(\mu_y, \sigma_y^2)$ respectively, then $Z = aX + bY$, where a and b are constants, is also a normal random variable and has mean $a\mu_x + b\mu_y$, variance $a^2\sigma_x^2 + b^2\sigma_y^2$.

Sums and differences of independent normal variables are of particular interest:

$Z = X + Y$ has the distribution $N(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)$,
$Z = X - Y$ has the distribution $N(\mu_x - \mu_y, \sigma_x^2 + \sigma_y^2)$.

If $X_1, X_2, \ldots, X_n$ are independent observations of a random variable with distribution $N(\mu, \sigma^2)$, then $Z = X_1 + X_2 + \ldots + X_n$ is normally distributed with mean $n\mu$, variance $n\sigma^2$.

Example 4 Brand X beefburgers have masses normally distributed about a mean of 65 g with variance 15 g^2. The masses of Brand Y beefburgers are normally distributed with mean 100 g, variance 25 g^2. Find the probability that the mass of 3 Brand X beefburgers is greater than the mass of 2 Brand Y beefburgers.

[It is tempting to think that what we need to find here is $P(3X - 2Y > 0)$. However, in the expression $3X - 2Y$, $3X$ represents 3 times the mass of one beefburger, not the mass of three beefburgers.]

Let $D = (X_1 + X_2 + X_3) - (Y_1 + Y_2)$, where X_1, X_2, X_3 represent the masses (in grammes) of 3 Brand X beefburgers and Y_1, Y_2 represent the masses of 2 Brand Y beefburgers, then

$$E(D) = E(X_1) + E(X_2) + E(X_3) - E(Y_1) - E(Y_2)$$
$$= 3 \times 65 - 2 \times 100 = -5$$
$$\text{Var}(D) = \text{Var}(X_1) + \text{Var}(X_2) + \text{Var}(X_3) + \text{Var}(Y_1) + \text{Var}(Y_2)$$
$$= 3 \times 15 + 2 \times 25 = 95$$

Letting Z be the standardised normal variable $(D + 5)/\sqrt{95}$,

$$P(D > 0) = P(Z > 5/\sqrt{95})$$
$$= P(Z > 0{\cdot}5130)$$
$$= 1 - \Phi(0{\cdot}5130) = 1 - 0{\cdot}6961 = 0{\cdot}3039$$

$\therefore$ the required probability is approximately 0·304.

Two further results are sometimes useful.

If independent random variables X and Y have distributions $B(m, p)$ and $B(n, p)$, then $X + Y$ has distribution $B(m + n, p)$.

If independent random variables X and Y have Poisson distributions with means λ and μ, then $X + Y$ has a Poisson distribution with mean $\lambda + \mu$.

Methods of proof are indicated in Exercise 16.6 questions 23 and 25.

Exercise 16.6

1. A discrete random variable X takes the values 1, 2, 3, 4, 5 with probabilities $p(1) = p(2) = 0{\cdot}1, p(3) = 0{\cdot}15, p(4) = 0{\cdot}4, p(5) = 0{\cdot}25$. Find $E(X)$, $E(X^2)$ and $E(1/X)$. Hence find $E(2X^2 - 1)$ and $E[(X + 1)^2/X]$.

2. A discrete random variable X takes the values $-1, 0, 1, 2$ with probabilities $p(-1) = 0{\cdot}2, p(0) = 0{\cdot}3, p(1) = 0{\cdot}4, p(2) = 0{\cdot}1$. Find $E(X)$, $E(X^2)$ and $E(X^4)$. Hence find $\mathrm{Var}(X)$ and $\mathrm{Var}(X^2)$.

3. A continuous random variable X has probability density function

$$f(x) = \tfrac{1}{2}(3 - x) \text{ for } 1 \leqslant x \leqslant 3 \text{ and } f(x) = 0 \text{ otherwise.}$$

Find $E(X)$, $E(X^2)$ and $E(X^3)$. Hence find $E(3X^3 + 2X)$ and $\mathrm{Var}(X)$.

4. A continuous random variable X has probability density function

$$f(x) = 6x(1 - x) \text{ for } 0 \leqslant x \leqslant 1 \text{ and } f(x) = 0 \text{ otherwise.}$$

Find $E(X)$, $E(X^2)$, and $E(\sqrt{X})$. Hence find $E[X(X - 1)]$ and $\mathrm{Var}(\sqrt{X})$.

5. A random variable X takes the values 5, 8, 4, 7 with equal probability and an independent variable Y takes the values 4, 7, 1 with equal probability. Find the mean and variance of the following:

(a) X (b) Y, (c) $4X + 3$, (d) $5 - 2Y$,
(e) $X + Y$, (f) $X - Y$, (g) $3X - Y$, (h) $2X + 3Y$.

6. Independent random variables X and Y are such that $E(X) = 10$, $\mathrm{Var}(X) = 9$ and $E(Y) = 7$, $\mathrm{Var}(Y) = 1$. Find (a) the mean and variance of $5X - 3Y + 2$, (b) the values of $E(X^2)$, $E(Y^2)$ and $E(X^2 + Y^2)$.

7. A factory produces nuts with masses distributed about a mean of 5 g with a standard deviation of 0·2 g. These are screwed on to bolts whose masses have mean

20 g and standard deviation 0·1 g. If each bolt is fitted with two nuts, find the mean and standard deviation of the resulting total mass.

8. In a can containing baked beans and 5 small sausages, the mass of each sausage is distributed about a mean of 20 g with standard deviation 2 g. The mean mass of the baked beans is 100 g with standard deviation 5 g. Find the mean mass of the total contents of a can and the standard deviation.

9.

X \ Y	0	1	2
1	0·06	0·12	0·02
2	0·15	0·3	0·05
3	0·09	0·18	0·03

The given table shows the joint probability distribution of two discrete random variables X and Y.

(a) Use the marginal distributions of X and Y to find $E(X)$, $\mathrm{Var}(X)$ and $E(Y)$, $\mathrm{Var}(Y)$.

(b) Are there any values of x and y for which

$$P(X = x, Y = y) \neq P(X = x) \times P(Y = y)?$$

Use your answer to determine whether or not X and Y are independent.

(c) Find the distribution of XY i.e. $P(XY = 0)$, $P(XY = 1)$, $P(XY = 2)$,... Hence determine whether or not $E(XY) = E(X)E(Y)$.

(d) Find the distribution of $X + Y$. Hence determine whether or not

$$\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y).$$

10. Repeat question 9 for the following joint distributions.

(i)

X \ Y	0	1	2	3
0	0·03	0·1	0.05	0·02
1	0·05	0	0·25	0·1
2	0·02	0·2	0·1	0·08

(ii)

X \ Y	1	2	3
0	0·05	0·2	0·05
1	0·2	0	0·2
2	0·05	0·2	0·05

11. A random variable X takes the values 0 and 2 each with probability 0·5. Find $E(X)$, $E(X^2)$, $E(X^3)$, $E(X^4)$. Use the formula $\mathrm{Var}(Y) = E(Y^2) - [E(Y)]^2$ to verify that (a) if $Y = 3X - 5$, $\mathrm{Var}(Y) = 9\,\mathrm{Var}(X)$,

(b) if $Y = X^2 + X$, $\mathrm{Var}(Y) \neq \mathrm{Var}(X^2) + \mathrm{Var}(X)$.

12. The daily demand, x kilogrammes, for fresh fish at a certain fishmonger's shop is a continuous random variable with probability density function f, where f is given by $f(x) = (30 - x)/450$, $0 \leqslant x \leqslant 30$, $f(x) = 0$, otherwise. The fishmonger receives a delivery of 20 kg of fresh fish each morning. He makes a profit of 40 pence on each kilogramme of fish sold on the day the fish is delivered; fish left unsold on the day it is

delivered is sold to a fertiliser manufacturer at a loss of 5 pence per kg. Show that on a day when x is less than 20, the fishmonger's profit is $(45x - 100)$ pence and that on a day when x is 20 or more, the fishmonger's profit is £8. Hence determine the fishmonger's expected daily profit from the sale of fresh fish. (JMB)

13 A sample of four mice is to be chosen at random from a litter of ten mice, of which six are male and four are female. Let X denote the number of male mice in the sample. Calculate the values of $P(X = r)$ for $r = 0, 1, 2, 3, 4$. Hence find the mean and the standard deviation of X. Deduce the mean and the standard deviation of the number of female mice in the sample. Independently of the above, a sample of four mice is to be chosen at random from another litter of ten mice, of which four are male and six are female. Calculate the mean and the standard deviation of the total number of male mice in the combined sample of eight mice. (W)

14. The joint probability distribution of two discrete random variables X and Y is displayed in the following table, the entries in the body of the table being $P(X = x, Y = y)$.

		x		
		0	1	2
y	0	0·1	0·1	0
	1	0·1	0·2	0·1
	2	0·1	0·2	0·1

(i) Find $E(X)$ and $E(Y)$. (ii) Determine the distribution of $Z = 2X - Y$ and verify that $E(Z) = 2E(X) - E(Y)$. (iii)Let (X_1, Y_1) and (X_2, Y_2) denote two independent observations of (X, Y). By using the distribution of Z, or otherwise, find the probability that

$$2(X_1 - X_2) = Y_1 - Y_2.$$ (W)

15. On a piece-work operation a company pays a bonus to any employee who processes in excess of 300 kg of raw material in a day. The daily amounts processed by two employees A and B are independent and normally distributed, the mean and standard deviation of the amounts processed by A being 291 kg and 9 kg, respectively, and of the amounts processed by B being 282 kg and 12 kg, respectively. Calculate, to two significant figures in each case, (i) the proportion of days on which both A and B will earn the bonus, (ii) the probability that the combined daily amount processed by A and B will exceed 600 kg, (iii) the proportion of days on which the amount processed by B will exceed that processed by A. (W)

16. The diameters of axles supplied by a factory have a mean value of 19·92 mm and a standard deviation of 0·05 mm. The inside diameters of bearings supplied by another factory have a mean of 20·04 mm and a standard deviation of 0·03 mm. What is the mean and standard deviation of the random variable defined to be the diameter of a bearing less the diameter of an axle? Assuming that both dimensions are normally distributed, what percentage of axles and bearings taken at random will not fit? (O&C)

17. The time of arrival of a bus at a bus stop varies in a normal distribution with a mean of 09.00 a.m. and a standard deviation of 2 minutes. Independently a second bus departs from this stop at a time which varies in a normal distribution with a mean of 09.01 a.m. (i.e. one minute past 09.00 a.m.) and a standard deviation of 1 minute. Find the probability that (a) the first bus arrives before the second bus leaves; (b) this happens on 5 given consecutive days. (O&C)

18. The weight of a large loaf of bread is a normal variable with mean 420 g and standard deviation 30 g. The weight of a small loaf of bread is a normal variable with mean 220 g and standard deviation 10 g. (i) Find the probability that 5 large loaves weigh more than 10 small loaves. (ii) Find the probability that the total weight of 5 large loaves and 10 small loaves lies between 4·25 kg and 4·4 kg. (C)

19. The mass of a cheese biscuit has a normal distribution with mean 6 g and standard deviation 0·2 g. Determine the probability that (i) a collection of twenty-five cheese biscuits has a mass of more than 149 g, (ii) a collection of thirty cheese biscuits has a mass of less than 180 g, (iii) twenty-five times the mass of a cheese biscuit is less than 149 g. The mass of a ginger biscuit has a normal distribution with mean 10 g and standard deviation 0·3 g. Determine the probability that a collection of seven cheese biscuits has a mass greater than a collection of four ginger biscuits. (C)

20. Four sprinters A, B, C and D each run 100 m. The times (in seconds) that they take may be regarded as independent observations from a normal distribution with mean 14 and standard deviation 0·2. The athlete E runs 400 m. The time (in seconds) that this athlete takes may be regarded as an observation from a normal distribution with mean 58 and standard deviation 1·0 and is independent of the times of the sprinters. Determine the probability that (i) the time taken by E is less than 3 seconds greater than the sum of the times taken by the four sprinters, (ii) the time taken by E is less than 4 times as great as the time taken by A. For the four sprinters A, B, C and D, find the probability that A is the fastest sprinter of the four. Find also the probability that C and D (in either order) are the two slowest sprinters. (C)

21. P is a fixed point on the circumference of a circle, centre O, of radius r and Q is a point on the circumference such that $\angle POQ = \theta$, where θ is a random variable with a rectangular distribution in $[0, 2\pi]$. Find the mean and median values for the length of the shorter arc PQ. The length of the chord PQ is X. Find $E(X)$ and show that $\mathrm{Var}(X) = 2r^2(1 - 8/\pi^2)$. Find also $P(X > r\sqrt{3})$. (C)

22. X and Y are independent random variables with means and variances μ_x, σ_x^2 and μ_y, σ_y^2 respectively, and you may assume that for any such independent variables $E(XY) = E(X)E(Y)$. Find expressions for (i) the expectation of the area of a square of side X, (ii) the expectation and variance of the area of a rectangle of sides X and Y, (iii) the expectation of the volume of a cuboid with a square base of side X and with height Y. (C)

23. Two independent random variables X and Y have Poisson distributions with means λ and μ respectively. Obtain an expression for $P(X = r, Y = n - r)$. Hence express $P(X + Y = n)$ as the sum of a series. Deduce that $X + Y$ has a Poisson distribution with mean $\lambda + \mu$.

24. Telephone calls reach a secretary independently and at random, internal ones at a mean rate of 2 in any 5 minute period, and external ones at a mean rate of 1 in any 5 minute period. Calculate the probability that there will be more than 2 calls in any period of 2 minutes. (O&C)

25. (a) Verify that if a random variable X has probability generating function $G_X(t)$, then $G_X(t) = E(t^X)$.
    (b) Two independent random variables X and Y have distributions $B(m, p)$ and $B(n, p)$ respectively. Write down expressions for $E(t^X)$ and $E(t^Y)$. Assuming that t^X and t^Y can be regarded as independent random variables, write down $E(t^X \times t^Y)$ and hence find the p.g.f. of the variable $Z = X + Y$. Deduce that Z has the distribution $B(m + n, p)$.

Exercise 16.7 (miscellaneous)

1. A random variable R takes the integer values $1, 2, \ldots, n$ each with probability $1/n$. Find the mean and variance of R. A pack of 15 cards bearing the numbers 1 to 15 is shuffled. Find the probability that the number on the top card is larger than that on the bottom card, giving reasons for your answer. If the sum of these two numbers is S, find (a) the probability that $S \leqslant 4$, (b) the expected value of S. (O&C)

2. The number X of radios sold per week by a certain shop is a binomially distributed random variable with $n = 10$ and $p = 0{\cdot}3$. The number Y of television sets sold per week by the same shop may be 0, 1 or 2 with $P(Y = 0) = 0{\cdot}6$, $P(Y = 1) = 0.3$, $P(Y = 2) = 0{\cdot}1$. The shopkeeper makes a profit of £2 on each radio sold and a profit of £20 on each television set sold. Find the shopkeeper's expected weekly profit from the sale of radios and television sets. (JMB)

3. (a) In a certain manufacturing process, it is known that approximately 10% of the items produced are defective. A quality control scheme is set up, by selecting 20 items out of a large batch, and rejecting the whole batch if 3 or more are defective. Find the probability that the batch is rejected.
    (b) Two boys, John and David, play a game with a die. The die will be thrown 4 times. David will give John £x if there is an odd number of sixes, otherwise John will give David £1. If the game is to be a fair one to both John and David, find the value of x. (SU)

4. The probability of there being X unusable matches in a full box of Surelite matches is given by $P(X = 0) = 8k$, $P(X = 1) = 5k$, $P(X = 2) = P(X = 3) = k$,

$P(X \geqslant 4) = 0$. Determine the constant k and the expectation and variance of X. Two full boxes of Surelite matches are chosen at random and the total number Y of unusable matches is determined. Calculate $P(Y > 4)$, and state the values of the expectation and variance of Y. (C)

5. Fergus Lightfingers, an educated thief, has broken into a house and has come upon a large safe. Lying upon the table is a bunch of k similar keys, only one of which will open the safe. He considers two possible strategies: (i) randomly select keys, one at a time *without replacement*, until successful; (ii) randomly select keys, one at a time *with replacement*, until successful. Let N, a random variable, be the number of keys including the successful one tried by Fergus in order to open the safe. Derive the probability distribution of N for both strategies. Calculate the expected value of N in each case. Use your statistical judgment in order to select the strategy that Fergus should use. (You may assume that $\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}$ for $|x| < 1$.) (AEB 1978)

6. A batsman's performance is described by the following probability distribution. For each ball that he receives he has probability $\frac{1}{2}$ of scoring no runs, probability 1/5 of scoring one run and otherwise he is equally likely to score two runs, four runs, or to be bowled out. In the last case he receives no further balls.

(i) Determine the probability that the batsman's total score is 0.
(ii) Given that the batsman is not bowled by a particular ball, calculate the expectation of his score from that ball.
(iii) Calculate the expectation of the number of balls that the batsman receives.
(iv) Deduce the expectation of the batsman's total score. (C)

7. In a series of N independent trials the probability of success in any trial is constant and equal to p. Show that the probability of r successes is given by

$$P(r) = \binom{N}{r} p^r (1-p)^{N-r}.$$

Find $P(3)$ if $N = 5$ and $p = 0{\cdot}2$. Write a flow diagram for a program to print values of $P(r)$ if $N = 25$ and $p = 0{\cdot}4$ for $r = 0(1)25$. (L)

8. In a certain factory 70% of the microprocessors produced are found to be imperfect and have to be discarded. Find the probability that, in a randomly chosen sample of 5 microprocessors produced, the number of perfect specimens will be (i) 0, (ii) 1, (iii) 2, (iv) 3, (v) 4, (vi) 5. Verify that the probability of more than 3 perfect specimens is less than 1/30. If a sample of 200 is taken from the same set of microprocessors, find the probability that there are at least 50 perfect specimens. (AEB 1980)

9. Derive the mean and variance of the binomial distribution. Mass production of miniature hearing aids is a particularly difficult process and so the quality of these products is monitored carefully. Samples of size six are selected regularly and tested for correct operation. The number of defectives in each sample is

recorded. During one particular week 140 samples are taken and the distribution of the number of defectives per sample is given in the following table.

Number of defectives per samples (x)	0	1	2	3	4	5	6
Number of samples with x defectives (f)	27	36	39	22	10	4	2

Find the frequencies of the number of defectives per sample given by a binomial distribution having the same mean and total as the observed distribution.

(AEB 1978)

10. Show that the probability generating function for the binomial distribution arising from n independent trials, in each of which the probability of a success is p, is $(1 - p + pt)^n$. Show that the probability generating function for the number of trials needed to obtain a success is $pt/(1 - t + pt)$, and find the mean and variance of this distribution.

When a gun first fires at a target the probability of a hit is 1/5. Firing continues until the first hit, after which the gun is resighted and the probability of a hit becomes 2/5. Find the probability generating function for the number of shots required to obtain 2 hits and hence find the probability that fewer than 5 shots will be needed.

(O&C)

11. A random variable X takes values in the intervals 0 to 1 and 2 to 3, and has probability density function given by

$$f(x) = \begin{cases} 1 - x & (0 \leqslant x \leqslant 1), \\ x - a & (2 \leqslant x \leqslant 3), \\ 0 & \text{elsewhere.} \end{cases}$$

Determine (i) the value of a, (ii) the expectation, μ, of X, (iii) the variance of X, (iv) the (cumulative) distribution function $F(x)$, (v) the probability that $|X - \mu|$ exceeds 1.

(C)

12. The random variable X is the distance, in metres, that an inexperienced tight-rope walker has moved along a given tight-rope before falling off. It is given that

$$P(X > x) = 1 - x^3/64, \quad 0 \leqslant x \leqslant 4.$$

(i) Show that $E(X) = 3$. (ii) Find the standard deviation, σ, of X. (iii) Show that $P(|X - 3| < \sigma) = \frac{69}{80}\sqrt{\frac{3}{5}}$. (C)

13. Find the mean and the variance of the distribution whose probability density function f is given by

$$\begin{aligned} f(x) &= x, && 0 \leqslant x < 1, \\ f(x) &= 2 - x, && 1 \leqslant x \leqslant 2, \\ f(x) &= 0, && \text{otherwise.} \end{aligned}$$

A rectangle is constructed with adjacent sides of lengths x cm and $(2 - x)$ cm where x is a random value from the above distribution. Find the probability that the

area of the rectangle will exceed $0{\cdot}75\,\text{cm}^2$. Also find the expected value of the area of the rectangle. (JMB)

14. The operational lifetime in hundreds of hours of a battery-operated minicalculator may be regarded as a continuous random variable having probability density function

$$f(x) = cx(10 - x), \qquad 5 \leqslant x \leqslant 10,$$
$$f(x) = 0, \qquad \text{otherwise.}$$

(a) Find the value of c and of the expected operational lifetime of such a minicalculator.

(b) The purchase price of such a minicalculator is £20 and its running cost (for batteries) amounts to 20 pence per hundred hours operation. Thus, the overall average cost in pence per hundred hours operation of a minicalculator whose operational lifetime is X hundred hours is given by $Y = 20 + (2000/X)$. (i) Evaluate $E(Y)$, the expected overall average cost per hundred hours. (ii) Find the probability that the overall average cost per hundred hours will exceed £2.70. (W)

15. A random variable X has (cumulative) distribution function

$$F(x) = \begin{cases} 0 & (x \leqslant 0) \\ \frac{1}{4}x^2 & (0 \leqslant x \leqslant 1) \\ \alpha x + \beta & (1 \leqslant x \leqslant 2) \\ \frac{1}{4}(5 - x)(x - 1) & (2 \leqslant x \leqslant 3) \\ 1 & (3 \leqslant x) \end{cases}$$

Find (i) the values of the constants α and β, (ii) the probability that $1{\cdot}5 \leqslant X \leqslant 2{\cdot}5$, (iii) the probability density function $f(x)$. Sketch the graph of $f(x)$, and hence, or otherwise, deduce the mean of X. Determine also the variance of X. (C)

16. A failure time, X, has density function $\lambda e^{-\lambda x}$ for $x \geqslant 0$, $\lambda > 0$. Find (a) the distribution function; (b) the mean and variance; (c) the median; (d) the time below which it is 95% certain that there will be no failure. (O)

17. Henri de Lade regularly travels from his home in the suburbs to his office in Paris. He always tries to catch the same train, the 08.05 from his local station. He walks to the station from his home in such a way that his arrival times form a normal distribution with mean 08.00 hours and standard deviation 6 minutes. (i) Assuming that his train always leaves on time, what is the probability that on any given day Henri misses his train? (ii) If Henri visits his office in this way 5 days each week and if his arrival times at the station each day are independent, what is the probability that he misses his train once and only once in a given week? (AEB 1980)

18. The score, S, gained by an expert rifleman with a single shot, is a random variable with the following probability distribution:

$$P(S = 8) = 0{\cdot}01, \quad P(S = 9) = 0{\cdot}29, \quad P(S = 10) = 0{\cdot}70.$$

(i) Use a normal approximation to determine the probability that the rifleman obtains six or more scores of 8 in a series of 900 independent shots. (ii) Find the expectation and variance of S. (iii) Use a normal approximation to determine the probability that the rifleman scores less than 96 with ten independent shots. (C)

19. The prices of houses on an estate agent's books are approximately normally distributed with a mean price of £18 000. Given that 90% of the houses are priced at less than £30 000, find the standard deviation of the house prices and the percentage of the houses with a price of less than £10 000. Find the mean price and the standard deviation of the prices when the price of each house is increased by (a) £2000, (b) 10%. (L)

20. The weights of grade A oranges are normally distributed with mean 200 g and standard deviation 12 g. Determine, correct to 2 significant figures, the probability that (i) a grade A orange weighs more than 190 g but less than 210 g, (ii) a sample of 4 grade A oranges weighs more than 820 g. The weights of grade B oranges are normally distributed with mean 175 g and standard deviation 9 g. Determine, correct to 2 significant figures, the probability that (iii) a grade B orange weighs less than a grade A orange, (iv) a sample of 8 grade B oranges weighs more than a sample of 7 grade A oranges. (C)

21. Mass-produced right circular cylindrical pipes have internal diameters that are normally distributed with a mean of 10 cm and a standard deviation of 0·4 cm. (i) Find the probability that a randomly chosen pipe will have an internal diameter greater than 10·3 cm. (ii) Find the expected number of pipes in a random sample of 100 pipes that have internal diameters in the range from 9·7 cm to 10·3 cm. (iii) Find the expected value of the internal cross-sectional area of a randomly chosen pipe; give your answer correct to three significant figures. (iv) Mass-produced pistons have diameters that are normally distributed with a mean of 9·9 cm and a standard deviation of 0·3 cm. Find the probability that a randomly selected piston will have a diameter less than the internal diameter of a randomly selected pipe. (W)

22. Two gauges are used to test the thickness of manufactured metal sheets. Over a long period it is found that 1·25% of the sheets will pass through the smaller gauge of 1·4 mm while 95·4% will pass through the larger gauge of 1·6 mm. Assuming that the distribution of sheet thickness is normal, find its mean and standard deviation. At the next stage of manufacture 2 sheets are clamped together to make a thick plate. Find the mean and the standard deviation of the thickness of these plates. If a plate will pass through a gauge of 3·2 mm but not through one of 2·8 mm, find the probability that both the sheets of which it is composed will pass through the 1·6 mm gauge but not through the 1·4 mm gauge. Find also the size of a gauge which will allow 99·5% of these plates to pass through. (O&C)

23. Next May, an ornithologist intends to trap one male cuckoo and one female cuckoo. The mass M of the male cuckoo may be regarded as being a normal random variable with mean 116 g and standard deviation 16 g. The mass F of the female cuckoo may be regarded as being independent of M and as being a normal random

variable with mean 106 g and standard deviation 12 g. Determine (i) the probability that the mass of the two birds together will be more than 230 g. (ii) the probability that the mass of the male will be more than the mass of the female. By considering $X = 9M - 16F$, or otherwise, determine the probability that the mass of the female will be less than nine-sixteenths of that of the male. Suppose that one of the two trapped birds escapes. Assuming that the remaining bird will be equally likely to be the male or the female, determine the probability that its mass will be more than 118 g. (C)

24. (a) The number of accidents notified in a factory per day over a period of 200 days gave rise to the following table:

No. of accidents	0	1	2	3	4	5
No. of days	127	54	14	3	1	1

(i) Calculate the mean number of accidents per day. (ii) Assuming that this situation can be represented by a suitable Poisson distribution, calculate the corresponding frequencies.

(b) Of items produced by a machine, approximately 3% are defective, and these occur at random. What is the probability that, in a sample of 144 items, there will be at least two which are defective? (SU)

25. Define a Poisson distribution giving the general probability formula; state the relationship between its mean and variance. The distribution of the number of vehicles observed passing under a motorway bridge in 100 successive intervals of 12 seconds, at a time when the traffic flow is 720 vehicles per hour, may be considered to be Poissonian. During how many of the intervals would one expect (i) no cars to have passed? (ii) 2 cars to have passed, (iii) more than 2 cars to have passed? Find the smallest value of N such that the probability of at most N cars in an interval is greater than 0·999. (AEB 1978)

26. Each play of a one-armed bandit machine has probability 0·04 of yielding a win. It may be assumed that this probability applies independently for each play of the machine. (a) Find the probability that in ten plays of the machine there will be (i) at least two wins, (ii) exactly two wins. (b) For a sequence of n plays of the machine write down an expression for the probability that there will be no win. Hence find the smallest number of plays of the machine for there to be a probability of at least 0·95 of winning at least once. (c) Use an appropriate method to find an approximate value for the probability that there will be six or fewer wins in 80 plays of the machine. (W)

27. Gnat larvae are distributed at random in pond water so that the number of larvae contained in a random sample of 10 cm^3 of pond water may be regarded as a random variable having a Poisson distribution with mean 0·2. Ten independent random samples, each of 10 cm^3, of pond water, are taken by a zoologist. Determine (correct to three significant figures) (i) the probability that none of the samples contain larvae, (ii) the probability that one sample contains a single larva and the remainder contain no larvae, (iii) the probability that one sample contains two or

more larvae and the remainder contain no larvae, (iv) the expectation of the total number of larvae contained in the ten samples, (v) the expectation of the number of samples containing no larvae. (C)

28. A prospector for gold examines exactly 800 pans of material every month. The contents of each pan may be assumed to be independent of the others with the probability of a pan containing gold being 0·005 for each pan. The prospector has a good month when 4 or more of the pans that he examines contain gold. Show that, to three significant figures, the probability that a randomly chosen month is a good month is 0·567. Determine the probability that (i) a randomly chosen period of four months contains more than two good months, (ii) a randomly chosen period of 24 months contains more than 12 good months. (C)

29. The number of organic particles suspended in a volume V cm^3 of a certain liquid follows a Poisson distribution with mean $0{\cdot}1V$. (a) Find the probabilities that a sample of 1 cm^3 of the liquid will contain (i) at least one organic particle, (ii) exactly one organic particle. (b) Use an appropriate approximate procedure to find the probability that a sample of 1000 cm^3 of the liquid will contain at least 90 organic particles. (c) The liquid is sold in vials, each vial containing 10 cm^3 of the liquid. The vials are dispatched for sale in boxes, each box containing 100 vials. Find the probability that a vial will contain at least one organic particle. Hence find the mean and the standard deviation of the number of vials per box of 100 vials that contain at least one organic particle. (W)

30. When a number is rounded off to its nearest integer value, the rounding off error X may be regarded as a random variable which is uniformly distributed over the interval $(-\frac{1}{2}, \frac{1}{2})$. Find the variance of X. If n numbers are rounded off to their nearest integer values and then summed, the total error in the sum is given by $Y = X_1 + X_2 + \ldots + X_n$, where $X_1, X_2, \ldots, X_n$ are the individual errors in the n numbers, which may be assumed to be independent and uniformly distributed over $(-\frac{1}{2}, \frac{1}{2})$. Write down the mean and the variance of Y in terms of n. Suppose n is large enough to justify approximating the distribution of Y by a normal distribution. Show that there is a probability in excess of 0·9 that Y will be numerically less than $\frac{1}{2}\sqrt{n}$. Find the largest integer value of n for which there is a probability of at least 0·6 that Y will be numerically less than 1·0. (W)

31. The joint probability function of two discrete random variables X and Y is given by $P(X = r, Y = s) = c|r + s|$ for $r = -1, 0, 1$ and $s = -2, -1, 0, 1, 2$,
$P(X = r, Y = s) = 0$, otherwise.
(a) Show that (i) $c = 0{\cdot}05$, (ii) $\mathrm{Var}(Y) = 2 + \mathrm{Var}(X)$.
(b) Evaluate $\mathrm{Var}(Y - X)$. (W)

32. A continuous random variable X is distributed with probability density function $f(x) = 3x^{-4}$, $x \geqslant 1$, $f(x) = 0$, otherwise. Find (i) the cumulative distribution function of X, (ii) the mean and the variance of X. The random variable Y is defined by $Y = X^{-1}$. By first finding the cumulative distribution function of Y, or otherwise, determine the probability density function of Y. Verify that $E(X)E(Y)$

and $E(XY)$ are not equal and explain why they have different values in this particular example. (W)

33. A continuous random variable X has probability density function

$$f(x) = 4x(1 - x^2), \qquad 0 \leqslant x \leqslant 1,$$
$$f(x) = 0, \qquad \text{otherwise.}$$

(i) Verify that $f(x)$ satisfies the requirements to be a probability density function. (ii) Find the mean value of X. (iii) Find the cumulative distribution function of X. (iv) If X denotes the area of a square of side Y find the probability density function of Y and use it to verify that $E(Y^2) = E(X)$. (W)

34. The probability density function of a continuous random variable X is given by

$$f(x) = \tfrac{2}{3}x, \qquad 1 \leqslant x \leqslant 2,$$
$$f(x) = 0, \qquad \text{otherwise.}$$

(a) Find (i) $E(X^2)$, (ii) $E(X^4)$, (iii) the cumulative distribution function of X.
(b) Suppose X is the length of the side of a square. (i) Find the mean and the variance of the area of the square. (ii) Show that the area of the square has a uniform distribution over a certain interval which should be specified. (W)

Formulae for reference

Series

Sum of A.P. $= \frac{1}{2}n\{2a + (n-1)d\}$, sum of G.P. $= \dfrac{a(1-r^n)}{1-r}$

For $|r| < 1$, sum to infinity of G.P. $= a/(1-r)$

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + b^n,$$

where n is a positive integer.

$$\sum_1^n r = \tfrac{1}{2}n(n+1); \quad \sum_1^n r^2 = \tfrac{1}{6}n(n+1)(2n+1); \quad \sum_1^n r^3 = \tfrac{1}{4}n^2(n+1)^2$$

Trigonometry

$$\cos^2\theta + \sin^2\theta = 1; \quad 1 + \tan^2\theta = \sec^2\theta; \quad \cot^2\theta + 1 = \operatorname{cosec}^2\theta.$$

$$\begin{aligned} \sin(A+B) &= \sin A\cos B + \cos A\sin B \\ \sin(A-B) &= \sin A\cos B - \cos A\sin B \\ \cos(A+B) &= \cos A\cos B - \sin A\sin B \\ \cos(A-B) &= \cos A\cos B + \sin A\sin B \end{aligned} \qquad \begin{aligned} \tan(A+B) &= \frac{\tan A + \tan B}{1 - \tan A\tan B} \\ \tan(A-B) &= \frac{\tan A - \tan B}{1 + \tan A\tan B} \end{aligned}$$

$$\text{Area of } \triangle ABC = \tfrac{1}{2}bc\sin A$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R, \quad a^2 = b^2 + c^2 - 2bc\cos A$$

Length of circular arc $= r\theta$, area of sector $= \frac{1}{2}r^2\theta$ (θ in radians)

Calculus

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}, \qquad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

y	$\sin x$	$\cos x$	$\tan x$	$\cot x$	$\sec x$	$\operatorname{cosec} x$
dy/dx	$\cos x$	$-\sin x$	$\sec^2 x$	$-\operatorname{cosec}^2 x$	$\sec x \tan x$	$-\operatorname{cosec} x \cot x$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int e^{kx}dx = \frac{1}{k}e^{kx} + c, \qquad \int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + c,$$

$$\int \frac{dx}{\sqrt{(a^2 - x^2)}} = \sin^{-1}\frac{x}{a} + c, \qquad \int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c.$$

Mechanics

Straight line motion with constant acceleration

$$v = u + at; \quad s = ut + \tfrac{1}{2}at^2; \quad s = \tfrac{1}{2}(u+v)t; \quad v^2 = u^2 + 2as.$$

Hooke's law: $T = \dfrac{\lambda}{l}x$; energy stored in string $= \dfrac{\lambda x^2}{2l}$.

Uniform body	Length, area or volume	Centre of mass
Triangular lamina, height h	$\frac{1}{2}h \times$ base	$\frac{1}{3}h$ from base
Circular arc, radius r, angle at centre 2α	$2r\alpha$	$\dfrac{r\sin\alpha}{\alpha}$ from centre
Sector of circle, radius r, angle at centre 2α	$r^2\alpha$	$\dfrac{2r\sin\alpha}{3\alpha}$ from centre
Solid cone or pyramid, height h	$\frac{1}{3}h \times$ base area	$\frac{1}{4}h$ from base
Solid hemisphere, radius r	$\frac{2}{3}\pi r^3$	$\frac{3}{8}r$ from centre
Hemispherical shell, radius r	$2\pi r^2$	$\frac{1}{2}r$ from centre

Probability

$$P(A \cup B) = P(A) + P(B) - P(A \cap B), \quad P(A \cap B) = P(A) \times P(B \mid A)$$

$$E(X) = \mu = \begin{cases} \sum x_r p(x_r) & X \text{ discrete} \\ \int_{-\infty}^{\infty} x f(x)\,dx & X \text{ continuous} \end{cases}$$

$$\mathrm{Var}(X) = \sigma^2 = \begin{cases} \sum (x_r - \mu)^2 p(x_r) = \sum x_r^2 p(x_r) - \mu^2, & X \text{ discrete} \\ \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2, & X \text{ continuous} \end{cases}$$

Binomial distribution B(n, p):

$$P(X = r) = \binom{n}{r} p^r (1-p)^{n-r} \quad (r = 0, 1, 2, \ldots, n)$$

$$E(X) = np, \quad \mathrm{Var}(X) = np(1-p)$$

Poisson distribution, parameter μ:

$$P(X = r) = \frac{\mu^r}{r!} e^{-\mu} \quad (r = 0, 1, 2, \ldots)$$

$$E(X) = \mu, \quad \mathrm{Var}(X) = \mu$$

Properties of expectation and variance:

$$E[g(X)] = \begin{cases} \sum g(x_r) p(x_r) & X \text{ discrete} \\ \int_{-\infty}^{\infty} g(x) f(x)\,dx & X \text{ continuous} \end{cases}$$

$$E(aX + b) = aE(X) + b, \qquad E(aX + bY) = aE(X) + bE(Y)$$

$$\mathrm{Var}(X) = E(X^2) - [E(X)]^2, \quad \mathrm{Var}(aX + b) = a^2 \mathrm{Var}(X)$$

For independent random variables $E(XY) = E(X)E(Y)$ and
$\mathrm{Var}(aX + bY) = a^2 \mathrm{Var}(X) + b^2 \mathrm{Var}(Y)$

$$\mathrm{Cov}(X, Y) = E[\{X - E(X)\}\{Y - E(Y)\}] = E(XY) - E(X)E(Y)$$

Statistics

Mean, $\bar{x} = \frac{1}{n}\sum x$

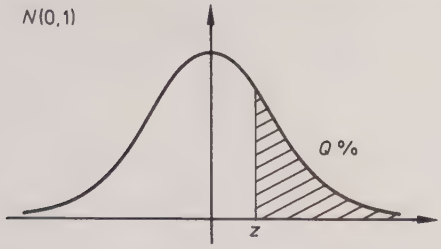
Variance, $\mathrm{var}(x) = \frac{1}{n}\sum (x - \bar{x})^2 = \frac{1}{n}\sum x^2 - \bar{x}^2$

Table 1 The normal distribution function $\Phi(z)$

z	·00	·01	·02	·03	·04	·05	·06	·07	·08	·09
0·0	·5000	·5040	·5080	·5120	·5160	·5199	·5239	·5279	·5319	·5359
01	·5398	·5438	·5478	·5517	·5557	·5596	·5636	·5675	·5714	·5753
0·2	·5793	·5832	·5871	·5910	·5948	·5987	·6026	·6064	·6103	·6141
0·3	·6179	·6217	·6255	·6293	·6331	·6368	·6406	·6443	·6480	·6517
0·4	·6554	·6591	·6628	·6664	·6700	·6736	·6772	·6808	·6844	·6879
0·5	·6915	·6950	·6985	·7019	·7054	·7088	·7123	·7157	·7190	·7224
0·6	·7257	·7291	·7324	·7357	·7389	·7422	·7454	·7486	·7517	·7549
0·7	·7580	·7611	·7642	·7673	·7704	·7734	·7764	·7794	·7823	·7852
0·8	·7881	·7910	·7939	·7967	·7995	·8023	·8051	·8078	·8106	·8133
0·9	·8159	·8186	·8212	·8238	·8264	·8289	·8315	·8340	·8365	·8389
1·0	·8413	·8438	·8461	·8485	·8508	·8531	·8554	·8577	·8599	·8621
1·1	·8643	·8665	·8686	·8708	·8729	·8749	·8770	·8790	·8810	·8830
1·2	·8849	·8869	·8888	·8907	·8925	·8944	·8962	·8980	·8997	·9015
1·3	·9032	·9049	·9066	·9082	·9099	·9115	·9131	·9147	·9162	·9177
1·4	·9192	·9207	·9222	·9236	·9251	·9265	·9279	·9292	·9306	·9319
1·5	·9332	·9345	·9357	·9370	·9382	·9394	·9406	·9418	·9429	·9441
1·6	·9452	·9463	·9474	·9484	·9495	·9505	·9515	·9525	·9535	·9545
1·7	·9554	·9564	·9573	·9582	·9591	·9599	·9608	·9616	·9625	·9633
1·8	·9641	·9649	·9656	·9664	·9671	·9678	·9686	·9693	·9699	·9706
1·9	·9713	·9719	·9726	·9732	·9738	·9744	·9750	·9756	·9761	·9767
2·0	·97725	·97778	·97831	·97882	·97932	·97982	·98030	·98077	·98124	·98169
2·1	·98214	·98257	·98300	·98341	·98382	·98422	·98461	·98500	·98537	·98574
2·2	·98610	·98645	·98679	·98713	·98745	·98778	·98809	·98840	·98870	·98899
2·3	·98928	·98956	·98983	·99010	·99036	·99061	·99086	·99111	·99134	·99158
2·4	·99180	·99202	·99224	·99245	·99266	·99286	·99305	·99324	·99343	·99361
2·5	·99379	·99396	·99413	·99430	·99446	·99461	·99477	·99492	·99506	·99520
2·6	·99534	·99547	·99560	·99573	·99585	·99598	·99609	·99621	·99632	·99643
2·7	·99653	·99664	·99674	·99683	·99693	·99702	·99711	·99720	·99728	·99736
2·8	·99744	·99752	·99760	·99767	·99774	·99781	·99788	·99795	·99801	·99807
2·9	·99813	·99819	·99825	·99831	·99836	·99841	·99846	·99851	·99856	·99861
3·0	·99865	·99869	·99874	·99878	·99882	·99886	·99889	·99893	·99896	·99900

z	3·1	3·2	3·3	3·4	3·5	3·6	3·7	3·8	3·9	4·0
Φ	·99903	·99931	·99952	·99966	·99977	·99984	·99989	·99993	·99995	·99997

Table 2 Percentage points of the normal distribution



The shaded area represents $Q\%$ of the total distribution.

$$Q(z) = 100P(Z \geqslant z) = 100\{1 - \Phi(z)\}$$

Q	z	Q	z	Q	z
50	0·000	1·5	2·170	0·2	2·878
40	0·253	1·0	2·326	0·15	2·968
30	0·524	0·9	2·366	0·1	3·090
20	0·842	0·8	2·409	0·05	3·291
10	1·282	0·7	2·457	0·01	3·719
5	1·645	0·6	2·512	0·005	3·891
4	1·751	0·5	2·576	0·001	4·265
3	1·881	0·4	2·652	0·0005	4·417
2·5	1·960	0·3	2·748	0·0001	4·753
2	2·054	0·25	2·807	0·00005	4·892

Answers

Exercise 1.1 (p. 4)

1. (a) **PS**, (b) **AE**, (c) **AB**, (d) **0**, (e) **MR**, (f) **PS**.
2. (a) false, (b) true, (c) true, (d) true, (e) false, (f) true.
3. (a) true, (b) false, (c) true, (d) true, (e) false, (f) false.
4. (a) 3 km E, (b) 10 km N, (c) 1 km N 30° E, (d) 1 km S.
5. (a) $2\sqrt{2}$ km NE, (b) $2\sqrt{2}$ km NE, (c) $2\sqrt{2}$ km SE, (d) $2\sqrt{2}$ km NW.
6. (a) 1 km S 30° E, (b) $\sqrt{3}$ km S, (c) $\sqrt{3}$ km S 60° E, (d) $\sqrt{3}$ km N 60° E.
7. $\mathbf{p}+\mathbf{q}$, $\mathbf{q}-\mathbf{p}$, $\frac{1}{2}(\mathbf{p}+\mathbf{q})$, $\frac{1}{2}(\mathbf{q}-\mathbf{p})$.
8. $-\mathbf{a}$, $\mathbf{a}-\mathbf{b}$, $2\mathbf{b}$, $2\mathbf{b}-\mathbf{a}$. 9. $2\mathbf{a}+\mathbf{b}$, $\mathbf{a}+\mathbf{b}$.

Exercise 1.2 (pp. 7–8)

1. (a) $2\mathbf{b}$, (b) $-3\mathbf{a}$, (c) $2\mathbf{b}-6\mathbf{a}$, (d) $\mathbf{b}-\mathbf{a}$, (e) $\mathbf{a}$, (f) $2\mathbf{b}-\mathbf{a}$.
2. (a) $\mathbf{z}$, (b) $\mathbf{x}$, (c) $-\mathbf{x}$, (d) $\mathbf{z}$, (e) $-\mathbf{x}+\mathbf{z}$, (f) $\mathbf{y}-\mathbf{z}$.
3. (a) $-\mathbf{x}+\mathbf{y}$, (b) $\mathbf{x}+\mathbf{z}$, (c) $-\mathbf{x}+\mathbf{y}-\mathbf{z}$, (d) $2\mathbf{x}-\mathbf{y}$, (e) $2\mathbf{x}-\mathbf{y}+2\mathbf{z}$, (f) $-\mathbf{y}+2\mathbf{z}$.
4. (a) $5\mathbf{i}+3\mathbf{j}$, (b) $-\mathbf{i}+6\mathbf{j}$, (c) $-4\mathbf{i}+2\mathbf{j}$, (d) $2\mathbf{i}-\mathbf{j}$, (e) $3\mathbf{i}+4\mathbf{j}$, (f) $\mathbf{i}+5\mathbf{j}$.
5. (a) $-4\mathbf{i}+5\mathbf{j}+2\mathbf{k}$, (b) $3\mathbf{j}+2\mathbf{k}$, (c) $\mathbf{i}-\mathbf{j}+\mathbf{k}$, (d) $\mathbf{i}+2\mathbf{j}+3\mathbf{k}$.
6. (a) 13, (b) $\sqrt{5}$, (c) 10, (d) 3, (e) 7, (f) $5\sqrt{3}$.
7. (a) $\frac{8}{9}\mathbf{i}-\frac{4}{9}\mathbf{j}-\frac{1}{9}\mathbf{k}$, (b) $\frac{8}{9}\mathbf{i}-\frac{4}{9}\mathbf{j}-\frac{1}{9}\mathbf{k}$, (c) $-\frac{8}{9}\mathbf{i}+\frac{4}{9}\mathbf{j}+\frac{1}{9}\mathbf{k}$.
8. (a) 5, (b) 3, (c) 6. 9. (a) $\sqrt{29}$, (b) $\sqrt{29}$, (c) 5.
10. (a) $\frac{1}{\sqrt{2}}\mathbf{i}-\frac{1}{\sqrt{2}}\mathbf{j}$, (b) $-\frac{7}{9}\mathbf{i}+\frac{4}{9}\mathbf{j}+\frac{4}{9}\mathbf{k}$, (c) $\mathbf{i}$.
11. $\mathbf{b}-\mathbf{a}$, $\mathbf{b}-2\mathbf{a}$; $2\mathbf{b}-2\mathbf{a}$, $2\mathbf{b}-3\mathbf{a}$, $\mathbf{b}-2\mathbf{a}$.
12. $\mathbf{a}-\mathbf{b}$, $\mathbf{c}-\mathbf{b}$, $\mathbf{a}-2\mathbf{b}+\mathbf{c}$; $2\mathbf{a}-3\mathbf{b}+2\mathbf{c}$.
13. (a) 6, $-\frac{1}{3}\mathbf{i}+\frac{2}{3}\mathbf{j}+\frac{2}{3}\mathbf{k}$, $2\mathbf{i}-3\mathbf{j}+\mathbf{k}$,
    (b) 22, $\frac{6}{11}\mathbf{i}-\frac{9}{11}\mathbf{j}-\frac{2}{11}\mathbf{k}$, $-\mathbf{i}+4\mathbf{j}-2\mathbf{k}$,

(c) $\sqrt{10}, \frac{3}{\sqrt{10}}\mathbf{j} - \frac{1}{\sqrt{10}}\mathbf{k}, \mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{5}{2}\mathbf{k}$,

(d) $3\sqrt{6}, -\frac{2}{\sqrt{6}}\mathbf{i} - \frac{1}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}, 2\mathbf{i} + \frac{5}{2}\mathbf{j} - \frac{1}{2}\mathbf{k}$.

14. 50.

Exercise 1.3 (pp. 8–9)

1. $\mathbf{a}+\mathbf{b}+\mathbf{c}, \mathbf{b}+\mathbf{c}-\mathbf{a}, \mathbf{a}+\mathbf{c}-\mathbf{b}, \mathbf{a}+\mathbf{b}-\mathbf{c}$.
2. (a) $\mathbf{b}-\mathbf{a}$, (b) $3\mathbf{b}-2\mathbf{a}$, (c) $3\mathbf{b}-\mathbf{c}$, (d) $\mathbf{a}-\frac{3}{2}\mathbf{b}$, (e) $-\mathbf{a}-\frac{1}{2}\mathbf{b}$,
   (f) $\mathbf{a}+\frac{3}{2}\mathbf{b}$.
3. (a) $2\sqrt{6}, 5\sqrt{2}, 20$, (b) $\frac{2}{3}\mathbf{i}+\frac{1}{3}\mathbf{j}-\frac{2}{3}\mathbf{k}$.
4. (a) $6\sqrt{3}$, (b) $\frac{1}{9}\sqrt{3}\mathbf{i} - \frac{5}{9}\sqrt{3}\mathbf{j} + \frac{1}{9}\sqrt{3}\mathbf{k}$, (c) $-2\mathbf{j}+3\mathbf{k}$.
5. 1, 4.
7. (a) $-\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$, (b) $\frac{3}{2}\sqrt{29}$.
8. $\mathbf{a}+k(2\mathbf{b}-\mathbf{a})$; 3, $6\mathbf{b}-2\mathbf{a}$.
9. $\frac{1}{3}\mathbf{a}+\frac{1}{3}\mathbf{b}$.
10. (4, −7, 8): 54 sq. units.

Exercise 2.3 (pp. 17–18)

1. (a) 13, 22·6°, (b) 8·06, 60·3°, (c) 9·85, 75·3°,
   (d) 14·5, 33°, (e) 14·1, 37·3°, (f) 8·86, 17·9°.
2. 133·5°.
3. 14·3 N at 81·4° to **G**.
4. 3·81 N, 15·7 N.
5. 9·43 N, N 12° W.
6. 13·0 N, S $37\frac{1}{2}$° E.
7. 16·8 N, N 57·6° E.
8. (a) 5, N 53·1° E, (b) 7, E, (c) 6·71, S 63·4° E, (d) 7·07, SW.
9. (a) 6, (b) 17, (c) 9·95.
10. 15·3 N.
11. $\sqrt{5}$.
12. $4\sqrt{10}, 6\sqrt{5}$.

Exercise 2.4 (pp. 21–22)

1. (a) 15·3 N, 12·9 N, (b) 1·46 N, 6·85 N, (c) 7·05 N, 9·71 N.
2. (a) $-W, N\cos\alpha, T\sin(\alpha+\beta)$, (b) $-W\sin\alpha, 0, T\cos\beta$,
   (c) $W\cos\alpha, -N, -T\sin\beta$.
3. 11·5 N, N 15·5° W.
4. 29·3 N, N 15·8° W.
5. 344·7°.
6. 15·6, 76·6°.
7. 15·2 N, 66·8°.
8. 4·03 N.
9. (a) 48 N, 40 N, 46·2 N, (b) 14 N, 30 N, 19·2 N, (c) 134·2 N,
   (d) 20·6 N, 157·4°.

Exercise 2.5 (pp. 27–28)

1. 31·4 N, S 84·6° W.
2. 12·9, 6·39.
3. $5\sqrt{3}$.
4. 6, 8.
5. (a) $20\sqrt{3}$, (b) 20, (c) $10\sqrt{3}$.
6. $3\frac{1}{2}$ N, $12\frac{1}{2}$ N.
7. 1·87 N, 4·41 N.
8. 37·1 N, 19·7 N.
9. (a) $5W$, $5\sqrt{3}W$, (b) $10W/\sqrt{3}$, $10W/\sqrt{3}$.
10. 4·14 N, 43°.
11. $8W$. $6W$.
12. $10\sqrt{3}$ N, $20\sqrt{3}$ N at 30° to vertical.
13. (a) $\sqrt{5}W$, $2\sqrt{2}W$, (b) 63·4°, 45°.
14. $2\sqrt{2}W$, $\sqrt{5}W$, 26·6°.
15. (a) $5{\cdot}80W$, $4{\cdot}24W$, $5{\cdot}02W$, (b) $7{\cdot}07W$, $5{\cdot}39W$, $5{\cdot}39W$.

Exercise 2.6 (pp. 28–30)

1. 69·6°.
2. $\sqrt{10}$, 51·3°.
3. $0 \leqslant \theta \leqslant 36{\cdot}9°$.
4. $\sqrt{65}$, N 60·3° E.
6. $2\sqrt{5}$ N, $4\sqrt{2}$ N.
7. $5\sqrt{3}$ N.
8. 154·9°.
9. 6·32 N, S 71·6° W.
10. 15·4 N, S 37·6° W.
11. $2\sqrt{5}W$.
12. 12·9 N, 45·2 N.
13. 46·8°, 11·5 N.
15. (a) 25/8 N, (b) 2·62 N.
16. $W/\sqrt{3}$.
19. 16·6 N, 6·50 m.

Exercise 3.1 (pp. 34–35)

1. 13, $25\mathbf{i}+6\mathbf{j}$.
2. $-2\mathbf{i}+4\mathbf{j}$.
3. $24\mathbf{i}+3\mathbf{j}+12\mathbf{k}$; $8\mathbf{i}+\mathbf{j}+4\mathbf{k}$, 11.
4. 85 m.
5. $5\mathbf{i}+8\mathbf{j}$; 7.
6. $-3\mathbf{i}+6\mathbf{j}$.
7. (a) 13, N 67·4° E, (b) $2\sqrt{10}$, N 71·6° E, (c) $4\sqrt{5}$, S 63·4° E, (d) $\sqrt{97}$, S 24° W.
8. (a) 13·6 m s^{-1}, N 45·2° E, (b) 59·7 km/h, S 74·4° E, (c) 16·4 m s^{-1}, N 6·6° W, (d) 98·0 km/h, S 48·8° E.
9. (a) 4·18 m s^{-1}, (b) 30 m.
10. 14·6 km/h, N 87° E.

Exercise 3.2 (pp. 37–38)

1. 9·43 km/h from S 58° W.
2. 41·6 km/h, S 70·1° E.
3. 15, 126·9°.
4. $\mathbf{i}-4\mathbf{j}$, $4\mathbf{i}+5\mathbf{j}$.
5. 33·1 km/h, S 65° W.
6. $20\sqrt{3}$ km/h, N 60° W.
7. 10·7 km/h from N 37·5° W.
8. 90 km/h.
9. N 73·7° W; 2 min.
10. 2·25 m s^{-1}; at 63·3° to the bank.
11. N 47·3° W; 1 h 30 min.
12. 76·8 km/h; S 64·8° W.

Exercise 3.3 (pp. 41–42)

1. 151·8°, 1 h 47 min.
2. $4\sqrt{3}$ km/h, N 11·4° E, 6 km/h.
3. 28·8 km/h on S 33·7° W; 8·32 km.
4. 5·57 m s^{-1} on 321·1°, 115 s.
5. 25 km/h on S 36·9° E; 60 m.
6. 576 km/h, N 51·3° E; 31·2 km; 10.04, N 38·7° W.
7. $500\sqrt{2}$ km/h, SE; 25·9 km.
8. (ii) $3\mathbf{i}+4\mathbf{j}$.
9. 3, 500 s; $1000\sqrt{10}$ m.
10. N 73·1° E; 2·9 km.

Exercise 3.4 (pp. 43–45)

1. 203°; 157°; 10·4 knots E.
2. $6\sqrt{3}$ knots, 120°.
3. (i) due north, 100 s, 583 m, (ii) N 36·9° W, 125 s, 500 m.
4. 95·2 s; 92·3 s; 125 s.
5. 032·8°, 237·2°; 64·4 min.
6. 65 km h^{-1} from N 32·6° E; N 5·8° E, S 54·2° W.
7. 97 min, 69 min, 137 min.
8. 14.36, N 45·2° W; 13.42.
9. N; 26·8 min.
10. (a) 3 m s^{-1}, (b) $\mathbf{i}+3\mathbf{j}$; $\sqrt{181}$ m s^{-1}.
11. (i) 50 km h^{-1} towards N 73·7° W, (ii) 112 m, (iii) 31·36 m, 107·52 m.
12. 42 km/h, S 8·2° E; 13.06$\frac{1}{4}$, 8·66 km; 13.25.
13. N 41·4° E, 45 min; N 48·2° E, 7·45 km.
14. 18·0 km/h; 13.12; 4·16 km/h.

Exercise 4.1 (pp. 49–50)

1. (a) $9t^2-5$, $18t$, (b) $4t+3$, 4.
2. (a) $6t^2+7$, $2t^3+7t$, (b) $20-10t$, $5+20t-5t^2$,
   (c) $8t^3-12t^2+4$, $2t^4-4t^3+4t-2$.
3. 0, 3; 4.
4. 24; 32 m.
5. 36 m, 9 m s^{-1}.
6. 1 s, 9 s; 16 m s^{-1}.
7. 29, 52 m.
8. -12 m s^{-1}, 16 m; 48 m.
9. (a) $2\frac{1}{2}$, (b) 5 m.
10. (a) -15 m s^{-1}, -4 m s^{-2}, (b) 0 s, 5 s, (c) 72 m.
11. (a) $9t-3t^2$, $\frac{9}{2}t^2-t^3$, (b) 27 m.
12. (a) 0 s, 6 s, (b) 32 m, (c) 64 m, (d) 36 m s^{-1}.
13. (a) 1 s, 4 s, (b) 10 m, (c) $18\frac{1}{2}$ m, (d) 7 m s^{-1}.
14. (a) 0 s, $2\frac{1}{4}$ s, 6 s, (b) $60\frac{3}{4}$ m, (c) $171\frac{1}{2}$ m, (d) 90 m s^{-1}.
15. $5\frac{1}{3}$, 10, $14\frac{2}{3}$; 6 m s^{-1}.
16. (a) 27 m, (b) 16 m s^{-1}, (c) 12 m s^{-2}.

Exercise 4.2 (pp. 54–56)

1. 11, 24.
2. 12, 2.
3. 16, -3.
4. -7, -4.
5. $\frac{1}{2}$, 10.
6. 0, 24.
7. 10, 10 or 15, -10.
8. 1, 60.
9. $-2\frac{1}{2}$, 2.
10. 0, $1\frac{1}{2}$.
11. $\frac{1}{4}$ km.
12. 1/48.
13. $31\frac{1}{4}$ m, 5 s.
14. 14 m s^{-1}, 8 m.
15. 5 m s^{-1}, 10 s.
16. 34 m, 20 m s^{-1}.
17. $2\frac{1}{2}$ m s^{-2}, 1 s.
18. 40 m s^{-1}, 80 m.
19. 1750 m; 48, 100, 32, 45.
20. 30, 28·8 km.
21. 12 min.
22. 2, 10·05.
23. 20.
24. 1 min, 900 m.
25. 3 s, 15 m.
26. 20 m.

Exercise 4.3 (pp. 60–62)

1. $1\frac{1}{4}$ s; $2\frac{1}{2}$ s, 53·0 m.
2. 60 m; 35·8 m s^{-1} at 33·2° to horizontal.
3. 50 m; $y = 2x - x^2/50$.
4. 5, 2·5; 30·8 at 54·2° to the horizontal.
5. 20 m s^{-1}, 19·6 m s^{-1}; 44·1 m.
6. 11·2 m s^{-1}; 3·2 m.
8. 4·5 km; 26·6°
9. $y = \frac{5}{3}x - \frac{49x^2}{90u^2}$; 2·1, 13·5 m.
10. 14, 36·9°.
11. 28; $15° < \alpha < 75°$.
12. $6u^2/g$; $2u/g$, $4u/g$.
13. $u/(g \sin \alpha)$.
14. $2V \sin \theta(U + V \cos \theta)/g$.
15. $12V^2/13g$; $2V/(g\sqrt{13})$.
16. $\sqrt{(V^2 - 2gh)}$, $2\sqrt{(V^2 \sin^2 \theta - 2gh)}/g$; 60°.
18. 63·4°, 26·6°; 4 : 5.
19. $81\frac{2}{3}$; 69·3 m s^{-1}.
20. $2u_2/g$.

Exercise 4.4 (p. 64)

1. (a) $3\mathbf{i} + 6t^2\mathbf{k}$, $12t\mathbf{k}$, (b) $2t\mathbf{i} + 4t^3\mathbf{j} - 6\mathbf{k}$, $2\mathbf{i} + 12t^2\mathbf{j}$.
2. -3 m s^{-1}, 13 m s^{-1}.
3. $3(t-2)\mathbf{i} + 2(t^2 - 1)\mathbf{j} + t(1 - t^2)\mathbf{k}$.
4. $(3t^2 + 1)\mathbf{i} + \mathbf{j} - 4t\mathbf{k}$, $(t^3 + t)\mathbf{i} + (t + 3)\mathbf{j} - 2t^2\mathbf{k}$.
5. $3\mathbf{i} + (2 - 10t)\mathbf{j}$, $3t\mathbf{i} + (2t - 5t^2)\mathbf{j}$.
6. $31\frac{1}{4}$ m, 200 m, 5 s.
7. $10\mathbf{i} + (10\sqrt{3} - 10t)\mathbf{j}$, $10t\mathbf{i} + (10\sqrt{3}t - 5t^2)\mathbf{j}$.
8. 48 m, 24 m; 32·0 m s^{-1}.

Exercise 4.5 (pp. 65–67)

1. $25\frac{11}{27}$ m, 20 m s^{-1}; $t < 2\frac{1}{6}$.
2. $343\frac{3}{4}$ m; 5 s; 49 m s^{-2}; 108 m s^{-1}.

3. (a) 7, (b) $2t(35+3t-2t^2)$, (c) $0 < t < 5$.
6. (i) $b+\frac{1}{2}ft^2-ut$, (ii) $b+(V-u)t-V^2/2f$;
   $b-u^2/2f$; car overtakes bus after time $(2fb-V^2)/2f(u-V)$.
7. (a) U/g, (b) $2U/g$.
8. 18·3 m, 61·6 m, 3·83 s.
9. (i) 40 m, (ii) 10 m, (iii) 20 m s^{-1} at 45° to horizontal,
   (iv) $2\sqrt{2}$ s; $(20+5\sqrt{2}, 8\frac{3}{4})$, $(20-5\sqrt{2}, 8\frac{3}{4})$.
10. 40, 45°; 15°.
11. $D^2\tan^2\theta-20D\tan\theta+20h+D^2=0$.
15. $\frac{3}{5}\mathbf{i}-\frac{4}{5}\mathbf{k}$; 2 s, $2\mathbf{j}+16\mathbf{k}$.
16. 5, 25.

Exercise 5.1 (pp. 71–72)

1. 48 N.
2. 0·2 m s^{-2}.
3. 12 kg.
4. 40 N.
5. $2\mathbf{i}-5\mathbf{j}$.
6. $12\mathbf{i}+4\mathbf{j}$.
7. 6·4 m s^{-2} at 17·4° to the 9 N force.
8. 84·9°.
9. 6·25 m s^{-2}, S 53·1° E.
10. 2·05 m s^{-2}, N 61·6° W.
11. 1·8 s.
12. 15 N.
13. 10 kg.
14. 10 m s^{-1}.
15. 8, 540 m.
16. 20 m s^{-1}; $6\frac{2}{3}$, 10.

Exercise 5.2 (pp. 74–75)

1. (a) 98 N, (b) 14·7 N, (c) 2·45 N.
2. 6·2 m s^{-2}.
3. 62·4 N.
4. (a) 440 N, (b) 392 N, (c) 360 N.
5. (a) decelerating at 1·2 m s^{-2}, (b) accelerating at 0·8 m s^{-2},
   (c) moving with uniform speed.
6. 33·1 N, 280 N.
7. 1·71 m s^{-2}, 87 N.
8. 5 N, 11·5°.
9. 8·49 m s^{-2}, 24·5 N.
10. 2·44 s.
11. (a) 8·28 N, (b) 14·3 N.
12. 2·12 m s^{-2}, 55·4 N.

Exercise 5.3 (pp. 79–81)

1. (a) 3 m s^{-2}, (b) 350 N.
2. (a) 240 N, 80 N, (b) 2640 N, 880 N.
3. 6000 N, 1200 N, 600 N; 0·25 m s^{-2}, 7200 N, 3600 N.
4. 4·2 m s^{-2}, 28 N.
5. 4·9 m s^{-2}, 44·1 N.
6. $\frac{M-m}{M+m}g, \frac{2Mm}{M+m}g$.
7. $24mg/5$.
8. 2·45 m s^{-2}, 8·82 N.
9. 0·392 m; $mg/\sqrt{2}$ N = 6·93m N at 45° to vertical.
10. 3·92 m s^{-2}, 17·64 N; 30·6 N at 30° to vertical.

11. $m \sin \alpha > M$.
12. 0·5 s.
13. 3·75 m.
14. 0·98 m s^{-2}, 34·3 N; 1·4 m s^{-1}; $1\frac{1}{6}$ m.

Exercise 5.4 (pp. 83–84)

1. 7·84 m s^{-2}, upwards; 17·64 N.
2. 2·8 m s^{-2}, downwards; 28 N.
3. 6·2 m s^{-2} upwards, 5·8 m s^{-2} downwards; 32 N, 16 N.
4. 1·96 m s^{-2} down, 5·88 m s^{-2} up; 31·4 N, 15·7 N.
5. 1·51 m s^{-2} down, 2·26 m s^{-2} down; 41·5 N, 15·1 N.
6. 1·89 m s^{-2}.
7. $49 \sin \alpha/(4 + \sin^2 \alpha)$ m s^{-1}.
8. 0·8 m s^{-2} vertically, 3·2 m s^{-2} down plane.

Exercise 5.5 (pp. 84–87)

1. 3·70 kg.
2. 24·1, 242·5°.
3. 1250, 40 s.
4. $10\mathbf{i} + 20\mathbf{j}$, $92\mathbf{i} + 10\mathbf{j}$.
5. 9·16 N.
6. (a) 791 N, (b) 651 N.
7. 2 m s^{-1}; 1980 N, 1960 N, 1920 N.
8. (a) 784 N, 0°, (b) 809 N, 14·3°.
9. $\sqrt{(2gx \sin \alpha)}$.
10. 12·5 m.
11. (a) 2, (b) 56·4°
12. 1071 N, 315 N.
13. 3·15 m.
14. (a) $2\frac{1}{3}$ m s^{-1}, (b) $1\frac{13}{15}$ m s^{-1}.
15. 1·70 m s^{-2}, 83·4 N at 25° to the vertical.
16. 47·04 N, 23·52 N; 2·06 m.
17. 30°; $mg\sqrt{3}$ at 30° to vertical; $\frac{3}{2}$mg, $\frac{1}{4}$g, $\frac{3\sqrt{3}}{2}$mg at 30° to vertical.
18. $\frac{2}{3}$mg.
19. 48 N, 24 N.
20. 525 N, 11·5°.

Exercise 6.1 (p. 90)

1. (a) 10 N, (b) 0, (c) 5 N.
2. (a) 5 N down plane, (b) 0, (c) 5 N up plane.
3. (a) 3, (b) $12\sqrt{2}/5 = 3{\cdot}39$, (c) $4\sqrt{2} = 5{\cdot}66$.
4. $\tan 25° = 0{\cdot}466$.
5. (a) 3·98, (b) 2·45.
6. 0·733 m s^{-2}.
7. 5·75 m s^{-2}.
8. 3·20 m s^{-2}.
9. 0·147.
10. 1·34 m s^{-2}.

Exercise 6.2 (pp. 93–94)

1. 9·8 N, $\frac{1}{4}$.
2. 1·5 s.
3. (a) 6·21 m s^{-2}, (b) 2·79 m s^{-2}.
4. 7·38.

5. (a) $\tan\alpha$, (b) $(g\sin\alpha - a)/g\cos\alpha$.
6. $23{\cdot}04\,\mathrm{m\,s^{-2}}$, 61·0 N.
7. 36·3 N, 0·48.
8. $\mu < 1$, 5/13.
9. $4{\cdot}2\,\mathrm{m\,s^{-2}}$, 22·4 N.
10. $\frac{2}{3}$; 34·6 N at 45° to horizontal.
11. $1{\cdot}94\,\mathrm{m\,s^{-2}}$ up the plane, 39·3 N.
12. (a) $1{\cdot}86\,\mathrm{m\,s^{-2}}$ up the plane, (b) $1{\cdot}25\,\mathrm{m\,s^{-2}}$ down the plane.

Exercise 6.3 (pp. 95–96)

1. 14°.
2. (a) 12·5 N, (b) 5·47 N.
3. (a) $\mu \geqslant \frac{1}{3}$, (b) $\mu \geqslant \frac{1}{5}\sqrt{3}$, (c) $\mu \geqslant \frac{1}{7}\sqrt{3}$.
5. $\frac{1}{6}\sqrt{3}$.
6. $2{\cdot}87 \leqslant P \leqslant 4{\cdot}79$.
7. (a) 16, (b) $24\frac{8}{13}$.
8. $M \geqslant 3m$.

Exercise 6.4 (p. 98)

1. (a) 6, (b) 5·66, (c) 9·74.
2. 12·5 N, 36·9°.
3. 1·94 N at 14° to horizontal.
4. $W\sin\lambda$ at λ to horizontal.
5. (a) 1·20 N at 18·4° to plane, i.e. at 11·6° above horizontal,
   (b) 4·49 N at 18·4° to plane, i.e. at 48·4° above horizontal.
6. (a) $W\sin(\lambda - \alpha)$ at λ to plane. (b) $W\sin(\lambda + \alpha)$ at λ to plane.

Exercise 6.5 (pp. 101–102)

1. 5 N.
2. 9 N.
3. 2·4 N.
4. 1·2 m.
5. 26·6°, $a(3 + \sqrt{5})$.
6. $6{\cdot}2\,\mathrm{m\,s^{-2}}$.
7. 1·8 m.
8. $2{\cdot}5W/8$.
9. 4·8 N in PA, 2 N in PB; 0·9 m.
10. 20.
11. 18 N; 4·5 cm, 7·5 cm, 12 cm.
12. 1·25 m, 1·75 m.

Exercise 6.6 (pp. 102–106)

1. (a) 140 N, (b) 135 N.
2. $\frac{1}{2}W$.
3. (a) 15·7, (b) 7·84; $3{\cdot}92\,\mathrm{m\,s^{-2}}$.
4. 65, $\frac{1}{2}$.
5. (i) $\dfrac{\mu W}{1+2\mu^2}$, (ii) $\dfrac{3\mu W}{\sqrt{(1+4\mu^2)}}$.
8. 99·0, 33·0 N.
9. (a) $4{\cdot}5\,\mathrm{m\,s^{-2}}$; 42 N, 24 N, (b) $2\,\mathrm{m\,s^{-2}}$; 42 N, 24 N.

11. (a) $\mu < \frac{1}{8}$, (b) $\mu < \frac{1}{8}$. 12. (ii) $\frac{1}{8}g(2-\sqrt{3}), \frac{1}{8}Mg(6+\sqrt{3})$.
14. 14°, $2{\cdot}60\,\mathrm{m\,s^{-2}}$.
15. (a) 0·09 N at 1·3° to horizontal,
(b) 1·45 N at 21·3° to horizontal.
16. $\lambda(2\sec\theta - 1)$, $W + \lambda(2 - \cos\theta)$, $\lambda(2\tan\theta - \sin\theta)$.
17. (i) $5mg$, $4\sqrt{2}mg$, (ii) 53·1°, 45°.

Exercise 7.1 (pp. 109–110)

1. 0, 0, Ps, $-Fs$.
2. $-Wx$, Px.
3. $Ws\sin\alpha$, 0; Fs.
4. Ts, 0; Fs, $Ws\sin\alpha$.
5. 105 J, 15 m.
6. −9000 J, 450 N.
7. 45 kN.
8. 770 N.
9. 62·5 J.
10. 300 J; 5/8.
11. 200 J; 45.
12. 270 J, 180 J; $6\,\mathrm{m\,s^{-1}}$.

Exercise 7.2 (pp. 113–114)

1. $10{\cdot}5\,\mathrm{m\,s^{-1}}$.
2. (a) 21·6 m above A, (b) 40 m below A.
3. $v^2/2g$.
4. 4·05 m.
5. $5\,\mathrm{m\,s^{-1}}$.
6. $14\,\mathrm{m\,s^{-1}}$.
7. $2{\cdot}8\,\mathrm{m\,s^{-1}}$.
8. 214 J.
9. 147 J; 7 m.
10. $(M+m)v^2/2(M-m)g$.
11. $1{\cdot}4\,\mathrm{m\,s^{-1}}$.
12. 10 m.
13. 8 m.
14. (a) $3{\cdot}5\,\mathrm{m\,s^{-1}}$, (b) 2·6 m.
15. 19·6 J; $\frac{1}{4}$.

Exercise 7.3 (pp. 117–118)

1. (a) 0·2 J, (b) 5 J.
2. (a) 0·25 J, (b) 9 J.
3. 2·25 J.
4. 15 N.
5. 7·5 J.
6. $12k$, $3ka$.
7. 0·56 m.
8. $2\,\mathrm{m\,s^{-1}}$.
9. $\frac{1}{2}\sqrt{(ag)}$.
10. 4 m.
11. $7\,\mathrm{m\,s^{-1}}$.
12. 60 cm; $1{\cdot}4\,\mathrm{m\,s^{-1}}$.
13. 2·5 N.
14. 1·2 m; (a) $\frac{7}{5}\sqrt{3}$, (b) 0·7 m.

Exercise 7.4 (pp. 121–122)

1. 392 W.
2. 40 s.
3. 400 W.
4. 500 kg.
5. 28·8 kW.
6. 6·8 kW.
7. 335 W.
8. 72 kN.
9. 1·2 kW; $0{\cdot}1\,\mathrm{m\,s^{-2}}$.

10. (a) 12 kW, (b) 14·7 kW, (c) 3 kW.
11. 21 kW.
12. $30\,\mathrm{m\,s^{-1}}$.
13. (a) $33\frac{1}{3}$ kW, (b) 52 kW, (c) $14\frac{2}{3}$ kW.
14. $6\frac{5}{6}$ kN, $13\frac{1}{3}\,\mathrm{m\,s^{-1}}$.
15. (a) 96 km/h, (b) 36 km/h; $1/6\,\mathrm{m/s^2}$.
16. $0{\cdot}98\,\mathrm{m\,s^{-2}}$.
17. 2, $1{\cdot}4\,\mathrm{m/s^2}$.
18. 20 kW; (i) 6·93 kW, (ii) $0{\cdot}201\,\mathrm{m\,s^{-2}}$.

Exercise 7.5 (pp. 122–127)

1. 1/8.
2. 30/49.
3. 10 N.
4. 53·1°.
5. 68·6 J, 7/12.
6. 2·4 m.
7. $2{\cdot}8\,\mathrm{m\,s^{-1}}$.
8. $7\,\mathrm{m\,s^{-1}}$, 2·4 m.
9. 3, $2{\cdot}57\,\mathrm{m\,s^{-1}}$.
10. $\frac{3}{2}\sqrt{(ga)}$, $2a$.
11. 30 mg/19.
13. (i) $2\sqrt{(2gb)}$, (ii) $\sqrt{(10gb)}$, (iii) 26·6°, (iv) $4b$; $-\frac{1}{2}c$.
14. (i) $2b-l$, (ii) b^2/l.
15. $4{\cdot}5\times10^4$ N, 450 kW; $24\,\mathrm{m\,s^{-1}}$.
16. 1800 N, $804\frac{1}{6}$ N.
17. 320 kW.
18. 113 kg, 22·3 kW.
19. (i) $19\frac{7}{11}\,\mathrm{km\,h^{-1}}$, (ii) $0{\cdot}53\,\mathrm{m\,s^{-2}}$; 60.
20. 500 N, 20 m/s.
21. (a) $(1000H-MVg\sin\alpha)/MV$, (b) $(1000H+MVg\sin\alpha)/MV$; $1000H/3MVg$, $3V/4$.
22. 400 kW, 15 m/s, 20 000 N.
23. $0{\cdot}395\,\mathrm{m\,s^{-2}}$.
24. (i) 0·25 m/s, (ii) 22·5 m/s; $233\frac{1}{3}$ m, $15\frac{5}{9}$ s.
25. 1200 N; 1/40; 10·5 kW.
26. $R+mg\sin\alpha+2mU^2\sin\alpha/h$, $2U(R+mg\sin\alpha+2mU^2\sin\alpha/h)$, $mgh+2mU^2+Rh\operatorname{cosec}\alpha$.

Exercise 8.1 (pp. 130–131)

1. $\frac{3\pi}{2}\,\mathrm{rad\,s^{-1}}$, $0{\cdot}42\,\mathrm{m\,s^{-1}}$.
2. $4\,\mathrm{rad\,s^{-1}}$.
3. $7{\cdot}3\times10^{-5}\,\mathrm{rad\,s^{-1}}$, $470\,\mathrm{m\,s^{-1}}$.
4. $2\times10^{-7}\,\mathrm{rad\,s^{-1}}$, 30 km/s.
5. 57 cm.
6. (a) $\dfrac{1}{1+t^2}$, (b) $\dfrac{6}{9+4t^2}$, (c) $\dfrac{2t}{1+t^4}$.
7. (a) $r(\cos\omega t\mathbf{i}+\sin\omega t\mathbf{j})$, (b) $r(-\sin\omega t\mathbf{i}+\cos\omega t\mathbf{j})$.
8. (a) $2\,\mathrm{rad\,s^{-1}}$, $6\,\mathrm{m\,s^{-1}}$, (b) $1\,\mathrm{rad\,s^{-1}}$, $5\,\mathrm{m\,s^{-1}}$, (c) $-2\pi\,\mathrm{rad\,s^{-1}}$, $2\pi\,\mathrm{m\,s^{-1}}$, (d) $k\,\mathrm{rad\,s^{-1}}$, $ak\,\mathrm{m\,s^{-1}}$.
9. $a(\cos\omega t\mathbf{i}+\sin\omega t\mathbf{j})$, $a(-\sin\omega t\mathbf{i}+\cos\omega t\mathbf{j})$, $a\omega(\sin\omega t-\cos\omega t)\mathbf{i}-a\omega(\cos\omega t+\sin\omega t)\mathbf{j}$.

Exercise 8.2 (pp. 133–134)

1. (a) $9\,\mathrm{m\,s^{-2}}$, (b) $4\,\mathrm{m\,s^{-2}}$, (c) $20\,\mathrm{m\,s^{-2}}$, (d) $63{\cdot}2\,\mathrm{m\,s^{-2}}$.
2. $22\frac{1}{2}$ N.
3. 12 N.
4. 375 N, 49 N.
5. $6\,\mathrm{m\,s^{-1}}$.
6. 0·4.
7. 0·7 m.
8. $1{\cdot}75\,\mathrm{rad\,s^{-1}}$.

Exercise 8.3 (pp. 137–139)

1. 49 N, $2{\cdot}8\,\mathrm{m\,s^{-1}}$.
2. 127·4 N, 3 m.
3. $mg\,\mathrm{cosec}\,\theta$; $rg\cot\theta$.
4. 2·45 m.
5. $3{\cdot}15\,\mathrm{m\,s^{-1}}$.
6. $\sqrt{(3ag)}$.
7. $5m(8u^2+9ag)/72a$ in AP, $5m(8u^2-9ag)/72a$ in BP.
8. $17\frac{1}{2}\,\mathrm{m\,s^{-1}}$.
9. 17·7°.
10. $35\,\mathrm{m\,s^{-1}}$, $5\sqrt{7}\,\mathrm{m\,s^{-1}}$.
11. $7\sqrt{5}\,\mathrm{m\,s^{-1}}$; 12·7°.

Exercise 8.4 (pp. 139–142)

1. $10\pi/9\,\mathrm{rad\,s^{-1}}$, $0{\cdot}42\,\mathrm{m\,s^{-1}}$.
2. $25/\pi$, 50.
3. $-6/(5t^2-6t+9)$.
4. $6\,\mathrm{m\,s^{-1}}$ S 30° W.
5. $5/36 \leqslant \mu < 5/9$.
6. 60 N, 75 N.
7. (i) 60°, (ii) $2mg$, (iii) $\pi\sqrt{(2a/g)}$ s.
8. 60°.
9. (i) $\sqrt{5}$, (ii) 1/6 N.
10. $\frac{1}{2}ma(2\omega^2+\Omega^2)$, $\frac{1}{2}ma(\Omega^2-2\omega^2)$.
11. $\sqrt{\{g(a^2-h^2)/h\}}$.
12. $\frac{2}{3}mg\sqrt{3}$, $m(3u^2-gl\sqrt{3})/3l$.
13. $m(v^2\cos\alpha+gl\sin^2\alpha)/l\sin\alpha$, $m(v^2-gl\cos\alpha)/l$.
14. 36·3°; $97{\cdot}8\,\mathrm{m\,s^{-1}}$.
15. $mg(\sqrt{3}-1)$.
16. $\sqrt{(2rg/11)}$ to $\sqrt{(2rg)}$.
17. $\sqrt{(45ag/8)}$.

Exercise 9.1 (pp. 147–149)

1. -42 N m, 14 N m, -21 N m, 0 N m.
2. 42 N m, -30 N m, -42 N m, 30 N m, 0 N m.
3. $(4+\sqrt{3})$ N m, $3\sqrt{3}$ N m, $(2+5\sqrt{3})$ N m.
4. (a) 6 N, 2 m, (b) 8 N, 3 m.
5. (a) 1 N, 10 m, (b) 1 N, 15 m.
6. 50 N m.
7. $F = G = 4$.
8. (a) $5\sqrt{2}$ N, 4 m, (b) 15 N, 3 m.
9. (a) $4\sqrt{2}$ N, 5 m, (b) 6·5 N, 12 m.
10. $6\sqrt{2}$ N along BD.
11. $20\sqrt{3}$ N m, $10\sqrt{3}$ N m, 0 N m, 0 N m, $10\sqrt{3}$ N m, $20\sqrt{3}$ N m; 10 N along CD

Exercise 9.2 (pp. 154–156)

1. 4 N at 90° to AB, 4 m.
2. 2 N at 90° to AB, 5 m.
3. $6\sqrt{3}$ N at 60° to AB, 1 m.
4. 6 N at 60° to AB, 2 m.
5. (a) $5P$ at 53·1° to AB, (b) $6a$.
6. $3x - 7y + 7 = 0$.

7. $2x - 5y + 10 = 0$. 8. *a*, *b*. 9. *c*. 10. *a*, *c*. 11. *b*, *c*.
12. *c*. 13. *c*. 14. *a*, *c*. 15. *b*, *c*. 16. *c*. 17. *a*.
18. *b*. 19. *a*, *c*. 20. 6 N m clockwise; $y = x + 1$.
21. (a) 5, 13, (b) 2·5, 19·5. 22. $7\sqrt{5}$ N, 1 m; 28 N m in sense *CBA*.
23. (a) 10; 28 N along *AD*, (b) 18 N, $14\sqrt{3}$ N m in sense *ABC*.
24. (a) $25P$, $24y = 7x$, (b) (i) $S = -18P$, $T = 15P$, (ii) $72Pa$, (iii) clockwise.

Exercise 9.3 (pp. 162–164)

1. 18 cm from *A*. 2. 1·05 m from *P*. 3. 5 cm. 4. 3 cm, 4 cm.
5. (2, 1). 6. $(3, -\frac{1}{2})$. 7. $3\mathbf{i} + \mathbf{j}$. 8. $2\mathbf{i} + 2{\cdot}5\mathbf{j}$. 9. (3, 0).
10. (0·4, 1·6). 11. 10 cm, 3 cm. 12. 10 cm. 13. $1\frac{2}{3}$ cm from *AB*.
14. $6\frac{1}{3}$ cm from *AD*, $4\frac{4}{5}$ cm from *AB*. 15. 11 cm from both *AB* and *AF*.
16. $3\frac{1}{2}$ cm from *AE*, 4 cm from *AB*. 17. 4 cm from *AD*, 3·5 cm from *AB*.
18. $7a/5$ above base, $16a/11$ above base.

Exercise 9.4 (pp. 170–171)

3. $\dfrac{2a(14 + 3\pi)}{3(8 + \pi)}$. 4. $\dfrac{2a(26 - 3\pi)}{3(16 - \pi)}$. 5. $\dfrac{28r}{9\pi}$.

6. $\dfrac{27\sqrt{3}a}{3\sqrt{3} + 4\pi}$. 7. $\dfrac{11h}{28}$.

8. (a) $\dfrac{7\sqrt{3}r}{8\pi + 3\sqrt{3}}$, (b) $\dfrac{7\sqrt{3}r}{4\pi - 3\sqrt{3}}$. 9. $28\pi a^3/3$, $12a/7$. 11. $3r$.

Exercise 9.5 (pp. 175–176)

1. 6, 103/15; (9/7, 103/70). 2. 153/4, 414/5; (17/8, 23/5).
3. ln 2, 7/48; (2 ln 2, 7/24). 4. 1/4, 2/5; (3/8, 3/5). 5. (20/7, 0).
6. (3/5, 0). 7. $(0, \pi/8)$. 8. (0, 7/3). 9. $\left(\dfrac{1}{e-1}, \dfrac{e+1}{4}\right)$.
10. $(1, \pi/8)$. 11. (2, 0). 12. (2/3, 0). 13. (2 ln 2, 0).
14. $\left(\dfrac{e^2 - 3}{2(e^2 - 1)}, 0\right)$. 15. (9/7, 0). 16. (0, 14/5).
17. $2\ln 2 - 1$, $2\ln 2 - \frac{3}{4}$, $2(\ln 2)^2 - 4\ln 2 + 2$; (1·65, 0·24).
18. 1/6; (1/2, 2/5).

Exercise 9.6 (pp. 176–179)

1. (a) $13P$, $12/5$, (b) $3a$; $12Pa$, sense ABC. 2. $\lambda = 5/2$; $5a/2$; $5a/7$.
3. (c) (i) $3\sqrt{2}$ along AC, $\sqrt{2}$ along BD; $7a$;
   (ii) 7 along AD, $4\sqrt{2}$ along CA, $6\sqrt{2}$ along DB.
4. (a) $4\sqrt{3}P$, $\frac{\sqrt{3}}{3}P$; $\frac{7\sqrt{3}}{3}P$; (b) $\frac{5\sqrt{3}}{3}P$, $\frac{\sqrt{3}}{3}P$; $\frac{7\sqrt{3}}{3}Pa$ in sense ABC.
5. $p = 11$, $q = 14$; (a) $2\sqrt{19}$ N, (b) $12\sqrt{3}$ N m.
6. (i) $m = 2$, $n = 6$, (iii) $x = 3a\sqrt{3}$; $\sqrt{31}$, $5Pa\sqrt{3}$.
7. $(7a/18, 4a/9)$.
8. 4·8 cm from OA, 8·8 cm from OC.
9. $\frac{11}{24}\pi a^3$, $\frac{5}{24}\pi a^3$; $\frac{21}{88}a$, $\frac{27}{40}a$ from centre of hemisphere.
11. $\frac{3a}{8}$. 12. $\frac{32}{15}\sqrt{2}$; $\frac{8\pi}{5}$. 13. 16/3; 8π; $(0, 20/3)$.
14. (0·864, 0·333). 15. 4/3, $104\pi/15$; (15/13, 0). 16. $3\pi a^2$, $(\pi a, 5a/6)$.

Exercise 10.1 (pp. 186–188)

1. 360 N, 2·5 m. 2. 41 N at A, 33 N at B. 3. 1·6 m.
4. 0·6 m, 20 N. 5. (a) 1200 N (b) 3 m. 6. 0·4 m.
7. $5W/8$ at A, $3W/8$ at B. 8. 5. 9. $5a/3$, $5a/2$; $\tan^{-1} 10/21 \approx 25{\cdot}5°$.
10. (a, a); 1/2. 11. $2/(\pi + 2)$. 12. $1/\pi$.
13. $\tan^{-1} 3/4 \approx 36{\cdot}9°$; $3aW$. 14. $\tan^{-1} 11/13 \approx 40{\cdot}2°$; $26aW/9$.
15. $\sqrt{2}$. 16. $2W/3$.
17. $h + \left(\frac{3a^2 - h^2}{8a + 4h}\right)$ from vertex of cone; (a) 17°, (b) $h = a\sqrt{3}$.
18. 30°; $1/\sqrt{3}$. 19. 36·9°. 20. (a) 53·1°, (b) 26·6°.

Exercise 10.2 (pp. 195–197)

1. (a) 10, (b) $10\sqrt{5}$ N at 26·6° to vertical.
2. $\frac{\sqrt{3}}{2}mg$ in AC, $\frac{1}{2}mg$ in BC; $\frac{1}{2}a$. 3. 2 N at A, $2\sqrt{17}$ N at C.
4. 30 N at 60° to vertical. 5. 45 cm, 16 N.
6. (a) $20\sqrt{3}$ N, 90°, (b) $15\sqrt{5}$ N, 63·4°. 7. 5/6; 5W/12, 13W/12.
8. (a) $W\sqrt{3}$ at both A and C, (b) $2a$.
9. (a) 15 N, (b) 9 N at 53·1° to vertical. 10. $5W$, 75 cm. 11. $\frac{1}{2}$; 30 cm.
12. 5/24. 13. (a) $\frac{1}{2}mg\cot\alpha$, (b) $\frac{1}{2}\cot\alpha$.
14. 45°, $\frac{1}{2}W$; half way up, all the way. 15. (a) 30°, (b) $1/\sqrt{3}$.
17. 1/3.

Exercise 10.3 (pp. 202–205)

1. (a) 24 N, (b) $12\sqrt{3}$ N, 13 N.
2. (a) 60 N, (b) 48·2 N at 41·6° to horizontal.
3. (a) 15, (b) 47·4 N at 71·6° to horizontal.
4. $4\sqrt{3}$ N perpendicular to rod; $2\sqrt{3}$ N, 10 N.
5. (a) 14 N, 30 N (b) 1·92 m. 6. (a) 4/7, (b) $\tan^{-1}\frac{2}{3} \approx 33{\cdot}7°$.
7. 5 N, 1/2. 8. 121 N, 17·5 N; 7/48. 9. $\frac{1}{2}\tan\alpha, 2W\tan\alpha$.
10. $3W/10$, $W/10$; 0·8. 11. 1/5. 12. $8W/15$; $24W/75$, $43W/75$.
13. $(5\sqrt{3})/18$; $(W\sqrt{3})/2$. 14. $5a/3$. 15. $\tan^{-1}(4/3)$; 40, 900 N.
16. 10, 6/17. 17. $W/4$, 1/3. 18. $11W/4$. 19. 5 m.

Exercise 10.4 (pp. 209–211)

1. (a) $5\sqrt{10}$ N, $5\sqrt{5}$ N (b) 21·8°, 45°. 2. $\frac{1}{2}W\sqrt{5}$, 63·4°.
3. $\frac{1}{3}W\sqrt{3}, \frac{1}{2}W$.
4. (ii) $W/\sqrt{(1+\mu^2)}$ at $\tan^{-1}\mu$ to vertical, (iii) $\mu W = \frac{1}{2}W\cot\theta$.
5. (a) 7/10, (b) $5W/2$, $3W/4$. 6. C; 67·4°. 7. C will slip.
8. (a) $9W/5$, $12W/5$, (b) 3·2°. 9. $\frac{3}{5}\sqrt{3}$. 10. $2-\sqrt{3}$.
11. $\frac{3}{4}W, \frac{1}{4}W\tan\alpha$; $1/\sqrt{3}$.
13. (i) $\frac{9w}{25}$, (ii) $W+\frac{27w}{125}$, (iii) $\frac{36w}{125}$.

Exercise 10.5 (pp. 213–215)

1. AB: $\sqrt{3}$ kN tension; BC: $4\sqrt{3}$ kN tension; CD: $2\sqrt{3}$ kN thrust; AD: $2\sqrt{3}$ kN thrust; BD: $2\sqrt{3}$ kN tension.
2. AB and BC: 5 kN thrust; BD: no stress; CD: 4 kN tension; AD: 3 kN tension.
3. AB: 2 kN thrust; BC: $2\sqrt{2}$ kN tension; CD: 4 kN tension; AC: $2\sqrt{2}$ kN thrust.
4. AB: 8 kN thrust; BC: $4\sqrt{3}$ kN thrust; CD: $2\sqrt{3}$ kN tension; BD: 4 kN tension.
5. AB: $10\sqrt{2}$ kN thrust; BC: $5\sqrt{2}$ kN thrust; CD: 5 kN tension; DE: $5\sqrt{2}$ kN tension; AE: 10 kN tension; BD: 5 kN thrust; BE: 5 kN tension.
6. AB: 4 kN tension; BC: 3 kN thrust; CD: 3 kN thrust; AD: 4 kN tension; AC: 15 kN tension.
7. AB: $4\sqrt{3}$ kN tension; BC: $2\sqrt{3}$ kN tension; CD: $4\sqrt{3}$ kN thrust; DE: $3\sqrt{3}$ kN thrust; AE: $8\sqrt{3}$ kN thrust; BD: $2\sqrt{3}$ kN tension; BE: $2\sqrt{3}$ kN thrust.
8. AB and BC: 6 kN tension; CD and AE: $6\sqrt{3}$ kN thrust; BD and BE: $2\sqrt{3}$ kN tension; DE: $4\sqrt{3}$ kN thrust.
9. $P = \frac{1}{3}W\sqrt{2}$; AB: $\frac{2}{3}W$ thrust; BC: $\frac{1}{3}W\sqrt{2}$ tension; CD: $\frac{1}{3}W$ thrust; AC: $\frac{2}{3}W\sqrt{2}$ tension; AD: $\frac{2}{3}W$ thrust.
10. (ii) AB: 390 N; BC: 650 N; CD: 150 N; AD: 250 N; AC: $120\sqrt{5}$ N (iii) AC and CD.

Exercise 10.6 (pp. 215–222)

2. $4/11$; $7:2$. 4. $5a/2$, $5a/2$; $\tan^{-1}(1/6) \approx 9{\cdot}5°$; $W/6$.
7. (a) $12\frac{1}{2}$ N; $7\frac{1}{2}$ N in the direction CA, (b) $6\frac{1}{4}$ N; $6\frac{1}{4}$ N at 53·1° to CA produced; 8 N acting 0·625 m from A.
8. $\frac{1}{2}W\sqrt{5}$; $7l/12$.
9. (a) 15 N, (b) 8/3 m, (c) 42·7 N at 73·7° to the horizontal.
10. (ii) $\frac{1}{4}\sqrt{3}$, (iii) $24c/7$. 11. $3/46$. 12. $3/8$, $5l/3$.
14. $\frac{3}{2}mg\cot\alpha$; $mg\cot\alpha$. 15. $\cos^{-1}(2m/M)$.
16. $W(1 - \sin 2\theta)$; 18·4°.
17. $AX = \frac{2}{3}a$, $13W$ at 67·4° to XA; $CY = 11a/12$.
18. (i) $25W/37$, $20W/37$, (iii) 13/40.
20. $(8a/15, 3a/16)$; $128/45 \approx 2{\cdot}84$.
21. $h + \left(\dfrac{6l^2 - h^2}{4h + 12l}\right)$ from vertex of cone. 22. $\sin^{-1}\left(\dfrac{k+3}{3k-24}\right)$.
23. (i) $35\sqrt{2}$ N, (ii) $5\sqrt{2}$ N.
24. Force at the ground. 25. $\frac{1}{4}Wl\sqrt{3}$.
26. $\frac{1}{2}$ W$\sqrt{5}$; $\frac{1}{2}W$, 0. 27. Slipping at A.
28. (i) $\frac{1}{2}W$, (ii) $\frac{3}{2}W$, W, (iv) $\frac{1}{2}$.
29. $2W\sqrt{3}$ along BA; $\frac{1}{3}W\sqrt{3}$, $3W/2$ at A. $\frac{1}{3}W\sqrt{3}$, $\frac{1}{2}W$ at C.
30. $\frac{2}{3}W\sqrt{3}$ between cylinders, $\frac{1}{3}W\sqrt{3}$ at both walls. 31. $1/\sqrt{3}$.
32. (i) $35W/4$, (ii) 0, $5W$ at A; $7W$, $5W/4$ at B, (iii) $21aW/2$.
33. (i) 2400 N, (ii) 2600 N at 67·4° to vertical,
(iii) AB: 2400 N; BC: 700 N; CD: $700\sqrt{2}$ N; DE: 1400 N; BD: $700\sqrt{2}$ N; BE: $1000\sqrt{2}$ N; rods AB, BC, BD in compression.
34. (i) 14, (ii) AB: $4/\sqrt{3}$ kN tension; BC: $8/\sqrt{3}$ kN thrust; CD: $8/\sqrt{3}$ kN thrust; DE: $16/\sqrt{3}$ kN thrust; AE: $2/\sqrt{3}$ kN thrust; BD: $4\sqrt{3}$ kN tension; BE: $4\sqrt{3}$ kN thrust, (iii) 2 kN vertically downwards.

Exercise 11.1 (pp. 227–228)

1. $6\mathbf{i} - 8\mathbf{j}$, $4\mathbf{i} + 3\mathbf{j}$; $10\mathbf{i} - 5\mathbf{j}$.
2. $3\sqrt{2}$ N, $5\sqrt{2}$ N. 3. 27 N.
4. (a) $\mathbf{r} = (1 + \lambda)\mathbf{i} + (1 + 2\lambda)\mathbf{j}$, (b) $\mathbf{r} = -2\lambda\mathbf{i} + (5\lambda + 2)\mathbf{j} + (3\lambda - 1)\mathbf{k}$,
(c) $\mathbf{r} = 3\mathbf{i} - 2\lambda\mathbf{j} + 3\lambda\mathbf{k}$.
5. $\mathbf{r} = (2 + 3\lambda)\mathbf{i} + (\lambda - 1)\mathbf{j} - 2\lambda\mathbf{k}$, $\mathbf{r} = (9 + \mu)\mathbf{i} - \mu\mathbf{j} + (3\mu - 1)\mathbf{k}$; $8\mathbf{i} + \mathbf{j} - 4\mathbf{k}$; $\mathbf{r} = (8 + 4t)\mathbf{i} + \mathbf{j} + (t - 4)\mathbf{k}$.
6. 11 N, $\mathbf{r} = (3 + 7\lambda)\mathbf{i} + (5 + 6\lambda)\mathbf{j} + 6\lambda\mathbf{k}$.
7. $3x + y = 26$.
8. $3\sqrt{13}$ N, $\frac{2}{3}$; $2x - 3y + 2 = 0$.
9. (a) 15**YX**, where $BX : XC = 3:2$, $AY : YB = 2:1$,
(b) 24**XY**, where $AX : XC = 1:3$, $AY : YB = 1:5$,
(c) 6**YX**, where $AX : XC = 2:1$ and $AY = YB$,
(d) 21**XY**, where $AX : XC = 1:2$, $AY : YB = 3:4$.

Exercise 11.2 (pp. 231–232)

1. $\mathbf{r}_A = (3 + 5t)\mathbf{i} + (4 + 2t)\mathbf{j}$, $\mathbf{r}_B = (5 + 2t)\mathbf{i} + (8 + t)\mathbf{j}$; $\mathbf{AB} = (2 - 3t)\mathbf{i} + (4 - t)\mathbf{j}$; $\sqrt{10}$ m.
2. $4\mathbf{i} - 6\mathbf{j}$; $(4t - 2)\mathbf{i} + (3 - 6t)\mathbf{j}$; $t = \frac{1}{2}$.
3. $(2 - 5t)\mathbf{i} + (4t - 18)\mathbf{j}$; $2\sqrt{41}$ km; $16\mathbf{i} + 19\mathbf{j}$, $8\mathbf{i} + 9\mathbf{j}$.
4. $8\cos\omega t\mathbf{i} + 8\sin\omega t\mathbf{j}$. 5. $\mathbf{i} - 2\mathbf{j}$, 3 m; 6 m s^{-1}.
6. (a) $\pi/4$, (b) $5\pi/4$. 7. (a) 1/3, (b) 3.
8. (a) $8t\mathbf{i} + (9t^2 - 2)\mathbf{j}$, $m(8\mathbf{i} + 18t\mathbf{j})$, (b) $-6t^2\mathbf{j} + 2t\mathbf{k}$, $m(-12t\mathbf{j} + 2\mathbf{k})$,
   (c) $(2 - 2t)\mathbf{i} + (6\cos 2t)\mathbf{j}$, $m\{-2\mathbf{i} - (12\sin 2t)\mathbf{j}\}$,
   (d) $e^{-t}(1 - t)\mathbf{i} - 2\mathbf{j}$, $me^{-t}(t - 2)\mathbf{i}$.
9. (a) $\sqrt{19}$ m s^{-1}, (b) $20\mathbf{i}$. 10. $12\mathbf{i} + 30\mathbf{j}$, $18\mathbf{i} + 45\mathbf{j}$.
11. 40 m s^{-1}, $18\mathbf{i} - 35\mathbf{j}$. 12. $5\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$.
13. $2\mathbf{i} + 6\mathbf{j}$; $9\sqrt{10}$ m.
14. $(16t - 5t^2)\mathbf{j}$, $12t\mathbf{i} + (25 - 5t^2)\mathbf{j}$; 15 m.

Exercise 11.3 (pp. 238–240)

1. 31·6°, 5 m s^{-1}. 2. $\sqrt{41}$ N $\approx$ 6·40 N, 50·6°.
3. (a) 2 N, (b) -3 N, (c) $\sqrt{2}$ N, (d) $-5/3$ N.
4. (a) $8\mathbf{i} - 6\mathbf{k}$, $3\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$, (b) $-10\mathbf{j} + 24\mathbf{k}$, $5\mathbf{i} - 12\mathbf{j} - 5\mathbf{k}$,
   (c) $-6\mathbf{i} + 18\mathbf{j} - 27\mathbf{k}$, $24\mathbf{i} + 17\mathbf{j} + 6\mathbf{k}$, (d) $\frac{2}{3}(4\mathbf{i} - 8\mathbf{j} + \mathbf{k})$, $\frac{1}{3}(\mathbf{i} + 4\mathbf{j} + 28\mathbf{k})$.
5. (a) 2 J, (b) 0 J, (c) 21 J, (d) -11 J.
6. (a) $8m$ J, (b) $24m$ J, (c) $360m$ J, (d) 0 J.
7. (a) $4m\mathbf{i}$, $4t\mathbf{i} - 5\mathbf{k}$; $16mt$, (b) $6m(t\mathbf{j} - \mathbf{k})$, $3t^2\mathbf{j} - 6t\mathbf{k}$; $18mt(t^2 + 2)$,
   (c) $-me^{-t}\mathbf{j}$, $\mathbf{i} + e^{-t}\mathbf{j}$; $-me^{-2t}$.
   (d) $-m(\sin t\mathbf{j} + 3\cos t\mathbf{k})$, $\cos t\mathbf{j} - 3\sin t\mathbf{k}$; $4m\sin 2t$.
8. 14 J; 8 m s^{-1}. 9. -114 J; 6; -96 W, -18 W.
10. $t^2\mathbf{i} + (1 - 3t)\mathbf{j} + \mathbf{k}$; 20 J; 31 W.
11. $(4t + 6)\mathbf{i} - 2\sin t\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$, $t^3 + 32t + 48 + 4\sin 2t$. 12. 24 J.
13. $10t^4 + 6t^2 + 2t$; 544 J. 14. $\int_1^2 \mathbf{F}.(\mathbf{i} - 2\lambda\mathbf{j})d\lambda$; 12.

Exercise 11.4 (pp. 240–242)

1. (i) 26·7 N in the direction of the vector $3\mathbf{i} + 13\mathbf{j}$; $-36/65$, (ii) $6\mathbf{i} + 26\mathbf{j}$.
2. $(2T^4 - 2)\mathbf{i} + (3T^2 + 3)\mathbf{j}$; $94\mathbf{i} + 30\mathbf{j}$.
3. $6\mathbf{i} + 8\mathbf{j}$, 6; $6\mathbf{i} + 8\mathbf{j} + 6(\mathbf{i}\cos(\pi t/12) + \mathbf{j}\sin(\pi t/12))$; $-\pi^2/24$, $-\pi^2\sqrt{3}/24$;
   $(9 - \sqrt{3}\pi)\mathbf{i} + (8 + 3\sqrt{3} + \pi)\mathbf{j}$. 4. $1/(1 + \cos\theta)$; towards O.
5. $2/\sqrt{13}$. 6. (i) $(10 - 2t)\mathbf{i} + (5 + 4t)\mathbf{j}$, (ii) 14.00 hrs, (iii) $4\sqrt{5}$ km.
7. $(260 - 35t)\mathbf{i} - 5\sqrt{3}t\mathbf{j}$; 62 km, 19.00 hrs.
8. (a) $\mathbf{v} - \mathbf{u}$, (b) $\mathbf{p} + t(\mathbf{v} - \mathbf{u})$; $\left\{p^2 - \dfrac{[\mathbf{p}.(\mathbf{u} - \mathbf{v})]^2}{|\mathbf{u} - \mathbf{v}|^2}\right\}^{1/2}$

9. (i) $\mathbf{F}.\mathbf{v}$, (ii) $\int_0^T \mathbf{F}.\mathbf{v}\,dt$; $-4a\sin 2t\mathbf{i} + 2a\cos 2t\mathbf{j}$,
$-8ma\cos 2t\mathbf{i} - 4ma\sin 2t\mathbf{j}$; $12ma^2\sin 4t$; $6ma^2$, $\frac{1}{4}\pi$.
10. $\frac{1}{2}ma^2(2 - 2\cos\theta)\dot{\theta}^2 + mga(1 + \cos\theta) = 2mga$.

Exercise 12.1 (pp. 248–250)

1. (a) $8\mathbf{i} - 28\mathbf{j}$, (b) $2\mathbf{i} - \frac{2}{3}\mathbf{j} + \mathbf{k}$, (c) 65, (d) 4.
2. (a) $10\mathbf{i} + 6\mathbf{j}$, (b) $2\mathbf{i} - 52\mathbf{j} + 8\mathbf{k}$, (c) $3\mathbf{j}$.
3. (a) 480 N s, (b) 30 000 N s, (c) 12 N s.
4. (a) $12\mathbf{j}$, (b) $-2\mathbf{i} + 6\mathbf{k}$, (c) $-60\mathbf{k}$. 5. 54 N s. 6. 4 N s.
7. 600 N. 8. (a) 16·8 m s^{-1}, (b) 7·2 m s^{-1}. 9. 7·5 s.
10. $12\mathbf{i} + 15\mathbf{j}$; $2\mathbf{i} + 2{\cdot}5\mathbf{j}$. 11. $8\mathbf{i} + 4\mathbf{j}$; 10 J
12. (a) 14 m s^{-1}, 240 J gain, (b) 2 m s^{-1}, 48 J loss, (c) 10 m s^{-1}, 96 J gain.
13. 2·5 N s. 14. $20\sqrt{3}$ N s. 15. 600 N. 16. 12 m s^{-1}.
17. 2 N at 53·1° to horizontal
18. (a) 2 m s^{-1}, $6\sqrt{3}$ N s, (b) $(4\sqrt{3})/3$ m s^{-1}, $4\sqrt{6}$ N s.
19. 1·5 m s^{-1} along AB, 5·22 m s^{-1} at 73·3° to AB; 4·5 N s.
20. (a) $2\sqrt{2}$ m s^{-1}, $2\sqrt{2}$ m s^{-1}, 4 m s^{-1}, (b) 2 m s^{-1}, $2\sqrt{3}$ m s^{-1}, 4 m s^{-1}.

Exercise 12.2 (pp. 252–253)

1. 5 m s^{-1}. 2. 720 m s^{-1}. 3. 7·5 m s^{-1} towards A; 78·75 J.
4. 2·5 m s^{-1}, 2512·5 J. 5. 5 m s^{-1}. 6. 7 m s^{-1}. 7. $2\mathbf{i} - 5\mathbf{j}$.
8. $5\mathbf{i} + \frac{5}{2}\mathbf{j}$. 9. 28 J. 10. $3u$, $2u$. 11. 6 m s^{-1}, 26 kJ.
12. 3·5 m s^{-1}, 2000 N. 13. 0·6 m s^{-1}, 720 N s. 14. 30.
15. (a) 14 m s^{-1}, (b) 23·8 m s^{-1}. 16. $u/2g\sin\alpha$.

Exercise 12.3 (pp. 258–260)

1. 4800 N, 5 s. 2. 9000 N, 1/600 s. 3. 2 m s^{-1}, 25 m. 4. 1.
5. (a) 70 N s, (b) 77 N s. 6. 10/7 s, $14\sqrt{2}$ m s^{-1}.
7. 4·2 m s^{-1}, 78 400 N. 8. 90 160 N, 1/7 s. 9. 1·25 m.
10. 469 m s^{-1}. 11. h, $\frac{1}{4}h$. 12. 0·1 m. 13. 2·1 m s^{-1}, 6·3 N s.
14. 7·35 m s^{-1}, $2\frac{1}{4}$ s. 15. 3 s, 4·2 m s^{-1}. 16. (a) $\frac{7}{3}h$, (b) $\frac{2}{3}m\sqrt{(\frac{2}{3}gh)}$.
17. 2·8 m s^{-1}, 20·6 J. 18. (i) $\frac{3}{4}mgh$, (ii) $9m^2gh/4M$.
19. (ii) $\frac{12}{5}m\sqrt{(gc)}$, (iii) $\frac{33}{25}c$, (iv) $\frac{12}{5}mgc$.
20. 40 m s^{-1}, 80 J; 1200 J; $(d + 14)$ cm; 1, 8000 N.

Exercise 12.4 (pp. 265–267)

1. (a) 2·5 m, (b) 14·7 J, (c) 4·2 N s. 2. $\frac{3}{4}$, 4·05 m. 3. $\frac{2}{3}$.
4. $8\sqrt{5}$ m s^{-1}, 60 N s. 5. $1/\sqrt{3}$; 45°. 6. 60°.
7. $4\frac{1}{6}$ m s^{-1} at 53·1° to horizontal. 8. $2u$, $4u$; $18mu^2$. 9. $\frac{1}{2}$.

10. (a) $1\,\mathrm{m\,s^{-1}}$, $4\,\mathrm{m\,s^{-1}}$, same direction.
(b) $\frac{1}{2}\,\mathrm{m\,s^{-1}}$, $5\frac{1}{2}\,\mathrm{m\,s^{-1}}$, opposite directions.
11. $\frac{1}{4}$, 2:3. 12. $5\,\mathrm{m\,s^{-1}}$, $-4\,\mathrm{m\,s^{-1}}$, $16\,\mathrm{m\,s^{-1}}$; yes.
13. $u/4$, $3u/16$, $9u/16$; yes. 14. 2/9. 16. $u(1-e)$, u.
17. $\frac{1}{2}\sqrt{(2ag)}$, $\frac{1}{4}\sqrt{(2ag)}$ in same direction.
18. (a) $\frac{1}{3}V(1-2e)$, $\frac{1}{3}V(1+e)$, 0 (b) $\frac{1}{3}V(1-2e)$, $\frac{1}{9}V(1+e)(2-e)$, $\frac{2}{9}V(1+e)^2$.

Exercise 12.5 (pp. 267–272)

1. (i) $9mu$, (ii) $81mu^2/4$. 2. (a) $12\mathbf{i}+4\mathbf{j}$, (b) $5\mathbf{i}-2\mathbf{j}$, (c) 32 J.

3. $-\frac{mv}{2}\mathbf{i}+\frac{mv}{2\sqrt{3}}\mathbf{j}$; $\frac{v}{3\sqrt{3}}$ at 30° to the x-axis in a clockwise sense.

4. (i) 283 kg, (ii) 11 500 J, (iii) 39 kW, (iv) 56 kW; 2500 N.
5. (ii) $u/6$, $(u\sqrt{3})/4$; $mu/6$. 6. $|m_1u_1-m_2u_2|/(m_1+m_2)$.
8. $\frac{1}{2}mgh$, $m(1+\sqrt{2})\sqrt{(gh)}$. 9. $1{\cdot}6\,\mathrm{m\,s^{-1}}$, $0{\cdot}4\,\mathrm{m\,s^{-1}}$; 131 mm, 8 mm.
10. (i) $5{\cdot}6\,\mathrm{m\,s^{-1}}$, (ii) $3{\cdot}2\,\mathrm{m\,s^{-1}}$, (iii) 13 440 J, (iv) 123·9 kN.
11. (i) $\frac{6}{5}m\sqrt{(2ga)}$, (ii) $\frac{6}{5}mga$, (iii) $\frac{2}{5}\sqrt{(17ga)}$.
12. $1{\cdot}96\,\mathrm{m\,s^{-2}}$; $1{\cdot}12\,\mathrm{m\,s^{-1}}$, 3·36 N s.
14. (i) $g/5$, $12mg/5$, (iii) $9mv/8$, (iv) $15mv^2/16$, (v) $1/\sqrt{3}$.
15. $mMu^2/2a(m+M)$. 16. (b) 16°, (c) 5:6.

17. $\frac{u\sin\alpha}{\sin\beta}$; $\frac{1}{2}u\left|\frac{\sin(\alpha-\beta)}{\sin\beta}\right|$; $\frac{u^2}{g\sqrt{3}}$.

19. (a) $\cot\alpha\tan\beta$, (b) $u\cos\alpha\sec\beta$, (c) $mu\sin(\alpha+\beta)\sec\beta$.
20. (a) $\frac{1}{2}mu^2(1-e^2)\cos^2\alpha$, (b) $u^2e\sin 2\alpha/g(1+e)$.
21. $\frac{1}{2}$; 2:7; $2\,\mathrm{m\,s^{-1}}$, $7\,\mathrm{m\,s^{-1}}$ in opposite directions.
22. (i) u along $\vec{BA}$, (ii) 4/5, (iii) $3mu^2$,
(iv) u, $4u/3$ along $\vec{BC}$, (v) 1/9.
23. (i) $\frac{1}{2}$, (ii) $\frac{1}{4}mu^2$, (iii) $\frac{2}{3}$, (iv) u. 24. $4u/3$, $u/6$; $7mu^2/12$.
26. $\frac{1}{3}(7-8e)u$, $\frac{1}{3}(7+16e)u$ in direction $\vec{AB}$; 1/8;
$3u/(1+k)$, $k>\frac{1}{2}$.

27. $e=1,\frac{1}{3},\frac{1}{5},\frac{1}{7},\ldots,\frac{1}{1+2n},\ldots$.

Exercise 13.1 (pp. 275–276)

1. $\ln 3\,\mathrm{m\,s^{-1}}$. 2. $\ln 5$ seconds. 3. $6\,\mathrm{m\,s^{-1}}$. 4. $0{\cdot}585\,\mathrm{m\,s^{-1}}$.
5. $2a$. 6. $\sqrt{(gR/3)}$. 7. (a) $2/x$, (b) $\sqrt{(4t+h^2)}$. 8. $-\frac{1}{2}e^x$.

Exercise 13.2 (pp. 279–282)

1. $\ln 3$. 2. $\frac{1}{2}\ln(1+u^2)$. 3. $-\frac{1}{k}\{v+\ln(1-v)\}$.

4. $\frac{a}{b^2}(2 - \ln 3)$. 5. $\frac{2u}{2 + su}$; $\frac{s^2}{4} + \frac{s}{u}$. 6. (i) 482 m, (ii) 24·1 s.

8. $\left[\frac{g}{k}(1 - e^{-2ks}) + u^2 e^{-2ks}\right]^{1/2}$. 9. (iii) 11·1 s.

10. $\frac{dv}{dt} = \frac{k(U^2 - v^2)}{mv}$; $v\frac{dv}{dx} = \frac{k(U^2 - v^2)}{mv}$.

11. $\frac{m}{2k}\tan^{-1}(\frac{1}{2}u)$; $\frac{m}{2k}ln(1 + \frac{1}{4}u^2)$. 12. $\frac{P^2}{k}\left(1 - \frac{1}{e}\right)$.

15. $\frac{m}{R^2}\left\{H\ln\left(\frac{H - Ru}{H - 2Ru}\right) - Ru\right\}$;

$\frac{m}{R^3}\left\{H^2\ln\left(\frac{H - Ru}{H - 2Ru}\right) - HRu - \frac{3}{2}R^2u^2\right\}$.

16. $\frac{mU^2}{2S}\ln\left(\frac{U^2}{U^2 - V^2}\right)$; $\frac{mU^2}{2S}\left\{U\ln\left(\frac{U + V}{U - V}\right) - 2V\right\}$.

Exercise 13.3 (pp. 286–287)

1. $5\,m\,s^{-1}$, $10\,m\,s^{-2}$. 2. (a) 2 m, (b) $\frac{2}{3}\pi$ s, (c) $3\sqrt{3}\,m\,s^{-1}$.
3. 6π s; 2·5 N. 4. (i) $270/\pi$, (ii) $\frac{2}{3}\sqrt{13}$ m, (iii) $540\sqrt{13}$ N.
5. $6\,m\,s^{-1}$, 2 m; $5{\cdot}9\,m\,s^{-1}$. 6. (i) π s, (ii) 2π s, (iii) $3\pi/2$ s.

7. (i) $\frac{5}{3\pi}$ m, (ii) $\frac{5}{2}\sqrt{3}\,m\,s^{-1}$, (iii) $\frac{1}{2\pi}$ m, (iv) $\frac{1}{18}$ s.

8. $2\sqrt{3}\,m\,s^{-1}$; 1/6 s; 8 J.
9. (a) $x = 3\cos 2t + 3\sin 2t$, (b) -3, 6, (c) $3\pi/8$.
10. (a) $x = 3\cos 4t + \sqrt{3}\sin 4t$, (b) $\pi/24$, (c) $2\sqrt{3}$.

Exercise 13.4 (pp. 293–295)

1. $5a/3$. 2. (i) $2l$, (ii) $\frac{1}{2}\pi\sqrt{(3l/g)}$, (iii) $15l/8$.
3. $\frac{1}{2}mg$, $\frac{1}{2}mg$; (i) $3l/2$, (ii) $\pi(2l/g)^{1/2}$. 4. 4.
5. $2\pi\sqrt{(2a/g)}$, a; $\frac{1}{2}a$ above 0. 6. $\sqrt{(2a\lambda/m)}$.

7. $5l/2$, $\pi(lm/\lambda)$, $\sqrt{(\lambda l/m)}$. 8. $2l$; $\pi\sqrt{\left(\frac{2l}{g}\right)}$, $\left(\frac{2u^2l + gl^2}{4g}\right)^{1/2}$.

9. $u\sqrt{\left(\frac{lm}{\lambda}\right)}$; $\frac{\pi}{2}\sqrt{\left(\frac{lm}{\lambda}\right)}$; $\pi\sqrt{\left(\frac{lm}{\lambda}\right)} + \frac{2l}{u}$. 10. $\frac{16}{9}a$; $\sqrt{(5ag)}$; $\frac{\pi}{6}\sqrt{\left(\frac{a}{g}\right)}$.

11. (a) 994 mm, (b) 991 mm, (c) 996 mm, (d) 164 mm.
12. (a) 143 cm, (b) 56 cm, (c) 36 cm, (d) 14 cm.

Exercise 13.5 (pp. 300–302)

1. $49\cos\theta - 24$, $58{\cdot}8\cos\theta - 19{\cdot}2$; 3·68, 25·8; 61°.
2. $6{\cdot}18\,\mathrm{m\,s^{-1}}$, 19·35 N. 3. $\frac{1}{3}$, $\frac{1}{3}ag$.
5. $mg|3\cos\theta - 2|$; $\frac{2}{3} < \cos\theta < 1$. 6. $\sqrt{(2ag/3)}$.
7. $\dfrac{mu^2}{a} - 2mg + 3mg\cos\theta$; mg. 8. (ii) $\frac{7}{2}ag$, $\sqrt{(\frac{1}{2}ag)}$.
9. $\cos\alpha\sqrt{(2gl\sin\alpha)}$; no. 10. $\{2ag(1+\sin\theta)\}^{1/2}$; (b) $\pi/6$, $(3ag)^{1/2}$; $(a/g)^{1/2}$.
11. $\{u^2 - 6ag(1-\cos\theta)\}^{1/2}$; $\dfrac{mu^2}{2a} - 3mg(1+\sin\phi)$.
12. $\frac{2}{3}mg(1+\cos\theta)$, $\frac{1}{3}mg(5\sin\theta - 4\theta)$.

Exercise 13.6 (pp. 302–307)

1. Particle escapes from the earth.
2. $mv\dfrac{dv}{dx} = Ke^{-\alpha x} - \lambda m$; $\left\{\dfrac{2K}{\alpha m}\left(1 - e^{-\alpha x}\right) - 2\lambda x\right\}^{1/2}$.
3. $\frac{1}{2}(\mu/a)^{1/2} \leqslant V \leqslant \frac{3}{4}(\mu/a)^{1/2}$.
4. $5\cos 3t + 12\sin 3t + 2$; 15, 39; $t \approx 0{\cdot}967$. 5. $\dfrac{2k^2}{g}\left(1 - e^{-gt/k}\right) - kt$.
6. $\frac{1}{800}$ s; $\dfrac{dv}{dt} = -16 \times 10^3(2v)^{1/2}$, $\frac{1}{400}$ s; $v = Ae^{-kt}$, A and k positive constants.
7. (i) $2uT$, (ii) $T\ln 3$, (iii) $4mu^2$. 8. $v = gTe^{-x/gT^2}$
9. $v^2 = u^2\tan\left(\dfrac{\pi}{4} - \dfrac{2gz}{u^2}\right)$. 10. (ii) $u\sqrt{(1 - e^{-2kx})}$.
13. (b) (i) $3\pi/8$ s, (ii) $26\frac{2}{3}\,\mathrm{m\,s^{-1}}$, (iii) $142\frac{2}{9}\,\mathrm{m\,s^{-2}}$, (iv) $\pi/16$ s.
14. $\dfrac{\pi}{4}$ m/s, $\dfrac{\pi^2}{8}$ m/s^2; $\dfrac{\pi\sqrt{3}}{8}$ m/s, $\frac{2}{3}$ s.
15. (i) 12 s, $24/\pi$ m, (ii) $t = 5$; $12\sqrt{2}$ s, $\sqrt{2}\,\mathrm{m\,s^{-1}}$.
16. $\ddot{x} = -\omega^2 x$, $2\pi/\omega$; $t = 12$ s, $t = 24$ s. 17. $4u/\pi$.
18. (i) 5·09 m, (ii) 3·60 m, (iii) $\dfrac{4}{\pi}\sin^{-1}\left(\dfrac{\pi}{16}\right)$; 2π N, $\mu \geqslant \pi/g$.
19. $\frac{400}{3}$ N; $\dfrac{2}{\sqrt{3}}$ m/s. 20. $2mg + \dfrac{2\lambda}{3l}x$. 21. $\sqrt{(\frac{1}{3}gl)}$.
23. $\dfrac{5\sqrt{2}}{24}l$. 25. $\{ag(2\cos\theta - 1)\}^{1/2}$, $2mg(3\cos\theta - 1)$; $8a/25$.
28. $\sqrt{(3ga/2)}$. 29. $\ddot{\mathbf{r}} = \mathbf{g} - k\dot{\mathbf{r}}$.

Exercise 14.1 (pp. 310–311)

1. 720.
2. 40 320.
3. 20 160.
4. 151 200.
5. 13 800.
6. 10 080.
7. 720.
8. (a) 48. (b) 180.
10. (a) 210, (b) 1140, (c) 1287, (d) 12.
11. 1260.
12. (a) 462, (b) 5775, (c) 10 395.
13. 15 840
14. 38, 16.
15. 238.
16. (a) 15 015, (b) 37 037.

Exercise 14.2 (pp. 316–317)

1. (a) $\{x\}$, (b) $\{x, y, p, q, r\}$, (c) $\{x, y, z, p, q, r\}$, (d) $\{x, p\}$, (e) $\{x\}$, (f) $\{x, y, p, q, r\}$.
2. (a) $\{x \in \mathbb{R} : -2 < x < 7\}$, $\{x \in \mathbb{R} : 0 < x < 4\}$,
   (b) $\{x \in \mathbb{R} : x \leqslant 9\}$, $\{x \in \mathbb{R} : 2 \leqslant x \leqslant 5\}$,
   (c) $\{x \in \mathbb{R} : |x| < 10\}$, $\{x \in \mathbb{R} : -4 < x < 3\}$,
   (d) $\{x \in \mathbb{R} : -3 < x \leqslant 6\}$, $\varnothing$.
3. (a) $\varnothing$, (b) $\{4\}$, (c) $\{3, 4, 5\}$, (d) $\{1, 2\}$, (e) $\{1, 2, 3\}$, (f) $\{4, 5\}$.
4. (a) X', (b) $X \cap Y$, (c) $X' \cap Y'$, (d) $X \cup Y'$, (e) $X' \cup Y'$, (f) $(X \cup Y) \cap (X' \cup Y')$ or $(X \cap Y') \cup (X' \cap Y)$.
9. (a) A, (b) $\varnothing$, (c) A, (d) A.
10. 27.
11. $n(A)+n(B)+n(C)-n(A \cap B)-n(B \cap C)-n(C \cap A)+n(A \cap B \cap C)$.
12. $n(M \cap F) \geqslant 3$, $n(M \cap P) \geqslant 10$, $n(M \cap F \cap P) \leqslant 12$.
14. (a) A, (b) B, (c) $A \cap C$, (d) $B \cup C$.

Exercise 14.3 (pp. 321–322)

1. $\frac{1}{6}, \frac{1}{2}, \frac{2}{3}$.
2. $\frac{1}{2}, \frac{1}{13}, \frac{3}{13}$.
3. $\frac{1}{8}, \frac{1}{2}, \frac{3}{4}$.
4. $\frac{1}{21}, \frac{2}{7}, \frac{10}{21}$.
5. (a) $\frac{2}{3}$, (b) 0, (c) $\frac{5}{6}$, (d) $\frac{1}{6}$.
6. (a) $\frac{7}{13}$, (b) $\frac{1}{26}$, (c) $\frac{1}{13}$, (d) $\frac{5}{13}$.
7. (a) 0·8, (b) 0·07, (c) 0·1, (d) 0·43.
8. (a) $\frac{1}{4}$, (b) $\frac{5}{9}$, (c) $\frac{7}{36}$.
9. (a) $\frac{1}{14}$, (b) $\frac{3}{28}$, (c) $\frac{3}{7}$, (d) $\frac{4}{7}$.
10. 0·32.
11. (a) $\frac{1}{9}$, (b) $\frac{2}{3}$.
12. (a) $\frac{1}{16}$, (b) $\frac{3}{8}$, (c) $\frac{5}{8}$, (d) $\frac{7}{8}$.
13. (a) $\frac{1}{3}$, (b) $\frac{2}{9}$, (c) $\frac{8}{27}$.

14. (a) $\frac{3}{10}$, (b) $\frac{3}{10}$, (c) $\frac{3}{10}$, (d) $\frac{1}{5}$.

15. (a) $\frac{3}{20}$, (b) $\frac{11}{20}$.

16. $\frac{11!}{3!2!}$; (a) $\frac{2}{11}$, (b) $\frac{1}{77}$.

17. $\frac{15}{28}$.

18. (a) $\frac{1}{2}$, (b) $\frac{1}{4}$, (c) $\frac{1}{2}$.

19. (a) 0·204, (b) 0·052, (c) 0·398.

20. $\frac{21}{55}, \frac{27}{55}, \frac{27}{220}, \frac{1}{220}$.

21. (a) 0·108, (b) 0·374.

22. (a) $\frac{1}{42}$, (b) $\frac{1}{7}$, (c) $\frac{2}{3}$, (d) $\frac{1}{2}$.

Exercise 14.4 (pp. 326–327)

1. $\frac{1}{2}$.

2. 0·33.

3. $\frac{1}{12}$.

4. (i) (a) A and B, A and C, (b) A and B;
   (ii) (a) 0, (b) $\frac{3}{4}$, (c) $\frac{1}{4}$, (d) $\frac{1}{2}$, (e) $\frac{1}{4}$, (f) 1.

5. (i) (a) none, (b) A and C;
   (ii) (a) $\frac{1}{9}$, (b) 1, (c) $\frac{7}{36}$, (d) $\frac{5}{6}$, (e) 0, (f) $\frac{8}{9}$.

6. $0{\cdot}6 \leqslant P(A') \leqslant 0{\cdot}9$.

7. $\frac{1}{2}, \frac{1}{52}, \frac{3}{13}, 0, \frac{1}{52}, \frac{3}{26}$; A and C.

8. $\frac{1}{3}, \frac{1}{4}, \frac{1}{6}, \frac{1}{12}, \frac{1}{12}, \frac{1}{6}$; A and B.

9. (a) $\frac{3}{20}$, (b) $\frac{7}{10}$, (c) $\frac{1}{10}$, (d) $\frac{17}{20}$.

10. $\frac{5}{6}, \frac{2}{5}$.

11. $\frac{9}{10}, \frac{9}{10}$.

12. (a) $\frac{9}{40}$, (b) $\frac{1}{28}$.

13. (a) $\frac{1}{16}$, (b) $\frac{7}{16}$.

14. (a) $\frac{1}{27}$, (b) $\frac{1}{8}$, (c) $\frac{91}{216}$.

15. (a) $\frac{1}{36}$, (b) $\frac{1}{36}$, (c) $\frac{1}{2}$.

16. (a) $\frac{5}{9}$, (b) 0, (c) $\frac{20}{81}$.

17. 0·302.

18. (a) $\frac{1}{16}$, (b) $\frac{9}{64}$, (c) $\frac{1}{4}$, (d) $\frac{1}{16}$.

19. 0·590.

20. (a) $\frac{5}{32}$, (b) $\frac{5}{16}$, (c) $\frac{13}{16}$.

21. (a) $\frac{1}{64}$, (b) $\frac{2}{9}$, (c) $\frac{5}{36}$.

Exercise 14.5 (pp. 330–332)

1. $\frac{1}{4}, \frac{1}{4}, \frac{4}{7}$.

2. $\frac{1}{7}, \frac{1}{7}, \frac{1}{28}, \frac{1}{28}$.

3. $\frac{1}{4}, \frac{4}{7}, \frac{1}{4}, \frac{4}{7}$.

4. $\frac{1}{7}, \frac{3}{28}, \frac{3}{7}$.

5. $\frac{3}{14}, \frac{15}{28}, \frac{3}{14}$.

6. $\frac{2}{13}, \frac{1}{2}, \frac{1}{4}$.

7. $\frac{1}{2}, \frac{1}{4}, \frac{1}{13}, \frac{1}{26}$.

8. $0, 0, \frac{2}{3}, \frac{1}{2}$.

9. $0, \frac{1}{2}, \frac{1}{4}$.

10. $\frac{3}{26}, \frac{11}{26}, \frac{33}{52}$.

11. (a) $\frac{4}{81}$, (b) $\frac{1}{15}$.

12. (a) $\frac{27}{5000}$, (b) $\frac{1}{140}$.

13. $\frac{2}{3}$.

14. $\frac{3}{4}$.

15. $\frac{1}{5}$.

16. $\frac{7}{12}$.

17. (a) 0·7, (b) 0·6, (c) 0·75.

18. (a) $\frac{25}{72}$, (b) $\frac{19}{36}$.

19. (a) $\frac{18}{35}$, (b) $\frac{5}{12}$.

20. $\frac{1}{2}, \frac{1}{9}$.

Exercise 14.6 (pp. 332–336)

1. (a) 1260, (b) 8232.
2. 30 240; (a) 840, (b) 10 080, (c) 13 440.
3. 63.
4. 20.
5. 280, 36.
6. (a) 243, (b) 21.
7. $K \cap R = \{\text{squares}\}$; $K \cap P = \{\text{rhombuses}\}$.
8. (a) true, (b) false, (c) false, (d) false.
9. 1200.
10. (a) $\frac{1}{7}$, (b) $\frac{3}{35}$.
11. (a) (i) $\frac{5}{18}$, (ii) $\frac{1}{3}$, (b) $\frac{12}{35}$, (c) 294.
12. (i) $\frac{7}{20}$, (ii) $\frac{4}{13}$.
13. (a) $\frac{6}{77}$, (b) 0, (c) $\frac{2}{15}$, (d) $\frac{2}{33}$.
14. $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}$.
15. (i) $\frac{5}{12}$, (ii) $\frac{8}{15}$, (iii) $\frac{1}{12}$.

16. $\frac{5}{8}$; (a) $\frac{55}{64}$, (b) $\frac{25\pi}{256}$. 17. $4{\cdot}12 \times 10^{-6}$.

18. (a) $\frac{1}{22}$, (b) $\frac{41}{55}$, (c) $\frac{3}{11}$, (d) $\frac{3}{44}$.

19. (a) $\frac{1}{3}, \frac{1}{2}$, (b) (i) $\frac{63}{64}$, (ii) $\frac{5}{16}$, (iii) $\frac{5}{72}$.

20. (a) $\frac{81}{256}$, (b) $\frac{1}{256}$, (c) $\frac{3}{128}$, (d) $\frac{3}{32}$.

21. (i) 0·3, (ii) 0·3, (iii) 0·5.

22. (a) $P(9) = \frac{25}{216}$, $P(10) = \frac{27}{216}$,

(b) $P(\text{at least one six}) = 1 - \left(\frac{5}{6}\right)^4 = 0{\cdot}518$;

$P(\text{at least one double six}) = 1 - \left(\frac{35}{36}\right)^{24} = 0{\cdot}491$.

(c) $P(\text{at least one six}) = 1 - \left(\frac{5}{6}\right)^6 = 0{\cdot}665$;

$P(\text{at least two sixes}) = 1 - \left(\frac{5}{6}\right)^{12} - 2\left(\frac{5}{6}\right)^{11} = 0{\cdot}619$.

23. (a) (i) $\frac{1}{4}$, (ii) $\frac{1}{4}$, (iii) $\frac{1}{17}$, (iv) $\frac{15}{34}$; (b) $\frac{13}{30}$;

(c) (i) $\frac{95}{253}$, (ii) $\frac{285}{506}$.

24. (a) $\frac{33}{16\,660}$, (b) $\frac{6}{4165}$; $\frac{11}{1081}$.

26. (a) $\frac{1}{221}$, (b) $\frac{32}{221}$, (c) $\frac{13}{17}$; $\frac{25}{33}$.

27. (a) (i) $\frac{3}{8}$, (ii) $\frac{1}{2}$, (iii) $\frac{1}{6}$; (b) 0·14.

28. (a) 0·01, (b) 0·87; (a) 0·9606, (b) 0·0847.

29. (a) $\frac{1}{36}$, (b) $\frac{5}{12}$; $\frac{73}{648}$, $\frac{25}{81}$.

30. (i) $\frac{2}{3}$, (ii) $\frac{1}{6}$, (iii) $\frac{5}{18}$, (iv) $\frac{1}{4}$, (v) $\frac{3}{5}$.

31. (i) $\frac{31}{120}$, (ii) $\frac{91}{240}$, (iii) 1; $\frac{2}{7}$.

32. (i) $P(A \cap B) = P(A).P(B)$. (ii) $P(A \cap C) = 0$; $\frac{13}{25}, \frac{1}{2}, \frac{9}{35}; \frac{3}{10}$.

33. (i) 0·22, (ii) 0·30, (iii) 0·44.

Exercise 15.1 (pp. 343–345)

5. (a) step diagram, (b) ogive.

6. One possibility is 32·5, 37·5, 42·5, ..., 67·5, 72·5; 88%.
7. 9/14, excluding the 10 late entrants (possibly transfers from schools in other areas).

Exercise 15.2 (pp. 349–350)

1. (a) 3, 3, $3\frac{1}{2}$. (b) no mode, 24, 20, (c) 20 and 21, 21, $20\frac{2}{3}$.
2. (a) 3, 3, 5, (b) no mode, 22, 20, (c) 20, $20\frac{1}{2}$, $20\frac{3}{5}$.
3. (a) 5, 5, 5, (b) 5, $5\frac{1}{14}$, 5, (c) 5, $5\frac{4}{7}$, $5\frac{1}{2}$.
4. 3, 3, 3·64.
5. 6·84.
6. (i) $52{\cdot}5 \leqslant x < 57{\cdot}5$, $54\frac{3}{8}$, 55, (ii) 56, 56, 54·175.
7. (i) 4·96, 5·14, (ii) 4·93, 4·95.
8. 26·6; (a) 27, (b) 27·2, (c) 27·1.

Exercise 15.3 (pp. 353–354)

1. (a) 7; $5\frac{1}{2}$, $8\frac{1}{2}$; $1\frac{1}{2}$, (b) 29; 27, 33; 3, (c) 2; −4, 10; 7.
2. (a) 7·25, 5·19, 2·28, (b) 31, 71·8, 8·47, (c) 3, 39·8, 6·31.
3. (a) (i) 5, 6, (ii) 1; (b) (i) 4·8, 5·7, (ii) 0·8; (c) (i) 5·3, 6·2, (ii) 0·8.
4. 6·31, 2·51.
5. (i) (a) 5, 7, 9, (b) 4·9, 6·9, 8·8; (ii) 2·45.
6. (i) 49·5, 59·5, (ii) 48·9, 60·3, (iii) 49, 60.
7. (i) 4 yrs 10 months, 5 yrs 1 month, (ii) 46·5 months^2.
8. I: 38−, II: 28−37, III: 19−27, IV: 0−18.

Exercise 15.4 (p. 357)

1. 5·5, 6·25, 2·5; (a) 8·5, 6·25, 2·5, (b) 22, 100, 10, (c) 4·55, 0·0625, 0·25.
2. 73·4°F, 9°F.
3. 5, 1·5, 1·22; 5·5, 1·5, 1·22.
4. (i) 7·84, (ii) 7·98.
5. 27·2, 9·69; 1·5, 9·2.
6. Mathematics: $72\frac{1}{2}$, 77, $57\frac{1}{2}$; English: $60\frac{1}{2}$, $51\frac{1}{2}$, $87\frac{1}{2}$; Combined: 133, $128\frac{1}{2}$, 145.

Exercise 15.5 (pp. 358–361)

1. 86·6, 44·1; 188·1.
2. 6·5; 4·2, 9·0.
3. (i) 57 min, (ii) 71·5 min, (iii) 32%.
4. mean 55 kg, s.d. 3·2 kg.
5. 0, 1.
6. (i) 100·7, 14·5, (ii) 72.
7. (b) 8·95, 9·95, (c) 13·53, (d) 13·55.
8. (i) £28·87, (ii) £54·04, (iii) £30·47.
9. 52°, 7·5°; 53·2°, 10·7°.
10. 172·1 cm, 2·608 cm; 174·6 cm, 6·8 cm^2; 1·054 in^2.
11. £49, £12·85.
12. (a) 5, 3·6, (b) 5·4, 5·04, (c) $4{\cdot}5 < \bar{x} < 5{\cdot}0$, (d) $x_{10} \geqslant 8$ or $x_{10} \leqslant 2$.

13. (ii) 6·8299; 39·23, 6·6335.
14. (a) (i) 3·4, 2, (ii) 0·4, 0·276; (b) $3\frac{1}{3}$, 19·6%.

Exercise 16.1 (pp. 366–369)

1. 10. 2. $3\frac{1}{2}$. 3. 14; 20 p. 4. (a) 1, (b) 9/8.
5. $p(2) = 2/15$, $p(3) = 8/15$, $p(4) = 1/3$; 3·2. 6. 0·8 p loss.
7. $\frac{125}{216}, \frac{75}{216}, \frac{15}{216}, \frac{1}{216}$; 2 p loss. 8. (a) £122.50, (b) £112·672.
9. $p(x) = 1/6$, $x = 1, 2, 3, 4, 5$; $p(6) = 0$; $p(x) = 1/36$, $x = 7, 8, 9, 10, 11, 12$; $4\frac{1}{12}$; 6/17.
10. (i) 24/625, (ii) 24/625, (iii) 2. 11. 28·5, 721·05. 12. 2·53, 0·305.
13. $1\frac{1}{2}, \frac{3}{4}; \frac{1}{20}, \frac{9}{20}, \frac{9}{20}, \frac{1}{20}$; 60%.
14.

r	0	1	2	3	4
$P(Y=r)$	0·09	0·24	0·34	0·24	0·09

r	0	1	2	3	4
$P(Z=r)$	0·447	0·232	0·222	0·072	0·027

$E(Z) = 1$, $\mathrm{Var}(Z) = 1{\cdot}2$.

15. $4\frac{1}{3}, 2\frac{2}{9}; 8\frac{4}{3}, 8\frac{8}{9}$. 16. $\frac{1}{2}(n+1)$, 0·42.
17. $2/n(n+1)$; $\frac{1}{3}(2n+1)$, $\frac{1}{18}(n-1)(n+2)$.
18. $x = 1, 2, 3, \ldots, 9$; $p(x) = \frac{9}{45}, \frac{8}{45}, \frac{7}{45}, \ldots, \frac{1}{45}$; $\frac{11}{3}, \frac{2}{3}\sqrt{11}$; 1;

$x = 1, 2, 3, \ldots$; $p(x) = \dfrac{10-x}{45}$.

19. (a) $1 + 2x + 3x^2 + \ldots + rx^{r-1} + \ldots$, (b) $p(1-p)^{r-1}$; $1/p$.
20. (i) $\frac{1}{3}$, (ii) $\frac{2}{9}$, (iii) $\frac{16}{243}$; $\frac{32}{243}$; 50 p; $\frac{16}{81}$; 0·177. 21. 3/16.

Exercise 16.2 (pp. 373–376)

1. (a) 0·279, (b) 0·457; 1·5, 1·25. 2. (a) 0·250, (b) 0·526; 2·5, 1·875.
3. 0·657; 7. 4. (a) 0·393, (*b*) 0·000 085 2; $n \geqslant 18$.
5. 24, 0·4; $(0{\cdot}6 + 0{\cdot}4t)^{24}$.
6. $(0{\cdot}55 + 0{\cdot}45t)^4$; (i) 4, (ii) $4(0{\cdot}55)(0{\cdot}45)^3$. 7. 0·228, 0·210.
8. 0·0464; 4, 2·4.
9. $\frac{4}{3}, \frac{356}{405}$;

No. of males	0	1	2	3	4
Expected no. of litters	160	320	240	80	10

10. 0·991, 0·977;

No. of ones	0	1	2	3	4	5	6
Expected no. of throws	335	402	201	54	8	1	0

11. (a) 0·315, (b) (i) 0·53, (ii) 5, 22, 37, 28, 8. 12. 26·4%, 62·4%.
13. (i) 0·201; (ii) (a) 0·0556, (b) 0·0367, (c) 0·6.
14. (i) 4·8, 0·98, (iii) 0·737, (iv) 0·388.
15. (a) $\frac{1}{27}$, (b) $\frac{2}{9}$, (c) $\frac{4}{9}$, (d) $\frac{8}{27}$, (e) 0·299, (f) 0·903, (g) 0·328.
16. (ii) 1/5. 17. $\dfrac{(n-r+1)p}{r(1-p)}$; (a) 8, (b) 2, 3.
18. (a) $\dfrac{135}{2048}$; (b) $\dfrac{459}{512}$; $\binom{n-1}{2}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^{n-3}$; 8, 9.
19. (a) $(r+1)/2^{r+2}$, (b) 2, (c) 4. 20. $t/(6-5t)$; 6, 30.

Exercise 16.3 (pp. 381–384)

1. (a) 1/2, 1/12, (b) 3, 4/3, (c) $k/2$, $k^2/12$, (d) $(a+b)/2$, $(b-a)^2/12$.
2. 4/27; 9/5, 2, 3/5. 3. (a) 2·4, (b) 20; 1/3, 0·178.
4. (i) −3/16, (ii) 19/80, (iv) 2. 5. 4, 8/15, 11/225; 0·541.
6. 3/8, 0, 2/5; (i) 85/512, (ii) 85/152. 7. (i) 12, 1, (ii) 0·0523.
8. (i) 2/5, (ii) 13/5, (iii) 3/2.
9. 0·6; $F(x) = 0$ for $x \leqslant 0$, $F(x) = 4x^3 - 3x^4$ for $0 \leqslant x \leqslant 1$, $F(x) = 1$ for $x \geqslant 1$.
10. 28, 12, 6, 2, 1, 1.
11. (i) 2/3, (ii) 4/3, $(\sqrt{14})/6$, (iii) $F(x) = 0$ for $x \leqslant 0$, $F(x) = \frac{1}{3}x^2$ for $0 \leqslant x \leqslant 1$, $F(x) = \frac{1}{6}(-3 + 6x - x^2)$ for $1 \leqslant x \leqslant 3$, $F(x) = 1$ for $x \geqslant 3$, (iv) 13/24.
12. (iii) $F(x) = -\frac{1}{12}x^3$ for $x \leqslant -1$, $F(x) = \frac{1}{2} + \frac{1}{2}x - \frac{1}{12}x^3$ for $-1 \leqslant x \leqslant 1$, $F(x) = 1 - \frac{1}{12}x^3$ for $x \geqslant 1$, (iv) $E(X) = 0$, $\mathrm{Var}(X) = 11/15$.
13. (ii) 1, 5/6, (iii) 1/4.
14. (i) $k = 2$, (ii) $f(x) = 2$ for $0 \leqslant x \leqslant 0{\cdot}5$, $f(x) = 0$ otherwise, (iii) 0·25, (iv) 0·144.
15. (i) 3/32, (ii) 11/16, (iii) $f(x) = 3(4 - x^2)/32$ for $-2 \leqslant x \leqslant 2$, $f(x) = 0$ otherwise, (iv) 0, 4/5.
16. $F(x) = 0$ for $x \leqslant 1$, $F(x) = \frac{1}{3}(x^2 - 1)$ for $1 \leqslant x \leqslant 2$, $F(x) = 1$ for $x \geqslant 2$; $G(y) = 0$ for $y \leqslant 1$, $G(y) = \frac{1}{3}(y^4 - 1)$ for $1 \leqslant y \leqslant \sqrt{2}$, $G(y) = 1$ for $y \geqslant \sqrt{2}$; $g(y) = 4y^3/3$ for $1 \leqslant y \leqslant \sqrt{2}$, $g(y) = 0$ otherwise.
17. $1/(2\sqrt{y})$; $(1/\sqrt{w}) - 1$.
18. $f(x) = 1/(b-a)$ for $a \leqslant x \leqslant b$, $f(x) = 0$ otherwise; $F(x) = 0$ for $x \leqslant a$, $F(x) = (x-a)/(b-a)$ for $a \leqslant x \leqslant b$, $F(x) = 1$ for $x \geqslant b$; $P(V \leqslant v) = 3 - 72/v$; (i) $f(v) = 72/v^2$, (ii) 0·4, (iii) 28·8 m.p.h.
19. (a) $a^2/18$, (b) $F(x) = 0$ for $x \leqslant a$, $F(x) = x^2/a^2$ for $0 \leqslant x \leqslant a$, $F(x) = 1$ for $x \geqslant a$, (c) (i) $1/(a+1)^2$, (ii) $2(1-y)$ for $0 \leqslant y \leqslant 1$, 0 otherwise.
20. $g(y) = 1$ for $0 \leqslant y \leqslant 1$, $g(y) = 0$ otherwise.

Exercise 16.4 (pp. 389–392)

1. (a) 0·6179, (b) 0·1151, (c) 0·97725, (d) 0·01287, (e) 0·1760, (f) 0·5558.
2. (a) 0·9378, (b) 0·1944, (c) 0·9715, (d) 0·02239, (e) 0·2065, (f) 0·2976.
3. (a) 0·9554, (b) 0·2119, (c) 0·6612.
4. (a) 0·3085, (b) 0·5793, (c) 0·0796.
5. (a) 0·9487, (b) 0·9750, (c) 0·04412
6. (a) 0·44, (b) 0·583, (c) −0·878.
7. (a) 0·524, (b) 2·326, (c) 3·291.
8. (a) 0·3%, (b) 6·6%, (c) 44%, (d) 1·2%.
9. 11·51%, 81%, 7·49%; £6528.64.
10. (i) 4·8%, (ii) 97·7%; 1111·5 to 1188·5.
11. (a) 37·82%, (b) 125·5 hrs to 194·5 hrs, (c) 0·405.
12. (a) 4 men, (b) 0·0446, (c) 1·746 m.
13. (a) 3, 62·3, (b) 15, 9·24, (c) 84·5, 111, (d) 8·31, 1·74.
14. 0·0122, 0·0102. 15. 104, 38·8.
16. (a) 0·205, 0·205, (b) 0·120, 0·119, (c) 0·0419, 0·0416, (d) 0·190, 0·179, (e) 0·0985, 0·0932, (f) 0·0493, 0·0474.
17. 0·0287. 18. 0·210. 19. (i) 0·026, (ii) 0·932, (iii) 0·923.
20. (a) 4320/15625; (b) 7120/15625; (a) 0·026, (b) 0·853.
21. (a) Negligible, (b) negligible, (c) 0·806.
22. (i) 0·648, (ii) 92·65%, (iii) 0·654.

Exercise 16.5 (pp. 395–397)

1. (a) 0·156, (b) 0·663. 2. (a) 0·353, (b) 0·285.
3. 0·181; $e^{-0·02n}$; 5. 4. 0·337; 1536.

5.

r	0	1	2	3	4	5	6	7	8	9	10
(a)				0·001	0·005	0·015	0·037	0·074	0·120	0·160	0·176
(b)			0·002	0·008	0·019	0·038	0·063	0·090	0·113	0·125	0·125
(c)				0·001	0·005	0·015	0·037	0·073	0·119	0·160	0·177
(d)	0·001	0·002	0·005	0·011	0·021	0·036	0·057	0·080	0·103	0·120	0·126

r	11	12	13	14	15	16	17	18	19	20	21
(a)	0·160	0·120	0·074	0·037	0·015	0·005	0·001				
(b)	0·114	0·095	0·073	0·052	0·035	0·022	0·013	0·007	0·004	0·002	0·001
(c)	0·160	0·119	0·073	0·037	0·015	0·005	0·001				
(d)	0·120	0·103	0·080	0·057	0·036	0·021	0·011	0·005	0·002	0·001	

6.

r	0	1	2	3	4	5	6	7	8	9
(a)	0·122	0·270	0·285	0·190	0·090	0·032	0·009	0·002		
(b)	0·135	0·271	0·271	0·180	0·090	0·036	0·012	0·003	0·001	
(c)	0·132	0·223	0·291	0·223	0·101	0·027	0·004			
(d)	0·144	0·217	0·276	0·217	0·106	0·032	0·006	0·001		

7.

r	0	1	2	3	4	5	6	7	8	9
(a)	0·005	0·029	0·078	0·139	0·181	0·185	0·154	0·108	0·064	0·033
(b)	0·007	0·034	0·084	0·140	0·175	0·175	0·146	0·104	0·065	0·036
(c)	0·017	0·033	0·070	0·121	0·167	0·186	0·167	0·121	0·070	0·033
(d)	0·022	0·037	0·073	0·119	0·160	0·177	0·160	0·119	0·073	0·037

r	10	11	12	13
(a)	0·015	0·006	0·002	0·001
(b)	0·018	0·008	0·003	0·001
(c)	0·012	0·004	0·001	
(d)	0·015	0·005	0·001	

8.

r	0	1	2	3	4	5	6	7	8	9
(a)		0·002	0·009	0·025	0·054	0·090	0·123	0·144	0·146	0·129
(b)		0·003	0·011	0·029	0·057	0·092	0·122	0·140	0·140	0·124
(c)	0·003	0·005	0·013	0·027	0·050	0·080	0·112	0·137	0·146	0·137
(d)	0·004	0·007	0·015	0·030	0·052	0·080	0·110	0·132	0·140	0·132

r	10	11	12	13	14	15	16	17	18
(a)	0·102	0·073	0·047	0·028	0·015	0·007	0·003	0·001	0·001
(b)	0·099	0·072	0·048	0·030	0·017	0·009	0·005	0·002	0·001
(c)	0·112	0·080	0·050	0·027	0·013	0·005	0·002	0·001	
(d)	0·110	0·080	0·052	0·030	0·015	0·007	0·003	0·001	

9. (a) 0·368, (b)0·456. 10. (a) 0·258, (b) 0·567.
11. (a) Poisson approximation: 0·442,
(b) Poisson approximation: 0·353,
(c) Normal approximation: 0·119.
14. (i) 0·19, (ii) 0·21, (iii) 0·012.
15. (a) (i) 1·04, (ii) 53, 55, 29, 10, 3; (b) 0·121.
16. 0·647, 0·185. 17. (i) 0·55, (ii) 0·18.
18. £266*C*, £415*C*. 19. 0·460; 33.
20. 0·323; 172 to 228.

Exercise 16.6 (pp. 403–407)

1. 3·6, 14·5, 0·35, 28, 5·95. 2. 0·4, 1, 2·2; 0·84, 1·2.
3. $1\frac{2}{3}$, 3, $5\frac{4}{5}$; $20\frac{11}{15}$, $\frac{2}{9}$.
4. $\frac{1}{2}$, $\frac{3}{10}$, $\frac{24}{35}$; $-\frac{1}{5}$, $\frac{73}{2450} \approx 0{\cdot}0298$.
5. (a) 6, 2·5, (b) 4, 6, (c) 27, 40, (d) −3, 24,
(e) 10, 8·5, (f) 2, 8·5, (g) 14, 28·5, (h) 24, 64.
6. (a) 31, 234, (b) 109, 50, 159. 7. 30 g, 0·3 g. 8. 200 g, 6·71 g.
9. (a) 2·1, 0·49; 0·8, 0·36, (b) no; independent,

(c)

xy	0	1	2	3	4	6
$p(xy)$	0·3	0·12	0·32	0·18	0·05	0·03

; yes,

(d)

$x + y$	1	2	3	4	5
$p(x + y)$	0·06	0·27	0·41	0·23	0·03

; yes.

10. (i) (a) 1·2, 0·56; 1·7, 0·81, (b) all values; not independent,

(c)

xy	0	1	2	3	4	6
$p(xy)$	0·27	0	0·45	0·1	0·1	0·08

; no,

(d)

$x + y$	0	1	2	3	4	5
$p(x + y)$	0·03	0·15	0·07	0·47	0·2	0·08

; no.

(ii) (a) 1, 0·6; 2, 0·6, (b) all values, not independent,

(c)

xy	0	1	2	3	4	6
$p(xy)$	0·3	0·2	0·05	0·2	0·2	0·05

; yes,

(d)

$x + y$	1	2	3	4	5
$p(x + y)$	0·05	0·4	0·1	0·4	0·05

; yes.

11. 1, 2, 4, 8. 12. £3.33.
13. $\frac{1}{210}, \frac{4}{35}, \frac{3}{7}, \frac{8}{21}, \frac{1}{14}$; 2·4, 0·8; 1·6, 0·8; 4, 1·13.
14. (i) 0·9, 1·2,

(ii)

z	−2	−1	0	1	2	3	4
$p(z)$	0·1	0·1	0·3	0·2	0·2	0·1	0

,

(iii) 0·2.

15. (i) 0·011, (ii) 0·036, (iii) 0·27. 16. 0·12 mm, 0·0583 mm; 1·98%
17. (a) 0·673, (b) 0·138. 18. (i) 0·089, (ii) 0·661.
19. (i) 0·841, (ii) 0·500, (iii) 0·421; 0·994.
20. (i) 0·823, (ii) 0·059; 1/4, 1/6. 21. $\frac{1}{2}r\pi, \frac{1}{2}r\pi$; $4r/\pi$; 1/3.
22. (i) $\mu_x^2 + \sigma_x^2$; (ii) $\mu_x\mu_y$; $\mu_x^2\sigma_y^2 + \mu_y^2\sigma_x^2 + \sigma_x^2\sigma_y^2$; (iii) $\mu_y(\mu_x^2 + \sigma_x^2)$.
23. $\dfrac{\lambda^r\mu^{n-r}}{r!(n-r)!}e^{-(\lambda+\mu)}$; $\displaystyle\sum_{r=0}^{n}\frac{\lambda^r\mu^{n-r}}{r!(n-r)!}e^{-(\lambda+\mu)}$. 24. 0·121.
25. (b) $(1 - p + pt)^m$, $(1 - p + pt)^n$; $(1 - p + pt)^{m+n}$.

Exercise 16.7 (pp. 407–414)

1. $(n + 1)/2$, $(n^2 - 1)/12$; 1/2; (a) 2/105, (b) 16.

2. £16. 3. (a) 0·323, (b) $£1\frac{32}{65} \approx £1.49$.
4. 1/15; 2/3, 34/45; 1/75; 4/3, 68/45.
5. (i) $P(N = r) = \frac{1}{k}(1 \leqslant r \leqslant k)$, $E(N) = \frac{1}{2}(k + 1)$,

   (ii) $P(N = r) = \frac{1}{k}\left(\frac{k-1}{k}\right)^{r-1}$ $(r \geqslant 1)$, $E(N) = k$; first strategy.
6. (i) 1/5, (ii) 8/9, (iii) 10, (iv) 8. 7. 0·0512.
8. (i) 0·168, (ii) 0·360, (iii) 0·309, (iv) 0·132, (v) 0·028, (vi) 0·002; 0·947.
9.

No. of defectives	0	1	2	3	4	5	6
Expected frequency	16	42	45	26	8	1	0

10. $\frac{1}{p}, \frac{1-p}{p^2}; \frac{2t^2}{(5-3t)(5-4t)}; \frac{194}{625} = 0{\cdot}3104$.
11. (i) 2, (ii) $1\frac{1}{2}$, (iii) $1\frac{5}{12}$,
    (iv) $F(x) = 0$ for $x \leqslant 0$, $F(x) = x - \frac{1}{2}x^2$ for $0 \leqslant x \leqslant 1$, $F(x) = \frac{1}{2}$ for $1 \leqslant x \leqslant 2$,
    $F(x) = \frac{1}{2}x^2 - 2x + \frac{5}{2}$ for $2 \leqslant x \leqslant 3$, $F(x) = 1$ for $x \geqslant 3$, (v) $\frac{3}{4}$.
12. (ii) $\sqrt{(3/5)}$. 13. 1, 1/6; 3/4; 5/6.
14. (a) 3/250, $6\frac{7}{8}$, (b) (1) £3.20, (ii) 0·792.
15. (i) 1/2, −1/4, (ii) 7/16,
    (iii) $f(x) = \frac{1}{2}x$ for $0 \leqslant x \leqslant 1$, $f(x) = \frac{1}{2}$ for $1 \leqslant x \leqslant 2$,
    $f(x) = \frac{3}{2} - \frac{1}{2}x$ for $2 \leqslant x \leqslant 3$, $f(x) = 0$ otherwise; 3/2; 5/12.
16. (a) $1 - e^{-\lambda x}$, (b) $\frac{1}{\lambda}, \frac{1}{\lambda^2}$, (c) $\frac{1}{\lambda}\ln 2$ (d) $\frac{1}{\lambda}\ln\left(\frac{20}{19}\right)$.
17. (i) 0·202, (ii) 0·410.
18. (i) 0·880, (ii) 9·69, 0·2339, (iii) 0·180.
19. £9360, 19·6%; (a) £20 000, £9360; (b) £19 800, £10 296.
20. (i) 0·60, (ii) 0·20, (iii) 0·95, (iv) 0·50.
21. (i) 0·227, (ii) 54·7, (iii) 78·7 cm², (iv) 0·579.
22. 1·514 mm, 0·051 mm; 3·028 mm, 0·072 mm; 0·895; 3·214 mm.
23. (i) 0·345, (ii) 0·692; 0·0033; 0·304.
24. (a) (i) 0·5, (ii)

No. of accidents	0	1	2	3	4	5
Expected no. of days	121	61	15	3	0	0

    (b) Binomial model: 0·932, Poisson approximation: 0·929.
25. (i) 9, (ii) 26, (iii) 43; 8.
26. (a) (i) 0·0581, (ii) 0·0519; (b) $(0{\cdot}96)^n$, 74; (c) 0·955.
27. (i) 0·135, (ii) 0·271, (iii) 0·0290, (iv) 2, (v) 8·19.
28. (i) 0·419, (ii) 0·676.
29. (a) (i) 0·0952, (ii) 0·0905; (b) 0·853; (c) 0·632; 63·2, 4·82.
30. 1/12; 0, $n/12$; 16. 31. (b) 2·2.

32. (i) $F(x) = 0$ for $x \leqslant 1$, $F(x) = 1 - x^{-3}$ for $x \geqslant 1$;
(ii) 3/2, 3/4; p.d.f. of Y: $g(y) = 3y^2$ for $0 \leqslant y \leqslant 1$, $g(y) = 0$ otherwise.
33. (ii) 8/15,
(iii) $F(x) = 0$ for $x \leqslant 0$, $F(x) = x^2(2 - x^2)$ for $0 \leqslant x \leqslant 1$, $F(x) = 1$ for $x \geqslant 1$;
(iv) $g(y) = 8y^3(1 - y^4)$ for $0 \leqslant y \leqslant 1$, $g(y) = 0$ otherwise.
34. (a) (i) 5/2, (ii) 7,
(iii) $F(x) = 0$ for $x \leqslant 1$, $F(x) = \frac{1}{3}(x^2 - 1)$ for $1 \leqslant x \leqslant 2$, $F(x) = 1$ for $x \geqslant 2$;
(b) (i) 5/2, 3/4, (ii) interval from 1 to 4.

Index

Acceleration, 46, 69
  as a function of displacement, 273
  as a function of velocity, 276
  as a vector, 31, 62, 229
  constant, 50, 63
  due to gravity, 53, 63, 67
Accelerations, related, 81
Addition formulae, 415
Addition law of probability, 323
Amplitude (of S.H.M.), 283
Angle between two vectors, 235
Angle of friction, 97
Angular acceleration, 128
Angular velocity, 128
Arc, length of circular, 415
Area of a sector, 415
Area of a triangle, 415
Arithmetic progression, 415
Arrangements, 308
Array of statistical data, 346
Assumed mean, 355
Average speed, 31
Average velocity, 31

Banked roads, 136
Bernouilli, Jacques, 369
Bernouilli trial, 369
Binomial distribution, 369, 417
  normal approximation, 388
  Poisson approximation, 394
  probability generating function, 373
  sum of binomial variables, 403
Binomial theorem, 415

Cardinal number, 313
Cartesian components, 5, 16
Cartesian coordinates in three dimensions, 6
Central tendency, measures of, 345
Centre of gravity, 157, 162
  composite bodies, 160
  derivation of standard results, 164
  set of particles, 158
  triangular lamina, 157
Centre of mass, 162
  standard results, 416
Centrøid, 162
  plane area, 172, 174
  volume of revolution, 174
Change of variable (statistics), 354
Circular motion, 128, 132, 229, 292, 295
Class boundaries, 341
Class limits, 341
Closest approach, 40
Coding of statistical data, 356
Coefficient of friction, 88
Combinations, 309
Complementary event, 318
Complement of a set, 312
Component of a force, 18
Components, Cartesian, 5, 16
Component vectors, 5, 233
Composite bodies, 160, 182
Conditional probability, 328
Conical pendulum, 134
Connected particles, 76
Conservation of energy, 111, 116, 255
Conservation of linear momentum, 250, 263
Conservative force, 112
Constant acceleration formulae, 51, 63, 416
Contingency table, 329
Continuity correction, 389
Continuous probability distribution, 377
Continuous variable, 339, 362
Coplanar forces, 149
  see also Equilibrium conditions

Couple, 145, 147
Covariance, 401, 417
Cumulative distribution function, 379
  standard normal variable, 386, 418
Cumulative frequency, 339, 340, 343

Density function, *see* Probability density function
Differentiation, standard formulae, 416
Differentiation of position vectors, 62
Direct impact, 261, 263
Direction vector, 223
Discrete probability distribution, 363
Discrete variable, 339, 362
Disjoint sets, 313
Dispersion, measures of, 350
Displacement, as a vector, 2, 31
  relative, 38
  using a definite integral, 48
Displacement–time graph, 46
Distance between two points, 6
Distribution function, (cumulative), 379
  standard normal variable, 386, 418
Dot product, 233

Elastic impact, 260
Elastic potential energy, 116
Elastic string, 99
  energy stored in, 117, 416
  in S.H.M., 287
Empty set, 311
Energy, 108, 110, 116
  conservation of, 111, 116
Energy equations, 237, 254, 255
Equilibrium, 11
  limiting, 94
  of a particle, 22, 94
Equilibrium conditions, 180, 197, 201
  three forces, 188, 190
  two forces, 181
Equivalent statements, 311
Equivalent systems of forces, 151
Event, 317
Expectation, continuous variable, 377, 397, 417
  discrete variable, 364, 397, 417
  function of a random variable, 397, 417
  linear combination of random variables, 400, 417
  linear function of a random variable, 398, 417
  product of independent random variables, 400, 417
  sum of random variables, 400
  using probability generating function, 373
Expectation algebra, 397, 417
Expected frequency, 372
Expected value, 364

Factorial notation, 308
Finite series, 415
First moment of area, 172
Forces, 11
  polygon of, 23
  triangle of, 24
Frameworks, 211
Frequency, 339, 340
  density, 340
  distribution, 364
Frequency of S.H.M., 283
Friction, 12, 88, 112
  angle of, 97
  coefficient of, 88
  cone of, 97
  laws of, 88
  limiting, 88

Geometric distribution, 364
Geometric progression, 415
Graphical representation of data, 337
Gravity, force of, 12
  acceleration due to, 53, 63, 67
Grouping of data, 341

Histogram, 340
Hooke, Robert, 99
Hooke's law, 99, 416

Impact, 245, 250
  direct, 261, 263
  indirect, oblique, 261, 262
Implication signs, 311
Impulse–momentum equation, 244, 254
Impulse of a force, 243
Impulsive tension, 246
Independent events, 324
Independent random variables, 400
Indirect impact, 261, 262
Inelastic impact, 261
Inextensible string, 11
Integration, standard forms, 416
Interception, 38
Intersection of sets, 312

Joint probability distribution, 399, 417
Joule, unit of work, 108

Karnaugh map, 314, 329
Kinetic energy, 108, 110

Lami, Bernard, 25

Lami's theorem, 25, 191
Lamina, 11
Least force problems, 95, 97
Length of circular arc, 415
Light frameworks, 211
Light rods, 211
Light string, 11, 12
Limiting equilibrium, 94
Limiting friction, 88
Linear combinations, of normal variables, 402
of random variables, 400, 401, 417
Linear interpolation, 347, 386
Line of action of resultant, 143, 149
parallel forces, 144, 149
vector equation, 224
Location, measures of, 345

Magnitude of a vector, 1, 5
Marginal distribution, 112
Mass, 12
Mathematical model, 10
Mean, 346
binomial distribution, 371, 417
continuous probability distribution, 377, 397
discrete probability distribution, 364, 397
frequency distribution, 364
grouped data, 348
Poisson distribution, 393, 417
using probability generating function, 373
see also Expectation
Mean absolute deviation, 351
Mechanical energy, 110
conservation of, 111, 116
Median of a distribution, 346, 378
grouped data, 347
Mid-point, position vector of, 6
Modal class, 347
Mode, 346, 378
grouped data, 347
Models, mathematical, 10
Modulus of elasticity, 99
Moment, of a couple, 147
of a force, 145
of an area, 172
Moments, principle of, 147
Momentum, 69, 244
conservation of, 250, 263
continuous change in, 245
Mutually exclusive events, 323

Natural length, 99
Newton, Sir Isaac, 68
Newton, unit of force, 13, 69
Newton's experimental law, 261
Newton's law of restitution, 261, 263
Newton's laws of motion, 68, 70
Newton's second law, vector form, 230
Non-conservative force, 112
Normal approximation,
to binomial distribution, 388
to Poisson distribution, 395
Normal curve, 385
Normal distribution, 384
as a limit of the binomial distribution, 388
linear combination of normal variables, 402
sum and difference of normal variables, 402
table of percentage points, 419
Normal distribution function, 386
table of values, 418
Normal reaction, 12, 88
Null set, 311
Null vector, 1

Oblique impact, 261, 262
Ogive, 340
Oscillations, simple harmonic, 282, 287

Parallel forces, 144, 149, 180
Parallelogram law, 3, 14
Particle, 11
Pendulum, 292
Percentage points of the normal distribution, 387, 419
Percentiles, 350
Period of S.H.M., 283
Permutations, 308
Perpendicular vectors, 233, 235
Poisson, Siméon Denis, 392
Poisson approximation to the binomial distribution, 394
Poisson distribution, 392, 417
as a limit of the binomial distribution, 392
normal approximation, 395
probability generating function, 396
sum of Poisson variables, 403
Polygon of forces, 23
Position vector, 6
differentiation of, 62
Possibility space, 317
Potential energy, 110
elastic, 116
gravitational, 110
Power, 118
vector approach, 238

Probability, 317
  conditional, 328
Probability mapping, 320
Probability space, 320
Probability density function, 377
  normal variable, 384
  standard normal variable, 385
Probability distributions, 417
  continuous, 377
  discrete, 363
  joint, 399
  marginal, 400
Probability function, 363
Probability generating function, 373, 396
Probability model, 362
Product law (probability), 324, 328
Progressions, arithmetic and geometric, 415
Projectile, 56
  range, 58
  time of flight, 58
  trajectory, 57
  vector equations, 63
Projection of a vector, 233
Proper subset, 311
Pulley systems, 81

Quartiles, 350

Random sample, 338
Random trial, 362
Random variables, 362
  independent, 400
  see also Expectation, Variance
Range of a distribution, 350
Ratio theorem, 226
Reaction, normal, 12, 87
  total, 12, 87, 96
Rectangular distribution, 363, 378
Relative displacement, 38
Relative frequency, 364
Relative motion, 13, 228
Relative velocity, 35, 228
Resolution of, forces, 18
  vectors, 5
  velocities, 33
Resolved part, 19
Restitution, coefficient of, 261
  Newton's law of, 261, 263
Resultant, displacement, 2
  force, 14, 151
  line of action, 143, 149, 224
  using ratio theorem, 226
  velocities and accelerations, 33
Rigid body, 11
Rough surface, 12
  equilibrium on, 94
  motion on, 91

Sample space, 317, 362
  diagram, 319
Scalar product, 233
Scalar quantity, 1
Scaling of marks, 357
Seconds pendulum, 293
Sector, area of, 415
Selections, 309
Semi-interquartile range, 350
Series, 415
Sets, 311
Simple harmonic motion, 282
  forces producing S.H.M., 287
  standard formulae, 283
Simple pendulum, 292
SI units, 13
Smoothly hinged rods, 206
Smooth surface, 11, 12, 180
Speed, 31
Spread, measures of, 350
Springs, 99, 287
Standard deviation, 352
Standardisation of marks, 357
Standard normal distribution, 385
Statistical tables, 418
Step function, 339
Straight line, vector equation, 223
Straight line motion, 46, 50
Subset, 311
Sum and product laws, 323, 324, 328
Systems, of coplanar forces, 149
  of forces in vector form, 225
  of rigid bodies, 205

Tables, statistical, 418
Tension, 12
  in a light rod, 212
Three-dimensional coordinate system, 6
Thrust in a light rod, 212
Toppling and sliding, 183, 185
Torque, 145
Total reaction, 12, 88, 96
Trajectory of a projectile, 56
Tree diagram, 319, 329
Triangle, area of, 415
Triangle law, 3, 14
Triangle of forces, 24
Trigonometric formulae, 415

Uniform acceleration, 51, 63
Uniform distribution, 363, 378
Uniform motion in a circle, 128

Union of sets, 312
Units, SI, 13
Unit vector, 5
Universal set, 311

Variables, continuous, 339
  discrete, 339
Variance, 352, 364, 417
  binomial distribution, 371
  continuous probability distribution, 377, 397
  discrete probability distribution, 364, 397
  for grouped data, 353
  in terms of expectation, 399
  linear combination of random variables, 401
  linear function of a random variable, 399
  Poisson distribution, 393
  sum and differences of random variables, 401
Variation, measures of, 350
Vector equation of a line, 223
Vectors, 1
  addition, 3
  angle between, 235
  differentiation of, 62
  polygon, 15, 33
  position, 6
  scalar product, 233
  subtraction, 3
  triangle, 14, 35
  unit, 5
Velocity, 46
  as a vector, 31, 62, 228
  relative, 35, 228
Velocity–time graph, 47
  for constant acceleration, 51
Venn, John, 312
Venn diagrams, 312
Vertical circle, motion in a, 295

Watt, unit of power, 118
Weight, 12, 72, 156
Work done, by a constant force, 107
  by a variable force, 115
  in extending an elastic string, 115
  using vector methods, 235
Work–energy equations, 108, 109, 112, 113, 237, 254

Advanced Mathematics: An Applied Course provides a complete course in applied mathematics for A level courses which include both pure and applied mathematics. It will also be particularly useful for students studying mechanics as part of a course in engineering.

Topics are presented in a logical sequence and are arranged in short chapters to allow flexibility. The text contains many worked examples and numerous exercises providing both routine practice and opportunities to develop understanding.

Advanced Mathematics: A Pure Course (ISBN 0 00 322239 X) provides the companion course in pure mathematics.

Advanced Mathematics: Book 1 (ISBN 0 00 322269 1) and *Advanced Mathematics: Book 2* (ISBN 0 00 322299 3) integrate the pure and applied chapters from these two books to form a *combined* course in pure mathematics, statistics, and mechanics which includes chapters on numerical methods and probability.

Collins Educational

ISBN 0-00-322270-5

9 780003 222708

A COMPREHENSIVE COURSE IN APPLIED MATHEMATICS

ADVANCED MATHEMATICS

AN APPLIED COURSE

PERKINS & PERKINS

Mark Foster

ADVANCED MATHEMATICS

AN APPLIED COURSE

Martin and Patricia Perkins

Unwin Hyman

Published in 1987 by Unwin Hyman Ltd
15–17 Broadwick Street
London W1V 1FP

© Martin and Patricia Perkins 1987

All rights reserved. No part of this publication may be reproduced, stored in a retrieval system, or transmitted in any form or by any means, electronic, mechanical, photocopying, recording or otherwise, without the prior permission of Unwin Hyman.

British Library Cataloguing in Publication Data

Perkins, Martin
Advanced mathematics: an applied course.
1. Mathematics ——— 1961–
I. Title II. Perkins, Patricia
510 QA39.2

ISBN 0 7135 2820 6

Printed in Great Britain at
The Alden Press Ltd, Oxford

Preface

This is the second of two books designed to cover most Advanced Level single subject Mathematics syllabuses. This volume provides a thorough treatment of the Mechanics required at this level, together with chapters on probability and elementary statistics. We hope that the inclusion of these additional topics will enable students and teachers to plan a course of study to suit individual syllabus requirements.

We are most grateful for permission to use questions from past G.C.E. Advanced Level examinations. These are acknowledged as follows:

University of London University Entrance and Schools Examination Council (L)
The Associated Examining Board (AEB)
Joint Matriculation Board (JMB)
University of Cambridge Local Examinations Syndicate (C)
Oxford and Cambridge Schools Examination Board (O&C)
Oxford Delegacy of Local Examinations (O)
Southern Universities' Joint Board for School Examinations (SU)
Welsh Joint Education Committee (W)

We must again thank the staff of Messrs. Unwin Hyman, especially Caroline Paines and Antonia Murphy, for both their enthusiasm and their patience. We have also been helped and encouraged by many friends and colleagues.

M.L.P., P.P.

Contents

Preface ii

List of symbols vi

1 Vectors 1
1.1 Introduction to vectors 1
1.2 Cartesian components and position vectors 4

2 Forces and equilibrium 10
2.1 Mathematical models 10
2.2 The action of forces 11
2.3 Resultant forces 14
2.4 Resolution into components 18
2.5 A particle in equilibrium 22

3 Relative motion 31
3.1 Velocity and acceleration as vectors 31
3.2 Relative velocity 35
3.3 Relative displacement 38

4 Velocity and acceleration 46
4.1 Elementary kinematics 46
4.2 Motion with constant acceleration 50
4.3 Projectiles 56
4.4 Differentiation of position vectors 62

5 Newton's laws of motion 68
5.1 Newton's laws 68
5.2 Further applications 72
5.3 Connected particles 75
5.4 Related accelerations 81

6 Friction; Hooke's law 88
6.1 The laws of friction 88
6.2 Motion on a rough surface 91
6.3 Equilibrium on a rough surface 94
6.4 Angle of friction 96
6.5 Hooke's law 99

7 Work, energy and power 107
7.1 Work and kinetic energy 107
7.2 Conservation of mechanical energy 110
7.3 Work done by a variable force 115
7.4 Power 118

8 Uniform motion in a circle 128
8.1 Angular velocity 128
8.2 Acceleration in circular motion 131
8.3 Further problems 134

9 Coplanar forces; centre of gravity 143
9.1 Parallel forces, moments and couples 143
9.2 Systems of coplanar forces 151
9.3 Centre of gravity; centre of mass 156
9.4 Standard results by integration 164
9.5 Centroids of areas and volumes 172

10 Equilibrium of rigid bodies 180
10.1 Parallel forces in equilibrium 180
10.2 Three force problems 188
10.3 General conditions for equilibrium 197
10.4 Rigid body systems 205
10.5 Light frameworks 211

11 Vector methods 223
11.1 Forces and their lines of action 223
11.2 Use of vectors in kinematics and dynamics 228
11.3 Scalar product and its applications 233

12 Impulse and momentum 243
12.1 The impulse of a force 243
12.2 Conservation of momentum 250
12.3 Work, energy and momentum 253
12.4 Newton's law of restitution 260

13 Further particle dynamics 273
13.1 Acceleration as a function of displacement 273
13.2 Acceleration as a function of velocity 276
13.3 Simple harmonic motion 282

13.4 Forces producing simple harmonic motion 287
13.5 Motion in a vertical circle 295

14 Probability 308
14.1 Permutations and combinations 308
14.2 The language of sets 311
14.3 Elementary theory of probability 317
14.4 Sum and product laws 323
14.5 Conditional probability 328

15 Elementary statistics 337
15.1 Graphical representation of data 337
15.2 Mode, median and mean 345
15.3 Measures of spread 350
15.4 Change of variable 354

16 Probability distributions 362
16.1 Discrete probability distributions 362
16.2 The binomial distribution 369
16.3 Continuous probability distributions 377
16.4 The normal distribution 384
16.5 The Poisson distribution 392
16.6 The algebra of expectation and variance 397

Formulae for reference 415

Statistical tables 418

Answers 420

Index 452

List of symbols

s.f.	significant figures
d.p.	decimal places
$\approx$	is approximately equal to
$<$	is less than
$\leqslant$	is less than or equal to
$>$	is greater than
$\geqslant$	is greater than or equal to
$\lvert x \rvert$	the modulus of x
$[x]$	the integral part of x
e^x, $\exp x$	exponential function of x
$\log_a x$	logarithm to the base a of x
$\ln x$	the natural logarithm of x, $\log_e x$
$\lg x$	the common logarithm of x, $\log_{10} x$
$\sin^{-1}$	inverse function of sin with range $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$
$\cos^{-1}$	inverse function of cos with range $[0, \pi]$
$\tan^{-1}$	inverse function of tan with range $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$
$\lim_{x \to a} f(x)$	limit of $f(x)$ as x tends to a
∞	infinity
δ	small increment
$f'(x)$ or $\frac{dy}{dx}$	first derivative of $y = f(x)$
$f''(x)$ or $\frac{d^2y}{dx^2}$	second derivative of $y = f(x)$
$\dot{x}, \dot{y}, \dot{\theta}, \ldots$	first derivative of $x, y, \theta, \ldots$ with respect to time
$\ddot{x}, \ddot{y}, \ddot{\theta}, \ldots$	second derivative of $x, y, \theta, \ldots$ with respect to time
PQ	vector represented by line segment $\overrightarrow{PQ}$
$\lvert\mathbf{PQ}\rvert$, PQ	magnitude of vector **PQ**
$\lvert\mathbf{a}\rvert$, a	magnitude of vector **a**

$\hat{\mathbf{a}}$	unit vector in the direction of **a**
$\mathbf{i}, \mathbf{j}, \mathbf{k}$	unit vectors in the directions of the x-, y- and z-axes
$\mathbf{a} . \mathbf{b}$	scalar product of **a** and **b**
g	acceleration due to gravity (taken as $9{\cdot}8\ \text{m s}^{-2}$ unless otherwise stated)
μ	coefficient of friction
λ	angle of friction or modulus of elasticity
ω	angular velocity
$\varnothing$	empty set
$\mathscr{E}$	universal set
A'	complement of set A
$n(A)$	number of elements in the set A
$\cup$	union
$\cap$	intersection
$\in$	is an element of
$\mathbb{Z}$	set of integers
$\mathbb{Z}^+$	set of positive integers
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	set of real numbers
$n!$	n factorial
${}_nP_r$	number of permutations of r objects chosen from n
${}_nC_r$	number of combinations of r objects chosen from n
$P(A)$	probability of event A
$P(A')$	probability that A does not occur
$P(A \cup B)$	probability of event A or event B (or both)
$P(A \cap B)$	probability of event A and event B
$P(A\|B)$	probability of event A given B
$\sum_{r=1}^{n} f(r)$	$f(1)+f(2)+\ldots+f(n)$
$\binom{n}{r}$	the binomial coefficient: ${}_nC_r = \dfrac{n!}{r!(n-r)!}$ for $n \in \mathbb{Z}^+$, $\dfrac{n(n-1)\ldots(n-r+1)}{r!}$ for $n \in \mathbb{Q}$
$x_1, x_2, \ldots$	observations
$f_1, f_2, \ldots$	frequencies with which the observations $x_1, x_2, \ldots$ occur
$\bar{x}$	mean of observations $x_1, x_2, \ldots$
var (x)	variance of observations $x_1, x_2, \ldots$
s.d.	standard deviation
$X, Y, R, \ldots$	random variables
$x, y, r, \ldots$	values of the random variables $X, Y, R, \ldots$

$p(x)$	value of the probability function $P(X = x)$ of the discrete random variable X
$p_1, p_2, \ldots$	probabilities of the values $x_1, x_2, \ldots$ of the discrete random variable X
$f(x), g(x), \ldots$	value of the probability density function of the continuous random variable X
$F(x), G(x), \ldots$	value of the (cumulative) distribution function $P(X \leqslant x)$ of the continuous random variable X
$E(X), \mu$	expectation (or mean) of the random variable X
$\mathrm{Var}(X), \sigma^2$	variance of the random variable X
$G(t)$	probability generating function for a random variable which takes the values $0, 1, 2, \ldots$
$B(n, p)$	binomial distribution, parameters n and p
$N(\mu, \sigma^2)$	normal distribution, mean μ and variance σ^2
Z	the standardised normal variable with distribution $N(0, 1)$
ϕ	probability density function for $N(0, 1)$
Φ	cumulative distribution function for $N(0, 1)$

The use of the symbols $=$ and $\approx$

In numerical work it is important to distinguish between exact values and approximate values. When giving exact answers we use the symbol $=$. For approximate results we use either the symbol $=$ (together with an appropriate phrase, such as 'by calculator', 'to 3 s.f.'), or the symbol $\approx$. The use of the symbol $\approx$ may show that a result has been obtained by an approximate method, but often indicates that a more accurate result obtained by calculator has been rounded to the given number of significant figures (or, in the case of angles, to the nearest tenth of a degree).

1 Vectors

1.1 Introduction to vectors

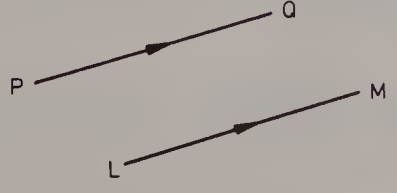
Some physical quantities, such as temperature, may be completely specified by a number referred to some unit of measurement, e.g. 25°C, 350°F. Such quantities, which have magnitude but which are not related to any definite direction in space, are called *scalar quantities*. Similarly, a number representing the magnitude of some physical quantity is called a *scalar*.

When measuring other quantities, such as wind velocity, it is necessary to give a direction as well as a number and a unit of measurement, e.g. 40 km/h from the north-east. These quantities, which have both magnitude and a definite direction in space, are called *vector quantities*. In general, a *vector* may be described as a number associated with a particular direction in space.

Any vector may be represented by a *directed line segment*, whose direction is that of the vector and whose length represents the magnitude of the vector.

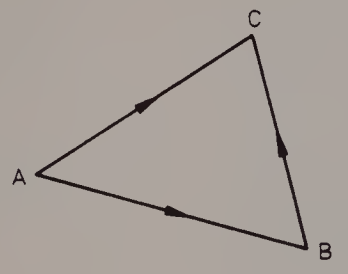
The vector represented here is denoted by **PQ** in bold type (or $\underset{\sim}{PQ}$ in manuscript). The *magnitude* or modulus of the vector is written $|\mathbf{PQ}|$ or PQ. The directed line segment joining P to Q is denoted by $\overrightarrow{PQ}$. However, since it is not usually necessary to distinguish between the line segment and the vector it represents, the notation $\overrightarrow{PQ}$ is in common use for both vector and line segment.

The magnitude of a vector is usually positive but may be zero, in which case the vector is called the *zero* or *null vector* and written **0** in bold type. The zero vector is the only vector with indeterminate direction.



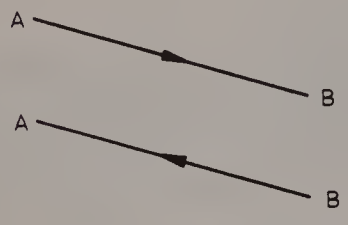
Two vectors are *equal* if they have the same magnitude and direction. Since PQ and LM are parallel and equal in length, $\mathbf{PQ} = \mathbf{LM}$.

A *displacement* is one of the simplest examples of a vector quantity. For instance, a car journey from a town A to a town B may be represented by a displacement vector **AB**. Its magnitude is the distance between A and B. Its direction is that of the straight line joining A to B.



Let us suppose that a man drives from A to B and then from B to C. On another occasion he drives directly from A to C. Since the result of these two journeys is the same, we may write $\mathbf{AB}+\mathbf{BC}=\mathbf{AC}$.

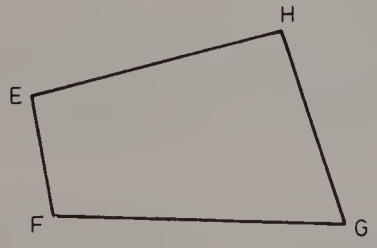
The displacement **AC** is the *sum* or *resultant* of the displacements **AB** and **BC**.



Consider now a journey from A to B, followed by the return journey from B to A. Since the result of these two journeys is a zero displacement, it is reasonable to write, $\mathbf{AB}+\mathbf{BA}=\mathbf{0}$ and $\mathbf{BA}=-\mathbf{AB}$.

All vectors represented by directed line segments may be manipulated in the same way as displacement vectors. For instance, given any three vectors **AB**, **BC** and **AC** represented by the sides of triangle ABC, we may write $\mathbf{AB}+\mathbf{BC}=\mathbf{AC}$. A vector which has the same magnitude as **AB**, but the opposite direction, is denoted by **BA** or $-\mathbf{AB}$.

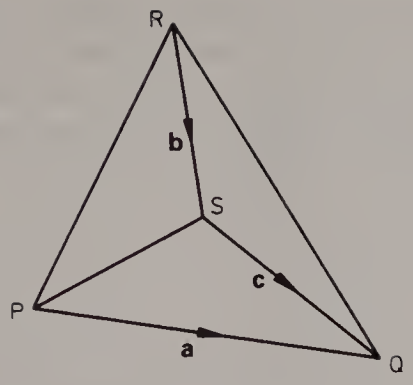
Example 1 Simplify $\mathbf{EF}+\mathbf{FG}-\mathbf{HG}$.



$$\begin{aligned}\mathbf{EF}+\mathbf{FG}-\mathbf{HG} &= \mathbf{EG}-\mathbf{HG}\\ &= \mathbf{EG}+\mathbf{GH}\\ &= \mathbf{EH}.\end{aligned}$$

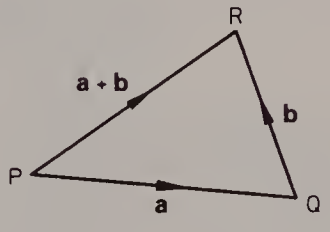
It is often convenient to denote a vector by a single letter such as **a** in bold type (or $\underset{\sim}{a}$ in manuscript). Its magnitude is then written $|\mathbf{a}|$ or a.

Example 2 In the given diagram $\mathbf{PQ}=\mathbf{a}$, $\mathbf{RS}=\mathbf{b}$ and $\mathbf{SQ}=\mathbf{c}$. Find expressions for **RQ**, **PS** and **PR** in terms of **a**, **b** and **c**.



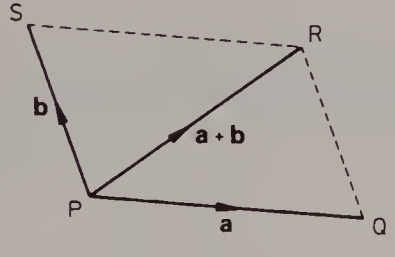
In $\triangle QRS$, $\mathbf{RQ} = \mathbf{RS} + \mathbf{SQ} = \mathbf{b} + \mathbf{c}$
In $\triangle PQS$, $\mathbf{PS} = \mathbf{PQ} + \mathbf{QS} = \mathbf{PQ} - \mathbf{SQ}$
$= \mathbf{a} - \mathbf{c}$
In $\triangle PQR$, $\mathbf{PR} = \mathbf{PQ} + \mathbf{QR} = \mathbf{PQ} - \mathbf{RQ}$
$= \mathbf{a} - (\mathbf{b} + \mathbf{c})$
$= \mathbf{a} - \mathbf{b} - \mathbf{c}$.

More formally, *vector addition* is defined by the *triangle law*, as follows:



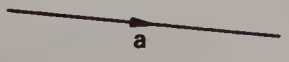
If two vectors **a** and **b** are represented by the sides $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ of a triangle, then $\mathbf{a} + \mathbf{b}$ is represented by the third side $\overrightarrow{PR}$.

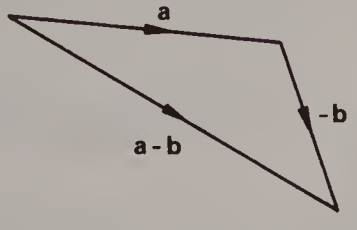
In physics and mechanics it is sometimes more convenient to use the *parallelogram law* for vector addition, which states:

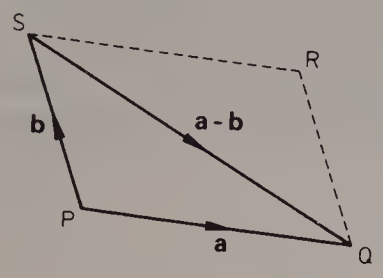


If two vectors **a** and **b** are represented by the sides $\overrightarrow{PQ}$ and $\overrightarrow{PS}$ of a parallelogram, then $\mathbf{a} + \mathbf{b}$ is represented by the diagonal $\overrightarrow{PR}$. [Since the line segments $\overrightarrow{PS}$ and $\overrightarrow{QR}$ are equal in magnitude and direction, the two addition laws are equivalent.]

The *subtraction* of **b** from **a** may be defined as the addition of $-\mathbf{b}$ to **a**, i.e. $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$.







If the vectors **a** and **b** are represented by the sides $\overrightarrow{PQ}$ and $\overrightarrow{PS}$ of a parallelogram, then **a** − **b** is represented by the diagonal $\overrightarrow{SQ}$.

Exercise 1.1

1. Simplify (a) **PQ** + **QR** + **RS**, (b) **AB** + **DE** + **CD** + **BC**,
(c) **AC** − **BC**, (d) **XY** + **YX**,
(e) **MN** − **RP** − **PN**, (f) **QT** − **QR** + **PR** − **ST**.

2. Given that *ABCDEF* is a regular hexagon, decide whether the following statements are true or false.
(a) **BC** = **EF**, (b) **BF** = **CE**, (c) |**BE**| = |**CF**|, (d) |**AB**| = |**DE**|,
(e) **AC** and **DF** are in the same direction, (f) **AD** is parallel to **EF**.

3. Given that *ABCDPQRS* is a cuboid lettered so that *AP*, *BQ*, *CR* and *DS* are edges, decide whether the following statements are true or false.
(a) **PQ** = **DC**, (b) **BR** = **PD**, (c) |**RA**| = |**BS**|, (d) |**PA**| = |**CR**|,
(e) **DQ** and **AR** are in the same direction, (f) **AQ** is parallel to **CS**.

If **a**, **b** and **c** are displacements of 1 km due east, 2 km due north and 1 km in the direction S 30° W respectively, find the magnitudes and directions of the displacements given in questions 4 to 6. (You may leave your answers in surd form.)

4. (a) 3**a**, (b) 5**b**, (c) −**c**, (d) $-\frac{1}{2}$**b**.

5. (a) 2**a** + **b**, (b) **b** + 2**a**, (c) 2**a** − **b**, (d) **b** − 2**a**.

6. (a) **a** + **c**, (b) **a** + 2**c**, (c) 2**a** + **c**, (d) **a** − **c**.

7. The diagonals of a parallelogram *ABCD* intersect at *M*. If **AB** = **p** and **AD** = **q**, express in terms of **p** and **q** the vectors **AC**, **BD**, **AM** and **MD**.

8. In a regular hexagon *ABCDEF* **AB** = **a** and **BC** = **b**. Find expressions in terms of **a** and **b** for **DE**, **DC**, **AD** and **BD**.

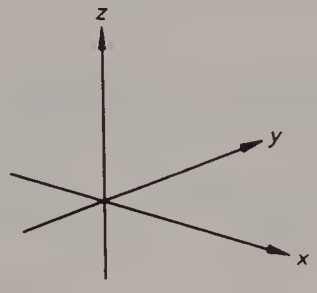
9. In the pentagon *PQRST* **PQ** = **a** and **RS** = **b**. If **PT** = 2**RS** and **TS** = 2**PQ**, find expressions for **PR** and **QR** in terms of **a** and **b**.

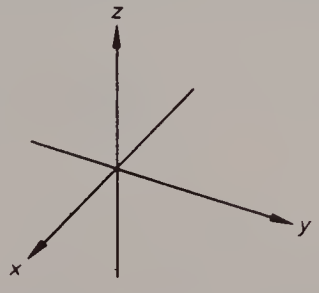
1.2 Cartesian components and position vectors

Resolving a vector into *component vectors* means expressing it as a sum of two or more non-parallel vectors.

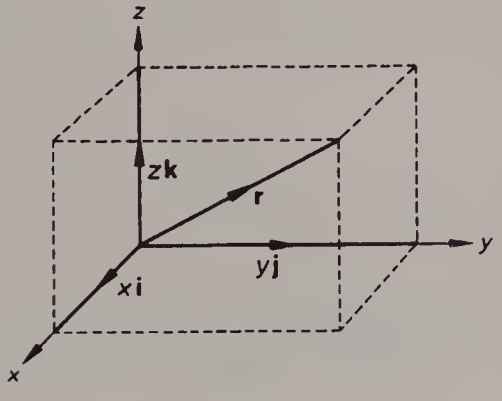
It is often convenient to resolve vectors into component vectors in the directions of the Cartesian coordinate axes. In three-dimensional work the third axis is the z-axis, which is constructed perpendicular to the x- and y-axes to form a *right-handed system.* This means that if we point the thumb of the right hand in the direction of the z-axis, then bend the fingers slightly, they will indicate the direction of a rotation from the x-axis to the y-axis.

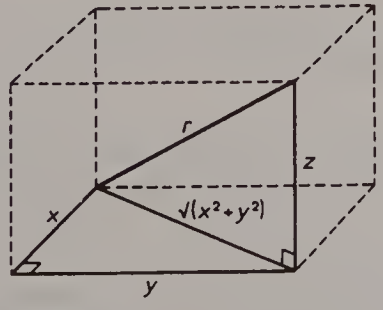
Two standard ways of drawing these axes are shown below.





Any vector with magnitude 1 is called a *unit vector.* The unit vectors in the directions of the x-, y- and z-axes are denoted by **i**, **j** and **k** respectively.





By constructing a cuboid with one diagonal representing **r** and sides parallel to the coordinate axes, we can demonstrate that there is a unique expression for any vector **r** of the form $x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$. The scalars x, y and z are called the *components* of **r** in the directions of **i**, **j** and **k**, or simply the *Cartesian components* of **r**.

Since **i**, **j** and **k** are unit vectors, x, y and z are the lengths of the sides of the cuboid constructed here. Applying Pythagoras' theorem in the two right-angled triangles illustrated, we find that the magnitude of **r** is given by

$$r = \sqrt{(x^2+y^2+z^2)}.$$

Example 1 If $\mathbf{a} = 3\mathbf{i}+2\mathbf{j}+7\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i}+4\mathbf{j}-5\mathbf{k}$, find $|\mathbf{a}|$ and $|\mathbf{a}-\mathbf{b}|$.

$|\mathbf{a}| = \sqrt{(3^2+2^2+7^2)} = \sqrt{(9+4+49)} = \sqrt{62}$

$\mathbf{a}-\mathbf{b} = (3\mathbf{i}+2\mathbf{j}+7\mathbf{k})-(2\mathbf{i}+4\mathbf{j}-5\mathbf{k}) = \mathbf{i}-2\mathbf{j}+12\mathbf{k}$

$\therefore \quad |\mathbf{a}-\mathbf{b}| = \sqrt{(1^2+\{-2\}^2+12^2)} = \sqrt{(1+4+144)} = \sqrt{149}.$

Since a represents the magnitude of a vector $\mathbf{a}$, the vector $\mathbf{a}/a$ has magnitude 1 and the same direction as $\mathbf{a}$. Thus the unit vector in the direction of a given vector $\mathbf{a}$ (sometimes denoted by $\hat{\mathbf{a}}$) may be expressed in the form $\mathbf{a}/a$.

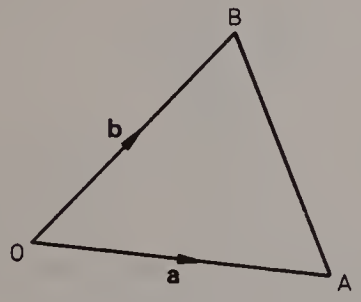
Example 2 Find the unit vector in the direction of $\mathbf{a} = 2\mathbf{i}+2\mathbf{j}-\mathbf{k}$.

$a = \sqrt{(2^2+2^2+\{-1\}^2)} = \sqrt{(4+4+1)} = \sqrt{9} = 3$

$\therefore$ the unit vector in the direction of $\mathbf{a}$

$$= \tfrac{1}{3}(2\mathbf{i}+2\mathbf{j}-\mathbf{k}) = \tfrac{2}{3}\mathbf{i}+\tfrac{2}{3}\mathbf{j}-\tfrac{1}{3}\mathbf{k}.$$

Taking a fixed point O as origin, the position of any point P can be specified by giving the vector $\mathbf{OP}$, which is then called the *position vector* of P.



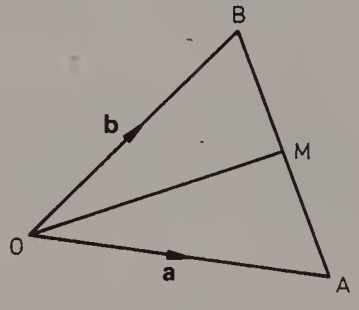
If A and B are two points with position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, then

$$\mathbf{AB} = \mathbf{AO}+\mathbf{OB} = -\mathbf{OA}+\mathbf{OB} = \mathbf{OB}-\mathbf{OA}$$
$$\therefore \quad \mathbf{AB} = \mathbf{b}-\mathbf{a}.$$

Hence the distance between the points A and B is given by $|\mathbf{b}-\mathbf{a}|$.

Consider now the mid-point M of AB.

$$\mathbf{AM} = \mathbf{MB}$$
$$\Rightarrow \mathbf{OM}-\mathbf{OA} = \mathbf{OB}-\mathbf{OM}$$
$$\Rightarrow \quad 2\mathbf{OM} = \mathbf{OA}+\mathbf{OB}$$
$$\Rightarrow \quad \mathbf{OM} = \tfrac{1}{2}(\mathbf{OA}+\mathbf{OB})$$



$\therefore$ the position vector of the mid-point of AB is $\frac{1}{2}(\mathbf{a}+\mathbf{b})$.

When working in a Cartesian coordinate system, the point P with coordinates (x, y, z) has position vector $x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors in the directions of the coordinate axes.

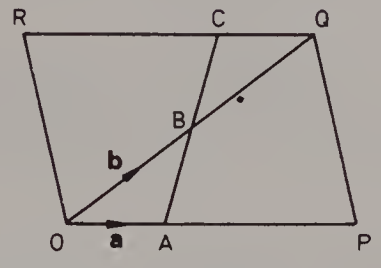
Example 3 The position vectors of points A and B with respect to an origin O are $\mathbf{a} = \mathbf{i}-2\mathbf{j}+7\mathbf{k}$ and $\mathbf{b} = 5\mathbf{i}+\mathbf{j}-5\mathbf{k}$. Find (i) the distance between the origin and the point A, (ii) the distance between the points A and B, (iii) the position vector of the mid-point of AB.

(i) the distance between the origin and the point A
$= |\mathbf{a}| = \sqrt{(1^2+\{-2\}^2+7^2)} = \sqrt{(1+4+49)} = \sqrt{54}$.

(ii) $\mathbf{AB} = \mathbf{b}-\mathbf{a} = (5\mathbf{i}+\mathbf{j}-5\mathbf{k})-(\mathbf{i}-2\mathbf{j}+7\mathbf{k}) = 4\mathbf{i}+3\mathbf{j}-12\mathbf{k}$
$\therefore$ the distance between the points A and B
$= |\mathbf{b}-\mathbf{a}| = |4\mathbf{i}+3\mathbf{j}-12\mathbf{k}| = \sqrt{(16+9+144)} = 13.$

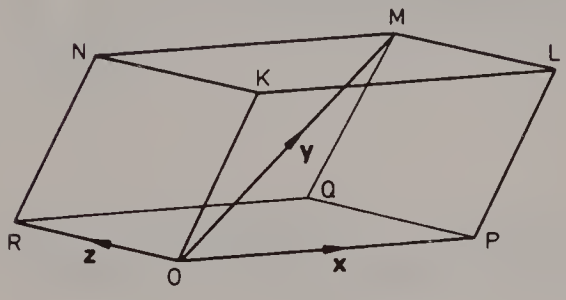
(iii) the position vector of the mid-point of AB
$= \frac{1}{2}(\mathbf{a}+\mathbf{b}) = \frac{1}{2}(1+5)\mathbf{i}+\frac{1}{2}(-2+1)\mathbf{j}+\frac{1}{2}(7-5)\mathbf{k} = 3\mathbf{i}-\frac{1}{2}\mathbf{j}+\mathbf{k}.$

Exercise 1.2



1. In the given diagram $\mathbf{OA} = \mathbf{a}$ and $\mathbf{OB} = \mathbf{b}$. The points P, Q and R are constructed so that $\mathbf{AP} = 2\mathbf{OA}$, $\mathbf{BQ} = \mathbf{OB}$ and $OPQR$ is a parallelogram. Express the following vectors as sums of component vectors in the directions of $\mathbf{a}$ and $\mathbf{b}$

(a) **OQ**, (b) **QR**, (c) **PR**, (d) **AB**, (e) **CQ**, (f) **OC**.



In the parallelepiped shown in the diagram $\mathbf{OP} = \mathbf{x}$, $\mathbf{OM} = \mathbf{y}$ and $\mathbf{OR} = \mathbf{z}$. In questions 2 and 3 resolve the given vectors into component vectors in the directions of of x, y and z.

2. (a) **PQ**. (b) **RQ**, (c) **MN**, (d) **LM**, (e) **PR**, (f) **RM**.

3. (a) **ON**, (b) **KM**, (c) **OK**, (d) **NP**, (e) **KQ**, (f) **LR**.

4. In quadrilateral $ABCD$, $\mathbf{AB} = 2\mathbf{i}-\mathbf{j}$, $\mathbf{BC} = 3\mathbf{i}+4\mathbf{j}$, $\mathbf{AD} = \mathbf{i}+5\mathbf{j}$ and M is the mid-point of CD. Express in terms of $\mathbf{i}$ and $\mathbf{j}$ the vectors
(a) **AC**, (b) **BD**, (c) **CD**, (d) **DM**, (e) **AM**, (f) **BM**.

5. In tetrahedron $PQRS$, $\mathbf{PQ} = 3\mathbf{i}-\mathbf{j}-\mathbf{k}$, $\mathbf{PR} = -\mathbf{i}+4\mathbf{j}+\mathbf{k}$ and $\mathbf{SQ} = 3\mathbf{i}-4\mathbf{j}-3\mathbf{k}$. The point T lies on RS produced such that $RS = ST$. Express in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ the vectors
(a) **QR**, (b) **PS**, (c) **RS**, (d) **PT**.

6. Find the magnitude of each of the following vectors
(a) $5\mathbf{i}-12\mathbf{j}$, (b) $-\mathbf{i}+2\mathbf{k}$, (c) $8\mathbf{j}+6\mathbf{k}$,
(d) $2\mathbf{i}+\mathbf{j}-2\mathbf{k}$, (e) $2\mathbf{i}-6\mathbf{j}+3\mathbf{k}$, (f) $-\mathbf{i}+7\mathbf{j}-5\mathbf{k}$.

7. Find unit vectors in the direction of
(a) $8\mathbf{i}-4\mathbf{j}-\mathbf{k}$, (b) $16\mathbf{i}-8\mathbf{j}-2\mathbf{k}$, (c) $-8\mathbf{i}+4\mathbf{j}+\mathbf{k}$.

In questions 8 to 10 take $\mathbf{a} = \mathbf{i}-2\mathbf{j}+\mathbf{k}$, $\mathbf{b} = 3\mathbf{i}-\mathbf{j}-\mathbf{k}$, $\mathbf{c} = \mathbf{i}+3\mathbf{j}-2\mathbf{k}$.

8. Find (a) $|\mathbf{a}+\mathbf{b}|$, (b) $|\mathbf{a}-\mathbf{b}|$, (c) $|2\mathbf{b}-2\mathbf{a}|$.

9. Find (a) $|\mathbf{b}+\mathbf{c}|$, (b) $|\mathbf{a}+\mathbf{b}+\mathbf{c}|$, (c) $|5\mathbf{a}-\mathbf{b}+3\mathbf{c}|$.

10. Find unit vectors in the directions of
(a) $2\mathbf{b}-\mathbf{c}$, (b) $5\mathbf{a}-5\mathbf{b}+3\mathbf{c}$, (c) $5\mathbf{a}-\mathbf{b}+3\mathbf{c}$.

11. In a regular hexagon $OABCDE$ the position vectors of A and B relative to O are $\mathbf{a}$ and $\mathbf{b}$ respectively. Find expressions in terms of $\mathbf{a}$ and $\mathbf{b}$ for the vectors $\mathbf{AB}$ and $\mathbf{BC}$. Find also the position vectors of the points C, D and E.

12. In a quadrilateral $ABCD$ the point M is the mid-point of the diagonal BD. Given that the position vectors of A, B and C are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively and that $ABCM$ is a parallelogram, find in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ the vectors $\mathbf{BA}$, $\mathbf{BC}$ and $\mathbf{BM}$. Hence find the position vector of the point D.

13. The points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. In each of the following cases, find the distance between A and B, the unit vector in the direction of $\mathbf{AB}$ and the position vector of the mid-point M of AB.
(a) $\mathbf{a} = 3\mathbf{i}-5\mathbf{j}-\mathbf{k}$, $\mathbf{b} = \mathbf{i}-\mathbf{j}+3\mathbf{k}$,
(b) $\mathbf{a} = -7\mathbf{i}+13\mathbf{j}$, $\mathbf{b} = 5\mathbf{i}-5\mathbf{j}-4\mathbf{k}$,
(c) $\mathbf{a} = \mathbf{i}-\mathbf{j}+3\mathbf{k}$, $\mathbf{b} = \mathbf{i}+2\mathbf{j}+2\mathbf{k}$,
(d) $\mathbf{a} = 5\mathbf{i}+4\mathbf{j}+\mathbf{k}$, $\mathbf{b} = -\mathbf{i}+\mathbf{j}-2\mathbf{k}$.

14. The points A, B and C have position vectors $\mathbf{a} = -\mathbf{i}+2\mathbf{j}+3\mathbf{k}$, $\mathbf{b} = 8\mathbf{i}+7\mathbf{j}-9\mathbf{k}$ and $\mathbf{c} = 2\mathbf{i}-3\mathbf{j}-\mathbf{k}$. Prove that the triangle ABC is right-angled and find its area.

Exercise 1.3 (miscellaneous)

1. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are vectors representing the edges of a parallelepiped, find vectors representing the four diagonals.

2. In a tetrahedron $OABC$ $\mathbf{OA} = \mathbf{a}$, $\mathbf{OB} = \mathbf{b}$ and $\mathbf{OC} = \mathbf{c}$. The points P and Q are constructed such that $\mathbf{OA} = \mathbf{AP}$ and $2\mathbf{OB} = \mathbf{BQ}$. The point M is the mid-point of PQ. Find, in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, expressions for (a) $\mathbf{AB}$, (b) $\mathbf{PQ}$, (c) $\mathbf{CQ}$, (d) $\mathbf{QM}$, (e) $\mathbf{MB}$, (f) $\mathbf{OM}$.

3. If $\mathbf{a} = 2\mathbf{i}-4\mathbf{j}+2\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i}+4\mathbf{j}-5\mathbf{k}$, find (a) the magnitudes of $\mathbf{a}$, $\mathbf{b}$ and $5\mathbf{a}+2\mathbf{b}$, (b) the unit vector in the direction of $\mathbf{a}+2\mathbf{b}$.

4. If P and Q are the points with position vectors

$$\mathbf{p} = -\mathbf{i}+3\mathbf{j}+2\mathbf{k} \text{ and } \mathbf{q} = \mathbf{i}-7\mathbf{j}+4\mathbf{k}$$

respectively, find (a) the distance PQ, (b) the unit vector in the direction of PQ, (c) the position vector of the mid-point of PQ.

5. Given that $\mathbf{a} = 6\mathbf{i}+(p-10)\mathbf{j}+(3p-5)\mathbf{k}$ and that $|\mathbf{a}| = 11$, find the possible values of p.

6. Given that the points A, B and C have position vectors $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 3\mathbf{i} - \mathbf{k}$ and $\mathbf{c} = -\mathbf{i} + \mathbf{j} + 4\mathbf{k}$, prove that the triangle ABC is isosceles.

7. In the parallelogram $OPQR$, the position vectors of P and R with respect to O are $6\mathbf{i} + \mathbf{j}$ and $3\mathbf{i} + 4\mathbf{j}$ respectively. M and N are the mid-points of the sides PQ and QR respectively. Find (a) the unit vector in the direction of $\overrightarrow{MN}$, (b) the magnitude of the vector $\overrightarrow{OM}$. (L)

8. In triangle OAB, $\mathbf{OA} = \mathbf{a}$ and $\mathbf{OB} = \mathbf{b}$. Points X, Y and Z are constructed so that $2\mathbf{AX} = 3\mathbf{AB}$, $\mathbf{OB} = \mathbf{BY}$ and Z is the point of intersection of OX and AY. If $\mathbf{AZ} = k\mathbf{AY}$, express $\mathbf{OZ}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and k. Show that $\mathbf{OZ}$ is a scalar multiple of $3\mathbf{b} - \mathbf{a}$. Hence find the value of k and the position vector of Z.

9. In a parallelogram $OABC$, M is the mid-point of AB and P is the point of intersection of OM and AC. If the position vectors of A and B with respect to O as origin are $\mathbf{a}$ and $\mathbf{b}$, find the position vector of P.

10. The points $A(3, -2, 0)$, $B(-2, 5, -4)$ and $C(-1, 0, 4)$ are three vertices of a rhombus. Use vector methods to find the fourth vertex and the area of the rhombus.

2 Forces and equilibrium

2.1 Mathematical models

When applying mathematics to a practical problem, we try to set up what is called a mathematical model. In many cases this will take the form of a set of equations. However, it may also include diagrams, graphs or flow charts. As real situations are often very complicated, a mathematical model is usually a simplification of the original problem. Many scientific laws used in setting up systems of equations represent no more than good approximations to the true relationships between the quantities involved. Some factors governing a situation may be neglected altogether if their effect is likely to be small.

To set up a mathematical model we must first decide what factors are to be taken into account and what mathematical relationships are to be assumed, bearing in mind the degree of accuracy required in the result. We then find ways of giving numerical values to the variables involved, choosing suitable units of measurement.

To illustrate the process we now consider the problem of estimating the amount of petrol which will be consumed during a car journey. For a rough estimate we can use the distance, d miles, to be travelled and an approximate rate of petrol consumption for the car, c miles per gallon, based on past experience. We then assume that the amount of petrol used, P gallons, is proportional to the distance travelled and write $P = d \div c$. Clearly this single equation represents one of the simplest possible models which can be constructed to solve this problem. Many factors have been ignored, such as the load the car will be carrying, the type of road to be used, the traffic and weather conditions.

In elementary mechanics we set up simple models of practical situations in order to study the action of forces on various types of body. In early work we simplify the bodies involved as much as possible and take into account only the most important forces on them. For example, walls, floors and table tops are regarded as flat surfaces. A light rope or cable is assumed to be straight when pulled taut, whereas in practice, unless it is vertical, it hangs in a slight curve. We now list some of the other conventions observed when solving problems in mechanics.

A *particle* is a body of dimensions so small that its position in space can be represented by a single point.

A *lamina* is a flat object whose thickness is negligible but whose area is not.

A *rigid body* is one in which any pair of particles of matter remain a fixed distance apart. This means that rigid bodies are assumed to retain their shape when touching or colliding. For instance, when a sphere rests on a plane surface the area of contact is regarded as a single point.

A *light* string is taken to be weightless. If a body is suspended from one or more light strings, these can be represented by straight lines. Unless otherwise stated strings are assumed *inextensible*, i.e. of constant length.

A *smooth* surface is one which offers negligible frictional resistance to the motion of a body sliding across it. Wheels, pulleys, pivots and joints can also be described as smooth or frictionless.

For most purposes the surface of the earth is assumed to be flat and horizontal. Effects due to air currents and air resistance are usually ignored.

Exercise 2.1

Discuss the setting up of mathematical models in relation to various practical situations. Some problems for consideration are suggested below.

1. To decide which is the quickest way to travel from home to work or college in the rush hour.

2. To estimate the weekly cost of running a car.

3. To budget for the household bills expected in the next three months.

4. To plan orders of crisps, fruit, etc., for a college snack bar.

5. To organise a school time-table.

6. To plan a revision course for candidates taking a mathematics examination.

7. To decide whether buying a domestic appliance such as a freezer or dish-washer will save a family (a) time, (b) money.

8. To evaluate possible ways of saving energy at home or school, such as installing double glazing or switching off unnecessary lights.

2.2 The action of forces

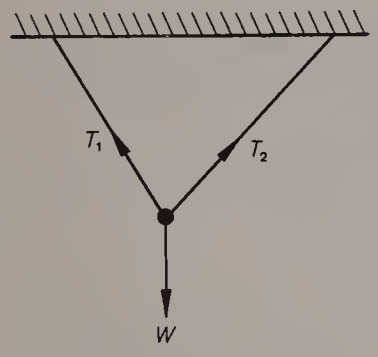
The effect of a *force* is to set a body in motion or to change the speed and direction of its motion. If the forces on a stationary body cancel each other out, then the body is said to be in *equilibrium*. The two main branches of mechanics

are *dynamics*, which is the study of the action of forces on bodies in motion, and *statics*, which is the study of forces in equilibrium. Closely related to dynamics is *kinematics*, which is the study of motion without reference to the forces involved.

A force may be completely specified by stating its magnitude, its direction and either its point of application or its line of action. Thus a force is a vector quantity associated with a particular point or line in space.

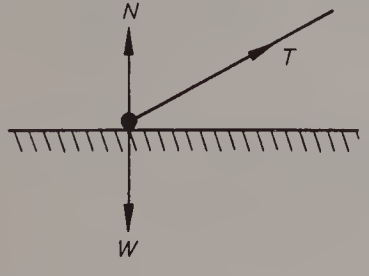
A force which is important in most problems is the gravitational attraction between the earth and any object close to it. This is the force which causes a body to fall to the ground and is called the *weight* of the body. Weight must not be confused with mass, which is not a force. The *mass* of a body is the amount of matter it contains. The weight of an object of fixed mass takes slightly different values at different points on the earth's surface. The gravitational pull on an object is much smaller on the surface of the moon, although its mass is unchanged.

When a body is attached to a string then the force acting along the string is called the *tension* in the string. The tension in a light string is assumed constant throughout its length. It is also possible to produce tension in rods and springs.



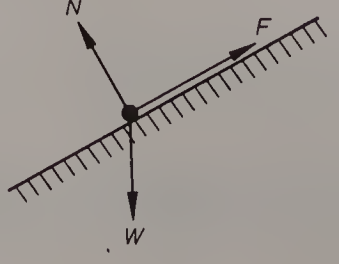
This diagram shows the forces on a particle of weight W, suspended from a beam by two light strings.

Another type of force called a *reaction* must be taken into account when two bodies are in contact. For a body resting on a smooth surface this force is referred to as the *normal reaction*, because its line of action is normal (i.e. perpendicular) to the surface. When two rough surfaces are in contact the total reaction is the sum of a normal reaction and a frictional force. [Friction will be considered further in Chapter 6.]



Shown here are the forces on a particle being pulled by a string across a smooth horizontal surface. The normal reaction is denoted by N.

Consider now a particle at rest on a rough inclined plane. As well as the normal reaction there will be a frictional force opposing the tendency of the particle to slide down the plane.



Force and other basic quantities are now usually measured in SI units, i.e. according to the Système International d'Unités. The SI unit of force is the newton, defined as the force needed to give a mass of 1 kilogramme an acceleration of 1 metre per second per second. [Students not familiar with the unit may find it helpful to remember that a cricket ball weighs approximately 1·5 N and a heavyweight boxer 800 N or more.]

We now give a table showing some of the SI units which are being used in this course.

Quantity	*Name of unit*	*Symbol*
length	metre	m
mass	kilogramme	kg
time	second	s
area	square metre	m^2
volume	cubic metre	m^3
velocity	metre per second	$m\,s^{-1}$ or m/s
acceleration	metre per second squared	$m\,s^{-2}$ or m/s^2
force	newton	N

[Note that 's' should never be added to a symbol to make a plural, because it is the symbol for second. For instance, 3 metres cannot be written as 3 ms because this means 3 milliseconds. Similarly the space in symbols for derived units is important, since without the space the symbol may have a different meaning. For example, $m\,s^{-1}$ means 'metre per second' but ms^{-1} means 'per millisecond'.]

Exercise 2.2

Draw diagrams to show the forces acting on a particle in each of the following situations.

1. At rest on a smooth horizontal plane.

2. At rest suspended from a fixed point by a string.

3. Sliding down a smooth inclined plane.

4. Sliding down a rough inclined plane.

5. Pulled across a rough horizontal plane by a string at an angle of 30° to the plane.

6. Held at rest on a smooth plane inclined at 25° to the horizontal by a string parallel to the plane.

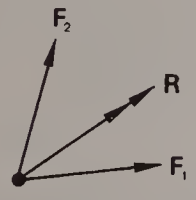
7. Pulled up a rough plane inclined at 25° to the horizontal by a string parallel to the plane.

8. Swinging at the end of a string, when the string is at an angle of 35° to the vertical.

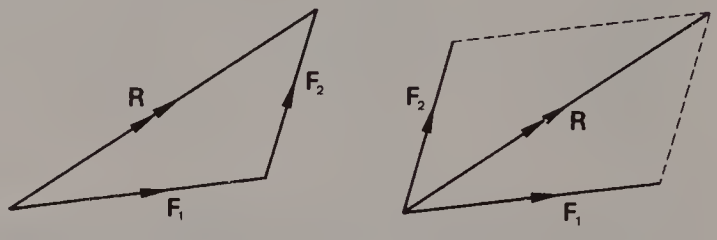
9. Sliding down the smooth inner surface of a hemispherical bowl.

2.3 Resultant forces

Since forces are vector quantities, they can be represented by directed line segments and added using either the triangle law or the parallelogram law as stated in §1.1. Thus, if $\mathbf{F}_1$ and $\mathbf{F}_2$ are two forces acting on a particle, their sum or resultant $\mathbf{R}$ may be represented by either the third side of a vector triangle or the diagonal of a parallelogram. A double arrow is used to emphasise that $\mathbf{R}$ is a resultant force rather than a third force acting on the particle.



Space diagram (showing positions in space)

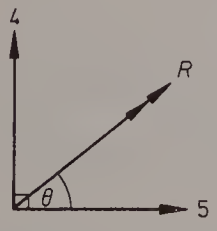


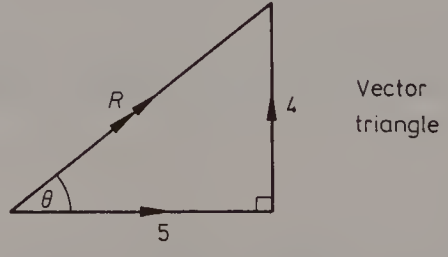
Vector diagrams

The magnitude and direction of the resultant can be found either by accurate scale drawing or by calculation. Although it is usually more convenient to use a vector triangle for this purpose, the parallelogram form of the vector diagram has the advantage that it shows the two forces and their resultant acting at a single point.

Example 1 Find the resultant of two forces of magnitudes 5 N and 4 N acting at right angles.

Let R be the magnitude of the resultant and θ the angle it makes with the 5 N force.



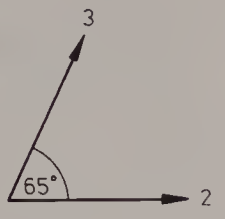


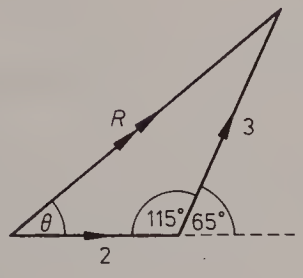
Using Pythagoras' theorem in the vector triangle,

$$R^2 = 5^2 + 4^2 = 25 + 16 = 41 \qquad \therefore \quad R \approx 6{\cdot}4$$
$$\tan\theta = 4/5 = 0{\cdot}8 \qquad \therefore \quad \theta \approx 38{\cdot}7°$$

The resultant is a force of magnitude 6·4 N at an angle 38·7° to the 5 N force.

Example 2 Two forces which act on a particle have magnitudes 2 N and 3 N. If the angle between their directions is 65°, find the resultant force on the particle.





Vector triangle

By the cosine rule, $R^2 = 2^2 + 3^2 - 2.2.3\cos 115°$

$= 13 - 12\cos 115°$

$\therefore \quad R \approx 4{\cdot}25$

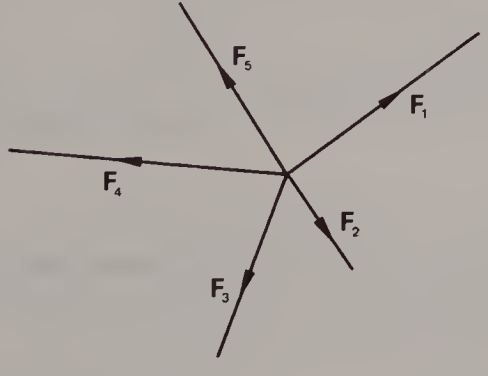
By the sine rule, $\dfrac{\sin\theta}{3} = \dfrac{\sin 115°}{R}$

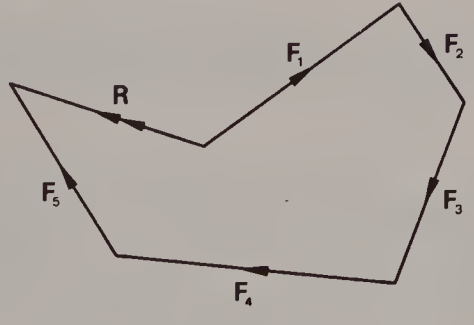
$\therefore \quad \sin\theta = \dfrac{3\sin 115°}{R}$ and $\theta \approx 39{\cdot}8°$

Hence the resultant is a force of magnitude 4·25 N at an angle 39·8° to the 2 N force.

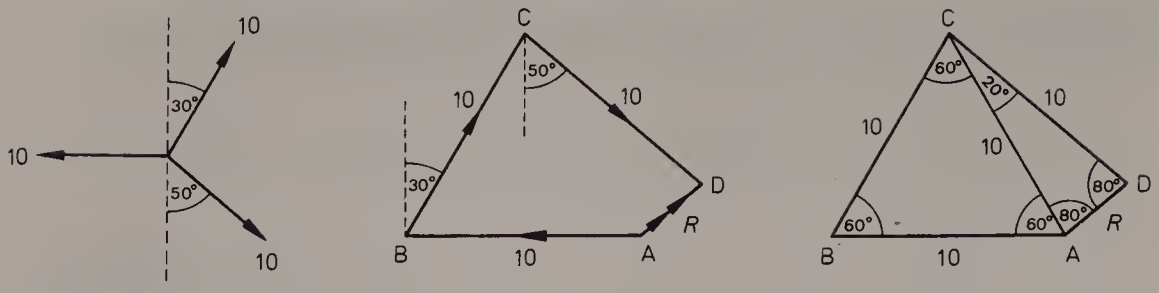
[Note that to achieve maximum accuracy the unrounded value of R should be used to find θ. The given results were obtained using a calculator, the value $R = 4{\cdot}251048811$ being stored in the memory.]

When the resultant of more than two forces is required a vector polygon can be constructed.





Example 3 Three forces, each of magnitude 10 N, act horizontally in the directions due west, N 30° E and S 50° E. Find the magnitude and direction of their resultant.



In the vector diagram, $\angle ABC = 60°$ and $AB = BC = 10$

$\therefore$ $\triangle ABC$ is an equilateral triangle.

Hence in $\triangle ACD$, $\angle ACD = 20°$ and $AC = CD = 10$

$\therefore$ $\triangle ACD$ is an isosceles triangle in which $\angle A = \angle D = 80°$ and $R = 2 \cdot 10 \cos 80° \approx 3{\cdot}47$.

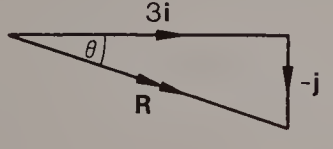
Thus the resultant force has magnitude 3·47 newtons and direction N 50° E.

The resultant of a set of forces can be found more quickly if the forces are expressed in terms of their Cartesian components. (See §1.2.)

Example 4 Find the magnitude and direction of the resultant of the forces represented by $2\mathbf{i} - 5\mathbf{j}$, $3\mathbf{j}$ and $\mathbf{i} + \mathbf{j}$, where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the directions due east and due north respectively.

Let $\mathbf{R}$ be the resultant of the given forces and let θ be the angle between $\mathbf{R}$ and $\mathbf{i}$.

$$\mathbf{R} = (2\mathbf{i} - 5\mathbf{j}) + 3\mathbf{j} + (\mathbf{i} + \mathbf{j}) = (2+1)\mathbf{i} + (-5+3+1)\mathbf{j}$$
$$= 3\mathbf{i} - \mathbf{j}$$



$$R^2 = 3^2 + (-1)^2 = 10$$
$$\therefore \quad R = \sqrt{10} \approx 3{\cdot}16$$
$$\tan\theta = \tfrac{1}{3} \qquad \therefore \quad \theta \approx 18{\cdot}4°$$

Hence the resultant is represented by $3\mathbf{i} - \mathbf{j}$. Its magnitude is 3·16 and its direction S 71·6° E.

Example 5 If $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors parallel to the x-, y- and z-axes, find the magnitude of the resultant of the forces represented by $3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, $\mathbf{i} - 3\mathbf{k}$ and $2\mathbf{i} + 5\mathbf{j} + \mathbf{k}$.

The resultant is given by

$(3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}) + (\mathbf{i} - 3\mathbf{k}) + (2\mathbf{i} + 5\mathbf{j} + \mathbf{k})$, i.e. $6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$

$\therefore$ the magnitude of the resultant is

$$|6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}| = \sqrt{(6^2 + 3^2 + 2^2)} = \sqrt{49} = 7.$$

Example 6 Two forces **P** and **Q** act in the directions of the vectors $4\mathbf{i}+3\mathbf{j}$ and $\mathbf{i}-2\mathbf{j}$ respectively and the magnitude of **P** is 25 N. Given that the magnitude of the resultant of **P** and **Q** is also 25 N, find the magnitude of **Q**.

$|4\mathbf{i}+3\mathbf{j}| = \sqrt{(4^2+3^2)} = \sqrt{25} = 5$

$\therefore$ the force **P** can be represented in magnitude and direction by the vector $5(4\mathbf{i}+3\mathbf{j})$, i.e. $20\mathbf{i}+15\mathbf{j}$.

Since **Q** can be represented by a vector of the form $\lambda(\mathbf{i}-2\mathbf{j})$, we may write

$\mathbf{P}+\mathbf{Q} = (20\mathbf{i}+15\mathbf{j})+\lambda(\mathbf{i}-2\mathbf{j}) = (20+\lambda)\mathbf{i}+(15-2\lambda)\mathbf{j}$

$\therefore$ the magnitude of the resultant of **P** and **Q**

$= \sqrt{\{(20+\lambda)^2+(15-2\lambda)^2\}} = 25$

$\therefore \quad 400+40\lambda+\lambda^2+225-60\lambda+4\lambda^2 = 625$

$5\lambda^2-20\lambda = 0$

$\therefore$ assuming λ is non-zero, $\lambda = 4$

Hence $\mathbf{Q} = 4(\mathbf{i}-2\mathbf{j})$ and $|\mathbf{Q}| = 4\sqrt{\{1^2+(-2)^2\}} = 4\sqrt{5}$.

Thus the magnitude of the force **Q** is $4\sqrt{5}$ N.

Exercise 2.3

1. The resultant of two forces **P** and **Q** is of magnitude R newtons and acts at an angle θ to the direction of **P**. Find R and θ, given that the magnitudes of **P** and **Q** and the angle between their directions are respectively

(a) 12 N, 5 N; 90°, (b) 4 N, 7 N; 90°,
(c) 8 N, 11 N; 120°, (d) 6 N, 10 N; 52°,
(e) 14 N, 9 N; 108°, (f) 2 N, 7 N; 23°.

2. Find the angle between the lines of action of two forces of magnitudes 7 N and 11 N, given that their resultant is of magnitude 8 N.

3. The resultant of two forces **F** and **G** is a force of 20 N acting at an angle of 45° to the force **G**. Given that the magnitude of **G** is 12 N, find the magnitude and direction of **F**.

4. Two forces of magnitudes 7 N and 9 N act at an angle θ where $\sin\theta = 0{\cdot}4$. Find the magnitudes of the two possible resultants.

5. Find the resultant of three horizontal forces whose magnitudes and directions are 16 N in the direction N 70° W, 8 N in the direction N 20° E and 11 N in the direction S 70° E.

6. Four forces, each of magnitude 12 N, act horizontally in the directions N 30° E, S 60° E, S 15° E and S 75° W. Find the magnitude and direction of their resultant.

7. Horizontal forces of magnitudes 12 N, 9 N and 6 N act on a particle in the directions due north, due east and S 60° E respectively. Find the magnitude and direction of their resultant.

8. Find the magnitude and direction of the resultant of the forces represented by each of the following sets of vectors, where **i** and **j** are unit vectors in the directions due east and due north respectively.

(a) $2\mathbf{i}-\mathbf{j}$, $3\mathbf{i}$, $-\mathbf{i}+4\mathbf{j}$,
(b) $5\mathbf{i}-2\mathbf{j}$, $2\mathbf{i}+3\mathbf{j}$, $-\mathbf{j}$,
(c) $7\mathbf{i}+4\mathbf{j}$, $2\mathbf{i}-5\mathbf{j}$, $-3\mathbf{i}-2\mathbf{j}$,
(d) $\mathbf{i}+\mathbf{j}$, $-3\mathbf{i}-2\mathbf{j}$, $-3\mathbf{i}-4\mathbf{j}$.

9. If **i**, **j** and **k** are unit vectors parallel to the x-, y- and z-axes, find the magnitude of the resultant of the forces represented by

(a) $2\mathbf{i}-3\mathbf{j}$, $5\mathbf{i}+\mathbf{j}-3\mathbf{k}$ and $-3\mathbf{i}+4\mathbf{j}-\mathbf{k}$,
(b) $\mathbf{i}+6\mathbf{j}-3\mathbf{k}$, $8\mathbf{i}-\mathbf{j}-5\mathbf{k}$ and $3\mathbf{i}+3\mathbf{j}-\mathbf{k}$,
(c) $2\mathbf{j}-5\mathbf{k}$, $5\mathbf{i}-\mathbf{k}$ and $2\mathbf{i}+3\mathbf{j}+\mathbf{k}$.

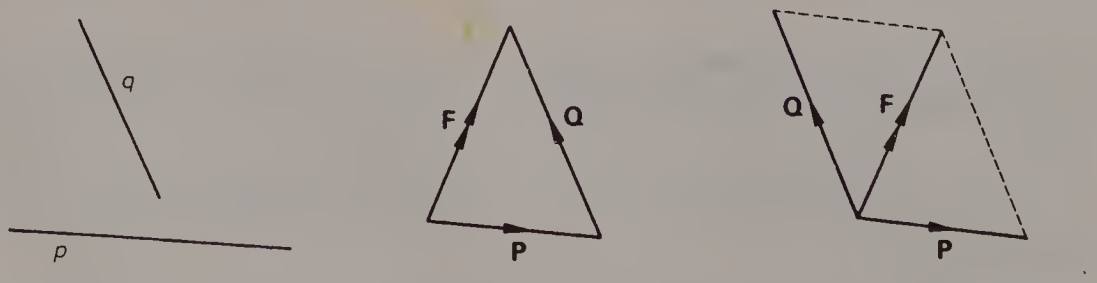
10. Find the magnitude of the resultant of two forces of magnitudes 15 N and $4\sqrt{2}$ N acting in the directions of the vectors $3\mathbf{i}-4\mathbf{j}$ and $\mathbf{i}+\mathbf{j}$ respectively.

11. Two forces, each of magnitude $5P$ newtons, act in the directions of the vectors $7\mathbf{i}+\mathbf{j}$ and $\mathbf{i}-\mathbf{j}$. Show that their resultant acts in the direction of the vector $3\mathbf{i}-\mathbf{j}$. Given that the magnitude of the resultant is 20 newtons, find the value of P in surd form.

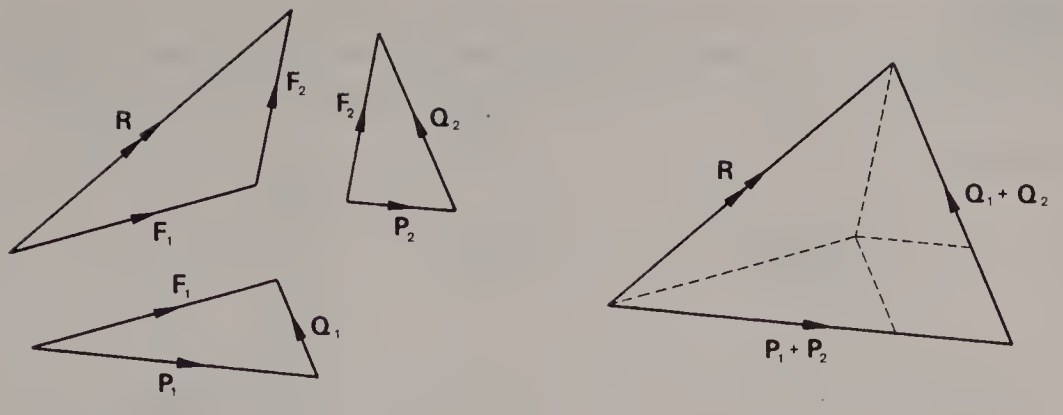
12. Two forces **F** and **G** act parallel to the vectors $3\mathbf{i}+\mathbf{j}$ and $\mathbf{i}+2\mathbf{j}$ respectively. If the resultant is a force of 10 newtons in the direction of the vector $3\mathbf{i}-4\mathbf{j}$, find the magnitudes of **F** and **G** in surd form.

2.4 Resolution into components

If p and q are two non-parallel lines in a plane, then any force **F** acting in that plane can be resolved into component forces **P** and **Q** in directions parallel to p and q by constructing a vector triangle or parallelogram as shown below.



Consider now two forces and their resolution into components parallel to the lines p and q. Let $\mathbf{F}_1 = \mathbf{P}_1 + \mathbf{Q}_1$, $\mathbf{F}_2 = \mathbf{P}_2 + \mathbf{Q}_2$ and let the resultant of $\mathbf{F}_1$ and $\mathbf{F}_2$ be **R**.



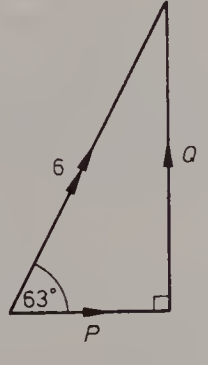
As shown in the diagrams,

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 = (\mathbf{P}_1 + \mathbf{Q}_1) + (\mathbf{P}_2 + \mathbf{Q}_2) = (\mathbf{P}_1 + \mathbf{P}_2) + (\mathbf{Q}_1 + \mathbf{Q}_2).$$

Thus the components of a resultant force are the sums of the corresponding components of the original forces. This property of forces and their components is often used to find resultant forces.

For most practical purposes forces are resolved into two components at right angles.

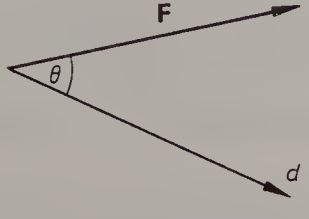
Example 1 Find the magnitudes of the horizontal and vertical components of a force of 6 N acting at an angle of 63° to the horizontal.



Let P and Q newtons be the magnitudes of the required components,

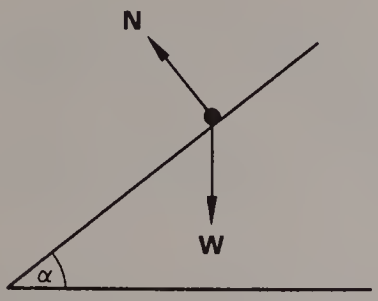
then $P = 6\cos 63° \approx 2{\cdot}72$
and $Q = 6\sin 63° \approx 5{\cdot}35$

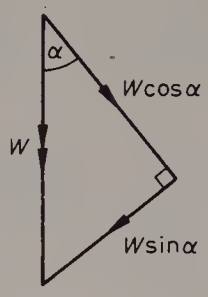
$\therefore$ the magnitudes of the horizontal and vertical components are 2·72 N and 5·35 N respectively.



If a force **F** acts at an angle θ to a given direction d, then the quantity $F\cos\theta$ is called the *resolved part* of **F** in the direction d, or simply the component of **F** in the direction d. If θ is acute, $F\cos\theta$ is positive, but if θ is obtuse, $F\cos\theta$ is negative. Thus the sign of the resolved part indicates whether the direction of the component force is the same as d or opposite to d.

Example 2 A particle is sliding down a smooth plane inclined at an angle α to the horizontal. Find the components of the weight **W** and the normal reaction **N** down the plane and in the direction of an upward perpendicular to the plane.



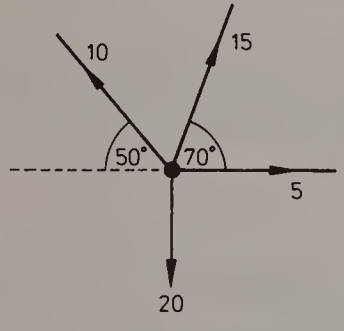


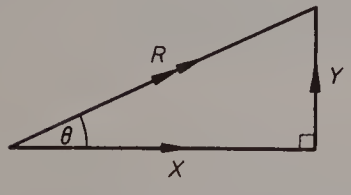
The components of **W** are $W\sin\alpha$ down the plane and $-W\cos\alpha$ along an upward perpendicular to the plane.

Since **N** acts at right angles to the plane, its component down the plane is 0 and its component along an upward perpendicular is N.

The resultant of a set of forces can now be found by resolving in two perpendicular directions.

Example 3 A small package of weight 20 N is attached to three light strings lying in the same vertical plane. If the tensions in the strings are as shown in the given diagram, find the resultant force on the package.





Let the resultant **R** have horizontal and vertical components X and Y. Resolving horizontally and vertically,

$\rightarrow X = 5 + 15\cos 70° - 10\cos 50°.$

$\uparrow Y = 15\sin 70° + 10\sin 50° - 20$

By calculator $X = 3{\cdot}70242$,

$Y = 1{\cdot}755835.$

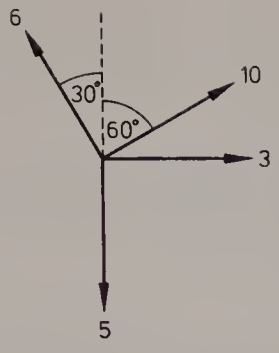
$\therefore \quad R = \sqrt{(X^2 + Y^2)} \approx 4{\cdot}10$

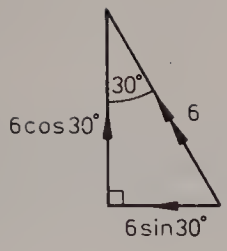
and $\tan\theta = Y/X, \quad \theta \approx 25{\cdot}4°.$

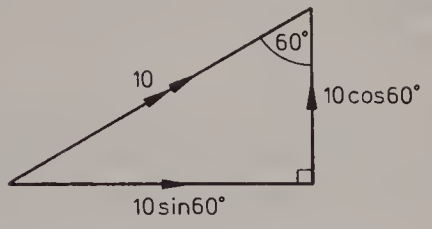
Hence the resultant is a force of magnitude 4·10 N at an angle 25·4° to the horizontal.

It is sometimes convenient to express a set of forces in terms of perpendicular unit vectors **i** and **j**.

Example 4 A set of horizontal forces of magnitudes 3 N, 5 N, 6 N and 10 N act on a particle in directions due east, due south, N 30° W and N 60° E respectively. Find the magnitude and direction of the resultant.





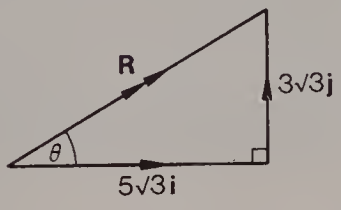


Let **i** and **j** be unit vectors in the directions due east and due north. The given set of forces can then be represented by

$$3\mathbf{i},\ -5\mathbf{j},\ -6\sin 30°\mathbf{i}+6\cos 30°\mathbf{j} \quad \text{and} \quad 10\sin 60°\mathbf{i}+10\cos 60°\mathbf{j}.$$

Hence the resultant **R** is given by

$$\mathbf{R} = 3\mathbf{i}+(-5\mathbf{j})+(-3\mathbf{i}+3\sqrt{3}\mathbf{j})+(5\sqrt{3}\mathbf{i}+5\mathbf{j}) = 5\sqrt{3}\mathbf{i}+3\sqrt{3}\mathbf{j}$$



$$R^2 = (5\sqrt{3})^2+(3\sqrt{3})^2 = 75+27 = 102$$

$$\therefore \quad R \approx 10{\cdot}1$$

$$\tan\theta = \frac{3\sqrt{3}}{5\sqrt{3}} = 0{\cdot}6 \qquad \therefore \quad \theta \approx 31°.$$

The resultant is a force of magnitude 10·1 N and direction N 59° E.

Exercise 2.4

1. Find the magnitudes of the horizontal and vertical components of
(a) a force of 20 N acting at 40° to the horizontal,
(b) a force of 7 N acting at 78° to the horizontal,
(c) a force of 12 N acting at 36° to the vertical.

2. A particle is being pulled up a smooth plane, inclined at an angle α to the horizontal, by a string which makes an acute angle β with the plane. Find the resolved parts of the weight of the particle **W**, the normal reaction **N** and the tension in the string **T** (a) vertically upwards, (b) up the plane, (c) in the direction of a downward perpendicular to the plane.

3. A set of coplanar forces of magnitudes 12, 9, 15 and 16 newtons act on a particle in directions south-west, north, north-west and east respectively. Find the magnitude and direction of their resultant.

4. A particle is being pulled across a smooth horizontal plane by three strings. The directions of the strings are N 60° W, due north and N 30° E. If the tensions in the strings are of magnitude 15 N, 12 N and 10 N respectively, find the resultant force on the particle.

5. Four horizontal forces of equal magnitude act on a particle in the directions whose bearings are 050°, 100°, 250° and 290°. Calculate the bearing of the resultant.

6. *ABCDEF* is a regular hexagon. Forces of magnitude 2, 3, 4, 5 and 6 newtons act in the directions of **AB**, **AC**, **AD**, **AE** and **AF** respectively. If their resultant is a force of R newtons acting at an angle θ to **AB**, find R and θ.

7. *ABCD* is a rectangle in which *AB* is 3 m, *BC* is 4 m. Forces 3 N, 2 N, 10 N, 5 N act in the directions of **AB**, **AD**, **AC**, **BD** respectively. Find their resultant, giving the angle it makes with **AB**.

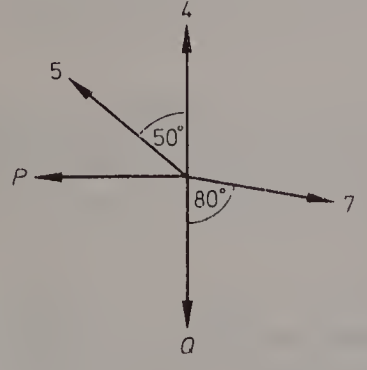
8. *ABCDE* is a regular pentagon. Find the magnitude of the resultant of forces 2 N, 3 N, 1 N, 4 N, 5 N in the directions of **AB**, **BC**, **CD**, **DE**, **EC** respectively.

9. A vertical flagpole of height 24 m is held in position by three straight wires fixed to its top and to points on the horizontal ground. The wires are of lengths 25 m, 30 m and 26 m and their fixing points are at bearings 060°, 150° and 270° respectively from the base of the pole. If the tensions in the wires are each 50 N, find (a) the vertical components of the tensions, (b) the horizontal components of the tensions. Hence find (c) the resultant force downwards caused by the wires, (d) the resultant horizontal force on the flagpole.

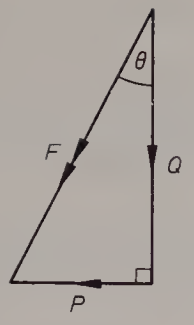
2.5 A particle in equilibrium

Forces on a particle are said to be in *equilibrium* if their resultant is zero. Thus when forces in equilibrium are resolved into components in two fixed directions, the sums of the components in each direction will be zero. This fact provides a basic approach to problem solving.

Example 1 A set of horizontal forces of magnitudes 5, 4 and 7 newtons act on a particle in the directions N 50° W, due north and S 80° E respectively. Find the magnitude and direction of a fourth force which holds the particle in equilibrium.



Let the components in the directions due west and due south of the fourth force F be P and Q respectively. Resolving in the directions west and south.



$\leftarrow P + 5\sin 50° - 7\sin 80° = 0$

$\downarrow Q + 7\cos 80° - 5\cos 50° - 4 = 0$

$\therefore \quad P = 7\sin 80° - 5\sin 50°$

$Q = 4 + 5\cos 50° - 7\cos 80°.$

By calculator $P = 3{\cdot}063436$,

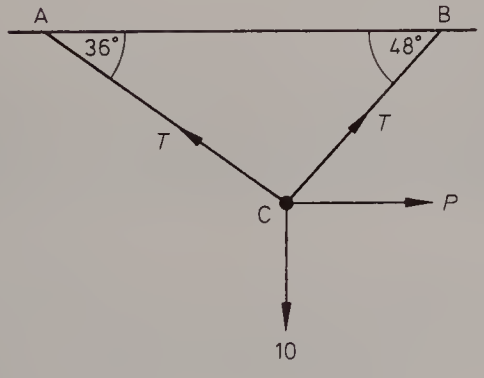
$Q = 5{\cdot}998404$

$\therefore \quad F = \sqrt{(P^2 + Q^2)} \approx 6{\cdot}74$ and $\tan\theta = P/Q, \quad \theta \approx 27{\cdot}1°.$

Hence the fourth force is of magnitude 6·74 newtons in the direction S 27·1° W.

Example 2 A light inextensible string is attached to two points A and B on a horizontal beam. A smooth ring C of weight 10 N, which can move freely on the string, is held in equilibrium by a horizontal force P acting in the vertical plane containing A, B and C. Given that $\angle BAC = 36°$ and $\angle ABC = 48°$, find the magnitude of P and the tension T in the string.

[Note that since the ring is smooth the tension is the same in both parts of the string.]



Resolving horizontally and vertically,

$\rightarrow P + T\cos 48° - T\cos 36° = 0$

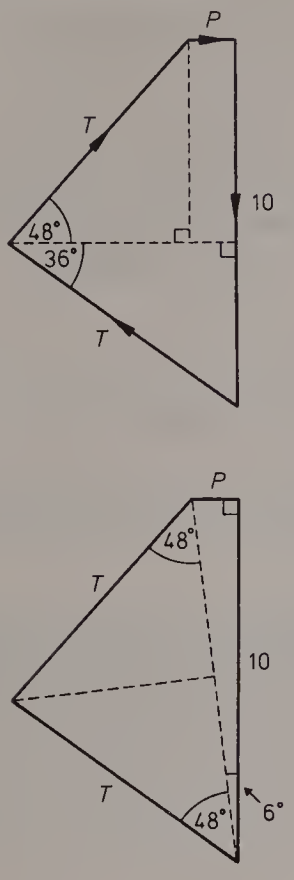
$\uparrow T\sin 48° + T\sin 36° - 10 = 0$

$$\therefore \quad T = \frac{10}{\sin 48° + \sin 36°} \approx 7{\cdot}51$$

and $P = T(\cos 36° - \cos 48°) \approx 1{\cdot}05.$

Hence the magnitude of the horizontal force P is 1·05 N and the tension in the string is 7·51 N.

In §2.3 we saw that forces can be added by constructing a vector polygon. If a set of forces is in equilibrium the length of the side representing their resultant must be zero. Hence a set of forces in equilibrium can be represented by the sides of a closed polygon, often referred to as the *polygon of forces.*



Example 2 could have been solved by drawing a polygon of forces. Using the right-angled triangles shown in the diagram,

$$T\sin 48° + T\sin 36° = 10$$

and $T\cos 48° + P = T\cos 36°$

$\therefore$ as before we find that

$$T = \frac{10}{\sin 48° + \sin 36°},$$

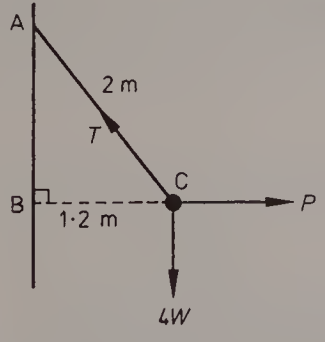
$P = T(\cos 36° - \cos 48°)$.

However, considering instead the angle properties of the figure, we could obtain the equivalent results,

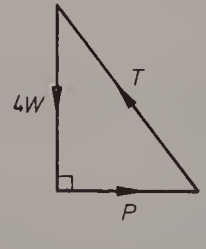
$P = 10\tan 6°$ and
$2T\cos 48°\cos 6° = 10$.

Of special interest are problems involving three forces in equilibrium, because in this case the vector diagram is simply a *triangle of forces*.

Example 3 A particle of weight $4W$ is attached to the end of a light inextensible string of length 2 m. The other end of the string is fixed to a point on a vertical wall. The particle is being held 1·2 m away from the wall by a horizontal force. Express the magnitudes of the horizontal force and the tension in the string in terms of W.



Space diagram



Triangle of forces

Let T be the tension in the string and P the horizontal force.

By Pythagoras' theorem in $\triangle ABC$,

$$AB^2 = AC^2 - BC^2 = 2^2 - (1{\cdot}2)^2 = 2{\cdot}56 \qquad \therefore \quad AB = 1{\cdot}6$$

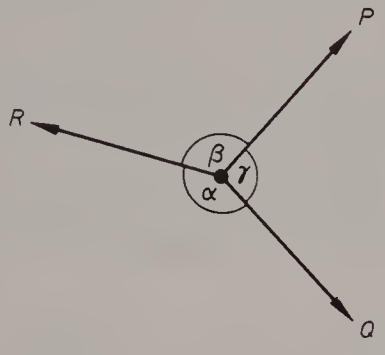
As shown in the diagrams, $\triangle ABC$ is similar to the triangle of forces

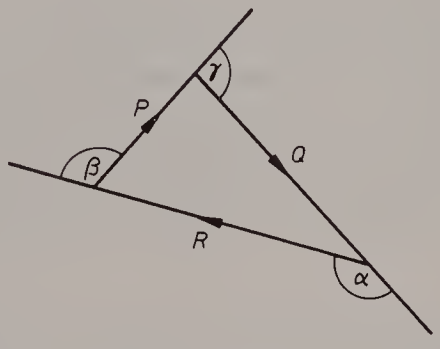
$$\therefore \quad \frac{T}{2} = \frac{P}{1{\cdot}2} = \frac{4W}{1{\cdot}6}.$$

Hence $T = 5W$ and $P = 3W$.

The magnitudes of the horizontal force and the tension in the string are $3W$ and $5W$ respectively.

Consider now three forces of magnitudes P, Q and R acting on a particle. If the angles between the forces are α, β and γ, then the exterior angles of the triangle of forces are also α, β and γ.



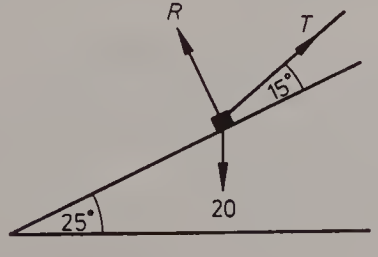


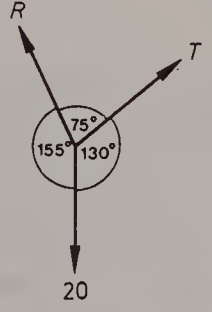
By the sine rule, $\dfrac{P}{\sin(180°-\alpha)} = \dfrac{Q}{\sin(180°-\beta)} = \dfrac{R}{\sin(180°-\gamma)}$

$$\therefore \quad \boxed{\frac{P}{\sin\alpha} = \frac{Q}{\sin\beta} = \frac{R}{\sin\gamma}}$$

This result is called *Lami's*† *Theorem.* It is often useful in solving three force problems in which angles rather than lengths are known.

Example 4 A small block of weight 20 N is at rest on a smooth plane inclined at an angle of 25° to the horizontal. The block is being held in position by a light rope. Given that the angle between the rope and the plane is 15°, find the magnitudes of the tension T in the rope and of the reaction R with the plane.





† *Lami, Bernard* (1640–1715) French scholar. He made a substantial contribution to the philosophical and mathematical theories of his time. His work on equilibrium appears in the *Traité de Mécanique* (1679).

By Lami's theorem, $\dfrac{T}{\sin 155°} = \dfrac{R}{\sin 130°} = \dfrac{20}{\sin 75°}$

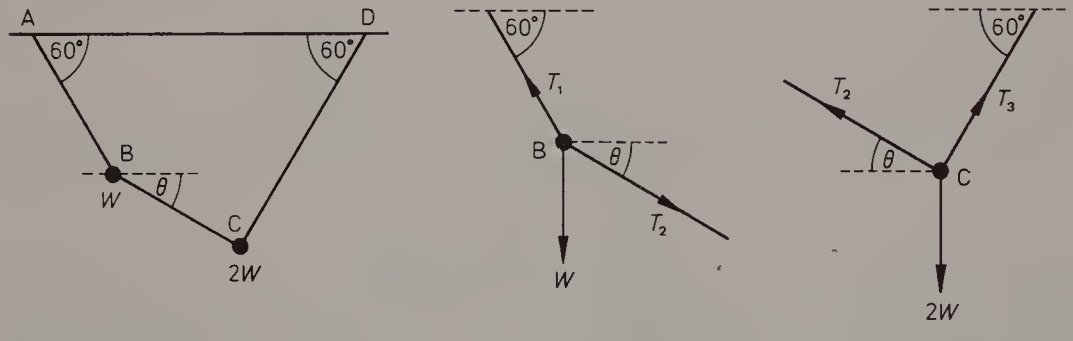
$\therefore \quad T = \dfrac{20 \sin 155°}{\sin 75°} \approx 8{\cdot}75 \quad$ and $\quad R = 20 \sin 130°/\sin 75° \approx 15{\cdot}9$

The tension in the rope is 8·75 N and the normal reaction with the plane is 15·9 N.

[This problem could, of course, also be solved by resolving the forces parallel and perpendicular to the plane or by drawing a triangle of forces.]

Problems involving more than one particle are often solved by considering the forces on each particle separately.

Example 5 A light inextensible string is suspended from two points A and D in the same horizontal plane. Particles of weight W and $2W$ are attached to the string at points B and C respectively. If AB and CD are inclined at 60° to the horizontal, find in terms of W the tensions T_1, T_2, T_3 in the strings AB, BC and CD respectively. Find also the angle θ, which BC makes with the horizontal.



Resolving the forces on the particle at B

horizontally $\quad \rightarrow T_2 \cos\theta - T_1 \cos 60° = 0 \quad (1)$

vertically $\quad \uparrow T_1 \sin 60° - T_2 \sin\theta - W = 0 \quad (2)$

Resolving the forces on the particle at C

horizontally $\quad \rightarrow T_3 \cos 60° - T_2 \cos\theta = 0 \quad (3)$

vertically $\quad \uparrow T_3 \sin 60° + T_2 \sin\theta - 2W = 0 \quad (4)$

Adding (1) to (3): $\quad T_3 \cos 60° - T_1 \cos 60° = 0 \quad \therefore \quad T_1 = T_3$

Adding (2) to (4): $\quad T_1 \sin 60° + T_3 \sin 60° - 3W = 0$

$\therefore \quad T_1 + T_3 = 3W/\sin 60° = 2\sqrt{3}W \quad$ and $\quad T_1 = T_3 = \sqrt{3}W$

Using (1) $\quad T_2 \cos\theta = \sqrt{3}W \,.\cos 60° = \dfrac{\sqrt{3}}{2} W$

Using (2) $\quad T_2 \sin\theta = \sqrt{3}W \,.\sin 60° - W = \dfrac{1}{2} W$

By squaring and adding these results:

$$T_2^2 = T_2^2(\cos^2\theta + \sin^2\theta) = \tfrac{3}{4}W^2 + \tfrac{1}{4}W^2 = W^2$$

Dividing: $\tan\theta = T_2\sin\theta / T_2\cos\theta = \frac{1}{2}W \Big/ \frac{\sqrt{3}}{2}W = 1/\sqrt{3}$

$$\therefore \quad T_2 = W \quad \text{and} \quad \theta = 30°.$$

Hence the tensions in the strings AB, BC and CD are $\sqrt{3}W$, W and $\sqrt{3}W$ respectively and BC is inclined at 30° to the horizontal.

Exercise 2.5

1. A set of horizontal forces of magnitude 20 N, 12 N and 30 N act on a particle in the directions due south, due east and N 40° E respectively. Find the magnitude and direction of a fourth force which holds the particle in equilibrium.

2. A set of four horizontal forces in equilibrium act in directions whose bearings are 020°, 110°, 230° and 320°. If the magnitudes of the forces are 8, 12, P and Q newtons respectively, find the values of P and Q.

3. A smooth ring C of weight 15 N can move freely on a light inextensible string attached to two fixed points A and B in the same horizontal line. A horizontal force of magnitude P holds the ring in equilibrium, such that AC is vertical and $\angle ABC = 30°$. Find the value of P using a polygon of forces.

4. A set of six forces in equilibrium act in the directions of the sides $\vec{AB}$, $\vec{BC}$, $\vec{CD}$, $\vec{DE}$, $\vec{EF}$ and $\vec{FA}$ of a regular hexagon. If the magnitudes of the forces are 7, 4, 10, 5, P and Q newtons respectively, use a polygon of forces to find the values of P and Q.

5. A body of weight 20 newtons is held at rest on a smooth plane inclined at 60° to the horizontal by a force of magnitude P newtons. Find P given that the force is (a) horizontal, (b), vertical, (c) parallel to the plane.

6. One end of a light inextensible string of length 75 cm is fixed to a point on a vertical pole. A particle of weight 12 N is attached to the other end of the string. The particle is held 21 cm away from the pole by a horizontal force. Find the magnitude of this force and the tension in the string.

7. A small object of weight 4 N is suspended from a fixed point by a string. The object is held in equilibrium, with the string at an angle of 25° to the vertical, by a horizontal force. Find the magnitude of this force and the tension in the string.

8. A particle P of weight 50 N is hanging in equilibrium supported by two strings inclined at 20° and 40° to the vertical. Find the tensions in the strings.

9. A particle of weight $10W$ is held at rest on a smooth plane inclined at 30° to the horizontal by a light string. Find the tension in the string and the reaction

between the particle and the plane if (a) the string is parallel to the plane, (b) the string is at an angle of 30° to the plane.

10. A particle of weight 2 N hangs on a string. The particle is pulled to one side by a force of magnitude 3 N acting at an angle of 20° below the horizontal. Find the tension in the string and the inclination of the string to the vertical.

11. A particle P of weight $10W$ is supported by two light inextensible strings attached to fixed points X and Y which lie in the same horizontal plane. Given that $PX = 36$ cm, $PY = 48$ cm and $XY = 60$ cm, find the tension in each string.

12. A light inextensible string passes through a smooth ring fixed at a point P. Particles A and B of weights 10 N and 20 N respectively are attached to the ends of the string. The system is held in equilibrium by a horizontal force acting on the particle A. Find the magnitude of this force. Find also the magnitude and direction of the resultant force on the ring due to the tension in the string.

13. A light inextensible string ABC is fixed at the point A. Two particles each of weight W are attached to the string at B and C. The system is held in equilibrium by a horizontal force of magnitude $2W$ acting on the particle at C. Find (a) the tensions in BC and AB, (b) the inclinations of BC and AB to the vertical.

14. Two smooth straight rods XY and XZ are fixed at 45° to the vertical and at right angles to each other. Two small beads A and B, each of weight W, are free to slide on the rods XY and XZ respectively. The beads are connected by a light inextensible string, to the mid-point of which is attached a particle C of weight $2W$. Find the reaction between the bead A and the rod XY, the tension in the string AC and its inclination to the horizontal when the system is in equilibrium.

15. $ABCD$ is a light inextensible string. The ends A and D are fixed so that the line AD is horizontal. Particles of weight $3W$ and $4W$ are attached to the string at the points B and C respectively. Find the tensions in the strings AB, BC and CD given that (a) $\angle DAB = 45°$ and $\angle ABC = 150°$, (b) $\angle DAB = 45°$ and the strings BC and DC are equally inclined to the horizontal.

Exercise 2.6 (miscellaneous)

1. Find the angle between the lines of action of two forces of magnitudes 15 N and 9 N, given that their resultant is a force of magnitude 20 N.

2. Two forces of magnitude $3P$ and $4P$ newtons act on a particle. The magnitude of their resultant is 20 newtons. When one of the forces is reversed the magnitude of the resultant is halved. Calculate the value of P and the original angle between the forces.

3. Two forces of magnitude 5 N and 3 N act on a particle. Their resultant acts at an angle θ to the direction of the 5 N force. Find the range of possible values of θ.

4. Given that **i** and **j** are unit vectors in the directions due east and due north respectively, find the magnitude and direction of the resultant of the forces represented by the vectors $3\mathbf{i}-\mathbf{j}$, $-2\mathbf{i}$, $5\mathbf{i}+7\mathbf{j}$ and $\mathbf{i}-2\mathbf{j}$.

5. Two concurrent forces **X** and **Y** have resultant **R**. If $X = R$ and the angle between **R** and **Y** is α, prove (a) $Y = 2X\cos\alpha$, (b) the angle between **X** and **Y** is $180° - \alpha$.

6. A force of magnitude 10 N parallel to the vector $4\mathbf{i}+3\mathbf{j}$ is the resultant of two forces parallel respectively to the vectors $2\mathbf{i}+\mathbf{j}$, $\mathbf{i}+\mathbf{j}$. Find the magnitudes of these two forces. (L)

7. The point O is the centre of a regular hexagon $ABCDEF$. Forces of magnitude 7, 3, 10, 9, 6 and 2 newtons act on a particle at O in the directions of **OA**, **OB**, **OC**, **OD**, **OE** and **OF** respectively. Find the magnitude of the resultant and show that it acts in the direction of **AC**.

8. Find the bearing of the resultant of five horizontal forces of equal magnitude which act in the directions whose bearings are 070°, 130°, 160°, 220° and 310°.

9. A set of four forces on a particle are in equilibrium. If **i** and **j** are unit vectors in the directions due east and due north respectively, three of the forces are represented by the vectors $\mathbf{i}+\mathbf{j}$, $3\mathbf{i}-2\mathbf{j}$ and $2\mathbf{i}+3\mathbf{j}$. Find the magnitude and direction of the fourth force.

10. A particle is in equilibrium under the action of a set of five horizontal forces. If four of the forces are of magnitude 6 N, 11 N, 8 N and 15 N acting in the directions N 72° W, N 15° W, N 35° E and S 63° E respectively, find the magnitude and direction of the fifth force.

11. A particle P of weight $10W$ is hanging in equilibrium from two strings PA and PB. If $\angle APB = 90°$ and the tensions in the strings are T and $2T$, find T in terms of W.

12. A particle P of weight 50 N is supported by two light inextensible strings attached to fixed points A and B. Given that A and B lie 2 m apart in the same horizontal plane and that $AP = 2$ m, $PB = 1$ m, find the tension in each string.

13. A small block of weight 20 N is held at rest on a smooth plane inclined at 20° to the horizontal by a string which makes an angle θ with the plane. If the tension in the string is 10 N, find the value of θ and the magnitude of the reaction between the block and the plane.

14. Show that the least force required to keep a particle of weight W in equilibrium on a smooth plane inclined at an angle α to the horizontal is of magnitude $W\sin\alpha$ and parallel to the plane.

15. The ends of a light inextensible string of length 80 cm are fixed to points A and B, which are 48 cm apart. A smooth ring C, of weight 5 N, which can move freely along the string, hangs in equilibrium. Find the tension in the string if (a) AB is horizontal, (b) AB is inclined at 60° to the horizontal.

16. One end of a light inextensible string is attached to a fixed point on a smooth straight wire inclined at 30° to the horizontal. The other end of the string is attached to a light ring which slides freely on the wire. A smooth bead of weight W, which is threaded on the string, is also free to slide. Find, in terms of W, the tension in the string when the system is in equilibrium.

17. A particle of weight 2 N is attached to a light string ABC at the point B. The ends A and C are threaded through two fixed smooth rings at points X and Y respectively, where the line XY makes an angle θ with the horizontal. A particle of weight 3 N is then attached to the string at A and a particle of weight 4 N at C. Show that equilibrium is possible only if Y lies above X and $1/4 < \sin\theta < 11/16$.

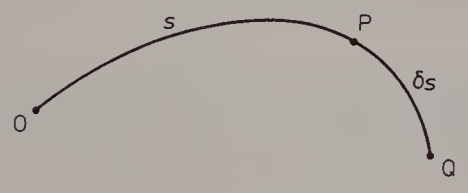
18. Two rods OA, OB are fixed in a vertical plane with O uppermost, each rod making an acute angle α with the vertical. Two smooth rings of equal weight W, which can slide one on each rod, are connected by a light inextensible string, upon which slides a third smooth ring of weight $2W$. Show that, in the symmetrical position of equilibrium, the angle between the two straight pieces of the string is 2β, where $\tan\beta = 2\cot\alpha$. Show that the tension in the string is $W\sqrt{(1+4\cot^2\alpha)}$ and that the reaction between the rings and the rods is $2W\operatorname{cosec}\alpha$. (W)

19. Four packages, each of weight 20 N, are fastened to a light rope at points B, C, D and E. The rope is attached to two fixed points A and F, which lie in the same horizontal plane. The points B and C are 1·2 m and 2·2 m respectively below the line AF and $AB = BC = CD = DE = EF$. Find the tension in the rope CD and the length of the rope.

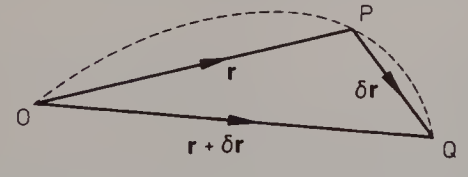
20. Construct a flow chart for a procedure to find the resultant of a set of n horizontal forces, printing out its magnitude and bearing. Assume that the data is provided in the form of a list giving the magnitude x newtons and the bearing $y°$ of each force, where $x > 0$ and $0 \leqslant y < 360$.

3 Relative motion

3.1 Velocity and acceleration as vectors

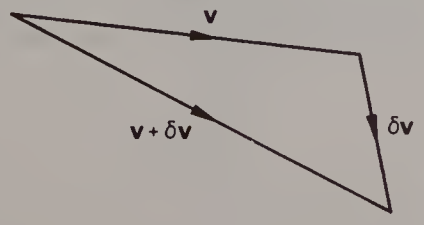


The diagram shows the path of a moving particle. The motion begins at the point O and after time t the particle reaches the point P. The *distance* travelled is the length s of the path OP. The *speed* of the particle is the rate at which it moves along its path. If the particle moves a further distance δs to a point Q in time δt, then its average speed in the motion from P to Q is $\delta s/\delta t$. Hence the speed of the particle as it passes through $P = \lim_{\delta t \to 0} \frac{\delta s}{\delta t} = \frac{ds}{dt}$.



The *displacement* of the particle from O at time t is the vector $\mathbf{r}$ represented by the line segment $\overrightarrow{OP}$. The *velocity* of the particle is the rate of change of its displacement. Thus the average velocity of the particle in its motion from P to Q can be expressed as $\delta\mathbf{r}/\delta t$. Hence the velocity of the particle as it passes through $P = \lim_{\delta t \to 0} \frac{\delta \mathbf{r}}{\delta t} = \frac{d\mathbf{r}}{dt}$.

By considering the magnitude and direction of $\overrightarrow{PQ}$ as $\delta t \to 0$, we see that the magnitude of the velocity is the speed of the particle and the direction of the velocity is the direction of motion.



The *acceleration* of a particle is defined as the rate of change of its velocity. If the velocities of a particle at times t and $t+\delta t$ are $\mathbf{v}$ and $\mathbf{v}+\delta\mathbf{v}$ respectively, then the acceleration at time $t = \lim_{\delta t \to 0} \frac{\delta \mathbf{v}}{\delta t} = \frac{d\mathbf{v}}{dt}$.

[Note that unless a particle is moving in a straight line the magnitude of its acceleration does *not* correspond to the rate of change of its speed and the direction of the acceleration is *not* the same as the direction of motion.]

It is clear from these definitions that distance and speed are scalar quantities, whereas displacement, velocity and acceleration are vector quantities. We now give some examples illustrating the relationships between them. Unless otherwise stated we will assume that the magnitudes of vector quantities are measured in the appropriate SI units.

Example 1 A particle moving with uniform velocity $4\mathbf{i}+3\mathbf{j}$ passes through the point A whose position vector with respect to the origin O is $3\mathbf{i}-\mathbf{j}$. Find (a) the speed of the particle, (b) the distance of the particle from the origin 3 seconds after it passes through A.

(a) The speed of the particle
$= |4\mathbf{i}+3\mathbf{j}| = \sqrt{(4^2+3^2)} = 5.$
(b) After 3 seconds the displacement of the particle from A
$= 3(4\mathbf{i}+3\mathbf{j}) = 12\mathbf{i}+9\mathbf{j}$
$\therefore$ after 3 seconds its position vector
$= (12\mathbf{i}+9\mathbf{j})+(3\mathbf{i}-\mathbf{j}) = 15\mathbf{i}+8\mathbf{j}.$
Hence the distance of the particle from the origin
$= |15\mathbf{i}+8\mathbf{j}| = \sqrt{(15^2+8^2)} = \sqrt{289} = 17.$

Example 2 A particle moves for 2 seconds with velocity $6\mathbf{i}+3\mathbf{j}-2\mathbf{k}$, then for 3 seconds with velocity $\mathbf{i}-2\mathbf{j}-2\mathbf{k}$. Find (a) the average velocity and (b) the average speed of the particle during the motion.

(a) The total displacement of the particle
$= 2(6\mathbf{i}+3\mathbf{j}-2\mathbf{k})+3(\mathbf{i}-2\mathbf{j}-2\mathbf{k}) = 15\mathbf{i}-10\mathbf{k}$
$\therefore$ the average velocity during the 5 seconds of the motion is $3\mathbf{i}-2\mathbf{k}$.
(b) The distance moved in the first 2 seconds
$= 2|6\mathbf{i}+3\mathbf{j}-2\mathbf{k}| = 2\sqrt{(6^2+3^2+2^2)} = 14.$
The distance moved in the next 3 seconds
$= 3|\mathbf{i}-2\mathbf{j}-2\mathbf{k}| = 3\sqrt{(1^2+2^2+2^2)} = 9$
Hence the particle moves 23 m in 5 seconds
$\therefore$ the average speed during the motion is $4{\cdot}6\ \mathrm{m\,s^{-1}}$.

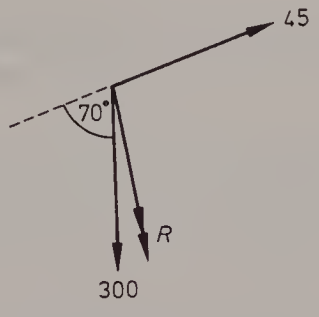
Example 3 A particle moves in a plane with constant acceleration given by $-2\mathbf{i}+\mathbf{j}$. If the initial velocity of the particle is $3\mathbf{i}$, find its speed after 1 second and after 3 seconds.

The velocity of the particle after 1 second
$= 3\mathbf{i}+(-2\mathbf{i}+\mathbf{j}) = \mathbf{i}+\mathbf{j}$
$\therefore$ the speed of the particle after 1 second
$= |\mathbf{i}+\mathbf{j}| = \sqrt{(1^2+1^2)} = \sqrt{2}.$

The velocity of the particle after 3 seconds
$= 3\mathbf{i}+3(-2\mathbf{i}+\mathbf{j}) = -3\mathbf{i}+3\mathbf{j}$
$\therefore$ the speed of the particle after 3 seconds
$= |-3\mathbf{i}+3\mathbf{j}| = 3|-\mathbf{i}+\mathbf{j}| = 3\sqrt{2}.$

Resultant velocities and accelerations can be found by the methods used to add forces in Chapter 2.

Example 4 An aircraft is heading due south with a velocity of 300 km/h. It is also being blown by a wind with velocity 45 km/h from the direction S 70°W. Find the velocity of the aircraft relative to the ground.



Vector diagram

Let R km/h be the resultant velocity and θ the angle it makes with due south.

By the cosine rule, $R^2 = 300^2+45^2-2.300.45\cos 70°$

$\therefore$ $R \approx 288$

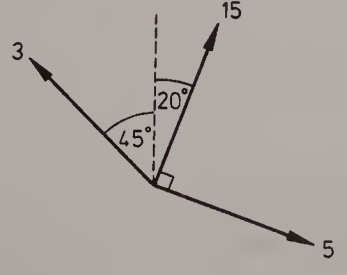
By the sine rule, $\dfrac{\sin\theta}{45} = \dfrac{\sin 70°}{R}$

$\therefore$ $\sin\theta = \dfrac{45\sin 70°}{R}$ and $\theta \approx 8{\cdot}5°$.

Hence the velocity of the aircraft relative to the ground is 288 km/h in the direction S 8·5° E.

Sums of three or more velocities can be found using a vector polygon. However, it is often more convenient to resolve the velocities in suitable directions and then add the components.

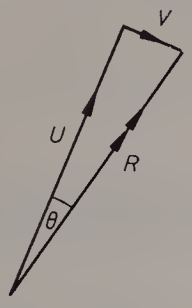
Example 5 A ship travelling through the water at 15 km/h is steering a course N 20°E. It is also being carried by a current with velocity 3 km/h to the north west. If a man is walking across the deck of the ship directly from port to starboard at 5 km/h, find his velocity relative to the shore.



Let U and V be the components of the man's velocity along the perpendicular to the course steered by the ship, then resolving in those directions,

$\uparrow$ $U = 15+3\cos 65°$

$\rightarrow$ $V = 5-3\sin 65°$



$R = \sqrt{(U^2 + V^2)} \approx 16{\cdot}4$
$\tan\theta = V/U \quad \therefore \quad \theta \approx 8°$
Hence the man's velocity relative to the shore is 16·4 km/h in the direction N 28° E.

Exercise 3.1

1. A particle starts from a point A with position vector $\mathbf{i}-4\mathbf{j}$ and moves with velocity $12\mathbf{i}+5\mathbf{j}$. Find the speed of the particle and its position vector after 2 seconds.

2. A particle moves from A to B in 5 seconds. If the position vectors of A and B are $7\mathbf{i}-12\mathbf{j}$ and $-3\mathbf{i}+8\mathbf{j}$ respectively, find the average velocity of the particle.

3. A particle starts from the origin and moves with velocity $6\mathbf{i}-3\mathbf{j}+6\mathbf{k}$ for 2 seconds, then with velocity $12\mathbf{i}+9\mathbf{j}$ for 1 second. Find the position vector of the particle. Find also the average velocity and the average speed of the particle during its motion.

4. A particle with position vector $40\mathbf{i}+10\mathbf{j}+20\mathbf{k}$ moves with constant speed $5\,\mathrm{m\,s^{-1}}$ in the direction of the vector $4\mathbf{i}+7\mathbf{j}+4\mathbf{k}$. Find its distance from the origin after 9 seconds.

5. A particle moves with constant acceleration given by the vector $\mathbf{i}+\mathbf{j}$. If the initial velocity of the particle is $2\mathbf{i}+5\mathbf{j}$, find its velocity after 3 seconds. Given that the speed of the particle after k seconds is $15\,\mathrm{m\,s^{-1}}$, find the value of k.

6. The initial velocity of a particle moving with constant acceleration is $2\mathbf{i}-4\mathbf{j}$. After 2 seconds the particle is moving in the opposite direction with double the speed. Find the acceleration of the particle.

7. If $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the directions due east and due north respectively, find the magnitude and direction of the resultant of the velocities represented by each of the following sets of vectors
(a) $5\mathbf{i}+3\mathbf{j}$, $4\mathbf{i}$, $3\mathbf{i}+2\mathbf{j}$,
(b) $\mathbf{i}-2\mathbf{j}$, $3\mathbf{i}-\mathbf{j}$, $2\mathbf{i}+5\mathbf{j}$,
(c) $7\mathbf{i}-5\mathbf{j}$, $-\mathbf{i}+2\mathbf{j}$, $2\mathbf{i}-\mathbf{j}$,
(d) $-2\mathbf{i}+\mathbf{j}$, $\mathbf{i}-6\mathbf{j}$, $-3\mathbf{i}-4\mathbf{j}$.

8. Find the magnitudes and directions of the resultants of the given velocities
(a) $7\,\mathrm{m\,s^{-1}}$ due north and $10\,\mathrm{m\,s^{-1}}$ N 75° E;
(b) 50 km/h N 40° E and 60 km/h S 25° E;
(c) $12\,\mathrm{m\,s^{-1}}$ N 20° E, $6\,\mathrm{m\,s^{-1}}$ due west and $5\,\mathrm{m\,s^{-1}}$ due north;
(d) 30 km/h S 45° W, 50 km/h S 30° E and 70 km/h due east.

9. A stretch of river has straight parallel banks 100 m apart. A boat is being rowed at $4\,\mathrm{m\,s^{-1}}$ in a direction perpendicular to the banks. If the river is flowing at $1{\cdot}2\,\mathrm{m\,s^{-1}}$, find (a) the magnitude of the resultant velocity of the boat, (b) the distance downstream that the boat is carried as it crosses from one bank to the other.

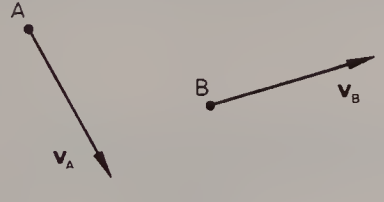
10. A ship travelling through the water at 12 km/h is steering a course S 75° E. It is also being carried by a current with velocity 2 km/h due E. If a passenger is walking across the deck of the ship directly from starboard to port at 4 km/h, find his velocity relative to the shore.

3.2 Relative velocity

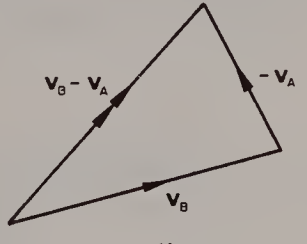
If A and B are two moving objects, then the apparent velocity of B when observed from A is called the velocity of B relative to A, i.e.

velocity of B relative to A = velocity of B − velocity of A.

Denoting the velocities of A and B by $\mathbf{v}_A$ and $\mathbf{v}_B$ respectively, the velocity of B relative to A is $\mathbf{v}_B - \mathbf{v}_A$ and may be found using a vector triangle.

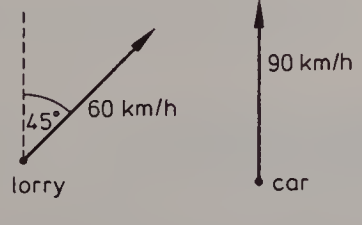


Space diagram

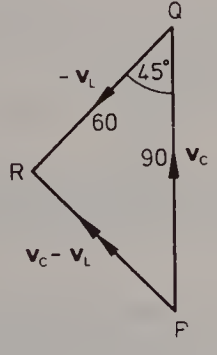


Vector diagram

Example 1 A car is travelling due north on a straight stretch of motorway at 90 km/h. The car is observed by the driver of a lorry travelling north-east on an approach road at 60 km/h. Find the apparent speed of the car.



Space diagram



Vector diagram

If the velocities of the car and the lorry are $\mathbf{v}_C$ and $\mathbf{v}_L$ respectively, then the velocity of the car relative to the lorry is $\mathbf{v}_C - \mathbf{v}_L$.

Using the cosine rule in the vector triangle PQR,

$$PR^2 = 90^2 + 60^2 - 2.90.60\cos 45° \qquad \therefore \quad PR \approx 63{\cdot}7.$$

Hence the apparent speed of the car is 63·7 km/h.

It follows from the definition of relative velocity that for moving objects A and B,

velocity of B = velocity of B relative to A + velocity of A.

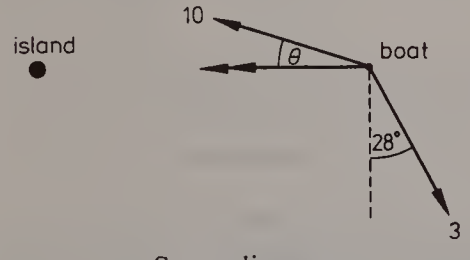
The advantage of using this result when solving problems is that it involves addition rather than subtraction of vectors.

Example 2 A man in a boat wishes to reach a small island due west of his present position. He knows that there is a current with a velocity of 3 km/h in the direction S 28°E. Given that the boat moves at 10 km/h through still water, find the course the man must steer to reach the island.

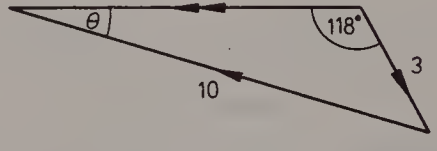
The true velocity of the boat

= velocity of boat relative to water + velocity of current.

If the boat is to reach the island its true velocity must be in the direction due west. Relative to the water the velocity of the boat is of magnitude 10 km/h. Let θ be the angle between these velocities.



Space diagram



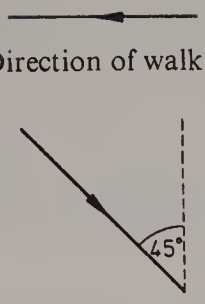
Vector triangle

By the sine rule, $\dfrac{\sin\theta}{3} = \dfrac{\sin 118°}{10}$

$$\therefore \quad \sin\theta = \frac{3\sin 118°}{10} \quad \text{and} \quad \theta \approx 15{\cdot}4°$$

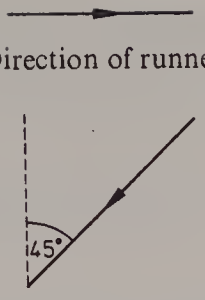
Hence the man must steer the boat in the direction N 74·6° W.

Example 3 To a man walking due west at 5 km/h the wind appears to be blowing from the north-west. However, to a man running due east at 10 km/h the wind appears to be blowing from the north-east. Find the true velocity of the wind.



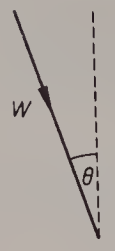
Direction of walker

Apparent direction of wind to walker



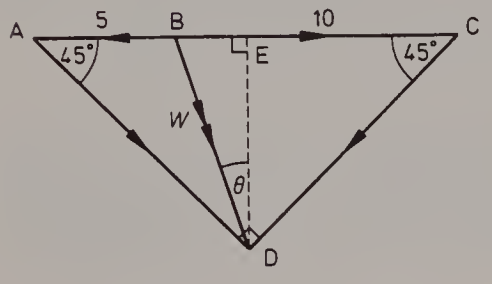
Direction of runner

Apparent direction of wind to runner



True magnitude and direction of wind

Velocity of wind
= velocity of wind relative to walker + velocity of walker
= velocity of wind relative to runner + velocity of runner



Vector diagram

From the symmetry of the diagram,

$$BE = 2{\cdot}5 \quad \text{and} \quad DE = 7{\cdot}5.$$

Using Pythagoras' theorem in $\triangle BDE$,

$$W^2 = (2{\cdot}5)^2 + (7{\cdot}5)^2$$

$$\therefore \quad W \approx 7{\cdot}9$$

$$\tan\theta = \frac{2{\cdot}5}{7{\cdot}5} = \frac{1}{3} \qquad \therefore \quad \theta \approx 18{\cdot}4°$$

Hence the velocity of the wind is 7·9 km/h from the direction N 18·4° W.

Exercise 3.2

1. A man is walking due south at 5 km/h. The wind is blowing from the west at 8 km/h. Find the magnitude and direction of the velocity of the wind relative to the man.

2. A ship is sailing south-east at 20 km/h and a second ship is sailing due west at 25 km/h. Find the magnitude and direction of the velocity of the first ship relative to the second.

3. If particles A and B have velocity vectors $7\mathbf{i}-5\mathbf{j}$ and $-2\mathbf{i}+7\mathbf{j}$ respectively, find the magnitude of the velocity of B relative to A and the angle its direction makes with the direction of $\mathbf{i}$.

4. Particles A and B have velocity vectors $3\mathbf{i}-11\mathbf{j}$ and $5\mathbf{i}+\mathbf{j}$ respectively. The velocity of a particle C relative to A is $-2\mathbf{i}+7\mathbf{j}$. Find, in vector form, the velocity of C and the velocity of B relative to C.

5. A boat A travels due west at a speed of 30 km/h. The velocity of a boat B relative to A is 14 km/h due south. Find the speed of boat B and the direction in which it is moving.

6. To an observer on a ship P steaming at 20 km/h in the direction due west, a ship Q appears to be steaming at 20 km/h in the direction N 30°W. Find the true velocity of Q.

7. To a cyclist riding due east at 15 km/h the wind appears to be blowing from the north-east at 12 km/h. Find the magnitude and direction of the true velocity of the wind.

8. To the driver of a car travelling due north at 40 km/h the wind appears to be blowing at 50 km/h from the direction N 60°E. The wind velocity remains constant, but the speed of the car is increasing. Find its speed when the wind appears to be blowing from the direction N 30°E.

9. A river with straight parallel banks 400 m apart flows due south at 3·5 km/h. Find the direction in which a boat, travelling at 12·5 km/h relative to the water, must be steered in order to cross the river from east to west along a course perpendicular to the banks. Find also the time taken to make the crossing.

10. A stretch of river has straight parallel banks 120 m apart. A man rowing at $5\,\mathrm{m\,s^{-1}}$ relative to the water and heading directly across the river reaches the other bank 54 m downstream. Find the speed of the current and the direction in which the man would have to row in order to cross the river at right angles to the banks.

11. A helicopter whose speed in still air is 60 km/h is to fly to a village 80 km away in the direction N 40°W. The wind is blowing at 10 km/h from the west. Find the direction in which the helicopter must fly and the time taken to reach the village to the nearest minute.

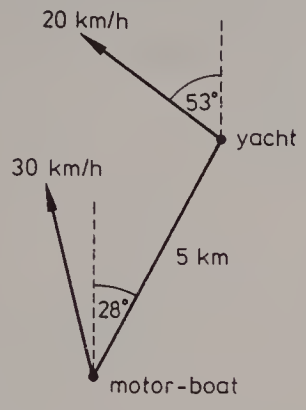
12. A plane flies in a straight line from A to B, where $AB = 580$ km and the bearing of B from A is N 50°E. The speed of the plane in still air is 300 km/h. Given that the wind is blowing from the direction N 40°W and that the flight takes 2 hours, calculate the speed of the wind and the course that should be set for the return journey, if the wind velocity remains unchanged.

3.3 Relative displacement

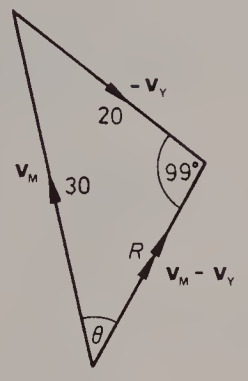
Some problems involve the positions as well as the velocities of objects in motion. In such cases it is usually best to select one of the objects and work with displacements and velocities relative to it.

If A and B are two moving objects then the displacement of B relative to A is given by the displacement vector **AB**. The velocity of B relative to A can be considered to be the rate of change of **AB**. The two objects can meet only if the relative velocity is parallel to the relative displacement.

Example 1 A motor-boat capable of 30 km/h wishes to intercept a yacht 5 km away on a bearing N 28° E. If the yacht is travelling at 20 km/h in the direction N 53° W, what course should the motor-boat take? Find also the time taken to reach the yacht.



Space diagram



Vector diagram

If the velocities of the yacht and the motor-boat are $\mathbf{v}_Y$ and $\mathbf{v}_M$ respectively, then the velocity of the motor-boat relative to the yacht is $\mathbf{v}_M - \mathbf{v}_Y$ and must lie in the direction N 28° E. Let R be the magnitude of this relative velocity and θ the angle it makes with the course of the motor-boat.

By the sine rule, $\dfrac{\sin\theta}{20} = \dfrac{\sin 99°}{30}$

$$\therefore \quad \sin\theta = \frac{20\sin 99°}{30} \quad \text{and} \quad \theta \approx 41{\cdot}182°.$$

Hence the motor-boat must be steered in the direction N 13·2° W.
The remaining angle in the vector triangle is 39·818°.

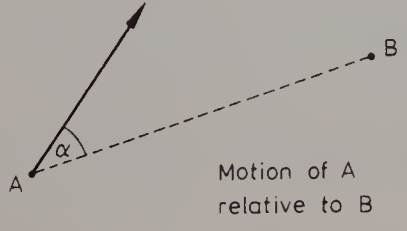
$$\therefore \quad \text{by the sine rule,} \quad \frac{R}{\sin 39{\cdot}818°} = \frac{30}{\sin 99°}$$

$$\therefore \quad R = \frac{30\sin 39{\cdot}818°}{\sin 99°} \approx 19{\cdot}45.$$

Thus on the course N 13·2° W the motor-boat can approach the yacht at a relative speed of 19·45 km/h

$$\therefore \quad \text{the time taken in minutes} = \frac{5}{19{\cdot}45} \times 60 \approx 15{\cdot}4.$$

Hence the motor-boat can reach the yacht in 15·4 minutes.

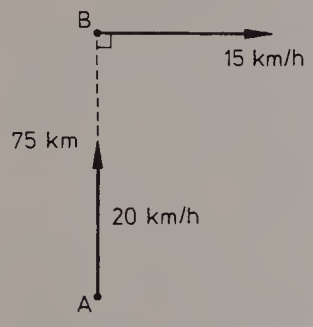


Motion of A relative to B

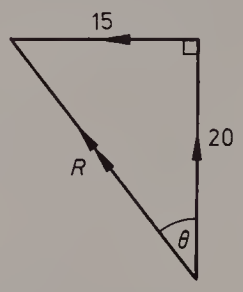
We consider now two objects A and B with given speeds and positions. We assume that the direction of B is such that the objects can never meet. If α is the angle between the velocity of A relative to B and the initial displace-

ment vector **AB**, then A will approach B as closely as possible when α takes its smallest possible value.

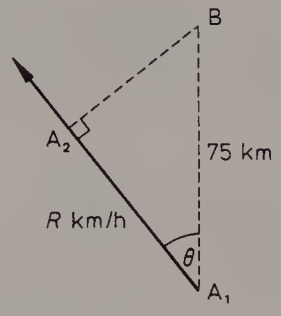
Example 2 A ship A is sailing due north at 20 km/h and a ship B is sailing due east at 15 km/h. At 12.00 hours A is 75 km due south of B. Find the shortest distance between the ships and the time at which they are closest, assuming that their velocities remain unchanged.



Space diagram



Vector diagram



Motion of A relative to B

Let the velocity of A relative to B be R km/h at an angle θ west of north.

Using the vector diagram, $R = \sqrt{(15^2 + 20^2)} = 25$

$$\therefore \quad \sin\theta = \frac{15}{25} = \frac{3}{5} \quad \text{and} \quad \cos\theta = \frac{20}{25} = \frac{4}{5}.$$

Let A_1 be the initial position of A and let A_2 be its position when closest to B. By considering the motion of A relative to B, we see that A_2B must be perpendicular to A's relative path

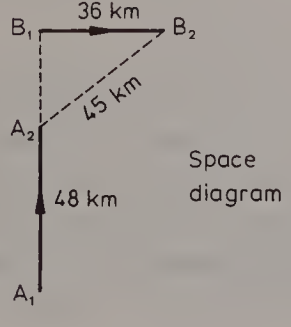
$\therefore$ the shortest distance between the two ships is $75 \sin\theta$ km, i.e. 45 km.

Relative to B the apparent distance between A_1 and A_2 is $75\cos\theta$ km, i.e. 60 km.

$\therefore$ the time A takes to move from A_1 to A_2

$$= \frac{\text{relative distance}}{\text{relative speed}} = \frac{60}{25} \text{ hours} = 2 \text{ hours } 24 \text{ min.}$$

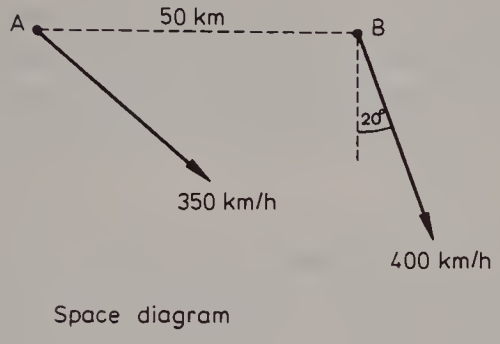
Hence the ships are closest together at 14.24 hours.



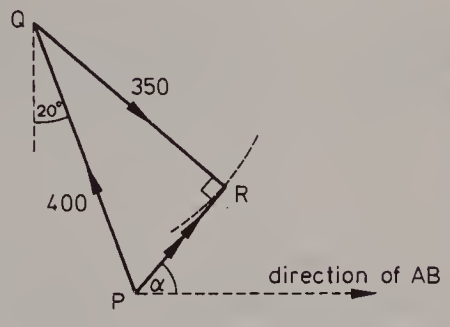
Space diagram

[A further diagram can be drawn to show the true paths of A and B. The true distance A_1A_2 is 48 km and the distance moved by B is 36 km.]

Example 3 An aircraft A flying at a speed of 350 km/h is 50 km due west of another aircraft B. If B is flying at the same height on a course S 20° E at a speed of 400 km/h, find the direction in which A must fly to approach B as closely as possible. Find also the shortest distance between the two aircraft.



Space diagram



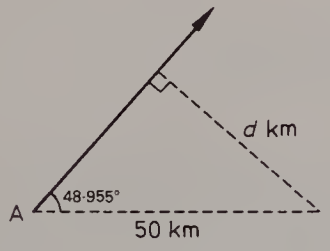
Vector diagram

In the vector diagram $\overrightarrow{QR}$ represents the velocity of A, $\overrightarrow{QP}$ represents the velocity of B and $\overrightarrow{PR}$ represents the velocity of A relative to B. Considering all possible directions for $\overrightarrow{QR}$, the smallest possible value of α occurs when $\angle PRQ$ is a right angle.

Hence $\angle PQR = \cos^{-1}(350/400) \approx 28{\cdot}955°$

$\therefore$ A must fly in the direction S 49° E to approach B as closely as possible.

Using the vector diagram, $\alpha = 49°$.



Let d km be the shortest distance between the aircraft, then by considering the direction of A's motion relative to B,

$d = 50 \sin 48{\cdot}955° \approx 37{\cdot}7$

$\therefore$ the shortest distance between the aircraft is 37·7 km.

Exercise 3.3

1. A destroyer must intercept a vessel 40 km away on a bearing of 110°. The vessel is sailing on a course of 200° at a speed of 20 km/h. If the speed of the destroyer is 30 km/h, find, to the nearest degree, the course it must steer. Find also the time taken for the destroyer to reach the vessel.

2. A man walking due north wishes to intercept a cyclist travelling due west at 12 km/h. The bearing of the cyclist from the man is N 60° E. Find the speed at which the man must walk if he does not change his direction. If the man walks at 8 km/h, find the direction in which he must go to intercept the cyclist as quickly as possible. What is the minimum speed at which the man can walk in order to intercept the cyclist?

3. A ship P is sailing due west at 16 km/h and a ship Q is sailing due north at 24 km/h. At a certain instant Q is 10 km due west of P. Find the velocity of P relative to Q and the shortest distance between the ships if they continue on these courses.

4. A cyclist A is riding along a straight level road on a bearing 030° at a steady speed of $5\,\mathrm{m\,s^{-1}}$. A second cyclist B is travelling due east along another straight level road at a steady speed of $6\,\mathrm{m\,s^{-1}}$. Find the magnitude and the direction of the velocity of A relative to B. At a certain instant A is 500 m due south of B. Find the time that elapses before A is due west of B. (JMB)

5. Two cyclists A and B are approaching a road junction at speeds of 20 km/h and 15 km/h respectively. A is 100 m south of the junction riding due north, when B reaches the junction riding due east. Find the velocity of B relative to A. Find also the shortest distance between A and B, assuming their velocities remain unchanged.

6. At 10.00 hours a plane A passes through a point X travelling due south at 360 km/h. At the same instant a plane B is 50 km due west of X travelling at 450 km/h towards X. Find the magnitude and direction of the velocity of B relative to A and the shortest distance between A and B. Find also the time at which the planes are closest and the bearing of B from A at that time.

7. Two planes A and B, flying at the same height, are travelling at 500 km/h in the directions due north and due east respectively. At a certain instant A is 100 km from B on a bearing of 150°. Find the velocity of B relative to A and the shortest distance between the planes if neither changes course.

8. (In this question the units of time and distance are the second and metre respectively.) Two particles A and B start simultaneously from points which have position vectors $3\mathbf{i}+4\mathbf{j}$ and $8\mathbf{i}+4\mathbf{j}$ respectively. Each moves with constant velocity, the velocity of B being $\mathbf{i}+4\mathbf{j}$. (i) If the velocity of A is $5\mathbf{i}+2\mathbf{j}$, show that the least distance between A and B subsequently is $\sqrt{5}$ metres. (ii) If the speed of A is 5 m/s and the particles collide, find the velocity of A. (L)

9. A destroyer sights a ship travelling with constant velocity $5\mathbf{j}$, whose position vector at the time of sighting is $2000(3\mathbf{i}+\mathbf{j})$ relative to the destroyer, distances being measured in m and speeds in $\mathrm{m\,s^{-1}}$. The destroyer immediately begins to move with velocity $k(4\mathbf{i}+3\mathbf{j})$, where k is a constant, in order to intercept the ship. Find k and the time to interception. Find also the distance between the vessels when half the time to interception has elapsed. (O&C)

10. A ship P sailing at 16 km/h is 10 km due south of a second ship Q. If Q is sailing in the direction S 70° E at a speed of 20 km/h, find the direction in which P must sail to approach Q as closely as possible. Find also the shortest distance between the ships.

Exercise 3.4 (miscellaneous)

[1 knot = 1 nautical mile per hour.]

1. During a race between two yachts, A and B, there is a wind of 9 knots blowing from due north. The resultant velocity of A is 6 knots on a bearing of 060°. Find the direction of the wind relative to A. At the same time, the resultant velocity of B is 6 knots on a bearing of 300°. Find, correct to the nearest degree, the direction of the wind relative to B and, in knots correct to one decimal place, the velocity of A relative to B. (L)

2. A yacht A is sailing due east at 9 knots and a second yacht B is sailing on a bearing of 030° at 6 knots. At a certain instant a third yacht C appears to an observer on A to be sailing due south and appears to an observer on B to be sailing on a bearing of 150°. Find the speed of the yacht C and the bearing on which it is sailing. (L)

3. A river flows at 5 m/s from west to east between parallel banks which are at a distance 300 metres apart. A man rows a boat at a speed of 3 m/s in still water. (i) State the direction in which the boat must be steered in order to cross the river from the southern bank to the northern bank in the shortest possible time. Find the time taken and the actual distance covered by the boat for this crossing. (ii) Find the direction in which the boat must be steered in order to cross the river from the southern bank to the northern bank by the shortest possible route. Find the time taken and the actual distance covered by the boat for this crossing. (AEB 1978)

4. A river flows at a constant speed of 5 m/s between straight parallel banks which are 300 m apart. A boat, which has a maximum speed of $3\frac{1}{4}$ m/s in still water, leaves a point A on one bank and sails in a straight line to the opposite bank. Find, graphically or otherwise, the least time the boat can take to reach a point B on the opposite bank where $AB = 500$ m and B is downstream from A. Find also the least time the boat can take to cross the river. Find the time taken to sail from A to B by the slowest boat capable of sailing directly from A to B. (L)

5. A helicopter flies with constant airspeed 100 knots from position A to position B, which is 50 nautical miles north east of A, and then flies back to A. Throughout the whole flight the wind velocity is 30 knots from the west. Find, by drawing or calculation, the course set for each of the two legs of the flight. Find also the total time of flight from A to B and back. (L)

6. A motorist A is travelling in a direction N 10° E at a constant speed of $60\,\text{km}\,\text{h}^{-1}$. The wind is blowing from the direction S 80° E at a constant speed of $25\,\text{km}\,\text{h}^{-1}$. Find the magnitude and direction of the velocity of the wind relative to the motorist. A second motorist B is travelling at a constant speed of $50\,\text{km}\,\text{h}^{-1}$. The velocity of B relative to A is in a direction S 30° W. Find graphically, or otherwise, the two possible directions in which B can be travelling. (C)

7. Three fixed buoys are at the vertices A, B and C of an equilateral triangle of side 12 km. The point B is due north of A, and C is to the east of the line AB. A steady current of speed $3\,\text{km h}^{-1}$ flows from west to east. A boat which has a top speed of $8\,\text{km h}^{-1}$ in still water does the triangular journey $ABCA$ at top speed. Find, graphically or otherwise, the time taken on each leg of the journey, giving your answers to the nearest minute. (C)

8. A helicopter, with a maximum speed in still air of $120\,\text{km h}^{-1}$, is to fly between airports A and B. B is 238 km from A in a direction Nθ°W where $\theta = \cos^{-1}\frac{12}{13}$. The helicopter leaves A at 12.00 hours in a wind blowing at $50\,\text{km h}^{-1}$ from the west. What is the earliest possible time of arrival at B, and the corresponding direction in which it must head? At 13.00 hours the helicopter receives a radio message not to proceed but to return to A as soon as possible. Neglecting the time to change course, what is the earliest time it can be back at A? (W)

9. A small boat fitted with an outboard engine is 10 nautical miles due east of a trawler. The trawler is travelling at 15 knots on a course Nθ°E $[\theta = \sin^{-1}(24/25)]$. By considering the trawler's velocity components due east and due north, or otherwise, show that the minimum speed necessary for the boat to intercept the trawler is 4·2 knots, and find the corresponding direction in which it must steer. If the boat travels at 9 knots, show that it can steer in either of two directions to reach the trawler and calculate the shorter time needed to do so. (W)

10. [In this question the units of distance and time are the metre and second respectively.] At time $t = 0$ a particle A is at the origin and a particle B is at the point with position vector $10\mathbf{i}$. Each particle is moving with constant velocity, the velocity of B being $-4\mathbf{i}+3\mathbf{j}$. Subsequently the particles collide. Find (a) the least speed at which A can travel, (b) the velocity of A if the collision occurs at time $t = 2$. A third particle C is moving in the same plane. When the velocity of A is $\mathbf{i}+3\mathbf{j}$ the velocity of C relative to A is in the direction of the vector $3\mathbf{i}+2\mathbf{j}$ and relative to B is in the direction of the vector $7\mathbf{i}+3\mathbf{j}$. Find the speed of C. (L)

11. A motorist A and a cyclist B are travelling along straight roads which cross at right angles at a point O. A is travelling at a constant speed of $48\,\text{km h}^{-1}$ towards the east, and B at $14\,\text{km h}^{-1}$ towards the north. At the moment that B is passing through O, A is 400 m from O and has not yet passed O. Calculate: (i) the velocity of B relative to A in magnitude and direction; (ii) the least distance between car and cycle; (iii) the distances of the car and cycle from O when they are nearest to one another. (SU)

12. A cruiser is moving due east at 30 km/h. Relative to the cruiser a frigate is moving on a course of 210° (S 30° W) at 48 km/h. Using a graphical method, or otherwise, find the magnitude and the direction of the velocity of the frigate relative to a coastguard who is recording the paths of these ships from a lighthouse. At 1300 hours the frigate is 10 km due east of the cruiser. If both ships

maintain their speeds and courses, find the time at which the distance between them is least and their actual distance apart at this instant. Find also the time at which the frigate is due south of the cruiser. (AEB 1977)

13. A ship A whose full speed is 40 km/h is 20 km due west of a ship B which is travelling uniformly with speed 30 km/h in a direction due north. The ship A travels at full speed on a course chosen so as to intercept B as soon as possible. Find the direction of this course and calculate to the nearest minute the time A would take to reach B. When half of this time has elapsed the ship A has engine failure and thereafter proceeds at half speed. Find the course which A should then set in order to approach as close as possible to B, and calculate the distance of closest approach (in kilometres to 2 decimal places). (JMB)

14. A ship A is travelling with constant velocity 20 km/h due east and a ship B has constant velocity 15 km/h in a direction 30° east of north. At noon B is 30 km due south of A. Find the magnitude of the velocity of B relative to A, and show that the direction of this relative velocity is approximately 44° west of north. Find, to the nearest minute, the time at which A and B are closest. At the time of closest approach a boat leaves A to intercept B. Find the least speed at which it must travel. (JMB)

15. Construct a flow chart for a procedure in which the input is the speeds and courses of two ships, A and B, in the form x km/h on a bearing y°. Assuming that initially ship A is 20 km due south of ship B, the following output is required:

(a) the time and distance of closest approach or an appropriate message if the ships are not moving closer to each other;
(b) the course A should steer to intercept B as soon as possible and the time of interception;
(c) if interception is not possible, the course A should steer to approach B as closely as possible.

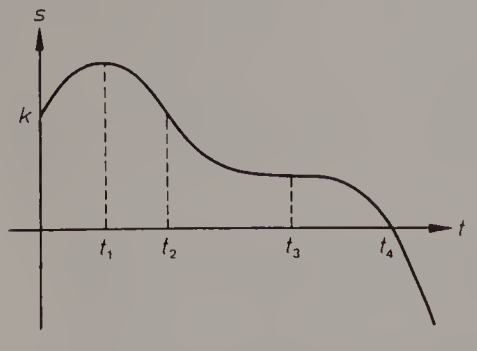
[It may be helpful to consider velocity components due east and due north.]

4 Velocity and acceleration

4.1 Elementary kinematics

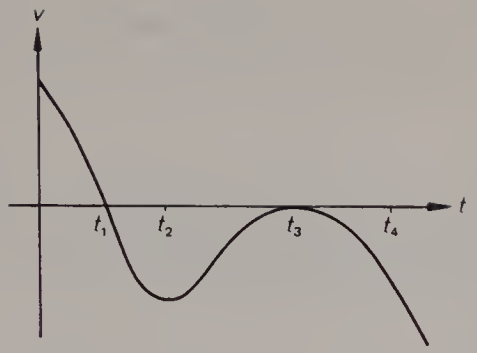
We consider now the motion of a particle in a straight line. Since in this type of motion the displacements, velocities and accelerations are parallel, there is no need to use vector notation. In general the displacement of the particle from a fixed point on the line after t seconds is denoted by s m, the velocity by v m s^{-1} and the acceleration by a m s^{-2}. This means that, $v = \frac{ds}{dt}$ and $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$. Thus if a displacement-time graph is plotted the velocity is represented by the gradient of the curve. When the gradient is positive the motion of the particle is in the positive sense along the straight line and when the gradient is negative motion is in the opposite direction. At stationary points on the curve the velocity is zero and the particle is instantaneously at rest. Similarly the gradient of the velocity-time graph represents the acceleration of the particle.

Consider, for instance, the motion of a particle in a straight line as represented by the graphs given below. The first graph shows the displacement s from a fixed point O at time t and the second is the corresponding velocity-time graph.



At the start of the motion the particle is at a distance k from O, moving away from O with positive velocity. It comes to rest at time t_1, then starts moving towards O. Its speed increases to a maximum value at $t = t_2$, then decreases so that the particle is again at rest when $t = t_3$. It then continues to move with negative velocity as t increases, passing through O at time t_4.

Considering the gradient of the velocity-time graph, we see that a negative acceleration produces the initial



reduction in speed, the change of direction at time t_1, then an increasing speed towards O. The minimum point on the graph at $t = t_2$ corresponds to a maximum speed of the particle in the negative direction. The positive acceleration from $t = t_2$ to $t = t_3$ brings the particle to rest. The motion then continues with velocity and acceleration both negative and hence with increasing speed in the negative direction.

Example 1 A particle is moving in a straight line, so that after t seconds its displacement s metres from a fixed point O is given by $s = 9t + 3t^2 - t^3$. Find (a) the initial displacement, velocity and acceleration of the particle, (b) the time at which the particle is instantaneously at rest, (c) the greatest speed attained by the particle in the first 4 seconds of the motion.

Differentiating, $$v = \frac{ds}{dt} = 9 + 6t - 3t^2$$

$$a = \frac{dv}{dt} = 6 - 6t.$$

(a) When $t = 0, s = 0, v = 9$ and $a = 6$

$\therefore$ initially the particle is at O moving with velocity $9\,\mathrm{m\,s^{-1}}$ and acceleration $6\,\mathrm{m\,s^{-2}}$ (both in the positive direction along the line of motion).

(b) $v = 0 \Leftrightarrow 9 + 6t - 3t^2 = 0 \Leftrightarrow 3(3 - t)(1 + t) = 0$

$\therefore$ $v = 0$ when $t = 3$ (assuming $t \geqslant 0$).

Hence the particle is instantaneously at rest after 3 seconds.

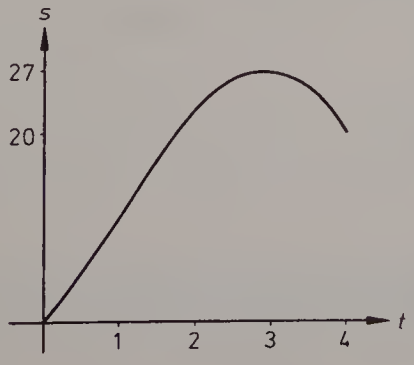
(c) $a = 0 \Leftrightarrow 6 - 6t = 0 \Leftrightarrow t = 1$.

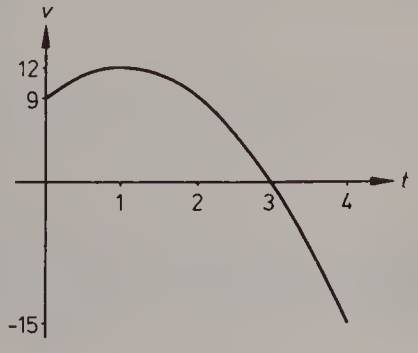
When $t < 1, a > 0$ and when $t > 1, a < 0$

$\therefore$ when $t = 1$, the velocity reaches a maximum value of $12\,\mathrm{m\,s^{-1}}$.

The velocity then decreases during the remainder of the motion. Since the particle is at rest when $t = 3$, it must move with increasing speed in the negative direction for $t > 3$. When $t = 4$, $v = -15$ $\therefore$ the greatest speed attained by the particle in the first 4 seconds of the motion is $15\,\mathrm{m\,s^{-1}}$.

[These results are illustrated in the sketch graphs below.]





When the acceleration of a particle is represented by a known function of the time t, then expressions for the velocity and displacement can often be found by integration.

Example 2 A vehicle moves in a straight line so that its acceleration after t seconds is given by $a = 2t$. Initially it has a velocity of $20\,\mathrm{m\,s^{-1}}$. Find an expression for the displacement s metres of the vehicle from its starting point.

$$a = \frac{dv}{dt} = 2t$$

Integrating, $v = t^2 + c$

but $v = 20$ when $t = 0$, thus $c = 20$

$\therefore$ $v = \dfrac{ds}{dt} = t^2 + 20.$

Integrating, $s = \frac{1}{3}t^3 + 20t + k$

but $s = 0$ when $t = 0$, thus $k = 0$

$\therefore$ $s = \frac{1}{3}t^3 + 20t.$

The next example shows that the total displacement of a particle in any given interval of time is represented by the area under the velocity-time graph and that this can be found using a definite integral.

Example 3 A particle moves in a straight line so that after t seconds its velocity in $\mathrm{m\,s^{-1}}$ is given by $v = (2t^2 - 25)(t - 1)$. Find (a) the displacement of the particle from its starting point after 3 seconds, (b) the total distance moved by the particle in the first 3 seconds of the motion.

(a) The displacement of the particle from its starting point after 3 seconds (in metres)

$$= \int_0^3 v\,dt = \int_0^3 (2t^2 - 25)(t - 1)\,dt$$

$$= \int_0^3 (2t^3 - 2t^2 - 25t + 25)\,dt$$

$$= \left[\frac{1}{2}t^4 - \frac{2}{3}t^3 - \frac{25}{2}t^2 + 25t\right]_0^3$$

$$= \left(\frac{1}{2}.81 - \frac{2}{3}.27 - \frac{25}{2}.9 + 25.3\right) - 0$$

$$= 40\tfrac{1}{2} - 18 - 112\tfrac{1}{2} + 75 = -15.$$

(b) Since $v = 0$ when $t = 1$, the particle is instantaneously at rest after 1 second.

For $0 < t < 1$, $v > 0$ and for $1 < t < 3$, $v < 0$.

Hence the direction of motion changes once in the first 3 seconds, i.e. when $t = 1$.

$$\int_0^1 v\,dt = \left[\frac{1}{2}t^4 - \frac{2}{3}t^3 - \frac{25}{2}t^2 + 25t\right]_0^1$$

$$= \left(\frac{1}{2} - \frac{2}{3} - \frac{25}{2} + 25\right) - 0 = \frac{1}{2} - \frac{2}{3} - 12\tfrac{1}{2} + 25 = 12\tfrac{1}{3}$$

$$\int_1^3 v\,dt = \left[\frac{1}{2}t^4 - \frac{2}{3}t^3 - \frac{25}{2}t^2 + 25t\right]_1^3$$

$$= \left(\frac{1}{2}.81 - \frac{2}{3}.27 - \frac{25}{2}.9 + 25.3\right) - \left(\frac{1}{2} - \frac{2}{3} - \frac{25}{2} + 25\right)$$

$$= -15 - 12\tfrac{1}{3} = -27\tfrac{1}{3}$$

$\therefore$ the total distance travelled in the first 3 seconds of the motion is $39\frac{2}{3}$ metres.

Exercise 4.1

Each of the questions in this exercise concerns a particle P moving along a straight line such that after t seconds its displacement from a fixed point A is s m, its velocity is $v\,\mathrm{m\,s^{-1}}$ and its acceleration is $a\,\mathrm{m\,s^{-2}}$.

1. Find expressions in terms of t for v and a, given that
(a) $s = 6-5t+3t^3$, (b) $s = (2t-1)(t+2)$.

2. Find expressions in terms of t for v and s, given that
(a) $a = 12t$ and when $t = 0$, $v = 7$ and $s = 0$,
(b) $a = -10$ and when $t = 0$, $v = 20$ and $s = 5$,
(c) $a = 24t(t-1)$ and when $t = 1$, $v = s = 0$.

3. If $v = 2t^3 - 9t^2$, find the values of t for which $a = 0$. Given that $s = 20$ when $t = 0$, find the value of s when $t = 2$.

4. If $v = 12+3t^2$, find the value of v when $a = 12$. Find also the distance travelled in the first two seconds of the motion.

5. The particle P starts from rest at A and moves such that $a = 6-2t$. If the particle later comes to rest at a point B, find the distance AB and the greatest speed attained while moving from A to B.

6. Given that the initial velocity of the particle P is $9\,\mathrm{m\,s^{-1}}$ and that $a = 2t-10$, find the times when the particle is instantaneously at rest and the maximum speed attained by the particle between these two times.

7. Given that $a = 3t$ and that $v = 5$ when $t = 0$, find the value of v when $t = 4$ and the distance travelled in the first four seconds of the motion.

8. If $s = t^3 - 12t$, find the initial velocity of the particle P and the distance between its positions when $t = 0$ and $t = 4$. Show that P changes direction between $t = 0$ and $t = 4$, then find the total distance travelled in this interval of time.

9. Given that $v = 25 - 4t^2$, calculate (a) the value of t when the particle is instantaneously at rest, (b) the distance travelled by the particle in the third second.

10. Given that $s = t^3 - 2t^2 - 15t$, find (a) the initial velocity and acceleration of P, (b) the times at which P passes through A, (c) the total distance travelled in the first five seconds of the motion.

11. The particle P starts from rest at the point A and is again instantaneously at rest after 3 seconds. If $a = k - 6t$, where k is constant, find (a) expressions for v and s in terms of t, (b) the distance P travels before returning to A.

12. Given that $s = 6t^2 - t^3$, sketch the displacement-time and velocity-time graphs for the motion of the particle P from $t = 0$ to $t = 6$. Find, in the interval $0 \leqslant t \leqslant 6$,
(a) the times at which the particle is at A,
(b) the greatest value of the distance AP,
(c) the total distance travelled,
(d) the greatest speed attained.

13. Repeat question 12 for $s = t^2 - 5t + 4$.

14. Repeat question 12 for $s = 4t^3 - 33t^2 + 54t$.

15. Given that when $t = 1$, $s = 0$ and that when $t > 1$, $v = t^2 + \frac{9}{t^2}$, find the values of a, v and s when $t = 3$. Find also the minimum speed of the particle for $t > 1$.

16. If the particle P starts from rest and moves such that $v = 4t^2(3 - t)$, find (a) the distance travelled by the particle before it next comes to rest, (b) the maximum speed attained by the particle in the first three seconds of its motion, (c) the maximum acceleration in the first two seconds of its motion.

4.2 Motion with constant acceleration

We now obtain general equations for the motion of a particle in a straight line with constant acceleration. These will then be used to solve a variety of problems.

Consider a particle with initial velocity u and constant acceleration a. Let its displacement from its initial position be s and its velocity v at time t.

$$\frac{dv}{dt} = a \quad \text{where } a \text{ is constant}$$

Integrating $v = at + c$

When $t = 0$, $v = u$ $\therefore \quad c = u$

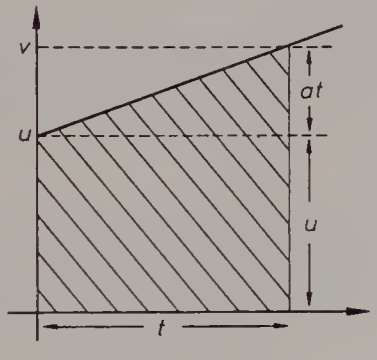
Thus $v = u + at$ (1)

and $\frac{ds}{dt} = u + at$

Integrating $s = ut + \frac{1}{2}at^2 + k$

When $t = 0$, $s = 0$ $\therefore \quad k = 0$

Thus $s = ut + \frac{1}{2}at^2$ (2)



Results (1) and (2) can also be obtained by considering the velocity-time graph for the motion. As shown in the sketch this is a straight line with gradient a. The displacement s is represented by the shaded area under the line.

Rearranging (1) $at = v - u$

Substituting in (2) $s = ut + \frac{1}{2}(v-u)t$

$\therefore \quad s = \frac{1}{2}(u+v)t$ (3)

Hence $2as = (u+v)at = (u+v)(v-u) = v^2 - u^2$

$\therefore \quad v^2 = u^2 + 2as.$ (4)

Eliminating u from equations (1) and (2),

$$s = vt - \tfrac{1}{2}at^2. \quad (5)$$

We now have five equations connecting the quantities s, u, v, a and t. When using them it is advisable to check that the acceleration is constant and that the units of measurement are consistent.

$$\begin{aligned} v &= u + at \\ s &= ut + \tfrac{1}{2}at^2 \\ s &= vt - \tfrac{1}{2}at^2 \\ s &= \tfrac{1}{2}(u+v)t \\ v^2 &= u^2 + 2as. \end{aligned}$$

Example 1 A particle moving in a straight line accelerates uniformly from rest to a speed of $60\,\mathrm{m\,s^{-1}}$ in 10 seconds. Find the distance travelled in this time.

Given: $u = 0, v = 60, t = 10.$

Using the equation $s = \frac{1}{2}(u+v)t$

$s = \frac{1}{2}(0+60)10 = 300.$

Hence the particle travelled a distance of 300 metres.

Example 2 A train travelling in a straight line is brought to rest in 1 minute with a uniform retardation of $0{\cdot}5\,\mathrm{m\,s^{-2}}$. Find the speed at which the train was travelling in km/h.

Given: $v = 0, a = -0{\cdot}5, t = 60.$

Using the equation $v = u+at$

$0 = u - 0{\cdot}5 \times 60 \quad \therefore \quad u = 30.$

Hence the speed of the train $= 30\,\mathrm{m\,s^{-1}}$

$$= \frac{30 \times 60 \times 60}{1000}\,\mathrm{km/h} = 108\,\mathrm{km/h}.$$

Example 3 A particle moving in a straight line with constant acceleration travels 10 m in 2 seconds, then a further 30 m in the following 2 seconds. Find the acceleration of the particle.

When $s = 10, t = 2$ and when $s = 40, t = 4$

$\therefore$ using the equation $s = ut + \frac{1}{2}at^2, \quad 10 = 2u + 2a$

$40 = 4u + 8a$

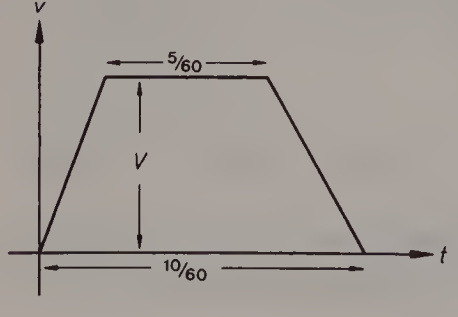
Simplifying $u + a = 5$

$u + 2a = 10 \quad \therefore \quad a = 5.$

Hence the acceleration of the particle is $5\,\mathrm{m\,s^{-2}}$.

Although the constant acceleration formulae are very useful, some problems are better solved by drawing a sketch of the velocity-time graph.

Example 4 A train takes 10 minutes to travel a distance of 12 km between two stations. It starts from rest at the first station, accelerating uniformly until it reaches a speed of V km/h. It travels at this speed for 5 minutes then decelerates uniformly, coming to rest at the second station. Find the value of V.



Consider the velocity-time graph for the motion, taking the units of distance and time as kilometres and hours.

Since the area of the trapezium under the graph represents the distance travelled,

$$\frac{1}{2}\left(\frac{5}{60}+\frac{10}{60}\right)V = 12, \quad \text{i.e. } \frac{1}{8}V = 12.$$

Hence the value of V is 96.

An object falling freely under gravity has a downward acceleration usually denoted by g. The magnitude of this acceleration varies slightly over the surface of the earth. In $\mathrm{m\,s^{-2}}$ the variation is from about 9·78 at the equator to 9·83 at the poles. [On the moon the value of g is 1·62.] Although the pull of gravity on an object decreases as its distance from the earth's surface increases, this decrease can, in general, be regarded as negligible. Thus in most problems g is taken to be a constant with value 9·8 or 10 depending on the accuracy required.

Example 5 A boy standing at the edge of a cliff throws a stone vertically upwards with a velocity of $10\,\mathrm{m\,s^{-1}}$. If the stone hits the sea below with a velocity of $40\,\mathrm{m\,s^{-1}}$, find the height of the cliff above sea-level and the maximum height above sea-level reached by the stone. [Take $g = 10$.]

Taking the upward direction as positive, $u = 10$, $v = -40$ and $a = -10$.

Using the equation $v^2 = u^2 + 2as$

$1600 = 100 + 2(-10)s$

$1500 = -20s$ $\quad \therefore \quad s = -75.$

Hence the height of the cliff above sea-level is 75 m.

When the stone reaches its maximum height its velocity is zero
$\therefore \quad u = 10$, $v = 0$ and $a = -10$.

Using the equation $v^2 = u^2 + 2as$

$0 = 100 + 2(-10)s$ $\quad \therefore \quad s = 5.$

Hence the maximum height reached by the stone is 5 m above the cliff, i.e. 80 m above sea-level.

Example 6 A particle is projected vertically upwards with a speed of $30\,\mathrm{m\,s^{-1}}$. After 3 seconds another particle is projected vertically upwards from the same point with a speed of $25\,\mathrm{m\,s^{-1}}$. Find the height at which the two particles collide. [Take $g = 10$.]

Let us assume that the collision occurs at a height of h m and T seconds after the projection of the second particle, then taking the upward direction as positive,

for the first particle: $s = h$, $u = 30$, $a = -10$, $t = T+3$
for the second particle: $s = h$, $u = 25$, $a = -10$, $t = T$.

Using the equation $s = ut + \frac{1}{2}at^2$ in both cases

$$h = 30(T+3) + \tfrac{1}{2}(-10)(T+3)^2$$
$$= 30T + 90 - 5T^2 - 30T - 45$$

i.e. $h = 45 - 5T^2$ (1)

and $h = 25T + \frac{1}{2}(-10)T^2$

i.e. $h = 25T - 5T^2$. (2)

Hence $25T = 45$ so that $T = 9/5$.

Substituting in (1) $h = 45 - 5\left(\frac{9}{5}\right)^2 = 28{\cdot}8$

$\therefore$ the particles collide at a height of 28·8 metres.

Exercise 4.2

[Take g as $10\,\mathrm{m\,s^{-2}}$.]

In questions 1 to 10 a particle is moving in a straight line with constant acceleration a and initial velocity u. At time t its displacement from its initial position is s and its velocity is v.

1. If $u = 5, a = 2, t = 3$, find v and s.

2. If $v = 7, u = 5, a = 1$, find s and t.

3. If $u = 10, v = -2, t = 4$, find s and a.

4. If $s = 15, t = 5, u = 13$, find v and a.

5. If $s = -5, v = 2, u = -3$, find a and t.

6. If $t = 3, a = -8, s = 36$, find v and u.

7. If $u = 50, s = 300, a = -4$, find t and v.

8. If $a = 0{\cdot}4, t = 15, v = 7$, find u and s.

9. If $v = -18, s = -64, t = 8$, find a and u.

10. If $s = 45, a = 40, v = 60$, find u and t.

11. A car accelerates uniformly from rest to 60 km/h in 30 seconds. Find the distance it travels in this time.

12. A cyclist travels 1·25 km as he accelerates uniformly at a rate of $k\,\mathrm{m\,s^{-2}}$ from a speed of 15 km/h to 30 km/h. Find the value of k.

13. A particle is projected vertically upwards from a point O with a speed of $25\,\mathrm{m\,s^{-1}}$. Find the maximum height reached by the particle and the time that elapses before it returns to O.

14. A boy on a bridge throws a pebble vertically upwards at a speed of $6\,\mathrm{m\,s^{-1}}$. After 2 seconds it hits the water below. Find the speed at which the pebble hits the water and its initial height above the water.

15. The points O, A, B and C lie in a straight line such that $AB = 28\,\mathrm{m}$ and $BC = 72\,\mathrm{m}$. A particle moving with constant acceleration starts from rest at O and passes through A, B and C, its velocities at B and C being $9\,\mathrm{m\,s^{-1}}$ and $15\,\mathrm{m\,s^{-1}}$ respectively. Find the velocity of the particle at A and the time it takes to travel from A to C.

16. A particle moving in a straight line with constant acceleration travels 10 m in 2 seconds, then a further 22 m in the next 2 seconds. Find the further distance travelled in 2 more seconds and the speed of the particle at the end of this 6-second interval of time.

17. A train, being brought to rest with uniform retardation, travels 30 m in 2 seconds, then a further 30 m in 4 seconds. Find the retardation of the train and the additional time it takes to come to rest.

18. A particle is projected vertically upwards from a point A. Given that it rises 15 m in the third second of its motion, find its initial speed and the maximum height above A that it reaches.

19. A bus sets off from a bus station P. It accelerates uniformly for T_1 seconds, covering a distance of 300 m. It travels at a speed of $V\,\mathrm{km/h}$ for T_2 seconds, covering a further distance of 1250 m. It then decelerates uniformly for T_3 seconds, coming to rest at a bus stop Q. If the total time taken is 3 minutes and $2T_1 = 3T_3$, find the distance from P to Q and the values of T_1, T_2, T_3 and V.

20. A train stops at a station A. It then accelerates at $0{\cdot}1\,\mathrm{m\,s^{-2}}$ for 5 minutes, reaching a speed of $V\,\mathrm{m\,s^{-1}}$. It continues at this speed for 12 minutes, then the brakes are applied for 3 minutes, bringing the train to rest with uniform retardation at a station B. Find the value of V and the distance AB.

21. A train stops at two stations 24 km apart. It takes 3 minutes to accelerate uniformly to a speed of $40\,\mathrm{m\,s^{-1}}$ then maintains this speed until it comes to rest with uniform retardation in a distance of 1200 m. Find the time taken for the journey.

22. Two stations A and B are 10 km apart. A train travelling at a constant speed of 144 km/h passes station A at 10.00 hours. At a distance d km from B the brakes are applied, producing a constant retardation of $0{\cdot}4\,\mathrm{m\,s^{-2}}$. If the train comes to rest at station B, find the value of d and the time at which the train reaches B.

23. A particle accelerates from rest at a constant rate of $3\,\mathrm{m\,s^{-2}}$ to a speed of $V\,\mathrm{m\,s^{-1}}$. It continues to move at that speed for a certain time, then decelerates at a constant rate of $1{\cdot}5\,\mathrm{m\,s^{-2}}$. If the total time taken is one minute and the total distance travelled is 1 km, find the value of V.

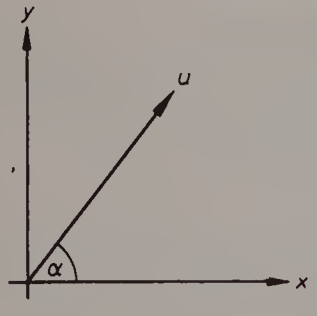
24. A car travelling along a straight level road at a constant speed of 54 km/h passes a second car as it starts to accelerate from rest at a uniform rate of $0{\cdot}5\,\mathrm{m\,s^{-2}}$. Find the time that elapses and the distance covered when the second car draws level with the first.

25. A ball is thrown vertically upwards from a point A with a speed of $20\,\mathrm{m\,s^{-1}}$. At the same instant a second ball is dropped from a point B which is 60 m vertically above A. Find the time which elapses before the two balls meet and their height above A at this instant.

26. A stone is dropped from the top of a tower. After one second another stone is thrown vertically downwards from the same point at a speed of $15\,\mathrm{m\,s^{-1}}$. If the stones reach the ground simultaneously, find the height of the tower.

4.3 Projectiles

Any particle which is given a non-zero initial velocity and then moves freely under gravity may be described as a *projectile*. In elementary work it is assumed that projectiles move in a vertical plane with constant downward acceleration g. The point of projection is taken as the origin O of a Cartesian coordinate system with the x-axis horizontal and the y-axis vertical. If at time t the projectile is at the point (x, y) then its displacement from O has horizontal component x and vertical component y. Thus the horizontal and vertical components of the velocity are dx/dt and dy/dt. Using a dot to indicate differentiation with respect to time these components are usually denoted by $\dot{x}$ and $\dot{y}$. Similarly the horizontal and vertical components of the acceleration are $\ddot{x}$ and $\ddot{y}$ respectively.



Consider a particle projected with initial velocity u at an angle α to the horizontal, the horizontal and vertical components of this velocity being $u\cos\alpha$ and $u\sin\alpha$ respectively. Since the particle has a constant downward acceleration of magnitude g, $\ddot{x} = 0$ and $\ddot{y} = -g$.

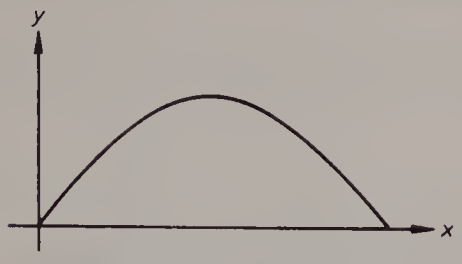
Thus the constant acceleration formulae derived in the previous section can now be applied to both horizontal and vertical motion. Using the equations $v = u + at$ and $s = ut + \frac{1}{2}at^2$,

$$\dot{x} = u\cos\alpha \qquad \dot{y} = u\sin\alpha - gt$$
$$x = ut\cos\alpha \qquad y = ut\sin\alpha - \tfrac{1}{2}gt^2.$$

Most projectile problems can be solved using one or more of these four equations.

The equation of the *trajectory* or path of the particle is obtained by eliminating from the expressions for x and y.

Writing $t = \dfrac{x}{u\cos\alpha}$, $y = u.\dfrac{x}{u\cos\alpha}.\sin\alpha - \dfrac{1}{2}g\left(\dfrac{x}{u\cos\alpha}\right)^2$



$$\therefore \quad y = x\tan\alpha - \frac{gx^2\sec^2\alpha}{2u^2}.$$

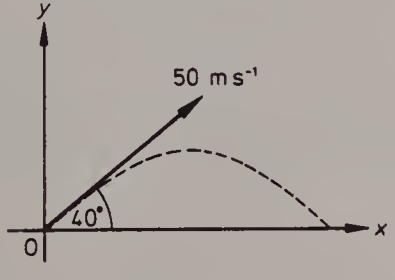
As shown in the sketch this is the equation of a parabola.

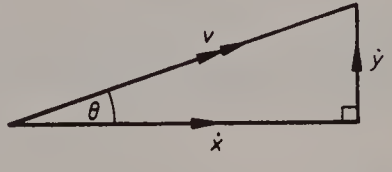
Example 1 A particle is projected from a point O on a horizontal plane with a speed of $50\,\mathrm{m\,s^{-1}}$ at an angle of 40° to the horizontal. Find (a) the velocity of the particle after 2 seconds, (b) the greatest height reached by the particle. [Take $g = 9{\cdot}8$.]

The equations for the motion of the particle are

$$\dot{x} = 50\cos 40° \qquad \dot{y} = 50\sin 40° - gt$$
$$x = 50t\cos 40° \qquad y = 50t\sin 40° - \tfrac{1}{2}gt^2.$$

(a)





Let the velocity of the particle after 2 seconds be $v\,\mathrm{m\,s^{-1}}$ at an angle θ to the horizontal.

When $t = 2$, $\dot{x} = 50\cos 40°$ $\dot{y} = 50\sin 40° - 9{\cdot}8 \times 2$

$\therefore \quad v = \sqrt{(\dot{x}^2 + \dot{y}^2)} \approx 40{\cdot}3$

and $\tan\theta = \dot{y}/\dot{x}$, $\theta \approx 18{\cdot}1°$.

[These results were obtained using a calculator, storing the value $\dot{y} = 12{\cdot}5394$ in the memory.]

Thus the velocity after 2 seconds is $40{\cdot}3\,\mathrm{m\,s^{-1}}$ at an angle 18·1° above the horizontal.

(b) When the particle reaches its greatest height, $\dot{y} = 0$

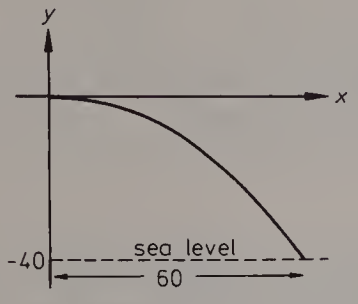
$\therefore \quad 50\sin 40° - gt = 0$ and $t = 50\sin 40°/g$

For this value of t,

$$y = \frac{(50\sin 40°)^2}{g} - \frac{1}{2}g\left(\frac{50\sin 40°}{g}\right)^2 = \frac{(50\sin 40°)^2}{2g} \approx 52{\cdot}7$$

$\therefore$ the greatest height reached by the particle is 52·7 m.

Example 2 A stone is thrown horizontally from the edge of a cliff 40 m above the sea. Given that the stone travels 60 m horizontally before it hits the water, find the time for which it is in the air and its initial speed. [Take $g = 9{\cdot}8$.]



Let the initial speed of the stone be $u\,\mathrm{m\,s^{-1}}$, then after t seconds,

$$x = ut \quad y = -\tfrac{1}{2}gt^2.$$

Let the time for which the stone is in the air be T seconds, then when $t = T$, $x = 60$ and $y = -40$

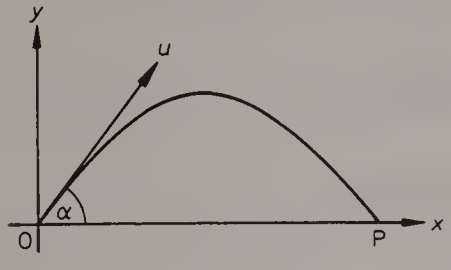
$\therefore \quad 60 = uT$ and $-40 = -\tfrac{1}{2} \times 9{\cdot}8T^2$.

Hence
$$T = \sqrt{\left(\frac{40}{4{\cdot}9}\right)} = \sqrt{\left(\frac{400}{49}\right)} = \frac{20}{7} = 2\frac{6}{7}$$

and
$$u = \frac{60}{T} = 60 \times \frac{7}{20} = 21$$

$\therefore$ the stone is in the air for $2\frac{6}{7}$ seconds and its initial speed is $21\,\mathrm{m\,s^{-1}}$.

If a particle is projected from a point O on a horizontal plane and strikes the plane again at the point P, then the distance OP is called the *range* of the projectile. The time which the particle takes to travel from O to P is called the *time of flight*.



Consider again a particle projected from O with initial velocity u and angle of projection α. Let the range of the particle on the horizontal plane through O be R and the time of flight be T. The horizontal and vertical displacements of the particle at time t are given by

$$x = ut\cos\alpha, \quad y = ut\sin\alpha - \tfrac{1}{2}gt^2$$

When $t = T$, $x = R$ and $y = 0$

$\therefore \quad R = uT\cos\alpha, \quad 0 = uT\sin\alpha - \tfrac{1}{2}gT^2$

Since $T \neq 0$, $u\sin\alpha - \tfrac{1}{2}gT = 0 \qquad \therefore \quad T = 2u\sin\alpha/g$

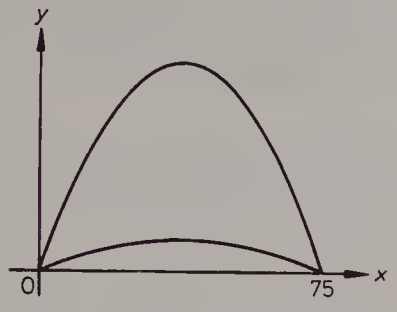
Thus
$$R = u\cos\alpha \cdot \frac{2u\sin\alpha}{g} = \frac{u^2}{g} \cdot 2\sin\alpha\cos\alpha = \frac{u^2\sin 2\alpha}{g}$$

Hence the range of the particle is $u^2\sin 2\alpha/g$ and its time of flight is $2u\sin\alpha/g$.

If u is fixed, but α can be varied, the maximum range of the particle is given by $\sin 2\alpha = 1$, i.e. $\alpha = 45°$.

Hence the particle has a maximum range of u^2/g when projected at 45° to the horizontal.

Example 3 A particle projected from a point O on horizontal ground, with speed $35\,\mathrm{m\,s^{-1}}$ and angle of projection θ, strikes the ground again at a point 75 m from O. Find the two possible values of θ. [Take $g = 9{\cdot}8$.]



Taking O as origin, after t seconds

$$x = 35t\cos\theta$$
$$y = 35t\sin\theta - \tfrac{1}{2}gt^2$$

$\therefore$ $y = 0$ when $35t\sin\theta - \frac{1}{2}gt^2 = 0$
i.e. when $t = 0$ or $t = 70\sin\theta/g$

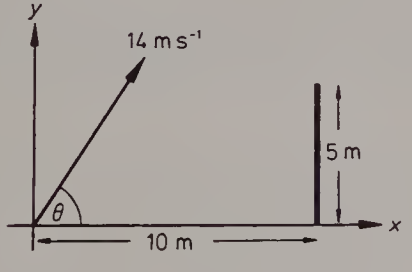
$\therefore$ the time of flight of the particle is $70\sin\theta/g$.
Since the range of the particle is 75 m,

$$75 = 35\cos\theta \cdot \frac{70\sin\theta}{g} = \frac{(35)^2}{9{\cdot}8} \cdot 2\sin\theta\cos\theta$$

Hence
$$\sin 2\theta = 2\sin\theta\cos\theta = \frac{75 \times 9{\cdot}8}{(35)^2} = 0{\cdot}6$$

Since θ is an acute angle, $2\theta = 36{\cdot}87°$ or $143{\cdot}13°$
$\therefore$ the two possible values of θ are $18{\cdot}4°$ and $71{\cdot}6°$.

Example 4 A particle is projected with speed $14\,\mathrm{m\,s^{-1}}$ at an angle θ to the horizontal. Find the values of θ for which the particle just clears an obstacle 5 m high and 10 m from the point of projection. [Take $g = 9{\cdot}8$.]



Taking the point of projection as origin, after t seconds

$$x = 14t\cos\theta$$
$$y = 14t\sin\theta - \tfrac{1}{2}gt^2$$

If the particle just clears the obstacle, then for some value of t, $x = 10$ and $y = 5$

$\therefore$ $10 = 14t\cos\theta$ and $5 = 14t\sin\theta - 4{\cdot}9t^2$.
Eliminating t from these equations,

$$5 = 14 \cdot \frac{10}{14\cos\theta} \cdot \sin\theta - 4{\cdot}9\left(\frac{10}{14\cos\theta}\right)^2$$

$$\Leftrightarrow \quad 5 = 10\tan\theta - \frac{5}{2}\sec^2\theta$$
$$\Leftrightarrow \quad 2 = 4\tan\theta - (1 + \tan^2\theta)$$
$$\Leftrightarrow \quad \tan^2\theta - 4\tan\theta + 3 = 0$$
$$\Leftrightarrow \quad (\tan\theta - 1)(\tan\theta - 3) = 0.$$

Thus either $\tan\theta = 1$ or $\tan\theta = 3$.
Hence the particle just clears the obstacle if $\theta = 45°$ or $\theta = 71{\cdot}6°$.

Exercise 4.3

[In numerical questions take g as $9{\cdot}8\,\mathrm{m\,s^{-2}}$.]

1. A particle is projected from a point on horizontal ground with speed $24{\cdot}5\,\mathrm{m\,s^{-1}}$ at an angle of 30° to the horizontal. Find the time it takes to reach its greatest height. Find also the time of flight and the range of the particle.

2. A particle is projected horizontally with speed $30\,\mathrm{m\,s^{-1}}$ from a point 19·6 m above horizontal ground. Find the horizontal distance it travels before striking the ground and its velocity as it strikes the ground.

3. A particle is projected from a point O on a horizontal plane with speed $35\,\mathrm{m\,s^{-1}}$ at an angle of elevation θ, where $\tan\theta = 2$. Find the maximum height above the plane of the particle and the equation of its trajectory.

4. A golf ball is given initial velocity $30\,\mathrm{m\,s^{-1}}$ at an angle α to the horizontal, where $\tan\alpha = 4/3$. The ball strikes the ground after T seconds at a horizontal distance of 90 m from the point of projection and k m below it. Find the values of T and k. Find also the velocity of the ball when it strikes the ground.

5. A particle is projected from a point O 24·5 m above a horizontal plane. After 5 seconds it hits the plane at a point whose horizontal distance from O is 100 m. Find the horizontal and vertical components of the initial velocity of the particle and the greatest height it reaches above the plane.

6. A boy throws a ball horizontally from a point 4·9 m above horizontal ground. What is the minimum speed at which the ball must be thrown to clear a fence 2·4 m high at a horizontal distance of 8 m from the point of projection? Find the distance beyond the fence of the point at which the ball strikes the ground if projected at this minimum speed.

7. A particle is projected from a point on a horizontal plane with velocity u and angle of projection α. The range of the particle is R and its time of flight T. Show that the particle reaches its greatest height h at time $\frac{1}{2}T$ and that $h = \frac{1}{4}R\tan\alpha$.

8. A gun fires shells with speed $210\,\mathrm{m\,s^{-1}}$. Find the maximum range of the gun over horizontal ground. Find also the smaller angle of elevation at which a shell should be fired to hit a target at a distance of 3·6 km.

9. A particle is projected from a point O on horizontal ground with an initial velocity whose horizontal and vertical components are $3u$ and $5u\,\mathrm{m\,s^{-1}}$ respectively. Find the equation of the trajectory of the particle. Given that it just clears an obstacle 5 m high and 9 m from O, find the value of u and the distance from O of the point at which the particle strikes the ground.

10. A ball thrown with speed $u\,\mathrm{m\,s^{-1}}$ and angle of projection α just clears the top of a fence 3·6 m high and a horizontal distance of 9·6 m from the point of

projection. Find u and α given that as the ball passes over the fence it is moving in a horizontal direction.

11. A projectile with initial speed $u\,\mathrm{m\,s^{-1}}$ has a maximum range of 80 m on a horizontal plane. Find the value of u. Given that the angle of projection is α, find the set of values of α for which the range is greater than 40 m.

12. The horizontal and vertical components of the initial velocity of a projectile are u and $3u$ respectively. Find in terms of u and g (a) the range of the projectile on the horizontal plane through the point of projection, (b) the times at which the particle is moving in a direction at 45° to the horizontal.

13. A particle is projected with speed u and angle of projection α. If after T seconds the velocity of the particle is perpendicular to its initial velocity, find an expression for T in terms of u, α and g. Comment on the significance of your answer in the cases $\alpha = 0°$ and $\alpha = 90°$.

14. A gun is fired from a vehicle moving along a straight horizontal road at speed U. Relative to the vehicle shells are fired with speed V and elevation θ. Find an expression for the range of the gun along the road ahead of the vehicle.

15. A particle is fired with speed V from a point O of a horizontal plane, the angle of elevation being $\arctan(\frac{2}{3})$. The particle strikes the plane at a point A. Find the distance OA. Show that the same point A could have been reached by firing the particle with the same speed from O at an angle of elevation $\arctan(\frac{3}{2})$. Find the difference in the times of flight for the two trajectories. (O&C)

16. A projectile is fired with initial speed V at an angle θ to the horizontal. It passes through two points A and B at the same height h above the point of projection. Find the speed of the projectile at A, and the time it takes to travel from A to B. Given that $AB = 2h$ and $V^2 = 4gh$, find the angle θ. (L)

17. Prove that two particles projected from the same point at the same time, but with different velocities, can never collide while moving freely under gravity.

18. Two particles are projected with the same speed from a point A, and land at B on the same horizontal level as A. If the greatest heights of the particles are in the ratio 4:1, find the angles of projection and the ratio of AB to the maximum range with the same speed of projection.

19. A projectile is fired from a point P with speed $V\,\mathrm{m\,s^{-1}}$ and angle of elevation θ, where $\cos\theta = 0{\cdot}8$. At the same instant a target moving vertically upwards with constant speed $13\,\mathrm{m\,s^{-1}}$ passes through a point Q in the same horizontal plane as P and at a distance of 480 m from P. Given that the projectile hits the target, find the value of V. Find also the speed at which the projectile is travelling when it hits the target.

20. A particle is projected from a point A on horizontal ground such that the horizontal and vertical components of its initial velocity are u_1 and u_2 respectively. At the same instant a particle is projected from a point B at a height h vertically above A. The horizontal and vertical components of its initial velocity are v_1 and v_2. Find the time of flight of the first particle and show that the particles will collide before hitting the ground, provided that $u_1 = v_1$ and $2u_2(u_2 - v_2) > gh$.

4.4 Differentiation of position vectors

In §3.1 it was shown that if a moving particle has position vector $\mathbf{r}$, velocity $\mathbf{v}$ and acceleration $\mathbf{a}$, then

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} \text{ or } \dot{\mathbf{r}} \quad \text{and} \quad \mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} \text{ or } \ddot{\mathbf{r}}.$$

In general, the basic rules for differentiation still hold when vectors are involved.

Thus
$$\frac{d}{dt}\left(\mathbf{r}_1 + \mathbf{r}_2\right) = \frac{d\mathbf{r}_1}{dt} + \frac{d\mathbf{r}_2}{dt},$$

and if $\mathbf{r}$ is of the form $f(t)\mathbf{c}$, where $\mathbf{c}$ is a constant vector,

then
$$\frac{d\mathbf{r}}{dt} = \frac{d}{dt}\left\{f(t)\mathbf{c}\right\} = \frac{df}{dt}\mathbf{c}.$$

Hence if a position vector $\mathbf{r}$ is expressed in terms of the standard set of unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, then

$$\begin{aligned} \mathbf{r} &= x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \\ \mathbf{v} = \dot{\mathbf{r}} &= \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k} \\ \mathbf{a} = \ddot{\mathbf{r}} &= \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k}. \end{aligned}$$

Using these results we can extend some of the earlier work in this chapter to motion in two and three dimensions.

Example 1 A particle moves so that after t seconds its position vector $\mathbf{r}$ is given by $\mathbf{r} = (4t-1)\mathbf{i} + t^2\mathbf{j} + (15t - t^3)\mathbf{k}$. Find vector expressions for its velocity and acceleration at time t. Find also its speed after 3 seconds.

$$\begin{aligned} \mathbf{r} &= (4t-1)\mathbf{i} + t^2\mathbf{j} + (15t - t^3)\mathbf{k} \\ \therefore \quad \mathbf{v} = \dot{\mathbf{r}} &= 4\mathbf{i} + 2t\mathbf{j} + (15 - 3t^2)\mathbf{k} \\ \therefore \quad \mathbf{a} = \ddot{\mathbf{r}} &= 2\mathbf{j} - 6t\mathbf{k}. \end{aligned}$$

Hence the velocity and acceleration at time t are given by

$$\mathbf{v} = 4\mathbf{i} + 2t\mathbf{j} + (15 - 3t^2)\mathbf{k} \quad \text{and} \quad \mathbf{a} = 2\mathbf{j} - 6t\mathbf{k}.$$

When $t = 3$, $\mathbf{v} = 4\mathbf{i}+6\mathbf{j}-12\mathbf{k}$

$\therefore \quad |\mathbf{v}| = \sqrt{\{4^2+6^2+(-12)^2\}} = \sqrt{(16+36+144)} = \sqrt{196} = 14.$

Hence after 3 seconds the speed of the particle is $14\,\mathrm{m\,s^{-1}}$.

Example 2 A particle moves so that at time t its velocity is given by $\mathbf{v} = 10t\mathbf{i}+(3-6t^2)\mathbf{k}$. Given that when $t = 1$ the particle is at the point with position vector $\mathbf{j}+\mathbf{k}$, find an expression for its position vector $\mathbf{r}$ at time t.

$$\mathbf{v} = \dot{\mathbf{r}} = 10t\mathbf{i}+(3-6t^2)\mathbf{k}$$

Integrating $$\mathbf{r} = 5t^2\mathbf{i}+(3t-2t^3)\mathbf{k}+\mathbf{c}$$

where $\mathbf{c}$ is a constant vector.

When $t = 1$, $\mathbf{r} = \mathbf{j}+\mathbf{k}$ $\quad \therefore \quad \mathbf{j}+\mathbf{k} = 5\mathbf{i}+\mathbf{k}+\mathbf{c}$

i.e. $\mathbf{c} = -5\mathbf{i}+\mathbf{j}$

Hence $$\mathbf{r} = 5t^2\mathbf{i}+(3t-2t^3)\mathbf{k}-5\mathbf{i}+\mathbf{j}$$

i.e. $$\mathbf{r} = (5t^2-5)\mathbf{i}+\mathbf{j}+(3t-2t^3)\mathbf{k}.$$

Consider now a particle with initial velocity $\mathbf{u}$ and constant acceleration $\mathbf{a}$.

$$\frac{d\mathbf{v}}{dt} = \mathbf{a}, \text{ where } \mathbf{a} \text{ is constant}$$

Integrating $\mathbf{v} = \mathbf{a}t+\mathbf{c}$, where $\mathbf{c}$ is constant

When $t = 0$, $\mathbf{v} = \mathbf{u}$ $\quad \therefore \quad \mathbf{c} = \mathbf{u}$.

Thus $$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{u}+\mathbf{a}t.$$

Integrating again and taking the initial position of the particle as origin,

$$\mathbf{r} = \mathbf{u}t+\tfrac{1}{2}\mathbf{a}t^2.$$

Hence $\mathbf{v} = \mathbf{u}+\mathbf{a}t$ $\qquad \mathbf{r} = \mathbf{u}t+\tfrac{1}{2}\mathbf{a}t^2$.

Eliminating $\mathbf{a}$, we also have $\mathbf{r} = \tfrac{1}{2}(\mathbf{u}+\mathbf{v})t$.

These results are the vector forms of the constant acceleration formulae derived in §4.2.

If the acceleration due to gravity is denoted by the constant vector $\mathbf{g}$, then by substituting $\mathbf{a} = \mathbf{g}$ we obtain the vector equations governing the motion of a projectile, namely

$$\mathbf{v} = \mathbf{u}+\mathbf{g}t \quad \text{and} \quad \mathbf{r} = \mathbf{u}t+\tfrac{1}{2}\mathbf{g}t^2.$$

Using the notation of §12.3 and horizontal and vertical unit vectors $\mathbf{i}$ and $\mathbf{j}$, let $\mathbf{r} = x\mathbf{i}+y\mathbf{j}$, $\mathbf{v} = \dot{x}\mathbf{i}+\dot{y}\mathbf{j}$, $\mathbf{u} = (u\cos\alpha)\mathbf{i}+(u\sin\alpha)\mathbf{j}$ and $\mathbf{g} = -g\mathbf{j}$, then

$$\dot{x}\mathbf{i}+\dot{y}\mathbf{j} = (u\cos\alpha)\mathbf{i}+(u\sin\alpha-gt)\mathbf{j}$$

and $$x\mathbf{i}+y\mathbf{j} = (ut\cos\alpha)\mathbf{i}+(ut\sin\alpha-\tfrac{1}{2}gt^2)\mathbf{j}.$$

Clearly in this form the vector equations correspond exactly to the results derived in the previous section.

Example 3 A particle is projected from a point O with initial velocity represented by the vector $\mathbf{i}+2\mathbf{j}$. Find vector expressions for its velocity $\mathbf{v}$ and its displacement $\mathbf{r}$ from O at time t. [Take $g = 10$.]

Using the equations $\mathbf{v} = \mathbf{u}+\mathbf{g}t$ and $\mathbf{r} = \mathbf{u}t+\frac{1}{2}\mathbf{g}t^2$,
where $\mathbf{u} = \mathbf{i}+2\mathbf{j}$ and $\mathbf{g} = -10\mathbf{j}$,

$$\mathbf{v} = (\mathbf{i}+2\mathbf{j})-10t\mathbf{j} = \mathbf{i}+(2-10t)\mathbf{j}$$

and

$$\mathbf{r} = (\mathbf{i}+2\mathbf{j})t+\tfrac{1}{2}(-10\mathbf{j})t^2 = t\mathbf{i}+(2t-5t^2)\mathbf{j}.$$

[Note that although vector equations can be used in any projectile problem, in elementary work many students will prefer the methods of §4.3.]

Exercise 4.4

[Use SI units and take $g = 10$.]

1. A particle moves such that its position vector at time t is $\mathbf{r}$. Find vector expressions for its velocity and acceleration at time t given that
(a) $\mathbf{r} = 3t\mathbf{i}-2\mathbf{j}+2t^3\mathbf{k}$, (b) $\mathbf{r} = (t^2+1)\mathbf{i}+t^4\mathbf{j}-6t\mathbf{k}$.

2. A particle moves such that at time t its position vector is $\mathbf{r} = t^2\mathbf{i}+(2-3t)\mathbf{j}+3t^2(t-2)\mathbf{k}$. Find the initial speed of the particle and its speed after 2 seconds.

3. A particle moves such that at time t its velocity is given by $\mathbf{v} = 3\mathbf{i}+4t\mathbf{j}+(1-3t^2)\mathbf{k}$. Given that when $t = 2$ the position vector of the particle is $6(\mathbf{j}-\mathbf{k})$, find an expression for its position vector at time t.

4. A particle moves such that at time t its acceleration is given by $\mathbf{a} = 6t\mathbf{i}-4\mathbf{k}$. Initially the particle has velocity vector $\mathbf{i}+\mathbf{j}$ and position vector $3\mathbf{j}$. Find expressions for its velocity and position vectors at time t.

In questions 5 to 8 the directions of the unit vectors $\mathbf{i}$ and $\mathbf{j}$ are horizontal and vertically upward respectively.

5. A particle is projected from a point O with initial velocity represented by the vector $3\mathbf{i}+2\mathbf{j}$. Find vector expressions for its velocity $\mathbf{v}$ and its displacement $\mathbf{r}$ from O at time t.

6. A particle is projected from a point O on a horizontal plane with initial velocity represented by the vector $40\mathbf{i}+25\mathbf{j}$. Find the greatest height reached by the particle, its range and time of flight.

7. A particle is projected from a point O with speed $20\,\mathrm{m\,s^{-1}}$ at an angle of $60°$ to the horizontal. Find vector expressions for its velocity $\mathbf{v}$ and its displacement $\mathbf{r}$ from O at time t.

8. A stone is thrown from the top of a cliff with initial velocity vector $8\mathbf{i}-\mathbf{j}$. If the stone hits the sea below after 3 seconds, find the height of the cliff and the horizontal distance travelled by the stone. Find also the speed at which the stone is travelling when it hits the water.

Exercise 4.5 (miscellaneous)

[In numerical questions take g as $10\,\mathrm{m\,s^{-2}}$.]

1. A particle is moving in a straight line such that its displacement from a fixed point after t seconds is s metres, where $s = 2t^3 - 13t^2 + 20t$. Find the total distance travelled and the maximum speed attained by the particle in the first 4 seconds of its motion. Find also the range of values of t for which the acceleration of the particle is negative.

2. A particle moving in a straight line starts from rest at a point A. Its acceleration at time t is $(45 + 12t - 9t^2)\,\mathrm{m\,s^{-2}}$. If the particle comes to rest instantaneously at a point B, find the distance AB and the time the particle takes to reach B. Find also the maximum acceleration and the maximum speed attained in that time.

3. Two particles A and B move along a straight line starting from a point O such that at time t their displacements from O are s_1 and s_2 respectively. If $s_1 = 5t^2(t+7)$ and $s_2 = t^3(t+3)$, find (a) the time at which the particles again meet, (b) an expression for the velocity of A relative to B, (c) the range of values of t for which A is travelling at a greater speed than B.

4. Due to track repairs a train retards uniformly, with retardation $1\,\mathrm{m\,s^{-2}}$, from a speed of $40\,\mathrm{m\,s^{-1}}$ at A to a speed of $10\,\mathrm{m\,s^{-1}}$ at B. The train travels from B to C, a distance of 3·5 km, at a constant speed of $10\,\mathrm{m\,s^{-1}}$ and then accelerates uniformly, with acceleration $0{\cdot}2\,\mathrm{m\,s^{-2}}$ so that its speed at D is $40\,\mathrm{m\,s^{-1}}$. Sketch the velocity-time graph for the journey from A to D, and show that the distance from A to D is 8 km. Show that the journey from A to D takes 330 s more than it would if the train travelled at a constant speed of $40\,\mathrm{m\,s^{-1}}$ from A to D. (C)

5. Two trains, P and Q, travel by the same route from rest at station A to rest at station B. Train P has constant acceleration f for the first third of the time, constant speed for the second third and constant retardation f for the last third of the time. Train Q has constant acceleration f for the first third of the distance, constant speed for the second third and constant retardation f for the last third of the distance. Show that the times taken by the two trains are in the ratio $3\sqrt{3}:5$. (L)

6. A bus starts from rest and moves along a straight road with constant acceleration f until its speed is V; it then continues at constant speed V. When the bus starts, a car is at a distance b behind the bus and is moving in the same direction with constant speed u. Find the distance of the car behind the bus at time t after the bus has started (i) for $0 < t < V/f$, (ii) for $t > V/f$. Show that the car cannot overtake the bus during the period $0 < t < V/f$ unless $u^2 > 2fb$. Find the least distance between the car and the bus in the case when $u^2 < 2fb$ and $u < V$. State briefly what will happen if $u^2 < 2fb$ and $u > V$. (JMB)

7. A man descending in a lift at constant speed $V\,\mathrm{m\,s^{-1}}$ throws a ball vertically upwards with speed $U\,\mathrm{m\,s^{-1}}$ relative to the lift. Find the time that elapses before the ball (a) has zero velocity relative to the lift, (b) returns to the man's hand.

8. A batsman hits a cricket ball with initial speed 25 m/s at an elevation of 50°. Find the greatest height attained, the range on level ground and the time of flight. Show that it will clear the roof of a pavilion 10 m high which is at a distance of 50 m from the batsman. (L)

9. A particle P, projected from a point A on horizontal ground, moves freely under gravity and hits the ground again at B. Referred to A as origin, AB as x-axis and the upward vertical at A as y-axis, the equation of the path of P is $y = x - x^2/40$, where x and y are measured in metres. Calculate (i) the distance AB, (ii) the greatest height above AB attained by P, (iii) the magnitude and the direction of the velocity of P at A, (iv) the time taken by P to reach B from A. Calculate the coordinates of the points on the path of P at the two instants when the speed of P is 15 m/s. (AEB 1978)

10. A particle P is projected from a point A on a horizontal plane with speed V m/s at an angle of elevation α. The particle strikes the horizontal plane again at B, where $AB = 160$ metres. Show that $V^2 \sin 2\alpha = 1600$. A second particle Q is projected from A with the same speed V m/s at an angle of elevation $\alpha - 30°$ and strikes the horizontal plane at C, where C is the mid-point of AB. Find the values of V and α and show that the time of flight of Q is $8 \sin 15°$ seconds. Given that P and Q are projected from A simultaneously, find, at the instant when Q reaches C, the angle between the tangent to the path of P and the horizontal. (AEB 1978)

11. A particle is projected from a point O with a speed of $10\,\mathrm{m\,s^{-1}}$ and angle of elevation θ. Given that the projectile just clears an obstacle of height h at a horizontal distance D from O, express in the form of a quadratic equation in $\tan\theta$ the relationship connecting θ, h and D. Show that provided that $D^2 + 20h < 100$, there are two distinct possible values of θ. Show further that if $h = 0$ the sum of these two values is 90°.

12. A point O lies on a horizontal plane, and the point A is at a height h vertically above O. A particle is projected from A with speed V at an angle α above the horizontal. Taking O as the origin and Oy vertically upwards, show that the equation of the path of the particle can be written in the form $y = h - \dfrac{gx^2}{2V^2} + x\tan\alpha - \dfrac{gx^2}{2V^2}\tan^2\alpha$. The particle hits the plane at the point $B(r, 0)$. In the case when $V^2 = gh$ derive a quadratic equation for $\tan\alpha$ in terms of h and r, and show that $r \leqslant \sqrt{3}h$. For the same value of V show that $r = \sqrt{3}h$ when $\alpha = 30°$. (JMB)

13. A particle is fired from the origin O with speed V at an angle of inclination α above the x-axis, which is horizontal. At the instant when the particle is at the highest point of its path, a second particle is fired from O with speed U at an angle of inclination β above the horizontal so that the particles collide at a point on the x-axis. Show that $\tan\beta = \frac{1}{4}\tan\alpha$. Show also that, if $\tan\alpha = 3$, then $8U^2 = 5V^2$.

14. A and B are points distant a apart on horizontal ground. A ball is thrown from A towards B with velocity u at an angle 2α ($< 90°$) to the horizontal, and simultaneously a second ball is thrown from B towards A with velocity v at an angle α to the horizontal. If the balls collide, show that $v = 2u\cos\alpha$ and hence that they collide after a time $t = a/u(2\cos 2\alpha + 1)$. Show also that (i) the two trajectories would have the same maximum height above AB, (ii) the range of the ball from B would always exceed the range of the ball from A. (W)

15. A particle P is set in motion at a point A and then moves such that at time t its position vector is $\mathbf{r} = 3(t-2)^2\mathbf{i} + 2\mathbf{j} - 4t(t-4)\mathbf{k}$. By considering the vector $\mathbf{AP}$, show that the particle moves in a straight line and find a unit vector parallel to the direction of motion. Find also the time that has elapsed and the position vector of P when it comes instantaneously to rest.

16. [In this question the directions of the unit vectors $\mathbf{i}$ and $\mathbf{j}$ are horizontal and vertically upwards.] A particle projected from the origin O with initial velocity represented by the vector $u\mathbf{i} + v\mathbf{j}$ passes through the points A and B with position vectors $20(\mathbf{i}+\mathbf{j})$ and $25\mathbf{i}$ respectively. Find the values of u and v. Show that a particle projected from O with velocity vector $v\mathbf{i} + u\mathbf{j}$ also passes through B.

5 Newton's laws of motion

5.1 Newton's laws

The theory of elementary dynamics is based on three laws first stated by Newton† in the seventeenth century. Although slight discrepancies can be detected when comparing the predictions of Newtonian mechanics with the true motion of atomic particles travelling at high speed, and with certain astronomical observations (when results based on the theory of relativity are more reliable), Newton's laws can be used to describe the behaviour of moving objects with great accuracy.

Newton's first law: Every body continues in its state of rest or of uniform motion in a straight line unless compelled to change that state by external forces.

This law implies that a force changes the velocity of the body it acts upon. However, if two or more forces act on a body there will be no change in velocity unless the forces have a non-zero resultant. Newton's first law also implies that no force is required to maintain uniform motion, but practical experience tells us that the thrust of an engine is needed to drive a car along a horizontal road with uniform velocity. The explanation of this apparent contradiction is that the resultant force on the car is zero, the thrust of the engine exactly balancing the various forces opposing motion.

Newton's second law: The rate of change of momentum of a body is proportional to the applied force and has the direction of that force.

† *Newton, Sir Isaac* (1642–1727) English scientist and mathematician. He invented, a little before the German mathematician Leibnitz, differential calculus (1665) and integral calculus (1666). He did significant work on the nature of light and the construction of telescopes. However, Newton is chiefly remembered for his statement of the laws of dynamics and his theory of gravitation, which are explained in *Philosophiae Naturalis Principia Mathematica* (1687).

The *momentum* of a body is defined as the product of its mass and its velocity, i.e. the momentum of a particle of mass m moving with velocity $\mathbf{v}$ is the vector quantity $m\mathbf{v}$. Thus for a particle of constant mass m,

$$\text{rate of change of momentum} = \frac{d}{dt}(m\mathbf{v}) = m\frac{d\mathbf{v}}{dt} = m\mathbf{a},$$

where $\mathbf{a}$ is the acceleration of the particle.

Hence Newton's second law states that, for a body of constant mass m, the quantity $m\mathbf{a}$ is proportional to the force $\mathbf{F}$ acting on the body, i.e. $\mathbf{F} = km\mathbf{a}$, where k is constant. The value of k depends on the units of measurement chosen for $\mathbf{F}$, m and $\mathbf{a}$. A newton is defined as the force which gives a mass of 1 kg an acceleration of $1\,\mathrm{m\,s^{-2}}$. Hence when using SI units, $k = 1$ and we obtain the fundamental equation of motion for a body of constant mass

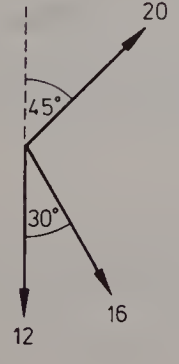
$$\boxed{\mathbf{F} = m\mathbf{a}.}$$

Example 1 A particle of mass 5 kg is moving with an acceleration of $3\,\mathrm{m\,s^{-2}}$. Find the magnitude of the resultant force acting on the particle.

If the resultant force on the particle is F newtons, by Newton's second law, $F = 5 \times 3$.
Hence there is a force of 15 N acting on the particle.

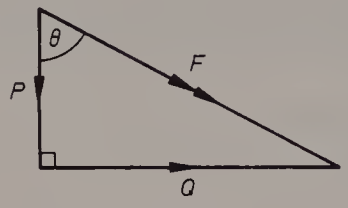
In Chapter 2 various ways of finding the resultant of a set of forces were discussed. Any of these methods, together with Newton's second law, can be used when finding the acceleration of a body moving under the action of two or more forces.

Example 2 A body of mass 8 kg on a smooth horizontal plane is acted upon by horizontal forces of magnitudes 20, 12 and 16 newtons in the directions N 45° E, due south and S 30° E respectively. Find the magnitude and direction of the acceleration of the body.



Let $\mathbf{F}$ be the resultant force on the body acting at an angle θ east of south and let $\mathbf{a}$ be the acceleration produced. If $\mathbf{P}$ and $\mathbf{Q}$ are the components of $\mathbf{F}$ in the directions south and east respectively, then resolving in these directions

$$\downarrow P = 12 + 16\cos 30° - 20\cos 45°$$
$$\rightarrow Q = 16\sin 30° + 20\sin 45°.$$



[By calculator $P = 11{\cdot}71426$, $Q = 22{\cdot}14214$.]

Using Newton's second law, $\mathbf{F} = 8\mathbf{a}$

$$\therefore \quad a = \frac{1}{8}F = \frac{1}{8}\sqrt{(P^2+Q^2)} \approx 3{\cdot}13$$

and $\tan\theta = Q/P$, $\theta \approx 62{\cdot}1°$.

Hence the body moves with an acceleration of $3{\cdot}13\,\mathrm{m\,s^{-2}}$ in the direction S $62{\cdot}1°$ E.

When a body is moving in a straight line under the action of constant forces then its acceleration is constant. Hence the motion is governed by the constant acceleration equations derived in §4.2.

Example 3 A particle of mass 2 kg starting from rest is acted on by a force of 4 N for 5 seconds. Find its acceleration and the speed attained. The particle is then brought to rest in a further 8 seconds by a constant resistance of R N. Find the value of R and the total distance travelled by the particle.

Let $a_1\,\mathrm{m\,s^{-2}}$ and $a_2\,\mathrm{m\,s^{-2}}$ be the accelerations of the particle in the two parts of its motion and let $V\,\mathrm{m\,s^{-1}}$ be the speed attained after 5 seconds.

By Newton's 2nd law, $\quad 4 = 2a_1 \quad \therefore \quad a_1 = 2.$

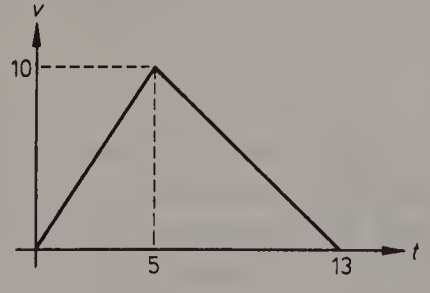
Using the formula $v = u + at$, substituting $u = 0$, $v = V$, $a = 2$, $t = 5$:
$V = 0 + 2 \times 5 = 10.$

Hence the initial acceleration of the particle is $2\,\mathrm{m\,s^{-2}}$ and the speed attained is $10\,\mathrm{m\,s^{-1}}$.

Using the formula $v = u + at$, substituting $u = 10$, $v = 0$, $a = a_2$, $t = 8$
$0 = 10 + 8a_2 \quad \therefore \quad a_2 = -1{\cdot}25.$

By Newton's 2nd law $\quad R = 2 \times 1{\cdot}25 = 2{\cdot}5.$

Hence the resistance is of magnitude 2·5 N.



The total distance travelled is represented by the area under the velocity-time graph for the motion

$\therefore$ total distance travelled

$= \frac{1}{2} \times 10 \times 13\,\mathrm{m} = 65\,\mathrm{m}.$

Newton's third law: To every action there is an equal and opposite reaction.

This means that if a body A exerts a force on a body B, then B exerts an equal and opposite force on A. For instance, consider a body of weight W. The gravitational pull of the earth on the body is W, but Newton's third law tells us that the body also exerts a force of magnitude W on the earth. Similarly, when a body is in contact with a smooth surface, the surface exerts a force called the normal reaction on the body. As stated in Newton's third law, the body exerts an equal and opposite force on the surface. This law will be particularly useful when examining systems involving two or more objects in contact.

Exercise 5.1

[Vector quantities are measured in SI units.]

1. A particle of mass 12 kg is moving with an acceleration of $4\,\mathrm{m\,s^{-2}}$. Find the magnitude of the resultant force on the particle.

2. Find the acceleration of a mass of 45 kg acted upon by a single force of 9 N.

3. A resultant force of 36 N acting on a particle produces an acceleration of $3\,\mathrm{m\,s^{-2}}$. Find the mass of the particle.

4. A particle of mass 8 kg is pulled along a smooth horizontal plane by a horizontal string. Find the tension in the string when the acceleration of the particle is $5\,\mathrm{m\,s^{-2}}$.

5. A particle of mass 15 kg is acted upon by a resultant force represented by the vector $30\mathbf{i} - 75\mathbf{j}$. Find, in vector form, the acceleration of the particle.

6. A particle of mass 4 kg is moving with acceleration vector $3\mathbf{i} + \mathbf{j}$. Find a vector representing the resultant force on the particle.

7. Forces of magnitude 5 N and 9 N act at an angle of 50° on a particle of mass 2 kg. Find the magnitude and direction of the acceleration produced.

8. Two forces of magnitude 16 N and 24 N act on a particle of mass 5 kg. Given that an acceleration of $6\,\mathrm{m\,s^{-2}}$ is produced, find the angle between the lines of action of the two forces.

9. Forces represented by the vectors $3\mathbf{i} + \mathbf{j}$, $-5\mathbf{i} + 2\mathbf{j}$, $4\mathbf{i} - 5\mathbf{j}$ and $2\mathbf{i} - \mathbf{j}$ act on a particle of mass 0·8 kg moving on a smooth horizontal plane. Given that $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the directions due east and due north respectively, find the magnitude and direction of the acceleration of the particle.

10. A body of mass 2 kg on a smooth horizontal plane is acted upon by horizontal forces of magnitudes 9, 4 and 7 newtons in the directions due west, N 20° W and S 75° E respectively. Find the magnitude and direction of the acceleration of the body.

11. A train of mass 300 000 kg travelling at $0{\cdot}3\,\mathrm{m\,s^{-1}}$ is brought to rest at a station by buffers exerting an average force of 50 000 N. Find the time taken to bring the train to rest.

12. A particle of mass 20 kg is brought to rest in a distance of 600 m from a speed of $30\,\mathrm{m\,s^{-1}}$ by a constant retarding force. Find the magnitude of this force.

13. A body is moving in a straight line with a speed of $40\,\mathrm{m\,s^{-1}}$. A retarding force of 25 N reduces its speed to $15\,\mathrm{m\,s^{-1}}$ in 10 seconds. Find the mass of the body.

14. A car of mass 800 kg is moving along a straight horizontal road under the action of a constant resultant force of 1000 N. Find its speed when it has moved 40 m from rest.

15. A particle of mass 5 kg starting from rest is acted on by a force of 12 N for 10 seconds. The particle continues to move with constant velocity for a further 10 seconds. It is then brought to rest by a constant resistance of R N in a further time of 15 seconds. Find the value of R and the total distance travelled.

16. A particle of mass 4 kg starts from rest and moves under the action of a constant force of F newtons for 12 seconds. It is then brought to rest in a further 8 seconds by a constant retarding force of R newtons. If the total distance travelled is 200 m, find the maximum speed attained and the values of F and R.

5.2 Further applications

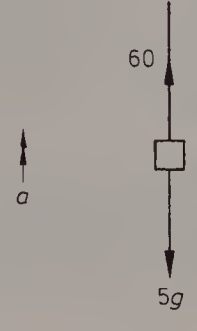
As stated in §4.2, an object falling freely under gravity has a constant downward acceleration denoted by g. Thus if a body of mass m kg has a weight of W N, then by Newton's second law,

$$W = mg.$$

Hence the force due to the gravitational pull of the earth on a body of mass m kg is of magnitude mg N.

[In this and later chapters the value of g will be taken as $9{\cdot}8\,\mathrm{m\,s^{-2}}$ unless otherwise stated.]

Example 1 A package of mass 5 kg is being lifted by means of a vertical cable. If the tension in the cable is 60 N, find the acceleration of the package.



Let the acceleration of the package be $a\,\mathrm{m\,s^{-2}}$ vertically upwards.
Applying Newton's 2nd law vertically

$$\uparrow 60 - 5g = 5a$$

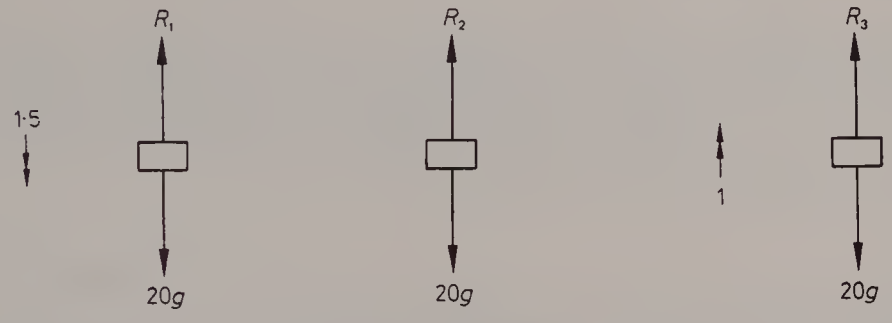
$$\therefore \quad a = 12 - g = 12 - 9{\cdot}8 = 2{\cdot}2.$$

Hence the acceleration of the package is $2{\cdot}2\,\mathrm{m\,s^{-2}}$.

[Note that to avoid confusion in diagrams showing both forces and accelerations, the forces are denoted by single-headed arrows and the accelerations by double-headed arrows.]

Example 2 A lift descends with an acceleration of $1{\cdot}5\,\mathrm{m\,s^{-2}}$, then moves with constant speed until it is retarded at $1\,\mathrm{m\,s^{-2}}$. A suitcase of mass 20 kg stands on the floor of the lift during the journey. Find the magnitude of the force it exerts on the floor of the lift at each stage of the journey.

Let R_1, R_2 and R_3 newtons be the reactions of the floor on the suitcase.



Applying Newton's 2nd law vertically in each case

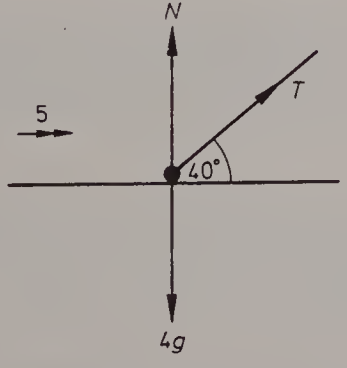
$\downarrow 20g - R_1 = 20 \times 1{\cdot}5 \quad \therefore \quad R_1 = 20g - 30 = 196 - 30 = 166$

$\uparrow R_2 - 20g = 0 \quad \therefore \quad R_2 = 20g = 196$

$\uparrow R_3 - 20g = 20 \times 1 \quad \therefore \quad R_3 = 20g + 20 = 196 + 20 = 216.$

By Newton's third law, the force exerted by the suitcase on the floor is equal in magnitude to the reaction of the floor on the suitcase. Hence the forces exerted by the suitcase on the floor of the lift are 166, 196 then 216 newtons.

Example 3 A particle of mass 4 kg is being pulled across a smooth horizontal plane by a string inclined at 40° to the horizontal. If the acceleration of the particle is $5\,\mathrm{m\,s^{-2}}$, find the tension in the string and the reaction between the particle and the plane.



Applying Newton's 2nd law horizontally and vertically

$\rightarrow T\cos 40° = 4 \times 5$

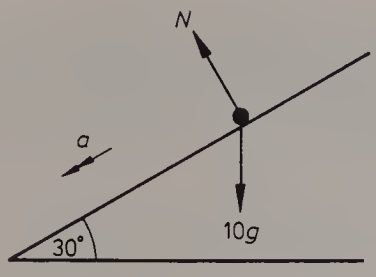
$\therefore \quad T = 20/\cos 40° \approx 26{\cdot}1$

$\uparrow N + T\sin 40° - 4g = 0$

$\therefore \quad N = 4 \times 9{\cdot}8 - 20\tan 40° \approx 22{\cdot}4.$

Hence the tension in the string is 26·1 N and the reaction between the particle and the plane is 22·4 N.

Example 4 A particle of mass 10 kg slides down a smooth plane inclined at 30° to the horizontal. If the particle starts from rest, find its speed after sliding a distance of 5 m.



Let the acceleration of the particle down the plane be $a\,\mathrm{m\,s^{-2}}$.
Applying Newton's 2nd law down the plane

$$\swarrow \quad 10g\sin 30° = 10a$$
$$\therefore \quad a = g\sin 30° = 4{\cdot}9$$

Using the formula $v^2 = u^2 + 2as$, where $u = 0$, $s = 5$, $a = 4{\cdot}9$,

$$v^2 = 0 + 2 \times 4{\cdot}9 \times 5 = 49 \qquad \therefore \quad v = 7$$

Hence the particle attains a speed of $7\,\mathrm{m\,s^{-1}}$.

Exercise 5.2

1. Find the weight in newtons of an object with mass
(a) 10 kg, (b) 1·5 kg, (c) 250 grammes.

2. A particle of mass 2 kg is being lifted by a vertical force of 32 N. Find its acceleration.

3. A package of mass 8 kg is being lowered by means of a vertical cable with a downward acceleration of $2\,\mathrm{m\,s^{-2}}$. Find the tension in the cable.

4. A boy of mass 40 kg is ascending in a lift. Find the force exerted by the floor of the lift on the boy when the lift is (a) accelerating at $1{\cdot}2\,\mathrm{m\,s^{-2}}$, (b) moving with constant speed, (c) decelerating at $0{\cdot}8\,\mathrm{m\,s^{-2}}$.

5. A man descends in a lift carrying a parcel of mass 3 kg. Describe the motion of the lift when the parcel has an apparent weight of (a) 33 N, (b) 27 N, (c) 29·4 N.

6. A box of mass 30 kg is being towed across a smooth horizontal surface by means of a light chain inclined at an angle of 25° to the horizontal. If the box is accelerating at $1\,\mathrm{m\,s^{-2}}$, find the tension in the chain and the reaction between the box and the surface.

7. A particle of mass 6 kg is being pushed across a smooth horizontal plane by a force of 30 N acting downwards at 20° to the vertical. Find the acceleration of the particle and the force the particle exerts on the plane.

8. A particle of mass 0·5 kg is suspended from a string. If the acceleration of the particle is horizontal and of magnitude $2\,\mathrm{m\,s^{-2}}$, find the tension in the string and its angle of inclination to the vertical.

9. A particle of mass 5 kg slides down a smooth plane inclined at an angle of 60° to the horizontal. Find the acceleration of the particle down the plane and the reaction between the particle and the plane.

10. A body of mass 12 kg slides down a smooth plane inclined at an angle of 20° to the horizontal. If it starts from rest, find the time it takes to slide a distance of 10 m.

11. A particle of mass 2 kg is being pulled up a smooth plane inclined at 25° to the horizontal by a string parallel to the plane. Find the tension in the string if the particle is (a) moving with uniform velocity, (b) accelerating at $3\,\mathrm{m\,s^{-2}}$.

12. An object of mass 4 kg is being pushed up a smooth plane inclined at an angle α to the horizontal by a horizontal force of 40 N. If $\sin\alpha = 0{\cdot}6$, find the acceleration of the particle and the reaction between the particle and the plane.

5.3 Connected particles

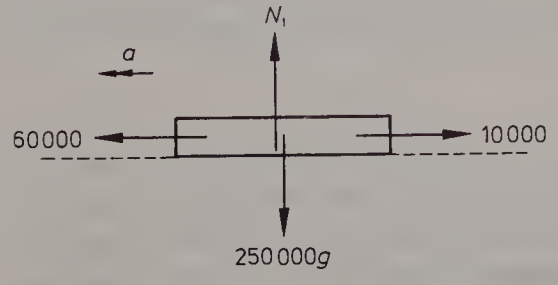
When considering a set of objects which are connected in some way, it is necessary to apply Newton's third law. Any force exerted by one of the objects upon another is matched by an equal and opposite reaction. For instance, in the case of a railway engine pulling a truck, the truck exerts a pull on the engine equal in magnitude to the pull of the engine on the truck. In problems involving such systems it is usually helpful to examine the forces on one or more of the objects separately.

Example 1 An engine of mass 50 tonnes is pulling several trucks of total mass 200 tonnes along a horizontal track. The resistance to the motion of the engine is 60 N per tonne and the resistance to the motion of the trucks is 35 N per tonne. If the tractive force exerted by the engine is 60 kN, find (a) the acceleration, (b) the tension in the coupling between the engine and the trucks.

[1 tonne = 1000 kg, 1 kN = 1000 N.]

Let $a\,\mathrm{m\,s^{-2}}$ be the acceleration and let T N be the tension in the coupling.

(a) Considering the forces on the whole train, the total resistance to motion is $(50 \times 60) + (200 \times 35)$ N, i.e. 10 000 N.



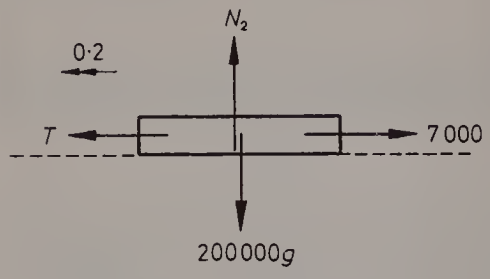
Applying Newton's 2nd law

$$\leftarrow 60\,000 - 10\,000 = 250\,000a$$

$$\therefore \quad a = \frac{50\,000}{250\,000} = 0{\cdot}2.$$

Hence the acceleration of the train is $0{\cdot}2\,\mathrm{m\,s^{-2}}$.

(b) Considering the forces on the trucks, the resistance to motion is (200×35) N, i.e. 7000 N.



Applying Newton's 2nd law

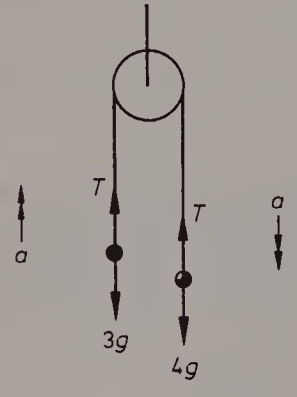
$\leftarrow \quad T - 7000 = 200\,000 \times 0{\cdot}2$

$\therefore \quad T = 40\,000 + 7000 = 47\,000.$

Hence the tension in the coupling is 47 kN.

Each of the remaining examples in this section involves two particles connected by a light inextensible string passing over a smooth fixed light pulley. Under these conditions we can assume that the tension in the string is constant throughout its length.

Example 2 Two particles of mass 3 kg and 4 kg are connected by a light inextensible string passing over a smooth fixed pulley. The particles are released from rest with the strings taut and vertical. Find the acceleration of the particles and the tension in the string.



Let the acceleration be $a\,\mathrm{m\,s^{-2}}$ and the tension T N.
Using Newton's 2nd law

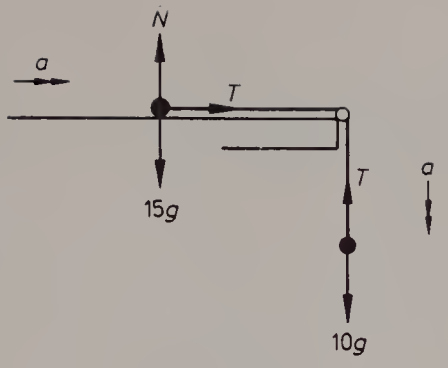
for the 3 kg mass: $T - 3g = 3a$ (1)
for the 4 kg mass: $4g - T = 4a$ (2)

Adding (1) to (2) $\quad g = 7a \quad \therefore \quad a = \frac{1}{7}g = 1{\cdot}4$

Substituting in (1) $\quad T - 3g = \frac{3}{7}g \quad \therefore \quad T = 33{\cdot}6.$

Hence the acceleration of the system is $1{\cdot}4\,\mathrm{m\,s^{-2}}$ and the tension in the string is 33·6 N.

Example 3 A particle of mass 15 kg rests on a smooth horizontal table. It is connected by a light inextensible string passing over a smooth pulley fixed at the edge of the table to a particle of mass 10 kg which hangs freely. Find the acceleration of the system when it is released from rest. Find also the force exerted by the string on the pulley.



Let $a\,\mathrm{m\,s^{-2}}$ be the acceleration and let $T\,\mathrm{N}$ be the tension in the string. Let $F\,\mathrm{N}$ be the force exerted by the string on the pulley.

Using Newton's 2nd law,

horizontally for the 15 kg mass: $T = 15a$ (1)

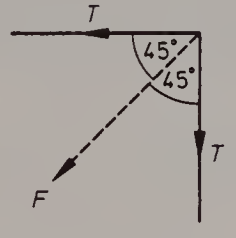
vertically for the 10 kg mass: $10g - T = 10a$ (2)

Adding (1) to (2), $10g = 25a \quad \therefore \quad a = \frac{2}{5}g = 3{\cdot}92.$

Hence the acceleration of the system is $3{\cdot}92\,\mathrm{m\,s^{-2}}$.

Substituting in (1) $T = 15.\frac{2}{5}g = 6g.$

The string exerts on the pulley a force T horizontally and a force T vertically.



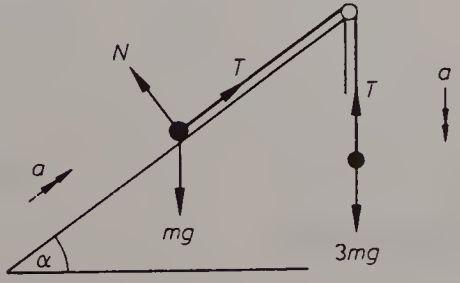
Since these forces are equal in magnitude, their resultant F acts along the bisector of the angle between the strings.

Resolving along this bisector:

$F = 2T\cos 45° = 12g\cos 45° \approx 83{\cdot}2$

Hence the string exerts a force on the pulley of 83·2 N at 45° to the horizontal.

Example 4 A particle of mass m rests on a smooth plane inclined at an angle α to the horizontal where $\sin\alpha = \frac{3}{5}$. It is connected by a light inextensible string passing over a smooth pulley fixed at the top of the plane to a particle of mass 3 m which hangs freely. Find the acceleration of the system when it is released from rest and the tension in the string.



Let a be the acceleration and let T be the tension in the string.

Using Newton's 2nd law,

parallel to the plane for the mass m: $T - mg\sin\alpha = ma$ (1)
vertically for the mass $3m$: $3mg - T = 3ma$ (2)

Adding (1) to (2) $3mg - mg\sin\alpha = 4ma$

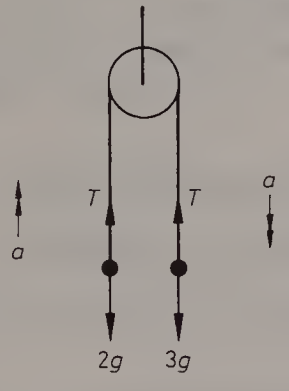
$$\therefore \quad a = \frac{1}{4}g\left(3 - \frac{3}{5}\right) = \frac{3}{5}g.$$

Substituting in (2) $3mg - T = 3m.\frac{3}{5}g$

$$\therefore \quad T = 3mg\left(1 - \frac{3}{5}\right) = \frac{6}{5}mg.$$

Hence the acceleration of the system is $\frac{3}{5}g$ and the tension in the string is $\frac{6}{5}mg$.

Example 5 Two particles of mass 2 kg and 3 kg are connected by a light inextensible string passing over a fixed smooth pulley. Initially the system is at rest with the strings taut and vertical with both particles at a height of 2 m above the ground. When the system is released, find the time which elapses before the 3 kg mass hits the ground and the maximum height reached by the 2 kg mass.



Let $a\,\mathrm{m\,s^{-2}}$ be the acceleration and let $T\,\mathrm{N}$ be the tension in the string.
Using Newton's 2nd law,

for the 2 kg mass: $T - 2g = 2a$ (1)
for the 3 kg mass: $3g - T = 3a$ (2)

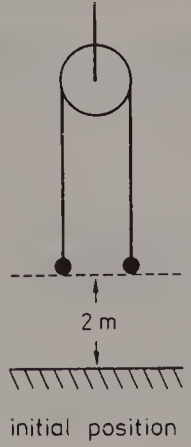
(1)+(2) $g = 5a \qquad \therefore \quad a = \frac{1}{5}g.$

Using the formula $s = ut + \frac{1}{2}at^2$

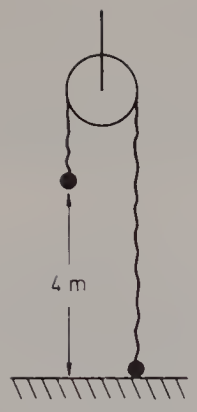
where $s = 2, u = 0, a = \frac{1}{5}g$

$$2 = 0 + \frac{1}{2}.\frac{1}{5}gt^2$$

$$\therefore \quad t = \sqrt{\left(\frac{20}{g}\right)} = \sqrt{\left(\frac{20}{9{\cdot}8}\right)} = \sqrt{\left(\frac{100}{49}\right)} = \frac{10}{7}.$$



Hence the 3 kg mass hits the ground after $1\frac{3}{7}$ seconds.



position when 3 kg mass hits the ground

Using the formula $v^2 = u^2 + 2as$,

where $u = 0, a = \frac{1}{5}g, s = 2,$

$$v^2 = 0 + 2.\frac{1}{5}g.2 = \frac{4}{5}g.$$

Hence when the 3 kg mass hits the ground both particles are travelling at $\sqrt{(4g/5)}\,\mathrm{m\,s^{-1}}$. Since the string is no longer taut, the 2 kg mass now moves freely under gravity with a downward acceleration of $g\,\mathrm{m\,s^{-2}}$.

Using the formula $v^2 = u^2 + 2as$, where $u = \sqrt{(4g/5)}, v = 0, a = -g$:

$$0 = \frac{4}{5}g - 2gs \qquad \therefore \quad s = \frac{2}{5}.$$

Hence the 2 kg mass rises a further 0·4 m, reaching a maximum height above the ground of 4·4 m.

Exercise 5.3

1. A motor-boat of mass 1000 kg is towing a water-skier of mass 75 kg by means of a horizontal cable. The resistance to the motion of the boat is 650 N and the resistance to the motion of the skier is 125 N. If the thrust of the boat's engine is 4000 N, find (a) the acceleration, (b) the tension in the cable.

2. A car of mass 800 kg tows a trailer of mass 400 kg. The resistance to motion for both car and trailer is 0·2 N per kg. Find the tractive force exerted by the car engine and the tension in the coupling between the car and the trailer when they are travelling (a) with uniform velocity, (b) with an acceleration of $2\,\mathrm{m\,s^{-2}}$.

3. A railway engine of mass 60 tonnes is pulling two trucks each of mass 12 tonnes along a horizontal track. The resistance to motion is 80 N per tonne for the engine and 50 N per tonne for the trucks. Given that the train is travelling at constant speed, find the tractive force exerted by the engine, the tension in the coupling between the engine and the first truck and the tension in the coupling between the two trucks. If the tractive force is increased to 27 kN, find the acceleration and the new tensions in the couplings.

4. Two particles of mass 2 kg and 5 kg are connected by a light inextensible string passing over a smooth fixed pulley. If the system is moving freely with the strings taut and vertical, find the acceleration of the particles and the tension in the string.

5. Repeat question 4 for particles of mass 9 kg and 3 kg.

6. Repeat question 4 for particles of mass m and M, where $m < M$.

7. Two particles of mass $2m$ and $3m$ are connected by a light inextensible string passing over a smooth fixed pulley. If the system is moving freely with the strings vertical, find the force exerted by the string on the pulley.

8. A particle of mass 3·6 kg rests on a smooth horizontal table. It is connected by a light inextensible string passing over a smooth pulley fixed at the edge of the table to a particle of mass 1·2 kg which hangs freely. Find the acceleration of the system when it is released from rest and the tension in the string.

9. A particle of mass m rests on a smooth horizontal table. It is connected by a light inextensible string passing over a smooth pulley fixed at the edge of the table to a second particle of mass m which hangs freely. If the system is released from rest find the distance travelled in the first 0·4 seconds of the subsequent motion. Find also the force exerted by the string on the pulley.

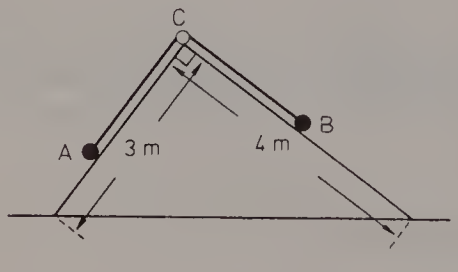
10. A particle of mass 2 kg rests on a smooth plane inclined at an angle of 30° to the horizontal. It is connected by a light inextensible string passing over a smooth pulley fixed at the top of the plane to a particle of mass 3 kg which hangs freely. Find the acceleration of the system when it is released from rest and the tension in the string. Find also the force exerted by the string on the pulley.

11. A particle of mass m rests on a smooth plane inclined at an angle α to the horizontal. It is connected by a light inextensible string passing over a smooth pulley fixed at the top of the plane to a particle of mass M which hangs freely. Find the condition that the mass m should slide down the plane when the system is released from rest.

12. Two particles of mass $3m$ and $5m$ are connected by a light inextensible string passing over a fixed smooth pulley. The system is released from rest with the strings taut and vertical. After 2 seconds the $5m$ mass hits the ground. Find the further time which elapses before the $3m$ mass reaches its greatest height.

13. Two particles of mass 0·5 kg and 0·7 kg are connected by a light inextensible string passing over a fixed smooth pulley. Initially both parts of the string are taut and vertical, and the 0·5 kg mass is moving vertically downwards with a speed of 3·5 m s^{-1}. Find the distance it travels before coming instantaneously to rest.

14.



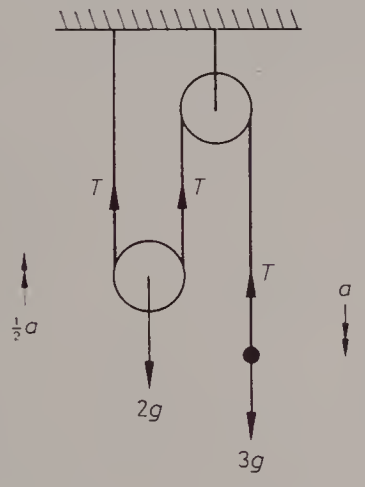
Two particles A and B, of mass 5 kg, rest on the smooth inclined faces of a fixed wedge as shown. They are connected by a light inextensible string passing over a smooth pulley C which lies in the same vertical plane as the particles. Find the acceleration of the

system when moving freely and the tension in the string. If the system is released from rest with the string taut and both particles 2 m from the pulley, find the speed of the particles when A hits the horizontal plane. Find also the distance that B has travelled when it first comes to rest.

5.4 Related accelerations

Some problems involve bodies moving with different but related accelerations. A system including both fixed and moving pulleys can be dealt with by the methods of the previous section when the relationships between the accelerations have been established.

Example 1 A light inextensible string attached to the ceiling passes under a smooth movable pulley of mass 2 kg and then over a smooth fixed pulley. A particle of mass 3 kg hangs freely from the end of the string. All parts of the string not touching the pulleys are vertical. If the system is released from rest, find the acceleration of the particle and the tension in the string.



If the particle moves a distance s m, then the movable pulley moves a distance $\frac{1}{2}s$ m in the opposite direction. Hence if the acceleration of the particle is $a\,\mathrm{m\,s^{-2}}$ downwards, then the acceleration of the movable pulley is $\frac{1}{2}a\,\mathrm{m\,s^{-2}}$ upwards.

Using Newton's 2nd law,

for the particle	$3g - T = 3a$	(1)
for the pulley	$2T - 2g = 2.\frac{1}{2}a$	(2)
$2\times(1)+(2)$	$4g = 7a$	

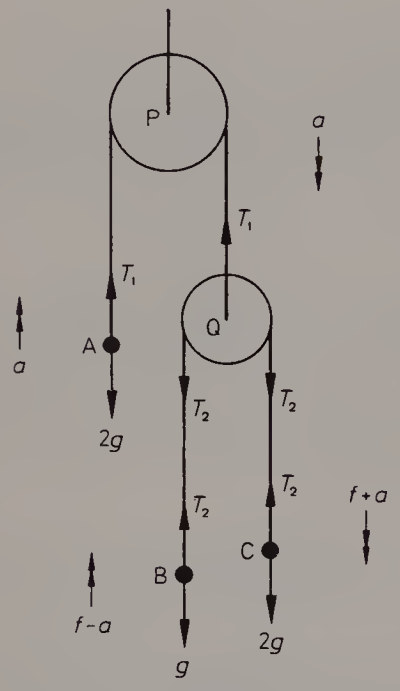
$$\therefore \quad a = \frac{4}{7}g = \frac{4}{7} \times 9{\cdot}8 = 5{\cdot}6.$$

Substituting in (1) $3g - T = 3.\frac{4}{7}g$

$$\therefore \quad T = \frac{9}{7}g = \frac{9}{7} \times 9{\cdot}8 = 12{\cdot}6.$$

Hence the acceleration of the particle is $5{\cdot}6\,\mathrm{m\,s^{-2}}$ and the tension in the string is 12·6 N.

Example 2 A light inextensible string which passes over a smooth fixed pulley P carries at one end a particle A of mass 2 kg and at the other end a smooth light pulley Q. A light inextensible string passes over pulley Q and carries at its ends a particle B of mass 1 kg and a particle C of mass 2 kg. Find the acceleration of particle A when the system is moving freely and the tensions in the two strings.



The tensions and accelerations in the different parts of the system are shown in the diagram, f being the acceleration of particles B and C relative to pulley Q.

Applying Newton's 2nd law, to pulley Q (which is assumed to have zero mass):

$$2T_2 - T_1 = 0 \qquad (1)$$

to A: $T_1 - 2g = 2a$ (2)

to B: $T_2 - g = f - a$ (3)

to C: $2g - T_2 = 2(f+a)$ (4)

(1)+(2) $2T_2 - 2g = 2a$ $\therefore \quad T_2 = a+g.$

Substituting in (3) $a+g-g = f-a \quad \therefore \quad f = 2a$

Substituting in (4) $2g-(a+g) = 2(2a+a)$

$$g - a = 6a$$

$$\therefore \quad a = \frac{1}{7}g = 1{\cdot}4.$$

Hence the acceleration of particle A is $1{\cdot}4\,\mathrm{m\,s^{-2}}$ upwards.

Thus $T_2 = a+g = 1{\cdot}4+9{\cdot}8 = 11{\cdot}2$

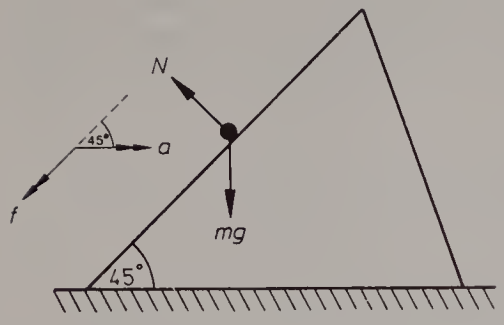
and $T_1 = 2T_2 = 22{\cdot}4.$

Hence the tensions in the strings over pulleys P and Q are 22·4 N and 11·2 N respectively.

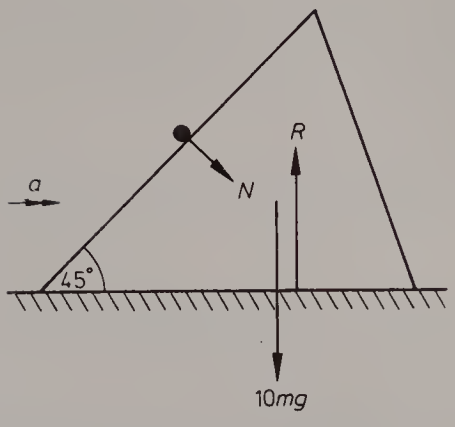
The last example concerns two bodies in contact, which are free to move with different accelerations. In problems of this kind it is helpful to consider the acceleration of one body relative to the other.

Example 3 A particle of mass m rests on a smooth face of a wedge which stands on a smooth horizontal plane. The face of the wedge is inclined at an angle of 45° to the horizontal and the mass of the wedge is $10m$. Find the acceleration of the wedge when the system is released.

Motion of the particle



Motion of the wedge



Let a be the acceleration of the wedge and let f be the acceleration of the particle relative to the wedge. [This means that the true acceleration of the particle has components of magnitudes a and f as shown in the diagram.]

Applying Newton's second law to the motion of the particle in a direction perpendicular to the wedge:

$$\searrow mg\cos 45° - N = ma\sin 45°.$$

Applying Newton's second law horizontally to the motion of the wedge:

$$\rightarrow N\sin 45° = 10ma$$

Hence
$$mg.\frac{1}{\sqrt{2}} - N = ma.\frac{1}{\sqrt{2}} \qquad (1)$$

and
$$N.\frac{1}{\sqrt{2}} = 10ma \qquad (2)$$

$\frac{1}{\sqrt{2}} \times (1)+(2)$
$$\tfrac{1}{2}mg = \tfrac{1}{2}ma + 10ma$$

$$\therefore \quad g = a + 20a$$

i.e.
$$a = \frac{g}{21} = \frac{9{\cdot}8}{21} = \frac{1{\cdot}4}{3} = \frac{7}{15}.$$

Hence the acceleration of the wedge is $7/15\ \mathrm{m\,s^{-2}}$ horizontally.

Exercise 5.4

1. A light inextensible string attached to the ceiling passes under a smooth movable pulley of mass 6 kg and then over a smooth fixed pulley. A particle of mass 1 kg hangs freely from the end of the string. All parts of the string not touching the pulleys are vertical. If the system is released from rest, find the acceleration of the particle and state its direction. Find also the tension in the string.

2. Repeat question 1 for a movable pulley of mass 5 kg and a particle of mass 4 kg.

3. A light inextensible string which passes over a smooth fixed pulley P carries at one end a particle A of mass 2 kg and at the other end a smooth light pulley Q. Particles B and C of masses 4 kg and 5 kg respectively are connected by a light inextensible string passing over pulley Q. Find the accelerations of particles A and B when the system is moving freely. Find also the tensions in the two strings.

4. Repeat question 3 for particles A, B and C of masses 4 kg, 1 kg and 2 kg.

5. Repeat question 3 for particles A, B and C of masses 5 kg, 2 kg and 1 kg. Assume in this case that pulley Q has a non-zero mass of 1 kg.

6. A particle of mass 2 kg is in contact with a smooth face of a wedge which stands on a smooth horizontal plane. The face of the wedge is inclined at 30° to the horizontal and the mass of the wedge is 4 kg. Find the acceleration of the wedge when the system is released.

7. A particle of mass m is placed on a smooth face of a wedge which stands on a smooth horizontal plane. The face of the wedge is inclined at an angle α to the horizontal and the mass of the wedge is $4m$. If the system is released from rest, find the speed of the particle relative to the wedge after 1 second.

8. A particle of mass 1 kg is placed on the smooth horizontal upper face of a wedge, which is itself standing on a smooth plane inclined at an angle α to the horizontal. The mass of the wedge is 3 kg and $\sin\alpha = \frac{1}{4}$. Find the accelerations of the particle and the wedge when the system is released.

Exercise 5.5 (miscellaneous)

1. Two forces of magnitude 10 N and 7 N act on a particle producing an acceleration of $4\,\mathrm{m\,s^{-2}}$. If the forces act at an angle of 60°, find the mass of the particle.

2. A body of mass 5 kg rests on a smooth horizontal plane. Horizontal forces of magnitudes 6, 12, 5 and F newtons act on the body in the directions 015°, 130°, 180° and 270° respectively. Given that the acceleration of the body is $3\,\mathrm{m\,s^{-2}}$, find the value of F and the direction of the acceleration.

3. A vehicle of mass 2000 kg is travelling along a straight horizontal road at 90 km/h. It is brought to rest by a constant force of F newtons in a distance of 500 m. Find F and the time taken for the car to come to rest.

4. A particle A of mass 2 kg is acted upon by a constant force of magnitude 10 N in the direction of the vector $3\mathbf{i}-4\mathbf{j}$ and a constant force of magnitude 50 N in the direction of the vector $7\mathbf{i}+24\mathbf{j}$. Find the acceleration of A. Initially the particle is at the origin moving with speed $39\,\mathrm{m\,s^{-1}}$ in the direction of the vector $12\mathbf{i}-5\mathbf{j}$. Find the position vector of the particle 2 s later. (O & C)

5. A particle of mass 3 kg is being pulled across a smooth horizontal surface by a string inclined at an angle of 35° to the horizontal. If the particle started from rest and moved a distance of 5 m in the first 2 seconds of its motion, find the constant tension in the string.

6. A man of mass 70 kg is travelling in a lift. Find the force between the man and the floor of the lift when the lift is (a) ascending with an acceleration of $1{\cdot}5\,\mathrm{m\,s^{-2}}$, (b) descending with an acceleration of $0{\cdot}5\,\mathrm{m\,s^{-2}}$.

7. A load of 200 kg is lifted by means of a vertical cable through a distance of 50 m. The load accelerates uniformly from rest for 20 seconds, then continues with constant speed for 10 seconds. It then decelerates uniformly to rest in a further 10 seconds. Find the maximum speed attained and the tensions in the cable at every stage of the motion.

8. A package of mass 80 kg is suspended by a rope from a helicopter. Find the tension in the rope and its angle of inclination to the vertical when the helicopter is flying (a) with uniform velocity, (b) with a constant acceleration of $2{\cdot}5\,\mathrm{m\,s^{-2}}$ horizontally.

9. A particle of mass m slides down a smooth plane inclined at an angle α to the horizontal. Find the speed attained by the particle when it has travelled a distance x from rest.

10. A particle of mass 5 kg rests on a smooth plane inclined at an angle α to the horizontal, where $\sin\alpha = 0{\cdot}2$. If the particle is projected up the plane with an initial speed of $7\,\mathrm{m\,s^{-1}}$, find the distance up the plane that it travels.

11. A ring of mass m kg slides down a smooth wire inclined at an angle α to the horizontal. The ring is released 4·9 m vertically above a horizontal plane and hits the plane t seconds later. Find (a) the value of t if $\alpha = 30°$, (b) the value of α if $t = 1{\cdot}2$.

12. A car of mass 1200 kg is pulling a trailer of mass 500 kg up a road inclined at an angle α to the horizontal where $\sin\alpha = 0{\cdot}1$. The resistance to motion for both car and trailer is 0·15 N per kg. Find the tractive force exerted by the car engine and the tension in the coupling between the car and the trailer if they are decelerating at $0{\cdot}5\,\mathrm{m\,s^{-2}}$.

13. Two particles of mass 1·5 kg and 2 kg are connected by a light inextensible string passing over a smooth fixed pulley. If the system is released from rest with the strings taut and vertical, find the distance moved by the 1·5 kg mass relative to the 2 kg mass in the first $1\frac{1}{2}$ seconds of the subsequent motion.

14. A particle of mass 4 kg rests on a smooth horizontal platform at a distance of 2·5 m from the edge. It is connected by a light inextensible string passing over a smooth pulley at the edge of the platform to a particle of mass 0·5 kg which is

hanging freely. If the particles are released from rest, find the speed of the 4 kg mass when it reaches the edge of the platform given that initially the 0·5 kg mass is (a) 3 m, (b) 1·6 m above the ground.

15. A particle of mass 10 kg rests on a smooth plane inclined at an angle of 40° to the horizontal. It is connected by a light inextensible string passing over a smooth pulley fixed at the top of the plane to a particle of mass 4 kg which is hanging freely. Find the acceleration of the system when it is released from rest and the force exerted by the string on the pulley.

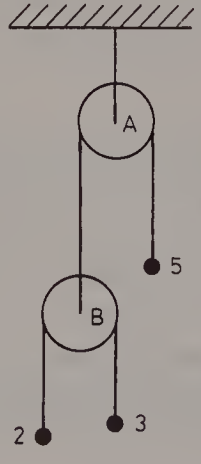
16. A particle A of mass 4 kg is connected by a light inextensible string passing over a smooth fixed pulley to a particle B of mass 3 kg. Particle B is connected by another light inextensible string to a particle C of mass 3 kg. The system is released from rest with the strings taut and vertical. Find the tensions in the strings. Given that B and C are initially 2 m and 1 m respectively above the ground, find the distance travelled by A before it first comes to rest.

17. A particle A of mass $2m$ is initially at rest on a smooth plane inclined at an angle α to the horizontal. It is supported by a light inextensible string which passes over a smooth light pulley P at the top edge of the plane. The other end of the string supports a particle B, of mass m, which hangs freely. Given that the system is in equilibrium, find α, and the magnitude and direction of the resultant force exerted by the string on the pulley.

A further particle of mass m is now attached to B and the system is released. Find, for the ensuing motion, the tension in the string, the acceleration of B and the magnitude and direction of the resultant force exerted by the string on the pulley. (L)

18. A light inextensible string has one end attached to a ceiling. The string passes under a smooth movable pulley of mass m and then over a smooth fixed pulley. A particle of mass $2m$ is attached at the free end of the string. The sections of the string not in contact with the pulleys are vertical. If the system is released from rest and moves in a vertical plane, show that the acceleration of the particle is $2g/3$ and find the tension in the string. (AEB 1977)

19.



The figure shows a light inextensible string which passes over a smooth light fixed pulley A and carries at one end a mass 5 kg and at the other end a smooth light pulley B. A light inextensible string passes over the pulley B and carries masses 3 kg and 2 kg at its ends. The system is in motion with the masses moving vertically. Find the tensions in the two moving strings stating the units. (C)

20. A crate of mass 50 kg is suspended by a light cable from a helicopter. The helicopter has a horizontal acceleration of $2\,m\,s^{-2}$. The crate is being winched aboard the helicopter with an acceleration relative to the helicopter of $0{\cdot}5\,m\,s^{-2}$. Find the tension in the cable and its angle of inclination to the vertical.

6 Friction; Hooke's law

6.1 The laws of friction

When two objects are in contact each exerts a force upon the other. As stated in Newton's third law, these forces are equal and opposite. The force exerted by a smooth surface on a body in contact with it is perpendicular to the surface. Thus there is no resistance to sliding motion across the surface. However, the force exerted by a rough surface on a body has a component perpendicular to the surface, but may also have a component which acts along or tangentially to the surface and which opposes sliding or a tendency to slide. These components are referred to as the *normal reaction* **N** and the *frictional force* **F**. Their resultant is called the *total reaction* **R**.

It can be shown experimentally that the following laws provide a fairly good approximation to the behaviour of frictional forces.

(1) When two rough surfaces are in contact friction acts to oppose the sliding of one surface relative to the other.

(2) When there is no sliding the frictional force is just sufficient to prevent the relative motion of the surfaces. However, the friction between two surfaces cannot exceed a maximum value called *limiting friction.*

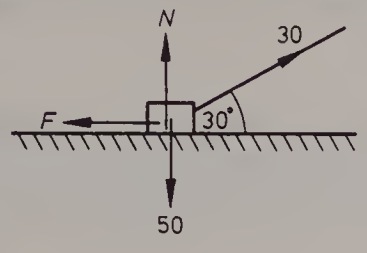
(3) The limiting value of the frictional force F is proportional to the normal reaction N. Thus the maximum value of F is μN, where μ is a constant called the *coefficient of friction.*

(4) The value of μ depends only on the nature of the two surfaces involved.

(5) When sliding occurs the frictional force takes its limiting value μN and acts in the direction that opposes the relative motion of the surfaces.

[In practice, the magnitude of the frictional force that acts during sliding is found to be slightly less than the limiting value reached just before sliding begins. Thus for any pair of surfaces the coefficient of sliding friction is slightly less than the coefficient of static friction. However, in elementary work the difference is assumed to be negligible and a single value of μ is used.]

Example 1 A block of weight 50 N rests in equilibrium on a rough horizontal plane. It is attached to a string inclined at 30° to the horizontal and the tension in the string is 30 N. Find the magnitude of the frictional force acting on the block.



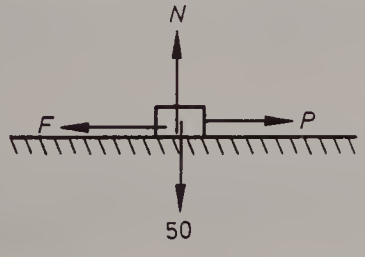
Resolving horizontally

$\leftarrow F - 30\cos 30° = 0$

$\therefore \quad F = 30 \times \frac{\sqrt{3}}{2} = 15\sqrt{3}.$

Hence the magnitude of the frictional force is $15\sqrt{3}$ N.

Example 2 A block of weight 50 N rests on a horizontal plane. The coefficient of friction between the block and the plane is 0·4. A horizontal force of P N is applied to the block. Find the value of P given that the block is just about to slide.



Resolving horizontally and vertically,

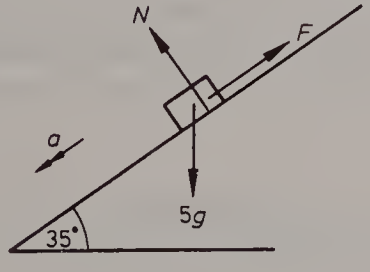
$\leftarrow F - P = 0, \quad \text{i.e.} \quad F = P$

$\uparrow N - 50 = 0, \quad \text{i.e.} \quad N = 50.$

When the block is just about to slide friction is limiting, i.e. $F = \mu N$

$\therefore \quad P = 0{\cdot}4 \times 50 = 20.$

Example 3 A block of mass 5 kg is sliding down a plane inclined at 35° to the horizontal. If the coefficient of friction between the block and the plane is 0·3, find the acceleration of the block.



Let the acceleration of the block be a m s^{-2} down the plane.

Applying Newton's second law parallel and perpendicular to the plane

$\swarrow \; 5g\sin 35° - F = 5a \qquad (1)$

$\nwarrow \; N - 5g\cos 35° = 0 \qquad (2)$

Since the block is sliding, $F = \mu N$

$\therefore$ using (2), $F = 0{\cdot}3 \times 5g\cos 35°.$

Substituting in (1) $5g\sin 35° - 0{\cdot}3 \times 5g\cos 35° = 5a$

$\therefore \quad a = g(\sin 35° - 0{\cdot}3\cos 35°) \approx 3{\cdot}21.$

Hence the acceleration of the block down the plane is 3·21 m s^{-2}.

Exercise 6.1

1. A block of weight 20 N rests in equilibrium on a rough horizontal plane under the action of a force of 10 N. Find the magnitude of the frictional force on the block, given that the external force acts (a) horizontally, (b) vertically downwards, (c) downwards at 30° to the vertical.

2. A block of weight 20 N rests in equilibrium on a rough plane inclined at 30° to the horizontal. Find the magnitude and direction of the frictional force on the block, given that there is a force on the block acting up the plane, which has magnitude (a) 15 N, (b) 10 N, (c) 5 N.

3. A block of weight 12 N rests on a horizontal plane. The coefficient of friction between the block and the plane is $\frac{1}{4}$. A force of P N is applied to the block, such that the block is just about to slide. Find the value of P, given that this force acts (a) horizontally, (b) upwards at 45° to the horizontal, (c) downwards at 45° to the horizontal.

4. A block of weight W rests on a rough plane inclined at 25° to the horizontal. Given that the block is just about to slide down the plane, find the coefficient of friction.

5. A particle of weight 5 N is at rest on a plane inclined at 40° to the horizontal under the action of a force of P N up the plane. The coefficient of friction is 0·2. Find the value of P if the particle is just about to slide (a) up the plane, (b) down the plane.

6. A particle of mass 2 kg is being pulled across a horizontal plane by a horizontal force of magnitude 8 N. Find the acceleration of the particle if the coefficient of friction is $\frac{1}{3}$.

7. A particle of mass 3 kg is being pulled across a rough horizontal plane by a string inclined at 35° to the horizontal. The coefficient of friction between the particle and the plane is 0·6. Find the acceleration of the particle when the tension in the string is 30 N.

8. A particle of mass 10 kg is sliding down a plane inclined at 30° to the horizontal. If the coefficient of friction is 0·2, find the acceleration of the particle.

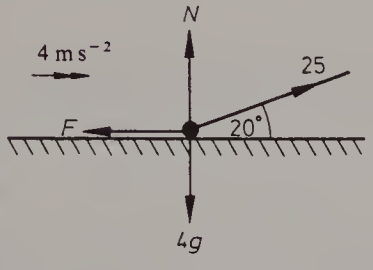
9. A particle of mass 4 kg is sliding down a plane inclined at 20° to the horizontal. If its acceleration is $2\,\mathrm{m\,s^{-2}}$, find the coefficient of friction between the particle and the plane.

10. A particle of mass 5 kg is being pulled up a plane inclined at 45° to the horizontal by a string parallel to the plane. If the coefficient of friction is $\frac{1}{4}$ and the tension in the string is 50 N, find the acceleration of the particle.

6.2 Motion on a rough surface

When a particle is in motion on a rough surface the frictional force F and the normal reaction N are connected by the equation $F = \mu N$. Many problems can be solved using this relationship together with the methods of Chapter 5.

Example 1 A particle of mass 4 kg is being pulled across a rough horizontal plane by a string inclined at an angle of 20° above the horizontal. When the tension in the string is 25 N the particle accelerates at $4\,\mathrm{m\,s^{-2}}$. Find the coefficient of friction between the particle and the plane.



Applying Newton's second law horizontally and vertically

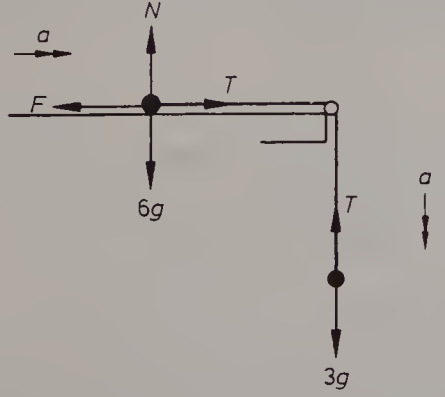
$$\rightarrow 25\cos 20° - F = 4 \times 4$$
$$\uparrow N + 25\sin 20° - 4g = 0$$
$$\therefore F = 25\cos 20° - 16$$
$$N = 4g - 25\sin 20°.$$

Since the particle is in motion, $F = \mu N$

$$\therefore \quad \mu = \frac{F}{N} = \frac{25\cos 20° - 16}{4 \times 9{\cdot}8 - 25\sin 20°} \approx 0{\cdot}244.$$

Hence the coefficient of friction is 0·244.

Example 2 A particle of mass 6 kg rests on a rough horizontal plane, the coefficient of friction between the particle and the plane being 0·4. The particle is connected by a light inextensible string passing over a smooth pulley at the edge of the plane to a second particle of mass 3 kg which hangs freely. If the system is released from rest, find the distance moved by each of the particles in the first 3 seconds of the subsequent motion.



Let the tension in the string be T N and the acceleration of the particles $a\,\mathrm{m\,s^{-2}}$.

Using Newton's second law for the 6 kg mass:

$$\rightarrow \quad T - F = 6a \qquad (1)$$
$$\uparrow N - 6g = 0 \qquad (2)$$

for the 3 kg mass:

$$\downarrow 3g - T = 3a \qquad (3)$$

When the 6 kg mass is sliding $F = \mu N$

$\therefore$ using (2) $F = 0{\cdot}4N = 0{\cdot}4 \times 6g = 2{\cdot}4g$

Substituting in (1) $T - 2{\cdot}4g = 6a$

Adding (3) $0{\cdot}6g = 9a$

$$\therefore \quad a = \frac{1}{9} \times \frac{3}{5}g = \frac{1}{15}g.$$

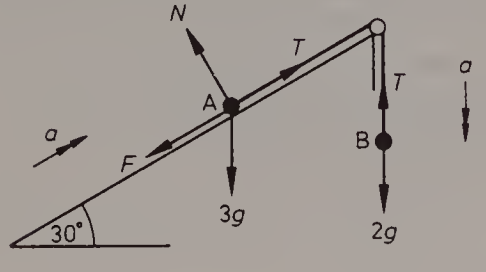
Using the formula $s = ut + \frac{1}{2}at^2$, where $u = 0$, $t = 3$, $a = \dfrac{g}{15}$

$$s = 0 + \frac{1}{2} \times \frac{g}{15} \times 9 = \frac{3g}{10} = 2{\cdot}94.$$

Hence the distance moved by the particles in the first 3 seconds is 2·94 m.

Example 3 A particle A of mass 3 kg rests on a rough plane inclined at 30° to the horizontal, the coefficient of friction between the particle and the plane being 1/10. It is connected by a light inextensible string passing over a smooth pulley at the top edge of the plane to a particle B of mass 2 kg which hangs freely. Find the acceleration of the system when it is released from rest and the tension in the string.

The component of the weight of particle A acting down the plane is $3g \sin 30°$, i.e. $\dfrac{3}{2}g$. Since this is less than the weight of particle B, A will have a tendency to move up the plane. Thus the force of friction will act down the plane.



Using Newton's second law, for particle A:

$\nearrow \quad T - F - 3g \sin 30° = 3a$ (1)

$\nwarrow \quad N - 3g \cos 30° = 0$ (2)

for particle B:

$\downarrow \quad 2g - T = 2a.$ (3)

When particle A is sliding $F = \mu N$

$\therefore$ using (2) $F = \dfrac{1}{10}N = \dfrac{3}{10}g \cdot \dfrac{\sqrt{3}}{2} = \dfrac{3\sqrt{3}}{20}g.$

Substituting in (1) $T - \dfrac{3\sqrt{3}}{20}g - 3g \cdot \dfrac{1}{2} = 3a.$

Adding (3) $2g - \dfrac{3\sqrt{3}}{20}g - \dfrac{3}{2}g = 5a.$

$$\therefore \quad a = \frac{1}{5}g\left(\frac{1}{2} - \frac{3\sqrt{3}}{20}\right) = \frac{g}{100}(10 - 3\sqrt{3}) \approx 0{\cdot}47.$$

Rearranging (3) $\qquad T = 2(g - a) \approx 18{\cdot}7.$

Hence the acceleration of the particles is approximately $0{\cdot}47\,\mathrm{m\,s^{-2}}$ and the tension in the string 18·7 N.

Exercise 6.2

1. A particle of mass 4 kg is projected across a horizontal plane with a speed of $7\,\mathrm{m\,s^{-1}}$. If it comes to rest in a distance of 10 m, find the magnitude of the frictional force acting on the particle and the coefficient of friction between the particle and the plane.

2. A body of mass 5 kg is at rest on a horizontal plane, the coefficient of friction being 0·3. A horizontal force of 21 N is applied to the body. If after $3\frac{1}{2}$ seconds this force ceases to act, find the further time that elapses before the particle comes to rest.

3. A block of mass 2 kg rests on a horizontal plane. The coefficient of friction is $\frac{1}{2}$. Find the acceleration of the block when a force of 20 N acts on it (a) upwards at an angle of 20° to the horizontal, (b) downwards at an angle of 20° to the horizontal.

4. A particle slides down a plane inclined at an angle of 30° to the horizontal. The coefficient of friction is 1/7. Find the distance the particle slides in 2 seconds from rest.

5. A particle is sliding down a plane inclined at an angle α to the horizontal. Find an expression for the coefficient of friction, given that the particle moves with (a) uniform velocity, (b) constant acceleration a.

6. A block of mass 2 kg rests on a rough plane inclined at an angle α to the horizontal, where $\sin\alpha = 7/25$. The coefficient of friction is $\frac{1}{2}$. Find the acceleration of the block when a force of 50 N acts on it down the plane. Find the magnitude of the force acting parallel to the plane that would give the block an equal acceleration up the plane.

7. A body of mass 8 kg is projected down a plane inclined at an angle of 15° to the horizontal with a velocity of $10\,\mathrm{m\,s^{-1}}$. If the body comes to rest after travelling a distance of 25 m, find the magnitude of the frictional force acting on the body during its motion and the coefficient of friction.

8. A particle is placed at a point O on a plane inclined at 45° to the horizontal. The particle is projected up the plane along a line of greatest slope with velocity $15\,\mathrm{m\,s^{-1}}$. Find the condition that must be satisfied by the coefficient of friction μ if the particle is to return through 0 after it comes to rest. Given that the particle does return through 0 with a velocity of $10\,\mathrm{m\,s^{-1}}$, find the value of μ.

9. A particle of mass 3 kg rests on a rough horizontal table, the coefficient of friction between the particle and the table being $\frac{1}{3}$. The particle is connected by a light inextensible string passing over a smooth pulley at the edge of the table to a second particle of mass 4 kg which hangs freely. If the system is released from rest, find the acceleration of the particles and the tension in the string.

10. A particle A of mass 6 kg rests on a rough horizontal plane. It is connected by a light inextensible string passing over a smooth pulley at the edge of the plane to a particle B of mass 2 kg which hangs vertically. If particle B is projected vertically downwards with a velocity of $3{\cdot}5\,\mathrm{m\,s^{-1}}$, it comes to rest after travelling a distance of 2·5 m. Find the coefficient of friction between particle A and the plane. Find also the force exerted by the string on the pulley during the motion.

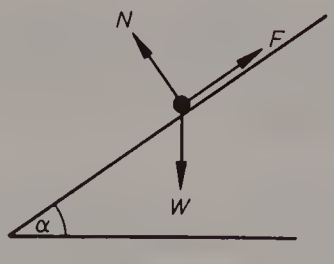
11. A particle of mass 5 kg rests on a rough plane inclined at 25° to the horizontal, the coefficient of friction between the particle and the plane being 0·2. The particle is connected by a light inextensible string passing over a smooth pulley at the top of the plane to a second particle of mass 5 kg which hangs freely. Find the acceleration of the system and the tension in the string.

12. A particle of mass 5 kg rests on a rough plane inclined at an angle α to the horizontal, the coefficient of friction between the particle and the plane being 1/8. The particle is connected by a light inextensible string passing over a smooth pulley at the top of the plane to a particle of mass 3 kg which hangs freely. Find the acceleration of the system when it is released from rest if (a) $\alpha = 10°$, (b) $\alpha = 60°$.

6.3 Equilibrium on a rough surface

When a particle is in equilibrium on a rough surface the frictional force F and the normal reaction N satisfy the inequality $F \leqslant \mu N$. When the particle is on the point of sliding $F = \mu N$. This state is sometimes called limiting equilibrium.

Example 1 A particle is placed on a rough plane inclined at an angle α to the horizontal. The coefficient of friction between the particle and the plane is μ. Find the condition that the particle should be in equilibrium.



If the particle is in equilibrium, resolving along and perpendicular to the plane,

$$\nearrow \quad F - W\sin\alpha = 0$$
$$\nwarrow \quad N - W\cos\alpha = 0$$

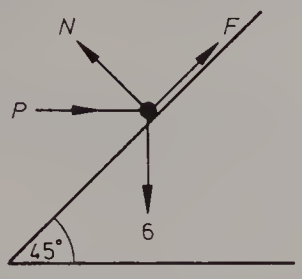
i.e. $F = W\sin\alpha$ and $N = W\cos\alpha$.

Since $F \leqslant \mu N$, $\quad W\sin\alpha \leqslant \mu W\cos\alpha \quad \therefore \quad \tan\alpha \leqslant \mu$.

Hence equilibrium will be possible if $\mu \geqslant \tan\alpha$.

In some problems it is necessary to find the least force required to maintain the equilibrium of a particle on a rough surface. As this least force just prevents motion, when it is applied the frictional force F takes its limiting value μN.

Example 2 A particle of weight 6 N is placed on a rough plane inclined at 45° to the horizontal, the coefficient of friction being $\frac{1}{2}$. Find the magnitude of the least horizontal force required to maintain equilibrium.



Consider a horizontal force of P N acting on the particle to maintain equilibrium.

Resolving along and perpendicular to the plane

$$\nearrow \; P\cos 45° + F - 6\sin 45° = 0$$
$$\nwarrow \; N - P\sin 45° - 6\cos 45° = 0$$

$$\therefore \quad F = 6\sin 45° - P\cos 45° = \frac{1}{\sqrt{2}}(6-P)$$
$$N = 6\cos 45° + P\sin 45° = \frac{1}{\sqrt{2}}(6+P).$$

The value of P is least when friction is limiting, i.e. when

$$F = \mu N$$

$$\therefore \quad \text{since} \quad \mu = \tfrac{1}{2}, \quad \frac{1}{\sqrt{2}}(6-P) = \frac{1}{2}\cdot\frac{1}{\sqrt{2}}(6+P)$$

i.e. $12 - 2P = 6 + P \qquad \therefore \quad P = 2$

Hence the least horizontal force required to maintain equilibrium is 2 N.

[A similar method can be used to find the least force required to move a particle across a rough surface.]

Exercise 6.3

1. A particle is at rest on a rough plane inclined at an angle α to the horizontal. The coefficient of friction is $\frac{1}{4}$. Given that the particle is on the point of sliding down the plane, find α.

2. A horizontal force of magnitude P N acts on a particle of weight 10 N at rest on a plane inclined at 40° to the horizontal. The coefficient of friction between the particle and the plane is 0·2. Find the value of P if the particle is just about to slide (a) up the plane, (b) down the plane.

3. A particle of weight 3 N rests on a rough horizontal plane. The coefficient of friction is μ. A force of 1 N acts on the particle. Find the condition that the particle should be in equilibrium if this force acts (a) horizontally, (b) upwards at 30° to the horizontal, (c) downwards at 30° to the horizontal.

4. A body of weight 4 N is placed on a rough plane inclined at an angle α to the horizontal. The coefficient of friction between the body and the plane is $\frac{1}{2}$. A force of 1·5 N acting downwards parallel to the plane is applied to the particle. Prove that the condition that the particle should be in equilibrium is $3\sec\alpha + 8\tan\alpha \leqslant 4$.

5. A body of weight 6 N is kept in equilibrium on a rough plane inclined at 30° to the horizontal by a force of P N acting up the plane. If the least value of P for which equilibrium is maintained is 1·5, find the coefficient of friction.

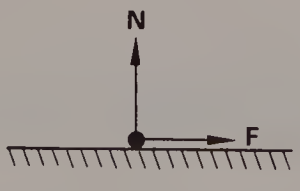
6. A block of weight 5 N is on a plane inclined at 50° to the horizontal. The coefficient of friction is 0·3. The block is kept in equilibrium by a force of P N acting up the plane. Find the range of possible values of P.

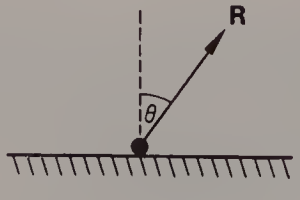
7. A particle of weight 20 N rests on a rough plane inclined at an angle α to the horizontal, where $\tan\alpha = \frac{3}{4}$. The coefficient of friction is $\frac{1}{4}$. The particle moves up the plane under the action of a force of P N. Find the least value of P, given that this force acts (a) up the plane, (b) horizontally.

8. A particle of mass M rests on a rough horizontal plane, the coefficient of friction being $\frac{1}{3}$. The particle is connected by a light inextensible string passing over a smooth pulley at the edge of the plane to a particle of mass m which hangs vertically. Find the condition that the system should remain at rest when released.

6.4 Angle of friction

The total reaction **R** of a surface on a body in contact with it is the resultant of the normal reaction **N** and the frictional force **F**.





If **R** acts at an angle θ to the normal to the surface then

$$F = R\sin\theta \quad \text{and} \quad N = R\cos\theta.$$

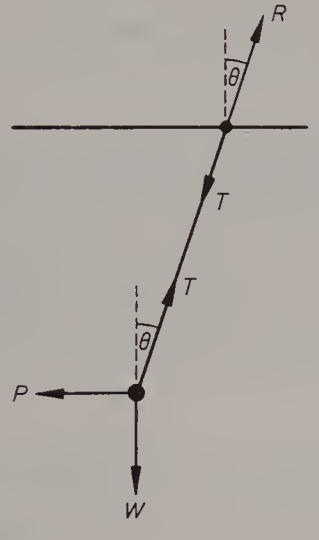
Since $F \leqslant \mu N$, $R\sin\theta \leqslant \mu R\cos\theta$, i.e. $\tan\theta \leqslant \mu$.

If λ is the acute angle such that $\tan\lambda = \mu$, then $\tan\theta \leqslant \tan\lambda$ and hence $\theta \leqslant \lambda$. The angle λ is called the *angle of friction*. It is the maximum angle which the total reaction can make with the normal to the surface. Thus the total reaction can lie anywhere on or within a 'cone of friction' with the normal to the surface as axis and λ as semi-vertical angle.

Some problems are simplified by using the total reaction **R** instead of its components **F** and **N**.

Example 1 A particle of weight **W** is suspended by a light inextensible string from a light ring. The ring can slide along a rough horizontal rod, the coefficient of friction being $\frac{1}{3}$. A horizontal force **P** is applied to the particle. Given that the system is at rest with the ring on the point of sliding along the rod, find P in terms of W.

Since the only forces acting on the ring are the tension in the string and the reaction of the rod on the ring, these forces must be equal and opposite. If the reaction makes an angle θ with the vertical, then since the ring is about to slide, $\theta = \lambda$.



Resolving the forces on the particle horizontally and vertically

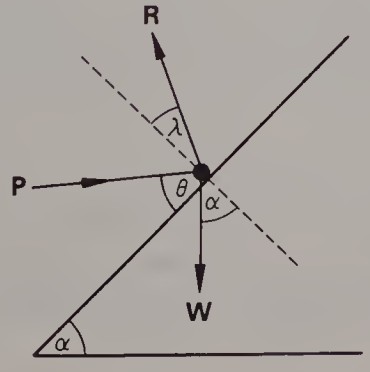
$$\leftarrow \quad P - T\sin\lambda = 0$$
$$\downarrow \quad W - T\cos\lambda = 0$$

$$\therefore \quad P = T\sin\lambda, \quad W = T\cos\lambda,$$

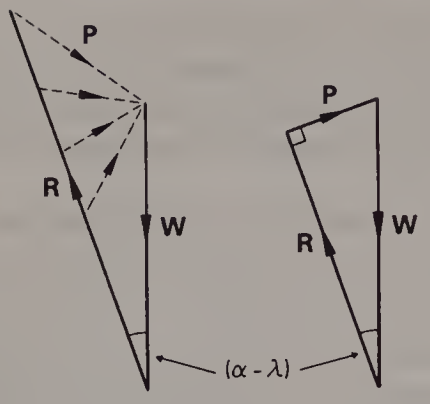
Hence $\dfrac{P}{W} = \tan\lambda = \frac{1}{3}$ so that

$$P = \tfrac{1}{3}W.$$

Example 2 A particle of weight **W** is placed on a rough plane inclined at an angle α to the horizontal. The coefficient of friction is μ, where $\mu < \tan\alpha$. Find the magnitude and direction of the least force required to prevent motion down the plane.



Suppose that sliding is prevented by a force of magnitude P acting at an angle θ to the plane. When P takes its least value, friction will be limiting and the reaction **R** will make an angle λ with the normal to the plane, where $\tan\lambda = \mu$.



Assuming that the particle is on the point of sliding, the value of P depends on the value chosen for θ. By considering the triangle of forces as θ varies, we see that the magnitude of **P** is least when its direction is perpendicular to that of **R**, i.e. when $\theta = \lambda$. Hence the least force required to prevent motion down the plane is of magnitude $W \sin(\alpha - \lambda)$ and acts at an angle λ to the plane where $\tan \lambda = \mu$.

Exercise 6.4

1. A particle of weight 12 N is suspended by a light inextensible string from a light ring. The ring can slide along a rough horizontal rod, the coefficient of friction being $\frac{1}{2}$. A force of P N is acting on the particle. The system is in equilibrium with the ring on the point of sliding along the rod. Find P given that the force is acting (a) horizontally, (b) upwards at 45° to the horizontal, (c) downwards at 30° to the horizontal.

2. One end of a light inextensible string is attached to a fixed point on a horizontal rod. The other end is fastened to a light ring which can slide on the rod. The coefficient of friction is $\frac{3}{4}$. A particle of weight 20 N is suspended from the mid-point of the string. Given that the system is in equilibrium with the ring on the point of sliding along the rod, find the tension in each part of the string and the angle of inclination to the vertical.

3. A block of weight 8 N rests on a horizontal plane. The coefficient of friction between the block and the plane is $\frac{1}{4}$. Find the magnitude and direction of the least force required to move the block.

4. A particle of weight W rests on a horizontal plane. If the angle of friction is λ, find the magnitude and direction of the least force required to move the block.

5. A particle of weight 6 N is placed on a rough plane inclined at 30° to the horizontal. The coefficient of friction is $\frac{1}{3}$. Find the magnitude and direction of the least force required (a) to prevent motion down the plane, (b) to move the particle up the plane.

6. A particle of weight W is placed on a rough plane inclined at an angle α to the horizontal. The angle of friction is λ, where $\lambda > \alpha$. Find the magnitude and direction of the least force required to move the particle (a) down the plane, (b) up the plane.

6.5 Hooke's† law

In this section we examine the behaviour of springs and elastic strings. The length of an elastic string when it is unstretched is called its *natural length*. When the string is stretched by forces exerted on its ends, its increase in length is called the *extension* of the string. It can be shown experimentally that as an elastic string is stretched the tension in the string is proportional to the extension. This result is called *Hooke's law* after the man who discovered it.

When a string is stretched beyond a certain point it ceases to obey Hooke's law. However, in elementary work with elastic strings it is assumed that conditions are such that Hooke's law can be used.

For a string of natural length l, tension T and extension x, the law is written in the form

$$T = \lambda \frac{x}{l}$$

where λ is a constant called the *modulus of elasticity*. The value of λ depends on the nature of the string in use. Since x/l is a ratio of two lengths it has a purely numerical value. Hence the modulus λ must be measured in the same units as T, i.e. in units of force. By writing $x = l$, we see that λ is equal to the tension required to double the length of an elastic string.

Example 1 An elastic string of natural length 1·5 m is stretched to 2 m. If its modulus of elasticity is 12 N, find the tension in the string.

By Hooke's law $$T = \frac{\lambda x}{l}.$$

Putting $\lambda = 12$, $l = 1{\cdot}5$, $x = 0{\cdot}5$:

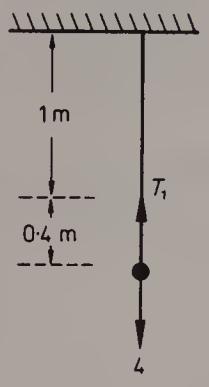
$$T = \frac{12 \times 0{\cdot}5}{1{\cdot}5} = 4$$

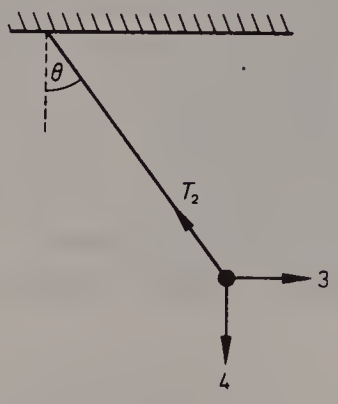
$\therefore$ the tension in the string is 4 N.

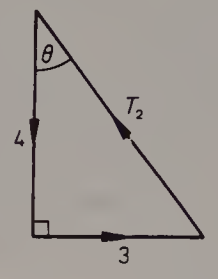
Example 2 A light elastic string of natural length 1 m is fixed at one end and a particle of weight 4 N is attached to the other end. When the particle hangs freely

†*Hooke, Robert* (1635–1703) English experimental physicist. He contributed to many of the important inventions of his time including the microscope. He worked on planetary motion and proposed a law of gravitation similar to Newton's. A pamphlet containing his theory of elasticity was published in 1678.

in equilibrium the length of the string is 1·4 m. The string is now held at an angle θ to the vertical by a horizontal force of 3 N acting on the particle. Find the value of θ and the new length of the string.







Triangle of forces

In the first equilibrium position $T_1 = 4$.

Using Hooke's law, $T = \dfrac{\lambda x}{l}$, where $T = 4$, $x = 0{\cdot}4$, $l = 1$,

$$4 = \frac{\lambda \times 0{\cdot}4}{1} \qquad \therefore \quad \lambda = 10.$$

Hence the modulus of elasticity for the string is 10 N.

Using the triangle of forces for the second equilibrium position

$$\tan\theta = \tfrac{3}{4} \qquad \therefore \quad \theta = 36^\circ 52'$$

$$T_2 = \sqrt{(3^2 + 4^2)} = 5.$$

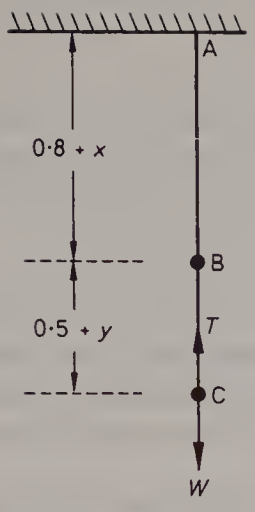
Using Hooke's law for $T = 5, \lambda = 10, l = 1$

$$5 = \frac{10x}{1} \qquad \therefore \quad x = 0{\cdot}5.$$

Hence the value of θ is 36° 52′ and the new length of the string is 1·5 m.

Hooke's law also applies to the stretching of springs.

Example 3 A spring AB of natural length 0·8 m and modulus 16 N is fixed at A. The other end is joined to a second spring BC of natural length 0·5 m and modulus 20 N. A particle of weight W N is then attached at the point C. When the system is hanging freely in equilibrium, the point C is a distance of 2·5 m vertically below A. Find the value of W.



Let the extensions in springs AB and BC be x m and y m respectively.

Applying Hooke's law to both springs

$$T = \frac{16x}{0{\cdot}8} = 20x \quad \text{and} \quad T = \frac{20y}{0{\cdot}5} = 40y$$

Since the tensions in the springs are equal,

$$20x = 40y, \quad \text{i.e.} \quad x = 2y \qquad (1)$$

Considering the two springs together, the natural length is 1·3 m and the stretched length is 2·5 m.

Hence $x + y = 2{\cdot}5 - 1{\cdot}3 = 1{\cdot}2$

Using (1) $3y = 1{\cdot}2$, i.e. $y = 0{\cdot}4$

$$\therefore \quad T = 40y = 40 \times 0{\cdot}4 = 16.$$

Since the particle is in equilibrium, $W = T = 16$.

Thus the value of W is 16.

A spring also obeys Hooke's law when compressed. We again write $T = \lambda x/l$, where in this case T represents a thrust and x a compression, i.e. a reduction in length.

Exercise 6.5

1. An elastic string of natural length 2 m is stretched to 2·5 m. If its modulus of elasticity is 20 N, find the tension in the string.

2. A spring of natural length 60 cm is stretched to a length of 1 m. If the tension in the spring is 6 N, find its modulus of elasticity.

3. A spring of natural length 10 cm is compressed to a length of 8 cm. If its modulus of elasticity is 12 N, find the thrust in the spring.

4. When the tension in an elastic string is 8 N, its length is 1·6 m. If the modulus of elasticity of the string is 24 N, find its natural length.

5. An elastic string of natural length $3a$ is fixed at one end and a particle of weight W is attached to the other end. When the particle hangs freely in equilibrium the length of the string is $5a$. If the string is held at an angle θ to the vertical by a horizontal force of magnitude $\frac{1}{2}W$, find the value of θ and the new length of the string.

6. A particle of mass 3 kg is suspended from an elastic string of natural length 0·5 m and modulus 48 N. If the particle is pulled vertically downwards and then

released when the length of the string is 1 m, find its acceleration at the instant that it is released.

7. An elastic string AB of natural length 1·2 m and modulus 10 N lies along a line of greatest slope of a smooth plane inclined at 30° to the horizontal. The end A is fixed and to the end B is attached a particle of weight 10 N. Find the length of the string when the particle at B rests in equilibrium on the plane.

8. The ends of an elastic string of natural length $4a$ are fixed to points A and B on the same horizontal level, where $AB = 3a$. A particle P of weight W is attached to the mid-point of the string and hangs in equilibrium at a depth of $2a$ below the level of AB. Find the modulus of elasticity of the string in terms of W.

9. A particle P of weight 5·2 N is hanging in equilibrium attached to two strings PA and PB, where A and B are fixed points on the same horizontal level. The string PA is inextensible, but the string PB is elastic with modulus of elasticity 6 N. Given that $AB = 1{\cdot}3$ m, $PA = 0{\cdot}5$ m and $\angle APB = 90°$, find the tension in each string and the natural length of the elastic string.

10. A spring PQ of natural length 1·5 m and modulus λ N is fixed at P. The other end is joined to a second spring QR of natural length 1 m and modulus 2λ N. A particle of weight 15 N is then attached to the end R of the second spring. When the system is hanging freely in equilibrium, the distance PR is 4 m. Find the value of λ.

11. Three light springs of natural lengths 80 cm, 100 cm and 120 cm whose moduli of elasticity are 320 N, 240 N and 180 N respectively are attached end to end and confined in a horizontal tube of length 276 cm. Calculate the thrust on the ends of the tube and the amount by which each spring is compressed.

12. A body of weight 50 N is suspended from a ceiling by a spring of natural length 1 m and modulus of elasticity 120 N. It is supported by another spring of natural length 2 m and modulus 160 N attached to the floor. The springs are vertical, and the distance from floor to ceiling is 3 m. Find the lengths of the springs.

Exercise 6.6 (miscellaneous)

1. A box of weight 400 N will just move across a rough horizontal floor when a horizontal force of 100 N is applied to it. Find the least force required to move the box if it acts (a) upwards at 60° to the horizontal, (b) downwards at 30° to the horizontal.

2. A particle of weight W rests on a horizontal plane, the coefficient of friction between the particle and the plane being $\frac{1}{3}$. When a horizontal force of magnitude P is applied to the particle its acceleration is a. When a horizontal force of magnitude $2P$ is applied, the acceleration is $4a$. Find P in terms of W.

3. A particle of mass 2 kg is placed on a rough plane inclined at an angle α to the horizontal, where $\sin\alpha = 3/5$. The coefficient of friction between the particle and the plane is 1/4. Initially the particle is at rest under the action of a force P N acting upwards parallel to the plane. Find the value of P given that the particle is about to move (a) up the plane, (b) down the plane. If the force is then removed, find the acceleration of the particle down the plane.

4. A block of weight W is placed on a rough plane inclined at an angle α to the horizontal, where $\tan\alpha = 5/12$. The block is on the point of sliding when a force of 5 N acts directly down the plane and when a force of 55 N acts directly up the plane. Find the value of W and the coefficient of friction between the block and the plane.

5. A particle of weight W rests on a rough plane inclined to the horizontal at an angle whose tangent is 2μ, where μ is the coefficient of friction between the particle and the plane. The particle is acted on by a force of magnitude P. (i) Given that the force acts horizontally in a vertical plane through a line of greatest slope, and that the particle is on the point of sliding down the plane, find the magnitude of P. (ii) Given that the force acts along a line of greatest slope and that the particle is on the point of sliding up the plane, find the magnitude of P. (C)

6. Two small light rings which are connected by a light inextensible string of length a can slide on a rough horizontal rod. A weight is attached to the mid-point of the string and the system hangs in equilibrium. If the coefficient of friction between the rings and the rod is μ, show that the greatest possible distance between the rings is $\mu a/(1+\mu^2)^{1/2}$. (O)

7. A mass of 5 kg is moved along a rough horizontal table by means of a light inextensible string which passes over a smooth light pulley at the edge of the table and is attached to a mass of 1 kg hanging over the edge of the table. The system is released from rest with the string taut, each portion being at right angles to the edge of the table. If the masses take twice the time to acquire the same velocity from rest that they would have done had the table been smooth, prove that the coefficient of friction is 1/10. After falling a distance 1/5 m, the 1 kg mass reaches the floor and comes to rest. Prove that the 5 kg mass then moves a further 1/6 m before coming to rest. [Assume throughout that the 5 kg mass does not leave the table.] (O & C)

8. Two particles A and B, of masses 5 kg and 10 kg respectively, lie with B above A on a rough slope inclined at an angle of 30° to the horizontal. The coefficient of friction between the particles and the slope is 0·2. A light inextensible string along a line of greatest slope joins A to B and a force of P N acts on B up the slope. If the system is in equilibrium with both particles just about to move up the slope, find the value of P and the tension in the string.

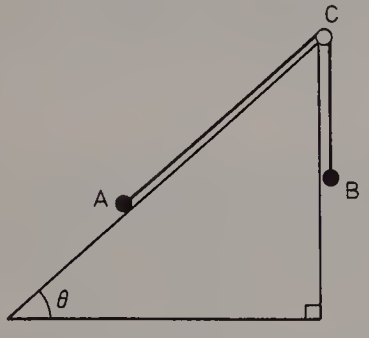
9. Three particles, A, B, C, are of masses 4, 4, 2 kg respectively. They lie at rest on a horizontal table in a straight line, with particle B attached to the mid-point of a light inextensible string. The string has particle A attached at one end and

particle C at the other, and is taut. A force of 60 N is applied to A in the direction CA produced, and a force of 15 N is applied to C in the opposite direction. Find the acceleration of the particles and the tension in each part of the string, (a) if the table is smooth, (b) if the coefficient of friction between each particle and the table is $\frac{1}{4}$. [Take g as $10\,\text{m/s}^2$.] (L)

10. A particle of mass m moves on a line of greatest slope of a plane inclined to the horizontal at an angle θ. The coefficient of friction between the particle and the plane is μ. The particle is released from rest at a point A and it slides down to B, where the distance $AB = d$. Prove that $\mu < \tan\theta$, and that V, the particle's speed at B, is given by $V^2 = 2gd(\sin\theta - \mu\cos\theta)$. The particle is projected up the plane from B with speed U, comes instantaneously to rest at A and then returns to B. If $U = 2V$, prove that (i) $\mu = \frac{3}{5}\tan\theta$, (ii) $V^2 = \frac{4}{5}gd\sin\theta$. (C)

11. A particle of mass $2m$ rests on a rough plane inclined at an angle α to the horizontal. It is connected by a light inextensible string passing over a smooth pulley at the top edge of the plane to a second particle of mass m which hangs freely. Given that when the system is released from rest it begins to move, find the condition that must be satisfied by μ, the coefficient of friction if (a) $\tan\alpha = 3/4$, (b) $\tan\alpha = 5/12$.

12.



The diagram shows a fixed wedge whose sloping face is inclined at an angle θ to the horizontal. Two particles A and B are connected by a light inextensible string which passes over a smooth pulley C at the top of the wedge.

The particle A rests on the sloping face of the wedge, the string between A and C being parallel to a line of greatest slope, and B hangs freely. Particle A has mass M, particle B has mass m and the coefficient of friction between A and the sloping plane is μ. The system is released from rest.

(i) Show that, if $\sin\theta - \mu\cos\theta \leqslant \frac{m}{M} \leqslant \sin\theta + \mu\cos\theta$, the masses remain at rest.

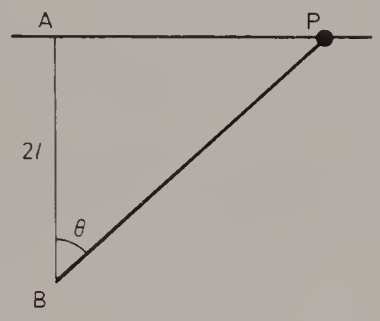
(ii) If $M = m$, $\theta = 30°$ and $\mu = \frac{1}{2}$, find expressions for the acceleration of the system and the tension in the string. (C)

13. The inclined faces of a fixed triangular wedge make angles α and β with the horizontal. Two particles of equal weight, connected by a light inextensible string which passes without friction over the vertex of the wedge, rest on the two faces of the wedge. The system rests in equilibrium with the plane containing the string perpendicular to both faces of the wedge. If both particles are in limiting equilibrium, show that $\mu = \tan\frac{1}{2}|\alpha - \beta|$, where μ is the coefficient of friction between each particle and the face of the wedge. (O)

14. A body of weight W N is being pulled across a rough horizontal plane by a string inclined at an angle θ to the horizontal. The tension in the string is $\frac{1}{2}W$N and the coefficient of friction between the body and the plane is $\frac{1}{4}$. Find the value of θ for which the acceleration of the body is greatest and the magnitude of the maximum acceleration.

15. A particle of weight 4 N is placed on a rough plane inclined at an angle of 10° to the horizontal. The coefficient of friction between the particle and the plane is 0·2. Find the magnitude and direction of the least force required to move the particle (a) down the plane, (b) up the plane.

16.

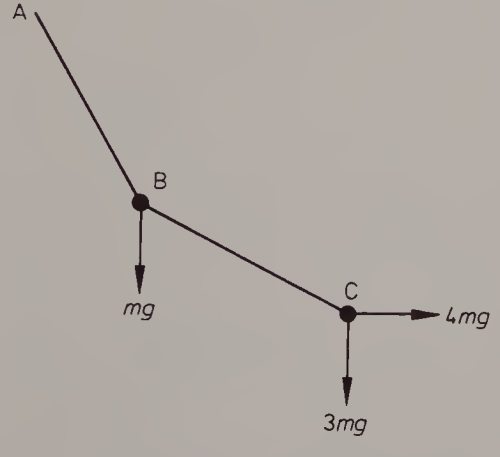


A small rough ring P of weight W is threaded on to a fixed horizontal straight wire which passes through a point A, as shown in the diagram. The ring is joined by a light elastic string of natural length l and modulus λ to a fixed point B which is at a distance $2l$ vertically below A. The ring is in equilibrium with the angle $ABP = \theta$. Find the tension in the string and the vertical and horizontal components of the force exerted by the wire on the ring. Show that

$$W \geqslant \lambda(2-\cos\theta)\left(\frac{\tan\theta}{\mu}-1\right),$$

where μ is the coefficient of friction between the ring and the wire. (JMB)

17. A particle of mass $3m$ is tied to the end C and a particle of mass m is tied at the mid-point B of a light unstretched elastic string ABC. The end A of the string is fixed and a horizontal force of magnitude $4mg$ is applied to the particle at C so that the system hangs in equilibrium as shown. Calculate (i) the tensions in BC and AB, (ii) the inclinations of BC and AB to the vertical. Given that the modulus of elasticity of the string is $6mg$, show that, for this position of equilibrium, $AB:BC = (6+4\sqrt{2}):11$. (AEB 1978)



18. A light elastic string is of natural length a, and is such that when it is stretched by a length a, the tension in it is mg.
(i) The string is fastened to two points, $2a$ apart, which are in the same

horizontal line. A particle of mass m is attached to the mid-point, and gently lowered until it comes to rest. Show that, in its equilibrium position, each of the parts of the string is inclined at θ to the horizontal, where θ is given by $2\tan\theta - \sin\theta = \frac{1}{2}$.

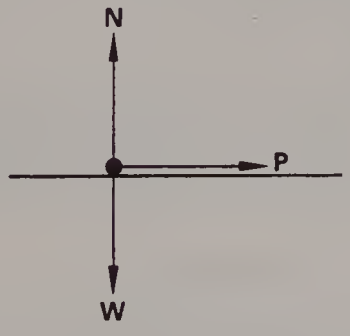
(ii) The same string is now fastened to two points, $2a$ apart, which are in the same vertical line. A particle of mass m is attached to the mid-point and gently released until it comes to rest. Show that, when the system comes to rest, the particle is $\frac{1}{4}a$ below its original position. (SU)

19. A wedge of mass M is resting on a smooth horizontal table. A particle of mass m is placed gently on a rough face of the wedge which is inclined at an angle θ to the horizontal, and slides down this face. Given that the coefficient of friction between the particle and the face is μ, where $\mu < \tan\theta$, prove that the acceleration of the wedge is $\dfrac{mg\cos\theta(\sin\theta - \mu\cos\theta)}{M + m\sin\theta(\sin\theta - \mu\cos\theta)}$. (O)

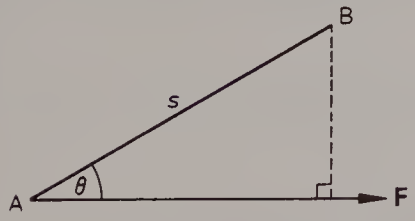
7 Work, energy and power

7.1 Work and kinetic energy

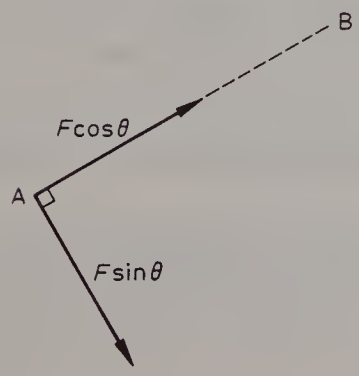
The *work done by a force* is the product of the force and the distance moved in the direction of the force by its point of application.



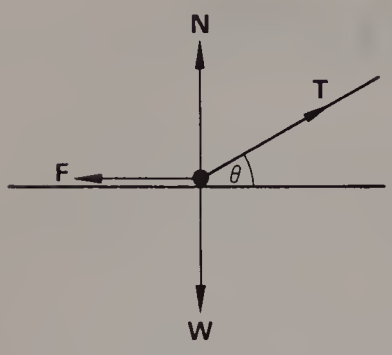
If a particle of weight **W** is pulled a distance s along a smooth horizontal plane by a horizontal force **P**, then the work done by this force is Ps. However, since the direction of motion is perpendicular to the directions of the weight **W** and the normal reaction **N**, these forces do no work.



Consider now a particle which moves a distance s from A to B under the action of a force **F**, where the angle between the displacement **AB** and the force **F** is θ.
The particle moves a distance $s\cos\theta$ in the direction of the force **F**. Hence the work done by the force is $Fs\cos\theta$.



Another way of obtaining the same result is to consider the force **F** as a sum of components along and perpendicular to **AB**. The component $F\sin\theta$ does no work but, since the particle moves a distance s in the direction of the component $F\cos\theta$, the work done is $F\cos\theta \times s$, i.e. $Fs\cos\theta$ as before.



If a particle is pulled a distance s along a rough horizontal plane by a string inclined at an angle θ to the horizontal, then the work done by the tension **T** in the string is $Ts\cos\theta$. The weight **W** and the normal reaction **N** do no work. However, the displacement of the particle in the direction of the frictional force **F** which opposes motion is $-s$. Hence the work done by the force **F** is $-Fs$. The quantity Fs is called the work done *against* friction.

The SI unit of work is the joule (J). When the point of application of a force of 1 newton moves 1 metre, the work done is 1 joule, i.e. 1 J = 1 N m.

To establish a relationship between the motion of a body and the work done on it, we consider a particle of mass m moving in a straight line under the action of a constant force F. We assume that the acceleration of the particle is a and that its initial velocity is u. If when the particle has moved a distance s its velocity is v, then, as shown in §4.2, $v^2 = u^2 + 2as$. By Newton's second law,

$$F = ma$$

$$\therefore \qquad Fs = mas \quad \text{and} \quad as = \tfrac{1}{2}(v^2 - u^2)$$

Hence
$$Fs = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2.$$

The quantity $\frac{1}{2}mv^2$ is called the *kinetic energy* (K.E.) of a particle of mass m moving with velocity v and is measured in the same units as work, i.e. in joules.

Thus when a particle is moving in a straight line under the action of a constant force,

work done by force = increase in kinetic energy.

Example 1 A body of mass 4 kg is moving in a straight line under the action of a constant force. Given that its speed increases from $3\,\mathrm{m\,s^{-1}}$ to $7\,\mathrm{m\,s^{-1}}$, find the work done by the force.

Work done = increase in K.E. $= \frac{1}{2}.4.7^2 - \frac{1}{2}.4.3^2$ J

$= 98 - 18$ J.

Hence the work done by the force is 80 J.

Example 2 A car of mass 1200 kg travelling along a straight horizontal road accelerates uniformly from $10\,\mathrm{m\,s^{-1}}$ to $15\,\mathrm{m\,s^{-1}}$ in a distance of 200 m. If the thrust of the engine is 500 N, find the resistance to motion.

Let the resistance to motion be R N, then the total horizontal force acting on the car is $(500 - R)$ N.

Work done = increase in K.E.

$$\therefore \qquad (500 - R)200 = \tfrac{1}{2}.1200.15^2 - \tfrac{1}{2}.1200.10^2 = 600(225 - 100)$$

$$\therefore \qquad 500 - R = 3 \times 125 = 375.$$

Hence there is resistance to motion of 125 N.

When the motion of a particle in a straight line is opposed by a constant force, it may be convenient to express the relationship between work and kinetic energy in the form:

Work done again resistance = loss of K.E.

Exercise 7.1 [Take $g = 10$.]

1. A particle of weight W is pulled a distance s across a rough horizontal plane by a horizontal force P. Find the work done by the weight W, the normal reaction N, the force P and the frictional force F.

2. A body of weight W is lifted through a distance x by a force P directed vertically upwards. Find the work done by the weight W and the force P.

3. A particle of weight W slides a distance s down a rough plane inclined at an angle α to the horizontal. Find the work done by the weight W and the normal reaction N. Find also the work done against the frictional force F.

4. A particle of weight W is pulled a distance s up a rough plane inclined at an angle α to the horizontal by a string parallel to the plane. Find the work done by the tension T and the normal reaction N. Find also the work done against the frictional force F and against gravity.

5. A body of mass 10 kg moving under the action of a constant force accelerates from $2\,\mathrm{m\,s^{-1}}$ to $5\,\mathrm{m\,s^{-1}}$. Find the work done by the force. Given that the magnitude of the force is 7 N, find the distance moved by the body.

6. A railway truck of mass 2 tonnes moving on a straight horizontal track is brought to rest from a speed of $3\,\mathrm{m\,s^{-1}}$ by a force of P N. Find the work done by this force. Given that the truck travels 20 m before coming to rest, find the value of P.

7. A bullet of mass 0·04 kg travelling at $300\,\mathrm{m\,s^{-1}}$ hits a fixed wooden block and penetrates a distance of 4 cm. Find the average resistance of the wood.

8. A car of mass 1000 kg travelling along a straight horizontal road accelerates uniformly from $15\,\mathrm{m\,s^{-1}}$ to $25\,\mathrm{m\,s^{-1}}$ in a distance of 320 m. If the resistance to motion is 145 N, find the driving force of the engine.

9. A box of mass 16 kg is pulled from rest a distance of 5 m across a smooth horizontal floor by a cable inclined at 60° to the horizontal. Find the work done by the tension in the cable given that its magnitude is 25 N.

10. A particle of mass 6 kg sliding across a rough horizontal plane comes to rest in a distance of 8 m. Given that its initial velocity was $10\,\mathrm{m\,s^{-1}}$, find the work done against friction. Find also the coefficient of friction between the particle and the plane.

11. Find the kinetic energy gained by a body of mass 2 kg falling freely from rest through a distance of 10 m. If a vertical force of P N brings the body to rest in a further distance of 8 m, find the value of P.

12. A particle of mass 5 kg slides a distance 9 m down a rough plane inclined at an angle α to the horizontal, where $\sin\alpha = \frac{3}{5}$. The coefficient of friction is $\frac{1}{2}$. Find the work done by gravity and the work done against friction. Find also the velocity attained by the particle.

7.2 Conservation of mechanical energy

The *energy* of a body can be regarded as its capacity for doing work. For instance, the kinetic energy of a body is its capacity to do work by virtue of its motion. As we saw in the previous section, a moving body does work against forces opposing motion as it loses kinetic energy. There are many other forms of energy such as heat, light, sound, chemical and electrical energy. However, our interest will be confined to forms of *mechanical energy*, i.e. energy by virtue of motion or position.

A body is said to have *potential energy* if its position is such that when released it begins to move and to gain kinetic energy. For instance, a body of mass m released at a height h above the ground will fall freely under gravity until it hits the ground with velocity v.

$$\text{Work done by gravity} = \text{gain in K.E.}$$
$$\therefore \quad mgh = \tfrac{1}{2}mv^2.$$

Thus, initially the body is said to have *gravitational potential energy* mgh by virtue of its height h above the ground.

In any given situation a *zero level* for gravitational potential energy must be decided, and then the potential energy of a body can be calculated relative to this level. Bodies above the zero level will have positive potential energy and bodies below it negative potential energy.

As stated earlier, if a particle is moving in a straight line under the action of a resultant force F then,

$$\text{work done by } F = \text{gain in K.E.}$$

If the only force doing work is gravity this result becomes,

$$\text{work done by gravity} = \text{gain in K.E.}$$

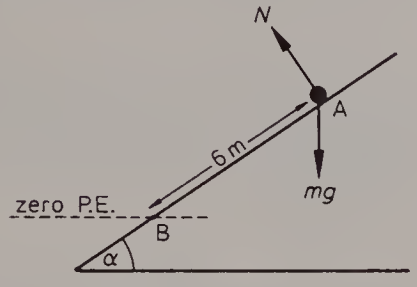
However, by definition, when a mass m falls from a height h the work done by gravity, mgh, is equal to the loss of potential energy.

Hence, $\quad$ loss of P.E. = gain in K.E.
so that $\quad$ K.E. + P.E. = constant.

This result is a simple form of the *principle of conservation of mechanical energy.*

If a particle is moving such that no external force other than gravity is doing work, then the total mechanical energy of the particle remains constant.

Example 1 A particle of mass m kg is released from rest at a point A on a smooth plane inclined at an angle α to the horizontal, where $\sin\alpha = 3/5$. Find the velocity acquired by the particle after travelling a distance of 6 m down the plane to a point B.



Let us assume that at B the particle has zero P.E. and velocity $v\,\mathrm{m\,s^{-1}}$, then
at A: K.E. $= 0$, P.E. $= mg \times 6\sin\alpha$
at B: K.E. $= \frac{1}{2}mv^2$, P.E. $= 0$.

Using the conservation of mechanical energy

$$\tfrac{1}{2}mv^2 + 0 = 0 + 6mg\sin\alpha$$

$$\therefore\quad v^2 = 2\times 6\times 9{\cdot}8\times\frac{3}{5} = \frac{6^2\times 7^2}{5^2}$$

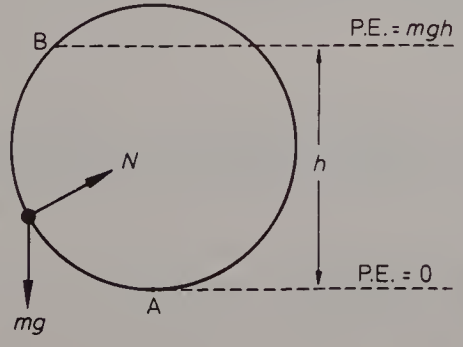
$$\therefore\quad v = \frac{6\times 7}{5} = 8{\cdot}4.$$

Hence the velocity acquired by the particle is $8{\cdot}4\,\mathrm{m\,s^{-1}}$.

[Note that although this problem could be solved using Newton's second law and one of the constant acceleration formulae, it is quicker to use an energy equation, since only velocity and distance moved are involved.]

It can be shown that the principle of conservation of mechanical energy also applies to the motion of a particle along a curved path, provided that no work is done by any external force other than gravity.

Example 2 A ring of mass m is threaded on a smooth circular wire fixed in a vertical plane. If the ring is projected from the lowest point A of the wire with velocity v and first comes to rest at a point B on the wire, express the vertical height h of B above A in terms of g and v.



Since the wire is smooth the force exerted by the wire on the ring is always perpendicular to the direction of motion and therefore does no work. Hence mechanical energy is conserved throughout the motion, i.e.

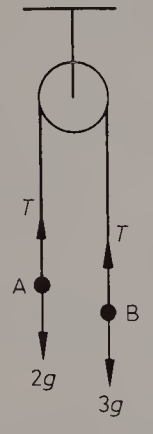
(K.E. + P.E.) at B = (K.E. + P.E.) at A

$$0 + mgh = \tfrac{1}{2}mv^2 + 0.$$

Thus $$h = v^2/2g.$$

When using energy equations to solve problems it is helpful to distinguish between *conservative* and *non-conservative* forces. When work is done against a conservative force such as gravity, there is a gain in potential energy. However, when work is done against a non-conservative force, such as friction, energy is produced in various forms, including heat and sound, but there is no gain in potential energy. Thus the principle of conservation of mechanical energy can be applied to systems of particles provided that (i) no work is done by external forces other than gravity, (ii) no work is done against non-conservative internal forces such as friction.

Example 3 A particle A of mass 2 kg is connected to a particle B of mass 3 kg by means of a light inextensible string which passes over a smooth light pulley. The system is released from rest with both parts of the string taut and vertical. Find the speed attained by each particle when A has risen 2 m.



Let the initial P.E. of the system be zero and let the final speed of each particle be $v\ \mathrm{m\,s^{-1}}$.
Thus initially the system has zero mechanical energy.
Final P.E. of $A = 2g \times 2\ \mathrm{J} = 4g\ \mathrm{J}$.
Final P.E. of $B = 3g \times (-2)\ \mathrm{J} = -6g\ \mathrm{J}$.

Final K.E. of $A = \frac{1}{2} \times 2 \times v^2\ \mathrm{J} = v^2\ \mathrm{J}$.

Final K.E. of $B = \frac{1}{2} \times 3 \times v^2\ \mathrm{J} = \frac{3}{2}v^2\ \mathrm{J}$.

Thus the final mechanical energy of the system

$$= \left(v^2 + \frac{3}{2}v^2 + 4g - 6g\right)\mathrm{J} = \left(\frac{5}{2}v^2 - 2g\right)\mathrm{J}.$$

Using conservation of mechanical energy, $\frac{5}{2}v^2 - 2g = 0$.

$$\therefore \qquad v = \sqrt{\left(\frac{4g}{5}\right)} = \sqrt{\left(\frac{4}{5} \times \frac{49}{5}\right)} = \frac{2 \times 7}{5} = 2{\cdot}8.$$

Hence each particle attains a speed of $2{\cdot}8\ \mathrm{m\,s^{-1}}$.

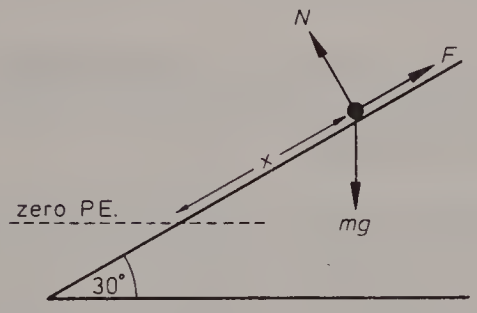
When the total mechanical energy of a particle or system of particles is changed by the action of external forces an energy equation similar to the following may be used,

initial (P.E. + K.E.) + work done = final (P.E. + K.E.).

If the mechanical energy of a system is reduced by work done against non-conservative forces a more convenient form may be,

initial (P.E. + K.E.) − work done against resistance = final (P.E. + K.E.).

Example 4 A small block of mass m slides down a rough plane inclined at an angle 30° to the horizontal. If the block acquires velocity v after travelling a distance x from rest, find an expression for the work done against friction.



Let the final P.E. of the block be zero, then the initial P.E. is $mg \times x \sin 30°$, i.e. $\frac{1}{2}mgx$.
The initial K.E. is zero and the final K.E. is $\frac{1}{2}mv^2$.

Initial (P.E. + K.E.) − work done against friction = final (P.E. + K.E.).
Work done against friction
$= \text{initial (P.E.} + \text{K.E.)} - \text{final (P.E.} + \text{K.E.)}$
$= (\frac{1}{2}mgx + 0) - (0 + \frac{1}{2}mv^2)$
$= \frac{1}{2}m(gx - v^2)$.

Exercise 7.2

1. A ball is thrown vertically downwards with a speed of $3{\cdot}5\,\text{m s}^{-1}$. Find its speed when it has travelled a distance of 5 m.

2. A particle is projected vertically upwards from a point A with a speed of $21\,\text{m s}^{-1}$. Find its position relative to A when its speed is (a) $4{\cdot}2\,\text{m s}^{-1}$, (b) $35\,\text{m s}^{-1}$.

3. A stone is thrown vertically upwards from a point A with velocity $v\,\text{m s}^{-1}$. Find an expression for the greatest height above A reached by the stone.

4. A small block is released from rest on a smooth plane inclined at 30° to the horizontal. Find the distance the particle has travelled down the plane when its speed is $6{\cdot}3\,\text{m s}^{-1}$.

5. A particle is projected with velocity $9\,\text{m s}^{-1}$ up a line of greatest slope of a smooth plane inclined at an angle α to the horizontal, where $\sin\alpha = 4/7$. Find the speed of the particle when it has travelled 5 m up the plane.

6. A tile slides from rest down a smooth roof inclined at an angle α to the horizontal, where $\sin\alpha = \frac{2}{3}$. It reaches the edge after travelling a distance of 4·5 m,

then falls to the ground 7 m below. Find the speed at which the tile is travelling as it hits the ground.

7. A particle is attached to a fixed point by means of a light inextensible string of length 0·8 m. If when the particle is released from rest the string makes an angle of 60° with the downward vertical, find the speed of the particle as it passes through its lowest point.

8. A force acts vertically upwards on a body of mass 4 kg. If the body rises vertically from rest through 5 m and acquires a speed of $3\,\mathrm{m\,s^{-1}}$, find the work done by the force.

9. Find the kinetic energy acquired by a body of mass 5 kg which falls freely from rest through a distance of 3 m. If the particle is then brought to rest by a vertical force of 70 N, find the further distance that it falls.

10. A particle of mass m is connected to a particle of mass M by means of a light inextensible string which passes over a smooth light pulley. The system is released from rest with both parts of the string taut and vertical. Given that $M > m$, find an expression for the distance moved by each particle when the velocity attained is v.

11. A particle of mass 3 kg standing on a smooth horizontal table is connected by means of a light inextensible string passing over a smooth pulley at the edge of the table to a particle of mass 2 kg hanging freely. Find the speed of the particles when they have travelled a distance of 25 cm from rest.

12. A cyclist travelling initially at $2\,\mathrm{m\,s^{-1}}$ free-wheels down a hill. At the bottom of the hill his speed is $10\,\mathrm{m\,s^{-1}}$. If the total mass of the cyclist and his machine is 90 kg and the work done against resistance is 4500 J, find the difference in level between the top and bottom of the hill.

13. In a fairground ride a car of mass 500 kg starts from rest at a point A, 30 m above the ground. After travelling 73 m along a track against a constant resistance of 600 N the car passes through a point B with a speed of $16\,\mathrm{m\,s^{-1}}$. Find the height of B above the ground.

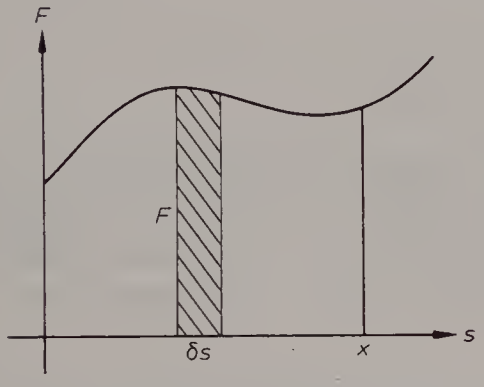
14. A particle is moving at a speed of $7\,\mathrm{m\,s^{-1}}$ when it begins to ascend a slope inclined at an angle α to the horizontal where $\sin\alpha = 5/13$. The coefficient of friction between the particle and the slope is 5/8. Find (a) the speed of the particle when it has travelled 1·95 m up the slope, (b) the total distance the particle travels up the slope before coming to rest.

15. A block of mass 5 kg is projected with velocity $5{\cdot}6\,\mathrm{m\,s^{-1}}$ up a line of greatest slope of a rough plane inclined at an angle α to the horizontal where $\sin\alpha = 0{\cdot}6$. If the block travels 2 m up the slope before coming to rest, find the work done against friction. Find also the coefficient of friction between the block and the plane.

7.3 Work done by a variable force

Consider a particle moving in a straight line under the action of a variable force **F** acting along the line. The work done by **F** in a small displacement δs is approximately equal to $F\delta s$. Thus an approximate value of the total work done in a displacement from $s = 0$ to $s = x$ is given by a sum of the form $\sum_{s=0}^{s=x} F\delta s$. Remembering that sums of this kind can be evaluated by integration, we write:

$$\text{work done} = \lim_{\delta s \to 0} \sum_{s=0}^{s=x} F ds = \int_0^x F ds.$$



Geometrically this quantity is represented by the area under the force-displacement graph for the motion.

An important application of this theory is to the stretching of a spring or elastic string. By Hooke's law, when the extension in an elastic string is x, then the tension T is given by

$$T = \frac{\lambda x}{l}$$

where λ is the modulus of elasticity and l is the natural length of the string. Hence the work done against the tension when the extension is increased from x_1 to x_2 is

$$\int_{x_1}^{x_2} T dx = \int_{x_1}^{x_2} \frac{\lambda x}{l} dx = \left[\frac{\lambda x^2}{2l} \right]_{x_1}^{x_2} = \frac{\lambda}{2l}\left(x_2^2 - x_1^2\right).$$

In particular, the work done when an elastic string is stretched from its natural length l to length $(l+x)$ is $\dfrac{\lambda x^2}{2l}$.

[A similar result can be obtained for the work done when a spring is compressed.]

Example 1 A particle of weight 10 N is suspended from a fixed point by a light elastic string of natural length 1·5 m. When the particle hangs in equilibrium the length of the string is 2 m. Find the modulus of elasticity of the string. Find also the work done in stretching the string from its natural length to a length of 2 m.

When the particle is in equilibrium the tension in the string is equal in magnitude to the weight of the particle.

Hence, using Hooke's law $T = \dfrac{\lambda x}{l}$, where $T = 10$, $l = 1{\cdot}5$ and $x = 0{\cdot}5$,

$$10 = \frac{\lambda \times 0{\cdot}5}{1{\cdot}5} \quad \therefore \quad \lambda = 30$$

$\therefore$ the modulus of elasticity of the string is 30 N.

The work done in extending a string from its natural length by an amount x is $\dfrac{\lambda x^2}{2l}$

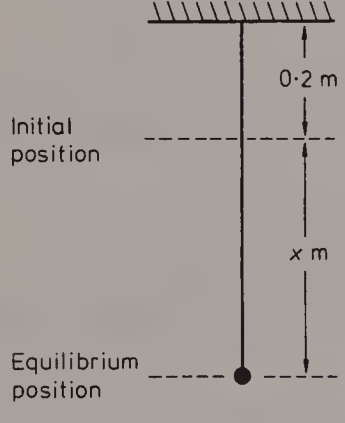
$\therefore$ the work done in extending this string by 0·5 m

$$= \frac{30 \times (0{\cdot}5)^2}{2 \times 1{\cdot}5} \text{J} = 2{\cdot}5 \text{J}.$$

Let us suppose that an elastic string is stretched and then released. As the tension acts to restore the string to its original length the work it does is equal to the work done against the tension in stretching the string. Thus, since no mechanical energy is lost, the tension in an elastic string is a conservative force. A spring or elastic string stretched from its natural length l to a length $(l+x)$ is said to have *elastic potential energy* of $\lambda x^2/2l$.

The principle of conservation of mechanical energy can be applied to systems involving springs and elastic strings, provided that no work is done by external forces (other than gravity).

Example 2 A particle of mass 2 kg is suspended from a fixed point by a spring of natural length 20 cm and modulus 10 N. Given that the extension in the spring when the particle is hanging in equilibrium is x m, find x. If the particle is released from rest with the spring vertical and unstretched, find the speed, v m s^{-1}, of the particle when it passes through its equilibrium position.



In the equilibrium position the tension in the spring is equal to the weight of the particle, i.e. $2g$ N.

Using Hooke's law, $T = \dfrac{\lambda x}{a}$

Substituting $T = 2g$, $\lambda = 10$, $a = 0{\cdot}2$

$$2g = \frac{10x}{0{\cdot}2}$$

$$\therefore \quad x = 0{\cdot}04g = 0{\cdot}04 \times 9{\cdot}8 = 0{\cdot}392.$$

Considering mechanical energy in the initial position,
let gravitational P.E. = 0,
energy stored in spring = 0,
kinetic energy = 0.

Considering mechanical energy in the equilibrium position,
gravitational P.E. $= 2g \times (-x)\text{J} = -0{\cdot}08\,g^2\,\text{J}$

energy stored in spring $= \dfrac{\lambda x^2}{2a}\text{J} = \dfrac{10(0{\cdot}04\,g)^2}{2+0{\cdot}2}\text{J} = 0{\cdot}04\,g^2\,\text{J}$

kinetic energy $= \frac{1}{2} \times 2 \times v^2\,\text{J} = v^2\,\text{J}$.

Using the principle of conservation of mechanical energy

$$-0{\cdot}08\,g^2 + 0{\cdot}04\,g^2 + v^2 = 0$$

$$\therefore \quad v^2 = 0{\cdot}04\,g^2, \text{ i.e. } v = 0{\cdot}2\,g = 1{\cdot}96.$$

Hence the speed with which the particle passes through its equilibrium position is $1{\cdot}96\,\text{m s}^{-1}$.

[Note that in practice it is virtually impossible to construct a system in which there is no loss of mechanical energy, since there are no truly frictionless pulleys, smooth surfaces nor perfectly elastic strings. In this chapter we are dealing with mathematical models of systems in which the work done against friction and other non-conservative forces is negligible compared with the total energy involved.]

Exercise 7.3

1. Find the work done in stretching a spring of natural length 1 m and modulus 10 N to a length of (a) 1·2 m, (b) 2 m.

2. Find the energy stored in an elastic string of natural length 4 m and modulus 2 N when its length is (a) 5 m, (b) 10 m.

3. Find the work done in stretching a spring of natural length 1·5 m and modulus 9 N from a length of 2 m to length 2·5 m.

4. The work done in stretching an elastic string of natural length 2·5 m from 3 m to 4 m in length is 6 J. Find the modulus of the string.

5. When an elastic string of natural length 1·5 m is stretched to the length 1·8 m the energy stored in it is 2·7 J. Find the energy stored in the string when its length is 2 m.

6. The work done in extending a spring from its natural length $6a$ to a length $7a$ is ka. Find the modulus of the spring in terms of k and the work done in extending the length of the spring from $7a$ to $8a$.

7. A particle of mass 8 kg is suspended from a fixed point by a spring of natural length 0·5 m and modulus 140 N. If the particle is released from rest with the spring vertical and unstretched, find the distance it falls before coming to rest instantaneously.

8. A particle of mass 5 kg on a smooth horizontal plane is attached to one end

of a string of natural length 1 m and modulus 20 N. The other end of the string is attached to a fixed point in the same plane. If the particle is released from rest with the string stretched to a length of 2 m, find its speed at the instant the string returns to its natural length.

9. A particle of mass m is attached by a light elastic string of natural length $4a$ and modulus $7mg$ to a fixed point. The particle is released from rest with the string taut and vertical, but unstretched. Find an expression for its speed when it has fallen a distance a.

10. A particle is suspended from a fixed point A by a light elastic string of natural length 2 m. When the particle hangs in equilibrium the length of the string is 2·5 m. Given that the particle is released from rest at A, find the distance it has fallen when it first comes to rest.

11. A particle of mass 7 kg is attached to one end of an elastic string of modulus $9g$ N and natural length 1·8 m. The other end of the string is fastened to a fixed point A. The point B is the position of the particle when it hangs in equilibrium. If the particle is released from rest at A, find its speed when it passes through B.

12. A particle is attached by a light elastic string of natural length 60 cm to a fixed point O. When the particle hangs in equilibrium its distance below O is 80 cm. If the particle is released from rest at a point 1 m vertically below O, find the distance of the particle from O when it next comes to rest. Find also the speed of the particle when it passes through its equilibrium position.

13. The ends of an elastic string of modulus 30 N and natural length 2 m are attached to fixed points A and B 2 m apart on the same horizontal level. A particle P is attached to the mid-point of the string then released at the mid-point of AB. If the particle falls 0·75 m before coming to rest instantaneously, find its weight.

14. A particle P of mass 2 kg is suspended from a point O by a light elastic string of length 1 m and modulus 98 N. Find the length of the string when P hangs in equilibrium. The particle is now pulled down and released from rest 1·6 m below O. Find (a) the speed of P as the string goes slack, (b) the depth below O at which P comes instantaneously to rest.

7.4 Power

Power is the rate at which a force does work. The SI unit of power is the watt (W). One watt is a rate of working of one joule per second, i.e. $1\,\text{W} = 1\,\text{J s}^{-1}$.

[For those who are familiar with the watt only as a unit of electrical power, it may be helpful to remember that a family saloon car has a maximum power output of approximately 40–50 kW. A sprinter may generate kinetic energy at an

average rate of about 1 kW for short periods, but most people produce mechanical energy at much lower rates.]

Example 1 Find the average power developed by a crane which lifts a load of 8 kN through a vertical distance of 15 m in 6 s.

Work done by the crane in 6 s is 8×15 kJ, i.e. 120 kJ
$\therefore$ average work done in 1 s is $120 \div 6$ kJ, i.e. 20 kJ.
Hence the average power of the crane is 20 kW.

Example 2 A pump raises water at a rate of 750 litres per minute through a vertical distance of 4 m and projects it into a reservoir at a speed of $6\,\mathrm{m\,s^{-1}}$. Find the power developed by the pump.

Since the mass of 1 litre of water is 1 kg, the mass of water raised per second $= \frac{750}{60}\text{kg} = \frac{25}{2}\text{kg}$.

Thus the P.E. given to the water each second

$$= \frac{25}{2}g \times 4\,\text{J} = 25 \times 9{\cdot}8 \times 2\,\text{J} = 490\,\text{J}.$$

The K.E. given to the water each second

$$= \frac{1}{2} \times \frac{25}{2} \times 6^2\,\text{J} = 25 \times 9\,\text{J} = 225\,\text{J}.$$

Hence the total power developed by the pump is

$(490 + 225)$ W, i.e. 715 W.

If the engine of a vehicle, such as a car or a train, produces a tractive force F newtons, then the power of the engine is the rate at which this force does work. When the vehicle is travelling at constant speed $v\,\mathrm{m\,s^{-1}}$, since the distance travelled each second is v m, the work done each second is Fv J. Hence the power of the engine is Fv watts.

A similar result is obtained for non-uniform motion by considering the work done when the vehicle moves a small distance δs in time δt. The rate of working in this interval is approximately $\frac{F\delta s}{\delta t}$, i.e. $F\frac{\delta s}{\delta t}$. However, as $\delta t \to 0$, $\frac{\delta s}{\delta t} \to \frac{ds}{dt}$. Thus the rate of working at the instant when the vehicle is moving with velocity $v\,\mathrm{m\,s^{-1}}$ must be Fv watts. Hence the power P watts, the tractive force F newtons and the velocity $v\,\mathrm{m\,s^{-1}}$ are connected by the formula

$$\boxed{P = Fv.}$$

Example 3 A car of mass 1200 kg is moving along a horizontal road against resistance to motion of 500 N. Find the power developed by the engine when the

car is travelling at $15\,\mathrm{m\,s^{-1}}$ and accelerating at $0{\cdot}25\,\mathrm{m\,s^{-2}}$. Find also the maximum speed reached by the car if the engine continues to work at the same rate.

If the pull of the engine is F newtons, then by Newton's second law,

$$F - 500 = 1200 \times 0{\cdot}25$$
$$\therefore \quad F = 300 + 500 = 800.$$

Hence the power of the engine

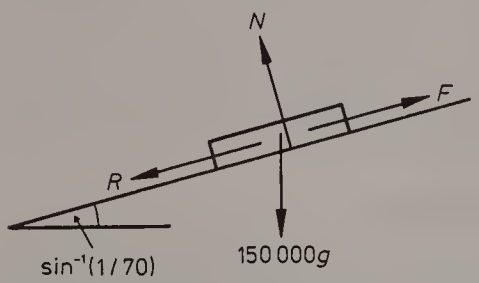
$$= 800 \times 15\,\mathrm{W} = 12\,000\,\mathrm{W} = 12\,\mathrm{kW}.$$

Let the maximum speed of the car be $v\,\mathrm{m\,s^{-1}}$. When maximum speed is reached the pull of the engine is equal in magnitude to the resistance to motion

$$\therefore \quad 12\,000 = 500\,v, \text{ i.e. } v = 24.$$

Hence the maximum speed when the engine is working at the rate of 12 kW is $24\,\mathrm{m\,s^{-1}}$.

Example 4 A train of mass 150 tonnes is ascending an incline of 1 in 70 at a constant speed of $25\,\mathrm{m\,s^{-1}}$. If the power being exerted by the engine is 750 kW, find the resistance to motion.



Let F N be the tractive force of the engine and R N the resistance to motion.
Using the formula $P = Fv$,

$$750\,000 = F \times 25$$
$$\therefore \quad F = 30\,000.$$

Applying Newton's second law up the incline

$$F - R - 150\,000\,g \times \frac{1}{70} = 0$$

$$\therefore \quad R = 30\,000 - \frac{150\,000 \times 9{\cdot}8}{70} = 30\,000 - 21\,000 = 9000.$$

Hence the resistance to motion is 9 kN.

Example 5 When a cyclist rides along a horizontal road at $v\,\mathrm{m\,s^{-1}}$ the resistance to motion is $(16 + \frac{1}{4}v^2)$ N. Find the rate at which the cyclist must work to maintain a steady speed of $6\,\mathrm{m\,s^{-1}}$.

When travelling at a steady speed the force exerted by the cyclist is equal to the resistance to motion,

$$\therefore \quad \text{his rate of work} = (16 + \tfrac{1}{4}v^2)v\,\mathrm{W}$$
$$= (16 + \tfrac{1}{4} \times 6^2)6\,\mathrm{W}$$
$$= (16 + 9)6\,\mathrm{W}$$

Hence to maintain a steady speed of $6\,\mathrm{m\,s^{-1}}$ the cyclist must work at a rate of 150 W.

Exercise 7.4

1. A man of mass 80 kg climbs a staircase of height 5 m in 10 s. Find his average rate of work against gravity.

2. If the effective power developed by a crane is 15 kW, find the time taken to lift a load of weight 50 kN through a vertical distance of 12 m.

3. Find the minimum power needed to pull an object of weight 500 N a distance of 8 m up a smooth slope inclined at 30° to the horizontal in 5 seconds.

4. A lift of mass 1500 kg is being raised by an engine working at the rate of 24·5 kW. If the lift rises 50 m in 40 s, find the load it is carrying.

5. Water from a river, flowing at $4\,m\,s^{-1}$, completely fills a pipe of cross-sectional area $3\,m^2$. If, when the water is used to drive a generator, 30% of its kinetic energy is converted into electrical energy, find the power supplied by the generator.

6. A pump raises 40 kg of water a second through a vertical distance of 10 m and delivers it in a jet with speed $12\,m\,s^{-1}$. Find the effective power developed by the pump.

7. A pump raises water at the rate of 250 litres per minute through a vertical distance of 8 m and projects it into a lake at $2\,m\,s^{-1}$. Find the power developed by the pump.

8. If the engine of a train travelling at a steady speed of $25\,m\,s^{-1}$ is working at a rate of 1800 kW, find the magnitude of the resistance to motion.

9. A motor-cyclist is riding along a horizontal road at a constant speed of $15\,m\,s^{-1}$ against a resistance of 80 N. Find the power at which the motor-cycle engine is working. Given that the total mass of the rider and his machine is 200 kg, find the initial acceleration when the power is increased to 1·5 kW.

10. A car of mass 900 kg is travelling at $20\,m\,s^{-1}$ along a horizontal road against a constant resistance of 600 N. Find the power developed by the engine if the car is (a) moving with constant speed, (b) accelerating at $0{\cdot}15\,m\,s^{-2}$, (c) decelerating at $0{\cdot}5\,m\,s^{-2}$.

11. A cable car of mass 1500 kg is pulled at a constant speed of $10\,m\,s^{-1}$ up a smooth slope inclined at an angle α to the horizontal where $\sin\alpha = 1/7$. Find the power exerted on the car.

12. A car of mass 1000 kg is travelling down an incline of 1 in 28 against a constant resistance of 750 N. Find the maximum speed that can be reached by the car when the engine is working at the rate of 12 kW.

13. A lorry of mass 2400 kg is moving at a steady speed of 60 km/h against a constant frictional resistance of 2 kN. Find the power developed by the engine if the lorry is travelling (a) along a horizontal road, (b) up a hill of inclination $\sin^{-1}(1/21)$, (c) down the same hill.

14. A train of mass 50 tonnes is ascending an incline of 1-in 60 with the engine working at the rate of 200 kW. When the train is travelling at $10\,\mathrm{m\,s^{-1}}$, its acceleration is $0{\cdot}1\,\mathrm{m\,s^{-2}}$. Find the frictional resistance to motion. Assuming that the resistance and the power of the engine remain constant, find the maximum speed the train can attain on this slope.

15. The constant non-gravitational resistance to the motion of a car of mass 1500 kg is 750 N. The engine of the car works at a constant rate of 20 kW. Find, in km/h, the maximum speed (a) on the level, and (b) directly up a road inclined at an angle arcsin (1/12) to the horizontal. Find also, in $\mathrm{m/s^2}$, the acceleration of the car when it is travelling at 72 km/h on a level road. [Take g as $10\,\mathrm{m/s^2}$.] (L)

16. The frictional resistance to the motion of a car of mass 1000 kg is $30v$ N where $v\,\mathrm{m\,s^{-1}}$ is the speed of the car. The car ascends a hill of inclination $\arcsin\left(\frac{1}{10}\right)$, the power exerted by the engine being 12·8 kW. If the car is moving at a steady speed, prove (do not merely verify) that this steady speed is $10\,\mathrm{m\,s^{-1}}$. On reaching the top of the hill, find the immediate acceleration of the car, assuming the road becomes level. (O&C)

17. The resistance to the motion of a car is $(160+cV^2)$ newtons, where c is a constant and V is the speed of the car in m/s. The mass of the car is 900 kg and, when the engine is working at the rate of 19·2 kW, the maximum speed on a level road is 20 m/s. Find the value of c. The car ascends a slope inclined to the horizontal at an angle θ, where $\sin\theta = 1/30$. If the engine continues to work at the same rate, find the acceleration of the car when its speed is 10 m/s. [Take g as $10\,\mathrm{m/s^2}$.] (L)

18. A car of mass 1·2 tonnes is travelling along a straight, horizontal road at a constant speed of 120 km/h against a resistance of 600 newtons. Calculate, in kW, the effective power being exerted. Given that the resistance is proportional to the square of the velocity, calculate also (i) the power required to go down a hill of 1 in 30 (along the slope) at a steady speed of 120 km/h, (ii) the acceleration of the car up this hill with the engine working at 20 kW at the instant when the speed is 80 km/h. (AEB 1978)

Exercise 7.5 (miscellaneous)

1. A particle is projected at a speed of $3{\cdot}5\,\mathrm{m\,s^{-1}}$ across a rough horizontal plane. Given that it comes to rest after travelling a distance of 5 m, find the coefficient of friction.

2. A particle released from rest on a smooth plane inclined at an angle α to the horizontal reaches a speed of $6\,\mathrm{m\,s^{-1}}$ after travelling 3 m down the plane. Find $\sin\alpha$.

3. A body of mass 2 kg is pulled from rest across a rough horizontal plane by a string inclined at an angle α to the horizontal where $\sin\alpha = 3/5$. The coefficient of friction is 1/3. After travelling 7·8 m the body attains a speed of $5{\cdot}2\,\mathrm{m\,s^{-1}}$. Assuming the tension in the string is constant, find its magnitude.

4. A particle is attached to a fixed point by means of a light inextensible string of length 1 m. If the particle is projected horizontally through its lowest point with a speed of $2{\cdot}8\,\mathrm{m\,s^{-1}}$, find the angle the string makes with the vertical when the particle first comes to rest.

5. A particle of mass 5 kg is projected with a speed of $2{\cdot}8\,\mathrm{m\,s^{-1}}$ down a rough plane inclined at an angle α to the horizontal, where $\sin\alpha = 5/13$. If the particle comes to rest after travelling 2·6 m, find the work done against friction. Find also the coefficient of friction between the particle and the plane.

6. The work done in stretching an elastic string from 3 m to 5 m in length is 4 J. If the modulus of the string is 3 N, find its natural length.

7. A particle of mass 2 kg on a smooth plane inclined at 30° to the horizontal is attached by means of a light inextensible string passing over a smooth pulley at the top edge of the plane to a particle of mass 3 kg which hangs freely. If the system is released from rest with both parts of the string taut, find the speed acquired by the particles when both have moved a distance of 1 m.

8. A curved chute with a smooth surface is fixed so that a point A on the chute is 2·5 m above a horizontal plane. A particle is released from A and is travelling horizontally when it leaves the chute at a point B 1·6 m above the horizontal plane. Find the speed of the particle as it hits this plane and the horizontal distance it has travelled since leaving the chute.

9. A particle P of mass m is attached to two elastic strings each of length 0·6 m and modulus kmg. The other ends of the strings are fixed to two points A and B 1·2 m apart on the same horizontal level. P is released from rest at M, the mid-point of AB and falls vertically 0·8 m before coming instantaneously to rest. Find k and also the speed of P as it passes through C, where $MC = 0{\cdot}45$ m.

10. A light elastic string obeys Hooke's law, so that when it is stretched by an amount x the tension in it is given by $T = kx$ for some constant k. Prove that the elastic energy stored in the string when its extension is x is given by $E = \frac{1}{2}kx^2$.

One end of such a string is attached to a fixed point A, and the other end is attached to a particle of mass m. The unstretched length of the string is a, and the particle hangs freely in equilibrium at a point B which is at a distance $\frac{5}{4}a$ below

A. Prove that $k = 4mg/a$. The particle is held at A, and released from rest. Find its velocity as it passes B, and the total distance it falls before coming instantaneously to rest. (C)

11. A bead of mass m slides on a smooth vertical rod and is attached to a light inextensible string which passes over a smooth peg at a distance a from the rod and has a particle of mass $2m$ fastened to its other end. The bead is held at the level of the peg and then released. Prove that it will descend $\frac{4}{3}a$ before coming to rest and find the tension in the string at this instant. (O)

12. A, B, C, D are fixed points situated at the mid-points of the sides of a square of side $2a$ with AC vertical (C above A) and BD horizontal. A weight W is held in the plane of the square by four springs, of modulus λ and natural length a, whose other ends are attached to A, B, C and D respectively. Show that in the equilibrium position the springs attached to B and D are inclined to the upward drawn vertical at an angle α where $2\cot\alpha - \cos\alpha = W/2\lambda$. Show that the amount of energy needed to raise the weight to the centre of the square must be at least $2a\lambda(\cot^2\alpha + \sin\alpha - 1)$. (W)

13. A small bead of mass m is threaded on a smooth wire which is shaped in the form of the curve $by = x^2 + b^2$ $(x \leqslant 0)$. The positive y-axis is vertically upwards and the x-axis lies in a horizontal plane. The bead is released from rest at the point where $x = -2b$ and slides down the wire to its end at the point $(0, b)$, after which the bead moves freely under gravity until it strikes the horizontal plane at a point on the x-axis. Find expressions for (i) the speed of the bead on leaving the wire, (ii) the speed of the bead on striking the x-axis, (iii) the angle that the direction of motion of the bead makes with the x-axis when it meets the x-axis, (iv) the x-coordinate of the point where the bead meets the x-axis.

Suppose instead that it is required that the bead should hit a small target which is to be placed on the x-axis at the point $(c, 0)$. Find the x-coordinate of the point on the curve at which the bead has to be released from rest if it is to achieve this. (JMB)

14. An elastic string is such that when stretched from its natural length l to a length $l+x$ the tension in the string becomes kx, where k is a positive constant. Show that the work done when the length of the string is increased from $l+x_1$ to $l+x_2$ is $\frac{1}{2}k(x_2^2 - x_1^2)$.

A particle of mass m is attached to one end of the string, the other end of which is attached to a fixed point O in a smooth inclined plane of inclination α. The particle rests in equilibrium on the plane and the extension of the string is $\frac{1}{2}l$. Show That $kl = 2mg\sin\alpha$. The particle is then pulled a further distance $b - \frac{1}{2}l$ down the line of greatest slope, and released. By considerations of energy or otherwise, determine the distance up the plane, measured from the point of release, that the particle will travel, distinguishing the cases (i) $b \leqslant l$, (ii) $l < b \leqslant 2l$. Describe briefly in words what happens if $b > 2l$. (JMB)

15. A train of mass 3×10^5 kg travels along a straight level track. The resistance to motion is $1{\cdot}5 \times 10^4$ N. Find the tractive force required to produce an acceleration of $0{\cdot}1\,\text{m s}^{-2}$, and the power in kW which is then developed by the engine when the speed of the train is $10\,\text{m s}^{-1}$. Find also the maximum speed attainable on the same track when the engine is working at a rate of 360 kW. (JMB)

16. A car of mass 1000 kg is travelling along a level road at a steady speed of 45 km/h with its engine working at $22\frac{1}{2}$ kW. Calculate in newtons the total resistance (assumed constant) due to friction, etc. The engine is disconnected and simultaneously the brakes are applied, bringing the car to rest in 30 metres. Find the force, assumed constant, exerted by the brakes. (O & C)

17. A cable car in the Alps gains height at an average rate of 900 m in $2\frac{1}{2}$ minutes. Given that the cable car has mass 1300 kg and carries 30 passengers of average mass 90 kg, determine the average power required if the efficiency of the machinery is 75%. [Take g to be $10\,\text{m s}^{-2}$.] (C)

18. A pump raises water through a vertical height of 15 m and delivers it at a speed of $10\,\text{m s}^{-1}$ through a circular pipe of internal diameter 12 cm. Find the mass of water raised per second and the effective power of the pump, giving your answers to 3 significant figures.

19. A motor lorry weighing 5 metric tonnes can develop up to 15 000 watts. It is moving up a hill with slope 1 in 20 $\left(\theta = \sin^{-1}\frac{1}{20}\right)$ against a frictional force of 300 N. This frictional force is independent of the speed. (i) Find the maximum steady speed in kilometres per hour at which the lorry can travel up this slope. (ii) Find the acceleration capable of being developed when the lorry travels up this slope at $10\,\text{km h}^{-1}$. The lorry travels on a straight horizontal road at a speed of v kilometres per hour against a braking force of $10v$ newtons and other resistances of 300 N. What is the maximum value of v? (W)

20. A locomotive of mass 15 000 kg, working at the rate of 220 kW, pulls a train of mass 35 000 kg up a straight track which rises 1 m vertically for every 50 m travelled along the track. When the speed is 10 m/s the acceleration is $0{\cdot}23\,\text{m/s}^2$. Find the frictional resistance at this speed. Given that the resistance is proportional to the speed of the train, find, in m/s, the greatest speed of the train up the slope if the rate of working is unchanged. [Take $g = 10\,\text{m/s}^2$.] (L)

21. The engine of a car, of mass M kg, works at a constant rate of H kW. The non-gravitational resistance to the motion of the car is constant. The maximum speed on level ground is V m/s. Find, in terms of M, V, H, α and g, expressions for the accelerations of the car when it is travelling at speed $\frac{1}{2}V$ m/s (a) directly up a road of inclination α, (b) directly down this road. Given that the acceleration in case (b) is twice that in case (a), find $\sin\alpha$ in terms of M, V, H and g. Find also, in terms of V alone, the greatest steady speed which the car can maintain when travelling directly up the road. (L)

22. An engine is travelling at a steady speed of 10 m/s on a level track against a constant resistance of 40 000 N. Calculate, in kW, the power output of the engine. The engine is then coupled to a carriage by a towbar. The constant resistance to the motion of the carriage is 20 000 N. If the power output of the engine is now 900 kW, calculate, in m/s, the maximum speed of the train on a level track. State the tension, in newtons, in the towbar. The train then ascends an incline of inclination arcsin (1/50) to the horizontal with the same power output, 900 kW, against the same constant frictional forces. If the total mass of the train is 340 tonne, show that the acceleration of the train when it is travelling at 5 m/s is $13/85\,\text{m/s}^2$. [Take g as $10\,\text{m/s}^2$.] (L)

23. A car of mass $2{\cdot}5 \times 10^3$ kg travels at a constant speed of $20\,\text{m s}^{-1}$ up a hill of inclination of $\sin^{-1}(1/20)$ to the horizontal, with the engine working at a power of 8×10^4 W. Given that the resistance is proportional to the cube of the speed show that the resistance at a speed of $24\,\text{m s}^{-1}$ is $4{\cdot}8 \times 10^3$ N, correct to 2 significant figures. Given that the engine is working at the same power, find the acceleration at an instant when the car is moving down a hill of inclination $\sin^{-1}(1/10)$ to the horizontal at a speed of $24\,\text{m s}^{-1}$. (C)

24. A motor-cyclist, whose total mass with his machine is 200 kg, is ascending a hill of inclination θ to the horizontal, where $\sin\theta = 1/8$. If the resistance forces total 150 N and the engine is working at 9 kW, calculate (i) the acceleration of the motor-cycle when the speed is 20 m/s, (ii) the maximum speed that can be attained by the motor-cycle up this hill. The motor-cyclist descends the same hill with a pillion rider whose mass is 80 kg. The resistive forces now total 170 N and the engine is switched off. Find the distance covered and the time taken as the speed of the motor-cycle increases from 10 m/s to 20 m/s. [Take the acceleration due to gravity to be $10\,\text{m/s}^2$.] (AEB 1977)

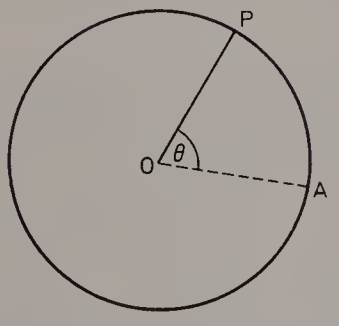
25. A car has a maximum speed of 108 km/h when moving along a horizontal road with the engine working at 36 kW. Calculate, in newtons, the total resistance to the motion of the car. The car, which is of mass 800 kg, can move down a road inclined at an angle α to the horizontal at a maximum speed of 108 km/h with the engine working at 30 kW against the same total resistance. Calculate $\sin\alpha$. Given that the total resistance varies as the square of the speed, find, in kW, the rate at which the engine is working at the instant when the car is moving along a horizontal road with an acceleration of $0{\cdot}5\,\text{m/s}^2$ at a speed of 54 km/h. [Take the acceleration due to gravity to be $10\,\text{m/s}^2$.] (AEB 1978)

26. Two points A and B are at the same level, and they are connected by a road consisting of two straight sections AC and CB, each of uniform gradient. The point C is at a height h vertically above the line AB, and the inclinations to the horizontal of AC and CB are α and β respectively. A car of mass m starts from rest at A, accelerates uniformly to reach a speed $2U$ at C, and then moves with constant retardation, attaining a speed U at B. Show that the time taken to move from A to B is $h(\operatorname{cosec}\alpha + \frac{2}{3}\operatorname{cosec}\beta)/U$. The resistance to the motion of the car is a constant force R. Find the force exerted by the engine during the climb from A

to C, the maximum power developed by the engine and the total work done by the engine. Show that the described motion from C to B can be achieved by using the brakes, with the engine switched off, only if $R \leqslant \frac{m \sin \beta}{2h}(2gh + 3U^2)$. (JMB)

8 Uniform motion in a circle

8.1 Angular velocity

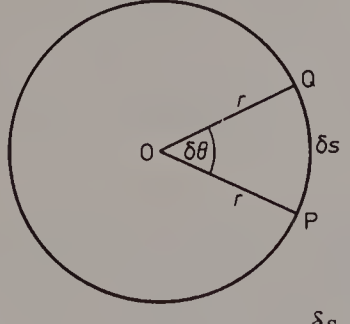


Consider a particle P moving round the circumference of a circle with centre O. If A is any fixed point on the circle and $\angle POA = \theta$, then the *angular velocity* of P is defined as the rate of change of θ and often denoted by ω.

Thus $\omega = \dot{\theta}$, i.e. $\omega = d\theta/dt$.

The value of ω may be positive or negative according to the direction of motion. Anti-clockwise rotations are usually taken to be positive. The *angular speed* of P is the magnitude of its angular velocity. The *angular acceleration* of P is the rate of change of the angular velocity, denoted by $\dot{\omega}$ or $\ddot{\theta}$.

The SI unit of angular velocity is one radian per second, i.e. $1\ \text{rad}\,\text{s}^{-1}$. Angular velocity can also be measured in revolutions per minute ($\text{rev}\,\text{min}^{-1}$).



Suppose now that a particle is moving round the circumference of a circle radius r such that when the angular velocity of the particle is ω, its speed is v. If the particle moves from P to Q in time δt, let arc $PQ = \delta s$ and $\angle POQ = \delta\theta$.

Since $\delta s = r\,\delta\theta$ we have $\dfrac{\delta s}{\delta t} = r\dfrac{\delta\theta}{\delta t}$.

As $\delta t \to 0$, $\dfrac{\delta s}{\delta t} \to \dfrac{ds}{dt}$ and $\dfrac{\delta\theta}{\delta t} \to \dfrac{d\theta}{dt}$.

Hence $$\frac{ds}{dt} = r\frac{d\theta}{dt}.$$

$\therefore$ $$\boxed{v = r\omega \quad \text{and} \quad \omega = \frac{v}{r}.}$$

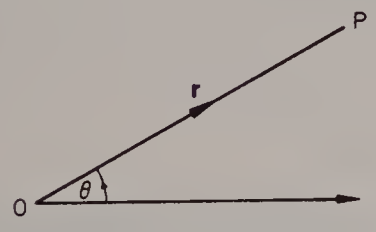
Example 1 The tip of the second hand of a large clock is 60 cm from the centre of the clock face. Find the angular speed of the second hand and the speed at which the tip is travelling.

The second hand makes 1 revolution per minute, i.e. it moves through 2π rad in 60 s

$\therefore$ its angular speed $= \frac{2\pi}{60}\text{rad s}^{-1} = \frac{\pi}{30}\text{rad s}^{-1}$.

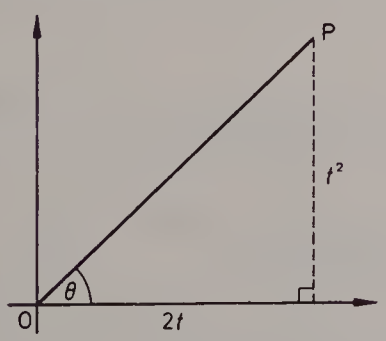
Hence the speed of the tip of the hand

$$= \frac{60}{100} \times \frac{\pi}{30}\text{m s}^{-1} = \frac{\pi}{50}\text{m s}^{-1} \approx 0{\cdot}063\,\text{m s}^{-1}.$$



The definition of angular velocity can be extended to the motion in a plane of a point P with position vector $\mathbf{r}$. If the angle between $\mathbf{r}$ and some fixed direction is θ, then the angular velocity of P about O (i.e. the angular velocity of **OP**) is $\dot{\theta}$.

Example 2 At time t, the position vector of a point P with respect to the origin O is $2t\mathbf{i} + t^2\mathbf{j}$. Find the angular velocity of **OP** at time t.



Let θ be the angle between **OP** and the direction of $\mathbf{i}$, then at time t,

$$\tan\theta = \frac{t^2}{2t} = \frac{t}{2}.$$

Differentiating with respect to t

$$\sec^2\theta\frac{d\theta}{dt} = \frac{1}{2}.$$

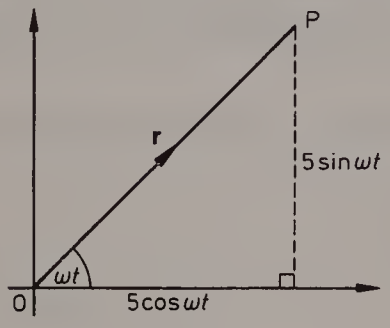
$\therefore$ $$(1 + \tan^2\theta)\frac{d\theta}{dt} = \frac{1}{2}$$

$$\left(1 + \frac{t^2}{4}\right)\frac{d\theta}{dt} = \frac{1}{2}$$

i.e. $$(4 + t^2)\frac{d\theta}{dt} = 2.$$

Hence the angular velocity at time t is given by $\dfrac{d\theta}{dt} = \dfrac{2}{4+t^2}$.

Example 3 A particle P is moving anti-clockwise with constant angular speed ω round a circle with centre O and radius 5. Given that initially the position vector of P is $5\mathbf{i}$, find its position vector at time t. Deduce vector expressions for the velocity and acceleration of P at time t.



Since the angular speed is ω, at time t **OP** makes an angle ωt with the direction of $\mathbf{i}$. Hence at time t the position vector of P is

$$\mathbf{r} = 5(\cos \omega t\mathbf{i} + \sin \omega t\mathbf{j}).$$

Differentiating with respect to t, the velocity of P is given by

$$\mathbf{v} = \dot{\mathbf{r}} = 5(-\omega \sin \omega t\mathbf{i} + \omega \cos \omega t\mathbf{j})$$

i.e. $$\mathbf{v} = 5\omega(-\sin \omega t\mathbf{i} + \cos \omega t\mathbf{j}).$$

Similarly the acceleration of P is given by

$$\mathbf{a} = \ddot{\mathbf{r}} = 5\omega(-\omega \cos \omega t\mathbf{i} - \omega \sin \omega t\mathbf{j})$$

i.e. $$\mathbf{a} = -5\omega^2(\cos \omega t\mathbf{i} + \sin \omega t\mathbf{j})$$

[We note that $v = 5\omega\sqrt{(\sin^2 \omega t + \cos^2 \omega t)} = 5\omega$, which is consistent with the result $v = r\omega$ obtained earlier in the section. We also see that $\mathbf{a} = -\omega^2\mathbf{r}$. Thus the acceleration of the particle is directed towards O and has magnitude $5\omega^2$.]

Exercise 8.1

1. A record of diameter 18 cm is rotating at 45 rev min^{-1}. Find its angular velocity in rad s^{-1} and the speed of a point on the rim of the record.

2. A point is travelling round a circle of radius 5 m at a speed of 20 m s^{-1}. Find the angular speed of the point about the centre of the circle.

3. Find the angular speed with which the earth is rotating about its axis. Hence, taking the radius of the earth to be 6400 km, find the speed of a point on the equator in m s^{-1}.

4. Find an approximate value for the angular speed at which the earth is moving about the sun. Assuming that the earth moves round the sun in an approximately circular orbit of radius 150 million kilometres, find the speed of the earth relative to the sun.

5. Given that the tip of the minute hand of a clock is moving at a speed of $0{\cdot}001\ \mathrm{m\,s^{-1}}$, find the length of the hand to the nearest cm.

6. Find the angular velocity of a point P about the origin O at time $t > 0$, given that the position vector of P is

(a) $\mathbf{i}+t\mathbf{j}$, (b) $2t\mathbf{i}-3\mathbf{j}$, (c) $\frac{1}{t}\mathbf{i}+t\mathbf{j}$.

7. A particle P moves anti-clockwise with constant angular speed ω round a circle with centre O and radius r. Write down the position vector of P at time t given that when $t = 0$ its position vector is (a) $r\mathbf{i}$, (b) $r\mathbf{j}$.

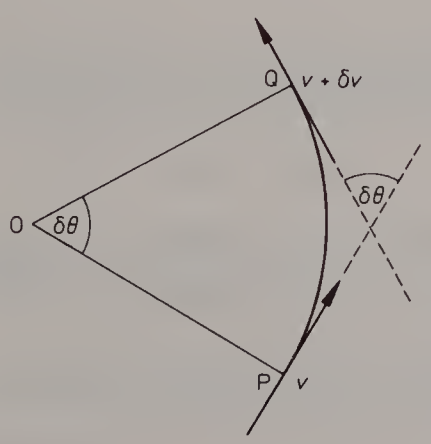
8. Use the results of question 7 to find the angular velocity of a particle P about the origin O and the speed at which the particle is travelling given that at time t its position vector is

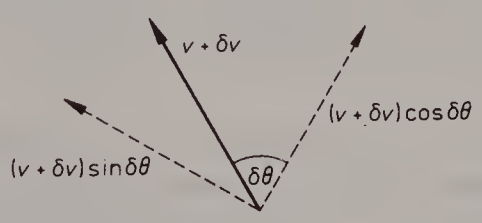
(a) $3(\cos 2t\mathbf{i}+\sin 2t\mathbf{j})$, (b) $5(-\sin t\mathbf{i}+\cos t\mathbf{j})$,
(c) $\cos 2\pi t\mathbf{i}-\sin 2\pi t\mathbf{j}$, (d) $a(\sin kt\mathbf{i}-\cos kt\mathbf{j})$.

9. The points S and T are moving anti-clockwise round a circle with centre O (the origin) and radius a, with the same constant speed $a\omega$. When T is passing through the point with position vector $a\mathbf{j}$, S is passing through the point with position vector $a\mathbf{i}$. Find, at time t later, the position vectors of S and T. Find also, in vector form, the velocity of T relative to S. (L)

8.2 Acceleration in circular motion

Consider a particle moving with angular velocity ω round the circumference of a circle centre O, radius r. Suppose that in time δt it moves from P to Q, where $\angle POQ = \delta\theta$. Let the speed of the particle be v at P and $v+\delta v$ at Q.





From the diagrams we see that the velocity of the particle at Q has components $(v+\delta v)\sin\delta\theta$ in the direction of $\vec{PO}$ and $(v+\delta v)\cos\delta\theta$ perpendicular to $\vec{PO}$.

Thus in the direction of $\overrightarrow{PO}$, the change in velocity in time δt

$$= (v+\delta v)\sin\delta\theta$$

$$\therefore \quad \text{average acceleration} = \frac{(v+\delta v)\sin\delta\theta}{\delta t} = (v+\delta v)\frac{\sin\delta\theta}{\delta\theta}\cdot\frac{\delta\theta}{\delta t}.$$

As $\delta t \to 0$, $\delta v \to 0$, $\dfrac{\sin\delta\theta}{\delta\theta} \to 1$ and $\dfrac{\delta\theta}{\delta t} \to \dfrac{d\theta}{dt}$

$$\therefore \quad \text{acceleration along } \overrightarrow{PO} = v\frac{d\theta}{dt} = v\omega = \frac{v^2}{r}.$$

Similarly, perpendicular to $\overrightarrow{PO}$, the change in velocity in time δt

$$= (v+\delta v)\cos\delta\theta - v$$

$$\therefore \quad \text{average acceleration} = \frac{(v+\delta v)\cos\delta\theta - v}{\delta t}$$

$$= \frac{v(\cos\delta\theta - 1)}{\delta t} + \frac{\delta v}{\delta t}\cos\delta\theta$$

$$= -\frac{v\,.\,2\sin^2\frac{1}{2}\delta\theta}{\delta t} + \frac{\delta v}{\delta t}\cos\delta\theta$$

$$= -v\frac{\delta\theta}{\delta t}\left(\frac{\sin\frac{1}{2}\delta\theta}{\frac{1}{2}\delta\theta}\right)\sin\tfrac{1}{2}\delta\theta + \frac{\delta v}{\delta t}\cos\delta\theta.$$

As $\delta t \to 0$, $\dfrac{\sin\frac{1}{2}\delta\theta}{\frac{1}{2}\delta\theta} \to 1$, $\dfrac{\delta\theta}{\delta t} \to \dfrac{d\theta}{dt}$, $\dfrac{\delta v}{\delta t} \to \dfrac{dv}{dt}$

but $\sin\frac{1}{2}\delta\theta \to 0$ and $\cos\delta\theta \to 1$

$$\therefore \quad \text{acceleration perpendicular to } \overrightarrow{PO} = \frac{dv}{dt}.$$

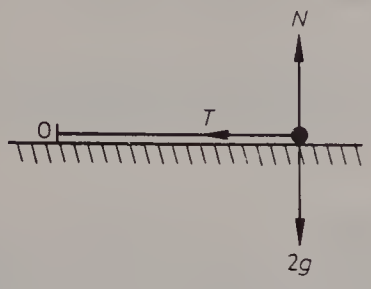
Hence the acceleration of a particle moving in a circle with variable speed v has, at any instant, components v^2/r towards the centre of the circle and dv/dt in the direction of motion.

If v is constant, then $dv/dt = 0$. Thus a particle moving in a circle with constant speed v has acceleration v^2/r towards the centre of the circle. In terms of the angular velocity ω the acceleration is $r\omega^2$.

[This last result can also be obtained by the vector methods used in Example 3 of the previous section.]

Using Newton's second law, we find that the resultant force acting on a particle of mass m, which moves in a circle radius r with constant speed v, is of magnitude mv^2/r and is directed towards the centre of the circle.

Example 1 One end of a light inextensible string of length 0·5 m is attached to a fixed point O on a smooth horizontal table and a particle of mass 2 kg is attached to its other end. If the particle moves in a horizontal circle, centre O, with a speed of $5\,\mathrm{m\,s^{-1}}$, find the tension in the string and the reaction of the table on the particle.

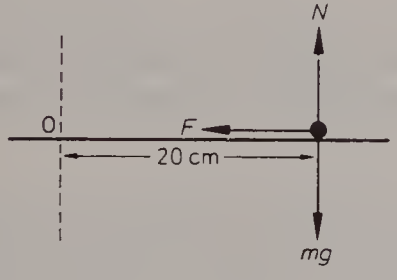


The acceleration of the particle is $5^2/0{\cdot}5\,\mathrm{m\,s^{-2}}$, i.e. $50\,\mathrm{m\,s^{-2}}$ towards O
$\therefore$ using Newton's second law horizontally and vertically,

$$\uparrow \quad N - 2g = 0$$
$$\leftarrow \quad T = 2 \times 50 = 100.$$
$$\therefore \quad N = 2g = 19{\cdot}6$$

Hence the tension in the string is 100 N and the reaction of the table is 19·6 N.

Example 2 A particle is placed at a point 20 cm from the centre O of a rough circular disc. When the disc rotates about a vertical axis through O at an angular speed of $3{\cdot}5\,\mathrm{rad\,s^{-1}}$, the particle is on the point of slipping. Find the coefficient of friction between the particle and the disc.



The acceleration of the particle is $0{\cdot}2 \times (3{\cdot}5)^2\,\mathrm{m\,s^{-2}}$ towards O. Using Newton's second law horizontally and vertically,

$$\uparrow \quad N - mg = 0 \qquad \therefore \quad N = mg$$

$$\leftarrow \quad F = m \times 0{\cdot}2(3{\cdot}5)^2 = m \times \frac{1}{5} \times \left(\frac{7}{2}\right)^2 = \frac{49m}{20}.$$

Since the particle is on the point of slipping, $F = \mu N$

$$\therefore \quad \frac{49m}{20} = \mu mg = 9{\cdot}8\,\mu m.$$

Hence $$\mu = \frac{49}{20 \times 9{\cdot}8} = \frac{1}{4}$$

$\therefore$ the coefficient of friction between the particle and the disc is $\frac{1}{4}$.

Exercise 8.2

1. Find the acceleration of a particle moving uniformly along a circular path
(a) radius 4 m at a speed of $6\,\mathrm{m\,s^{-1}}$,

(b) radius 25 m at a speed of 36 km/h,
(c) radius 80 cm with angular velocity 5 rad s^{-1},
(d) radius 10 m with angular velocity 24 rev/min.

2. Find the magnitude of the resultant force acting on a particle of mass 5 kg which is moving in a circle of radius 2 m with a constant speed of 3 m s^{-1}.

3. Find the magnitude of the resultant force acting on a particle of mass 1·5 kg which is moving in a circle of radius 50 cm with a constant angular velocity of 4 rad s^{-1}.

4. One end of a light inextensible string of length 3 m is attached to a fixed point O on a smooth horizontal surface and a particle of mass 5 kg is attached to its other end. If the particle is moving in a horizontal circle, centre O, at a speed of 15 m s^{-1}, find the tension in the string and the reaction of the surface on the particle.

5. One end of a light inextensible string of length 1·6 m is attached to a fixed point O on a smooth horizontal surface and a particle of mass 4 kg is attached to the other end. If the string will break when the tension in it exceeds 90 N, find the maximum speed at which the particle can move in a horizontal circle with centre O.

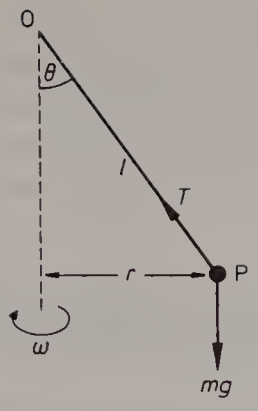
6. A particle is placed at a point 0·5 m from the centre O of a rough circular disc. When the disc rotates about a vertical axis through O at an angular speed of 2·8 rad s^{-1}, the particle is on the point of slipping. Find the coefficient of friction between the particle and the disc.

7. A small ring A is threaded on a horizontal rod PQ, the coefficient of friction between the rod and the ring being 2/7. When the rod rotates horizontally about P with a constant angular velocity of 2 rad s^{-1}, the ring does not slip along the rod. Find the maximum possible distance AP.

8. A particle A is placed on a rough horizontal platform, the coefficient of friction between the particle and the platform being 3/8. The platform rotates about a vertical axis through a fixed point O on the platform. If $OA = 1{\cdot}2$ m, find the angular velocity of the platform when the particle is on the point of slipping.

8.3 Further problems

Consider a particle P of mass m attached by means of a light inextensible string of length l to a fixed point O. The particle is set in motion so that it moves in a horizontal circle of radius r with constant angular velocity ω. Since the string traces out a cone in space, this system is called a *conical pendulum.*



Applying Newton's second law horizontally and vertically,

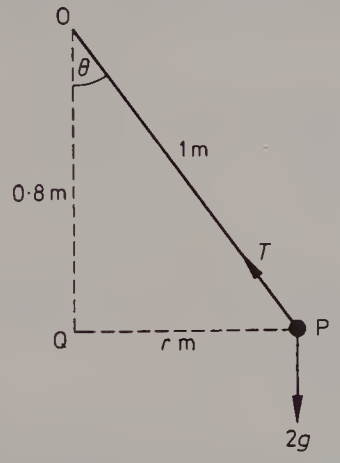
$$\leftarrow T\sin\theta = mr\omega^2$$
$$\uparrow T\cos\theta - mg = 0.$$

Substituting $r = l\sin\theta$,

$$T\sin\theta = ml\sin\theta\omega^2$$

$\therefore$ the tension in the string is $ml\omega^2$.

Example 1 A particle of mass 2 kg is attached by a light inextensible string of length 1 m to a fixed point O. The particle is made to move in a horizontal circle whose centre is 0·8 m vertically below O. Find the tension in the string and the speed of the particle.



Let the speed of the particle be $v\,\mathrm{m\,s^{-1}}$, then applying Newton's second law horizontally and vertically

$$\leftarrow T\sin\theta = \frac{2v^2}{r} \qquad (1)$$

$$\uparrow T\cos\theta - 2g = 0 \qquad (2)$$

In $\triangle OPQ$, by Pythagoras' theorem

$$r = \sqrt{\{1^2 - (0{\cdot}8)^2\}} = 0{\cdot}6$$

$\therefore$ $\sin\theta = 0{\cdot}6$ and $\cos\theta = 0{\cdot}8$

Using (2)
$$T = \frac{2\times 9{\cdot}8}{0{\cdot}8} = 24{\cdot}5$$

$\therefore$ the tension in the string is 24·5 N.

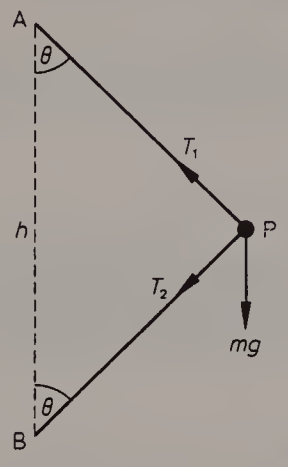
Using (1)
$$v^2 = 24{\cdot}5\times 0{\cdot}6\times\frac{0{\cdot}6}{2} = 7^2\times(0{\cdot}3)^2$$

$\therefore$
$$v = 7\times 0{\cdot}3 = 2{\cdot}1.$$

Hence the speed of the particle is $2{\cdot}1\,\mathrm{m\,s^{-1}}$.

The methods used in Example 1 can also be applied to a particle sliding round the inner surface of a smooth sphere or cone in a horizontal circle. In such problems the acceleration towards the centre of the circle is produced by the horizontal component of the normal reaction of the surface on the particle.

Example 2 One end of a light inextensible string of length l is attached to a fixed point A and a particle P is attached to the other end. The ends of a second string of the same length are attached to P and to a fixed point B at a distance h $(<2l)$ vertically below A. If the particle moves in a horizontal circle with uniform angular speed ω, find the least value of ω for which both strings are taut.



Let $\angle PAB = \theta$, then $h = 2l\cos\theta$ and the radius of the circle described by the particle is $l\sin\theta$.

Hence the acceleration of the particle

$$= \omega^2 l \sin\theta.$$

Applying Newton's second law horizontally and vertically

$$\uparrow T_1\cos\theta - T_2\cos\theta - mg = 0$$
$$\leftarrow T_1\sin\theta + T_2\sin\theta = m\omega^2 l\sin\theta$$

$$\therefore \quad T_1 + T_2 = m\omega^2 l$$

$$T_1 - T_2 = \frac{mg}{\cos\theta} = \frac{2mgl}{h}.$$

Subtracting

$$2T_2 = m\omega^2 l - \frac{2mgl}{h}$$

$$\therefore \quad T_2 = \frac{ml}{h}(h\omega^2 - 2g).$$

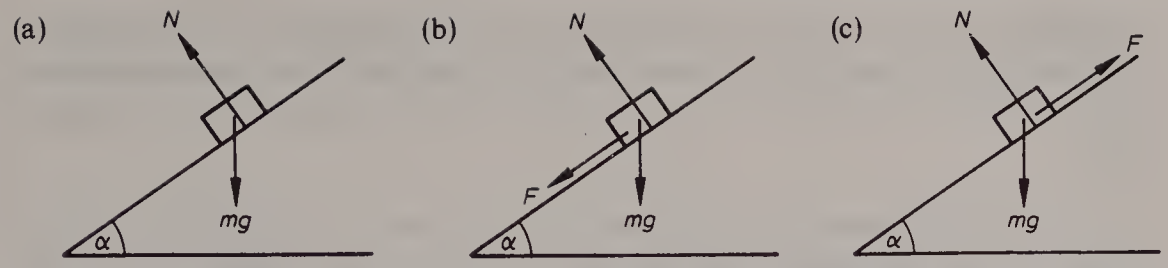
Since $T_1 > T_2$, both strings will be taut if $T_2 \geqslant 0$

i.e. if $h\omega^2 - 2g \geqslant 0$.

Hence the least value of ω for which both strings are taut is $\sqrt{(2g/h)}$.

When a vehicle travels at constant speed round a bend, regarded as a circular arc, its acceleration is directed towards the centre of the arc. If the road surface is horizontal then this acceleration is produced by the frictional force between the wheels and the ground. However, if the road is banked, the horizontal components of the frictional force and of the normal reaction between the vehicle and the ground together provide the required acceleration. [We will ignore at this stage the possibility that the vehicle could overturn.]

Example 3 A car of mass m travels at constant speed v round a bend of radius r on a road banked at an angle α. The coefficient of friction between the car's tyres and the road surface is μ. Find an expression for v^2 given that (a) the car travels with no tendency to slip, (b) the car is about to slip outwards, (c) the car is about to slip inwards.



(a) In this case there is no frictional force acting.
Applying Newton's second law horizontally and vertically

$$\leftarrow N\sin\alpha = mv^2/r$$
$$\uparrow N\cos\alpha - mg = 0$$

$$\therefore \quad v^2 = \frac{rN\sin\alpha}{m} = \frac{r\sin\alpha}{m}.\frac{mg}{\cos\alpha} = rg\tan\alpha.$$

Hence when there is no tendency to slip $v^2 = rg\tan\alpha$.

(b) The frictional force acts down the slope and takes its limiting value μN.
Applying Newton's second law horizontally and vertically

$$\leftarrow N\sin\alpha + \mu N\cos\alpha = mv^2/r$$
$$\uparrow N\cos\alpha - \mu N\sin\alpha - mg = 0$$

$$\therefore \quad v^2 = \frac{rN}{m}(\sin\alpha + \mu\cos\alpha) \quad \text{and} \quad N = \frac{mg}{\cos\alpha - \mu\sin\alpha}.$$

Hence when the car is about to slip outwards

$$v^2 = \frac{rg(\sin\alpha + \mu\cos\alpha)}{\cos\alpha - \mu\sin\alpha}, \quad \text{i.e.} \quad \frac{rg(\tan\alpha + \mu)}{1 - \mu\tan\alpha}.$$

(c) The frictional force acts up the slope and takes its limiting value μN.
Applying Newton's second law horizontally and vertically

$$\leftarrow N\sin\alpha - \mu N\cos\alpha = mv^2/r$$
$$\uparrow N\cos\alpha + \mu N\sin\alpha - mg = 0.$$

Hence when the car is about to slip inwards

$$v^2 = \frac{rg(\sin\alpha - \mu\cos\alpha)}{\cos\alpha + \mu\sin\alpha}, \quad \text{i.e.} \quad \frac{rg(\tan\alpha - \mu)}{1 + \mu\tan\alpha}.$$

[Note that if we substitute $\mu = \tan\lambda$, where λ is the angle of friction, the results in parts (b) and (c) may be written in the form $v^2 = rg\tan(\alpha+\lambda)$ and $v^2 = rg\tan(\alpha-\lambda)$ respectively.]

Exercise 8.3

1. A particle of mass 3 kg is attached by a light inextensible string of length 75 cm to a fixed point. The particle moves in a horizontal circle with constant speed such that the string makes an angle θ with the vertical, where $\tan\theta = 4/3$. Find the tension in the string and the speed of the particle.

2. A particle of mass 5 kg is attached by a light inextensible string of length 3·25 m to a fixed point. The particle moves in a horizontal circle with a constant angular velocity of 2·8 rad s^{-1}. Find the tension in the string and the radius of the circle.

3. A particle of mass m is moving with constant speed v in a horizontal circle of radius r around the smooth inner surface of a hollow cone of semi-vertical angle θ. Find the reaction between the particle and the cone and express v^2 in terms of r, θ and g.

4. A particle moving on the smooth inner surface of a sphere describes a horizontal circle with constant angular speed 2 rad s^{-1}. Find the depth of the circle below the centre of the sphere.

5. A particle of mass 2 kg is suspended from a fixed point by a light elastic string of natural length 1 m and modulus 19·6 N. The particle is moving in a horizontal circle with constant speed v m s^{-1}. Given that the length of the string is 2·25 m, find the value of v.

6. A light inextensible string of length $4a$ has one end fixed at a point A and the other end fixed at a point B which is at a distance $2a$ vertically below A. A smooth ring R of mass m is threaded on the string and moves in a horizontal circle with centre B such that the string is taut. Show that $AR = 5a/2$ and, assuming that the ring is free to move on the string, find its speed.

7. A light inextensible string of length $5a$ has one end fixed at a point A and the other end fixed at a point B which is vertically below A and at a distance $4a$ from it. A particle P of mass m is fastened to the midpoint of the string and moves with speed u, and with the parts AP and BP of the string both taut, in a horizontal circular path whose centre is the midpoint of AB. Find, in terms of m, u, a and g, the tensions in the two parts of the string, and show that the motion described can take place only if $8u^2 \geqslant 9ga$. (JMB)

8. A car is moving in a horizontal circle of radius 100 m on a track banked at an angle α, where $\tan\alpha = 5/16$. Find the speed of the car when there is no tendency for it to slip sideways.

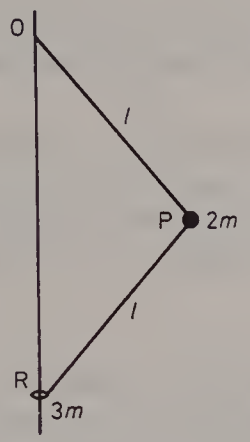
9. A car travels at 90 km/h round a bend of radius 200 m on a road banked at an angle α. Find α, given that there is no tendency to skid sideways.

10. A car travels round a bend of radius 125 m on a road banked at an angle α where $\tan\alpha = 0{\cdot}5$. The coefficient of friction between the car's tyres and the road is $\frac{1}{3}$. Find the maximum and minimum speeds at which the car can travel without slipping sideways.

11. A vehicle is approaching a bend of radius 50 m. The coefficient of friction between its wheels and the road is $\frac{1}{2}$. If the road surface is horizontal, find the

maximum speed at which the vehicle can negotiate the bend without skidding. If the vehicle is to drive round the bend at $20\,\mathrm{m\,s^{-1}}$ without skidding, find the least angle at which the road must be banked.

12.



The figure shows a particle P of mass $2m$ which is attached to a fixed point O by a light inextensible string of length l. A ring R of mass $3m$, which is attached to P by another light inextensible string of length l, is free to slide on a smooth vertical wire passing through O.

The plane OPR rotates about the wire with constant angular velocity ω, where $\omega^2 > 4g/l$. Show that $OR = 8g/\omega^2$. What happens if $\omega^2 \leqslant 4g/l$? (C)

Exercise 8.4 (miscellaneous)

1. A record is rotating at $33\frac{1}{3}$ rev/min. Find its angular speed in $\mathrm{rad\,s^{-1}}$. Find also the speed of a point on the record at a distance of 12 cm from the centre.

2. A van, whose wheels are of diameter 0·8 m, is travelling at 72 km/h. If the wheels do not slip on the road, find the number of revolutions made each second and hence the angular velocity of the wheels in $\mathrm{rad\,s^{-1}}$.

3. A point P moves such that at time t its position vector is $2t\mathbf{i} + (3-t)\mathbf{j}$. Find an expression for the angular velocity of P about the origin O at time t.

4. A particle A is moving in a horizontal circle of radius 3 m with constant speed $6\,\mathrm{m\,s^{-1}}$. A second particle B is moving in a concentric circle of radius 9 m with the same constant speed. At a given instant both particles are due east of the centre of the circles and are moving northwards. Find, in magnitude and direction, the velocity of A relative to B after a time $\frac{1}{4}\pi$ s. (O & C)

5. On an icy morning a car travelling at 42 km/h drives safely round a bend of radius 100 m but skids on a bend of radius 25 m. Assuming that the road surface is horizontal, find the range of possible values of the coefficient of friction between the car's tyres and the road.

6. One end of a light inextensible string of length 1·2 m is attached to a fixed point A on a smooth horizontal surface. A particle of mass 2 kg is attached to the other end C and a particle of mass 1 kg is attached to the mid-point B. Each

particle is moving in a horizontal circle centre A, such that the string remains straight. If the particle at C is moving at a constant speed of $6\,\mathrm{m\,s^{-1}}$, find the tensions in BC and AB.

7. Show that when a particle moves in a circle of radius r with constant speed u its acceleration is of magnitude u^2/r and is directed towards the centre of the circle.

A particle of mass m is attached to one end A of a light inelastic string of length a, the other end being attached to a fixed point O. A moves in a horizontal circle below O with constant speed $\sqrt{(3ag/2)}$. Find (i) the angle that OA makes with the vertical, (ii) the tension in the string, (iii) the time taken to complete one revolution. (SU)

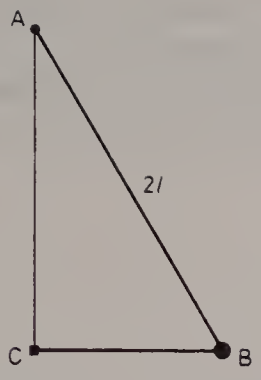
8. A particle P is attached by a light inextensible string of length l to a fixed point O. The particle is held with the string taut and OP at an acute angle α to the downward vertical, and is then projected horizontally at right angles to the string with speed u chosen so that it describes a circle in a horizontal plane. Show that $u^2 = gl\sin\alpha\tan\alpha$. The string will break when the tension exceeds twice the weight of P. Find the greatest possible value of α. (JMB)

9. A particle moves in a circle, of radius a and centre O, at constant speed u. Prove that the acceleration of the particle is u^2/a towards O. The end A of a light rigid rod AB of length 1 metre is smoothly hinged at a fixed point. A ring is threaded on AB and the coefficient of friction between the ring and the rod is $\frac{1}{3}$. The rod is made to rotate horizontally at 2 rad/s about A. Show that the ring can remain at rest relative to the rod at any position between A and C, where $AC = 5/6$ metres. The ring is placed at C and a second identical ring is threaded on the rod and placed at the mid-point of AB. The rings are connected by a taut, light inextensible string. When the rod rotates horizontally at ω rad/s, the rings are on the point of sliding on the rod. Given that the mass of each ring is 0·2 kg, calculate (i) the value of ω, (ii) the tension in the string. [Take the acceleration due to gravity to be $10\,\mathrm{m/s^2}$.] (AEB 1977)

10. One end of a light inelastic string of length a is attached to a particle P of mass m, and the other end of the string is attached to a fixed point O. P moves with constant speed in a horizontal circle whose centre C is a fixed point vertically below O. If the angular velocity at which CP rotates is ω, show that $OC = g/\omega^2$. A second string, also of length a, is fixed to P and C, and the new system rotates about OC with uniform angular velocity Ω and with both strings taut. Find expression in terms of m, a, ω and Ω for the tensions in OP and PC, and deduce that $\Omega > \omega\sqrt{2}$. (C)

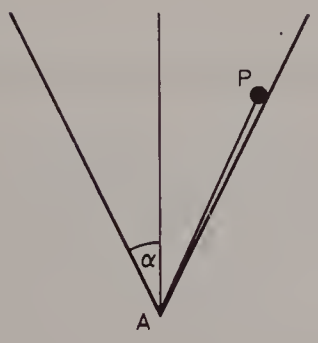
11. A particle moves in a horizontal circle on the smooth inner surface of a fixed spherical bowl of radius a. If the depth of the circle below the centre of the sphere is h, find an expression for the speed of the particle.

12.



The ends of a light inextensible string ABC of length $3l$ are attached to fixed points A and C, C being vertically below A at a distance $\sqrt{3}l$ from A. At a distance $2l$ along the string from A a particle B of mass m is attached. When both portions of the string are taut, B is given a horizontal velocity u, and then continues to move in a circle with constant speed. Find the tensions in the two portions of the string and show that the motion is possible only if $u^2 \geqslant \frac{1}{3}gl\sqrt{3}$. (JMB)

13.



A circular cone of semi-vertical angle α is fixed with its axis vertical and its vertex, A, lowest, as shown in the diagram. A particle P of mass m moves on the inner surface of the cone, which is smooth. The particle is joined to A by a light inextensible string AP of length l. The particle moves in a horizontal circle with constant speed v and with the string taut. Find the reaction exerted on P by the cone. Find the tension in the string and show that the motion is possible only if $v^2 \geqslant gl\cos\alpha$. (JMB)

14. A car moves in a horizontal circle of radius 500 m on a track which is banked at an angle α to the horizontal. When the speed is 60 m/s there is no sideways force on the wheels. Find the angle α. Find also the maximum speed round the track to avoid slipping if the coefficient of friction is 0·5. (AEB 1978)

15. A light elastic string AB of natural length a and modulus of elasticity mg is joined to a light inextensible string BC of length a. The ends A and C of the string are fastened to two fixed points with A vertically above C and $AC = a$. A particle of mass m is fixed at B and rotates with speed v in a horizontal circle. Show that if AB makes an angle $\pi/6$ with the downward vertical then $v^2 = \frac{1}{2}\sqrt{3}ag$, and find the tension in BC. (O)

16. A car moves with constant speed in a horizontal circle of radius r on a track which is banked at an angle α to the horizontal, where $\tan\alpha = \frac{3}{4}$. The coefficient of friction between the tyres and the track is $\frac{1}{2}$. Find, in terms of r and g, the range of speeds at which the car can negotiate this bend without the tyres slipping on the road surface. Show that the greatest possible speed is $\sqrt{11}$ times the least possible speed. [It may be assumed that the car will not overturn at these speeds.] (L)

17. A small ring of mass m can slide on a smooth fixed vertical rod. One end of a light inextensible string of length $2a$ is fastened to the ring, and its other end is fastened to a point of the rod. A particle of mass $4m$ attached to the mid-point of the string is moving in a horizontal circle. Show that the angular velocity of the particle in this circle is greater than $\sqrt{(3g/2a)}$. Find the speed of the particle if the tension in the lower half of the string is $4mg$. (L)

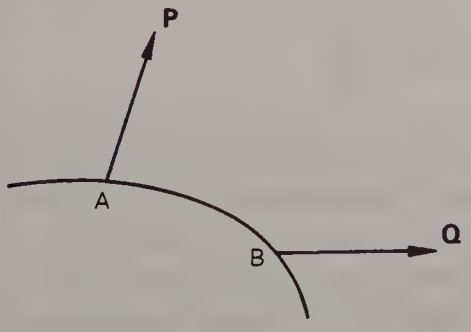
9 Coplanar forces; centre of gravity

9.1 Parallel forces, moments and couples

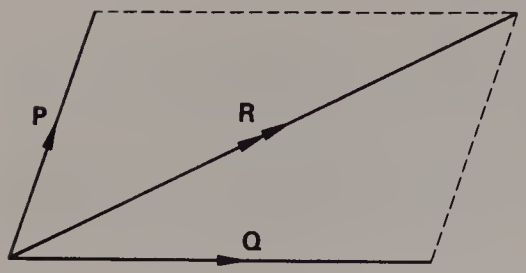
In earlier chapters we considered forces acting on bodies small enough to be treated as particles and represented by single points in space. Applying Newton's laws of motion, we found that if the forces acting on such a body have a non-zero resultant then the effect produced is an acceleration in the direction of the resultant. However, if the vector sum of the forces on the body is zero, the body either remains at rest or continues to move with unchanged velocity.

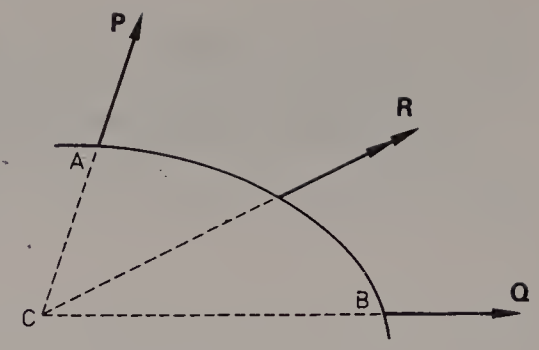
There are many situations in which this elementary approach is inadequate. From experience we know that the forces on a body may tend to produce rotation as well as the types of motion discussed in earlier work. A projectile, such as a cricket ball, may spin as it moves. A block placed on an inclined plane may topple rather than slide. Thus, when solving a problem in which the rotational effects of forces could be important, we must use a more realistic representation of the bodies involved. We will then find that the effect produced by a set of forces depends not only on the magnitude and direction of the resultant, but also on its line of action.

We now develop these ideas by considering the effect of coplanar forces on a variety of rigid bodies i.e. bodies whose shapes are not changed by the forces acting on them.

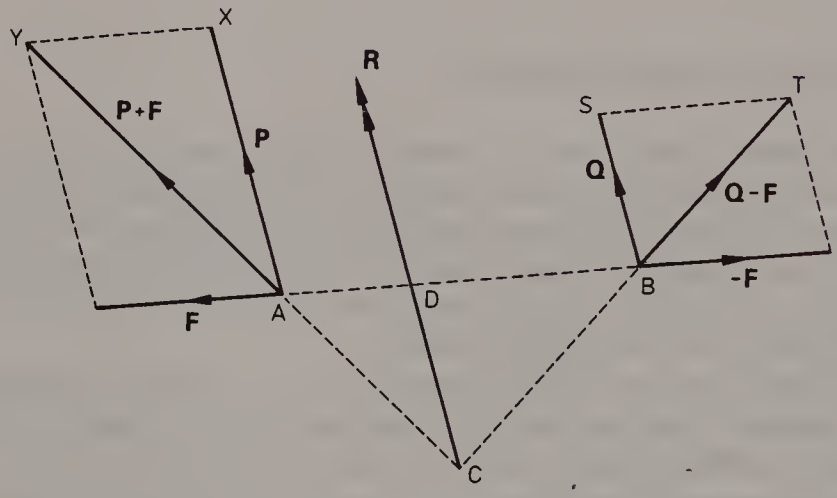


Let **P** and **Q** be *non-parallel* coplanar forces acting at points A and B of a rigid body. The magnitude and direction of their resultant **R** can be found using the parallelogram law. If the lines of action of **P** and **Q** intersect at the point C, then the line of action of **R** also passes through C.





To find the line of action of the resultant of *parallel* forces **P** and **Q** acting at points A and B of a rigid body, we introduce forces **F** and $-\mathbf{F}$ acting along AB. The application of these equal and opposite forces will have no effect on the body and leave the line of action of the resultant unaltered.



The diagram shows how the parallelogram law is used to construct a point C on the line of action of the resultant **R**, when **P** and **Q** are *like parallel forces* i.e. forces in the same sense.

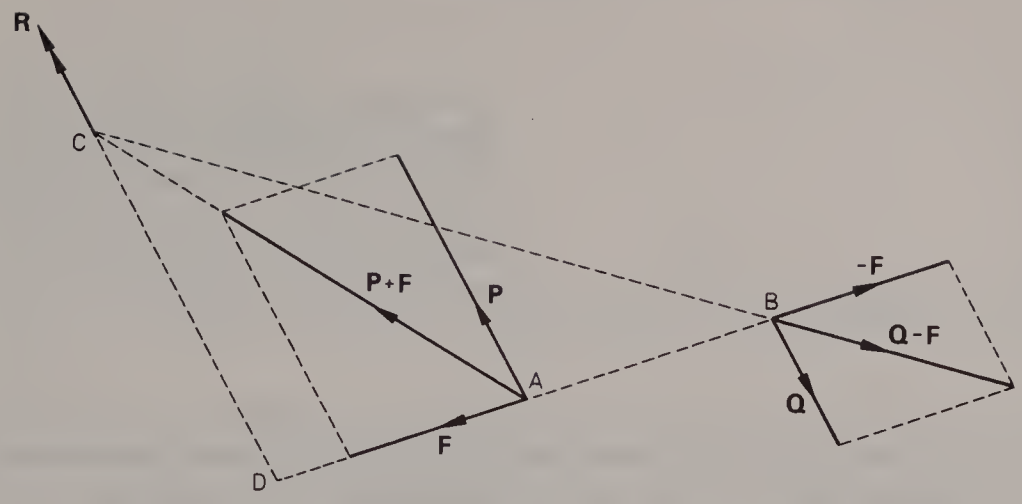
In similar triangles ADC, YXA: $\dfrac{AD}{DC} = \dfrac{YX}{XA} = \dfrac{F}{P}$

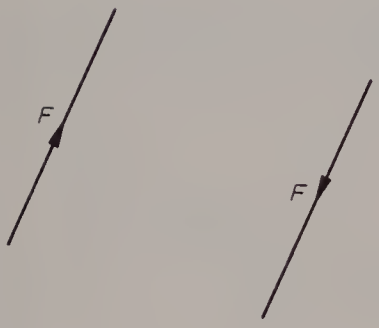
In similar triangles DBC, STB: $\dfrac{DC}{DB} = \dfrac{SB}{ST} = \dfrac{Q}{F}$

$$\therefore \quad \frac{AD}{DB} = \frac{AD}{DC} \times \frac{DC}{DB} = \frac{F}{P} \times \frac{Q}{F} = \frac{Q}{P}$$

By vector addition, the magnitude of **R** is $P + Q$.
Hence the resultant of the like parallel forces **P** and **Q** is of magnitude $P + Q$ and divides AB internally in the ratio $Q:P$.

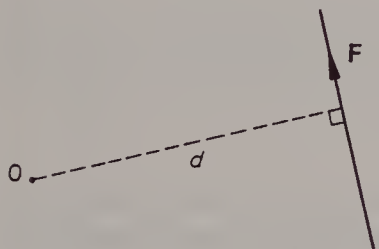
Using similar arguments it can be shown that the resultant of *unlike parallel forces* **P** and **Q**, where $P > Q$, is of magnitude $P - Q$ and divides AB externally in the ratio $Q:P$.





A pair of equal unlike parallel forces has zero resultant, but clearly does have a turning effect. Such a system is called a *couple*.

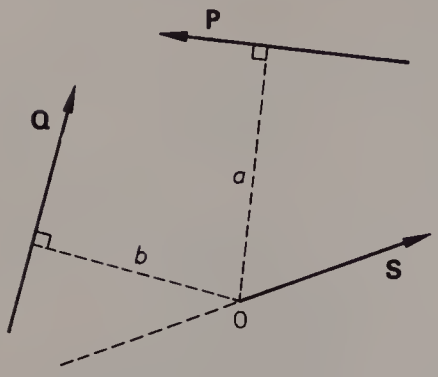
A quantity called the *moment of a force* or *torque* is used as a measure of "turning effect". The moment of a force about a given point is defined as the product of the magnitude of the force and its distance from the point. If the line of action of a force **F** is at a perpendicular distance d from a point O, then the moment of **F** about O is Fd.



[In three-dimensional work the quantity Fd is more accurately described as the moment or torque about an axis through O perpendicular to the plane containing O and the force **F**.]

The SI unit of torque is the newton metre (N m). Thus, if a force of magnitude 40 N acts along a line whose perpendicular distance from a point A is 2 m, then the moment of the force about A is 80 N m.

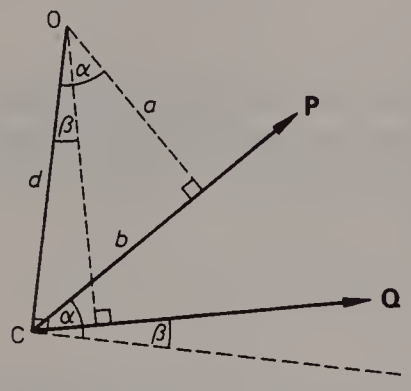
When finding the sum of the moments of a set of coplanar forces about a point, the moment of each force is taken to be positive or negative according to the sense of the rotation it would produce. The moment of a force about any point on its line of action is zero.

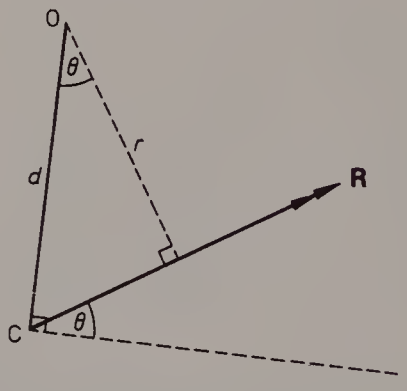


For instance, if forces **P**, **Q** and **S** act as shown in the diagram, then the sum of the anti-clockwise moments of **P**, **Q** and **S** about O is $Pa - Qb$, i.e. the combined turning effect of the system about O is $Pa - Qb$ anti-clockwise.

We now show that the sum of the moments of two coplanar forces about any point in their plane is equal to the moment of their resultant about the same point.

Suppose that **P** and **Q** are forces whose lines of action intersect at a point C. Let **R** be their resultant and let O be a point in the same plane such that $OC = d$.





Using the notation of the above diagrams, the sum of the moments of **P** and **Q** about O

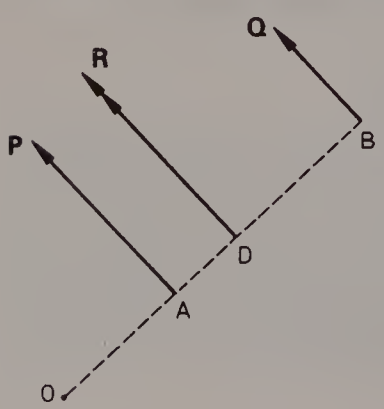
$$= Pa + Qb = Pd\cos\alpha + Qd\cos\beta = d(P\cos\alpha + Q\cos\beta)$$

The moment of **R** about O

$$= Rr = Rd\cos\theta = d(R\cos\theta).$$

Considering the components of **P**, **Q** and **R** in a direction perpendicular to OC, we have

$$P\cos\alpha + Q\cos\beta = R\cos\theta.$$



Hence the sum of the moments about O of **P** and **Q** is equal to the moment about O of **R**.

If **P** and **Q** are like parallel forces with resultant **R**, as shown in the diagram, then

$$\frac{AD}{DB} = \frac{Q}{P}$$

$$\therefore \qquad P \times AD = Q \times DB$$

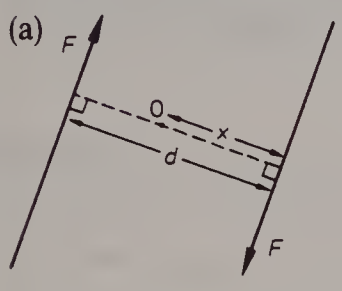
Thus the sum of the moments of **P** and **Q** about O

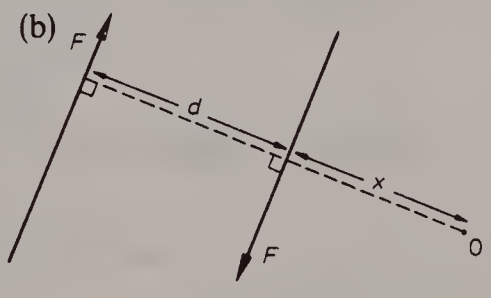
$= P \times OA + Q \times OB$
$= P(OD - AD) + Q(OD + DB)$
$= (P + Q)OD - P \times AD + Q \times DB$
$= R \times OD$
= the moment of **R** about O.

By extending these arguments to any set of coplanar forces with a non-zero resultant it is possible to establish the following general result called the *principle of moments.*

> The sum of the moments of a system of coplanar forces about any point in their plane is equal to the moment of their resultant about the same point

A couple has zero resultant but non-zero moment. Consider a couple consisting of a pair of equal and opposite forces of magnitude F with lines of action at a distance d apart. Suppose that O is any point in the plane of this couple.





In diagram (a) the sum of the moments about O

$= F(d - x) + Fx = Fd - Fx + Fx = Fd.$

In diagram (b) the sum of the moments about O

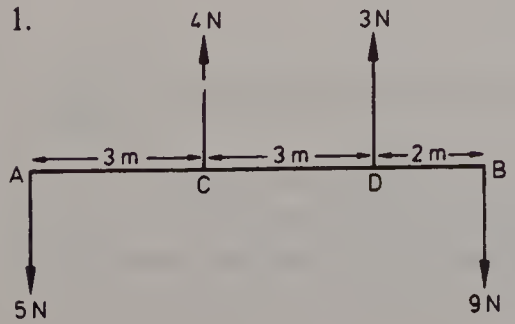
$= F(d + x) - Fx = Fd + Fx - Fx = Fd.$

Since both results are independent of x, the couple has the same moment, i.e. Fd clockwise, for all positions of O.

Hence the *moment of a couple* about any point in its plane is the product of the magnitude of one of the forces and the perpendicular distance between the forces.

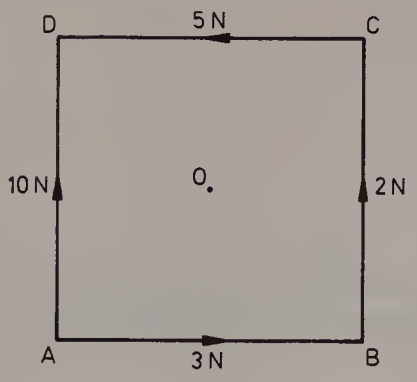
Exercise 9.1

1.



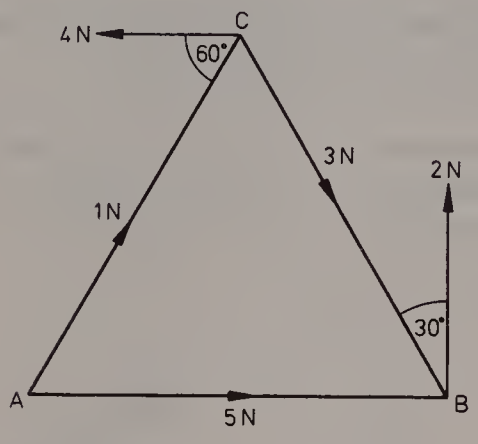
Forces act on a rod AB of length 8 m as shown in the diagram. Find the sum of the anti-clockwise moments of these forces about each of the points A, B, C and D.

2.



ABCD is a square of side 6 m with centre *O*. Forces act along the sides *AB*, *BC*, *CD*, *AD* as shown in the diagram. Find the sum of the anti-clockwise moments of these forces about each of the points *A*, *B*, *C*, *D* and *O*.

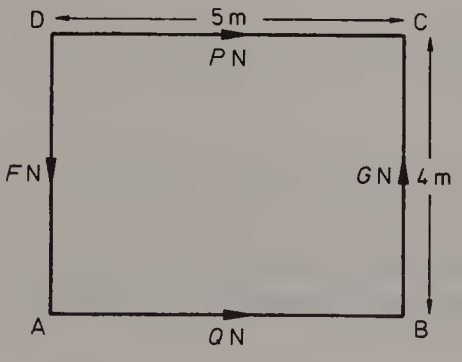
3.



ABC is an equilateral triangle of side 2 m. Find the sum of the anti-clockwise moments of the forces shown in the diagram about each of the points *A*, *B* and *C*. [You may leave your answers in surd form.]

In questions 4 to 9 forces **F**, **G**, **P** and **Q** act along the sides of the rectangle *ABCD* as shown in the diagram.

4.



The resultant of **P** and **Q** acts through a point *X* on *BC*. Find the magnitude of the resultant and the length of *BX* if (a) $P = Q = 3$, (b) $P = 6, Q = 2$.

5. The resultant of **F** and **G** acts through a point *Y* on *AB* produced. Find the magnitude of the resultant and the length of *AY* if
(a) $F = 1, G = 2$, (b) $F = 2, G = 3$.

6. Find the moment of the couple formed by forces **F** and **G** if $F = G = 10$.

7. If the forces **F** and **G** form a couple of moment 20 N m, find *F* and *G*.

8. The resultant of **F** and **P** acts through a point *S* on *AB*. Find the magnitude of the resultant and the length of *AS* if (a) $F = P = 5$, (b) $F = 12, P = 9$.

9. The resultant of **G** and **Q** acts through a point T on DA produced. Find the magnitude of the resultant and the length of AT if (a) $G = Q = 4$, (b) $G = 6, Q = 2.5$.

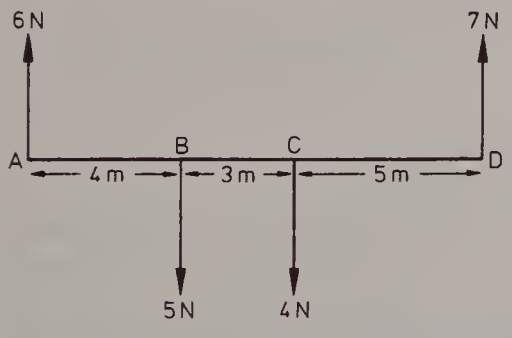
10. Four forces each of magnitude 3 N act along the sides BA, BC, CD, AD of a square $ABCD$, in the directions indicated by the order of the letters. Find the magnitude and the line of action of their resultant.

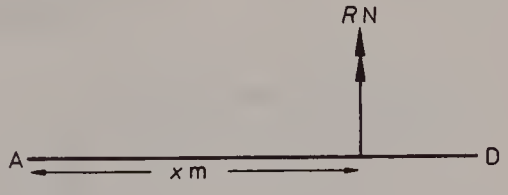
11. Three forces each of magnitude 10 N act along the sides AB, DE, AF of a regular hexagon $ABCDEF$ of side 2 m. Find the sum of the moments of these forces about each of the points A, B, C, D, E and F. Find also the magnitude and the line of action of the resultant of the three forces.

9.2 Systems of coplanar forces

We consider first a system of parallel coplanar forces with non-zero resultant. The magnitude and direction of the resultant is found by vector addition. Its line of action can be determined using the principle of moments.

Example 1 Find the magnitude and line of action of the resultant of the system of parallel forces shown in the diagram.





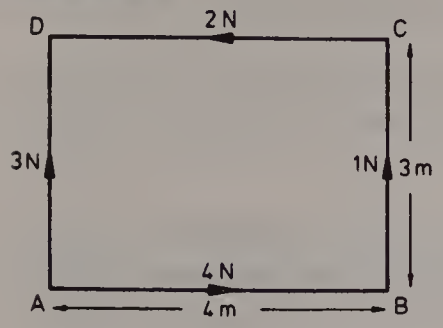
Let the resultant of the given system of forces be of magnitude R N acting at a distance x m from A.
Comparing magnitudes and moments about A)

$$\uparrow \quad R = 6 - 5 - 4 + 7 = 4$$
$$A) \quad Rx = 0 - 5 \times 4 - 4 \times 7 + 7 \times 12 = 36$$
$$\therefore \quad R = 4 \quad \text{and} \quad x = 9.$$

Hence the resultant has magnitude 4 N and its line of action cuts AD at a point 9 m from A.

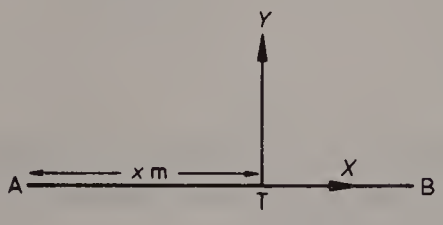
Similar methods can be applied to any set of coplanar forces with a non-zero resultant. The magnitude and direction of the resultant is found by means of a vector polygon or by resolving in two perpendicular directions. As before the line of action of the resultant is determined using the principle of moments.

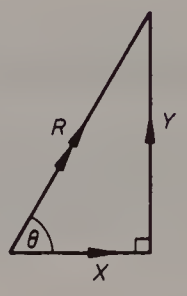
Example 2



The resultant of the system of forces shown in the diagram is a force **R** acting through a point T on AB. Find the magnitude and direction of **R** and the length of AT.

Let X and Y be the components of **R** parallel and perpendicular to AB.





Resolving parallel and perpendicular to AB

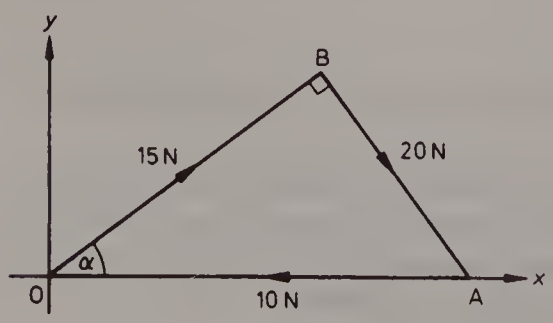
$$\rightarrow \qquad X = 4 - 2 \qquad \therefore \quad X = 2$$
$$\uparrow \qquad Y = 3 + 1 \qquad \therefore \quad Y = 4$$

Taking moments about A

$$Yx = 1 \times 4 + 2 \times 3 = 10 \qquad \therefore \quad x = 2{\cdot}5$$
$$\therefore \quad R = \sqrt{(X^2 + Y^2)} = \sqrt{(4 + 16)} = \sqrt{20} = 2\sqrt{5} \quad \text{and}$$
$$\tan\theta = \frac{Y}{X} = \frac{4}{2} = 2, \quad \theta \approx 63{\cdot}4°.$$

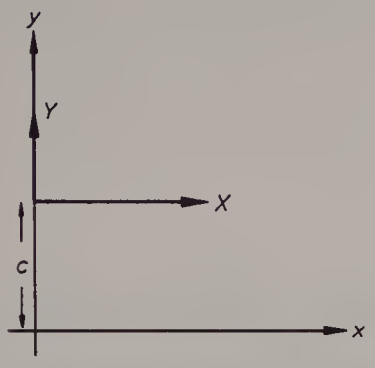
Hence the resultant has magnitude $2\sqrt{5}$ N and acts at 63·4° to AB. The length of AT is 2·5 m.

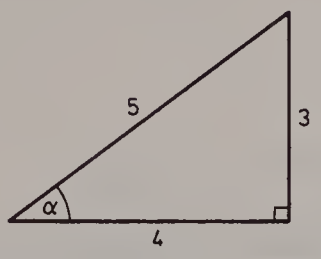
Example 3



Find the equation of the line of action of the resultant of the forces shown in the diagram, given that $\tan\alpha = \frac{3}{4}$ and that A is the point (7, 0).

Let X and Y be the components of the resultant in the directions of the x- and y-axes respectively. Let the line of action cut the y-axis at the point $(0, c)$.





$$\Rightarrow \quad \begin{aligned} \tan\alpha &= \tfrac{3}{4} \\ \sin\alpha &= \tfrac{3}{5}, \\ \cos\alpha &= \tfrac{4}{5}. \end{aligned}$$

Resolving in the directions of the x- and y- axes,

$$\rightarrow \qquad X = 15\cos\alpha + 20\sin\alpha - 10$$
$$\uparrow \qquad Y = 15\sin\alpha - 20\cos\alpha$$

$$\therefore \quad X = 12 + 12 - 10 = 14, \quad Y = 9 - 16 = -7.$$

Taking moments about O: $Xc = 20 \times 7\cos\alpha$

$$\therefore \quad c = \frac{20 \times 7\cos\alpha}{X} = \frac{20 \times 7 \times 4}{14 \times 5} = 8$$

Hence the line of action of the resultant has gradient $Y/X = -7/14 = -1/2$ and intercept on the y-axis $c = 8$,

$\therefore$ the equation of the line of action is

$$y = -\tfrac{1}{2}x + 8 \quad \text{i.e.} \quad x + 2y = 16.$$

Two systems of forces are said to be *equivalent* if their effects are the same. In particular, two sets of coplanar forces are equivalent if

(i) the sums of the components of the forces in any given direction are equal and

(ii) the sums of the moments of the forces about any given point are equal.

Thus, the *resultant* of a system of forces is the single force equivalent to the whole system.

It also follows from conditions (i) and (ii) that any two coplanar *couples* with the same moment are equivalent. Similarly a set of coplanar couples with moments $G_1, G_2, G_3, \ldots, G_n$ is equivalent to a single couple with moment

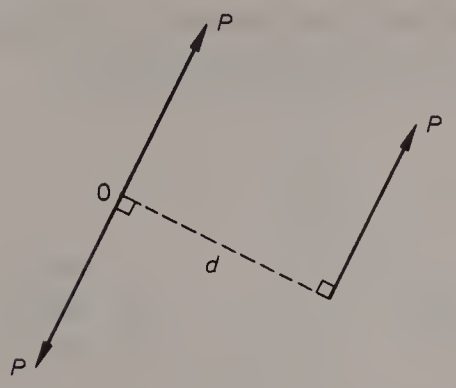
$$G_1 + G_2 + G_3 + \ldots + G_n.$$

We next consider the resultant of a force **P** acting at a point O and a couple of moment G in the same plane. The couple is equivalent to a pair of equal and opposite forces of magnitude P and perpendicular distance d apart, provided that

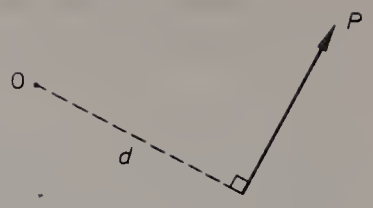
$$Pd = G.$$

Therefore the force and the couple are equivalent to the three forces shown in diagram (a) and hence to the single force shown in diagram (b).

(a)



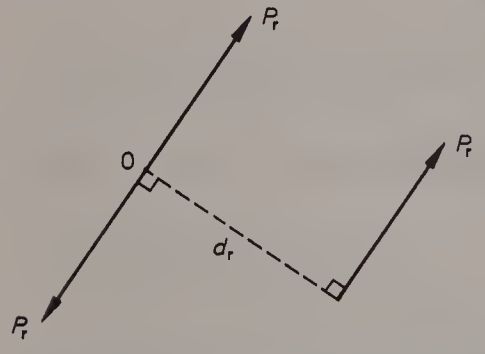
(b)



Thus the resultant of a force **P** acting at O and a couple of moment G is a force **P** acting at a distance d from O, where $d = G/P$.

Reversing the argument we may also deduce that a force **P** acting at a distance d from a point O is equivalent to a force **P** acting at O together with a couple of moment Pd.

This result can be used to find a simple equivalent system for any set of coplanar forces. Let $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$ be a system of coplanar forces acting at distances $d_1, d_2, \ldots, d_n$ respectively from a fixed point O in their plane. A typical force $\mathbf{P}_r$ acting at a distance d_r from O is equivalent to a force $\mathbf{P}_r$ acting at O together with a couple of moment $P_r d_r$. Applying this reasoning to all the forces in the system we obtain an equivalent system of forces $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$ acting at 0 together with couples of moments $P_1 d_1, P_2 d_2, \ldots, P_n d_n$.



If **R** is the vector sum of the forces and G is the sum of the moments of the original forces about O, then this new system can be replaced by a single force **R** acting at O and a couple of moment G.

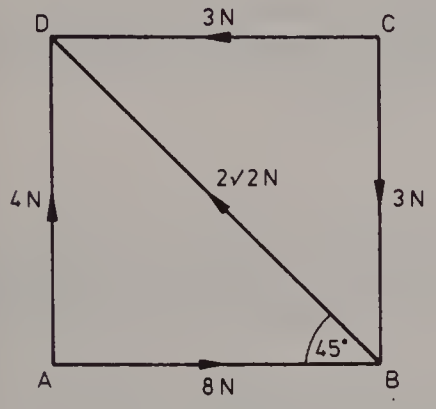
> Thus, in general, any system of coplanar forces is equivalent to a single force acting through a given point together with a couple.

We consider further four possible types of system.

(1) $R = 0$, $G = 0$. The system is in equilibrium.
(2) $R = 0$, $G \neq 0$. The system is equivalent to a couple of moment G.
(3) $R \neq 0$, $G = 0$. The system is equivalent to a single force **R** acting through O.
(4) $R \neq 0$, $G \neq 0$. The system is equivalent to a single force **R** acting at a distance G/R from O.

> Hence any system of coplanar forces which is not in equilibrium may be reduced either to a single force or to a couple.

Example 4 A square $ABCD$ of side 3 m has forces of magnitude 8, 3, 3, 4 and $2\sqrt{2}$ newtons acting along AB, CB, CD, AD and BD respectively. If the system is reduced to a force acting through A together with a couple, find the magnitude and direction of the force and the moment of the couple. If the system is reduced to a single force acting through a point E on AB at a distance d metres from A, find d.



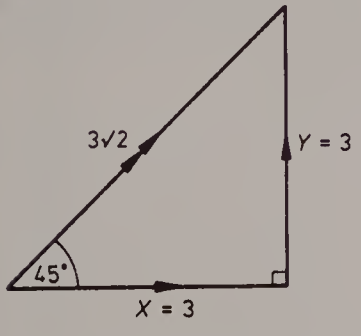
Suppose that the given system is equivalent to forces X and Y acting along AB and AD respectively, together with a couple of moment G in the sense $ABCD$.

Resolving in the directions of AB and AD:

$$\rightarrow \qquad X = 8 - 3 - 2\sqrt{2}\cos 45° = 3$$
$$\uparrow \qquad Y = 4 - 3 + 2\sqrt{2}\sin 45° = 3$$

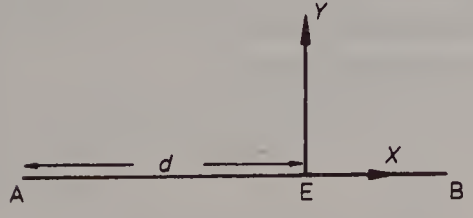
Taking moments about A ↻:

$$G = 3 \times 3 - 3 \times 3 + 2\sqrt{2} \times 3\sin 45° = 6$$



Hence the system is equivalent to a force of $3\sqrt{2}$ N acting along AC together with a couple of moment 6 N m in the sense $ABCD$.

The single equivalent force through E has moment G about A and components X and Y in the directions of AB and AD respectively.

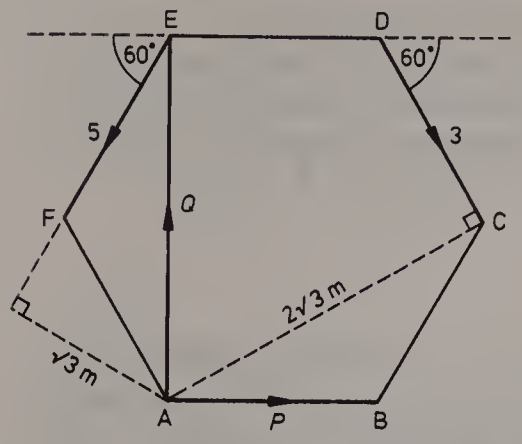


Taking moments about A ↻:

$$G = Yd$$
$$\text{i.e.} \quad 6 = 3d$$
$$\therefore \quad d = 2.$$

Example 5 $ABCDEF$ is a regular hexagon of side 2 m. Forces of magnitude P, 3, 5 and Q newtons act along AB, DC, EF and AE respectively. Show that this system of forces is not in equilibrium.

(a) If the system is equivalent to a couple, find its moment and the values of P and Q.
(b) If the system is equivalent to a single force through E, find P.



The sum of the moments of the forces about A (in N m)
$= 3 \times 2\sqrt{3} - 5 \times \sqrt{3} = \sqrt{3}$.
Since this sum is not zero, the forces are not in equilibrium.

(a) If the system is equivalent to a couple, it will have the same moment about any point in the plane,
$\therefore$ the moment of the couple is $\sqrt{3}$ N m.
The sum of the components of the forces in a couple is zero in any direction,
$\therefore$ resolving in the directions of AB and AE:

$$\rightarrow \qquad P + 3\cos 60° - 5\cos 60° = 0$$
$$\uparrow \qquad Q - 3\sin 60° - 5\sin 60° = 0$$

$\therefore \quad P = 2\cos 60° = 1 \quad$ and $\quad Q = 8\sin 60° = 4\sqrt{3}$.

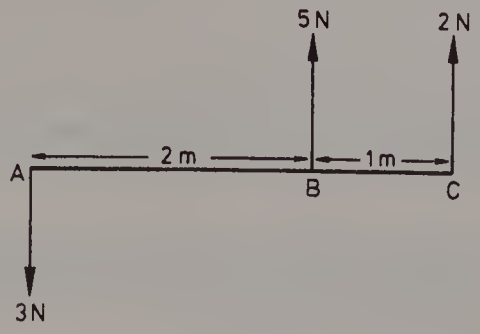
(b) If the system is equivalent to a single force through E, then the sum of the moments about E will be zero.
Taking moments about E: $P \times 2\sqrt{3} - 3 \times \sqrt{3} = 0$

$$\therefore \quad P = \frac{3\sqrt{3}}{2\sqrt{3}} = 1{\cdot}5.$$

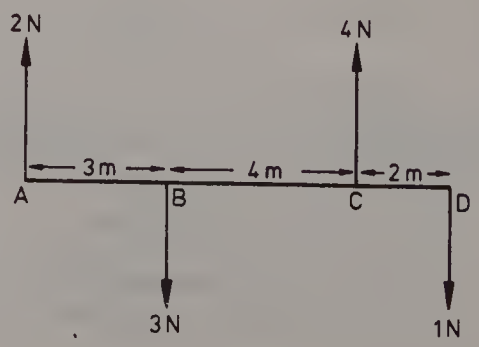
Exercise 9.2

In questions 1 to 4 the resultant of the given forces acts through a point T on AB or AB produced. In each case find the magnitude and direction of the resultant and the length of AT. [In question 4 $ABCDEF$ is a regular hexagon.]

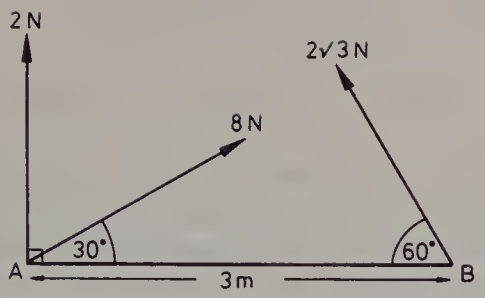
1.



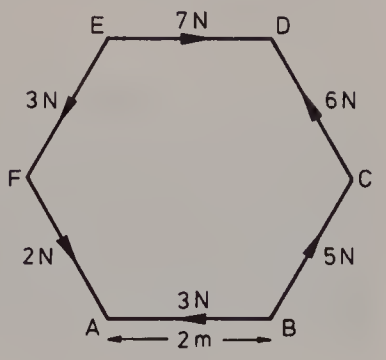
2.



3.



4.



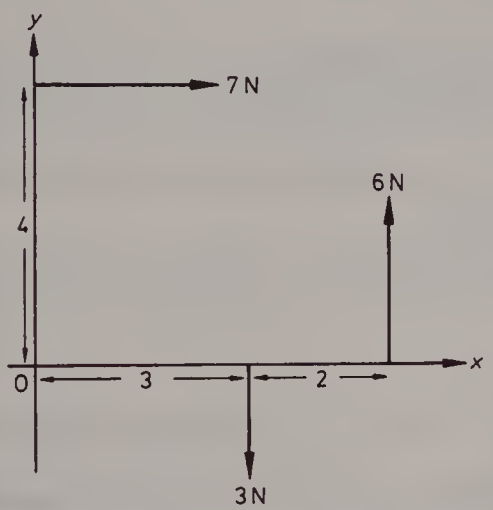
5. Forces of magnitude $8P, 7P, 5P$ and $3P$ act along the sides AB, BC, CD and DA respectively of a square $ABCD$ of side $2a$. The resultant of this system of forces cuts AB produced at E. Find

(a) the magnitude and direction of the resultant,

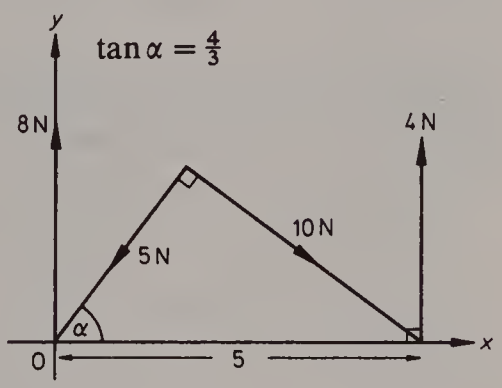
(b) the length of AE.

In questions 6 and 7 find the equation of the line of action of the resultant of the given forces.

6.



7.



In questions 8 to 19 a system of forces acts in the plane of perpendicular axes Ox and Oy. The sums of the components of the forces in the directions of Ox and Oy are X and Y respectively. The sums of the anti-clockwise moments of the forces about three non-collinear points A, B and C in the plane are G_A, G_B and G_C respectively. In each question use the given facts to decide which of the following statements could be true.

a: the system is in equilibrium,
b: the system is equivalent to a couple,
c: the system is equivalent to a single force.

8. $X = 0, Y = 0$.

9. $X = 0, Y \neq 0$.

10. $Y = 0, G_A = 0$.

11. $X = 0, G_B \neq 0$.

12. $Y \neq 0, G_C = 0$.

13. $G_A \neq 0, G_B = 0$.

14. $G_A = G_B = 0$. 15. $G_A = G_B \neq 0$. 16. $G_A = G_B \neq G_C$.

17. $G_A = G_B = G_C = 0$. 18. $G_A = G_B = G_C \neq 0$ 19. $X = 0, G_A = G_B = 0$.

20. In the square $OABC$ of side 2 m, O, A and B are the points (0, 0), (2, 0) and (2, 2) respectively. Forces of magnitude 5, 8, 3 and $5\sqrt{2}$ newtons act along OA, BA, OC and AC respectively. Show that this system of forces is equivalent to a couple and find its moment. If a fifth force of magnitude $6\sqrt{2}$ newtons acts along OB, find the equation of the line of action of the resultant of the enlarged system.

21. In a rectangle $ABCD$, $AB = 12$ m, $BC = 5$ m. Forces of magnitude 6, P, 18, 10 and Q newtons act along AB, CB, CD, AD and DB respectively. Show that this system of forces is not in equilibrium. Find P and Q given that the system reduces to (a) a couple, (b) a single force acting along DC.

22. In a rectangle $ABCD$, $AB = 3$ m, $BC = 2$ m. Forces of magnitude 5, 8, 12, 6 and P newtons act along BA, BC, DC, AD and EA respectively, where E is a point on CD. Given that the forces are in equilibrium, find P and the distance DE. If the force along EA is replaced by an equal force acting through C, show that the system now reduces to a couple and find its moment.

23. $ABCDEF$ is a regular hexagon of side 1 m. Forces of magnitude 4, 6, 10, 8, P and 2 newtons act along AB, BC, CD, ED, FE and AF respectively, directions being indicated by the order of the letters.

(a) If the resultant passes through the centre of the hexagon find the value of P and the magnitude and direction of the resultant.

(b) The force in FE is replaced by another so that the new system reduces to a couple. Find the magnitude of the force and the moment of the couple indicating its sense. (SU)

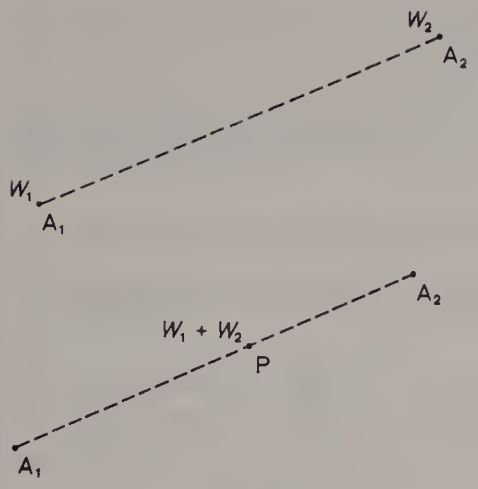
24. The points A, B and C have coordinates $(9a, 0), (0, -4a)$ and $(6a, 4a)$ respectively referred to the coordinate axes Ox and Oy.

(a) Forces $12P$, $15P$, $5P$ and P act along $\overrightarrow{OA}, \overrightarrow{BC}, \overrightarrow{CA}$ and $\overrightarrow{OB}$ respectively. Calculate the magnitude of the resultant of these forces and the equation of the line of action of this resultant.

(b) Forces $15P$, S and T act along $\overrightarrow{BC}, \overrightarrow{OA}$ and $\overrightarrow{CA}$ respectively.
Given that these forces reduce to a couple, calculate (i) the values of S and T, in terms of P, (ii) the magnitude of the couple, in terms of a and P, (iii) the sense of the couple. (AEB 1979)

9.3 Centre of gravity; centre of mass

A solid body may be regarded as a tightly packed collection of small particles. The *weight* of the body is the resultant of the weights of the particles. For relatively small bodies on or near the earth's surface the weights of the constituent particles are assumed to form a system of parallel forces acting vertically downwards.

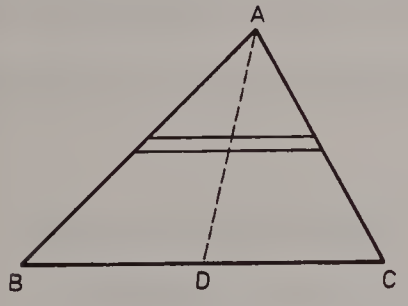


Suppose that at points A_1 and A_2 of a body there are particles of weights W_1 and W_2 respectively. As shown in §9.1, the resultant of the weights of these particles is a force of magnitude $W_1 + W_2$ acting through the point P which divides A_1A_2 in the ratio $W_2 : W_1$.

By extending this argument to include all the particles forming the body, we deduce that the resultant weight force is equal to the sum of the weights of the particles and acts through a fixed point in the body.

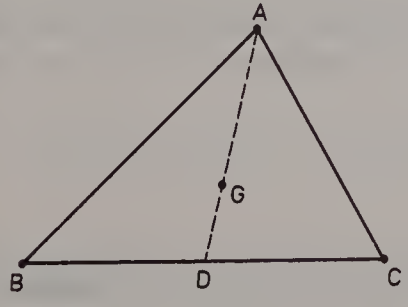
The *centre of gravity* of a body is the fixed point through which its weight acts.

The *symmetries* of a body can be used to determine its centre of gravity. If the particles in a body are evenly distributed about an axis or a plane of symmetry, then the centre of gravity of the body must lie on this axis or plane. Thus the centre of gravity of a thin uniform rod is at its mid-point and the centre of gravity of a uniform sphere is at its centre.



To find the centre of gravity of a *uniform triangular lamina ABC*, suppose that the triangle is divided into n strips parallel to the side BC. If the value of n is increased the strips become thin uniform rods with centres of gravity at their mid-points. Hence the centre of gravity of the whole lamina must lie on the line joining these mid-points i.e. on the median AD. Similarly the centre of gravity must also lie on the medians through B and C.

Thus the centre of gravity of a uniform triangular lamina is at the point of intersection of the medians.



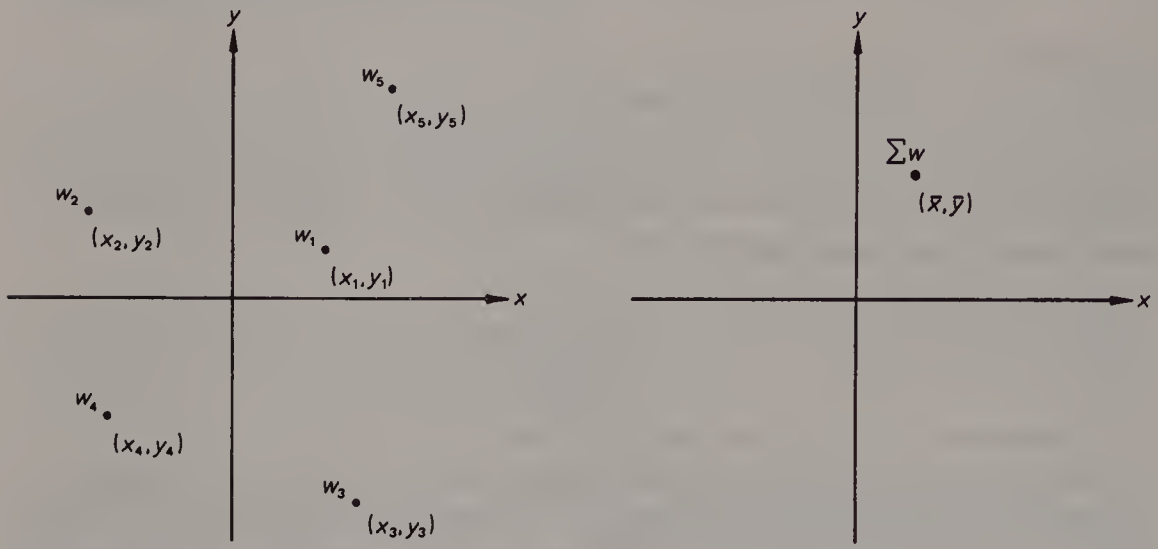
Consider now three particles each of weight W placed at the vertices of a triangle ABC. The resultant of the weights of the particles at B and C is a force of $2W$ acting at D, the mid-point of BC. Hence the resultant of the weights of all three particles is a force of $3W$ acting at the point G on AD which divides

AD in the ratio 2:1. Using similar arguments we find that *G* is also the point which divides the medians through *B* and *C* in the ratio 2:1.

Thus the centre of gravity of the three equal particles at *A*, *B* and *C* is at the point of intersection of the medians of triangle *ABC*, i.e. at the same point as the centre of gravity of the uniform lamina *ABC*.

[Note that we have also verified that this point divides each median in the ratio 2:1.]

In general the centre of gravity of a set of particles in a plane is found using the principle of moments.



Let particles with weights $w_1, w_2, \ldots, w_n$ have positions $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ relative to x- and y-axes in their plane. Suppose that the resultant weight of the set of particles acts through the point $(\bar{x}, \bar{y})$.

Assuming the x, y plane to be horizontal and applying the principle of moments we have,
for moments about the y-axis:

$$w_1x_1 + w_2x_2 + \cdots + w_nx_n = (w_1 + w_2 + \cdots + w_n)\bar{x}$$

and for moments about the x-axis:

$$w_1y_1 + w_2y_2 + \cdots + w_ny_n = (w_1 + w_2 + \cdots + w_n)\bar{y}.$$

Hence the centre of gravity of the particles is at the point $(\bar{x}, \bar{y})$,

where
$$\bar{x} = \frac{\sum wx}{\sum w}, \quad \bar{y} = \frac{\sum wy}{\sum w}.$$

This result can be expressed in vector notation by letting the position vectors of the particles be $\mathbf{r}_1 = x_1\mathbf{i} + y_1\mathbf{j}, \mathbf{r}_2 = x_2\mathbf{i} + y_2\mathbf{j}, \ldots$ and the position vector of the centre of gravity be $\bar{\mathbf{r}} = \bar{x}\mathbf{i} + \bar{y}\mathbf{j}$, so that

$$\bar{\mathbf{r}} = \frac{(w_1x_1 + w_2x_2 + \cdots)\mathbf{i}}{\sum w} + \frac{(w_1y_1 + w_2y_2 + \cdots)\mathbf{j}}{\sum w}$$

$$= \frac{w_1(x_1\mathbf{i} + y_1\mathbf{j}) + w_2(x_2\mathbf{i} + y_2\mathbf{j}) + \cdots}{\sum w}$$

Thus the position vector of the centre of gravity is $\bar{\mathbf{r}} = \dfrac{\sum w\mathbf{r}}{\sum w}$.

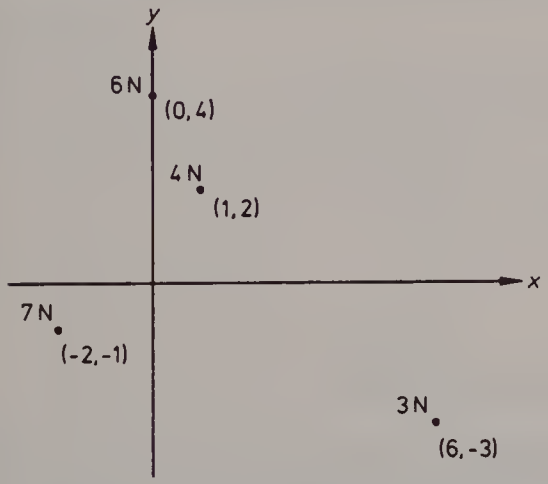
Note that in the case of three particles of weight W at the points $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ we have

$$\bar{x} = \frac{Wx_1 + Wx_2 + Wx_3}{3W} = \tfrac{1}{3}(x_1 + x_2 + x_3)$$

and similarly $\bar{y} = \tfrac{1}{3}(y_1 + y_2 + y_3)$.

Hence the centre of gravity of three equal particles at A, B, C and also of a uniform triangular lamina ABC has coordinates $(\tfrac{1}{3}\{x_1 + x_2 + x_3\}, \tfrac{1}{3}\{y_1 + y_2 + y_3\})$ and position vector $\tfrac{1}{3}\{\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3\}$.

Example 1 Find the coordinates of the centre of gravity of particles of weights 4 N, 7 N, 3 N and 6 N at the points (1, 2), (−2, −1), (6, −3) and (0, 4) respectively.



Let the centre of gravity of the particles be at the point $(\bar{x}, \bar{y})$, then the resultant weight of the particles has magnitude 20 N and acts through $(\bar{x}, \bar{y})$.

Taking moments about the y-axis,

$$20\bar{x} = 4 \times 1 + 7 \times (-2) + 3 \times 6 + 6 \times 0$$

$$\therefore \quad \bar{x} = \frac{4 - 14 + 18 + 0}{20} = \frac{8}{20} = \frac{2}{5}$$

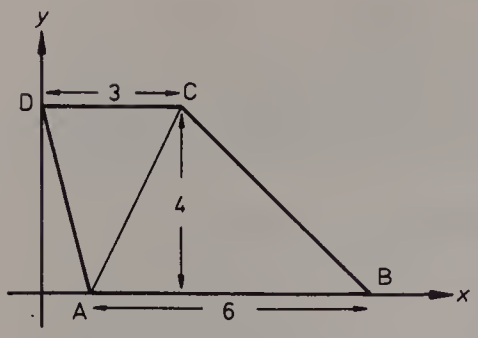
Taking moments about the x-axis,

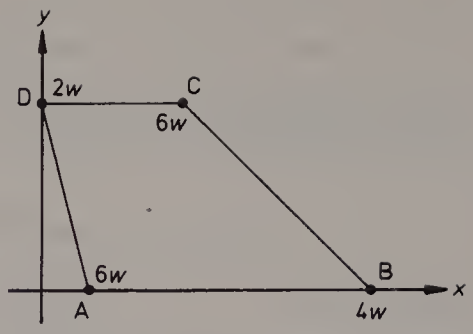
$$20\bar{y} = 4 \times 2 + 7 \times (-1) + 3 \times (-3) + 6 \times 4$$

$$\therefore \quad \bar{y} = \frac{8 - 7 - 9 + 24}{20} = \frac{16}{20} = \frac{4}{5}$$

Hence the centre of gravity has coordinates $\left(\dfrac{2}{5}, \dfrac{4}{5}\right)$.

Example 2 Find the coordinates of the centre of gravity of a uniform lamina in the form of a trapezium with vertices $A(1, 0)$, $B(7, 0)$, $C(3, 4)$ and $D(0, 4)$.





The weight of the lamina $ABCD$ is the resultant of the weights of lamina ABC and lamina ACD. Let w be the weight per unit area of the lamina, then the weight of ABC is $12w$ and is equivalent to three weights of $4w$ at points A, B and C.
Similarly the weight of ACD is $6w$ and is equivalent to three weights of $2w$ at points A, C and D.
Hence the centre of gravity $(\bar{x}, \bar{y})$ of the lamina $ABCD$ is the same as the centre of gravity of particles of weights $6w, 4w, 6w$ and $2w$ at the points A, B, C and D respectively.
Taking moments about the y-axis,

$$18w\bar{x} = 6w \times 1 + 4w \times 7 + 6w \times 3$$

Taking moments about the x-axis,

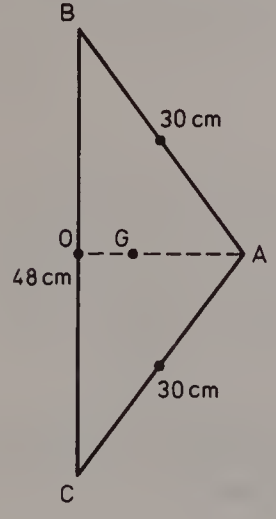
$$18w\bar{y} = 6w \times 4 + 2w \times 4$$

$$\therefore \bar{x} = \frac{52w}{18w} = \frac{26}{9}, \quad y = \frac{32w}{18w} = \frac{16}{9}.$$

Hence the centre of gravity has coordinates $\left(\frac{26}{9}, \frac{16}{9}\right)$.

The principle of moments is also used to find the centre of gravity of any body which can be divided into smaller bodies, each with known weight and centre of gravity.

Example 3 A thin uniform wire is bent to form a triangle ABC in which $AB = AC = 30$ cm and $BC = 48$ cm. Find its centre of gravity.



From symmetry the centre of gravity G lies on OA, where O is the mid-point of BC.
Each of the wires AB, AC and BC has its centre of gravity at its mid-point.
By Pythagoras' theorem, $OA = 18$ cm.

Letting w be the weight per cm of the wire, we have:

Body	Weight	Distance of centre of gravity from BC
Wire AB	$30w$	9
Wire AC	$30w$	9
Wire BC	$48w$	0
Whole body	$108w$	$\bar{x}$

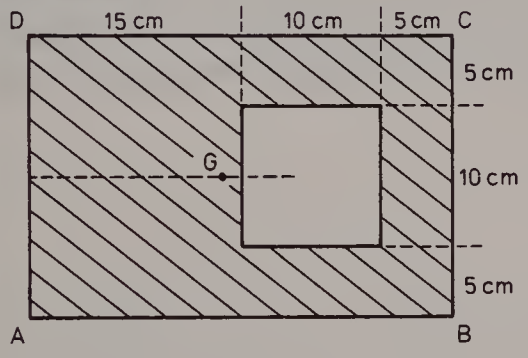
Taking moments about BC,

$$108w\bar{x} = 30w \times 9 + 30w \times 9 + 48w \times 0$$

$$\therefore \quad \bar{x} = \frac{2 \times 30w \times 9}{108w} = 5$$

$\therefore$ the centre of gravity is 5 cm from BC on the line OA.

Example 4



A square hole is cut in a uniform rectangular lamina as shown in the diagram. Find the centre of gravity of the resulting body.

From symmetry the centre of gravity G lies on the line joining the mid-points of AD and BC.

We use the positions of the centres of gravity of the complete rectangle and of the square removed to find the distance $\bar{x}$ of G from AD.
Letting w be the weight per unit area, we have:

Body	Weight	Distance of centre of gravity from AD
Rectangle	$600w$	15
Square	$100w$	20
Remainder	$500w$	$\bar{x}$

Taking moments about AD,

$$500w\bar{x} + 100w \times 20 = 600w \times 15$$

$$\therefore \quad 500w\bar{x} = 600w \times 15 - 100w \times 20$$

$$\therefore \quad \bar{x} = \frac{9000w - 2000w}{500w} = 14$$

$\therefore$ the centre of gravity is 14 cm from AD on the line joining the mid-points of AD and BC.

As shown earlier the centre of gravity of a set of particles in a plane is the point $\left(\frac{\sum wx}{\sum w}, \frac{\sum wy}{\sum w}\right)$. If $m_1, m_2, \ldots$ are the masses of the particles, their *centre of mass* is defined to be the point

$$\left(\frac{\sum mx}{\sum m}, \frac{\sum my}{\sum m}\right).$$

The weight w and the mass m of a typical particle are connected by the relationship $w = mg$, where g is the acceleration due to gravity. Thus, provided that g takes the same value for all the particles, $\frac{\sum wx}{\sum w} = \frac{\sum mgx}{\sum mg} = \frac{\sum mx}{\sum m}$ and similarly $\frac{\sum wy}{\sum w} = \frac{\sum my}{\sum m}$. It follows that, in general, the centre of gravity and the centre of mass of a body are at the same point.

[We note, however, that if a body is so large that the weights of its constituent particles are not parallel forces, then its centre of gravity cannot be found by elementary methods and does not necessarily coincide with the centre of mass.]

A *centroid* is a centre of area or volume. The centroid of a plane area coincides with the centre of mass of a uniform lamina of the same shape. The centroid of a three-dimensional figure coincides with the centre of mass of the corresponding uniform solid.

Exercise 9.3

1. Find the centre of gravity of two particles of weight 2 N and 3 N at points A and B respectively, given that $AB = 30$ cm.

2. Find the centre of gravity of two particles of weight 30 N and 70 N at points P and Q respectively, given that $PQ = 1{\cdot}5$ m.

3. A uniform triangular lamina ABC has $AB = AC = 17$ cm and $BC = 16$ cm. Find the distance of the centre of gravity from BC.

4. A uniform triangular lamina PQR has $PQ = 9$ cm, $QR = 15$ cm and $PR = 12$ cm. Find the distances of the centre of gravity from PR and PQ.

5. Find the coordinates of the centre of gravity of particles of weight 5 N, 7 N, 1 N and 3 N at the points (1, 0), (3, 1), (6, 3) and (0, 2) respectively.

6. Find the coordinates of the centre of mass of particles of mass 9 kg, 4 kg, 6 kg and 5 kg at the points (4, 3), (6, −6), (−3, 0) and (6, −3) respectively.

7. Find the position vector of the centre of mass of a uniform triangular lamina, given that the position vectors of its vertices are $3\mathbf{i} - 5\mathbf{j}$, $7\mathbf{i} + 2\mathbf{j}$ and $-\mathbf{i} + 6\mathbf{j}$.

8. Find the position vector of the centre of gravity of three particles of weight 7 N, 9 N and 4 N with position vectors $4\mathbf{i} + \mathbf{j}$, $3\mathbf{j}$ and $3\mathbf{i} + 4\mathbf{j}$.

9. A uniform lamina in the form of a quadrilateral has vertices $O(0,0)$, $A(4,-6)$, $B(7,0)$ and $C(0,6)$. If the weight per unit area of the lamina is w, show that it has the same centre of gravity as particles of weight $14w$, $7w$, $14w$ and $7w$ at O, A, B and C respectively. Hence find the centre of gravity of the lamina.

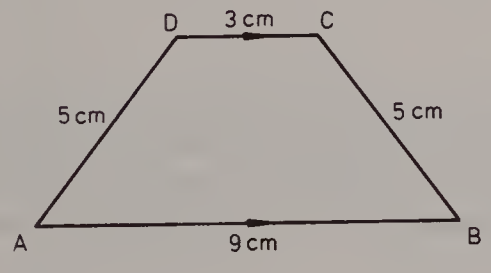
10. Use the method of the previous question to find the centre of gravity of a uniform lamina in the form of a pentagon with vertices $O(0,0)$, $A(3,0)$, $B(3,2)$, $C(0,4)$ and $D(-3,1)$.

11. A thin uniform wire is bent to form a triangle ABC in which $AB = 24\,\text{cm}$, $AC = 10\,\text{cm}$ and $\angle A = 90°$. Find the distances of the centre of gravity from AC and AB.

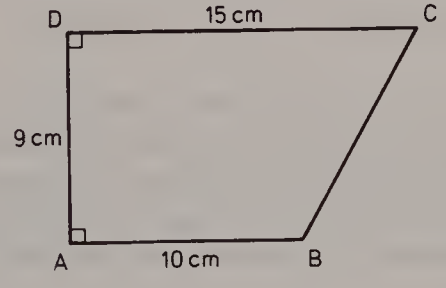
12. A thin uniform wire is bent into the shape of an isosceles trapezium $ABCD$ in which $AB = 24\,\text{cm}$, $AD = BC = 25\,\text{cm}$ and $CD = 10\,\text{cm}$. Find the distance of the centre of gravity from AB.

In questions 13 to 16 find the position of the centre of gravity of the uniform lamina shown in the given diagram.

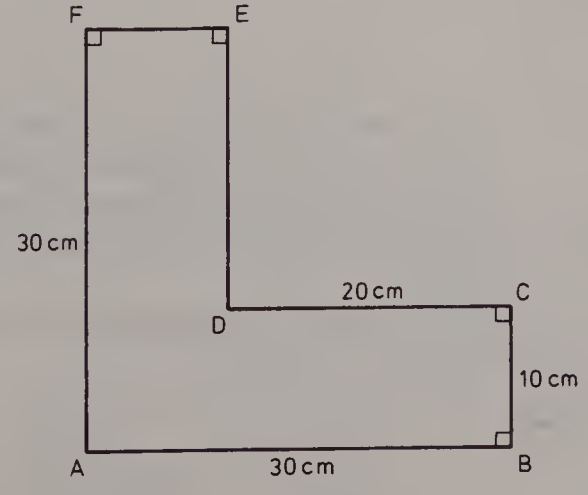
13.



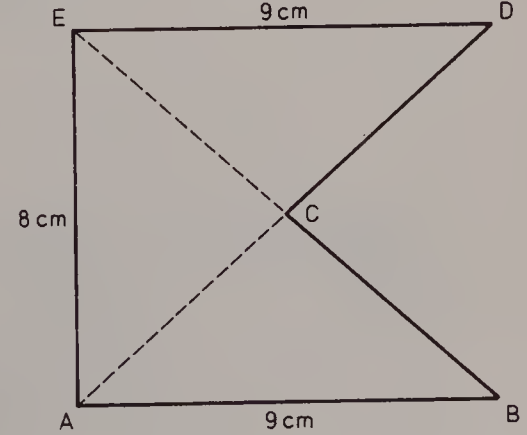
14.



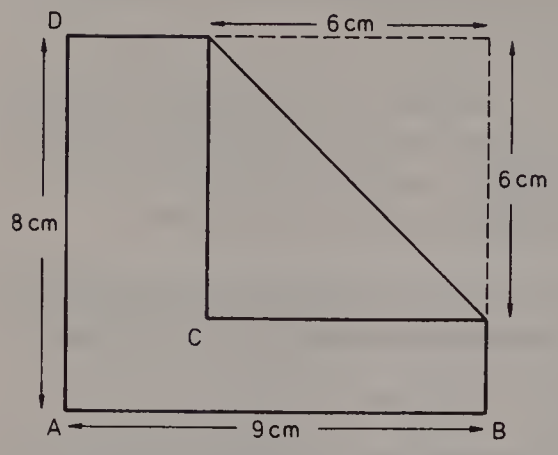
15.



16.

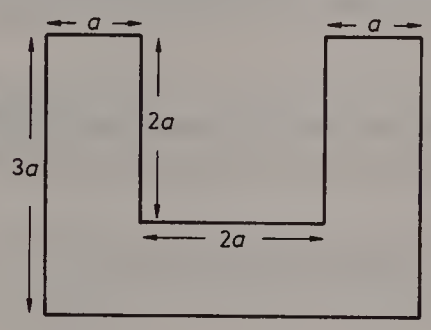


17.



Find the centre of gravity of a uniform rectangular lamina $ABCD$ folded as shown in the diagram.

18.

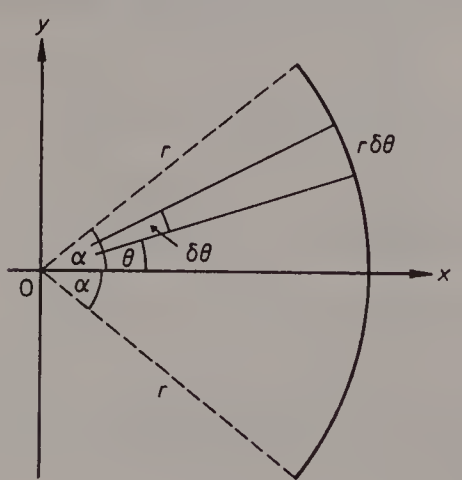


The diagram shows the central cross-section of a uniform cylindrical container weighing $2w$ per unit volume. Find its centre of gravity. If the container is now filled with a liquid weighing w per unit volume, find the new position of the centre of gravity.

9.4 Standard results by integration

As we saw in the previous section the principle of moments can be used to find the centre of gravity of any body which can be divided into smaller bodies with known weights and centres of gravity. In certain cases it is necessary to divide a body into a large number of small parts or elements. The sums of the moments of these elements about appropriate axes are then found by integration.

Example 1 Find the centre of gravity of a uniform wire in the form of an arc of a circle, radius r, subtending an angle 2α at the centre.



From symmetry the centre of gravity lies on the bisector of the angle 2α. As shown in the diagram, a typical element of arc subtends an angle $\delta\theta$ at O and is of length $r\,\delta\theta$.

Letting w be the weight per unit length of the wire, we have:

Body	Weight	Distance of centre of gravity from the y-axis
Element	$wr\,\delta\theta$	$r\cos\theta$
Whole arc	$wr\,2\alpha$	$\bar{x}$

Taking moments about the y-axis,

$$wr\,2\alpha\bar{x} \approx \sum(wr\,\delta\theta \times r\cos\theta)$$

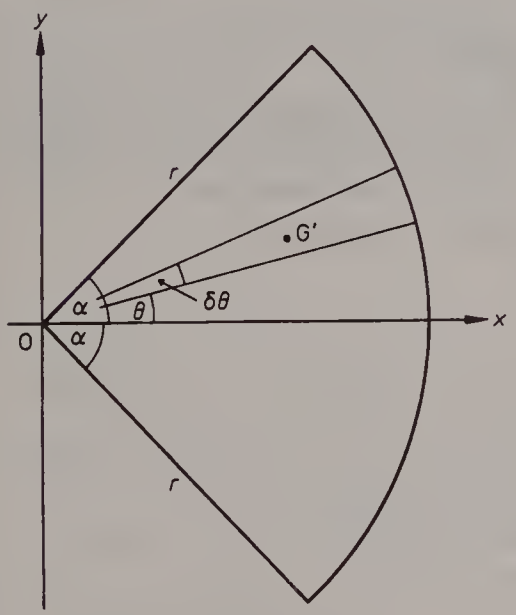
$$\therefore \quad 2wr\,\alpha\bar{x} = \int_{-\alpha}^{\alpha} wr^2\cos\theta\,d\theta$$

$$= wr^2\Big[\sin\theta\Big]_{-\alpha}^{\alpha} = 2wr^2\sin\alpha$$

$$\therefore \quad \bar{x} = \frac{r\sin\alpha}{\alpha}$$

Hence the centre of gravity of the wire is at a distance $(r\sin\alpha)/\alpha$ from the centre.

Example 2 Find the centre of gravity of a uniform lamina in the form of a sector of a circle, radius r, subtending an angle 2α at the centre.



From symmetry the centre of gravity lies on the bisector of the angle 2α. A typical element is a sector subtending an angle $\delta\theta$ at O and having area $\frac{1}{2}r^2\,\delta\theta$. This sector is approximately triangular with its centre of gravity G' at a distance $\frac{2}{3}r$ from O.

Letting w be the weight per unit area of the lamina, we have:

Body	Weight	Distance of centre of gravity from the y-axis
Element	$w\frac{1}{2}r^2\,\delta\theta$	$\frac{2}{3}r\cos\theta$
Whole sector	$w\frac{1}{2}r^2\,2\alpha$	$\bar{x}$

Taking moments about the y-axis,

$$w\tfrac{1}{2}r^2\,2\alpha\bar{x} \approx \sum(w\tfrac{1}{2}r^2\,\delta\theta \times \tfrac{2}{3}r\cos\theta)$$

$$\therefore \quad wr^2\,\alpha\bar{x} = \int_{-\alpha}^{\alpha} \tfrac{1}{3}wr^3\cos\theta\,d\theta$$

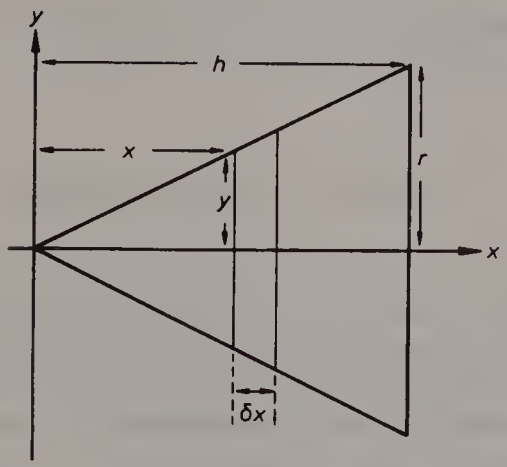
$$= \tfrac{1}{3}wr^3\Big[\sin\theta\Big]_{-\alpha}^{\alpha} = \tfrac{2}{3}wr^3\sin\alpha$$

$$\therefore \quad \bar{x} = \frac{2r\sin\alpha}{3\alpha}$$

Hence the centre of gravity of the lamina is at a distance $(\frac{2}{3}r\sin\alpha)/\alpha$ from the centre.

[This result can also be produced by showing that a uniform sector of radius r has the same centre of gravity as a uniform arc of radius $\frac{2}{3}r$.]

Example 3 Find the centre of gravity of a uniform solid right circular cone of base radius r, height h.



Let the axis of the cone lie along the x-axis, then from symmetry the centre of gravity is on the x-axis. If the cone is divided into elements by planes parallel to its base, then each element is approximately a disc, thickness δx, radius y, which has volume $\pi y^2\,\delta x$.

Letting w be the weight per unit volume of the cone, we have:

Body	Weight	Distance of centre of gravity from the y-axis
Element	$w\pi y^2\,\delta x$	x
Whole cone	$w\frac{1}{3}\pi r^2 h$	$\bar{x}$

Taking moments about the y-axis

$$w\tfrac{1}{3}\pi r^2 h\bar{x} \approx \sum(w\pi y^2\,\delta x \times x)$$

$$\therefore \quad \tfrac{1}{3}\pi r^2 hw\bar{x} = \int_0^h \pi w x y^2\,dx$$

From similar triangles, $\frac{y}{x} = \frac{r}{h}$

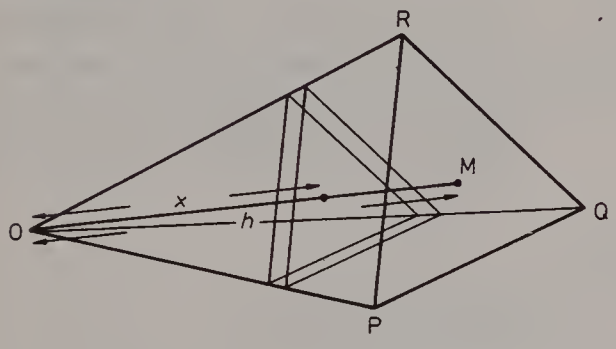
$$\therefore \quad \tfrac{1}{3}\pi r^2 hw\bar{x} = \int_0^h \frac{\pi r^2 w}{h^2} x^3 \, dx$$

$$= \frac{\pi r^2 w}{h^2}\left[\tfrac{1}{4}x^4\right]_0^h = \tfrac{1}{4}\pi r^2 h^2 w$$

$$\therefore \quad \bar{x} = \tfrac{3}{4}h$$

Hence the centre of gravity of the cone is on its axis at a distance $\frac{1}{4}h$ from its base.

Example 4 Find the centre of gravity of a uniform solid tetrahedron.



In tetrahedron $OPQR$, let M be the point of intersection of the medians of $\triangle PQR$. Let the length of OM be h and let the area of $\triangle PQR$ be A.

If the tetrahedron is divided into elements by planes parallel to the face PQR, then a typical element is a triangular lamina of thickness δx, which cuts OM at a distance x from O. Since this element is similar to $\triangle PQR$, its medians intersect on OM and its area is Ax^2/h^2.

Letting w be the weight per unit volume, we have:

Body	Weight	Distance of centre of gravity from 0
Element	$wAx^2\,\delta x/h^2$	x
Whole body	$w\frac{1}{3}Ah$	$\bar{x}$

Taking moments about an axis perpendicular to OM

$$w\tfrac{1}{3}Ah\bar{x} \approx \sum\left(\frac{wAx^2\,\delta x}{h^2} \times x\right)$$

$$\therefore \quad \tfrac{1}{3}Ahw\bar{x} = \int_0^h \frac{Aw}{h^2} x^3 \, \delta x$$

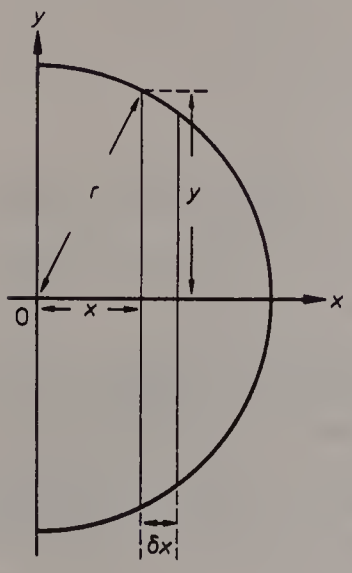
$$= \frac{Aw}{h^2}\left[\tfrac{1}{4}x^4\right]_0^h = \tfrac{1}{4}Awh^2$$

$$\therefore \quad \bar{x} = \tfrac{3}{4}h$$

Hence the centre of gravity of the tetrahedron lies one quarter of the way up a line joining the centroid of one face to the opposite vertex.

Example 5 Find the centre of gravity of a uniform solid hemisphere of radius r.

If we take x- and y-axes along and perpendicular to the axis of symmetry of the hemisphere, then its centre of gravity will be on the x-axis.



A typical element is approximately a disc of thickness δx, radius y at a distance x from the y-axis.
Letting w be the weight per unit volume of the hemisphere:

Body	Weight	Distance of centre of gravity from the y-axis
Element	$w\pi y^2\,\delta x$	x
Whole hemisphere	$w\frac{2}{3}\pi r^3$	$\bar{x}$

Taking moments about the y-axis,

$$w\tfrac{2}{3}\pi r^3\bar{x} \approx \sum(w\pi y^2\,\delta x \times x)$$

$$\therefore\quad \tfrac{2}{3}\pi r^3 w\bar{x} = \int_0^r \pi w x y^2\,dx$$

By Pythagoras' theorem, $x^2 + y^2 = r^2$,

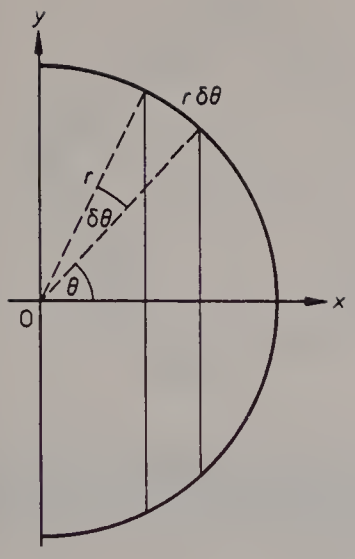
$$\therefore\quad \tfrac{2}{3}\pi r^3 w\bar{x} = \int_0^r \pi w(r^2x - x^3)\,dx$$

$$= \pi w\left[\tfrac{1}{2}r^2x^2 - \tfrac{1}{4}x^4\right]_0^r = \tfrac{1}{4}\pi r^4 w$$

$$\therefore\quad \bar{x} = 3r/8$$

Hence the centre of gravity of the hemisphere is on its axis of symmetry at a distance $3r/8$ from its plane face.

Example 6 Find the centre of gravity of a uniform hemispherical shell of radius r.



If we take x- and y-axes along and perpendicular to the axis of symmetry of the hemispherical shell, then its centre of gravity will be on the x-axis.
If the shell is divided into narrow bands by planes perpendicular to the x-axis, then a typical elemental band is approximately cylindrical with radius $r \sin\theta$ and width $r\,\delta\theta$. Letting w be the weight per unit area of the shell, we have:

Body	Weight	Distance of centre of gravity from the y-axis
Element	$w2\pi r^2 \sin\theta\,\delta\theta$	$r\cos\theta$
Whole body	$w2\pi r^2$	$\bar{x}$

Taking moments about the y-axis

$$w2\pi r^2\bar{x} \approx \sum(w2\pi r^2 \sin\theta\,\delta\theta \times r\cos\theta)$$

$$\therefore \quad 2\pi w r^2\bar{x} = \int_0^{\pi/2} 2\pi w r^3 \sin\theta\cos\theta\,d\theta$$

$$= \int_0^{\pi/2} \pi w r^3 \sin 2\theta\,d\theta$$

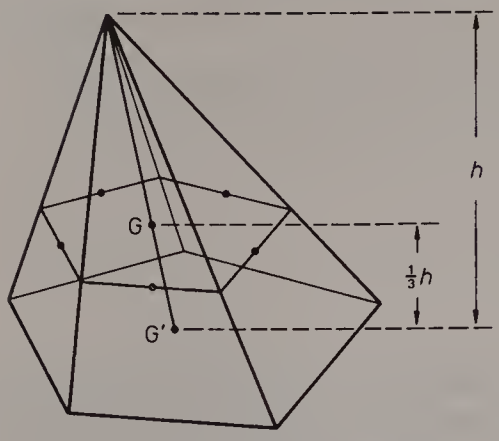
$$= \pi w r^3[-\tfrac{1}{2}\cos 2\theta]_0^{\pi/2} = \pi w r^3$$

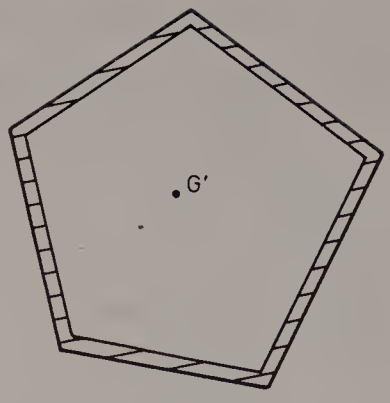
$$\therefore \quad \bar{x} = \frac{r}{2}$$

Hence the centre of gravity of the hemispherical shell is at a distance $\frac{1}{2}r$ from its base.

[For an alternative method see Exercise 9.6, question 11.]

It is not necessary to use integration to find the centres of gravity of hollow pyramids and cones.





Let us consider a thin hollow pyramid of height h, without base. Each of the faces is a triangular lamina with centre of gravity at a height $\frac{1}{3}h$ above the base. If the pyramid is divided into sections parallel to the base, we see that the centre of gravity must lie on the line joining the centres of gravity of these sections.

Hence the centre of gravity of the *hollow pyramid* lies one third of the way up a line joining the centroid of the base to the vertex.

As the number of sides of the base of the pyramid is increased indefinitely, it tends to become a cone. Hence the centre of gravity of a *hollow right circular cone*, without base, is on the axis of the cone at a distance of one third of the height of the cone from the base.

[Note that the results obtained in this section may be quoted without proof when finding the centres of gravity of composite bodies. A list of standard results will be found in the "Formulae for reference" section near the end of the book.]

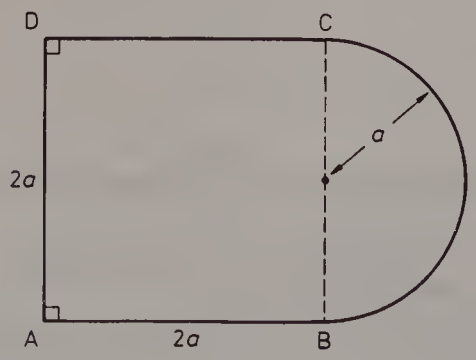
Exercise 9.4

1. Prove that the centre of gravity of a uniform wire in the form of a semi-circle of radius r is at a distance $2r/\pi$ from the centre.

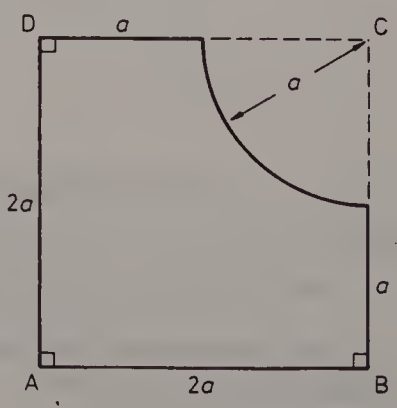
2. Prove that the centre of gravity of a uniform semi-circular lamina of radius r is at a distance $4r/3\pi$ from the centre.

In questions 3 to 6 find the distance from AD of the centre of gravity of the given uniform lamina.

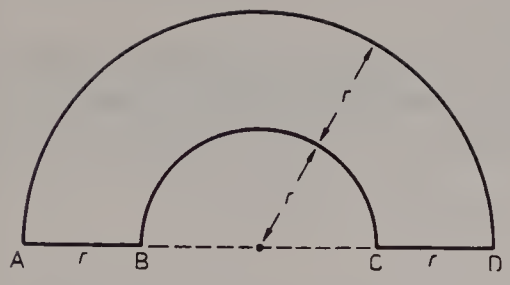
3.



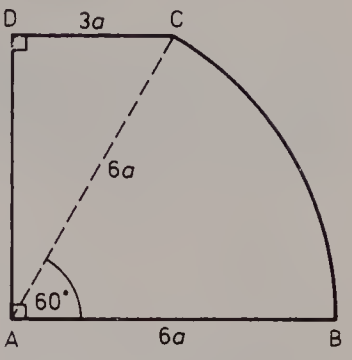
4.



5.



6.



7. A frustum is cut from a solid right circular cone of base radius r and height $2h$ by a plane parallel to the base and at a distance h from it. Find the distance of the centre of gravity of the frustum from the base.

8. A circular lamina of radius $2r$ is divided into two segments by a line at a distance r from the centre. Find the distance from the centre of the centre of gravity of (a) the larger segment, (b) the smaller segment.

9. A solid sphere of radius $3a$ is divided into two parts by a plane at a distance a from the centre. Find, by integration, the volume of the smaller part and the distance of its centre of gravity from the centre of the sphere.

10. Prove that the centre of gravity of a uniform solid pyramid with a square base lies one quarter of the height of the pyramid above the base and on the line joining the vertex to the point of intersection of the diagonals of the base.

11. Prove that the centre of gravity of a uniform solid hemisphere of radius r is a distance $3r/8$ from the centre.

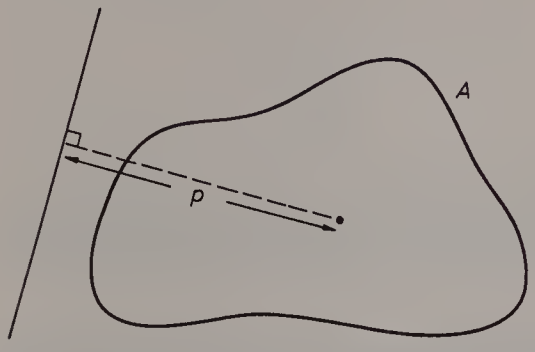
A child's toy is made up from a uniform and solid right circular cone and hemisphere. The radius of the cone is r, and its height $3r$. The radius of the hemisphere is r. The base of the cone and hemisphere are sealed together. The material from which the hemisphere is made is 3 times as heavy per unit volume as the cone material. Find the distance of the centre of gravity of the toy from the vertex of the cone. (SU)

12. Prove that the centre of gravity of a uniform thin hemispherical cup of radius r is at a distance $r/2$ from the centre.

A goblet consists of a uniform thin hemispherical cup of radius r, a circular base of the same material, thickness and radius as the cup, and an intervening stem of length r and whose weight is one-quarter of that of the cup. Show that the height of the centre of gravity above the base is $13r/14$. If the weight of the goblet is W and that of the amount of liquid that fills it is W', show that filling it raises the centre of gravity through a distance $\frac{39}{56}\left(\frac{W'}{W+W'}\right)r$. (W)

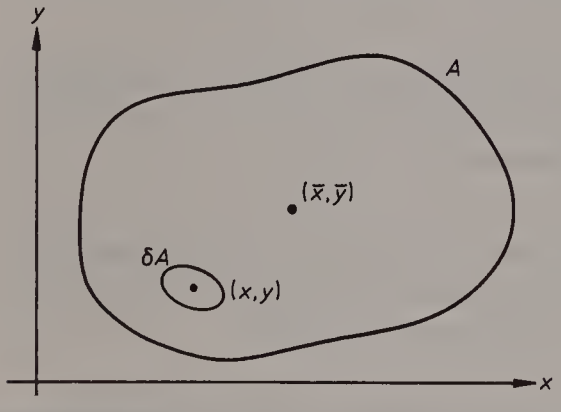
9.5 Centroids of areas and volumes

The *centroid* of a plane figure is its centre of area i.e. the point about which the area is evenly distributed. In simple cases we can use symmetry to find the centroid of a figure. However, to arrive at a more general method, we define a quantity called the *first moment of area.*



If the centroid of an area A is at a perpendicular distance p from an axis, then the first moment of A about this axis is pA.

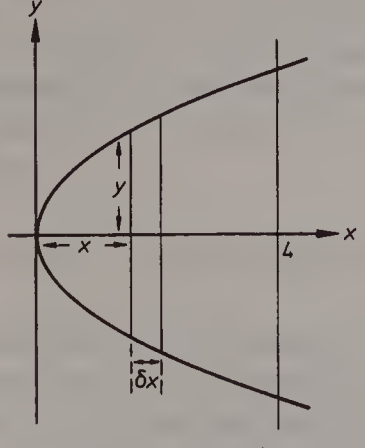
If an area is divided into a number of parts, then the sum of the first moments of the parts about any axis is equal to the first moment of the whole area about that axis.



Suppose that a region of the x, y plane has area A and centroid $(\bar{x}, \bar{y})$. If a typical element of area δA has centroid (x, y) then, comparing first moments about the coordinates axes, we have:

$$\bar{x}A = \sum x\,\delta A, \quad \bar{y}A = \sum y\,\delta A.$$

Example 1 Find the centroid of the region bounded by the curve $y^2 = x$ and the line $x = 4$.



A typical element of area is a strip of width δx, length $2y$ at a distance x from the y-axis. From symmetry the centroids of such strips and of the whole region lie on the x-axis.

Thus, we may write:

	Area	Coordinates of centroid
Element	$2y\,\delta x$	$(x, 0)$
Whole region	A	$(\bar{x}, 0)$

First moment of area about the y-axis

$$= \bar{x}A \approx \sum(x \times 2y\,\delta x)$$

$$\bar{x}A = \int_0^4 2xy\,dx$$

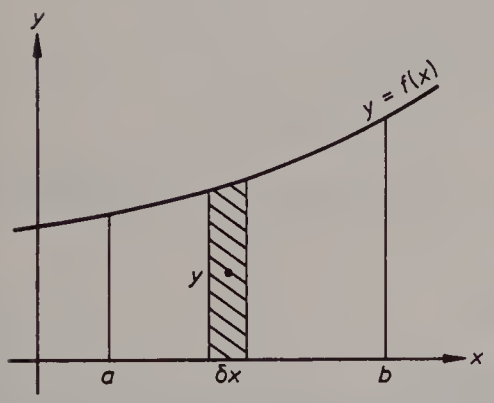
$$= \int_0^4 2x^{3/2}\,dx = 2\left[\frac{2}{5}x^{5/2}\right]_0^4 = \frac{128}{5}$$

But $$A = \int_0^4 2y\,dx$$

$$= \int_0^4 2x^{1/2}\,dx = 2\left[\frac{2}{3}x^{3/2}\right]_0^4 = \frac{32}{3}$$

$$\therefore \quad \bar{x} = \frac{128}{5} \Big/ \frac{32}{3} = \frac{128}{5} \times \frac{3}{32} = \frac{12}{5}$$

Hence the centroid of the region is the point (12/5, 0).



Let us consider now the area bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$, $x = b$. A typical element of area is an approximately rectangular strip of width δx and height y.

	Area	Coordinates of centroid
Element	$y\,\delta x$	$(x, \frac{1}{2}y)$
Whole region	A	$(\bar{x}, \bar{y})$

First moment of area about the y-axis

$$= \bar{x}A \approx \sum(x \times y\,\delta x)$$

$$\therefore \quad \bar{x}A = \int_a^b xy\,dx$$

First moment of area about the x-axis

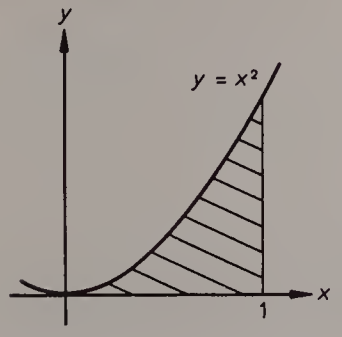
$$= \bar{y}A \approx \sum(\tfrac{1}{2}y \times y\,\delta x)$$

$$\therefore \quad \bar{y}A = \int_a^b \tfrac{1}{2}y^2\,dx$$

But $A = \int_a^b y\,dx$, so the centroid has coordinates

$$\bar{x} = \int_a^b xy\,dx \Big/ \int_a^b y\,dx \quad \bar{y} = \int_a^b \tfrac{1}{2}y^2\,dx \Big/ \int_a^b y\,dx$$

Example 2 Find the centroid of the area bounded by the curve $y = x^2$, the x-axis and the line $x = 1$.



Area of the given region

$$= \int_0^1 y\,dx = \int_0^1 x^2\,dx = \left[\frac{1}{3}x^3\right]_0^1 = \tfrac{1}{3}$$

First moment about the y-axis

$$= \int_0^1 xy\,dx = \int_0^1 x^3\,dx = \left[\frac{1}{4}x^4\right]_0^1 = \tfrac{1}{4}$$

First moment about the x-axis

$$= \int_0^1 \tfrac{1}{2}y^2\,dx = \int_0^1 \tfrac{1}{2}x^4\,dx = \left[\frac{1}{10}x^5\right]_0^1 = \frac{1}{10}$$

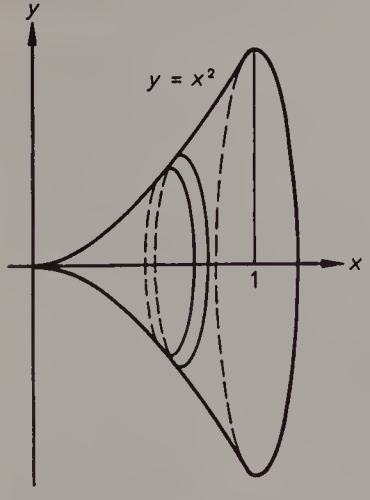
$$\therefore \quad \bar{x} = \int_0^1 xy\,dx \Big/ \int_0^1 y\,dx = \frac{1}{4} \div \frac{1}{3} = \frac{3}{4}$$

$$\therefore \quad \bar{y} = \int_0^1 \tfrac{1}{2}y^2\,dx \Big/ \int_0^1 y\,dx = \frac{1}{10} \div \frac{1}{3} = \frac{3}{10}$$

Hence the centroid of the area is the point $(\frac{3}{4}, \frac{3}{10})$.

The centroid of a solid three-dimensional figure is its centre of volume i.e. the point about which the volume is evenly distributed. The methods used to find centroids of similar to those used to find centroids of area.

Example 3 Find the centroid of the volume obtained by rotating completely about the x-axis the area bounded by the curve $y = x^2$, the x-axis and the line $x = 1$.



A typical element of volume is approximately a disc of thickness δx, radius y at a distance x from the y-axis. From symmetry the centroids of such discs and of the whole solid lie on the x-axis.

Thus we may write:

	Volume	Coordinates of centroid
Element	$\pi y^2\,\delta x$	$(x, 0)$
Whole solid	V	$(\bar{x}, 0)$

First moment of volume about the y-axis

$$= \bar{x}V \approx \sum(x \times \pi y^2\,\delta x)$$

$$\therefore \quad \bar{x}V = \int_0^1 \pi x y^2\,dx$$

$$= \pi \int_0^1 x^5\,dx = \pi\left[\frac{1}{6}x^6\right]_0^1 = \frac{\pi}{6}$$

But $$V = \int_0^1 \pi y^2\,dx$$

$$= \pi \int_0^1 x^4\,dx = \pi\left[\frac{1}{5}x^5\right]_0^1 = \frac{\pi}{5}$$

$$\therefore \quad \bar{x} = \frac{\pi}{6}\Big/\frac{\pi}{5} = \frac{\pi}{6} \times \frac{5}{\pi} = \frac{5}{6}$$

Hence the centroid of the solid is at the point $(5/6, 0)$.

Exercise 9.5

Find the first moments about the y-axis and the x-axis of the areas bounded by the following curves and straight lines. In each case use your results to find the coordinates of the centroid of the area.

1. $y = x^2 + 1$, $y = 0$, $x = 0$ and $x = 2$.

2. $y = x(x + 2)$ for $x \geqslant 0$, $y = 0$ and $x = 3$.

3. $y = \dfrac{1}{x^2}$, $y = 0$, $x = 1$ and $x = 2$.

4. $y = x^2$ for $x \geqslant 0$, $x = 0$ and $y = 1$.

Find the centroids of the areas bounded by the following curves and straight lines.

5. $y^2 = x^3$ and $x = 4$.

6. $y^2 = 4x$ and $x = 1$.

7. $y = \cos x$ for $-\frac{1}{2}\pi \leqslant x \leqslant \frac{1}{2}\pi$ and $y = 0$.

8. $x^2y = 4$, $y = 1$ and $y = 4$.

9. $y = e^x$, $y = 0$, $x = 0$ and $x = 1$.

10. $y = \sin x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, $y = 0$ and $x = \frac{1}{2}\pi$.

Find the centroids of the solids formed by rotating completely about the x-axis the regions defined by the following inequalities.

11. $y \geqslant x(x - 4)$, $y \leqslant 0$.

12. $y^2 \leqslant 9x$, $y \geqslant 0$, $x \leqslant 1$.

13. $xy \leqslant 4$, $y \geqslant 0$, $1 \leqslant x \leqslant 2$.

14. $0 \leqslant y \leqslant e^{-x}$, $0 \leqslant x \leqslant 1$.

15. Sketch the curve $y^2 = x(3 - x)^2$. Find the centroid of the area enclosed by the loop.

16. Find the centroid of the solid formed by rotating through π radians about the y-axis the region bounded by the curve $y = 4x^2$ and the straight lines $y = 1$ and $y = 4$.

17. Use integration by parts to evaluate

$$\int_1^2 \ln x \, dx, \quad \int_1^2 x \ln x \, dx \quad \text{and} \quad \int_1^2 (\ln x)^2 \, dx.$$

Hence find, correct to 2 decimal places, the coordinates of the centroid of the area bounded by the curve $y = \ln x$, the x-axis and the line $x = 2$.

18. Find the area of the region R bounded by the curve $y = x^2$ and the straight line $y = x$. Show that the first moments of R about the x- and y-axes are given by

$$\int_0^1 \tfrac{1}{2}(x^2 - x^4) \, dx \quad \text{and} \quad \int_0^1 (x^2 - x^3) \, dx \text{ respectively.}$$

Hence find the centroid of the region R.

Exercise 9.6 (miscellaneous)

1. All forces in this question act in the plane of a triangle ABC in which $AB = 4a$, $AC = 3a$ and the angle $A = 90°$. Forces of magnitude $17P$, $15P$, $3P$ act along AB, BC, AC respectively in the directions indicated by the order of the letters. Calculate
(a) the magnitude of the resultant of these three forces and the tangent of the angle made by its line of action with AB,
(b) the distance from A of the point where the line of action of the resultant cuts AB. A couple G is now added to the system and the resultant of this enlarged system acts through the point B. Calculate the magnitude and sense of G. (L)

2. A rigid square lamina $ABCD$ of side a is subject to forces of magnitude $1, 2, 3, 1, 3\sqrt{2}$ and $\lambda\sqrt{2}$ units acting along AB, BC, CD, AD, AC and DB respectively

in the directions indicated by the order of the letters. Given that the direction of the resultant force is parallel to AC, find λ. With this value of λ, find the total moment about A of the forces acting on the lamina. Hence, or otherwise, find AE in terms of a, where E is the intersection of AB with the line of action of the resultant force.

(JMB)

3. (a) State conditions which will ensure that a system of forces in a plane will reduce to a couple.
   (b) State conditions which will ensure that a system of forces in a plane is in equilibrium.
   (c) $ABCD$ is a square of side a. Forces of size 1, 3, 3 and 7 act respectively along AB, BC, CD and DA in the directions indicated by the letters. (i) Find the sizes of the forces which must act along AC and BD so that the six forces will be equivalent to a couple, and find the moment of the couple. (ii) If the forces in DA, AC and BD are to be replaced so that the system is in equilibrium, how will this have to be done? (SU)

4. An equilateral triangle ABC has side $2a$ and D, E are the mid-points of BC, CA respectively. Fixed forces of magnitude $P, 2P, 4P$ act along AB, BC, CA respectively and variable forces of magnitude x, y act along AD, BE respectively, the direction of each force being indicated by the order of the letters. Find (in terms of P) the values of x and y if

(a) the system is equivalent to a force through B parallel to AD, stating the magnitude of this force;
(b) the system is equivalent to a couple, stating the moment of the couple. (O&C)

5. Forces $2, 4, 6, 2p, 2q$ and 18 newtons act along the sides AB, CD, ED, EF and AF respectively of a regular hexagon $ABCDEF$, the directions of the forces being indicated by the order of the letters. If the system is in equilibrium, find, by resolving parallel and perpendicular to AB, the values of p and q. Check your results by finding the moment of the forces about O, the centre of the hexagon.

The forces along ED, EF and AF are now replaced by a coplanar force through O and a coplanar couple. If the resulting system is in equilibrium and if the length of each side of the hexagon is 2 metres, calculate

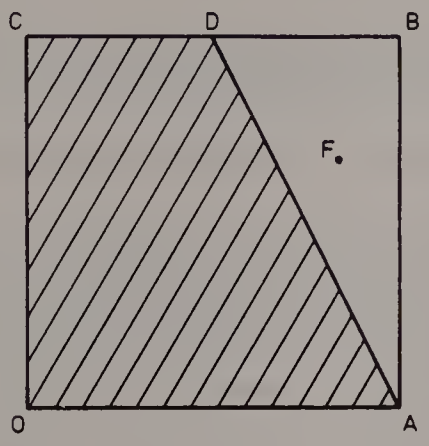
(a) the magnitude of this force through O,
(b) the magnitude of the couple. (L)

6. The centre of a regular hexagon $ABCDEF$ of side a is O. Forces of magnitude $P, 2P, 3P, 4P, mP$ and nP act along $\vec{AB}$, $\vec{BC}$, $\vec{CD}$, $\vec{DE}$, $\vec{EF}$, and $\vec{FA}$ respectively. Given that the resultant of these six forces is of magnitude $3P$ acting in a direction parallel to $\vec{EF}$,

(i) determine the values of m and n,
(ii) show that the sum of the moments of the forces about O is $9Pa\sqrt{3}$.
The mid-point of EF is M.
(iii) Find the equation of the line of action of the resultant referred to OM as x-axis and OA as y-axis.

The forces mP and nP acting along $\vec{EF}$ and $\vec{FA}$ are removed from the system. The remaining four forces and an additonal force Q, which acts through O, reduce to a couple. Calculate the magnitude of Q and the moment of the couple. (AEB 1979)

7.



The diagram shows a square $OABC$ of side a. The mid-point of BC is D. Show that, with respect to OA and OC as axes, the coordinates of the centroid F of the triangular region ABD are $(5a/6, 2a/3)$. Find the coordinates of the centre of mass of a uniform lamina in the form of the figure $OADC$. (JMB)

8. A uniform thin sheet of paper has the shape of a rectangle $OABC$, where $OA = 20\,\text{cm}$ and $OC = 12\,\text{cm}$. The side AB is folded down so as to lie entirely along OA. Find the distances of the centre of gravity of the folded sheet from OA and OC. (O&C)

9. A uniform solid hemisphere of radius a is cut into two parts by a plane parallel to the plane face of the hemisphere and at a distance $\frac{1}{2}a$ from it. Find the volume of each of the two parts and the position of the centre of mass of each of the two parts. (AEB 1980)

10. Prove that the centre of gravity of a uniform circular arc subtending an angle 2θ at the centre of a circle of radius a is $a \sin\theta/\theta$ from the centre. Deduce that the centre of gravity of a uniform sector bounded by that arc and the radii to its extremities is $\frac{2}{3}a \sin\theta/\theta$ from the centre.

Show also that the centre of gravity of a segment of a circular lamina cut off by a chord subtending a right angle at the centre of the circle is $\frac{2}{3}\sqrt{2}a/(\pi - 2)$ from the centre. (W)

11. Find by integration the distance of the centroid of a uniform solid hemisphere of radius a from its plane face.

A hollow sphere of external radius R and internal radius r is cut in half by a plane through its centre. Show that the distance of the centroid of each half from the centre of the sphere is

$$\frac{3(R + r)(R^2 + r^2)}{8(R^2 + Rr + r^2)}.$$

Deduce that the centroid of a hemispherical shell of radius R is at a distance $R/2$ from its centre. (L)

12. Find the area A of the loop of the curve $y^2 = x^2(2 - x)$. Find also the first moment about the y-axis of the volume generated when the area A is rotated through π radians about the x-axis. (AEB 1975)

13. Find the area of the region in the first quadrant enclosed between the curve $y = x^2 + 4$, the line $y = 8$ and the y-axis. The region is rotated through four right-angles about Oy to form a uniform solid. Find the volume and the coordinates of the centre of mass of the solid. (C)

14. Sketch on one diagram the graphs of $x^3y = 1$ and $y = \sqrt{x}$, for positive values of x. Show that the area of the finite region bounded by the straight lines $y = 0$ and $x = 2$ and an arc of each of the above curves is 25/24. Find also the coordinates of the centroid of the region. (C)

15. Find the area of the finite region between the curve $y = x^2 + 1$ and the line $y = 2x + 1$. The region is rotated through four right angles about Ox, to produce a uniform solid of revolution. Find the volume and the coordinates of the centre of gravity of this solid. (C)

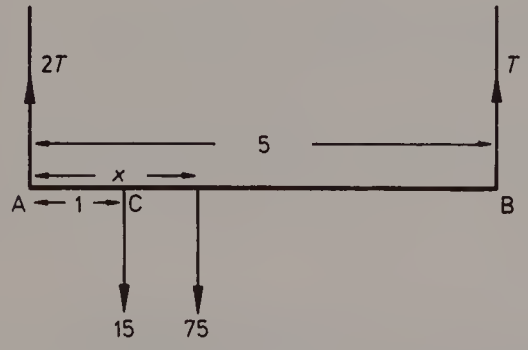
16. Sketch the arc of the curve given parametrically by $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, for which $0 \leqslant \theta \leqslant 2\pi$. Find the area of the finite region enclosed by this arc and the x-axis. Find also the coordinates of the centroid of the region. (L)

10 Equilibrium of rigid bodies

10.1 Parallel forces in equilibrium

If a set of coplanar forces is in equilibrium then
(i) the sum of the components of the forces in any given direction is zero and
(ii) the sum of the moments of the forces about any given point is zero.
These facts are used to find unknown quantities in many statics problems.

Example 1 A non-uniform rod AB of length 5 m and weight 75 N is supported in a horizontal position by two vertical strings attached to its ends. A weight of 15 N is attached to the rod at the point C, where $AC = 1$ m. If the tensions in the strings at A and B are $2T$ N and T N respectively, find the value of T. Find also the distance, x m, of the centre of gravity of the rod from A.



Considering the vertical forces on the rod,

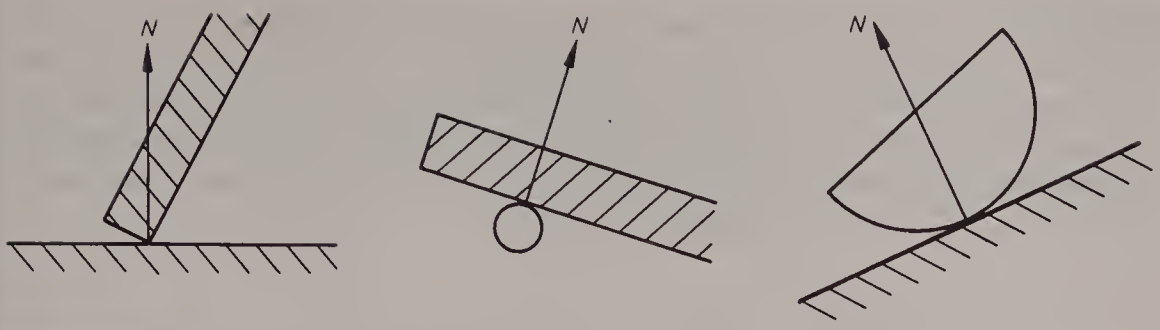
$\uparrow \qquad 2T - 15 - 75 + T = 0$
$\therefore \qquad 3T = 90$
Hence $\qquad T = 30.$

Taking moments clockwise about A,

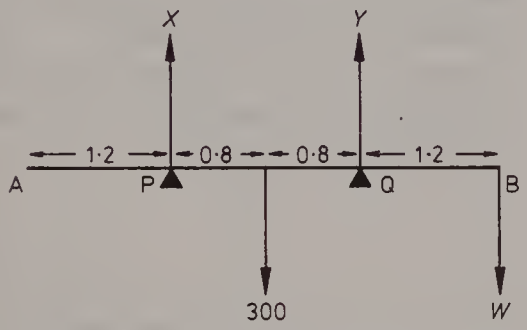
$\curvearrowright A \qquad 15 \times 1 + 75 \times x - T \times 5 = 0$

$\therefore \qquad 75x = 5T - 15 = 135$
Hence $\qquad x = 1{\cdot}8.$

When a rigid body rests on an object described as *smooth*, (e.g. a smooth support or a smooth table,) then the reaction on the body at any point of contact is assumed to be perpendicular to the surfaces in contact.



Example 2 A uniform plank AB of length 4 m and weight 300 N rests horizontally on smooth supports at P and Q, where $AP = QB = 1{\cdot}2$ m. Find the maximum weight that can be placed at B without tilting the plank.



Let us assume that the plank is in equilibrium with reactions of X N and Y N at P and Q respectively and a weight of W N at B.

Taking moments about Q,

$$X \times 1{\cdot}6 + W \times 1{\cdot}2 - 300 \times 0{\cdot}8 = 0$$

$\therefore \quad 1{\cdot}2W = 240 - 1{\cdot}6X$

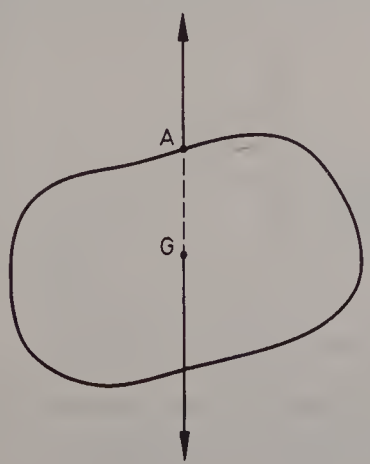
Since the reaction at P acts vertically upwards, $X \geqslant 0$,

$\therefore \quad 1{\cdot}2W \leqslant 240$ i.e. $W \leqslant 200$.

Hence the maximum weight that can be placed at B without tilting the plank is 200 N.

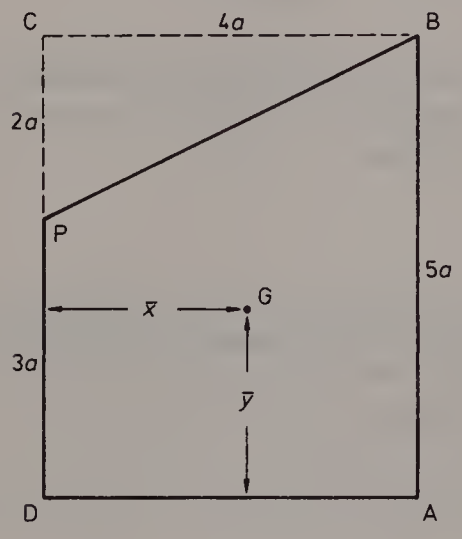
[Note that when a weight of 200 N is placed at B, the reaction at P is zero and the plank just fails to rotate about Q.]

When a body is in equilibrium under the action of *two forces*, these forces must be equal in magnitude and act in opposite directions along the same line.



In particular, when a body freely suspended from a point A is in equilibrium, the two forces acting on it, namely its weight and the force at A, must act along the same vertical line. We deduce that the body hangs in equilibrium with its centre of gravity G vertically below A.

Example 3 A uniform rectangular lamina $ABCD$ of weight $5W$ has $AB = 5a$ and $BC = 4a$. The point P lies on the edge DC and $DP = 3a$. The triangle BCP is cut away and the remaining trapezium is suspended from P. Find the angle that DP makes with the vertical when the trapezium hangs freely in equilibrium. Find also the moment of the couple which would be required to maintain the trapezium in equilibrium with D vertically below P.



We first find the position of G, the centre of gravity of the lamina $ABPD$. Rectangle $ABCD$ has area $20a^2$ and weight $5W$. Since triangle BCP has area $4a^2$, its weight must be W.

Body	Weight	Distance of centre of gravity from CD	Distance of centre of gravity from DA
Rectangle $ABCD$	$5W$	$2a$	$5a/2$
Triangle BCP	W	$4a/3$	$13a/3$
Remainder	$4W$	$\bar{x}$	$\bar{y}$

Taking moments about CD,

$$4W \times \bar{x} + W \times \frac{4a}{3} = 5W \times 2a$$

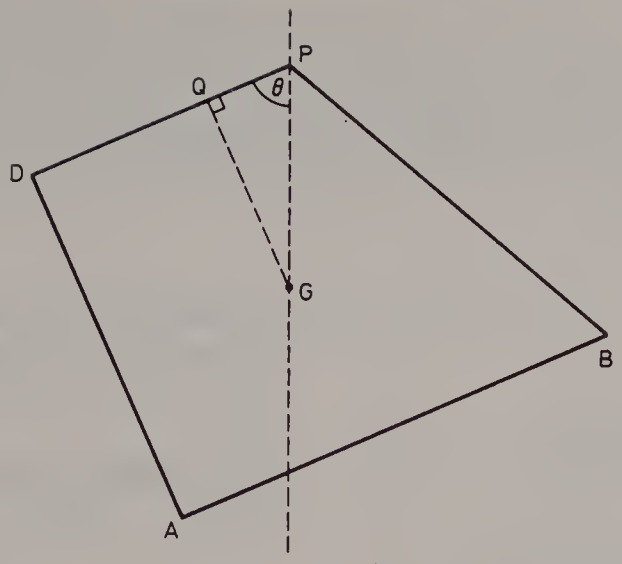
$$\therefore \quad \bar{x} = \frac{1}{4}\left(10a - \frac{4a}{3}\right) = \frac{13a}{6}$$

Taking moments about DA,

$$4W \times \bar{y} + W \times \frac{13a}{3} = 5W \times \frac{5a}{2}$$

$$\therefore \quad \bar{y} = \frac{1}{4}\left(\frac{25a}{2} - \frac{13a}{3}\right) = \frac{49a}{24}$$

When the trapezium hangs in equilibrium from P, the line PG must be vertical. Let θ be the angle DP then makes with the vertical and let Q be the foot of the perpendicular from G to DP.



$$GQ = \bar{x} = \frac{13a}{6}, \quad DQ = \bar{y} = \frac{49a}{24}, \qquad \therefore \quad QP = \frac{23a}{24}.$$

Thus $\tan\theta = \dfrac{GQ}{QP} = \dfrac{13}{6} \times \dfrac{24}{23} = \dfrac{52}{23}$ and $\theta \approx 66{\cdot}1°$.

Hence the angle made by DP with the vertical is 66·1°.

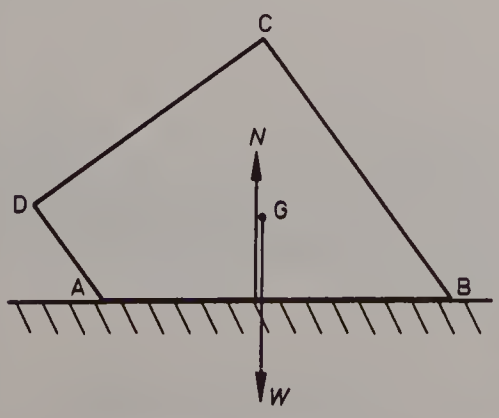
Suppose now that the trapezium is in equilibrium with D vertically below P and that a couple of moment M in the sense DAB is required to maintain this position. Taking moments anti-clockwise about P,

$$M - 4W \times \frac{13a}{6} = 0 \qquad \therefore \quad M = \frac{26Wa}{3}.$$

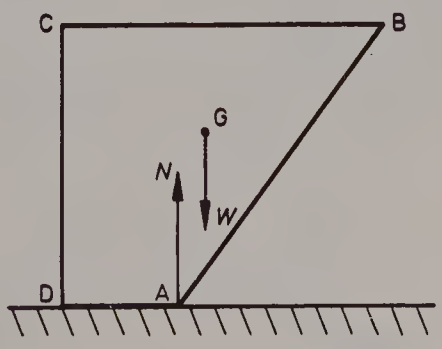
Hence the moment of the required couple is $26Wa/3$.

When a body rests in equilibrium on a horizontal plane, the weight of the body and the force exerted by the plane on the body must act along the same line, namely the vertical line through the centre of gravity.

In the case of a lamina with an edge AB resting on a smooth horizontal plane, the force N exerted by the plane on the lamina is the resultant of normal reactions at the points of contact along AB. Thus N must act through some point on AB. It follows that, if the vertical line through the centre of gravity G passes through a point on the edge AB, then the force N can act to maintain equilibrium. Otherwise equilibrium is not possible and the lamina will topple.

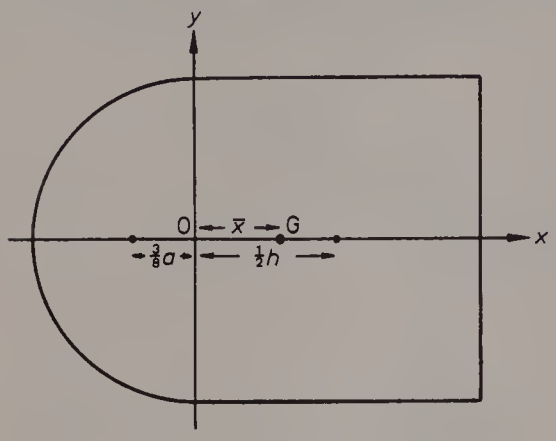


Lamina in equilibrium



Lamina will topple

Example 4 A uniform solid is formed by joining a hemisphere of radius a to a circular cylinder of radius a and height h, so that the plane face of the hemisphere coincides with a plane face of the cylinder. Show that this solid can rest in equilibrium on a horizontal plane with the curved surface of the cylinder touching the plane if $h^2 \geqslant \frac{1}{2}a^2$.



If we take x- and y-axes as shown in the diagram, then the centre of gravity G of the combined solid must lie on the x-axis.
Let w be the weight per unit volume of the solid.

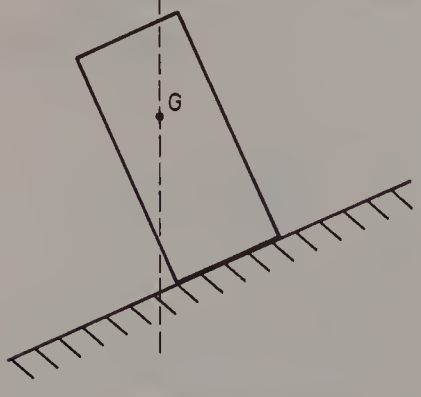
Body	Weight	x-coordinate of centre of gravity
Cylinder	$\pi a^2 hw$	$\frac{1}{2}h$
Hemisphere	$\frac{2}{3}\pi a^3 w$	$-\frac{3}{8}a$
Whole body	$(h + \frac{2}{3}a)\pi a^2 w$	$\bar{x}$

Taking moments about the y-axis,

$$(h + \tfrac{2}{3}a)\pi a^2 w\bar{x} = \pi a^2 hw \times \tfrac{1}{2}h - \tfrac{2}{3}\pi a^3 w \times \tfrac{3}{8}a$$
$$\therefore \quad (h + \tfrac{2}{3}a)\bar{x} = \tfrac{1}{2}h^2 - \tfrac{1}{4}a^2$$

When this solid is placed with the curved surface of the cylinder touching a horizontal plane, it will rest in equilibrium if the vertical line through G passes through a point of contact between the plane and the solid.
This condition will be satisfied if $\bar{x} \geqslant 0$ i.e. if $h^2 \geqslant \frac{1}{2}a^2$.

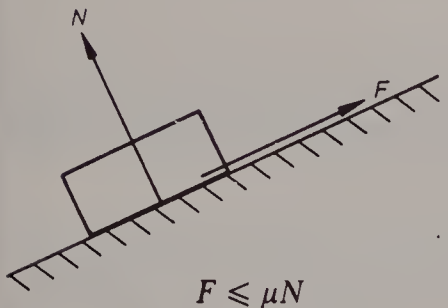
For a body to rest in equilibrium on a rough inclined plane the force exerted by the plane on the body must act along a vertical line through the centre of gravity of the body.



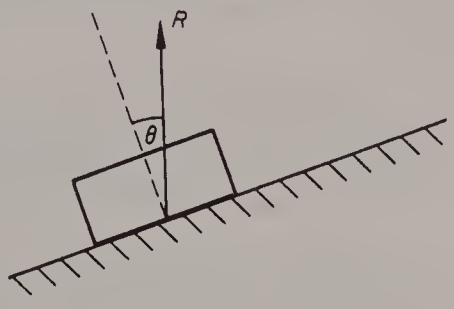
For the block shown in this diagram equilibrium is not possible. Since the vertical line through the centre of gravity G falls outside the base of the block, it will topple.

The reaction R between a body and a rough surface is the resultant of two components, the normal reaction N and a friction force F. According to the laws of friction, if μ is the coefficient of friction between the body and the surface, then $F \leqslant \mu N$. We can also express this condition in terms of λ, the angle of friction, and θ, the angle made by R with the normal to the surface, since $\tan \lambda = \mu$ and $\tan \theta = F/N$.

$$F \leqslant \mu N \Leftrightarrow \tan \theta \leqslant \mu \Leftrightarrow \theta \leqslant \lambda.$$



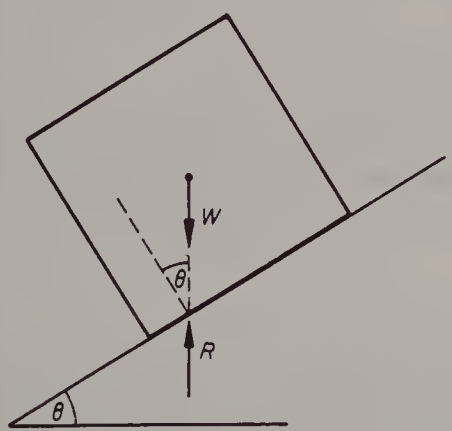
$F \leqslant \mu N$



$\tan \theta \leqslant \mu, \quad \theta \leqslant \lambda$

Example 5 A uniform solid cube is in equilibrium with one face resting on a rough inclined plane and one pair of opposite faces vertical. The coefficient of friction between the cube and the plane is μ. The angle of inclination of the plane is gradually increased. Show that equilibrium will be broken by sliding rather than toppling if $\mu < 1$.

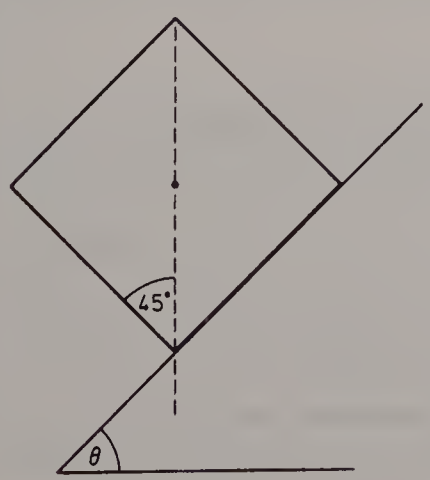
(1)



Let θ be the angle of inclination of the plane to the horizontal. When the cube is in equilibrium, as shown in diagram (1), the total reaction R acts vertically. The angle between its line of action and the normal to the plane is θ. When the cube is on the point of sliding, friction is limiting,

$$\therefore \qquad \tan \theta = \mu.$$

(2)



In diagram (2) the cube is on the point of toppling, since the vertical line through its centre of gravity passes through the lower edge of the base. Thus, when the cube is about to topple,

$$\tan \theta = \tan 45° = 1.$$

Hence equilibrium will be broken by sliding if $\tan\theta$ reaches the value μ before it reaches the value 1, i.e. if $\mu < 1$.

Exercise 10.1

1. A beam AB of length 4·5 m is supported in a horizontal position by two vertical cables attached to its ends. The tensions in the cables at A and B are 160 N and 200 N respectively. Find the weight of the beam and the distance of its centre of gravity from A.

2. A uniform rod AB of length 3 m and weight 50 N rests horizontally on smooth supports at A and B. A load of 24 N is attached to the rod at a point C, where $AC = 1$ m. Find the forces exerted on the rod by the supports.

3. A uniform bar of length 2·8 m and weight 80 N has loads of 20 N and 40 N attached to its ends A and B respectively. If the bar balances in a horizontal position when smoothly supported at C, find the distance of C from A.

4. A uniform rod $APQB$ of length 2 m rests horizontally on smooth supports at P and Q, where $PQ = 1{\cdot}2$ m. If the reaction at P is twice the reaction at Q, find the distance AP. Given that when a weight of 5 N is attached to the rod at B the reactions at P and Q are equal, find the weight of the rod.

5. A uniform plank ABC of weight 400 N is held in a horizontal position by a smooth support at B and a vertical rope attached at C, where $AB = 1$ m and $AC = 5$ m. If the rope is likely to break when the tension in it exceeds 750 N, find
(a) the maximum weight that can be placed on the plank at the mid-point of BC,
(b) the maximum distance that a man of weight 800 N can safely walk from B towards C.

6. A non-uniform beam $AXYB$ of length 4 m and weight 400 N is maintained in a horizontal position by smooth supports at X and Y, where $AX = YB = 1$ m. A boy of weight 500 N can just walk to the end A of the beam without overturning it. If the boy walks along the beam towards B, find his distance from B when the beam is just about to tip.

7. A uniform right circular solid cone of weight W is suspended by two vertical strings attached to the ends A and B of a diameter of its base. If the cone hangs in equilibrium with its vertex vertically below A, find the tensions in the strings.

8. $ABCD$ is a uniform rectangular lamina of weight 10 N. A particle of weight W N is attached to the lamina at B. When the system is freely suspended from A it rests in equilibrium with the mid-point of BC vertically below A. Find the value of W.

9. A uniform wire is bent into the form of a trapezium $ABCD$ in which $\angle ABC = \angle BCD = 90°$, $AB = 6a$, $BC = 4a$ and $CD = 3a$. Find the distances of

the centre of gravity from AB and BC. If the trapezium is freely suspended from A, find the angle which AB makes with the vertical.

10. A uniform triangular lamina ABC has $AB = 4a, BC = 5a, CA = 3a$. The lamina is folded through 180° along LN, where L and N are the mid-points of BC and AB respectively, so that the vertex B is at A. Find the coordinates of the centre of mass of the folded lamina relative to AN and AC as x-axis and y-axis respectively. The folded lamina is suspended from C and hangs in equilibrium under gravity. Find the tangent of the angle that CA makes with the vertical. (C)

11. A uniform wire is bent to form a semi-circular arc together with the diameter AB joining the ends of the arc. If the wire hangs in equilibrium from a smooth pivot at A, find the tangent of the angle that AB makes with the vertical.

12. A uniform wire AB of weight $2W$ is bent into a semi-circular arc. A particle of weight W is attached to the end B and the system is freely suspended from A. Find the tangent of the angle that AB makes with the vertical.

13. Three uniform rods AB, BC, CD, each of length $2a$ and weight W are rigidly joined together to form three sides of a square $ABCD$. If the system hangs freely from a smooth pivot at A, find the angle that AB makes with the vertical. Find also, in terms of W and a, the moment of the couple required to maintain the system in equilibrium with BC horizontal.

14. $ABCD$ is a uniform square lamina of side $6a$ and weight W. P is a point on the diagonal AC, such that $AP = 2PC$. The square having diagonal PC is cut away and the remaining body is freely suspended from B. Find the angle which BA makes with the vertical. Find also the moment of the couple required to maintain the body in equilibrium with BA horizontal.

15. $ABCD$ is a uniform rectangular lamina in which $BC = 2a$. P is the mid-point of BC and R is a point on AB such that $AR = a$ and $RB = ka$. The rectangle $BPQR$ is cut away and the remaining body is placed in a vertical plane with AR resting on a horizontal plane. Find the maximum value of k given that the lamina does not topple about R.

16. A uniform lamina of weight $3W$ in the form of a trapezium $ABCD$ has $\angle ABC = \angle BCD = 90°, AB = BC = a$ and $CD = 2a$. The lamina is placed in a vertical plane with AB resting on a horizontal plane. Find, in terms of W, the greatest weight that can be attached to the lamina at D without causing it to topple about A.

17. A solid hemisphere of radius a and a solid right circular cone of height h and base radius a are made from the same uniform material and are joined together with their plane faces completely in contact. O is the centre of the common base. Find the position of the centre of gravity of the whole body.

(a) The body is suspended freely from a point A on the edge of the common base and hangs in equilibrium under gravity. If $h = 3a$, find, correct to the nearest degree, the acute angle made by AO with the vertical.
(b) The body is found to rest in equilibrium when it is placed on a horizontal table with *any* point of its hemispherical surface in contact with the table. Find h in terms of a. (O&C)

18. A uniform thin hemispherical bowl rests in equilibrium with its axis of symmetry horizontal and its curved surface on a rough inclined plane. Find the angle between the plane and the horizontal. Find also the least possible value of the coefficient of friction between the bowl and the plane.

19. A uniform solid hemisphere rests in equilibrium with its curved surface on a rough plane inclined at an angle α to the horizontal, where $\sin\alpha = 0{\cdot}3$. Find the angle between the plane face of the hemisphere and the vertical.

20. A uniform right circular solid cone of radius a and height $3a$ rests in equilibrium with its base in contact with a rough inclined plane. The angle of inclination of the plane is then increased steadily.
(a) Assuming that the cone does not slide down the plane, find the angle of inclination of the plane to the horizontal when the cone is about to topple.
(b) If the coefficient of friction between the cone and the plane is $\frac{1}{2}$, find the angle of inclination of the plane to the horizontal when the cone is about to slide.

21. A uniform solid cylinder of radius a and height h is placed with one plane face resting on a rough inclined plane. The angle of inclination of the plane is then gradually increased. Show that the cylinder will slide before it topples if the coefficient of friction between the cylinder and the plane is less than $\frac{2a}{h}$.

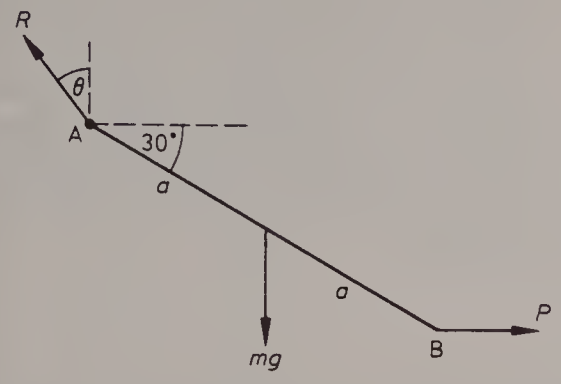
10.2 Three force problems

When a body is in equilibrium under the action of *three forces*, the vector sum of these forces must be zero. Hence three non-parallel forces in equilibrium can be represented in magnitude and direction by the sides of a triangle. We also know that the sum of the moments of coplanar forces in equilibrium about any point in their plane is zero.

Thus one approach to three force problems is to
(i) take moments about any suitable point, and
(ii) use a triangle of forces.

Example 1 A uniform rod AB of mass m, which is smoothly hinged at A, is maintained in equilibrium by a horizontal force P acting at B. Given that the rod is

inclined at 30° to the horizontal with B below A, find (a) an expression for P, (b) the magnitude and direction of the reaction at the hinge.



[Note that the reaction at a *smooth hinge* is a single force which may act in any direction. The system of reactions in a hinge which is *not smooth* may reduce to a single force together with a couple.]

Let the length of the rod be $2a$ and let the reaction at the hinge be of magnitude R acting at an angle θ to the vertical.

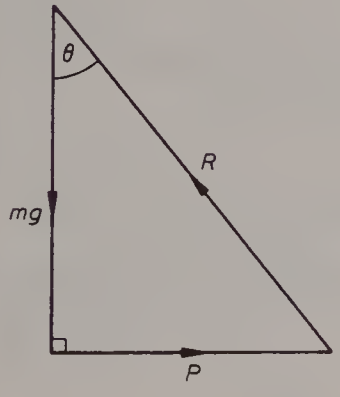
(a) The weight of the rod is mg and acts through the mid-point of the rod. Thus taking moments anti-clockwise about A, we have

$$P \times 2a \sin 30° - mg \times a \cos 30° = 0$$

$$\therefore \qquad Pa - \frac{\sqrt{3}}{2} mga = 0$$

Hence $P = \dfrac{\sqrt{3}}{2} mg.$

(b)



Using the triangle of forces,

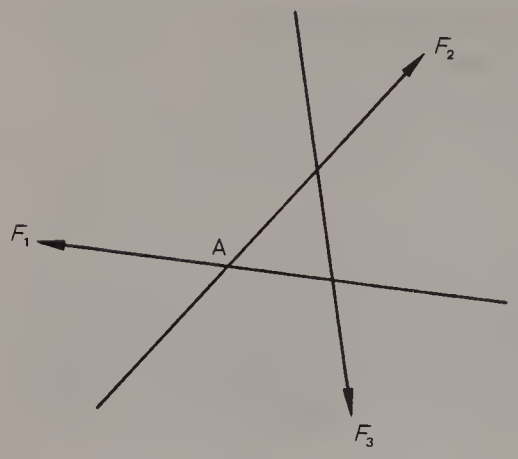
$$R = \sqrt{\{(mg)^2 + P^2\}} = mg\sqrt{(1 + \tfrac{3}{4})}$$

$$= \frac{\sqrt{7}}{2} mg$$

$$\tan\theta = \frac{P}{mg} = \frac{\sqrt{3}}{2} \qquad \therefore \quad \theta \approx 40{\cdot}9°$$

Hence the reaction at the hinge has magnitude $\dfrac{\sqrt{7}}{2} mg$ and makes an angle $\tan^{-1} \dfrac{\sqrt{3}}{2} \approx 40{\cdot}9°$ with the vertical.

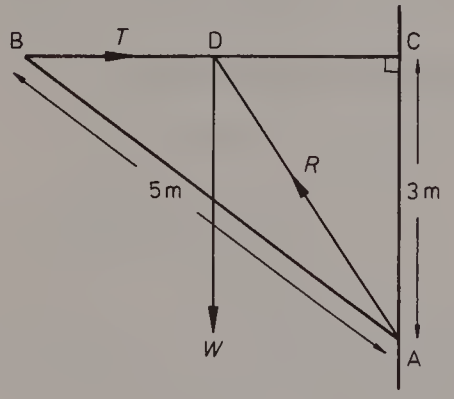
We now establish a property of three force systems which is a useful aid to problem solving. Suppose that a rigid body is in equilibrium under the action of three coplanar forces F_1, F_2 and F_3. Then either all three forces are parallel or the lines of action of two of them meet.



If F_1 and F_2 act along lines which intersect at a point A, then the moments of F_1 and F_2 about A are zero. Since F_1, F_2 and F_3 are in equilibrium, the moment of F_3 about A must also be zero. Hence the line of action of F_3 passes through A. We may deduce the following general principle.

> If a rigid body is in equilibrium under the action of three coplanar forces, then the lines of action of the forces are either parallel or concurrent.

Example 2 A uniform pole AB of length 5 m is smoothly hinged at A and supported at B by a light elastic string BC of natural length 3 m and modulus 60 N. The end C of the string is fixed 3 m vertically above A. Given that the system is in equilibrium with BC horizontal, find in newtons the tension T in the string, the weight W of the pole and magnitude R of the reaction at A.



The lines of action of T and W meet at D, the mid-point of BC,

$\therefore$ the third force R must also act through D.

By Pythagoras' theorem in $\triangle ABC$,

$$BC = 4\,\text{m},$$

$\therefore$ the extension in the elastic string

$$= (4 - 3)\,\text{m} = 1\,\text{m}.$$

By Hooke's law, $T = \dfrac{\lambda x}{l}$

where $\lambda = 60$, $l = 3$ and $x = 1$,

$$\therefore \quad T = \frac{60 \times 1}{3} = 20.$$

Hence the tension in the elastic string is 20 N.

Triangle of forces

Comparing the diagrams we see that the triangle of forces is similar to $\triangle ADC$.
By Pythagoras' theorem in $\triangle ADC$,

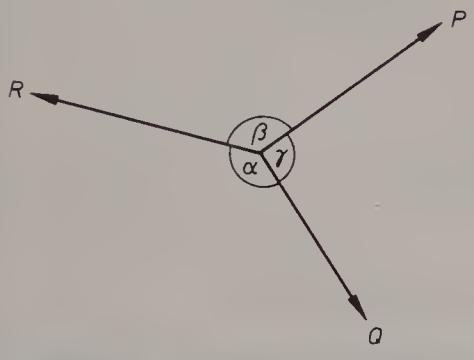
$$AD = \sqrt{(AC^2 + DC^2)} = \sqrt{(3^2 + 2^2)}\,\text{m} = \sqrt{13}\,\text{m},$$

$$\therefore \quad \frac{T}{2} = \frac{W}{3} = \frac{R}{\sqrt{13}}$$

$$\therefore \quad W = \frac{3}{2}T = 30$$

and $$R = \frac{\sqrt{13}}{2}T = 10\sqrt{13} \approx 36{\cdot}1.$$

Hence the weight of the pole is 30 N and the magnitude of the reaction at A is 36·1 N.



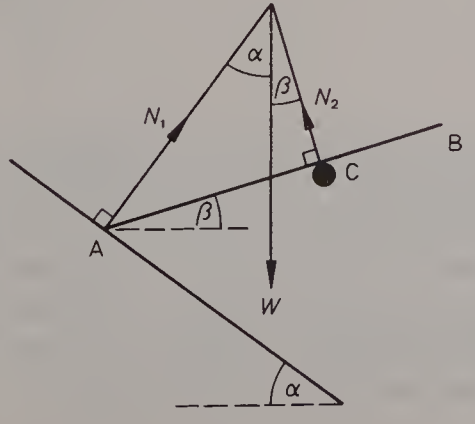
When solving three force problems in which angles rather than lengths are known, *Lami's theorem* may be useful. It states that for forces P, Q and R in equilibrium, as shown in the diagram.

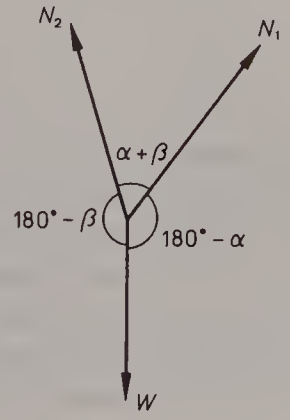
$$\frac{P}{\sin\alpha} = \frac{Q}{\sin\beta} = \frac{R}{\sin\gamma}$$

Example 3 A uniform rod AB of length $5a$ and weight W rests with the end A on a smooth plane inclined at an angle α to the horizontal, where $\sin\alpha = 3/5$. It is maintained in equilibrium by a smooth support at C. If the rod makes an angle β with the horizontal, where $\sin\beta = 7/25$, find

(a) the magnitudes of the reactions at A and C,
(b) the distance AC.

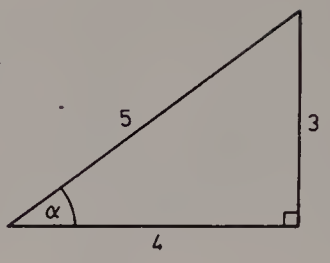
The reaction at A is perpendicular to the inclined plane and the reaction at C is perpendicular to the rod. Let their magnitudes be N_1 and N_2 respectively. Since the forces N_1, N_2 and W are in equilibrium, their lines of action are concurrent.





(a) By Lami's theorem, $\dfrac{N_1}{\sin(180° - \beta)} = \dfrac{N_2}{\sin(180° - \alpha)} = \dfrac{W}{\sin(\alpha + \beta)}$

$$\therefore \quad \frac{N_1}{\sin\beta} = \frac{N_2}{\sin\alpha} = \frac{W}{\sin(\alpha + \beta)} \qquad (1)$$



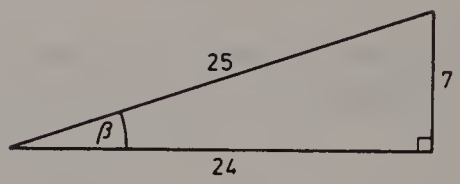
$$\sin\alpha = \frac{3}{5} \Rightarrow \cos\alpha = \frac{4}{5}$$

$$\sin\beta = \frac{7}{25} \Rightarrow \cos\beta = \frac{24}{25}$$

$$\therefore \quad \sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$= \frac{3}{5} \times \frac{24}{25} + \frac{4}{5} \times \frac{7}{25}$$

$$= \frac{72 + 28}{125} = \frac{100}{125} = \frac{4}{5}$$



Substituting these values in equation (1),

$$\frac{25}{7}N_1 = \frac{5}{3}N_2 = \frac{5}{4}W$$

$$\therefore \quad N_1 = \frac{7}{25} \times \frac{5}{4}W = \frac{7}{20}W \quad \text{and} \quad N_2 = \frac{3}{5} \times \frac{5}{4}W = \frac{3}{4}W$$

Hence the magnitudes of the reactions at A and C are $7W/20$ and $3W/4$ respectively.

(b) The distance of the centre of gravity of the rod from A is $5a/2$. Let the distance AC be x.

Taking moments anti-clockwise about A,

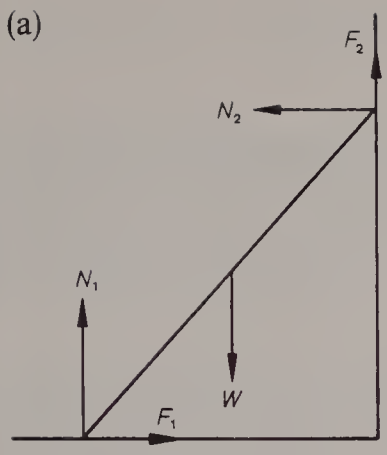
$$N_1 \times 0 + N_2 \times x - W \times \frac{5}{2}a\cos\beta = 0$$

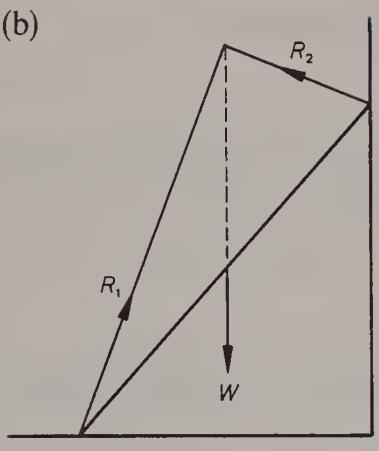
$$\therefore \quad \frac{3}{4}Wx - \frac{5}{2} \times \frac{24}{25}Wa = 0$$

$$\therefore \quad x = \frac{4}{3} \times \frac{5}{2} \times \frac{24}{25}a = \frac{16}{5}a$$

Hence the distance AC is $16a/5$.

In problems involving contact between rough surfaces the reaction at any point of contact can be treated as the resultant of the normal reaction and a frictional force or as a single force. For instance, a problem concerning a ladder resting on a rough horizontal floor and against a rough vertical wall may be solved using diagram (a) and the general methods to be described in §10.3. Alternatively, as indicated in diagram (b), the set of forces on the ladder can be reduced to a three force system by using the total reactions at the points of contact with the floor and the wall.

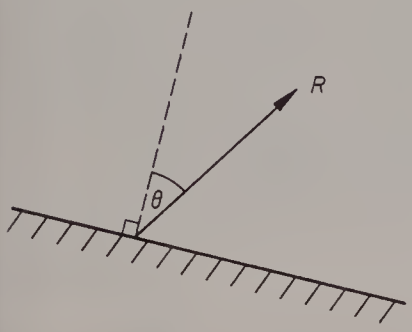




When using this latter approach, it is helpful to remember that if θ is the angle between the total reaction and the normal to the surfaces in contact, then

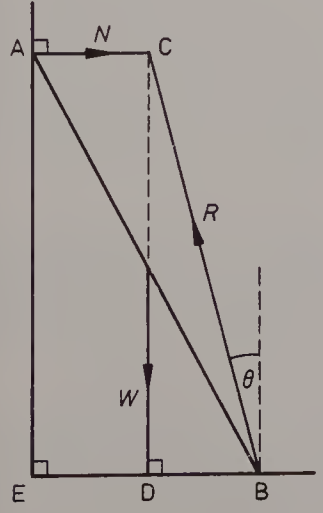
$$\tan\theta \leqslant \mu \quad \text{and} \quad \theta \leqslant \lambda,$$

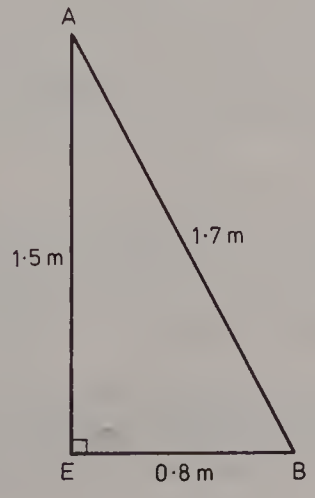
where μ is the coefficient of friction and λ is the angle of friction.

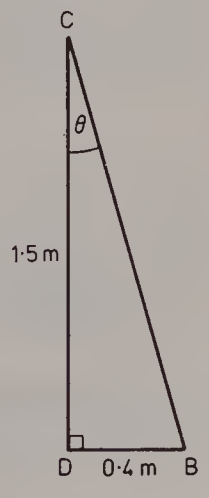


Example 4 A uniform rod AB of length 1·7 m is placed with the end A against a smooth vertical wall and the end B on rough horizontal ground at a distance of 0·8 m from the wall. If the coefficient of friction between the rod and the ground is μ, find the least value of μ for which equilibrium is possible.

If the rod is in equilibrium under the action of its weight W, the normal reaction N at A and the reaction R at B, then the lines of action of these three forces must be concurrent. Let C be their point of intersection and let θ be the angle the reaction R makes with the vertical.







By Pythagoras' theorem in $\triangle ABE$, $AE^2 + EB^2 = AB^2$

$\therefore \quad AE = \sqrt{\{(1{\cdot}7)^2 - (0.8)^2\}}\,\text{m} = 1{\cdot}5\,\text{m}.$

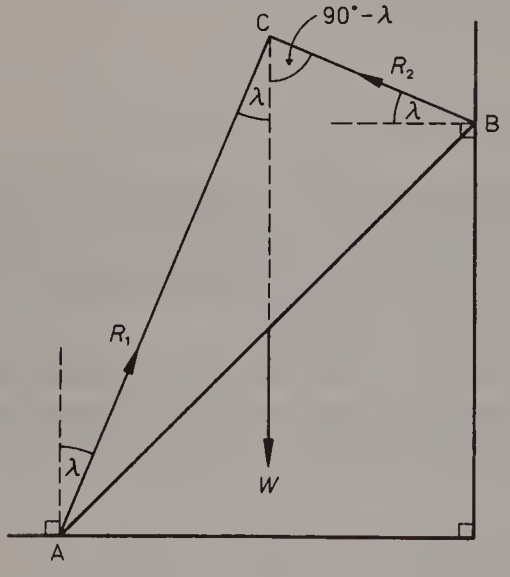
Since the weight of the rod acts vertically through its mid-point, D must be the mid-point of EB,

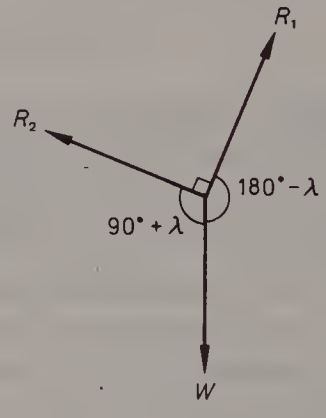
$$\therefore \quad \text{using } \triangle BCD, \quad \tan\theta = \frac{0{\cdot}4}{1{\cdot}5} = \frac{4}{15}.$$

Equilibrium is possible provided that $\tan\theta \leqslant \mu$.
Hence the least possible value of μ is 4/15.

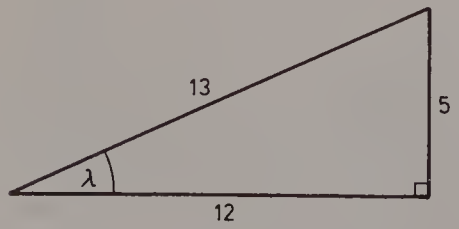
Example 5 A uniform ladder AB of weight W rests in limiting equilibrium with the end A on rough horizontal ground and the end B against a rough vertical wall. If the coefficient of friction at each end of the ladder is 5/12, find the magnitudes of the reactions at A and B.

Since the ladder is in limiting equilibrium, the reaction R_1 at A acts at an angle λ to the vertical and the reaction R_2 at B acts at an angle λ to the horizontal, where $\tan\lambda = 5/12$. If the lines of action of R_1 and R_2 meets at C, then the weight of the ladder also acts through C and $\angle ACB = \lambda + (90° - \lambda) = 90°$.





By Lami's theorem, $\dfrac{R_1}{\sin(90° + \lambda)} = \dfrac{R_2}{\sin(180° - \lambda)} = \dfrac{W}{\sin 90°}$



$$\therefore \quad \frac{R_1}{\cos\lambda} = \frac{R_2}{\sin\lambda} = W$$

Since $\tan\lambda = \frac{5}{12}$,

$\cos\lambda = \frac{12}{13}$ and $\sin\lambda = \frac{5}{13}$,

$\therefore \quad R_1 = W\cos\lambda = \frac{12}{13}W$ and $R_2 = W\sin\lambda = \frac{5}{13}W$.

Hence the reactions at A and B are $12W/13$ and $5W/13$ respectively.

Exercise 10.2

[Take g as $10\,\text{m s}^{-2}$.]

1. A uniform rod AB of weight 20 N is smoothly hinged at B and maintained in equilibrium by a horizontal force of P N acting at A. If the rod is inclined at 45° to the horizontal with A above B, find (a) the value of P, (b) the magnitude and direction of the reaction at the hinge.

2. A non-uniform bar AB of length $2a$ and mass m is maintained in a horizontal position by two strings AC and BC attached to a fixed point C above the bar. If $\angle CAB = 60°$ and $\angle ABC = 30°$, find expressions for the tensions in the strings. Find also the distance of the centre of gravity of the bar from A.

3. A uniform rod AB of length 40 cm and weight 8 N rests on a rough peg at C, where $AC = 15$ cm, and against a smooth vertical wall at A. If the rod is in equilibrium at an angle α to the horizontal, where $\sin\alpha = 4/5$, find the magnitudes of the reactions at A and at C.

4. A uniform bar AB of length 2 m and mass 3 kg rests on a smooth peg at X, where $AX = 0{\cdot}5$ m and against a rough vertical wall at A. If the bar is in equilibrium at 60° to the horizontal, find the magnitude and direction of the force exerted by the wall on the bar.

5. A sphere of radius 40 cm and weight 30 N rests in equilibrium on a smooth plane inclined at an angle α to the horizontal, where $\tan\alpha = 8/15$. A horizontal string joins a point on the sphere to a fixed point on the plane. Find the length of the string and the tension in the string.

6. A rod AB of length 40 cm, mass 6 kg and centre of mass G, is suspended by a light inextensible string AXB of length 80 cm, which passes over a smooth peg at X. Find the tension in the string and the angle the rod makes with the vertical when it hangs in equilibrium if (a) $AG = GB$, (b) $AG:GB = 3:5$.

7. A uniform rod of weight W is in equilibrium with one end resting against a smooth vertical plane and the other end against a smooth plane inclined at an angle α to the horizontal, where $\tan\alpha = 5/12$. The line of intersection of the planes is horizontal. If the rod makes an angle θ with the vertical, find $\tan\theta$. Find also the magnitudes of the reactions at the vertical and inclined planes.

8. A non-uniform rod AB of weight $3W$ rests with the end A on a smooth plane inclined at 30° to the horizontal. It is maintained in equilibrium at 30° to the horizontal by a smooth peg at C, where $AC = 3a$. Find (a) the magnitudes of the reactions at A and C, (b) the distance of the centre of gravity of the rod from A.

9. A uniform rod AB of length 1 m is smoothly hinged at A and the end B is attached by means of a light elastic string of natural length 1 m, modulus 20 N to a point C which is 1 m vertically above A. Given that the tension in the string is 12 N,

find (a) the weight of the rod, (b) the magnitude and direction of the reaction at the hinge.

10. A uniform rod AB of weight $4W$ rests in equilibrium with the end A against a smooth vertical wall. A light elastic string XY of natural length 24 cm, modulus $15W$ is attached to a point X vertically above A and a point Y on the rod. Given that $XY = 32$ cm and $\angle XYB = 90°$, find the tension in the string and the length of the rod.

11. A smooth hemispherical bowl of radius $5\sqrt{5}$ cm is fixed with its rim horizontal. A uniform rod AB is in equilibrium with the end A resting inside the bowl and a point C of the rod against the rim, where $AC = 20$ cm. If the rod is inclined at an angle α to the horizontal, find $\tan \alpha$. Find also the length of the rod.

12. A uniform ladder of length 6·5 m rests with one end against a smooth vertical wall and the other end on rough horizontal ground at a distance of 2·5 m from the wall. If the foot of the ladder is about to slip, find the coefficient of friction between the ladder and the ground.

13. A uniform pole of length $2a$ and mass m rests in equilibrium at an angle α to the horizontal, with one end against a smooth vertical wall and the other end on rough horizontal ground. Find expressions for (a) the force exerted on the pole by the wall, (b) the least possible value of the coefficient of friction between the pole and the ground.

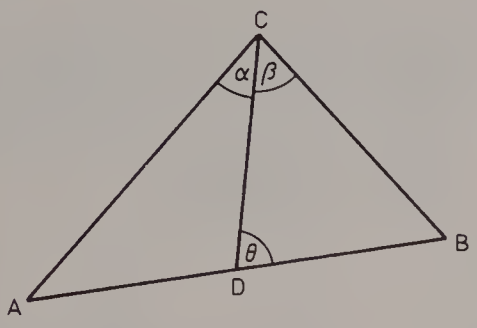
14. A uniform ladder of weight W and length $2l$ rests with its upper end against a smooth vertical wall. Its lower end stands on rough horizontal ground, and the coefficient of friction between the ladder and the ground is $\frac{1}{2}$. The ladder is in limiting equilibrium. Find the angle the ladder makes with the horizontal. Find also the reaction at the wall. A man of weight W climbs the ladder. Find how far up it he can go before the ladder will slip. Find also how far up the ladder he can go when a load of weight W is placed on the foot of the ladder. (L)

15. A uniform rod AB of length a and weight W has a light ring attached to the end A. The ring is free to slide along a straight wire inclined at 30° to the horizontal. The rod is maintained in equilibrium by a light inextensible string of length a attached to the end B. The other end of the string is attached to the wire at a fixed point C above A. (a) If the wire is smooth, find the angle θ which the string makes with the vertical. (b) If the string is vertical, find the least value of the coefficient of friction between the ring and the wire.

16. A heavy uniform rod of length $2a$ rests in equilibrium against a small smooth horizontal peg with one end of the rod on a rough horizontal floor. If the height of the peg above the floor is b and friction is limiting when the rod makes an angle $\frac{\pi}{6}$ with the floor, show that the coefficient of friction between the floor and the rod is $\frac{\sqrt{3}a}{(8b - 3a)}$. (O)

17. A uniform rod AB of length $10a$ rests in limiting equilibrium with the end A on horizontal ground and the end B against a vertical wall. The coefficient of friction at each end of the rod is μ. Given that A is at a distance $6a$ from the wall, find the value of μ.

18. (a)



In the given diagram D is the mid-point of AB. Using the sine rule or otherwise prove that

$$2\cot\theta = \cot\alpha - \cot\beta.$$

(b) A uniform ladder rests at an angle θ to the vertical, with one end on horizontal ground and the other end against a vertical wall. The coefficients of friction between the ladder and the ground and between the ladder and the wall are μ_1 and μ_2 respectively. Given that the ladder is on the point of slipping, use part (a) to prove that $\tan\theta = 2\mu_1/(1 - \mu_1\mu_2)$.

10.3 General conditions for equilibrium

We have already established that if a body is in equilibrium under the action of a set of coplanar forces, then the sum of their components in any direction is zero and the sum of their moments about any point is zero. In this section we consider sets of conditions that are sufficient to ensure equilibrium.

Suppose that a system of coplanar forces is such that

(i) the sum of the components in a direction d_1 is zero,

(ii) the sum of the components in a direction d_2, not parallel to d_1, is zero, and

(iii) the sum of the moments about a point A in the plane is zero.

This system of forces is not equivalent to a single force, because no force can have zero components in two directions. The system does not reduce to a couple, because a couple has the same non-zero moment about every point in its plane. Thus, if a set of forces acting on a rigid body satisfies conditions (i), (ii) and (iii), it follows that the body is in equilibrium.

We can use this set of equilibrium conditions to solve problems involving forces in a *vertical plane* by

(i) resolving horizontally,

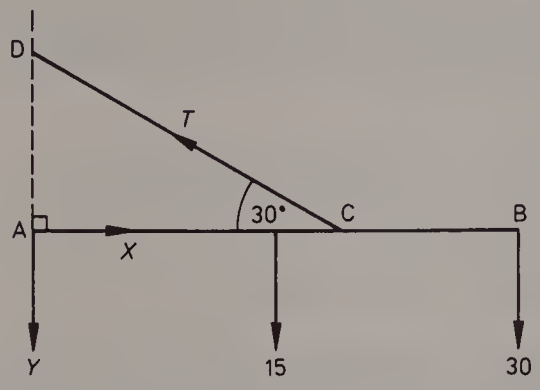
(ii) resolving vertically, and

(iii) taking moments about a suitable point.

In this way we obtain up to three equations which we then solve to find any unknown quantities.

Example 1 A uniform rod AB of weight 15 N and length 4 m is smoothly hinged at A and has a particle of weight 30 N attached to it at B. A light inextensible string,

attached to the rod at a point C, where $AC = 2{\cdot}5$ m, and to a point D vertically above A, keeps the rod in a horizontal position. If the angle between the rod and the string is 30°, find (a) the tension in the string, (b) the magnitude and direction of the reaction at the hinge.



Let the tension in the string be T N and let the horizontal and vertical components of the reaction at the hinge be X N and Y N respectively.

(a) Taking moments anti-clockwise about A,

$$T \times 2{\cdot}5 \sin 30° - 15 \times 2 - 30 \times 40 = 0$$

i.e. $$T \times \tfrac{5}{2} \times \tfrac{1}{2} - 30 - 120 = 0$$

$$\therefore \quad T = 150 \times \tfrac{4}{5} = 120.$$

Hence the tension in the string is 120 N

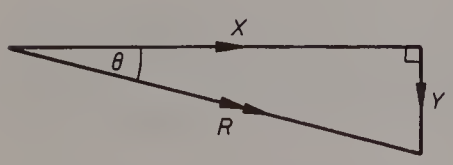
(b) Resolving horizontally and vertically,

$$\rightarrow \qquad X - T\cos 30° = 0 \qquad (1)$$

$$\downarrow \qquad Y + 15 + 30 - T\sin 30° = 0 \qquad (2)$$

From (1), $X = 120 \times \dfrac{\sqrt{3}}{2} = 60\sqrt{3}$

From (2), $Y = 120 \times \tfrac{1}{2} - 45 = 15$



If the resultant reaction R acts at an angle θ to the horizontal, then

$$R^2 = X^2 + Y^2, \tan\theta = \frac{Y}{X}.$$

$$\therefore \quad R = \sqrt{\{(60\sqrt{3})^2 + 15^2\}} = 15\sqrt{\{(4\sqrt{3})^2 + 1^2\}} = 15\sqrt{(48 + 1)} = 105,$$

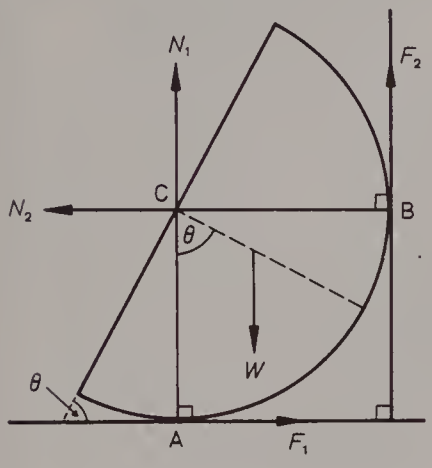
$$\tan\theta = \frac{15}{60\sqrt{3}} = \frac{1}{4\sqrt{3}} \quad \text{i.e.} \quad \theta \approx 8{\cdot}2°.$$

Hence the reaction at the hinge has magnitude 105 N and acts at 8·2° to the horizontal.

When a rigid body is kept in equilibrium by frictional forces, then at each point of contact with a rough surface the frictional force F acts along the surface, opposing any tendency to slide. It also satisfies the condition $F \leqslant \mu N$, where N is the normal

reaction and μ is the coefficient of friction. If the body is just about to slip, then friction is limiting at *every* point of contact and in each case $F = \mu N$.

Example 2 A uniform solid hemisphere rests in limiting equilibrium with its curved surface in contact with a horizontal floor and a vertical wall. The coefficient of friction at the floor is $\frac{1}{4}$ and at the wall is $\frac{1}{2}$. If θ is the angle of inclination of the plane face of the hemisphere to the horizontal, find $\sin\theta$.



Let the hemisphere have centre C and radius a. Let the forces on it be as shown in the diagram. Since the hemisphere is in limiting equilibrium, friction must be limiting at both A and B,

$$\therefore \quad F_1 = \tfrac{1}{4}N_1, \quad F_2 = \tfrac{1}{2}N_2.$$

Resolving horizontally and vertically,

$$\rightarrow \qquad F_1 - N_2 = 0 \qquad (1)$$
$$\uparrow \qquad N_1 + F_2 - W = 0 \qquad (2)$$

Substituting for F_1 in (1):

$$\tfrac{1}{4}N_1 - N_2 = 0 \qquad \therefore \quad N_1 = 4N_2$$

Substituting for N_1 and F_2 in (2):

$$4N_2 + \tfrac{1}{2}N_2 - W = 0 \qquad \therefore \quad N_2 = \tfrac{2}{9}W$$

Hence $F_1 = N_2 = \frac{2}{9}W$ and $F_2 = \frac{1}{2}N_2 = \frac{1}{9}W$.

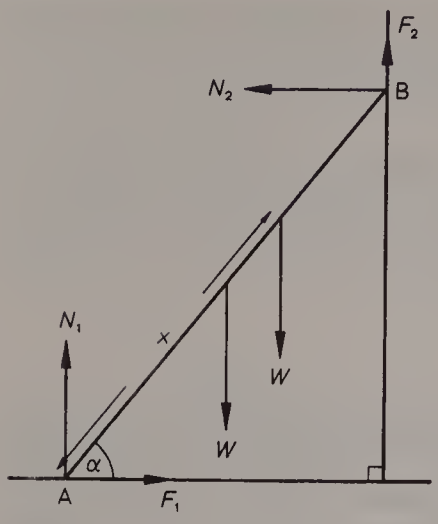
The centre of gravity of the hemisphere is on its axis of symmetry at a distance $\frac{3}{8}a$ from C.

Thus, taking moments clockwise about C,

$$W \times \tfrac{3}{8}a\sin\theta - F_1 \times a - F_2 \times a = 0$$
$$\therefore \qquad \tfrac{3}{8}aW\sin\theta - \tfrac{2}{9}aW - \tfrac{1}{9}aW = 0$$

Hence $\frac{3}{8}\sin\theta - \frac{3}{9} = 0$ i.e. $\sin\theta = \frac{8}{9}$.

Example 3 A uniform ladder AB of length $2a$ and weight W rests with the end A on horizontal ground and the end B against a vertical wall. The coefficients of friction at A and B are $\frac{2}{5}$ and $\frac{1}{2}$ respectively. The ladder makes an angle α with the horizontal, where $\tan\alpha = \frac{5}{4}$. If a man of weight W starts to climb the ladder, find his distance from A when the ladder begins to slip.



Let the reactions at A and B when the ladder is about to slip be as shown in the diagram. Since friction must be limiting at both A and B,

$$F_1 = \tfrac{2}{5}N_1, \quad F_2 = \tfrac{1}{2}N_2.$$

Resolving horizontally and vertically,

$$\rightarrow \qquad F_1 - N_2 = 0 \qquad (1)$$
$$\uparrow \qquad N_1 + F_2 - 2W = 0 \qquad (2)$$

Substituting for F_1 in (1),

$$\tfrac{2}{5}N_1 - N_2 = 0 \qquad \therefore \quad N_1 = \tfrac{5}{2}N_2$$

Substituting for N_1 and F_2 in (2),

$$\tfrac{5}{2}N_2 + \tfrac{1}{2}N_2 - 2W = 0 \qquad \therefore \quad N_2 = \tfrac{2}{3}W$$

Hence $F_2 = \tfrac{1}{2}N_2 = \tfrac{1}{3}W$

Let the man be a distance x from A when the ladder begins to slip. Taking moments clockwise about A,

$$W \times x\cos\alpha + W \times a\cos\alpha - F_2 \times 2a\cos\alpha - N_2 \times 2a\sin\alpha = 0$$

Dividing by $\cos\alpha$,

$$Wx + Wa - 2F_2a - 2N_2a\tan\alpha = 0$$
$$\therefore \quad Wx + Wa - \tfrac{2}{3}Wa - \tfrac{4}{3}Wa \times \tfrac{5}{4} = 0$$
$$\therefore \quad x = \tfrac{5}{3}a + \tfrac{2}{3}a - a = \tfrac{4}{3}a.$$

Hence the man is at a distance of $4a/3$ from A when the ladder begins to slip.

The set of conditions discussed earlier, for the equilibrium of a rigid body under the action of coplanar forces, involved the sums of components in two directions and the sum of moments about one point. We now consider whether any other combinations of similar conditions will also ensure equilibrium.

To be certain that a set of forces is not equivalent to a couple, we need to know that the system has zero moment about at least one point. A system with zero moments about two points A and B may have a resultant whose line of action passes through A and B. We can rule out this possibility if we find that the components of the forces in some direction, not perpendicular to AB, have zero sum or that the system has zero moment about a third point not on AB.

We have then three sets of equilibrium conditions. A system of coplanar forces must be in equilibrium if

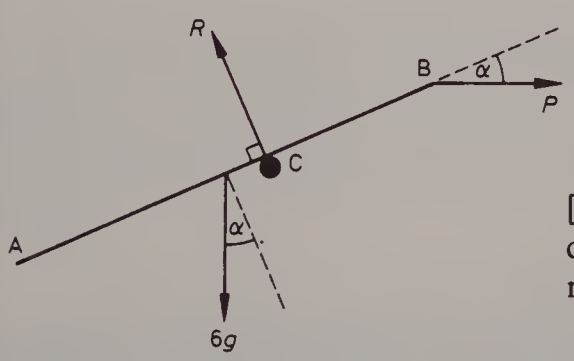
(a) the sums of the components in two non-parallel directions are each zero, and the sum of the moments about one point is zero;
or (b) the sums of the moments about two points A and B are each zero, and the sum of the components in a direction, not perpendicular to AB, is zero;
or (c) the sums of the moments about three points, not in a straight line, are each zero.

Thus when tackling problems we may
(a) resolve in two directions and take moments about one point;
or (b) take moments about two points A and B then resolve in a direction not perpendicular to AB;
or (c) take moments about three points which are not collinear.

Since each method involves putting three different restrictions on the system of forces concerned, we will obtain three *independent* equations connecting any unknown quantities i.e. three equations each giving information which could not be obtained from the other two. [Note that since these three equations ensure the equilibrium of the system, no new information can be gained from further equations formed by resolving or taking moments.]

The following examples show that, by careful choice of the points about which we take moments and of the directions in which we resolve, we may
(1) avoid equations involving quantities that we do not need to find,
(2) form equations containing as few unknowns as possible.

Example 4 A uniform rod AB of mass 6 kg rests on a smooth peg at a point C. The rod is kept in equilibrium at an angle α to the horizontal, where $\tan\alpha = 5/12$, by a horizontal force of P newtons acting at the upper end B of the rod. Find the value of P.
[Take g as $10\,\mathrm{m\,s^{-2}}$.]



[We try to avoid equations involving distances or the reaction at C, as we do not need to find these quantities.]

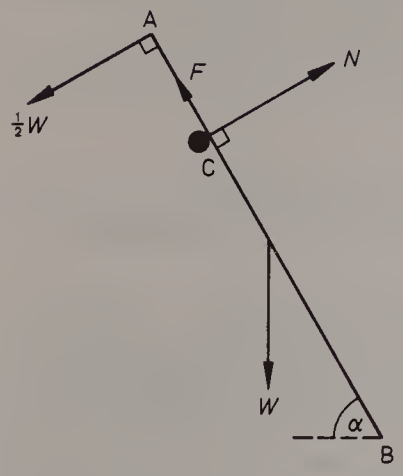
Resolving along the rod in the direction AB,

$$P\cos\alpha - 6g\sin\alpha = 0$$

$$\therefore \quad P = 6g\tan\alpha = 6 \times 10 \times \tfrac{5}{12} = 25.$$

Example 5 A uniform bar AB of weight W and length $4a$ rests on a rough peg at a point C, where $AC = a$. The bar is maintained in equilibrium, with A above B, by a force of magnitude $\frac{1}{2}W$ acting at A in a direction perpendicular to the bar. Find the

angle α, which the bar makes with the horizontal. Given that the bar is about to slip, find the coefficient of friction μ between the bar and the peg.



[In this example we try to form equations which simplify the working as much as possible.]

Taking moments clockwise about C,

$$W \times a\cos\alpha - \tfrac{1}{2}W \times a = 0 \qquad \therefore \cos\alpha = \tfrac{1}{2}$$

Hence $$\alpha = 60°.$$

Let F be the frictional force at C and let N be the normal reaction.
Taking moments clockwise about the mid-point of AB,

$$N \times a - \tfrac{1}{2}W \times 2a = 0 \qquad \therefore \quad N = W$$

Resolving along the bar in the direction BA,

$$F - W\sin\alpha = 0 \qquad \therefore \quad F = \frac{\sqrt{3}}{2}W$$

Since the bar is about to slip, $F = \mu N$.

Hence $$\mu = \frac{F}{N} = \frac{\sqrt{3}}{2}.$$

Exercise 10.3

[Take g as $10\,\mathrm{m\,s^{-2}}$.]

1. A uniform rod AB of weight 20 N and length 2 m is freely hinged at A. The rod is held in equilibrium at 30° to the horizontal, with B higher than A, by a light inextensible string of length 1·25 m. The string joins a point C on the rod to a fixed point D at a distance of 1·25 m vertically above A. If a particle of weight 5 N is suspended from B, find (a) the tension in the string, (b) the horizontal and vertical components of the reaction at A.

2. A uniform rod AB of mass 6 kg and length 3 m is freely hinged at A. A load of mass 2 kg is attached to the rod at B. The rod is kept in equilibrium by a light inextensible string CD of length 1·5 m, which is attached to a point C on the rod and

to a fixed point D on the same horizontal level as A. Given that $AC = 2\,\text{m}$ and $AD = 2{\cdot}5\,\text{m}$, find (a) the tension in the string, (b) the magnitude and direction of the reaction at the hinge.

3. A uniform square lamina $ABCD$ of mass 6 kg is hinged at A so that it is free to move in a vertical plane. It is maintained in equilibrium, with B vertically below A, by a horizontal force acting at C and a vertical force acting at D each of magnitude P newtons. Find (a) the value of P, (b) the magnitude and direction of the force exerted by the hinge on the lamina.

4. A uniform rod AB of weight 16 N is freely hinged at A. It is kept in equilibrium at an angle of 30° to the horizontal, with B lower than A, by a force acting at B. Find the magnitude and direction of the least force required at B to maintain equilibrium. Find also the corresponding horizontal and vertical components of the reaction at A.

5. A uniform rod AB of length 3 m and weight 40 N rests in equilibrium with the lower end A on a smooth plane inclined at an angle α to the horizontal, where $\tan\alpha = 3/4$, and a point C of the rod against a smooth peg. If the rod makes an angle θ with the horizontal, where $\tan\theta = 7/24$, find (a) the magnitudes of the reactions at A and C, (b) the distance AC.

6. A uniform rod AB of mass 13 kg is in limiting equilibrium with its lower end A on a rough horizontal plane and its upper end B attached to a light inextensible string inclined at an angle θ to the upward vertical, where $\tan\theta = 12/5$. The string passes over a smooth fixed pulley at a point vertically above A and supports at its other end a mass of 6·5 kg. Find (a) the coefficient of friction between the rod and the plane, (b) the angle the rod makes with the horizontal.

7. A smooth hemisphere of radius 60 cm is fixed with its plane face on a rough horizontal floor. A uniform rod AB of length 1 m and weight 10 N rests in equilibrium in a vertical plane with the end A on the floor and a point C of the rod in contact with the hemisphere, where $AC = 80\,\text{cm}$. Given that the rod is on the point of slipping, find the reaction at C and the coefficient of friction between the rod and the floor.

8. A uniform ladder AB of length 5 m and mass 12 kg rests with the end A on rough horizontal ground and the end B in contact with a smooth vertical wall. The foot of the ladder is at a distance of 1·4 m from the wall. Find the magnitudes of the reactions at A and B. If the ladder is about to slip, find the coefficient of friction between the ladder and the ground.

9. A uniform beam AB of length $2a$ and weight W rests with its end A against a smooth vertical wall and B on a rough horizontal plane. The vertical plane through AB is perpendicular to the wall. When it makes an angle of α with the wall the beam is on the point of slipping. Find the coefficient of friction between the beam and the horizontal plane. Find, also, what horizontal force applied to the mid-point of the

beam (in the vertical plane through AB) will just cause the beam to slip at B towards the wall. (SU)

10. A thin uniform hemispherical bowl, of weight W and radius a, rests in limiting equilibrium with its plane face at an angle θ to the horizontal. Its curved surface is in contact with a rough horizontal floor and a rough vertical wall. If the coefficient of friction at each point of contact is $\frac{1}{3}$, find the magnitudes of the frictional forces at the floor and the wall. Find also the value of $\sin\theta$.

11. A uniform solid hemisphere rests in limiting equilibrium with its plane face at an angle α to the horizontal, where $\sin\alpha = 8/13$. Its curved surface is in contact with rough horizontal ground and a rough vertical wall. If the coefficient of friction at both points of contact is μ, find the value of μ.

12. A uniform rod AB of weight W and length $4a$ rests with its end A on a rough horizontal plane and a point C of the rod, where $AC = 3a$, in contact with a smooth peg. Given that the rod makes an angle θ with the horizontal, where $\sin\theta = 3/5$, find the magnitude of the reaction at C and the horizontal and vertical components of the reaction at A. If μ is the coefficient of friction between the rod and the horizontal plane, show that $\mu \geqslant 24/43$.

13. A uniform ladder of weight W rests at 60° to the horizontal with its foot on rough horizontal ground and the other end against a smooth vertical wall. If the ladder is just about to slip when a man of weight $2W$ stands at the top, find the coefficient of friction between the ladder and the ground. Find the frictional force between the ladder and the ground when the man is standing at the mid-point of the ladder.

14. A uniform ladder of length $2a$ and weight W rests against a smooth vertical wall and has its foot on rough horizontal ground. The coefficient of friction between the ladder and the ground is $\frac{1}{2}$. If the ladder is placed at an angle α to the horizontal, where $\tan\alpha = \frac{3}{2}$, find the distance that a man of weight $3W$ can climb up the ladder before it slips.

15. A uniform pole AB of length 8 m and mass 60 kg rests in limiting equilibrium with the end A on horizontal ground and the end B against a vertical wall. If the coefficient of friction at each end of the pole is $\frac{1}{3}$, find the angle the pole makes with the horizontal. The pole is now placed at an angle α to the horizontal, where $\tan\alpha = 2$, and a body of mass m kg is attached to the pole at B without causing the pole to slip. Find the maximum value of m and the magnitude of the corresponding normal reaction at A.

16. A uniform circular hoop of weight 15 N hangs over a rough horizontal peg A. A horizontal force P N, acting in the plane of the hoop, is applied at the other end B of the diameter through A. If the hoop is on the point of slipping when AB is inclined at

an angle α to the downward vertical, where $\tan\alpha = 4/3$, find the value of P and the coefficient of friction between the hoop and the peg.

17. A uniform rod AB, of weight W and length $6a$, rests with its lower end A on a smooth horizontal plane and a point C of the rod against a smooth peg, where $AC = 4a$. The rod is kept in equilibrium at an angle α to the horizontal by a force of magnitude $\frac{1}{4}W$ acting along the rod. Find the magnitude of the reaction at A and the value of $\sin\alpha$.

18. A uniform ladder, of weight W and length l, rests in a vertical plane with one end against a smooth vertical wall, the wall being perpendicular to the vertical plane through the ladder. The other end of the ladder rests on horizontal ground, the coefficient of friction between the ladder and the ground being 1/4. The ladder is inclined at an angle θ to the horizontal, where $\tan\theta = 24/7$. A man of weight $10W$ climbs up the ladder. Show that the man can reach a height of $6l/7$ above the ground before the ladder begins to slip and calculate the force exerted by the ladder on the wall at this instant. (L)

19. A uniform ladder of length 7 m rests against a vertical wall with which it makes an angle of 45°, the coefficients of friction between the ladder and the wall and the ladder and the (horizontal) ground being $\frac{1}{3}$ and $\frac{1}{2}$ respectively. A boy whose weight is one half that of the ladder slowly ascends the ladder. How far along the ladder will he be when the ladder slips? [Assume the ladder is a uniform rod which is perpendicular to the line of intersection of the wall and ground.] (O&C)

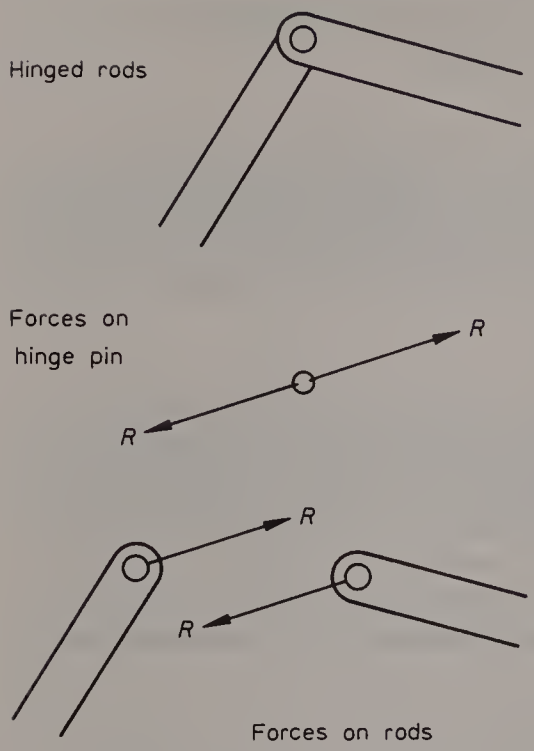
20. A uniform circular disc, of radius a and weight w, stands in a vertical plane upon a rough horizontal floor with the point P of its circumference in contact with an equally rough vertical wall and its plane at right angles to that of the wall. A particle of equal weight w is attached to a point Q of the disc between its centre O and P so that the disc is on the point of slipping. Draw a good diagram indicating clearly the normal reactions and the friction forces acting on the disc. Show that

$$OQ = \frac{2a\mu(1+\mu)}{1+\mu^2},$$

where μ is the coefficient of friction. (W)

10.4 Rigid body systems

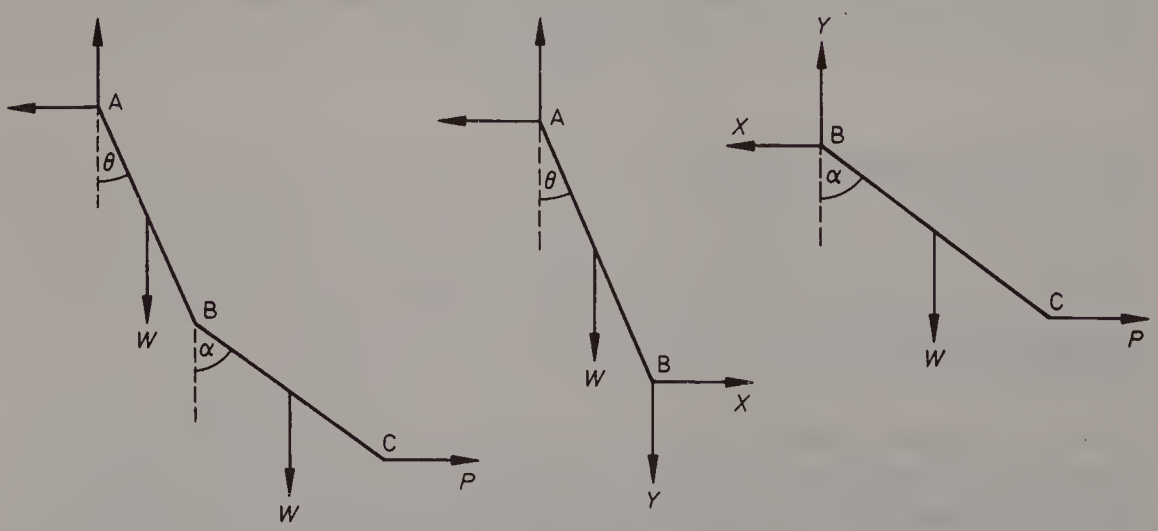
When two or more rigid bodies are in equilibrium under the action of a set of coplanar forces, then the whole system can be treated as a single body in equilibrium under the action of the external forces. Each separate body in the system is in equilibrium under the action of a set of forces which includes both the external forces on the body and the forces exerted on it by the other bodies in the system.



We first consider two rods connected by a smooth light hinge. Each rod exerts a force on the hinge and, according to Newton's third law, the hinge exerts an equal and opposite force on the rod. Thus, provided no external force acts on the hinge, the effect of the hinge is to set up equal and opposite forces on the two rods.

Example 1 Two equal uniform rods AB and BC, each of length $2a$ and weight W, are freely hinged together at B and hang freely from the point A. Equilibrium is maintained by a horizontal force P acting at C. Given that BC makes an angle α with the downward vertical, where $\tan\alpha = 4/3$, find the value of P and the horizontal and vertical components of the force exerted by the rod AB on the rod BC. Find also the angle made by AB with the vertical.

Let the horizontal and vertical components of the reaction at B be X and Y respectively and let θ be the angle AB makes with the vertical.



External forces Forces on AB Forces on BC

Consider first the forces on the rod BC.
Taking moments about B,

$$\circlearrowleft \quad P \times 2a\cos\alpha - W \times a\sin\alpha = 0$$
$$\therefore \quad P = \tfrac{1}{2}W\tan\alpha = \tfrac{2}{3}W.$$

Resolving horizontally and vertically,

$$\leftarrow \quad X - P = 0 \qquad \therefore \quad X = P = \tfrac{2}{3}W$$
$$\uparrow \quad Y - W = 0 \qquad \therefore \quad Y = W$$

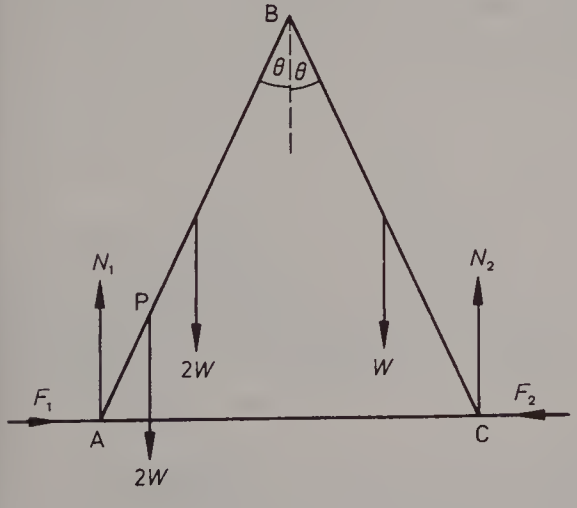
Hence the horizontal and vertical components of the force exerted by the rod AB on the rod BC are $\frac{2}{3}W$ and W respectively.
Consider now the forces on the rod AB.
Taking moments about A,

$$\circlearrowleft \quad W \times a\sin\theta + Y \times 2a\sin\theta - X \times 2a\cos\theta = 0$$
$$\therefore \quad Wa\sin\theta + 2Wa\sin\theta - \tfrac{4}{3}Wa\cos\theta = 0$$

$$\therefore \quad 3Wa\sin\theta = \tfrac{4}{3}Wa\cos\theta$$
$$\therefore \quad \tan\theta = \tfrac{4}{9} \quad \text{and} \quad \theta \approx 24°.$$

Hence AB makes an angle of 24° with the vertical.

Example 2 Two uniform rods AB and BC, each of length $2a$, have weights $2W$ and W respectively. The rods are freely jointed at B and rest in a vertical plane with A and C on a rough horizontal plane, the coefficient of friction at both points of contact being $\frac{1}{2}$. A particle of weight $2W$ is attached to a point P of the rod AB, where $AP = \frac{1}{2}a$. Given that the rods are inclined at an angle θ to the vertical, find the maximum value of θ for which equilibrium is possible.



Let the reactions at A and C be as shown in the diagram.

Consider first the forces on the whole system.
Resolving horizontally and vertically,

$$\rightarrow \quad F_1 - F_2 = 0 \qquad (1)$$
$$\uparrow \quad N_1 + N_2 - 5W = 0 \qquad (2)$$

Taking moments about A,

$$\circlearrowleft \quad N_2 \times 4a\sin\theta - W \times 3a\sin\theta - 2W \times a\sin\theta - 2W \times \tfrac{1}{2}a\sin\theta = 0$$
$$\therefore \quad 4N_2 - 3W - 2W - W = 0 \quad \text{giving} \quad N_2 = \tfrac{3}{2}W.$$
From (2), $\qquad N_1 = 5W - N_2 = \tfrac{7}{2}W.$

Using the laws of friction, $F_1 \leqslant \mu N_1$ and $F_2 \leqslant \mu N_2$
Since $\mu = \frac{1}{2}$, $F_1 \leqslant \frac{7}{2}W$ and $F_2 \leqslant \frac{3}{4}W$
However, using (1), we may write $F_1 = F_2 = F$
$\therefore$ in any equilibrium position, $F \leqslant \frac{3}{4}W$ (3)
Taking moments about B for the forces on rod BC,

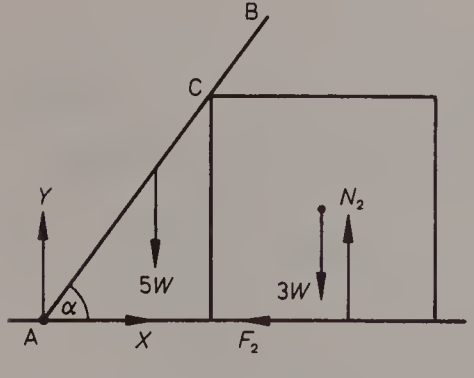
$$\circlearrowleft \quad N_2 \times 2a\sin\theta - W \times a\sin\theta - F_2 \times 2a\cos\theta = 0$$
$$\therefore \quad 3Wa\sin\theta - Wa\sin\theta - 2Fa\cos\theta = 0$$
$$\therefore \quad 2Wa\sin\theta = 2Fa\cos\theta$$
$$\text{Thus} \quad \tan\theta = F/W$$
$$\therefore \text{ using (3)}, \quad \tan\theta \leqslant 3/4$$

Hence the maximum value of θ for which equilibrium is possible is $\tan^{-1}(3/4) \approx 36{\cdot}9°$.

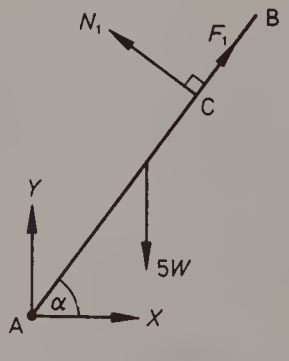
[Note that when θ takes this maximum value, slipping is about to occur at C.]

We now consider problems concerning two bodies in contact. By Newton's third law, the forces exerted by each body on the other are equal and opposite. Thus, as in the case of jointed bodies, unknown quantities can be found by resolving and taking moments, considering either the external forces acting on the system as a whole, or both internal and external forces acting on each body separately. In any given problem we try to simplify the working by avoiding equations involving quantities that we do not need to find.

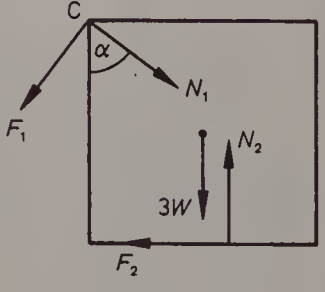
Example 3 A uniform cube of side $12a$ and weight $3W$ rests on a rough horizontal table. A uniform rod AB of length $20a$ and weight $5W$ is freely hinged at A to a fixed point on the table. A point C of the rod rests against the mid-point of one edge of the cube, this edge being perpendicular to the vertical plane containing the rod. When the rod is at an angle α to the horizontal, where $\sin\alpha = 4/5$, the cube is on the point of sliding across the table without toppling. If the coefficient of friction at C is $\frac{1}{2}$, find the coefficient of friction between the cube and the table.



Forces on whole system



Forces on rod



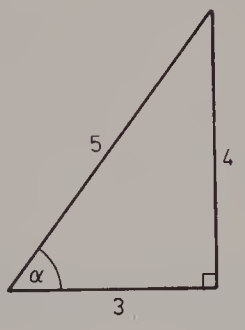
Forces on cube

Let F_1, N_1 and F_2, N_2 be the frictional forces and normal reactions at C and the base of the cube respectively. Consider first the forces on the rod.

Taking moments about A,

$$\circlearrowleft \qquad N_1 \times 12a \operatorname{cosec}\alpha - 5W \times 10a \cos\alpha = 0$$

Using a right-angled triangle,



if $\sin\alpha = \frac{4}{5}$, then $\operatorname{cosec}\alpha = \frac{5}{4}$, $\cos\alpha = \frac{3}{5}$

$\therefore \quad N_1 \times 12a \times \frac{5}{4} - 5W \times 10a \times \frac{3}{5} = 0$

Hence $N_1 = 2W$.
Since the cube is about to slide, $F_1 = \frac{1}{2}N_1 = W$.
Consider now the forces on the cube.
Resolving horizontally and vertically,

$$\leftarrow \qquad F_2 + F_1\cos\alpha - N_1\sin\alpha = 0 \qquad (1)$$

$$\uparrow \qquad N_2 - 3W - F_1\sin\alpha - N_1\cos\alpha = 0 \qquad (2)$$

From (1), $\quad F_2 + \frac{3}{5}W - \frac{8}{5}W = 0 \quad \therefore \quad F_2 = W$

From (2), $\quad N_2 - 3W - \frac{4}{5}W - \frac{6}{5}W = 0 \quad \therefore \quad N_2 = 5W$

Since the cube is on the point of sliding, the coefficient of friction between the cube and the table is given by F_2/N_2 and is equal to 1/5.

Exercise 10.4

1. Two uniform rods AB and BC, of weights 5 N and 10 N respectively, are freely hinged at B and hang freely from a fixed hinge at A. The system is kept in equilibrium by a horizontal force of magnitude 5 N acting at C. Find (a) the magnitudes of the reactions at the hinges A and B, (b) the angles the rods AB and BC make with the vertical.

2. Two equal uniform rods, each of weight W, are freely jointed at B and hang freely from a hinge at A. The end C is attached by means of a light inextensible string to a point D on the same horizontal level as A. When the system is in equilibrium the rod AB is inclined at 45° to the vertical and the rod BC is horizontal. Find the tension in the string and the angle it makes with the vertical.

3. Two uniform rods AB and BC, each of length $2a$, have weights W and $3W$ respectively. The rods are smoothly jointed at B. The ends A and C of the rods are smoothly hinged to fixed points at the same horizontal level and a distance $2a$ apart. If the system is in equilibrium with B below AC, find the horizontal and vertical components of the reaction at B.

4. Two equal uniform rods, AB and AC, each of length $2a$ and weight W, are freely jointed at A. The end C is freely hinged to a point on a rough horizontal plane, the end B rests on the plane and is about to slip, and both rods are in a vertical plane. If the coefficient of friction between B and the plane is μ and the angle ABC is θ, (i) prove that $\mu = \frac{1}{2}\cot\theta$, (ii) find, in terms of μ, the magnitude and direction of the reaction at the hinge at C, (iii) find the magnitude of the reaction at the joint A.
(SU)

5. Two uniform rods AB and BC, freely jointed together at B, are of equal length, but have weights $4W$ and W respectively. The end A is freely hinged to a fixed point on a rough vertical wall and the end C rests against the wall at a point vertically below A. The system rests in limiting equilibrium with each rod inclined at an angle α to the vertical, where $\tan\alpha = 2$. Find (a) the coefficient of friction between the rod BC and the wall, (b) the horizontal and vertical components of the reaction of AB on BC.

6. Two uniform rods AB and AC, of length $2a$, but of weights $2W$ and W respectively, are smoothly jointed at A. The rods rest in a vertical plane with B and C on a rough horizontal plane, the coefficient of friction between the rods and the plane being 0·4. Given that the system is in limiting equilibrium, determine whether slipping is about to occur at B or at C and find angle BAC.

7. Two equal uniform rods, AB and BC, each of mass $4m$, are smoothly hinged together at B and stand in a vertical plane with A and C on a rough horizontal floor. The angle ABC is 2α where $\tan\alpha = \frac{1}{2}$, and the coefficient of friction at both A and C is μ. A mass m is fixed to the mid-point of AB. Show that equilibrium is possible provided that $\mu \geqslant 9/34$. What happens if $\mu = \frac{1}{4}$?

8. Two smooth uniform spheres of weights W and $2W$ are each suspended from a fixed point O by a light inextensible string and rest against each other. The angle between the strings is $90°$ and the string supporting the heavier sphere makes an angle α with the vertical, where $\tan\alpha = \frac{3}{4}$. Find (a) the tensions in the strings, (b) the angle of inclination to the horizontal of the line joining the centres of the spheres.

9. A uniform rod AB of length $4a$ and weight W is freely hinged at A. A second uniform rod CD of length $5a$ and weight $2W$ is freely hinged at C, which lies at a distance $4a\sqrt{3}$ from A in the same horizontal plane. The rods rest in equilibrium in a vertical plane with the rod CD in contact with the end B of the rod AB. If the rods are on the point of slipping when $BC = 4a$, find the coefficient of friction between the rods.

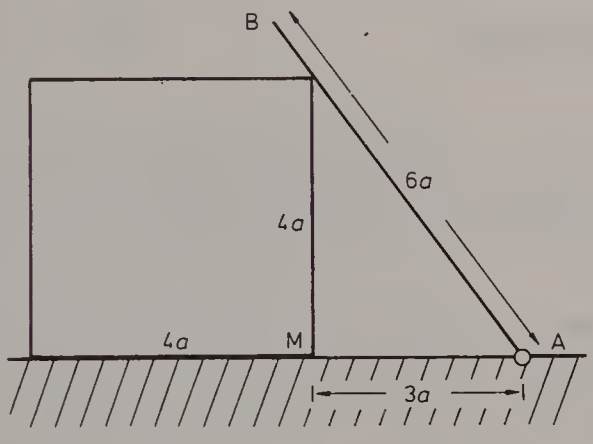
10. Two uniform circular cylinders rest in contact on a horizontal table. A third cylinder is supported by the first two. The cylinders are identical and their axes are horizontal and parallel. The coefficient of friction between any pair of cylinders and between the lower cylinders and the table is μ. Find the least value of μ for which equilibrium is possible.

11. A uniform rod OA of weight W and length $2a$ is pivoted smoothly to a fixed point at the end O. A light inextensible string, also of length $2a$, has one end attached to the end A of the rod, its other end being attached to a small heavy ring of weight $\frac{1}{2}W$ which is threaded on to a fixed rough straight horizontal wire passing through O. The system is in equilibrium with the string taut and the rod making an angle α with the downward vertical at O. By taking moments about O for the rod and ring

together, or otherwise, find the normal component R of the reaction exerted by the wire on the ring. Find also the frictional force F on the ring. The equilibrium is limiting when $\alpha = 60°$. Determine the coefficient of friction between the wire and the ring. (O&C)

12. Two uniform rods AB, AC, each of weight W and length $2a$, are smoothly jointed at A. The rods rest on a smooth circular cylinder of radius a with their plane perpendicular to the axis of the cylinder which is horizontal. When the rods are in equilibrium in a vertical plane with A vertically above the axis of the cylinder, angle $BAC = 2\theta$. Show that the force exerted by the cylinder on each rod is of magnitude $W \operatorname{cosec} \theta$. Show also that $\sin^3 \theta = \cos \theta$. Hence show that there is only one such position of equilibrium. (O&C)

13.

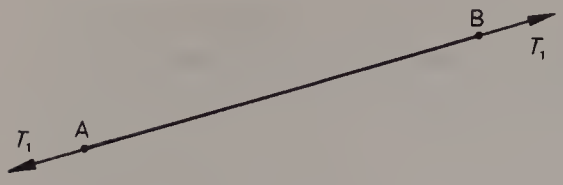


A uniform cube, of edge $4a$ and weight W, rests on a smooth horizontal surface (see diagram). A uniform smooth rod AB, of length $6a$ and weight w, has one end freely hinged to a point A of the surface. The rod rests against the mid-point of a horizontal edge of the top face of the cube. The rod is in a vertical plane through the centre of the cube. A light inelastic string of length $3a$ joins A to the mid-point M of the horizontal edge of the cube nearest to A. The system is in equilibrium. (i) Find the magnitude of the reaction between the rod and the cube. (ii) Find the magnitude of the reaction between the cube and the horizontal surface. (iii) Find the tension in the string AM. (iv) Show that $18w \leqslant 125W$. (C)

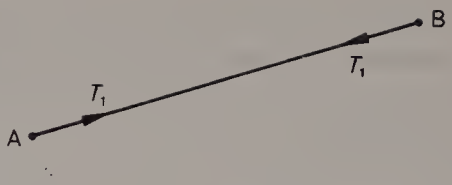
10.5 Light frameworks

The expression *light framework* is used to describe a system of rods whose weights are negligible compared with the other forces involved. It is also assumed that the rods are smoothly jointed together at their ends.

Let us consider the two resultant forces acting at the ends of a typical light rod AB in such a framework. If the rod is in equilibrium these forces must be equal and opposite. By taking moments about A and B we also find that both forces act along AB.



Forces on a rod in tension



Forces exerted by a rod in tension

Forces on a rod in thrust

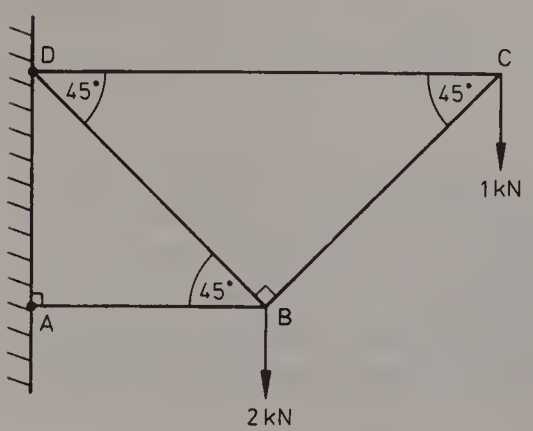
Forces exerted by a rod in thrust

As shown in the diagrams above, a rod may either be in *tension* or in *thrust* i.e. under *compression*. It follows from Newton's third law that at each end of the rod, the force exerted by the rod on the joint is equal and opposite to the force exerted by the joint on the rod.

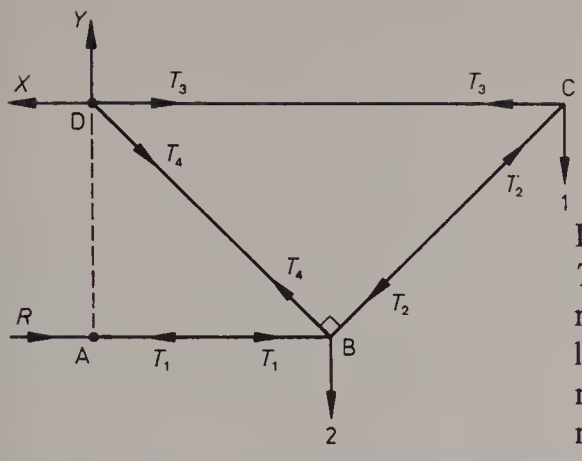
Elementary problems about light frameworks can be solved by considering the equilibrium of the external forces on the framework and the equilibrium of the forces at each joint. In general, unknown forces at joints are found by resolving in suitable directions. However, when dealing with three forces at a joint it is sometimes quicker to use a triangle of forces.

Whatever method is chosen, it is important to draw a large clear diagram showing the forces acting at each joint. Sometimes it is difficult to decide whether a particular rod is in tension or in thrust. In this case we may choose to indicate one type of stress in the diagram, but if calculation leads to a negative value for this stress, we deduce that it is of the other type.

Example 1



The diagram shows a framework of four smoothly jointed light rods. The framework is freely hinged at A and D to a vertical wall. Loads of 2 kN and 1 kN are hung from B and C respectively. Find the magnitudes of the reactions at A and B. Find also the stresses in the rods, stating whether they are tensions or thrusts.



Let the stresses in the rods be thrusts T_1, T_2 and tensions T_3, T_4 as shown. Let the magnitude of the reaction at A be R and let the horizontal and vertical components of the reaction at D be X and Y respectively.

Since there are only two forces acting at A, the reaction R must be a horizontal force such that

$$R - T_1 = 0 \tag{1}$$

Consider now the forces on the whole framework.
Let $AB = AD = a$, then taking moments about D,

↻ $\quad R \times a - 2 \times a - 1 \times 2a = 0 \quad \therefore \quad R = 4$

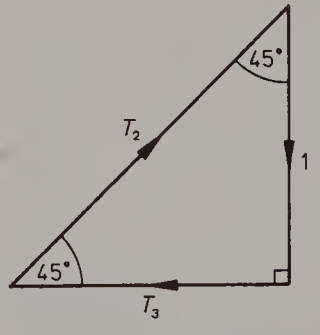
Hence the magnitude of the reaction at A is 4 kN.
Resolving horizontally and vertically,

$\leftarrow \quad X - R = 0 \quad \therefore \quad X = R = 4$
$\uparrow \quad Y - 2 - 1 = 0 \quad \therefore \quad Y = 3$

Hence the magnitude of the reaction at D

$$= \sqrt{(X^2 + Y^2)}\,\text{kN} = \sqrt{(4^2 + 3^2)}\,\text{kN} = 5\,\text{kN}$$

From (1), $T_1 = R = 4$.
Using the triangle of forces for the joint C,



$$T_2 = \sqrt{2}, \quad T_3 = 1.$$

Resolving the forces at B in the direction BD,

↖ $\quad T_4 - 2\cos 45° - T_1 \cos 45° = 0$

$$\therefore \quad T_4 = 2 \times \frac{1}{\sqrt{2}} + 4 \times \frac{1}{\sqrt{2}} = 3\sqrt{2}$$

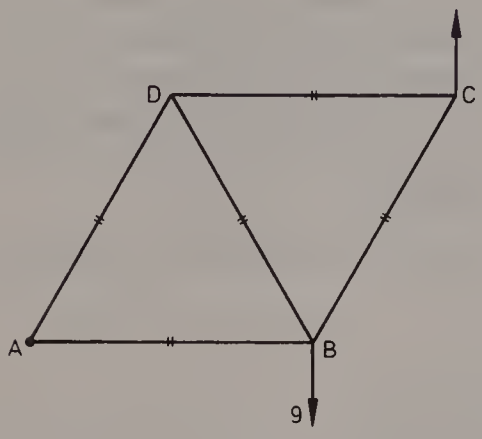
Hence the stresses in AB and BC are thrusts of magnitude 4 kN and $\sqrt{2}$ kN respectively. The stresses in BD and CD are tensions of magnitude $3\sqrt{2}$ kN and 1 kN respectively.

Exercise 10.5

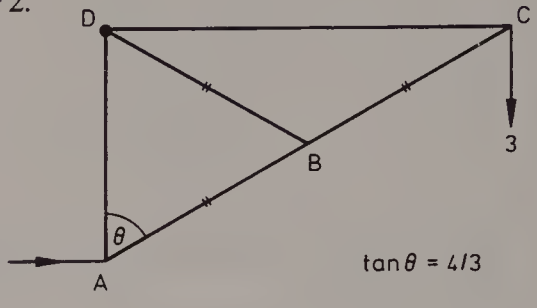
In questions 1 to 8 each diagram shows a light framework consisting of smooth jointed rods. The large dots represent points at which the framework is smoothly

hinged to a fixed point. External forces acting at other points are indicated by arrows and given in kilonewtons. Find the stresses in the rods, stating whether they are tensions or thrusts.

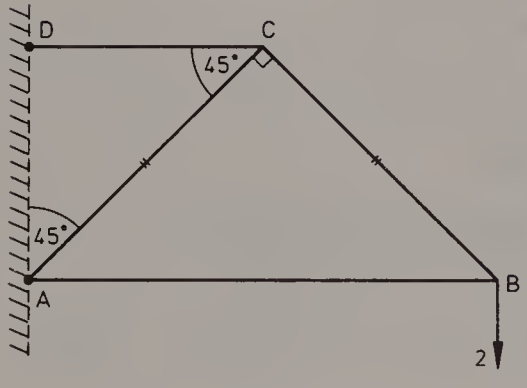
1.



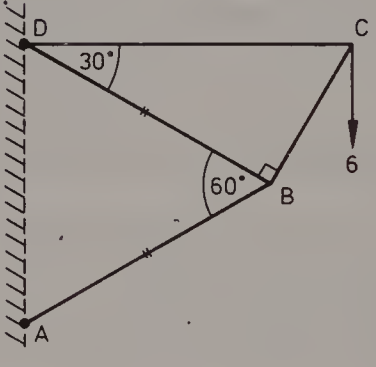
2.



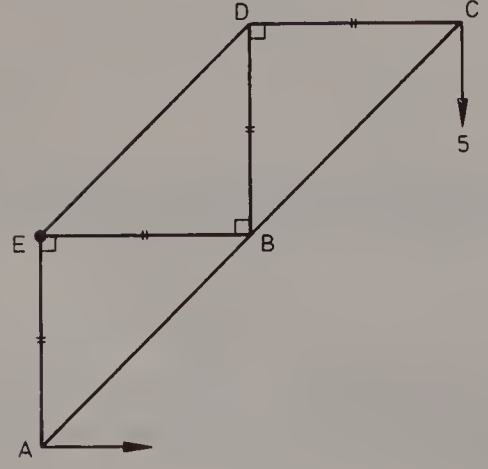
3.



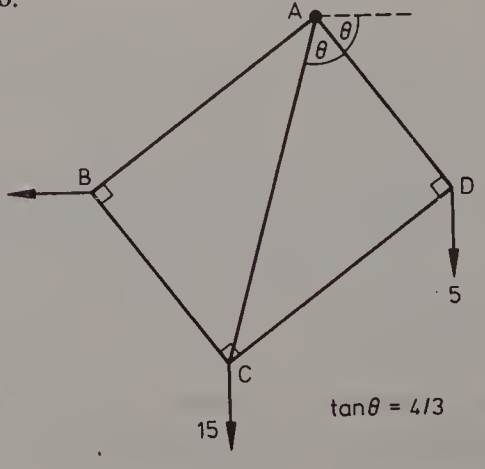
4.



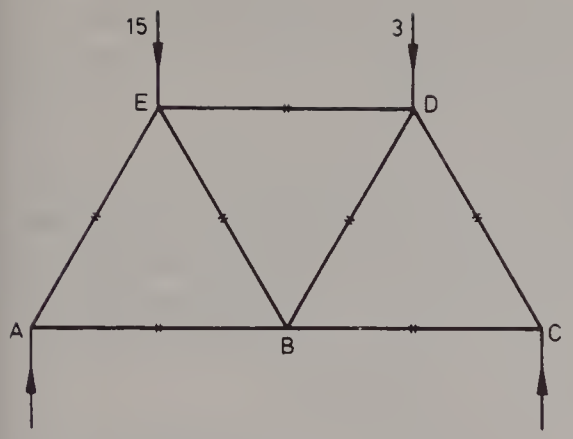
5.



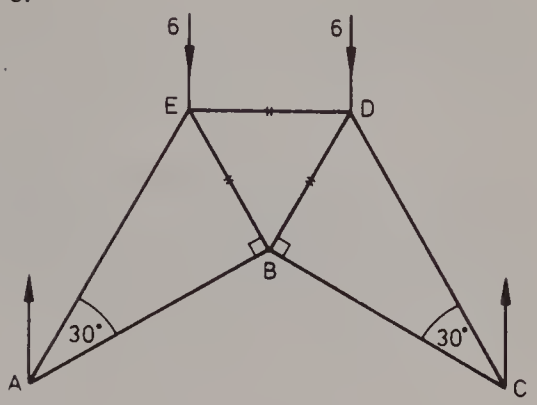
6.



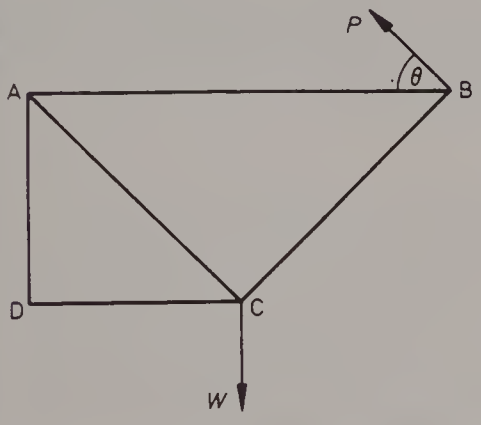
7.



8.



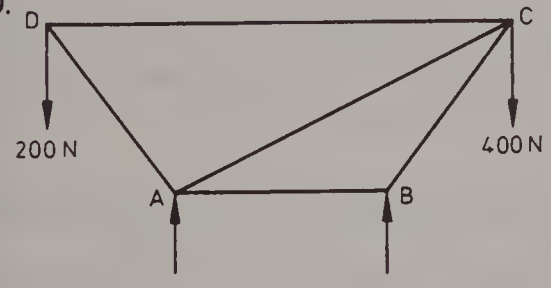
9.



The diagram represents a framework of five smoothly jointed light rods with $AB = \sqrt{2}BC = \sqrt{2}CA = 2CD = 2DA$. The framework is freely hinged to a support at D. A weight W is hung from C and the framework is supported with CD horizontal by a force P at B making an angle θ with the horizontal.

If the magnitude of the stress in CA is twice the magnitude of the stress in BC, find P and the stresses in the rods, specifying whether they are tensions or thrusts. (O)

10.



The diagram shows a smoothly jointed framework consisting of five light rigid rods with $AB = BC = AD = 5$ metres and $DC = 11$ metres. The framework is simply supported at A and B and is in equilibrium in a vertical plane with AB and DC horizontal and with loads of 400 N and 200 N at C and D respectively.
(i) Show that the magnitude of the force supporting the framework at B is 520 N.
(ii) Find the magnitude of the force acting in each member of the framework.
(iii) State which members of the framework are in tension. (AEB 1978)

Exercise 10.6 (miscellaneous)

[Take g as $10\,\mathrm{m\,s^{-2}}$.]

1. A uniform plank $ABCD$ of length $6a$ and weight W rests on two supports B and C; $AB = BC = CD = 2a$. A man of weight $4W$ wishes to stand on the plank at A, for

which purpose he places a counter-balancing weight Y at a point P between C and D at a distance x from D. Show that the least value of Y required is $7Wa/(4a - x)$. Show that, if the board is also to be in equilibrium when he is not standing on it, then the greatest value of Y that may be used is $Wa/(2a - x)$. (JMB)

2. A uniform lamina $ABCD$ is in the form of a trapezium in which $DC = AD = a$, $AB = 2a$ and $\angle BAD = \angle CDA = 90°$. When the lamina is freely suspended from a point E in BC the edge AB is vertical. Show that $BE:EC = 4:5$. Find the tangent of the angle which AB makes with the vertical when the lamina is freely suspended from B. The lamina is suspended by two vertical strings attached at C and D. Find the ratio of the tensions in the strings if CD is horizontal. (L)

3. Three equal uniform rods AB, BC, CA, each of weight w, are joined together to form a triangle ABC. A particle of weight W is attached to B and the system is suspended freely from the point A. In equilibrium the rod AB makes an angle θ with the vertical. Prove that $\tan\theta = \sqrt{3}w/(2W + 3w)$. (O)

4. A uniform thin sheet of cardboard of weight W is in the form of a square $ABCD$ of side $6a$. A cut is made along DO, where O is the point of intersection of the diagonals, and the triangular portion AOD is folded over along AO and stuck to the portion AOB with which it now coincides. Find the distances of the centre of gravity of the resulting object from AB and from BC and show that the centre of gravity lies on BD. If this object hangs freely from A, find the angle which BO makes with the horizontal. A particle of weight w is now attached to the object at D so that, in equilibrium, BO is horizontal. Find w. (L)

5. A uniform right circular solid cone of radius r and height h rests with its base on a rough plane which is initially horizontal. The inclination of the plane to the horizontal is gradually increased. Show that the cone will slide before it topples if the coefficient of friction between the plane and the cone is less than $4r/h$. (O)

6. If three non-parallel coplanar forces are in equilibrium, show that they must be concurrent.

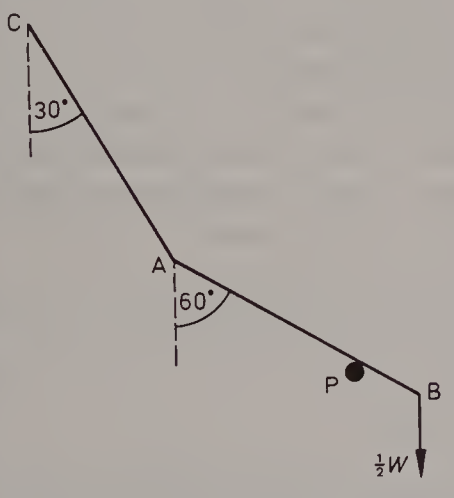
A uniform rod AB of length $2a$ and weight W has one end A in contact with a vertical wall and to the other end B is attached one end of a light string. The other end of the string is attached to the wall at C, the point at a height $2a$ vertically above A. The rod makes an angle α $(0° < \alpha < 90°)$ with the downward vertical. Show that equilibrium is impossible if the wall is smooth. If $\alpha = 60°$ and the rod is in equilibrium, show that the least possible value of the coefficient of friction between the rod and the wall is $1/\sqrt{3}$. (L)

7. A uniform rectangular lamina $ABCD$ of mass 1 kg has $AB = 0{\cdot}8$ m and $AD = 0{\cdot}6$ m. The lamina is smoothly pivoted at A. It is kept in equilibrium in a vertical plane with the diagonal AC horizontal and with B as the lowest point by a force P acting at right angles to CD. Find the magnitude of P and the magnitude and direction of the force exerted on the lamina by the pivot if (a) the force P acts through

the mid-point of CD, (b) the force P acts at C. Find also the magnitude and the line of action of P when the force exerted by the pivot acts in the direction BA. (L)

8. A uniform rod AB, of length $2l$ and weight W, is smoothly hinged to a vertical wall at A. It is held at rest in a horizontal position by a light inextensible string attached at B and to a point C which is vertically above and at a distance l from A. Calculate the tension in the string and show that the reaction at A is equal in magnitude to the tension. When a load of weight $4W$ is attached to the rod at a point D the horizontal and vertical components of the reaction at A are equal in magnitude. Find AD. (L)

9. A thin uniform rod AB of length 3 m and mass 5 kg is freely pivoted to a fixed point A. A light elastic string BC of modulus 30 N has one end attached to B and its other end C fixed to a point at the same level as A, where $AC = 5$ m. When the system is in equilibrium $BC = 4$ m. Calculate (a) the tension in the string. (b) the natural length of the string. (c) the resultant force, in magnitude and direction, exerted on the rod at A. (L)

10. 

The diagram shows a uniform rod AB, of length $4c$ and weight W, resting on a rough fixed peg P, where $AP = 3c$. A particle of weight $\frac{1}{2}W$ hangs from B. One end of a light string is tied at A and the other end is tied at a fixed point C. The rod and the string are in the same vertical plane and are inclined at angles of 60° and 30° respectively to the vertical. (i) Show that the tension in the string is $\frac{1}{6}W\sqrt{3}$. (ii) Given that equilibrium is limiting, calculate the coefficient of friction between the peg and the rod. (iii) Given that $AC = 4c$ and that the string is elastic with modulus of elasticity $W\sqrt{3}$, calculate the natural length of the string. (AEB 1979)

11. A uniform square lamina $ABCD$ of weight W and side $2a$ rests in a vertical plane with the vertex A in contact with a rough horizontal plane. The lamina is kept in equilibrium by a force P acting at the vertex C in the direction of DC produced. Given that the height of the vertex D above the horizontal plane is $6a/5$, show that $P = W/10$. Find the coefficient of friction between the lamina and the plane if the lamina is in limiting equilibrium. (L)

12. A uniform rod AB, of weight W and length $2l$, rests in equilibrium with the end A on rough horizontal ground and with the end B in contact with a smooth vertical wall, which is perpendicular to the vertical plane containing the rod. If AB makes an angle α with the horizontal, where $\tan\alpha = 4/3$, find the least possible value of μ, the coefficient of friction between the rod and the ground, for equilibrium to be

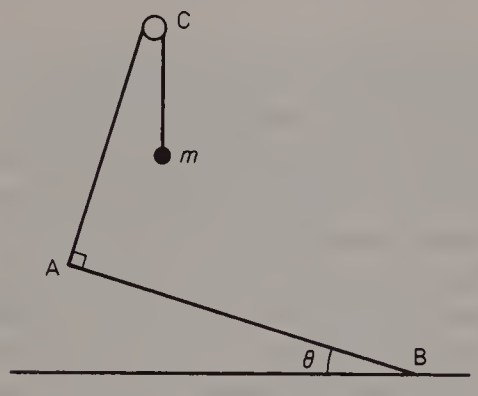
preserved. If $\mu = \frac{1}{2}$, find the distance from A to the highest point of the rod at which a load of weight W can be attached without equilibrium being disturbed. (L)

13. Prove that the position of the centre of gravity of a uniform solid hemisphere of radius a is distant $\frac{3}{8}a$ from the centre of its plane face.

A uniform solid hemisphere rests with its curved surface on a rough horizontal surface and also against a rough vertical surface. The coefficient of friction at both points of contact is $\frac{1}{4}$. If the hemisphere is just about to slip, show that its top surface is inclined at an angle of just under 52° to the horizontal. (SU)

14. A uniform bar, of length $3a$ and mass m, rests in limiting equilibrium at an angle α to the horizontal with one end A on rough horizontal ground and the other end B against a smooth vertical wall. A horizontal force P is now applied to the bar at a point C, where $AC = a$. Find in terms of m, α and g, the value of P which will just cause the bar to slip towards the wall and the corresponding reaction between the bar and the wall.

15.



AB is a uniform rod of mass M. The end B rests on a rough horizontal table (coefficient of friction μ), and a light inextensible string is attached to the end A, passes over a smooth pulley at C and carries a mass m at its other end. The system rests in equilibrium with the string at right angles to AB. Find an expression for θ in terms of M and m and show that equilibrium is not possible unless $2m < M$. Show also that $\mu \geqslant \dfrac{m\sqrt{(M^2 - 4m^2)}}{M^2 - 2m^2}$. (SU)

16. A hemisphere of radius a, whose curved surface is smooth, is fixed with its flat base on a rough horizontal table. A uniform rod AB of length $4a$ and weight W rests in equilibrium against the hemisphere, its point of contact with the hemisphere lying between A and B. The end A of the rod is in contact with the table and AB makes an acute angle θ with the horizontal. Prove that the frictional force F at A has magnitude $2W\sin^2\theta$ and find the normal reaction R at A. If the coefficient of friction between the end A and the table is $\frac{1}{2}$, prove that $3\tan^2\theta + 2\tan\theta - 1 \leqslant 0$ and deduce the maximum value of θ consistent with equilibrium. (O&C)

17. $ABCD$ is a rectangular lamina, freely pivoted at a point X on AB. The length of AB is $2a$ and of BC is a. Forces $3W$, $5W$, $2W$ and $7W$ act along the sides AB, BC, DC and AD respectively in the directions indicated by the order of the letters. Find the position of X so that the lamina is in equilibrium and, in this case, determine the reaction of the pivot on the lamina in magnitude and direction. If, instead, the lamina is pivoted at a point Y on CD and is in equilibrium, deduce from your results the position of Y. (O&C)

18. Explain, with the aid of a diagram, the meaning of the expression coefficient of friction.

A straight uniform rod AB of weight W rests in limiting equilibrium with the end A on horizontal ground and the end B against a vertical wall. The vertical plane containing AB is perpendicular to the wall. The coefficient of friction between the rod and the ground is 4/5. The coefficient of friction between the rod and the wall is 3/5. Given that the inclination of AB to the horizontal is α, calculate (i) the normal reactions at A and B in terms of W, (ii) the numerical value of $\tan\alpha$. (AEB 1980)

19. A uniform right circular heavy cylinder, of height h and radius a, rests with one plane face upon a rough horizontal plane. A gradually increasing horizontal force is applied to a point on the edge of the upper face, and the force passes through the axis of the cylinder. Show that the cylinder will slide before it topples if $\mu h < a$, where μ is the coefficient of friction between the cylinder and the horizontal plane. (O)

20. A uniform solid hemisphere of radius a has mass M. A concentric hemisphere of mass $M/8$ is removed. Show that the centre of mass of the remainder is at a distance $45a/112$ from the centre O of the plane base. A particle of mass M is attached to a point X on the outer edge of the plane base of the remainder. Find, relative to OX and the axis of symmetry of the hemisphere as axes, the coordinates of the centre of mass G of the combined solid (i.e. remainder together with attached mass). The combined solid is placed with its curved surface on a horizontal plane and rests in equilibrium. Find the tangent of the angle between the plane base of the solid and the horizontal. (C)

21. Prove by integration that the centre of mass of a uniform solid right circular cone of height h and base radius r is at a distance $\frac{3}{4}h$ from the vertex. Such a cone is joined to a uniform solid right circular cylinder, of the same material, with base radius r and height l, so that the plane base of the cone coincides with a plane face of the cylinder. Find the centre of mass of the solid thus formed. Show that, if $6l^2 \geqslant h^2$, this solid can rest in equilibrium on a horizontal plane, with the curved surface of the cylinder touching the plane. Given that $l = h$, show that the solid can rest in equilibrium with its conical surface touching the plane provided that $r \geqslant \frac{1}{4}h\sqrt{5}$. (JMB)

22. A toy consists of a solid hemisphere of radius a to which is glued a solid circular cylinder of radius a and height $2a$ so that the plane end of the hemisphere is in complete contact with a plane end of the cylinder. The cylinder is made of uniform material of density ρ, and the hemisphere is made of uniform material of density $k\rho$. The toy is designed so that if placed on a horizontal table with the hemisphere downwards and then tilted to one side, it will return to the vertical position. Show that $k > 8$. The toy is placed on a desk of slope α where $\sin\alpha = 1/8$, sufficiently rough to prevent slipping. It rests in equilibrium with the hemisphere in contact with the desk. Find an expression giving the (acute) angle β made by its axis of symmetry with the vertical. Hence deduce that $k \geqslant 13\frac{1}{2}$. (O&C)

23. Using a diagram, explain the meaning of the term angle of friction. A uniform straight pole AB of length 2·5 metres stands in limiting equilibrium with the end A on horizontal ground and the end B against a vertical wall. The vertical plane containing AB is perpendicular to the wall and A is at a distance 0·7 metres from the wall. The angles of friction between the pole and the ground and between the pole and the wall are equal. Show that the line of action of the resultant force exerted on the pole by the ground at A is perpendicular to the line of action of the resultant force exerted on the pole by the wall at B. Hence, or otherwise, show that the coefficient of friction at A and B is $1/7$. Given that the pole weighs 50 N, find the magnitude of the resultant reactions (i) between the pole and the ground, (ii) between the pole and the wall. (AEB 1978)

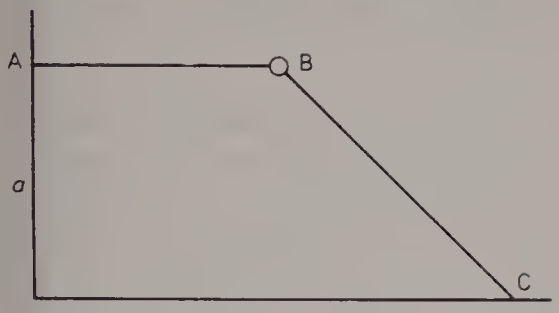
24. A uniform rod of weight W rests in contact with two parallel, horizontal, smooth pegs, passing over the higher peg and under the lower one. The lower end of the rod rests on a smooth horizontal plane. A force of magnitude P acts on the upper end of the rod in the direction of the rod and tending to move the rod upwards. The length of the rod is l, and its inclination to the horizontal is θ. If the upper and lower pegs are at distances a and b respectively from the lower end of the rod, prove that, so long as the rod remains in equilibrium, the reaction of the upper peg on the rod is $\dfrac{Wl\cos\theta - 2Pb\cot\theta}{2(a-b)}$. Given that $l > 2a$, determine which of the three forces exerted on the rod by the two pegs and the ground will first become zero as P is gradually increased from zero. (C)

25. A uniform rod AB, of length l and weight W, is smoothly hinged at A. The point C is vertically above A and is such that $AC = l$. An elastic string, of unstretched length $\frac{1}{2}l$, obeys Hooke's law and has modulus W. The ends of the string are attached to B and C so that the rod rests in equilibrium in an inclined position. Show that the tension in the string is $W/3$. Find the magnitude of the couple which when applied to the rod will maintain the system in equilibrium with $AB = BC$. (L)

26. A uniform rod AB of weight W and length $2a$ has the end A freely hinged to a fixed point. The other end B is freely hinged to a uniform rod BC of weight W and length $6a$. The ends A and C are joined by a light inelastic string whose length is such that AB is perpendicular to BC, and the system hangs in equilibrium. Show that AB is inclined at 45° to the vertical. Find the tension in AC and the horizontal and vertical components of the reaction on BC at B. (C)

27. Two uniform rods AB and BC of the same thickness and material, and of length 4 metres and 3 metres respectively, are freely hinged together and rest in a vertical plane with the ends A and C on a rough horizontal plane. The system is in limiting equilibrium when the angle ABC is 90°. Determine how equilibrium will be broken when the angle ABC is slightly increased beyond 90° and show that the coefficient of friction between the rods and the ground is $84/163$. (W)

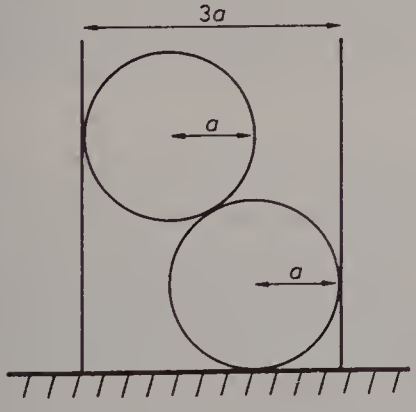
28.



A uniform rod AB of length a and weight W is smoothly jointed at B to a uniform rod BC of length $a\sqrt{2}$ and also of weight W. The system is in equilibrium with C resting on a rough horizontal floor at a distance $2a$ from a rough vertical wall, A resting against the wall, and AB horizontal at a height a above the floor. The plane of the rods is perpendicular to the floor and the wall. (i) Find the frictional force exerted by the wall at A. (ii) Find the vertical and horizontal components of the force exerted on BC at C. (iii) Show that equilibrium is possible only if the coefficient of friction at C is at least $\frac{2}{3}$. (iv) Find the least value of the coefficient of friction at A for equilibrium to be possible. (JMB)

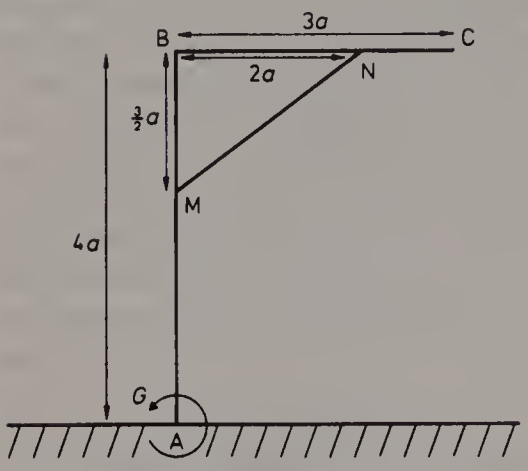
29. Three uniform straight rods, each of weight W, are smoothly jointed together to form an equilateral triangle ABC. The joint B is hinged to a fixed support and the system is free to rotate about B in a vertical plane. The system is held in equilibrium with BC horizontal and A uppermost by a horizontal light string which connects A with a fixed point. Show that the tension in the string is $W\sqrt{3}$ and find the magnitude and direction of the force exerted on the triangle by the support at B. Find the horizontal and vertical components of all the forces which are exerted on the rod AC, and indicate them clearly on your diagram. (JMB)

30. The diagram shows two parallel vertical walls at a distance $3a$ apart, with two uniform smooth cylinders, each of radius a and weight W. The axes of the cylinders are horizontal and parallel to the walls. Each cylinder is in contact with the other and with one wall; the lower cylinder rests on horizontal ground. Find the magnitudes of the reactions between the two cylinders and between each cylinder and the corresponding wall. (C)



31. A uniform sphere of radius a and weight $W/\sqrt{3}$ rests on a rough horizontal table. A uniform rod AB of weight $2W$ and length $2a$ is freely hinged at A to a fixed point on the table and leans against the sphere so that the centre of the sphere and the rod lie in a vertical plane. The rod makes an angle of $60°$ with the horizontal. Show that the frictional force between the rod and the sphere is $\frac{1}{3}W$. The coefficient of friction at each point of contact is μ. What is the smallest value of μ which makes equilibrium possible? (O&C)

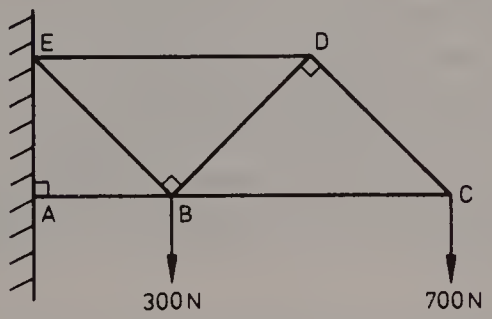
32.



The diagram shows a uniform post AB, of length $4a$ and weight W, smoothly hinged to the ground at A. A horizontal uniform beam BC, of length $3a$ and weight W, is smoothly hinged to AB at B. The beam is supported by a light inelastic rod MN smoothly jointed to AB at M and to BC at N, where $BM = \frac{3}{2}a$ and $BN = 2a$. A weight $3W$ hangs from C, and the system is maintained in equilibrium, with AB vertical, by an external couple of magnitude G applied to AB, as shown.

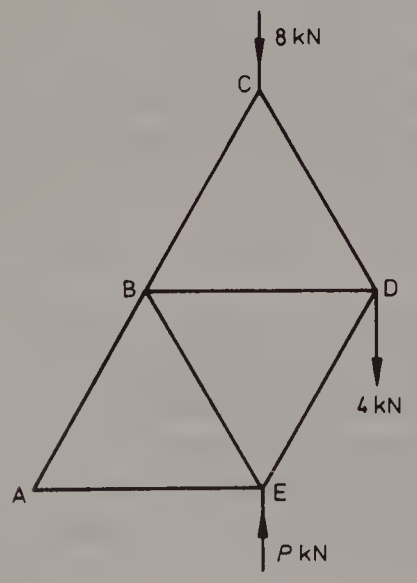
Find (i) the thrust in the rod MN, (ii) the horizontal and vertical components of the forces on the post AB at A and at B, (iii) the value of G. (C)

33.



The diagram shows a framework $ABCDE$, consisting of 6 light rigid rods which are smoothly jointed. The framework is smoothly hinged to a vertical wall at A and E and the triangles ABE, BED and BCD are all right-angled and isosceles. When loads of 300 N and 700 N are hung from B and C respectively, the framework is in equilibrium in a vertical plane. (i) Explain why the reaction on the hinge at A is horizontal and find its magnitude. (ii) Find the magnitude and the direction of the reaction on the hinge at E. (iii) Find the magnitude of the force acting in each member of the framework and state which rods are in compression. (AEB 1978)

34.

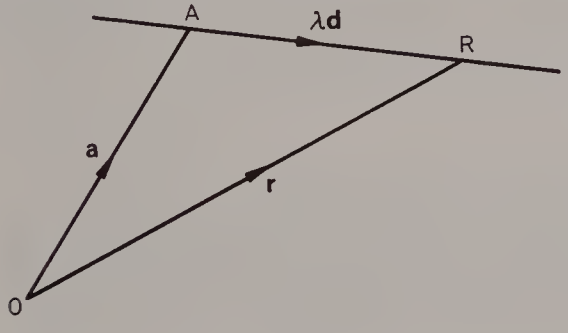


The diagram shows a framework of 7 light equal rods, smoothly jointed at their ends. It is freely hinged to a vertical wall at A. Forces of 8 kN and 4 kN are applied at C and D as shown. A force of size P kN is applied vertically upwards at E in order to keep the rod AE in a horizontal position. Calculate (i) the magnitude of P; (ii) the stresses in the rods, stating if they are in tension or under compression; (iii) the magnitude and direction of the reaction of the support at A on the frame. (SU)

11 Vector methods

11.1 Forces and their lines of action

In this chapter we consider the application of vector methods to systems of forces given in vector form, often in terms of unit vectors **i**, **j** and **k**.



The diagram shows a straight line which passes through a fixed point A with position vector **a** and which is parallel to a given vector **d**. Let **r** be the position vector of a point R on the line.

Since **AR** is parallel to **d**, then, for some scalar λ,

$$\mathbf{AR} = \lambda\mathbf{d}$$

$$\therefore \quad \mathbf{r} - \mathbf{a} = \lambda\mathbf{d}$$

$$\text{i.e.} \quad \mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$$

Thus every point on the line has a position vector of the form $\mathbf{a} + \lambda\mathbf{d}$. Moreover, each real value of the parameter λ corresponds to some point on the line.

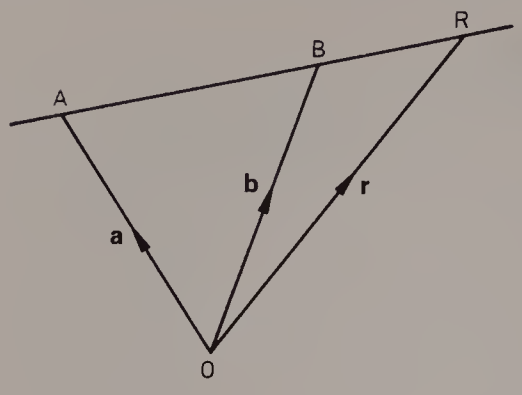
The equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$ is called the *vector equation* of the line through A parallel to **d**. The vector **d** is a *direction vector* of the line.

Example 1 Find the vector equation of the line which passes through the point with position vector $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and which is parallel to the vector $2\mathbf{j} - \mathbf{k}$.

The vector equation of the given line is

$$\mathbf{r} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k} + \lambda(2\mathbf{j} - \mathbf{k})$$

$$\text{i.e.} \quad \mathbf{r} = 2\mathbf{i} + (2\lambda - 1)\mathbf{j} + (3 - \lambda)\mathbf{k}.$$



Consider now the line through two points A and B with position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. If a point R, with position vector $\mathbf{r}$, lies on this line, then for some scalar λ,

$$\mathbf{AR} = \lambda\mathbf{AB}$$
$$\therefore \quad \mathbf{r} - \mathbf{a} = \lambda(\mathbf{b} - \mathbf{a})$$
$$\therefore \quad \mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}).$$

Hence the vector equation of the line through A and B is $\mathbf{r} = (1 - \lambda)\mathbf{a} + \lambda\mathbf{b}$.

Example 2 Find the vector equation of the straight line which passes through the points $A(1, 0, -2)$ and $B(2, 3, -1)$.

The position vectors of A and B are $\mathbf{i} - 2\mathbf{k}$ and $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ respectively.
Hence the vector equation of the line through A and B is

$$\mathbf{r} = (1 - \lambda)(\mathbf{i} - 2\mathbf{k}) + \lambda(2\mathbf{i} + 3\mathbf{j} - \mathbf{k})$$
i.e. $$\mathbf{r} = (\lambda + 1)\mathbf{i} + 3\lambda\mathbf{j} + (\lambda - 2)\mathbf{k}.$$

Example 3 A force $\mathbf{P}$ of magnitude 28 N acts at the point A with position vector $2\mathbf{j} + \mathbf{k}$ in the direction of the vector $3\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$. A second force $\mathbf{Q}$ of magnitude 27 N also acts at A, but in the direction of the vector $\mathbf{i} - 4\mathbf{j} - 8\mathbf{k}$. Find the magnitude of the resultant of the forces $\mathbf{P}$ and $\mathbf{Q}$. Obtain a vector equation for its line of action.

$$|3\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}| = \sqrt{\{3^2 + (-2)^2 + 6^2\}} = \sqrt{\{9 + 4 + 36\}} = 7$$
$$|\mathbf{i} - 4\mathbf{j} - 8\mathbf{k}| = \sqrt{\{1^2 + (-4)^2 + (-8)^2\}} = \sqrt{\{1 + 16 + 64\}} = 9$$

$\therefore$ the unit vectors in the directions of $\mathbf{P}$ and $\mathbf{Q}$ are $\frac{1}{7}(3\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})$ and $\frac{1}{9}(\mathbf{i} - 4\mathbf{j} - 8\mathbf{k})$ respectively.

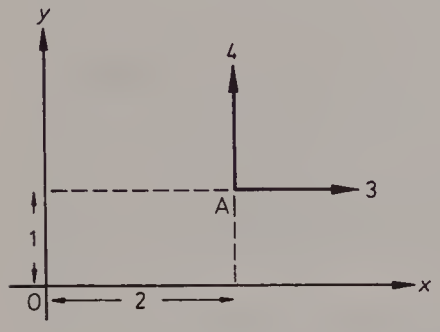
Thus $$\mathbf{P} = 28 \times \tfrac{1}{7}(3\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}) = 12\mathbf{i} - 8\mathbf{j} + 24\mathbf{k}$$
and $$\mathbf{Q} = 27 \times \tfrac{1}{9}(\mathbf{i} - 4\mathbf{j} - 8\mathbf{k}) = 3\mathbf{i} - 12\mathbf{j} - 24\mathbf{k}$$
$$\therefore \quad \mathbf{P} + \mathbf{Q} = 15\mathbf{i} - 20\mathbf{j} = 5(3\mathbf{i} - 4\mathbf{j})$$
$$\therefore \quad |\mathbf{P} + \mathbf{Q}| = 5\sqrt{\{3^2 + (-4)^2\}} = 5\sqrt{\{9 + 16\}} = 25$$

Hence the resultant of $\mathbf{P}$ and $\mathbf{Q}$ acts in the direction of the vector $3\mathbf{i} - 4\mathbf{j}$ and has magnitude 25 N.
Since $\mathbf{P}$ and $\mathbf{Q}$ act at A, their resultant must also act at A. Thus a vector equation for its line of action is

$$\mathbf{r} = 2\mathbf{j} + \mathbf{k} + \lambda(3\mathbf{i} - 4\mathbf{j})$$
i.e. $$\mathbf{r} = 3\lambda\mathbf{i} + (2 - 4\lambda)\mathbf{j} + \mathbf{k}.$$

The approach of Example 3 can also be used to find the magnitude and direction of the resultant of a set of coplanar forces with different points of application. However, to find the line of action of the resultant we must use the principle of moments introduction in §9.1.

[Note that unless otherwise stated we will assume that vector quantities are measured in the appropriate SI units.]



To illustrate how the moments of individual forces are calculated, we suppose that a force $\mathbf{F} = 3\mathbf{i} + 4\mathbf{j}$ acts at the point A with position vector $2\mathbf{i} + \mathbf{j}$.

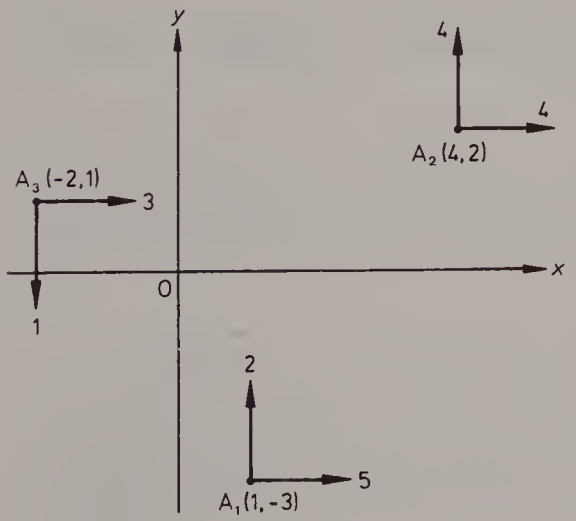
Taking anti-clockwise moments as positive, the moment of $\mathbf{F}$ about O

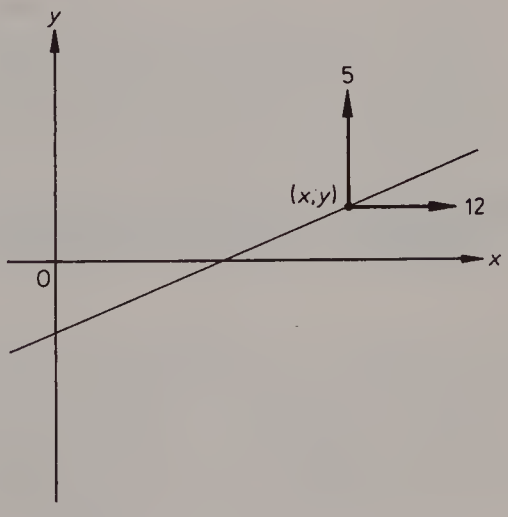
$$= (\text{moment about } O \text{ of the force } 3\mathbf{i}) + (\text{moment about } O \text{ of the force } 4\mathbf{j})$$
$$= -(3 \times 1) + (4 \times 2)\,\text{N m} = 5\,\text{N m}.$$

Example 4 Coplanar forces $\mathbf{F}_1 = 5\mathbf{i} + 2\mathbf{j}$, $\mathbf{F}_2 = 4\mathbf{i} + 4\mathbf{j}$ and $\mathbf{F}_3 = 3\mathbf{i} - \mathbf{j}$ act at points A_1, A_2 and A_3 with position vectors $\mathbf{i} - 3\mathbf{j}, 4\mathbf{i} + 2\mathbf{j}$ and $-2\mathbf{i} + \mathbf{j}$ respectively. Find the magnitude of the resultant of these forces and the Cartesian equation of its line of action.

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = (5 + 4 + 3)\mathbf{i} + (2 + 4 - 1)\mathbf{j} = 12\mathbf{i} + 5\mathbf{j}$$
$$\therefore\ |\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3| = \sqrt{(12^2 + 5^2)} = \sqrt{(144 + 25)} = 13.$$

Hence the magnitude of the resultant of the three forces is 13 N.





Let the point with position vector $x\mathbf{i} + y\mathbf{j}$ be some point on the line of action of the resultant, then the anti-clockwise moment of the resultant about O is $5x - 12y$.

Taking moments anti-clockwise about O for the original forces,

$$\text{moment of } \mathbf{F}_1 = 5 \times 3 + 2 \times 1 = 17$$
$$\text{moment of } \mathbf{F}_2 = -4 \times 2 + 4 \times 4 = 8$$
$$\text{moment of } \mathbf{F}_3 = -3 \times 1 + 1 \times 2 = -1$$

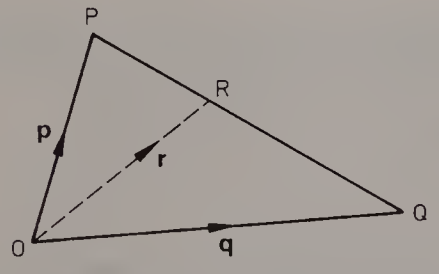
$\therefore$ using the principle of moments,

$$5x - 12y = 17 + 8 - 1$$

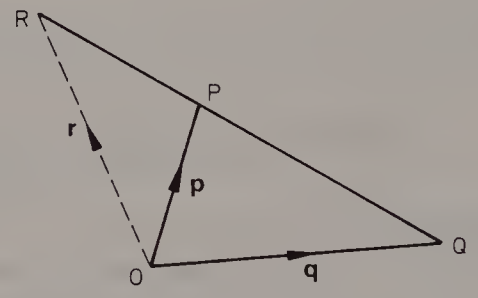
Hence the Cartesian equation of the line of action of the resultant is $5x - 12y = 24$.

Let us consider three points P, Q and R such that R divides PQ in the ratio $\mu:\lambda$. We will observe the convention that (i) if λ and μ have the same sign then R lies between P and Q, and (ii) if λ and μ have opposite signs then R does not lie between P and Q.

(i)



(ii)



In both cases $\quad \lambda\mathbf{PR} = \mu\mathbf{RQ}$

$\therefore \quad \lambda(\mathbf{OR} - \mathbf{OP}) = \mu(\mathbf{OQ} - \mathbf{OR})$

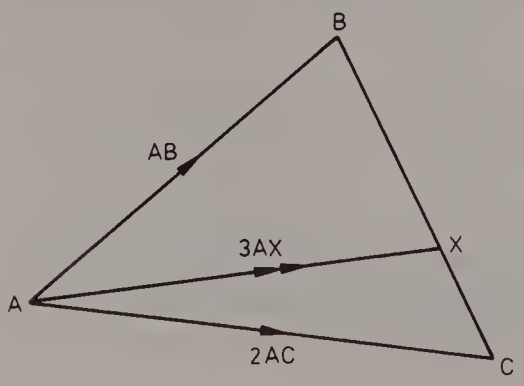
$\therefore \quad \lambda\mathbf{OR} - \lambda\mathbf{OP} = \mu\mathbf{OQ} - \mu\mathbf{OR}$

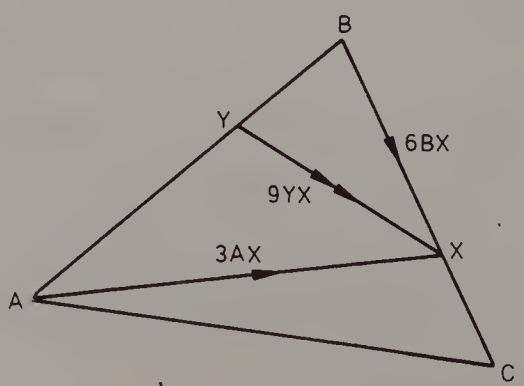
Hence $\quad (\lambda + \mu)\mathbf{OR} = \lambda\mathbf{OP} + \mu\mathbf{OQ}$

This result is called the *ratio theorem* and may also be stated as follows:

> If two forces acting at a point O are represented in magnitude and direction by $\lambda\mathbf{OP}$ and $\mu\mathbf{OQ}$, then their resultant is represented by $(\lambda + \mu)\mathbf{OR}$, where R is the point which divides PQ in the ratio $\mu:\lambda$.

Example 5 Forces represented by **AB**, 2**AC** and 4**BC** act along the sides of a triangle ABC. Find their resultant.





By the ratio theorem, $\mathbf{AB} + 2\mathbf{AC} = 3\mathbf{AX}$,
where X is the point which divides BC in the ratio 2:1.

Since $BX:XC = 2:1$, we have $BC = \frac{3}{2}BX$,

$\therefore \quad \mathbf{AB} + 2\mathbf{AC} + 4\mathbf{BC} = 3\mathbf{AX} + 6\mathbf{BX}$.

By the ratio theorem, $3\mathbf{AX} + 6\mathbf{BX} = 9\mathbf{YX}$,

where Y is the point which divides AB in the ratio 6:3 i.e. 2:1. Hence the resultant of the three forces acts through X and is represented by $9\mathbf{YX}$, where X divides BC in the ratio 2:1 and Y divides AB in the ratio 2:1.

Exercise 11.1

1. Forces of magnitude 10 N and 5 N act in the directions of the vectors $3\mathbf{i} - 4\mathbf{j}$ and $4\mathbf{i} + 3\mathbf{j}$ respectively. Express each force as a vector in terms of $\mathbf{i}$ and $\mathbf{j}$. Hence find the resultant of the forces as a vector.

2. A force $\mathbf{R}$ of magnitude $4\sqrt{2}$ N acting in the direction of the vector $\mathbf{i} - \mathbf{j}$ is the resultant of two forces $\mathbf{P}$ and $\mathbf{Q}$ which act in the directions of the vectors $\mathbf{i} + \mathbf{j}$ and $\mathbf{i} - 7\mathbf{j}$ respectively. Find the magnitudes of $\mathbf{P}$ and $\mathbf{Q}$.

3. In a parallelogram $OABC$ the position vectors of A and B relative to O are $\mathbf{a} = 4\mathbf{i} + 3\mathbf{j}$ and $\mathbf{b} = 3\mathbf{j} + 3\mathbf{k}$ respectively. Forces of magnitude 20 N, $9\sqrt{2}$ N and 5 N act along OA, OB and OC respectively. Find the magnitude of their resultant.

4. In each of the following cases obtain a vector equation for the line of action of the resultant of forces $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$ acting at a point with position vector $\mathbf{a}$.
(a) $\mathbf{F}_1 = 2\mathbf{i} - 3\mathbf{j}$, $\mathbf{F}_2 = \mathbf{i} + 5\mathbf{j}$, $\mathbf{F}_3 = -\mathbf{i} + 2\mathbf{j}$; $\mathbf{a} = \mathbf{i} + \mathbf{j}$.
(b) $\mathbf{F}_1 = -3\mathbf{i} + 2\mathbf{k}$, $\mathbf{F}_2 = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{F}_3 = 4\mathbf{j}$; $\mathbf{a} = 2\mathbf{j} - \mathbf{k}$.
(c) $\mathbf{F}_1 = \mathbf{i} - 5\mathbf{j} + 8\mathbf{k}$, $\mathbf{F}_2 = 2\mathbf{i} - \mathbf{j}$, $\mathbf{F}_3 = -3\mathbf{i} + \mathbf{k}$; $\mathbf{a} = 3\mathbf{i}$.

5. Forces $\mathbf{P} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{Q} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$ act through points with position vectors $2\mathbf{i} - \mathbf{j}$ and $9\mathbf{i} - \mathbf{k}$ respectively. Write down vector equations for the lines of action of $\mathbf{P}$ and $\mathbf{Q}$. Hence find the position vector of the point through which both forces act. Find also a vector equation for the line of action of the resultant of $\mathbf{P}$ and $\mathbf{Q}$.

6. Forces $\mathbf{F}_1 = \mathbf{i} + 7\mathbf{j} + 4\mathbf{k}$ and $\mathbf{F}_2 = 5\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$ act through the origin O. A third force $\mathbf{F}_3 = \mathbf{i} - 4\mathbf{j} + 6\mathbf{k}$ acts through the point A. with position vector $\mathbf{a} = 3\mathbf{i} + 5\mathbf{j}$. Show that the resultant of $\mathbf{F}_1$ and $\mathbf{F}_2$ also acts through A. Find the magnitude of the resultant of all three forces and obtain a vector equation for its line of action.

7. Coplanar forces $\mathbf{F}_1 = \mathbf{i} - 2\mathbf{j}$, $\mathbf{F}_2 = 3\mathbf{i} + 4\mathbf{j}$, $\mathbf{F}_3 = -5\mathbf{i} + \mathbf{j}$ act at points A_1, A_2, A_3 with position vectors $4\mathbf{i} + 5\mathbf{j}$, $3\mathbf{i} - 2\mathbf{j}$, $\mathbf{i} + 4\mathbf{j}$ respectively. Find the Cartesian equation of the line of action of the resultant of these forces.

8. Coplanar forces $\mathbf{F}_1 = 6\mathbf{i} + 2\mathbf{j}, \mathbf{F}_2 = -\mathbf{i} + 7\mathbf{j}, \mathbf{F}_3 = 4\mathbf{i} - 3\mathbf{j}$ act at points A_1, A_2, A_3 with position vectors $\mathbf{i} - \mathbf{j}, 4\mathbf{j}, 2\mathbf{i} + 3\mathbf{j}$ respectively. Find the magnitude of the resultant and the tangent of the angle it makes with the vector $\mathbf{i}$. Find also the Cartesian equation of its line of action.

9. The following sets of forces act along the edges of a triangle ABC. In each case find the resultant force, using a clear diagram to illustrate your answer.

(a) $2\mathbf{AB}, 3\mathbf{AC}, 6\mathbf{BC}$,
(b) $\mathbf{CB}, 3\mathbf{AB}, 5\mathbf{CA}$,
(c) $2\mathbf{BC}, \mathbf{BA}, 2\mathbf{AC}$,
(d) $6\mathbf{AB}, 4\mathbf{CA}, 3\mathbf{CB}$.

10. Three forces acting at a point P are represented by $\mathbf{PA}, \mathbf{PB}$ and $\mathbf{PC}$. Show that their resultant is given by $3\mathbf{PG}$, where G is the centroid of triangle ABC.

11. Forces represented by $3\mathbf{AB}, 2\mathbf{CB}, 4\mathbf{CD}$ and $6\mathbf{AD}$ act along the sides of a square $ABCD$. Show that their resultant is $15\mathbf{XY}$, where X divides AC in the ratio 2:3 and Y divides BD in the ratio 2:1.

12. Three forces represented by $\mathbf{PQ}, \mathbf{QR}$ and $\mathbf{RP}$ act along the sides of a triangle PQR. Show that these forces are equivalent to a couple whose moment is represented by twice the area of the triangle.

11.2 Use of vectors in kinematics and dynamics

Suppose that a particle P moving with constant velocity $\mathbf{v}$ has position vector $\mathbf{r}_0$ initially and position vector $\mathbf{r}$ at time t.

It follows that $$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}.$$

This equation gives the position vector of P at any instant and can be taken as the vector equation of the straight line path of the particle.

The next example shows the use of such equations in a problem concerning the *relative motion* of two ships.

Example 1 At noon ships A and B have position vectors $3\mathbf{i} + 2\mathbf{j}$ and $\mathbf{i} - 4\mathbf{j}$ respectively and constant velocities $4\mathbf{i} + 3\mathbf{j}$ and $3\mathbf{i} + 5\mathbf{j}$ respectively, where distances are measured in kilometres and times in hours. Find vectors to represent (a) the velocity of A relative to B, (b) the displacement of A relative to B, t hours after noon. Find also the least distance between the ships.

(a) The velocity of A relative to B

$$= (\text{velocity of } A) - (\text{velocity of } B)$$
$$= (4\mathbf{i} + 3\mathbf{j}) - (3\mathbf{i} + 5\mathbf{j}) = \mathbf{i} - 2\mathbf{j}.$$

(b) The position vector of A after t hours is given by

$$\mathbf{r}_A = 3\mathbf{i} + 2\mathbf{j} + t(4\mathbf{i} + 3\mathbf{j}).$$

The position vector of B after t hours is given by

$$\mathbf{r}_B = \mathbf{i} - 4\mathbf{j} + t(3\mathbf{i} + 5\mathbf{j})$$

$\therefore$ the displacement of A relative to B after t hours is

$$\mathbf{r}_A - \mathbf{r}_B = 2\mathbf{i} + 6\mathbf{j} + t(\mathbf{i} - 2\mathbf{j})$$

i.e. $$\mathbf{r}_A - \mathbf{r}_B = (2 + t)\mathbf{i} + (6 - 2t)\mathbf{j}.$$

The distance between the ships

$$\begin{aligned} = |\mathbf{r}_A - \mathbf{r}_B| &= \sqrt{\{(2 + t)^2 + (6 - 2t)^2\}} \\ &= \sqrt{\{4 + 4t + t^2 + 36 - 24t + 4t^2\}} \\ &= \sqrt{\{5t^2 - 20t + 40\}} \\ &= \sqrt{\{5(t - 2)^2 + 20\}} \end{aligned}$$

Since $(t - 2)^2$ is never negative, this expression takes its least value when $t = 2$. Hence the least distance between the ships is $\sqrt{20}$ km i.e. $5\sqrt{2}$ km.

[Note that the least distance can also be found by writing

$$f(t) = 5t^2 - 20t + 40$$

Differentiating, $$f'(t) = 10t - 20, \quad f''(t) = 10$$

$\therefore$ $$f'(t) = 0 \Leftrightarrow 10t - 20 = 0 \Leftrightarrow t = 2.$$

Since $f(2) = 20$ and $f''(2) > 0$, the function $f(t)$ has a minimum value of 20.]

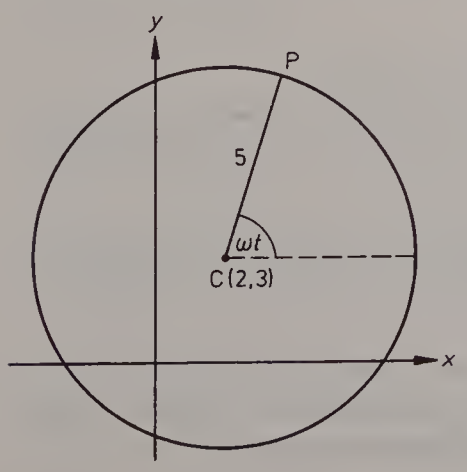
In general, if a moving particle P has position vector $\mathbf{r}$ at time t, then its velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ are given by

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} \quad \text{and} \quad \mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}$$

or, using a dot to indicate differentiation with respect to time,

$$\mathbf{v} = \dot{\mathbf{r}} \quad \text{and} \quad \mathbf{a} = \ddot{\mathbf{r}}.$$

Example 2 A particle P has position vector $\mathbf{r}$ at time t and moves in a circle with constant angular velocity ω. The centre of the circle is the point C with position vector $\mathbf{c} = 2\mathbf{i} + 3\mathbf{j}$, and when $t = 0$, $\mathbf{CP} = 5\mathbf{i}$. Find an expression for $\mathbf{r}$ and hence for the velocity $\mathbf{v}$ of the particle at time t. Find also the time at which P first moves parallel to the vector $\sqrt{3}\mathbf{i} - \mathbf{j}$.



Since P moves on a circle with centre C, the vector $\mathbf{CP}$ has constant magnitude and rotates with constant angular velocity ω. Initially $\mathbf{CP} = 5\mathbf{i}$ and $|\mathbf{CP}| = |5\mathbf{i}| = 5$. Thus, at time t, the

vector **CP** has magnitude 5 units and makes an angle ωt with the direction of **i**. It follows that

$$\mathbf{CP} = 5\cos\omega t\mathbf{i} + 5\sin\omega t\mathbf{j}$$
$$\therefore \quad \mathbf{r} - \mathbf{c} = 5\cos\omega t\mathbf{i} + 5\sin\omega t\mathbf{j}$$
$$\therefore \quad \mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + 5\cos\omega t\mathbf{i} + 5\sin\omega t\mathbf{j}$$

Differentiating with respect to t,

$$\mathbf{v} = \dot{\mathbf{r}} = -5\omega\sin\omega t\mathbf{i} + 5\omega\cos\omega t\mathbf{j}.$$

Thus when P is moving parallel to the vector $\sqrt{3}\mathbf{i} - \mathbf{j}$ there is a scalar λ such that

$$-5\omega\sin\omega t\mathbf{i} + 5\omega\cos\omega t\mathbf{j} = \lambda(\sqrt{3}\mathbf{i} - \mathbf{j})$$

i.e. $-5\omega\sin\omega t = \lambda\sqrt{3}$ and $5\omega\cos\omega t = -\lambda$

$$\therefore \quad \tan\omega t = \frac{\sin\omega t}{\cos\omega t} = \sqrt{3}$$

Hence P first moves parallel to the vector $\sqrt{3}\mathbf{i} - \mathbf{j}$ when $\omega t = \pi/3$ i.e. when $t = \pi/3\omega$.

[Note that at this instant P is not moving in the direction of the vector $\sqrt{3}\mathbf{i} - \mathbf{j}$, but in the opposite direction.]

In problems concerning the forces on moving objects *Newton's second law* is useful. It states that if a particle of mass m moves under the action of a force **F** with acceleration **a**, then

$$\mathbf{F} = m\mathbf{a}.$$

Example 3 A particle of mass 3 kg moves so that its position vector after t seconds is $\mathbf{r} = 8t\mathbf{i} - 5\mathbf{j} + 2t^3\mathbf{k}$. Find (a) the speed of the particle when $t = 1$, (b) the force acting on the particle when $t = 3$.

Let **v** and **a** be the velocity and acceleration of the particle after t seconds, then

$$\mathbf{v} = \dot{\mathbf{r}} = 8\mathbf{i} + 6t^2\mathbf{k}$$
and $\mathbf{a} = \ddot{\mathbf{r}} = 12t\mathbf{k}$.

(a) When $t = 1$, $\mathbf{v} = 8\mathbf{i} + 6\mathbf{k}$

$$\therefore \quad v = |8\mathbf{i} + 6\mathbf{k}| = \sqrt{(8^2 + 6^2)} = 10.$$

Hence, when $t = 1$, the speed of the particle is $10\,\mathrm{m\,s^{-1}}$.

(b) Let **F** be the force on the particle after t seconds, then by Newton's second law,

$$\mathbf{F} = 3\mathbf{a}.$$

When $t = 3$, $\mathbf{a} = 36\mathbf{k}$ $\quad\therefore\quad$ $\mathbf{F} = 108\mathbf{k}$.

Hence, when $t = 3$, the force acting on the particle is $108\mathbf{k}$ (in newtons).

Example 4 A particle of unit mass is acted upon at time t by a force **F**, where $\mathbf{F} = 4\mathbf{i} - 6t\mathbf{j}$. At time $t = 0$ the particle is at the point with position vector $\mathbf{i} + \mathbf{j}$ and its velocity is $5\mathbf{i}$. Find the position vector of the particle at time $t = 2$.

Let **r**, **v** and **a** be the position vector, velocity and acceleration of the particle at time t.
By Newton's second law, $\mathbf{F} = m\mathbf{a}$.
Since the particle has unit mass, $\mathbf{F} = \mathbf{a}$.

$$\therefore \qquad \mathbf{a} = \frac{d\mathbf{v}}{dt} = 4\mathbf{i} - 6t\mathbf{j}$$

Integrating, $\qquad \mathbf{v} = 4t\mathbf{i} - 3t^2\mathbf{j} + \mathbf{c},$

where **c** is a constant vector.
When $t = 0$, $\mathbf{v} = 5\mathbf{i}$, $\quad \therefore \quad \mathbf{c} = 5\mathbf{i}$

$$\therefore \qquad \mathbf{v} = \frac{d\mathbf{r}}{dt} = (4t + 5)\mathbf{i} - 3t^2\mathbf{j}$$

Integrating, $\qquad \mathbf{r} = (2t^2 + 5t)\mathbf{i} - t^3\mathbf{j} + \mathbf{d},$

where **d** is a constant vector.
When $t = 0$, $\mathbf{r} = \mathbf{i} + \mathbf{j}$, $\quad \therefore \quad \mathbf{d} = \mathbf{i} + \mathbf{j}$
$\therefore \quad \mathbf{r} = (2t^2 + 5t + 1)\mathbf{i} + (1 - t^3)\mathbf{j}$

Hence, when $t = 2$, the position vector of the particle is

$$\mathbf{r} = (8 + 10 + 1)\mathbf{i} + (1 - 8)\mathbf{j} = 19\mathbf{i} - 7\mathbf{j}.$$

Exercise 11.2

1. Particles A and B are moving with constant velocities $5\mathbf{i} + 2\mathbf{j}$ and $2\mathbf{i} + \mathbf{j}$ respectively. Initially the particle A has position vector $3\mathbf{i} + 4\mathbf{j}$ and the particle B has position vector $5\mathbf{i} + 8\mathbf{j}$. Write down expressions for $\mathbf{r}_A$ and $\mathbf{r}_B$, the position vectors of the particles at time t. Find the vector **AB** at time t. Hence find the minimum distance between the particles.

2. Particles P and Q are moving with constant velocities $3\mathbf{i} - 2\mathbf{j}$ and $-\mathbf{i} + 4\mathbf{j}$ respectively. Initially the particle P has position vector $\mathbf{i} + 5\mathbf{j}$ and the particle Q has position vector $3\mathbf{i} + 2\mathbf{j}$. Find the velocity of P relative to Q and the displacement of P relative to Q at time t. Show that the particles collide and state the value of t when the collision occurs.

3. At noon ships A and B have position vectors $10\mathbf{i} + 3\mathbf{j}$ and $12\mathbf{i} - 15\mathbf{j}$ respectively. Both ships move with constant velocity, the velocity of A being $3\mathbf{i} + 8\mathbf{j}$ and the velocity of B relative to A being $-5\mathbf{i} + 4\mathbf{j}$. The unit of distance is the kilometre and the unit of time is the hour. Find an expression for the vector **AB**, t hours after noon. Hence find the shortest distance between the ships. Find also the position vectors of the ships A and B when they are closest together.

4. A particle P is moving with constant angular velocity ω on a circle of radius 8 units, with centre at the origin. Given that initially P has position vector $8\mathbf{i}$, find its position vector **r** at time t. Hence show that $\ddot{\mathbf{r}} = -\omega^2\mathbf{r}$.

5. A particle P moves so that its position vector at time t is given by $\mathbf{r} = \mathbf{i} - 2\mathbf{j} + 3(\mathbf{i}\sin 2t + \mathbf{j}\cos 2t)$. Show that P is moving on a circle, stating the position vector of the centre of the circle and its radius. Find also the speed at which P is moving.

6. Two particles A and B are moving such that, at time t, their position vectors are $\mathbf{r}_A = 3(1 + \cos t)\mathbf{i} + (2\sin t)\mathbf{j}$ and $\mathbf{r}_B = (5 - 6t)\mathbf{i} + (3 + 4t)\mathbf{j}$. Find the value of t when the particles are first moving (a) in the same direction, (b) in opposite directions.

7. The position vector of a particle P at time t is $\mathbf{r} = (1 - t^2)\mathbf{i} + (3t - 5t^2)\mathbf{j}$. Find the time at which P is moving (a) directly towards the origin, (b) directly away from the origin.

8. A particle of mass m moves so that its position vector at time t is $\mathbf{r}$. Find its velocity $\mathbf{v}$ and the force $\mathbf{F}$ acting on it, given that

(a) $\mathbf{r} = 4t^2\mathbf{i} + (3t^3 - 2t)\mathbf{j}$,
(b) $\mathbf{r} = 5\mathbf{i} - 2t^3\mathbf{j} + (t^2 - 1)\mathbf{k}$,
(c) $\mathbf{r} = (2t - t^2)\mathbf{i} + (3\sin 2t)\mathbf{j}$,
(d) $\mathbf{r} = (te^{-t})\mathbf{i} - 2t\mathbf{j}$.

9. A particle of mass 5 kg moves so that its position vector after t seconds is $\mathbf{r} = (\cos 2t)\mathbf{i} + (4\sin 2t + 3)\mathbf{j}$. Find (a) the speed of the particle when $t = \pi/3$, (b) the force acting on the particle when $t = \pi/2$.

10. A particle of mass 2 kg starts from rest at the origin and moves under the action of the force $\mathbf{F}$, where $\mathbf{F} = 8\mathbf{i} + 20\mathbf{j}$. Find the velocity and the position vector of the particle after 3 seconds.

11. A particle of unit mass starts from rest at the point with position vector $2\mathbf{i} - 3\mathbf{j}$. At time t the force acting on the particle is $12t\mathbf{i} - 16\mathbf{j}$. Find the speed of the particle and its position vector at time $t = 2$.

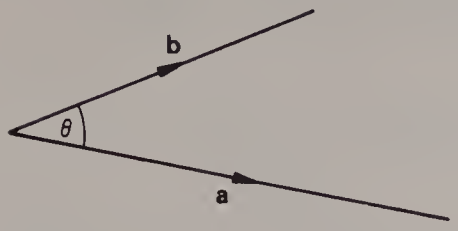
12. A particle of mass 3 kg is acted upon at time t by a force $\mathbf{F}$, where $\mathbf{F} = 6\mathbf{i} - 36t^2\mathbf{j} + 54t\mathbf{k}$. At time $t = 0$ the particle is at the point with position vector $\mathbf{i} - 5\mathbf{j} - \mathbf{k}$ and its velocity is $3(\mathbf{i} + \mathbf{j})$. Find the position vector of the particle at time $t = 1$.

13. A particle of mass 2 kg moves from rest at the origin under the action of two forces. One force $\mathbf{P}$ has magnitude 22 N and acts in the direction of the vector $2\mathbf{i} - 6\mathbf{j} + 9\mathbf{k}$. The other force $\mathbf{Q}$ has magnitude 30 N and acts in the direction of the vector $4\mathbf{j} - 3\mathbf{k}$. Find the acceleration of the particle and the distance it travels in the first 3 seconds of its motion.

14. A particle P is projected vertically upwards from a point O with a speed of $16\,\mathrm{m\,s^{-1}}$. At the same instant a second particle Q is projected horizontally from a point A, 25 m vertically above O, with speed $12\,\mathrm{m\,s^{-1}}$. Using $\mathbf{i}$ and $\mathbf{j}$ as unit vectors in the horizontal and upward vertical directions respectively, find expressions for the position vectors of P and Q with respect to O at time t after projection. Hence find the least distance between the particles.

[Take g as $10\,\mathrm{m\,s^{-2}}$.]

11.3 Scalar product and its applications



The *scalar product* (or dot product) of two vectors **a** and **b** is defined as the scalar $ab\cos\theta$, where θ is the angle between **a** and **b**.

The scalar product is denoted by **a . b**, which is read "**a** dot **b**".

Thus $$\mathbf{a}.\mathbf{b} = ab\cos\theta$$

The scalar product **a . a** is sometimes written as $\mathbf{a}^2$.

[Note that although a product of three real numbers a, b, c can be evaluated by writing $abc = (ab)c$, it is not possible to define a scalar product of three vectors in a similar way. Since **a . b** is a scalar, the expressions (**a . b**) **. c** and **a . b . c** are meaningless. Similarly, since **a . a** is a scalar, the expression $\mathbf{a}^3$ cannot be defined as (**a . a**) **. a**.]

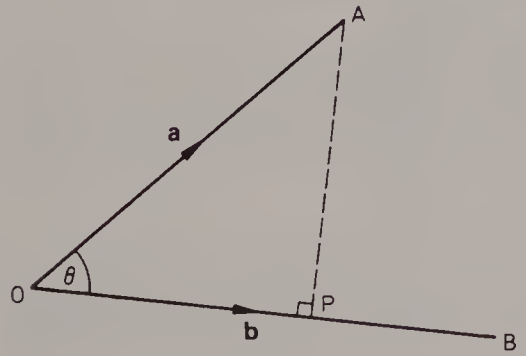
The definition of the scalar product gives rise to various useful properties.

(4) For any vector **a**, $$\mathbf{a}.\mathbf{a} = a^2\cos 0^\circ = a^2.$$

(5) If non-zero vectors **a** and **b** are perpendicular, then

$$\mathbf{a}.\mathbf{b} = ab\cos 90^\circ = 0.$$

Conversely, if non-zero vectors **a** and **b** are such that **a . b** = 0, then **a** is perpendicular to **b**.

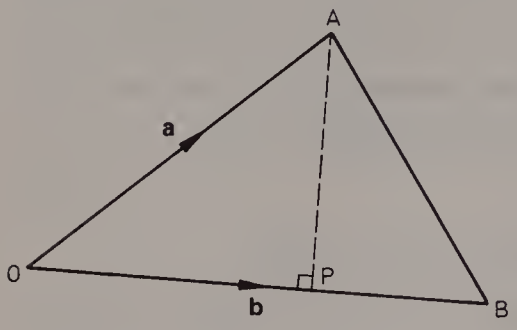


The diagram shows two vectors **a** and **b** represented by line segments $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively. If the vector **a** is resolved along and perpendicular to **b**, then its component in the direction of **b** is $a\cos\theta$.

Thus the component of **a** in the direction of **b** is $\dfrac{\mathbf{a}.\mathbf{b}}{b}$ or $\mathbf{a}.\hat{\mathbf{b}}$, where $\hat{\mathbf{b}}$ is the unit vector in the direction of **b**.

In geometrical work the quantity $\mathbf{a}.\hat{\mathbf{b}}$ is called the *projection* of **a** on **b**. In the diagram the length of this projection is represented by the distance OP.

Example 1 The points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ with respect to an origin O. Show that the area of triangle OAB is given by $\frac{1}{2}\sqrt{\{a^2b^2 - (\mathbf{a}.\mathbf{b})^2\}}$.



The projection of $\mathbf{a}$ on $\mathbf{b}$ is

$$(\mathbf{a}.\mathbf{b})/b,$$

$\therefore$ if P is the foot of the perpendicular from A to OB,

$$OP^2 = (\mathbf{a}.\mathbf{b})^2/b^2$$

Using Pythagoras' theorem in $\triangle OAP$, $OP^2 + AP^2 = OA^2$

$$\therefore \quad AP^2 = OA^2 - OP^2 = a^2 - \frac{(\mathbf{a}.\mathbf{b})^2}{b^2}$$

Hence the area of $\triangle OAB = \frac{1}{2} \times OB \times AP$

$$= \tfrac{1}{2}\sqrt{\left\{b^2 \times \left(a^2 - \frac{(\mathbf{a}.\mathbf{b})^2}{b^2}\right)\right\}} = \tfrac{1}{2}\sqrt{\{a^2b^2 - (\mathbf{a}.\mathbf{b})^2\}}.$$

When considering the unit vectors $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ the definition of scalar product leads to the results

$$\mathbf{i}.\mathbf{i} = \mathbf{j}.\mathbf{j} = \mathbf{k}.\mathbf{k} = 1, \quad \mathbf{i}.\mathbf{j} = \mathbf{j}.\mathbf{k} = \mathbf{k}.\mathbf{i} = 0$$

Thus for vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$,

$$\begin{aligned}\mathbf{a}.\mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}).(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\ &= a_1b_1\mathbf{i}.\mathbf{i} + a_1b_2\mathbf{i}.\mathbf{j} + a_1b_3\mathbf{i}.\mathbf{k} \\ &\quad + a_2b_1\mathbf{j}.\mathbf{i} + a_2b_2\mathbf{j}.\mathbf{j} + a_2b_3\mathbf{j}.\mathbf{k} \\ &\quad + a_3b_1\mathbf{k}.\mathbf{i} + a_3b_2\mathbf{k}.\mathbf{j} + a_3b_3\mathbf{k}.\mathbf{k}\end{aligned}$$

$\therefore$

$$\boxed{\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3}$$

Example 2 Find the component of the force $\mathbf{F} = 2\mathbf{i} + 13\mathbf{j} - 8\mathbf{k}$ in the direction of the vector $\mathbf{d} = 2\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}$. Hence resolve $\mathbf{F}$ into component forces parallel and perpendicular to $\mathbf{d}$.

Let $\hat{\mathbf{d}}$ be the unit vector in the direction of $\mathbf{d}$.

$$d = |2\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}| = \sqrt{\{2^2 + (-6)^2 + 3^2\}} = \sqrt{\{4 + 36 + 9\}} = 7$$

$$\therefore \quad \hat{\mathbf{d}} = \frac{\mathbf{d}}{d} = \tfrac{2}{7}\mathbf{i} - \tfrac{6}{7}\mathbf{j} + \tfrac{3}{7}\mathbf{k}$$

Hence the component of $\mathbf{F}$ in the direction of $\mathbf{d}$

$$= \mathbf{F}.\hat{\mathbf{d}} = (2\mathbf{i} + 13\mathbf{j} - 8\mathbf{k}).(\tfrac{2}{7}\mathbf{i} - \tfrac{6}{7}\mathbf{j} + \tfrac{3}{7}\mathbf{k})$$

$$= \tfrac{4}{7} - \tfrac{78}{7} - \tfrac{24}{7} = -\tfrac{98}{7} = -14.$$

Resolving **F** parallel and perpendicular to **d**, the component force parallel to **d**

$= (\mathbf{F}.\hat{\mathbf{d}})\hat{\mathbf{d}} = -14(\frac{2}{7}\mathbf{i} - \frac{6}{7}\mathbf{j} + \frac{3}{7}\mathbf{k}) = -4\mathbf{i} + 12\mathbf{j} - 6\mathbf{k}.$

Thus the component force perpendicular to **d**

$= (2\mathbf{i} + 13\mathbf{j} - 8\mathbf{k}) - (-4\mathbf{i} + 12\mathbf{j} - 6\mathbf{k}) = 6\mathbf{i} + \mathbf{j} - 2\mathbf{k}.$

If the angle between **a** and **b** is θ, then $\mathbf{a}.\mathbf{b} = ab\cos\theta$.

Thus
$$\boxed{\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{ab} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{ab}}$$

It follows that non-zero vectors **a** and **b** are perpendicular if and only if

$$a_1b_1 + a_2b_2 + a_3b_3 = 0.$$

Example 3 Find the angle θ between the vectors $\mathbf{a} = 4\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} - 5\mathbf{j} - 4\mathbf{k}$.

$\mathbf{a}.\mathbf{b} = 4 \times 3 + 5 \times (-5) + 3 \times (-4) = 12 - 25 - 12 = -25$

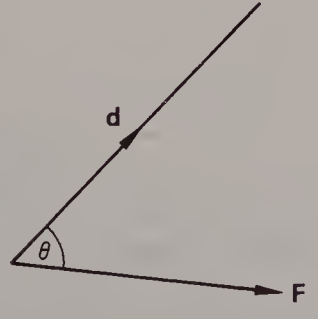
$a = \sqrt{\{4^2 + 5^2 + 3^2\}} = \sqrt{\{16 + 25 + 9\}} = \sqrt{50}$

$b = \sqrt{\{3^2 + (-5)^2 + (-4)^2\}} = \sqrt{\{9 + 25 + 16\}} = \sqrt{50}$

$$\therefore \quad \cos\theta = \frac{\mathbf{a}.\mathbf{b}}{ab} = \frac{-25}{\sqrt{50} \times \sqrt{50}} = -\frac{25}{50} = -\frac{1}{2}$$

Hence the angle between the vectors is 120°.

We next show that the *work done by a force* can be expressed as a scalar product. We recall that if the point of application of a constant force **F** moves a distance s in a direction which makes an angle θ with the direction of **F**, then the work done by the force is $Fs\cos\theta$.



If we now suppose that the point of application of a constant force **F** is given a displacement **d**, where the angle between the vectors **F** and **d** is θ, then,

work done by $\mathbf{F} = Fd\cos\theta = \mathbf{F}.\mathbf{d}$.

[We will assume that forces, displacements and work are measured in newtons, metres and joules respectively.]

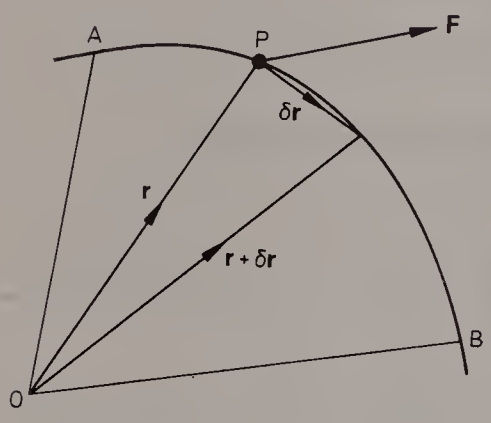
Example 4 A constant force $\mathbf{F} = 3\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}$ acts on a particle. Find the work done by $\mathbf{F}$ when the particle moves from the point with position vector $\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ to the point with position vector $4\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

The displacement of the particle

$$= (4\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) - (\mathbf{i} + 3\mathbf{j} - \mathbf{k}) = 3\mathbf{i} - \mathbf{j} + 4\mathbf{k}$$

$\therefore$ the work done by $\mathbf{F}$

$$= (3\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}).(3\mathbf{i} - \mathbf{j} + 4\mathbf{k}) = 9 + 4 - 8 = 5.$$



Consider now a variable force $\mathbf{F}$ acting on a particle P as it moves along a curve from a point A to a point B. If the path from A to B is regarded as a series of small displacements, then the work done by $\mathbf{F}$ in a typical displacement $\delta\mathbf{r}$ is approximately equal to $\mathbf{F}.\delta\mathbf{r}$.

$\therefore$ total work done by $\mathbf{F}$

$$\approx \sum_{A}^{B} \mathbf{F}.\delta\mathbf{r}$$

Letting $\delta\mathbf{r} \to 0$, we deduce that as the particle moves from A to B,

$$\boxed{\text{the work done by } \mathbf{F} \text{ is } \int_A^B \mathbf{F}.d\mathbf{r}} \tag{1}$$

If $\mathbf{F}$ and $\mathbf{r}$ are given as functions of the time t, then we write

$$\int_A^B \mathbf{F}.d\mathbf{r} = \int_{t_1}^{t_2} \mathbf{F}.\frac{d\mathbf{r}}{dt}dt = \int_{t_1}^{t_2} \mathbf{F}.\mathbf{v}\,dt \tag{2}$$

where t_1 and t_2 are the times at which the particle passes through A and B respectively.

Example 5 A particle moves along a curve so that its position vector at time t is given by $\mathbf{r} = t^3\mathbf{i} + 2t\mathbf{j}$. If one of the forces acting on the particle is a force $\mathbf{F}$, where $\mathbf{F} = 2\mathbf{i} - t\mathbf{j}$ at time t, find the work done by $\mathbf{F}$ in the interval from $t = 0$ to $t = 3$.

$$\mathbf{r} = t^3\mathbf{i} + 2t\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = 3t^2\mathbf{i} + 2\mathbf{j}$$

$\therefore$ the work done by $\mathbf{F}$ from $t = 0$ to $t = 3$

$$= \int_0^3 \mathbf{F} . \frac{d\mathbf{r}}{dt} dt$$

$$= \int_0^3 (2\mathbf{i} - t\mathbf{j}) . (3t^2\mathbf{i} + 2\mathbf{j})\, dt$$

$$= \int_0^3 (6t^2 - 2t)\, dt$$

$$= \Big[2t^3 - t^2 \Big]_0^3 = 54 - 9 = 45.$$

If a particle of mass m is moving with velocity $\mathbf{v}$ at time t, then its *kinetic energy* at that instant is $\frac{1}{2}mv^2$. We can establish the relationship between work and kinetic energy by considering the motion of this particle along a curved path AB. We will assume that the particle passes through A at time t_1 with velocity $\mathbf{v}_1$ and through B at time t_2 with velocity $\mathbf{v}_2$. If $\mathbf{F}$ is the resultant force acting on the particle at time t and W is the work done by $\mathbf{F}$ in the interval $t_1 \leqslant t \leqslant t_2$, then using our earlier results (1) and (2),

$$W = \int_A^B \mathbf{F} . d\mathbf{r} = \int_{t_1}^{t_2} \mathbf{F} . \mathbf{v}\, dt.$$

Using Newton's second law, $\mathbf{F} = m\mathbf{a} = m\dfrac{d\mathbf{v}}{dt}$

$\therefore$
$$W = \int_{t_1}^{t_2} m\frac{d\mathbf{v}}{dt} . \mathbf{v}\, dt$$

But, using the product rule proved in Exercise 7.5 question 25,

$$\frac{d}{dt}(v^2) = \frac{d}{dt}(\mathbf{v} . \mathbf{v}) = \mathbf{v} . \frac{d\mathbf{v}}{dt} + \frac{d\mathbf{v}}{dt} . \mathbf{v} = 2\frac{d\mathbf{v}}{dt} . \mathbf{v}$$

$\therefore$
$$\int \frac{d\mathbf{v}}{dt} . \mathbf{v}\, dt = \tfrac{1}{2}v^2 + \text{constant}.$$

Hence
$$W = \Big[\tfrac{1}{2}mv^2 \Big]_{t_1}^{t_2} = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2$$

Thus

work done by $\mathbf{F}$ = K.E. at B − K.E. at A

Example 6 A particle of mass 10 kg moves under the action of a force $\mathbf{F}$, so that its position vector after t seconds is given by $\mathbf{r} = 3t\mathbf{i} - \mathbf{j} + t^2\mathbf{k}$. Find the work done by $\mathbf{F}$ in the interval $1 \leqslant t \leqslant 5$.

$$\mathbf{r} = 3t\mathbf{i} - \mathbf{j} + t^2\mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = 3\mathbf{i} + 2t\mathbf{k}$$

$$\Rightarrow v^2 = 3^2 + (2t)^2 = 9 + 4t^2$$

$\therefore$ when $t = 1$, $v^2 = 13$ and when $t = 5$, $v^2 = 109$.

Work done by $\mathbf{F}$ = (K.E. when $t = 5$) − (K.E. when $t = 1$)
$= \frac{1}{2} \times 10 \times 109 - \frac{1}{2} \times 10 \times 13\,\text{J}$
$= 480\,\text{J}.$

The *power* of a force is the rate at which it does work. Let us suppose that at time t a force $\mathbf{F}$ acts at a point with position vector $\mathbf{r}$. If the point of application of $\mathbf{F}$ is given a displacement $\delta\mathbf{r}$ in a small interval of time δt, then the work done is approximately $\mathbf{F}.\delta\mathbf{r}$ and the average rate of working is $(\mathbf{F}.\delta\mathbf{r})/\delta t$.
$\therefore$ the power of the force $\mathbf{F}$ at time t

$$= \lim_{\delta t \to 0} \frac{\mathbf{F}.\delta\mathbf{r}}{\delta t} = \lim_{\delta t \to 0} \mathbf{F}.\frac{\delta\mathbf{r}}{\delta t} = \mathbf{F}.\frac{d\mathbf{r}}{dt}.$$

Hence if the point of application of a force $\mathbf{F}$ is moving with velocity $\mathbf{v}$, then

the power of $\mathbf{F}$ is $\mathbf{F}.\mathbf{v}$.

Example 7 At time t a particle of unit mass is moving with velocity $\mathbf{v}$ under the action of a force $\mathbf{F} = 4t\mathbf{i} + 3\mathbf{k}$. Given that when $t = 0$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$, find an expression for the velocity $\mathbf{v}$ at time t. Hence find an expression for the power of $\mathbf{F}$ at time t.

By Newton's second law, $\mathbf{F} = m\mathbf{a}$.
Since $\mathbf{F} = 4t\mathbf{i} + 3\mathbf{k}$, $m = 1$ and $\mathbf{a} = \dfrac{d\mathbf{v}}{dt}$,

we have $\dfrac{d\mathbf{v}}{dt} = 4t\mathbf{i} + 3\mathbf{k}$,

Integrating, $\mathbf{v} = 2t^2\mathbf{i} + 3t\mathbf{k} + \mathbf{c}$,

where $\mathbf{c}$ is a constant vector.

But when $t = 0$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$ $\quad\therefore\quad \mathbf{c} = \mathbf{i} + 2\mathbf{j}$.
Hence $\mathbf{v} = (2t^2 + 1)\mathbf{i} + 2\mathbf{j} + 3t\mathbf{k}$.
The power of $\mathbf{F}$ at time $t = \mathbf{F}.\mathbf{v}$
$= (4t\mathbf{i} + 3\mathbf{k}).(\{2t^2 + 1\}\mathbf{i} + 2\mathbf{j} + 3t\mathbf{k})$
$= 4t(2t^2 + 1) + 3 \times 3t$
$= 8t^3 + 4t + 9t = 8t^3 + 13t.$

Exercise 11.3

1. Two particles A and B are moving with constant velocities $8\mathbf{i} + \mathbf{j} - 4\mathbf{k}$ and $4\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$ respectively. Find the angle between the paths of the particles. Find also the speed of A relative to B.

2. Two forces $\mathbf{F}_1 = 4\mathbf{j} - 5\mathbf{k}$ and $\mathbf{F}_2 = 2\mathbf{i} - 5\mathbf{j} - \mathbf{k}$ act on a particle. Find the magnitude of the resultant force acting on the particle and the angle the resultant makes with the force $\mathbf{F}_1$.

3. Find the component of the force $\mathbf{F}$ in the direction of the vector $\mathbf{d}$ if
(a) $\mathbf{F} = 3\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$, $\mathbf{d} = 6\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$,
(b) $\mathbf{F} = \mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$, $\mathbf{d} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$,
(c) $\mathbf{F} = 7\mathbf{i} - \mathbf{j} - 4\mathbf{k}$, $\mathbf{d} = 3\mathbf{i} - 5\mathbf{j} + 4\mathbf{k}$,
(d) $\mathbf{F} = 3\mathbf{i} + 5\mathbf{j} - \mathbf{k}$, $\mathbf{d} = 4\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}$.

4. Resolve the force $\mathbf{F}$ into component forces parallel and perpendicular to the vector $\mathbf{d}$ if
(a) $\mathbf{F} = 11\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$, $\mathbf{d} = 4\mathbf{i} - 3\mathbf{k}$,
(b) $\mathbf{F} = 5\mathbf{i} - 22\mathbf{j} + 19\mathbf{k}$, $\mathbf{d} = -5\mathbf{j} + 12\mathbf{k}$,
(c) $\mathbf{F} = 18\mathbf{i} + 35\mathbf{j} - 21\mathbf{k}$, $\mathbf{d} = 2\mathbf{i} - 6\mathbf{j} + 9\mathbf{k}$,
(d) $\mathbf{F} = 3\mathbf{i} - 4\mathbf{j} + 10\mathbf{k}$, $\mathbf{d} = 4\mathbf{i} - 8\mathbf{j} + \mathbf{k}$.

5. Find the work done when the point of application of a force $\mathbf{F}$ moves through a displacement $\mathbf{d}$, given that
(a) $\mathbf{F} = 3\mathbf{i} - 7\mathbf{j} + 6\mathbf{k}$, $\mathbf{d} = 5\mathbf{i} + \mathbf{j} - \mathbf{k}$,
(b) $\mathbf{F} = -2\mathbf{i} + 3\mathbf{j} - 8\mathbf{k}$, $\mathbf{d} = 6\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}$,
(c) $\mathbf{F} = 15\mathbf{i} + 7\mathbf{k}$, $\mathbf{d} = 10\mathbf{j} + 3\mathbf{k}$,
(d) $\mathbf{F} = 3\mathbf{i} - 9\mathbf{j} - 4\mathbf{k}$, $\mathbf{d} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

6. At time t a particle of mass m is moving with position vector $\mathbf{r}$ and velocity $\mathbf{v}$ under the action of a force $\mathbf{F}$. In each of the following cases find an expression for $\mathbf{v}$ and hence determine the work done by $\mathbf{F}$ in the given interval of time.
(a) $\mathbf{r} = 3t\mathbf{i} + t^2\mathbf{j}$, $0 \leqslant t \leqslant 2$,
(b) $\mathbf{r} = 5\mathbf{i} - t^2\mathbf{j} + 3t\mathbf{k}$, $2 \leqslant t \leqslant 4$,
(c) $\mathbf{r} = t^3\mathbf{i} - 4t\mathbf{j} + 3\mathbf{k}$, $1 \leqslant t \leqslant 3$,
(d) $\mathbf{r} = t\mathbf{i} - \mathbf{j} + (t^2 - 7t)\mathbf{k}$, $2 \leqslant t \leqslant 5$.

7. At time t a particle of mass m is moving with position vector $\mathbf{r}$ and velocity $\mathbf{v}$ under the action of a force $\mathbf{F}$. Find expressions for $\mathbf{F}$, $\mathbf{v}$ and the power of $\mathbf{F}$ at time t, given that
(a) $\mathbf{r} = 2t^2\mathbf{i} - 5t\mathbf{k}$,
(b) $\mathbf{r} = \mathbf{i} + t^3\mathbf{j} - 3t^2\mathbf{k}$,
(c) $\mathbf{r} = t\mathbf{i} - \mathrm{e}^{-t}\mathbf{j}$,
(d) $\mathbf{r} = \sin t\mathbf{j} + 3\cos t\mathbf{k}$.

8. A particle of unit mass is moving under the action of a constant force $\mathbf{F} = \mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$. Find the work done by $\mathbf{F}$ when the particle moves from the point A with position vector $5\mathbf{j} + 2\mathbf{k}$ to the point $\mathbf{B}$ with position vector $3\mathbf{i} + \mathbf{k}$. If the speed of the particle at A is $6\,\mathrm{m\,s^{-1}}$, find the speed of the particle at B.

9. A particle of mass 3 kg moves under the action of a constant force $\mathbf{F}$ with acceleration $\mathbf{a} = 3\mathbf{i} + 2\mathbf{k}$. Find the work done by $\mathbf{F}$ as the particle moves from the point A with position vector $4\mathbf{i} - 9\mathbf{j} + 21\mathbf{k}$ to the point B with position vector $10\mathbf{i} + 15\mathbf{j} - 7\mathbf{k}$. If the particle passes through A with velocity $12\mathbf{j} - 16\mathbf{k}$ and through

B with velocity $\lambda(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$, find the value of λ and the power of $\mathbf{F}$ at the points A and B.

10. A particle of unit mass is acted upon at time t by a force $\mathbf{F} = 2t\mathbf{i} - 3\mathbf{j}$. Given that the initial velocity of the particle is $\mathbf{j} + \mathbf{k}$, find an expression for its velocity $\mathbf{v}$ at time t. Hence find the work done by $\mathbf{F}$ in the interval $0 \leqslant t \leqslant 2$ and the power of $\mathbf{F}$ when $t = 2$.

11. A particle of mass 2 kg is acted upon at time t by a force $\mathbf{F} = 8\mathbf{i} - 4\cos t\mathbf{j} + 2t\mathbf{k}$. When $t = 0$ the velocity of the particle is $6\mathbf{i}$. Find expressions for the velocity of the particle and the power of $\mathbf{F}$ at time t.

12. A particle moves along a curve so that its position vector at time t is $\mathbf{r} = (3t - 2)\mathbf{i} + t^4\mathbf{j}$. If one of the forces acting on the particle is $\mathbf{F} = 5t^2\mathbf{i} - \mathbf{j}$, find by integration the work done by $\mathbf{F}$ in the interval $0 \leqslant t \leqslant 2$.

13. A particle moves along a curve with velocity $\mathbf{v} = \mathbf{i} + 10t^2\mathbf{j} - 3t\mathbf{k}$ at time t. If one of the forces acting on the particle is $\mathbf{F} = 2t\mathbf{i} + t^2\mathbf{j} - 2t\mathbf{k}$, find the power of $\mathbf{F}$ in terms of t. Hence find the work done by $\mathbf{F}$ in the interval from $t = 1$ to $t = 3$.

14. A particle moves on a curve with vector equation $\mathbf{r} = (\lambda - 12)\mathbf{i} - \lambda^2\mathbf{j} + \mathbf{k}$, where λ is a parameter. One of the forces acting on the particle is a variable force $\mathbf{F}$. Write down, in the form of a definite integral, an expression for the work done by $\mathbf{F}$ as the particle moves from the point where $\lambda = 1$ to the point where $\lambda = 2$. Evaluate this integral given that $\mathbf{F} = -4\mathbf{r}$.

Exercise 11.4 (miscellaneous)

1. (i) The force $\mathbf{F}_1$, of magnitude 26 N, acts at the origin in the direction of the vector $(3\mathbf{i} + 4\mathbf{j} + 12\mathbf{k})$. A second force $\mathbf{F}_2$, of magnitude 30 N, also acts at the origin, but in the direction of the vector $(6\mathbf{j} - 8\mathbf{k})$. Calculate the magnitude and the direction of the resultant of the forces $\mathbf{F}_1$ and $\mathbf{F}_2$. Calculate also the cosine of the angle between $\mathbf{F}_1$ and $\mathbf{F}_2$.
   (ii) A particle of mass 2 kg starts from rest at the origin and the force $\mathbf{F}_3$, where $\mathbf{F}_3 = (6\mathbf{i} + 26\mathbf{j})$ N, acts on it as it moves. Find the position vector of the particle 2 seconds later. (L)

2. The force acting at time t $(0 \leqslant t < 2)$ on a particle of unit mass is $24t^2\mathbf{i} + 6\mathbf{j}$. At time $t = 0$ the particle is at rest at the point with position vector $-2\mathbf{i} + 3\mathbf{j}$. Find the position vector of the particle at time $t = T$ $(0 \leqslant T < 2)$. For time $t \geqslant 2$ the force acting on the particle is $6\mathbf{j}$. Find the position vector of the particle at time $t = 3$. (L)

3. State the centre and radius of the circle which has vector equation $\mathbf{r} = 6\mathbf{i} + 8\mathbf{j} + 6(\mathbf{i}\cos\theta + \mathbf{j}\sin\theta)$. A particle P of mass 2 kg moves on this circle with

constant angular velocity $\pi/12$ radians per second. Write down the position vector of P at time t given that, at $t = 0$, P is at the point corresponding to $\theta = 0$. Calculate the components parallel to **i** and **j** of the resultant vector force on the particle when $t = 4$. Find the position vector of the particle when $t = 8$ given that the forces on the particle cease to act when $t = 4$. (L)

4. The position vector, relative to the origin O, of an object of mass m moving in a horizontal plane in which **i** and **j** are perpendicular unit vectors is given by

$$\mathbf{r} = \frac{\cos\theta}{1+\cos\theta}\mathbf{i} + \frac{\sin\theta}{1+\cos\theta}\mathbf{j}, \quad -\pi < \theta < \pi.$$

Obtain an expression for r, the magnitude of **r**. Given that $r^2\dfrac{d\theta}{dt} = h$ where h is constant, show that the velocity of the object is $-h\sin\theta\mathbf{i} + h(1+\cos\theta)\mathbf{j}$. Show that the force acting on the object is of magnitude mh^2/r^2, and find the direction of this force. (JMB)

5. A particle is projected from the origin O with initial velocity $u\mathbf{i} + v\mathbf{j}$, where **i** and **j** are unit vectors in the horizontal and upward vertical directions respectively, and moves freely under gravity. Show that its velocity and position vector with respect to O, at time t after projection, are given by $\dot{\mathbf{r}} = u\mathbf{i} + (v - gt)\mathbf{j}$ and $\mathbf{r} = ut\mathbf{i} + (vt - \frac{1}{2}gt^2)\mathbf{j}$ respectively.

For the remainder of this question components of velocities are measured in $\mathrm{m\,s^{-1}}$ and components of displacements in m. Take g as $10\,\mathrm{m\,s^{-2}}$.

Two particles, A and B, are projected simultaneously from the origin with velocities $-30\mathbf{i} + 90\mathbf{j}$ and $30\mathbf{i} + 110\mathbf{j}$ respectively. Both particles impinge on a plane which contains the origin, meeting the plane on a line of greatest slope which has the equation $\mathbf{r} = \lambda(3\mathbf{i} + \mathbf{j})$, where λ is a scalar. Show that the times of flight of A and B are equal and that the distance between the points of impact is $400\sqrt{10}\,\mathrm{m}$. Find the cosine of the acute angle between the direction of motion of A at the instant of impact and the line of greatest slope of the plane. (JMB)

6. At noon two ships A and B have the following position and velocity vectors:

	Position vector	*Velocity vector*
Ship A	$10\mathbf{i} + 5\mathbf{j}$	$-2\mathbf{i} + 4\mathbf{j}$
Ship B	$2\mathbf{i} - \mathbf{j}$	$2\mathbf{i} + 7\mathbf{j}$

where **i** and **j** are unit vectors in the directions East and North respectively, and where the speeds are measured in kilometres per hour and the distances in kilometres. If they continue on their respective courses, (i) find the position vector of A after a time t hours; (ii) show that they will collide, and give the time of the collision; (iii) determine how far ship A will have travelled between noon and the collision. (SU)

7. A ship A is travelling on a course of 060° at a speed of $30\sqrt{3}\,\mathrm{km/h}$ and a ship B is travelling on a course of 030° at 20 km/h. At noon B is 260 km due east of A.

Using unit base vectors **i** and **j** pointing east and north respectively find $\overrightarrow{AB}$ in terms of **i**, **j** and t at time t hours in the afternoon. Hence, or otherwise, calculate the least distance between A and B, to the nearest kilometre, and the time at which they are nearest to one another. (L)

8. An aeroplane A moves in space with a variable velocity **v** and is attacked by a guided missile B which moves with constant speed u, where $u > |v|$. The guidance system of B is such that, if from any instant the velocity of A stayed constant, B would hit A without having to change direction. Show that the velocity of A relative to B is

$$\left[\frac{\mathbf{r}.\mathbf{v} - \sqrt{\{(\mathbf{r}.\mathbf{v})^2 + (u^2 - \mathbf{v}^2)\mathbf{r}^2\}}}{\mathbf{r}^2}\right]\mathbf{r},$$

where **r** is the position vector of A relative to B. (O)

9. At time t a particle is in motion with velocity **v** and is being acted upon by a variable force **F**. Write down expressions for (i) the power at time t, (ii) the work done by **F** during the time interval $0 \leqslant t \leqslant T$.

The particle, of mass m, moves in a plane where **i** and **j** are perpendicular unit vectors so that its position vector at time t is given by $\mathbf{r} = 2a\cos 2t\mathbf{i} + a\sin 2t\mathbf{j}$, where a is a positive constant. Derive expressions for the velocity **v** and the force **F** at time t. Obtain an expression in terms of t for the power at time t and show that the work done by **F** during the interval $0 \leqslant t \leqslant T$ is $3ma^2(1 - \cos 4T)$. If T varies, find the maximum value of the work done by **F** and determine also the smallest value of T for which this maximum value is reached. (JMB)

10. A smooth fixed wire is in the shape of the curve given by

$$\mathbf{r} = a(\theta - \sin\theta)\mathbf{i} + a(1 + \cos\theta)\mathbf{j},\ 0 \leqslant \theta \leqslant 2\pi,$$

where **i**, **j** are unit vectors in the horizontal and upward vertical directions respectively. A particle threaded on the wire is released from rest at the point A ($\theta = 0$). Write down, in terms of θ and its derivative, an equation expressing the conservation of energy during the motion. Show that the time taken for the particle to reach B ($\theta = 2\pi$) is $2\pi\sqrt{(a/g)}$. (JMB)

12 Impulse and momentum

12.1 The impulse of a force

When a constant force $\mathbf{F}$ acts for a time t, then the *impulse* of the force is defined as the vector quantity $\mathbf{F}t$.

The SI unit of impulse is the newton second (N s).

Example 1 A force $\mathbf{F} = 4\mathbf{i} - 3\mathbf{j}$ acts on a particle for 3 seconds. Find the magnitude of the impulse of the force.

If the impulse of the force is $\mathbf{I}$, then

$$\mathbf{I} = (4\mathbf{i} - 3\mathbf{j}) \times 3$$
$$\therefore \quad |\mathbf{I}| = 3|4\mathbf{i} - 3\mathbf{j}| = 3\sqrt{(4^2 + 3^2)} = 15.$$

Hence the magnitude of the impulse is 15 N s.

When a variable force $\mathbf{F}$ acts from time t_1 to time t_2, then the impulse of the force is defined as $\int_{t_1}^{t_2} \mathbf{F}\, dt$.

Example 2 A particle is moving under the action of a force $\mathbf{F} = 2\mathbf{i} - 3t^2\mathbf{j} + 4t\mathbf{k}$ at the time t. Find the impulse given to the particle in the interval $1 \leqslant t \leqslant 2$.

$$\text{Impulse} = \int_1^2 (2\mathbf{i} - 3t^2\mathbf{j} + 4t\mathbf{k})\, dt$$
$$= \Big[2t\mathbf{i} - t^3\mathbf{j} + 2t^2\mathbf{k}\Big]_1^2$$
$$= (4\mathbf{i} - 8\mathbf{j} + 8\mathbf{k}) - (2\mathbf{i} - \mathbf{j} + 2\mathbf{k})$$
$$= 2\mathbf{i} - 7\mathbf{j} + 6\mathbf{k}.$$

In some circumstances it is difficult to determine the exact nature of the forces involved. For example, in collisions between moving objects comparatively large

unknown forces act for very short intervals of time. We will now show that the impulse of such a force can be found by considering the effect it produces.

Suppose that at time t a particle of mass m is moving with velocity $\mathbf{v}$ under the action of a force $\mathbf{F}$. Let $\mathbf{I}$ be the impulse given to the particle by $\mathbf{F}$ in the interval $t_1 \leqslant t \leqslant t_2$. Whether $\mathbf{F}$ is constant or variable,

$$\mathbf{I} = \int_{t_1}^{t_2} \mathbf{F}\, dt$$

Using Newton's second law in its original form,

$$\mathbf{F} = \frac{d(m\mathbf{v})}{dt},$$

where $m\mathbf{v}$ is the *momentum* of the particle.

$$\therefore \quad \mathbf{I} = \int_{t_1}^{t_2} \frac{d(m\mathbf{v})}{dt} dt$$

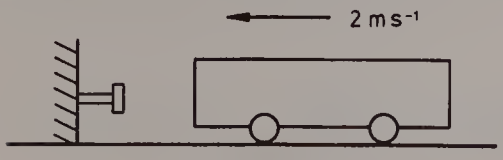
$$\mathbf{I} = \Big[m\mathbf{v} \Big]_{t_1}^{t_2}$$

$$\mathbf{I} = (m\mathbf{v})_{at\ t_2} - (m\mathbf{v})_{at\ t_1}$$

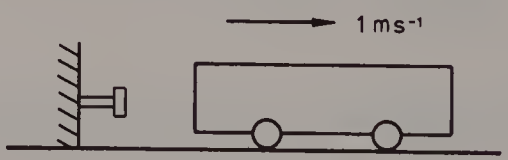
i.e. impulse = change in momentum

[Because of the relationship between impulse and momentum, both quantities are usually measured in newton seconds. An alternative unit of momentum is the kilogramme metre per second. However, since $1\,\mathrm{N} = 1\,\mathrm{kg\,m\,s^{-2}}$, it follows that $1\,\mathrm{N\,s} = 1\,\mathrm{kg\,m\,s^{-1}}$.]

Example 3 A truck of mass 1400 kg moving along a straight horizontal track at $2\,\mathrm{m\,s^{-1}}$ runs into fixed buffers and rebounds at a speed of $1\,\mathrm{m\,s^{-1}}$. If the truck and the buffers are in contact for 1·2 s, find the average force exerted by the buffers on the truck.



Before impact



After impact

Let the direction away from the buffers be positive and let F N be the average force exerted by the buffers.

Impulse = change in momentum

$$\therefore \quad F \times 1{\cdot}2 = 1400 \times 1 - 1400 \times (-2)$$

$$\therefore \quad F = \frac{1400 \times 3}{1{\cdot}2} = 3500$$

Hence the average force exerted by the buffers is 3500 N.

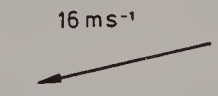
Example 4 A particle of mass 5 kg is moving with velocity $3\mathbf{i} - 7\mathbf{j}$ when it is given an impulse $\mathbf{I} = -7\mathbf{i} + 16\mathbf{j}$. If the velocity of the particle after the impulse is $\mathbf{v}$, find $\mathbf{v}$. Find also the change in kinetic energy.

$$\text{Impulse} = \text{change in momentum}$$
$$\therefore \quad -7\mathbf{i} + 16\mathbf{j} = 5\mathbf{v} - 5(3\mathbf{i} - 7\mathbf{j})$$
$$\therefore \quad 5\mathbf{v} = -7\mathbf{i} + 16\mathbf{j} + 15\mathbf{i} - 35\mathbf{j}$$
$$\text{i.e.} \quad \mathbf{v} = \tfrac{1}{5}(8\mathbf{i} - 19\mathbf{j})$$

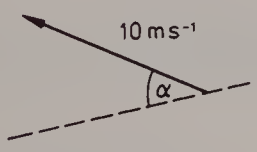
Initial speed $= \sqrt{(3^2 + 7^2)} = \sqrt{(9 + 49)} = \sqrt{58}$
New speed $= \frac{1}{5}\sqrt{(8^2 + 19^2)} = \frac{1}{5}\sqrt{(64 + 361)} = \frac{1}{5}\sqrt{425} = \sqrt{17}$.
$\therefore$ there is a loss in kinetic energy of

$$(\tfrac{1}{2} \times 5 \times 58 - \tfrac{1}{2} \times 5 \times 17)\,\text{J} \quad \text{i.e.} \quad 102{\cdot}5\,\text{J}.$$

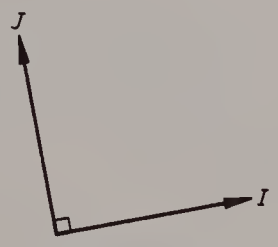
Example 5 A ball of mass 0·25 kg strikes a wall at a speed of $16\,\text{m s}^{-1}$ and rebounds with a speed of $10\,\text{m s}^{-1}$. If the angle between the velocities of the ball before and after impact is α, where $\tan\alpha = \frac{3}{4}$, find the magnitude of the impulse exerted by the wall on the ball.



Before impact



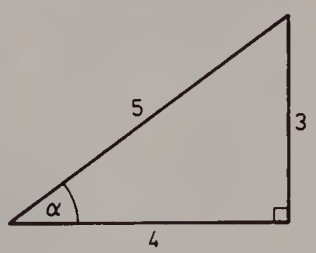
After impact



Components of the impulse

Resolving parallel and perpendicular to the initial velocity of the ball,

$$\nearrow \qquad I = 0{\cdot}25(-10\cos\alpha) - 0{\cdot}25(-16)$$
$$\nwarrow \qquad J = 0{\cdot}25(10\sin\alpha)$$



Using a right-angled triangle,

$$\sin\alpha = \tfrac{3}{5}, \quad \cos\alpha = \tfrac{4}{5}.$$

$$\therefore \quad I = 0{\cdot}25(-8) - 0{\cdot}25(-16) = 2$$
$$J = 0{\cdot}25 \times 6 = 1{\cdot}5$$

Hence the magnitude of the impulse exerted by the wall

$$= \sqrt{(2^2 + 1{\cdot}5^2)}\,\text{N s} = 2{\cdot}5\,\text{N s}.$$

When a jet of water is directed at a wall, the reaction of the wall on the water produces a *continuous change in momentum.* Provided that the water is being delivered at a steady rate and with constant velocity, the force exerted by the wall can be determined by considering the momentum lost by the water each second.

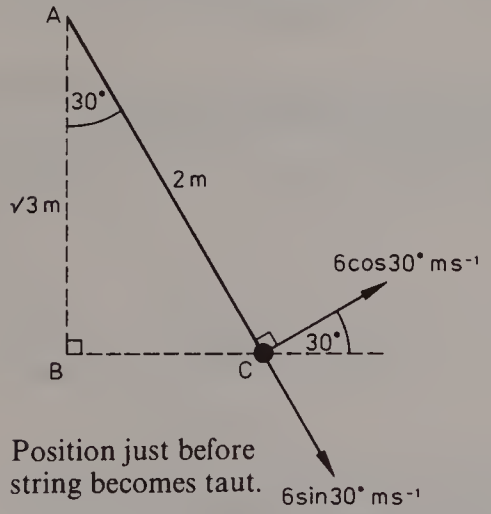
Example 6 A hose delivers water horizontally at a rate of 20 kg per second with a speed of $25\,\mathrm{m\,s^{-1}}$. The water strikes a vertical wall and does not rebound. Find the force exerted on the wall.

Let F N be the force exerted on the wall, then the force exerted by the wall is also F N,
$\therefore$ impulse exerted in 1 second $= F \times 1\,\mathrm{N\,s} = F\,\mathrm{N\,s}$.
Mass of water delivered in 1 second $= 20\,\mathrm{kg}$
$\therefore$ loss of momentum in 1 second $= 20 \times 25\,\mathrm{N\,s} = 500\,\mathrm{N\,s}$
However, impulse = change in momentum,
$\therefore \quad F = 500$
Hence the force exerted on the wall is 500 N.

An instantaneous impulse acting along a string is called an *impulsive tension.* It may produce changes in momentum in the direction of the string, but has no effect perpendicular to the string. In particular, if a stationary particle is jerked into motion by the sudden tightening of a string attached to it, then the initial velocity of the particle will be in the direction of the string.

Example 7 One end of a light inextensible string of length 2 m is fixed at a point A on a smooth horizontal plane. The other end is attached to a particle of mass 2 kg. The particle is placed at a point B on the plane, where $AB = \sqrt{3}\,\mathrm{m}$, then projected horizontally with speed $6\,\mathrm{m\,s^{-1}}$ in a direction perpendicular to AB. Find the speed of the particle immediately after the string has become taut. Find also the impulsive tension in the string.

If C is the point on the path of the particle at which the string first becomes taut, $\angle CAB = \cos^{-1}(\sqrt{3}/2) = 30°$.



Position just before string becomes taut.

As shown in the diagram, just before the string tightens the velocity of the particle has components $6\sin 30°\,\mathrm{m\,s^{-1}}$ along AC and $6\cos 30°\,\mathrm{m\,s^{-1}}$ perpendicular to AC. When the string becomes taut the particle loses its momentum in the direction of AC, but its momentum perpendicular to AC is unchanged.

Hence just after the string has become taut, the speed of the particle is $6\cos 30°\,\mathrm{m\,s^{-1}}$ i.e. $3\sqrt{3}\,\mathrm{m\,s^{-1}}$.
The loss of momentum along AC is produced by the impulsive tension in the string,

$\therefore \quad$ impulsive tension $= 2 \times 6\sin 30°\,\mathrm{N\,s} = 6\,\mathrm{N\,s}$.

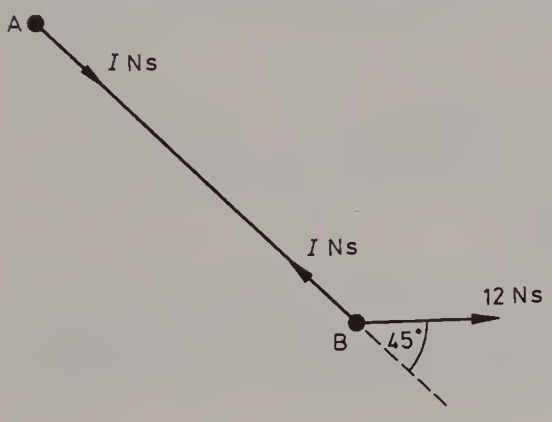
In problems which involve systems of two or more particles the impulse-momentum

relationship can be used both for the system as a whole and for particles taken separately. When considering the impulse given to the whole system, we will see that internal impulses occur in equal and opposite pairs and cancel each other out. Thus we may write:

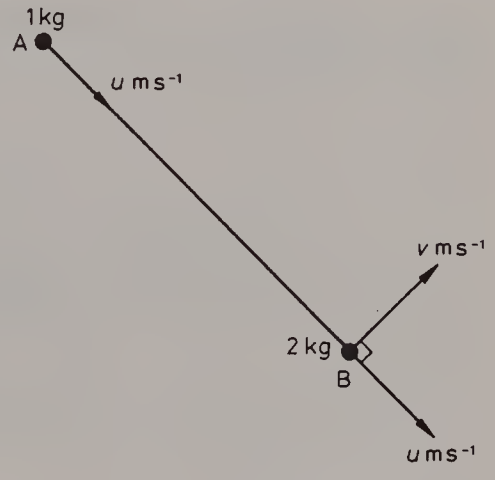
sum of external impulses = total change in momentum.

The next example concerns two particles connected by a light inextensible string. In this case the impulsive tensions acting on the particles are equal and opposite, so they can be disregarded when dealing with the system as a whole.

Example 8 Particles A and B, of masses 1 kg and 2 kg respectively, are connected by a light inextensible string. The particles are at rest on a smooth horizontal plane with the string taut. A horizontal impulse of magnitude 12 N s is applied to particle B in a direction inclined at 45° to the direction of AB. Find the velocities with which A and B start to move. Find also the impulsive tension in the string.



Impulse diagram



Velocity diagram

Let the initial velocity of B have components u and v along and perpendicular to AB respectively. Since the string is taut, A will start to move with speed u along AB.

For A and B together, impulse = change in momentum,

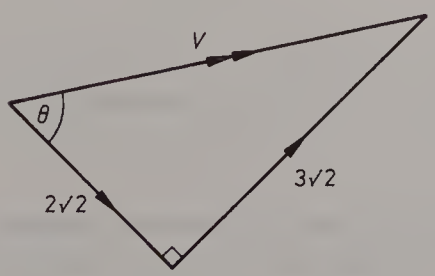
$\therefore$ resolving along and perpendicular to AB,

$$\searrow \qquad 12\cos 45° = 1 \times u + 2 \times u$$

$$\nearrow \qquad 12\sin 45° = 2 \times v$$

$$\therefore \qquad u = 2\sqrt{2} \quad \text{and} \quad v = 3\sqrt{2}$$

Thus the initial velocity of A is $2\sqrt{2}\,\text{m s}^{-1}$ along AB.



If the initial velocity of B is $V\,\text{m s}^{-1}$ at an angle θ to AB, then

$$V = \sqrt{\{(2\sqrt{2})^2 + (3\sqrt{2})^2\}} = \sqrt{26}$$

$$\theta = \tan^{-1}(3/2) \approx 56{\cdot}3°$$

Thus the initial velocity of B is $\sqrt{26}\,\text{m s}^{-1}$ at an angle of 56·3° to AB.

Let I N s be the impulsive tension in the string, then considering the motion of A,

$$I = 1 \times 2\sqrt{2}.$$

Hence the impulsive tension in the string is $2\sqrt{2}$ N s.

Exercise 12.1

1. When a constant force **F** acts for time t, the impulse of the force is **I**.
(a) If $\mathbf{F} = 2\mathbf{i} - 7\mathbf{j}$ and $t = 4$, find **I**.
(b) If $\mathbf{I} = 6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and $t = 3$, find **F**.
(c) If $\mathbf{F} = -12\mathbf{i} + 5\mathbf{j}$ and $t = 5$, find $|\mathbf{I}|$.
(d) If $\mathbf{I} = 6\mathbf{i} + 12\mathbf{j} - 4\mathbf{k}$ and $t = 3{\cdot}5$, find $|\mathbf{F}|$.

2. Find the impulse of the variable force **F** in the stated interval of time.
(a) $\mathbf{F} = 5t\mathbf{i} + 3\mathbf{j}$, $0 \leqslant t \leqslant 2$.
(b) $\mathbf{F} = \mathbf{i} - 6t^2\mathbf{j} + 2t\mathbf{k}$, $1 \leqslant t \leqslant 3$.
(c) $\mathbf{F} = \cos 2t\mathbf{i} + 3\sin 2t\mathbf{j}$, $0 \leqslant t \leqslant \frac{1}{2}\pi$.

3. Find the momentum of
(a) a man of mass 80 kg running at $6\,\mathrm{m\,s^{-1}}$,
(b) a car of mass 1200 kg travelling at $25\,\mathrm{m\,s^{-1}}$,
(c) a bullet of mass 15 g moving at $800\,\mathrm{m\,s^{-1}}$.

4. A particle of mass m moves so that its position vector at time t is **r**. Find, as a vector, the momentum of the particle at time T if
(a) $\mathbf{r} = 3\mathbf{i} + t^2\mathbf{j}$, $m = 2$, $T = 3$;
(b) $\mathbf{r} = -4t\mathbf{i} + t^3\mathbf{k}$, $m = 0{\cdot}5$, $T = 2$;
(c) $\mathbf{r} = 5\cos 3t\mathbf{i} + 2\mathbf{j} + 5\sin 3t\mathbf{k}$, $m = 4$, $T = \pi$.

5. A stone of mass 3 kg is falling at a speed of $18\,\mathrm{m\,s^{-1}}$ when it hits the ground. If the stone is brought to rest by the impact, find the impulse exerted by the ground.

6. A ball of mass 0·2 kg strikes a wall when moving horizontally at $12\,\mathrm{m\,s^{-1}}$. If the ball rebounds horizontally at $8\,\mathrm{m\,s^{-1}}$, find the impulse exerted by the wall.

7. A bullet of mass 0·03 kg is fired into a fixed block of wood at a speed of $400\,\mathrm{m\,s^{-1}}$. If the bullet is brought to rest in 0·02 s, find the average resistance exerted by the wood.

8. A particle of mass 5 kg is moving under the action of a constant force of magnitude 8 N. Initially the speed of the particle is $12\,\mathrm{m\,s^{-1}}$. Find its speed after 3 seconds if the force (a) acts in the direction of motion, (b) directly opposes motion.

9. A particle of mass 2 kg is projected across a rough horizontal plane with a speed of $21\,\mathrm{m\,s^{-1}}$. Find the time in which the particle comes to rest, given that the coefficient of friction is 2/7.

10. A particle of mass 0·5 kg is moving with velocity $\mathbf{u} = 6\mathbf{i} - 7\mathbf{j}$ when it is given an impulse $\mathbf{I} = 3\mathbf{i} + 11\mathbf{j}$. Find the new velocity of the particle and the average force required to bring it to rest in a further 3 seconds.

11. A particle of mass 0·8 kg is moving with velocity $4\mathbf{i} + 7\mathbf{j}$ when it strikes a fixed obstacle and rebounds with velocity $-6\mathbf{i} + 2\mathbf{j}$. Find the impulse exerted by the particle on the obstacle. Find also the kinetic energy lost by the particle in the collision.

12. A particle of mass 3 kg is moving with speed $6\,\mathrm{m\,s^{-1}}$ when it receives an impulse of magnitude 24 N s. Find the speed of the particle immediately after the impulse and the change in the kinetic energy of the particle if the direction of the impulse and the original direction of motion are (a) the same, (b) opposite, (c) perpendicular.

13. A ball of mass 0·2 kg strikes a wall at a speed of $12\,\mathrm{m\,s^{-1}}$ and rebounds with a speed of $3{\cdot}5\,\mathrm{m\,s^{-1}}$. If its path is diverted through 90° by the impact, find the magnitude of the impulse exerted on the wall.

14. An object of mass 4 kg strikes a barrier at a speed of $10\,\mathrm{m\,s^{-1}}$ and rebounds at a speed of $5\,\mathrm{m\,s^{-1}}$. If the angle between the velocities of the object before and after impact is 60°, find the magnitude of the impulse exerted on the barrier.

15. A hose is discharging water at the rate of 50 litres per second. The jet of water is travelling horizontally with a speed of $12\,\mathrm{m\,s^{-1}}$ when it strikes a vertical wall. Assuming that the water does not rebound, find the force exerted on the wall. [Take the mass of 1 litre of water to be 1 kg.]

16. A stream of water falling vertically from a pipe at the rate of 75 kg per second exerts a force of 900 N on the ground below. Assuming that the ground destroys the momentum of the water, find the speed at which the water hits the ground.

17. In a factory small screws are falling vertically onto a conveyor belt without bouncing at a rate of 48 kg per minute. Just before impact the screws are travelling at $2\,\mathrm{m\,s^{-1}}$. If the conveyor belt moves horizontally at $1{\cdot}5\,\mathrm{m\,s^{-1}}$, find the magnitude and direction of the average force it exerts on the screws.

18. One end of a light inextensible string of length $\sqrt{3}$ m is fixed at a point A on a smooth horizontal plane. The other end is attached to a particle of mass 3 kg. The particle is placed at a point B on the plane, where $AB = 1$ m, then projected horizontally with speed $4\,\mathrm{m\,s^{-1}}$. Find the speed of the particle immediately after the string has become taut and the impulsive tension in the string, given that the angle between the initial velocity of the particle and the direction BA is (a) 60°, (b) 90°.

19. Particles A and B, of masses 3 kg and 2 kg respectively, are connected by a light inextensible string. The particles are at rest on a smooth horizontal plane with the string taut. A horizontal impulse of magnitude 12·5 N s is applied to particle B in a direction inclined at an angle α to the direction of AB, where $\tan\alpha = 4/3$. Find the

velocities with which A and B start to move. Find also the impulsive tension in the string.

20. Particles A and B, both of mass m, are connected by a light inextensible string. A particle C of mass $2m$ is attached to the mid-point of the string. The particles are at rest on a smooth horizontal plane with the string taut and with AC at right angles to BC. A horizontal impulse of magnitude $12m$ is applied to C in a direction inclined at an angle α to AC, such that both parts of the string remain taut. Find the speeds with which the particles begin to move if (a) $\alpha = 45°$, (b) $\alpha = 60°$.

12.2 Conservation of momentum

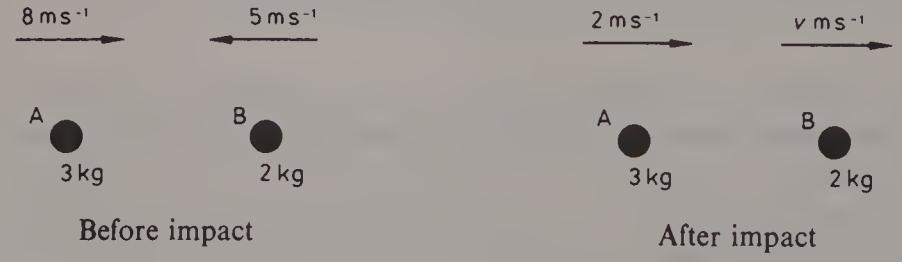
When two particles collide they exert equal and opposite impulses on each other. Therefore the changes in the momentum of the particles are also equal and opposite. It follows that the total momentum of the two particles together remains unchanged. A similar argument can be used in any situation in which equal and opposite impulses act on two bodies, e.g. when a bullet is fired from a gun or when a light inextensible string connecting two particles suddenly becomes taut.

These examples are special cases of a more general result called the *principle of conservation of linear momentum.*

> If no external force acts on a system of particles in a particular direction, then there can be no change in the total momentum of the system in that direction.

Example 1 Two particles A and B, of masses 3 kg and 2 kg respectively, are travelling in opposite directions along the same straight line when they collide. Just before the impact A is moving with speed $8\,\mathrm{m\,s^{-1}}$ and B with speed $5\,\mathrm{m\,s^{-1}}$. After the impact the speeds of A and B are $2\,\mathrm{m\,s^{-1}}$ and $v\,\mathrm{m\,s^{-1}}$ respectively. Find the value of v given that after the impact A and B are travelling in (a) the same direction, (b) opposite directions.

(a)



Momentum before impact = momentum after impact

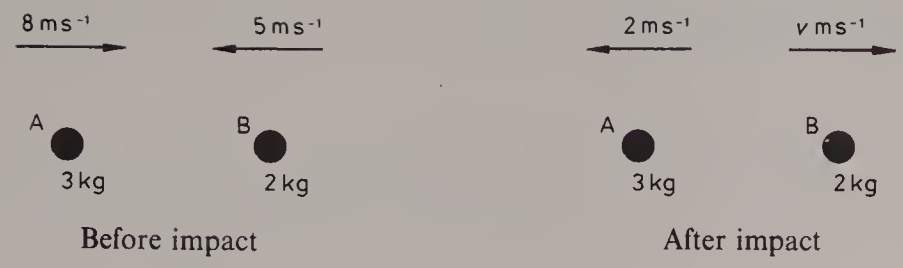
$\therefore$ taking the direction of motion of A as positive,

$$3 \times 8 + 2 \times (-5) = 3 \times 2 + 2 \times v$$

i.e. $$14 = 6 + 2v$$

Thus $$v = 4$$

(b)



Before impact After impact

Momentum before impact = momentum after impact

$\therefore \quad 3 \times 8 + 2 \times (-5) = 3 \times (-2) + 2 \times v$

i.e. $14 = -6 + 2v$

Thus $v = 10.$

Example 2 A bullet of mass 0·02 kg is fired from a gun of mass 6 kg with a speed of 450 m s^{-1}. Find the speed of the gun's recoil. Find also the constant force required to bring the gun to rest in 0·5 seconds.

Let v m s^{-1} be the speed of the gun and let F N be the force required to bring the gun to rest in 0·5 seconds.

Momentum after firing = momentum before firing

$\therefore \quad 6 \times v + 0{\cdot}02 \times (-450) = 0$

i.e. $6v - 9 = 0$

$v = 1{\cdot}5$

Hence the speed of the gun's recoil is 1·5 m s^{-1}.
As the gun is brought to rest,

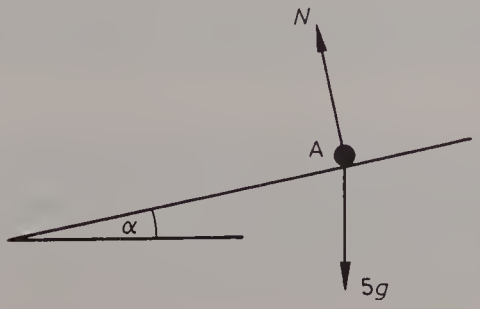
impulse = change in momentum

$\therefore \quad F \times 0{\cdot}5 = 6 \times 1{\cdot}5$

i.e. $F = 18$

Hence the constant force required is 18 N.

Example 3 A particle A of mass 5 kg slides from rest down a smooth plane inclined at an angle α to the horizontal, where $\sin\alpha = 1/6$. After 3 s it collides with a particle B of mass 1 kg which is sliding down the plane at 2·5 m s^{-1}. If the two particles coalesce, find their common speed immediately after impact.



The change in the momentum of A is equal to the impulse received.
$\therefore$ just before the collision the momentum of A down the plane

$= 5g \sin\alpha \times 3$ N s

$= 5 \times 9{\cdot}8 \times \frac{1}{6} \times 3$ N s

$= 24{\cdot}5$ N s

Momentum is conserved in the collision between A and B. Thus if $v\,\mathrm{m\,s^{-1}}$ is their common speed after impact,

$$24{\cdot}5 + 1 \times 2{\cdot}5 = (5 + 1)v$$
$$\therefore \qquad v = 4{\cdot}5$$

Hence the common speed of the particles after impact is $4{\cdot}5\,\mathrm{m\,s^{-1}}$.

Exercise 12.2

1. A railway truck of mass 12 000 kg is moving along a straight level track at $7\,\mathrm{m\,s^{-1}}$ when it collides with another truck of mass 8000 kg moving in the same direction at $2\,\mathrm{m\,s^{-1}}$. After the collision the trucks move on together. Find the common speed of the trucks after impact.

2. A bullet of mass 0·04 kg is fired horizontally into a block of wood of mass 1·4 kg which rests on a smooth horizontal surface. The bullet becomes embedded in the block and after impact they move on together at a speed of $20\,\mathrm{m\,s^{-1}}$. Find the speed at which the bullet is fired.

3. Two particles, A of mass 5 kg and B of mass 2 kg, are moving horizontally along the same straight line when they collide. Both come to rest instantaneously. If the velocity of A was $3\,\mathrm{m\,s^{-1}}$ just before the impact, find the velocity of B and the kinetic energy lost in the impact.

4. A gun of mass 4 kg fires a bullet of mass 20 g at a speed of $500\,\mathrm{m\,s^{-1}}$. Find the speed of the gun's recoil and the energy of the explosion in the gun.

5. Two particles A and B, of mass 3 kg and 5 kg respectively, collide. Just before impact A is moving with speed $8\,\mathrm{m\,s^{-1}}$ and B is moving with speed $4\,\mathrm{m\,s^{-1}}$ in the opposite direction. Immediately after the impact A rebounds with speed $7\,\mathrm{m\,s^{-1}}$. Find the speed of B immediately after impact.

6. Two particles X and Y, of mass 4 kg and 3 kg respectively, collide. Just before impact X is moving with speed $13\,\mathrm{m\,s^{-1}}$ and Y is moving with speed $5\,\mathrm{m\,s^{-1}}$ in the opposite direction. Immediately after the impact X moves on, with no change in direction, at $4\,\mathrm{m\,s^{-1}}$. Find the speed of Y immediately after impact.

7. A particle of mass m and velocity $4\mathbf{i} + \mathbf{j}$ collides with a particle of mass $2m$ and velocity $\mathbf{i} - 8\mathbf{j}$. If the particles coalesce on impact, find their common velocity just after the collision.

8. Two particles, A of mass $3m$ and B of mass $2m$, are moving with velocities $4\mathbf{i} - 6\mathbf{j}$ and $2\mathbf{i} + 7\mathbf{j}$ respectively when they collide. Immediately after the impact A moves on in the same direction as before, but with half its original speed. Find the velocity of B just after the collision.

9. A particle of mass 2 kg moving with speed $10\,\mathrm{m\,s^{-1}}$ collides with a stationary particle of mass 7 kg. Immediately after impact the particles move with the same speed but in opposite directions. Find the kinetic energy lost in the collision.

10. Two particles, of masses $3m$ and $2m$ moving in opposite directions with speeds u and $4u$ respectively, collide. Given that no kinetic energy is lost in the collision, find the speeds of the particles immediately after impact.

11. Three trucks A, B and C, each of mass 2 tonnes, are moving on a straight horizontal track in the same direction with speeds of 10, 5 and $3\,\mathrm{m\,s^{-1}}$ respectively. When any two trucks collide, they become coupled. Thus after two collisions the three trucks move on with a common speed. Find this speed and find also the total loss of kinetic energy in the collisions.

12. A truck A of mass 3000 kg moving at $4\,\mathrm{m\,s^{-1}}$ collides with a truck B of mass 2000 kg moving at $2\,\mathrm{m\,s^{-1}}$ in the same direction. If after impact truck A continues to move in the same direction but at a speed of $3\,\mathrm{m\,s^{-1}}$, find the new speed of truck B. If the trucks are in contact for 1·5 s, find the average force each exerts on the other.

13. A van of mass 1800 kg begins to tow a stationary car of mass 1200 kg. Just before the tow-rope tightens, the van is moving at a speed of $1\,\mathrm{m\,s^{-1}}$. Find the common speed of the vehicles just after the rope tightens. Find also the impulsive tension in the rope.

14. A bullet of mass 0·03 kg is fired from a gun with a horizontal velocity of $600\,\mathrm{m\,s^{-1}}$. If the gun is then brought to rest in 1·2 s by a horizontal force which rises uniformly from zero to P N and then falls uniformly to zero, find the value of P.

15. A particle of mass 2 kg slides from rest down a smooth plane inclined at an angle of 30° to the horizontal. After 5 seconds it collides with a particle of mass 3 kg which is sliding down the plane at a speed of $7\,\mathrm{m\,s^{-1}}$. If the two particles coalesce on impact find their common velocity (a) just after the collision, (b) 2 seconds after the collision.

16. Two small beads A and B, each of mass m, are threaded on a smooth straight wire inclined at an angle α to the horizontal. Bead A is released from rest and begins to slide down the wire. At same instant bead B is projected up the wire towards A with speed u. After a time t the two beads collide and are brought to rest by the impact. Find an expression for t in terms of u, g and α.

12.3 Work, energy and momentum

When faced with a problem in particle dynamics, we will now have to decide whether to consider work-energy relationships, impulse-momentum relationships or a

combination of both. The two fundamental equations are:

Work done = change in kinetic energy
Impulse = change in momentum

Example 1 A bullet of mass m is fired horizontally with velocity V into a fixed block of wood and penetrates a distance d into it. If the block exerts a constant resistance R, which stops the bullet in a time T, find expressions for R and T in terms of m, V and d.

Work done against resistance = loss of kinetic energy

$$\therefore \quad Rd = \tfrac{1}{2}mV^2$$

Hence $$R = mV^2/2d$$

Impulse = change in momentum

$$\therefore \quad RT = mV$$

Hence $$T = \frac{mV}{R} = mV \times \frac{2d}{mV^2} = \frac{2d}{V}.$$

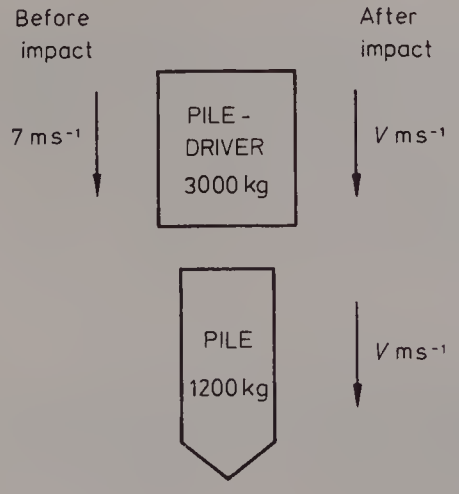
[Note that as the resistance is constant this problem could also be solved using Newton's second law and the constant acceleration formulae $v^2 = u^2 + 2as$, $v = u + at$.]

Example 2 A pile-driver of mass 3000 kg falls freely from a height of 2·5 m and strikes a pile of mass 1200 kg. After the blow the pile and the driver move on together. If the pile is driven a distance of 0·5 m into the ground, find the speed at which the pile starts to move into the ground and the average resistance of the ground to penetration.

Let v m s^{-1} be the downward velocity of the pile-driver just before impact. Using the formula $v^2 = u^2 + 2as$, where $u = 0, a = 9{\cdot}8, s = 2{\cdot}5$, we have

$$v^2 = 0 + 2 \times 9{\cdot}8 \times 2{\cdot}5 = 49$$

$$\therefore \quad v = 7$$



Let V m s^{-1} be the common velocity of the pile and the driver after impact. Using the principle of conservation of momentum,

$$7 \times 3000 + 0 \times 1200 = V(3000 + 1200)$$

$$\therefore \quad V = \frac{7 \times 3000}{4200} = 5$$

Hence the pile starts to move into the ground at 5 m s^{-1}.

Let R N be the average resistance of the ground, then the resultant force acting on pile and driver together is $(R - 4200g)$ N.

$$\text{Work done} = \text{change in kinetic energy}$$
$$\therefore \quad (R - 4200g) \times 0{\cdot}5 = \tfrac{1}{2} \times 4200 \times 5^2$$
$$\therefore \quad R - 4200g = 4200 \times 25$$
$$\therefore \quad R = 4200(25 + 9{\cdot}8) = 146\,160$$

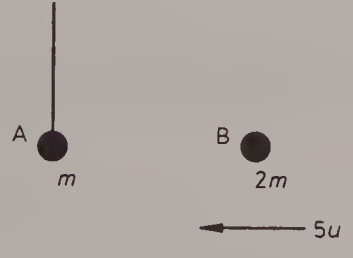
Hence the average resistance of the ground is 146 160 N.

When a particle moves in a vertical plane there are changes in its gravitational potential energy. Provided that no force other than gravity is doing work, we may then apply the principle of conservation of mechanical energy in the form:

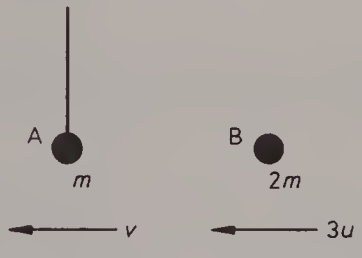
$$\boxed{\text{K.E.} + \text{P.E.} = \text{constant}}$$

Example 3 A particle A, of mass m, is attached to a fixed point by means of a light inextensible string and hangs in equilibrium. A particle B, of mass $2m$, is travelling horizontally with speed $5u$ when it collides with particle A. Immediately after the impact B continues to move in the same direction but with speed $3u$. If A swings to a height h above its original position before coming instantaneously to rest, find an expression for h in terms of u and g.

Let v be the speed of particle A after impact.



Just before impact



Just after impact

Using the principle of conservation of momentum,

$$m \times 0 + 2m \times 5u = m \times v + 2m \times 3u$$
$$10mu = mv + 6mu$$
$$\therefore \quad v = 4u$$

Hence the speed of A immediately after impact is $4u$.

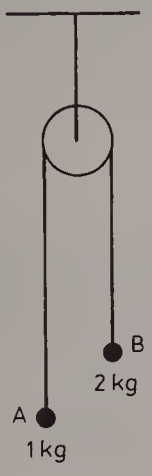
When A is swinging freely its direction of motion is always at right angles to the string. Thus the tension in the string does no work and mechanical energy is conserved.

P.E. after impact $= 0$, K.E. after impact $= \frac{1}{2}m(4u)^2 = 8mu^2$.
P.E. at height $h = mgh$, K.E. at height $h = 0$.

$$\therefore \quad 0 + 8mu^2 = mgh + 0$$
$$\text{Hence} \quad h = 8u^2/g.$$

Example 4 Particles A and B, of masses 1 kg and 2 kg respectively, are attached to the ends of a light inextensible string passing over a smooth light pulley. The system is released from rest with both parts of the string taut and vertical. After A has travelled 1·2 m it hits and coalesces with a stationary particle C, of mass 2 kg. Find the new speed of the system and the impulsive tension in the string at the instant when C is picked up.

Taking the initial P.E. of each particle as zero, after the system has travelled 1·2 m the P.E. of A is $1{\cdot}2g$ J and the P.E. of B is $-1{\cdot}2 \times 2g$ J i.e. $-2{\cdot}4g$ J.



The initial K.E. of the system is zero. If v m s^{-1} is the speed of the particles after they have travelled 1·2 m, the K.E. acquired is $\frac{1}{2}(1+2)v^2$ J i.e. $1{\cdot}5v^2$ J.
Since no external force is doing work, the mechanical energy of the system remains constant.

$$\therefore \quad 1{\cdot}5v^2 + (1{\cdot}2g - 2{\cdot}4g) = 0$$
$$\text{i.e.} \quad 1{\cdot}5v^2 = 1{\cdot}2 \times 9{\cdot}8$$
$$\therefore \quad v = 2{\cdot}8$$

Thus, just before C is picked up the system is travelling at 2·8 m s^{-1}.
Let V m s^{-1} be the new speed of the system and J N s the impulsive tension in the string when C is picked up.

Impulse = change in momentum

Thus for B: $J = 2 \times 2{\cdot}8 - 2 \times V$

i.e. $J = 5{\cdot}6 - 2V$ (1)

For A and C: $J = (1 \times V + 2 \times V) - (1 \times 2{\cdot}8 + 2 \times 0)$

i.e. $J = 3V - 2{\cdot}8$ (2)

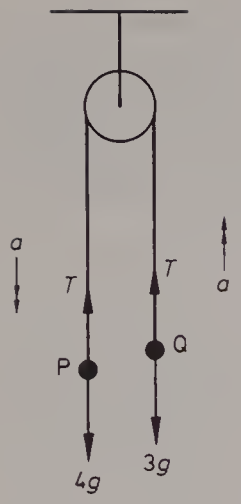
Subtracting (2) from (1): $0 = 8{\cdot}4 - 5V$

$\therefore \quad V = 1{\cdot}68$

Substituting in (1): $J = 5{\cdot}6 - 2 \times 1{\cdot}68 = 2{\cdot}24$.

Hence the new speed of the system is 1·68 m s^{-1} and the impulsive tension in the string is 2·24 N s.

Example 5 A particle P of mass 4 kg is connected to a particle Q of mass 3 kg by means of a light inextensible string which passes over a smooth light pulley. The system is released from rest with both parts of the string taut and vertical. After 3·5 s particle P hits the ground without rebounding. Assuming that particle Q does not hit the pulley, find the further time which elapses before the string is again taut. Find also the speed with which P leaves the ground.



Initial position

Let $a\,\mathrm{m\,s^{-2}}$ be the acceleration of the particles and let $T\,\mathrm{N}$ be the tension in the string.
Using Newton's second law,

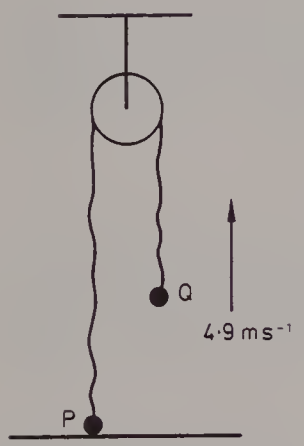
for P: $\quad 4g - T = 4a \qquad (1)$
for Q: $\quad T - 3g = 3a \qquad (2)$
(1) + (2): $\quad g = 7a$

$$\therefore \qquad a = \frac{g}{7} = 1{\cdot}4$$

Using the formula $v = u + at$, where $u = 0$, $a = 1{\cdot}4$, $t = 3{\cdot}5$,

$$v = 0 + 1{\cdot}4 \times 3{\cdot}5 = 4{\cdot}9$$

Hence when P hits the ground, Q has an upward velocity of $4{\cdot}9\,\mathrm{m\,s^{-1}}$.

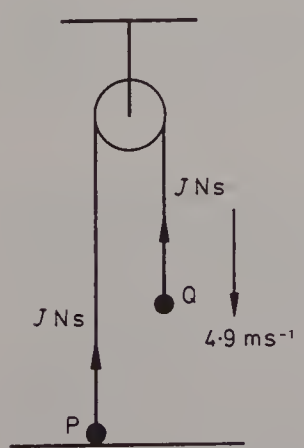


When P hits the ground

While P is on the ground Q moves freely under gravity. Hence the time which elapses before Q returns to the position where the string slackened is given by the formula $s = ut + \frac{1}{2}at^2$, where $s = 0$, $u = 4{\cdot}9$ and $a = -9{\cdot}8$.

$$\therefore \quad 0 = 4{\cdot}9t - 4{\cdot}9t^2$$
$$\therefore \quad t = 0 \quad \text{or} \quad t = 1.$$

Hence the string is again taut after a further time of 1 s.
Since there is no loss of energy in this time, it follows that as the string tightens Q has a downward velocity of $4.9\,\mathrm{m\,s^{-1}}$.



As the string becomes taut again

Let $V\,\mathrm{m\,s^{-1}}$ be the speed with which P is jerked into motion and let $J\,\mathrm{N\,s}$ be the impulsive tension in the string.

Impulse = change in momentum

Thus for P: $\quad J = 4V$
and for Q: $\quad J = 3 \times 4{\cdot}9 - 3V$
Subtracting: $\quad 0 = 7V - 3 \times 4{\cdot}9$
$\therefore \quad V = 2{\cdot}1$

Hence P leaves the ground at a speed of $2{\cdot}1\,\mathrm{m\,s^{-1}}$.

If we look again at the worked examples in this chapter we will see that a collision or the sudden tightening of a string often results in a loss of kinetic energy. This happens because some of the kinetic energy of the bodies involved is converted into other forms such as heat, sound or light. It is therefore important to stress that the principle of conservation of mechanical energy cannot usually be applied to a system when impulsive forces are acting.

Exercise 12.3

1. A car of mass 1200 kg travelling in a straight line at a speed of $20\,\mathrm{m\,s^{-1}}$ is brought to rest in a distance of 50 m by a constant braking force. Find the magnitude of this force and the time taken to stop the car.

2. A bullet of mass 25 g is fired horizontally with velocity $600\,\mathrm{m\,s^{-1}}$ into a fixed block of wood and penetrates a distance of 50 cm into it. Find the constant resistance exerted by the wood and the time taken to bring the bullet to rest.

3. A railway truck of mass 3000 kg moving at $4\,\mathrm{m\,s^{-1}}$ collides with a truck of mass 2000 kg moving in the opposite direction at $1\,\mathrm{m\,s^{-1}}$. If the trucks move on together after the collision, find their common speed just after impact. If a constant retarding force of 400 N acts on the trucks after impact, find the distance they travel before coming to rest.

4. A particle is projected at a speed of $7\,\mathrm{m\,s^{-1}}$ across a rough horizontal plane. After travelling a distance of 2 m it collides with a stationary particle of equal mass. The two particles coalesce and travel a further distance d m before coming to rest. If the coefficient of friction between the particles and the plane is 5/12, find the value of d.

5. A stone of mass 5 kg is dropped from a point 10 m above horizontal ground. Find the impulse exerted by the ground on the stone if (a) it comes to rest without rebounding, (b) it rises to a maximum height of 10 cm after impact.

6. A particle of mass 2 kg is projected vertically upwards from a point A at a speed of $21\,\mathrm{m\,s^{-1}}$. At the same instant a particle of mass 1 kg is dropped from a point 30 m vertically above A. Find the time which elapses before the two particles collide. If the particles coalesce on impact, find the speed at which they pass through A.

7. A pile-driver of mass 1500 kg falls freely from a height of 1·6 m and strikes a pile of mass 500 kg. After the blow the pile and the driver move on together. If the pile is driven a distance of 0·3 m into the ground, find the speed at which the pile starts to move into the ground and the average resistance of the ground to penetration.

8. A pile-driver of mass 1200 kg falls freely from a height of 3·6 m and strikes without rebounding a pile of mass 800 kg. The blow drives the pile a distance of 36 cm into the ground. Find the resistance of the ground, assumed to be constant. Find also the time for which the pile is in motion.

9. A particle P is released from rest at a point A on a smooth plane inclined at 30° to the horizontal. When it has reached a point B on the plane, P collides with a particle Q, of equal mass, moving up the plane at $7\,\mathrm{m\,s^{-1}}$. After the collision P moves up the plane coming to rest instantaneously at A. Given that Q is brought to rest by the impact, find the distance AB.

10. A bullet of mass 30 g is fired into a stationary block of wood of mass 5 kg and becomes embedded in it. The block is freely suspended from a fixed point and after impact swings to a height of 40 cm above its initial position. Find, correct to 3 significant figures, the velocity of the bullet just before impact.

11. Two beads, A of mass m and B of mass $2m$, are threaded on a smooth circular wire fixed in a vertical plane. Initially A is at rest at the lowest point O of the wire and B is released from rest at a height h above O. Just before B collides with A it is moving with velocity v and immediately after impact A starts to move with velocity v. Find, in terms of h, the height above O at which each bead first comes to rest.

12. A particle P is attached to one end of a light inextensible string of length 1·25 m, the other end being attached to a fixed point A. The string is taut and inclined at an angle of 60° to the downward vertical when P is released from rest. As P reaches its lowest point B, it strikes a stationary particle Q, of equal mass, which is free to move horizontally. If the speed of Q just after impact is $2{\cdot}1\,\mathrm{m\,s^{-1}}$, find the maximum height above B to which P rises after the collision.

13. Particles A and B, of masses 1 kg and 3 kg respectively are attached to the ends of a light inextensible string passing over a smooth light pulley. The system is released from rest with both parts of the string taut and vertical. After A has travelled 1·8 m it hits and coalesces with a stationary particle C, of mass 4 kg. Find the new velocity of the system and the impulsive tension in the string at the instant when C is picked up.

14. Particles A and B, of masses 1 kg and 2 kg respectively, are connected by a light inextensible string which passes over a smooth light pulley. The system is released from rest with the string taut and the hanging parts vertical. After 4·5 s A picks up a stationary particle of mass 3 kg. Find the velocity of the system immediately after impact and the further time which elapses before the system first comes instantaneously to rest.

15. A particle A of mass 5 kg is connected to a particle B of mass 2 kg by means of a light inextensible string which passes over a smooth light pulley. The system is released from rest with both parts of the string taut and vertical. After 3·5 s particle A hits the ground without rebounding. Assuming that particle B does not hit the pulley, find the further time which elapses before the string is again taut. Find also the speed with which A leaves the ground.

16. Particles P and Q, of masses m and $2m$ respectively, are connected by a light inextensible string which passes over a smooth light pulley. The system is released from rest with the string taut and both particles at a height h above the ground.

Assuming that Q hits the ground without rebounding and that P does not hit the pulley, find expressions for (a) the greatest height above the ground reached by P, (b) the impulsive tension in the string when Q leaves the ground again.

17. A particle A, of mass 4 kg, rests on a smooth plane inclined at an angle of 30° to the horizontal. It is connected by a light inextensible string passing over a smooth pulley fixed at the top of the plane to a particle B, of mass 3 kg, which hangs freely. The system is released from rest with both parts of the string taut. When B has fallen a distance of 6·3 m A picks up a particle C, of mass 3·5 kg, which was previously at rest. Find the velocity of the system immediately after C has been picked up. Find also the loss in kinetic energy as C is picked up, giving your answer correct to 3 significant figures.

18. A pile of mass M is being driven into the ground by a pile-driver of mass m. The pile-driver is released from rest at a height h above the top of the pile and falls freely under gravity on to the top of the pile. The pile-driver rebounds vertically from the top of the pile and, after the first impact, it comes first to rest at a point $\frac{3}{4}h$ below the original point of release. (i) Find the kinetic energy lost by the pile-driver in the first impact. (ii) Find the kinetic energy gained by the pile in the first impact, assuming that the momentum of the system is conserved in the impact. (iii) Given that the pile is driven a distance $\frac{1}{16}h$ into the ground by the first impact, show that the frictional resistance, assumed constant, between the pile and the ground is $(36m^2 + M^2)g/M$. (C)

19. Two particles, A and B, of masses $2m$ and $3m$ respectively, are attached to the ends of a light inextensible string of length c and are placed close together on a horizontal table. The particle A is projected vertically upwards with speed $\sqrt{(6gc)}$. (i) Show that, at the instant immediately after the string tightens, B is moving with velocity $\frac{4}{5}\sqrt{(gc)}$. (ii) State the impulse of the tension in the string. (iii) Find the height to which A rises above the table before it comes to instantaneous rest. (iv) Calculate the loss in kinetic energy due to the tightening of the string. (AEB 1979)

20. A cube of mass 100 g and side 14 cm is at rest on a smooth horizontal table. A bullet of mass 20 g is fired with velocity 400 m s^{-1} into the centre of a vertical face of the cube. The bullet travels through the cube in a direction perpendicular to this face and emerges undeflected with velocity 200 m s^{-1}. Find the velocity of the cube as the bullet emerges. Find the kinetic energy gained by the cube and the kinetic energy lost by the bullet. If the cube moves a distance of d cm before the bullet emerges, write down the distance moved by the bullet as it passes through the cube. By considering the work done on the cube and on the bullet, determine the value of d and the magnitude of the constant force exerted by the cube on the bullet.

12.4 Newton's law of restitution

In a collision between two elastic bodies the impulses exerted by each body on the other produce changes in momentum and the bodies bounce away from each other.

If both bodies are moving along the line of action of these impulses at the instant the collision occurs, then the impact is said to be *direct*. Otherwise it is described as *indirect* or *oblique*. Both types of elastic impact were investigated by Sir Isaac Newton and as a result of his experimental work he was able to formulate a law of impact which is known as *Newton's law of restitution* or as *Newton's experimental law*. In its general form the law concerns the indirect impact of two moving bodies. However, we first consider its application to the direct impact of one body with a fixed plane.

Our everyday experience tells us that if several objects made of different materials are dropped from a given height on to horizontal ground, then the various objects will probably rebound to different heights after impact. A lump of plasticine will become flattened and show little tendency to rebound, whereas a "superball" will regain most of its original height. The behaviour of most objects will come somewhere between these two extremes.

Further observation shows that when a body strikes a fixed plane there is a period of time during which the body is compressed and its speed is reduced to zero. This is followed by a period of restitution in which the body regains its original shape and starts to rebound. While the body is being compressed some of its original kinetic energy is stored as elastic potential energy and this is then released as the body rebounds. The remaining kinetic energy is converted into other forms such as heat energy. Thus, the behaviour of a particular body during impact depends on its elasticity.

Experimental evidence suggests that in a series of direct collisions between a given elastic body a fixed plane the ratio of the speed of the body after impact to its speed before impact is approximately constant. Thus, in its simplest form, Newton's law of restitution states that if the speed of the body before impact is u and its speed after impact is v, then

$$v = eu,$$

where e is constant for that particular body. The constant e is known as the *coefficient of restitution* between the body and the plane and its value always lies between 0 and 1. If $e = 0$, collisions between the body and the plane are said to be *inelastic*, but if $e = 1$ they are described as *perfectly elastic*.

Example 1 A ball dropped from a height of 9 m above horizontal ground rebounds to a height of 4 m. Find the coefficient of restitution between the ball and the ground.

Substituting $u = 0, a = g, s = 9$ in the formula $v^2 = u^2 + 2as$,
we have: $v^2 = 0 + 2g \times 9$
$\therefore \quad v = 3\sqrt{(2g)}$

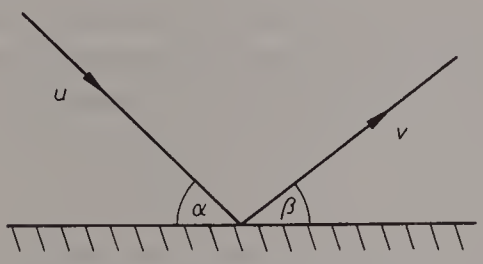
Hence the speed of the ball just before impact is $3\sqrt{(2g)}\,\mathrm{m\,s^{-1}}$.
Substituting $v = 0, a = -g, s = 4$ in the formula $v^2 = u^2 + 2as$,
we have: $0 = u^2 - 2g \times 4$
$\therefore \quad u = 2\sqrt{(2g)}$

Hence the speed of the ball just after impact is $2\sqrt{(2g)}\,\mathrm{m\,s^{-1}}$.

Thus, if e is the coefficient of restitution,

$$2\sqrt{(2g)} = e \times 3\sqrt{(2g)}$$
$$\therefore \quad e = \tfrac{2}{3}.$$

When considering indirect impacts between an elastic body and a smooth fixed plane, it is found that Newton's law of restitution can be applied to motion perpendicular to the plane.



Let us suppose that a body strikes a smooth fixed plane while travelling with speed u in a direction inclined at an angle α to the plane, then rebounds with speed v at an angle β to the plane.

Since the plane is smooth no force acts along the plane and therefore there is no change in the component of velocity along the plane

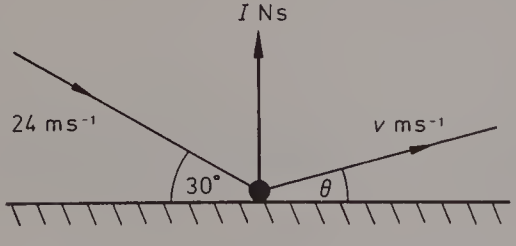
i.e. $$v\cos\beta = u\cos\alpha.$$

However, the impulse of the normal reaction of the plane changes the component of velocity perpendicular to the plane in accordance with Newton's law,

i.e. $$v\sin\beta = eu\sin\alpha,$$

where e is the coefficient of restitution.

Example 2 A particle of mass 2 kg strikes a fixed smooth plane while moving at a speed of $24\,\mathrm{m\,s^{-1}}$ in a direction making an angle of 30° with the plane. If the coefficient of restitution between the particle and the plane is $\frac{1}{4}$, find the speed at which the particle rebounds. Find also the impulse exerted by the plane on the particle.



Let the velocity of the particle after impact be $v\,\mathrm{m\,s^{-1}}$ at an angle θ to the plane and let the impulse exerted by the plane be I N s.

There is no change in velocity parallel to the plane

$$\therefore \quad v\cos\theta = 24\cos 30° = 12\sqrt{3}.$$

By Newton's law of restitution perpendicular to the plane,

$$v\sin\theta = \tfrac{1}{4} \times 24\sin 30° = 3$$

Squaring and adding: $v^2\cos^2\theta + v^2\sin^2\theta = (12\sqrt{3})^2 + 3^2$

$$\therefore \quad v = \sqrt{(432 + 9)} = 21.$$

Hence the speed of the particle after impact is $21\,\mathrm{m\,s^{-1}}$.

Considering the motion of the particle perpendicular to the plane,

impulse = change in momentum

$$\therefore \qquad I = 2 \times v\sin\theta + 2 \times 24\sin 30° = 30.$$

Hence the impulse exerted by the plane is 30 N s.

When applied to the direct impact of two moving bodies, Newton's law of restitution states that the ratio of the relative speed of separation after impact to the relative speed of approach before impact is constant.

Before impact ($u_1 > u_2$) After impact ($v_1 < v_2$)

Let us suppose that two spheres of masses m_1 and m_2 collide when moving along the same straight line with velocities u_1 and u_2 respectively. Let the velocities of the spheres after impact be v_1 and v_2 respectively. Provided that all velocities are measured in the same direction, as shown in the diagram, the relative speed of approach is $(u_1 - u_2)$ and the relative speed of separation is $(v_2 - v_1)$.

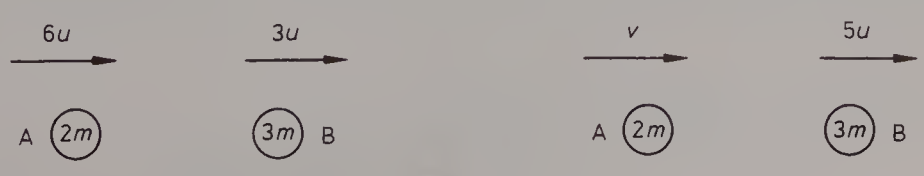
> By Newton's law of restitution
>
> $$v_1 - v_2 = -e(u_1 - u_2)$$
>
> where e is the coefficient of restitution.

Assuming that no external impulse acts during the impact, we can also apply the principle of conservation of momentum to give:

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

Example 3 A sphere A, of mass $2m$ and moving with speed $6u$, collides directly with a sphere B of mass $3m$ moving in the same direction with speed $3u$. After the impact the speed of B is $5u$. Find the coefficient of restitution between the spheres.

Before impact After impact



Let the speed of A after impact be v, as shown in the diagram.

Using the principle of conservation of momentum,

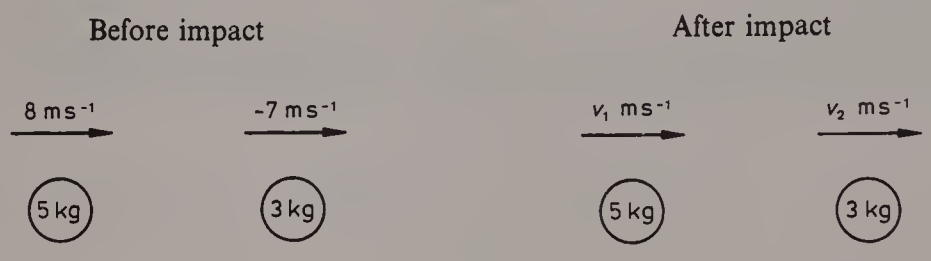
$$2m \times 6u + 3m \times 3u = 2m \times v + 3m \times 5u$$
$$\therefore \quad 12mu + 9mu = 2mv + 15mu$$
$$\therefore \quad v = 3u.$$

Using Newton's law of restitution,

$$v - 5u = -e(6u - 3u)$$
$$\therefore \quad 3u - 5u = -3eu$$
$$\therefore \quad e = \tfrac{2}{3}$$

Hence the coefficient of restitution between the spheres is $\frac{2}{3}$.

Example 4 A sphere of mass 5 kg, moving at $8\,\mathrm{m\,s^{-1}}$, collides directly with a sphere of mass 3 kg moving in the opposite direction at $7\,\mathrm{m\,s^{-1}}$. If the coefficient of restitution between the spheres is 0·6, find their speeds after impact. Find also the magnitude of the impulse exerted on each sphere during the collision.



Let the velocities of the spheres after impact be $v_1\,\mathrm{m\,s^{-1}}$ and $v_2\,\mathrm{m\,s^{-1}}$, as shown in the diagram.
Using conservation of momentum,

$$5 \times 8 - 3 \times 7 = 5v_1 + 3v_2$$
$$\therefore \quad 5v_1 + 3v_2 = 19 \qquad (1)$$

Using Newton's law of restitution,

$$v_1 - v_2 = -0{\cdot}6\{8 - (-7)\}$$
$$\therefore \quad v_1 - v_2 = -9 \qquad (2)$$

Adding 3 × (2) to (1): $8v_1 = -8$
Hence $v_1 = -1$ and $v_2 = 8$.

Thus after impact the speed of the 5 kg sphere is $1\,\mathrm{m\,s^{-1}}$ and the speed of the 3 kg sphere is $8\,\mathrm{m\,s^{-1}}$. The directions of both spheres are reversed in the collision.

Considering the motion of the 3 kg sphere,

impulse = change in momentum
$= 3 \times 8 - 3 \times (-7)\,\mathrm{N\,s} = 45\,\mathrm{N\,s}$

Hence the magnitude of the impulse exerted on each sphere is 45 N s.

Exercise 12.4

1. A ball of mass 0·2 kg is dropped from a point 10 m above horizontal ground. If the coefficient of restitution between the ball and the ground is $\frac{1}{2}$, find (a) the height above the ground to which the ball rebounds, (b) the loss of kinetic energy in the impact, (c) the impulse exerted by the ground.

2. A ball dropped from a height of 8 m above horizontal ground rebounds to a height of 4·5 m. Find the coefficient of restitution between the ball and the ground. If the ball is then dropped from a height of 12·8 m and allowed to bounce twice, find the maximum height it reaches after the second bounce.

3. A particle is projected vertically upwards with speed $7\,\mathrm{m\,s^{-1}}$ from a point A, which is 2 m above a horizontal plane. After striking this plane the particle first comes instantaneously to rest at A. Find the coefficient of restitution between the particle and the plane.

4. A particle of mass 3 kg strikes a fixed smooth plane while moving at a speed of $20\,\mathrm{m\,s^{-1}}$ in a direction making an angle α with the plane, where $\sin\alpha = 3/5$. If the coefficient of restitution between the particle and the plane is 2/3, find the speed at which the particle rebounds. Find also the impulse exerted by the plane on the particle.

5. A particle strikes a fixed smooth plane while travelling in a direction inclined at 45° to the plane. After impact its direction makes an angle of 30° with the plane. Find the coefficient of restitution between the particle and the plane. If on another occasion the particle is moving in a direction inclined at 60° to the plane, find its direction after impact with the plane.

6. A particle strikes a smooth fixed plane and rebounds in a direction at right angles to its direction just before impact. If the coefficient of restitution is $\frac{1}{3}$, find the acute angle between the initial direction of motion of the particle and the plane.

7. A particle is projected with speed $6{\cdot}5\,\mathrm{m\,s^{-1}}$ at an angle θ to the horizontal, where $\tan\theta = 12/5$, from a point A on a smooth horizontal plane. The particle strikes the plane again at B, then rebounds and strikes the plane for a second time at C. The coefficient of restitution between the particle and the plane is 5/9. Find the speed and direction with which the particle rebounds from the plane at B. Show that the distance AC is 100/21 m.

8. A sphere of mass $4m$ and moving with speed $5u$, collides directly with a sphere of mass $3m$, which is at rest. If the coefficient of restitution between the spheres is 0·4, find their speeds after impact. Find also the loss in kinetic energy due to the impact.

9. A sphere A, of mass 1 kg and moving with speed $12\,\mathrm{m\,s^{-1}}$, collides directly with a sphere B, of mass 4 kg, moving in the same direction with speed $2\,\mathrm{m\,s^{-1}}$. Given that sphere A is brought to rest by the collision, find the coefficient of restitution between the spheres.

10. A sphere moving with speed $7\,\mathrm{m\,s^{-1}}$ collides directly with an identical sphere moving in the opposite direction with speed $2\,\mathrm{m\,s^{-1}}$. Find the speeds of the spheres after impact if the coefficient of restitution is (a) $\frac{1}{3}$, (b) $\frac{2}{3}$. In each case state whether the spheres are travelling in the same direction or in opposite directions.

11. A sphere of mass m moving with speed $9u$ collides directly with a sphere of mass M moving in the same direction with speed $5u$. After impact the spheres move in the same direction with speeds $6u$ and $7u$. Find the coefficient of restitution between the spheres and the ratio $m:M$.

12. Three perfectly elastic particles A, B, C, with masses 4 kg, 2 kg, 3 kg respectively, lie at rest in a straight line on a smooth horizontal table. Particle A is projected towards B with speed $15\,\mathrm{m\,s^{-1}}$ and after A has collided with B, B collides with C. Find the velocities of the particles after the second collision and state whether there will be a third collision.

13. Three identical spheres A, B, C, lie at rest in a straight line on a smooth horizontal table. Particle A is projected towards B with speed u. After the collision between A and B, B moves on to hit C. The coefficient of restitution between any two of the spheres is $\frac{1}{2}$. Find the speeds of the spheres after the second collision and state whether there will be a third collision.

14. A sphere A of mass $2m$, moving on a smooth horizontal floor with speed $5u$, collides directly with a sphere B of mass m, which is at rest. If the coefficient of restitution between the spheres is 4/5, find the speed of B after impact. Sphere B goes on to strike a vertical wall, which is at right angles to the direction of motion of the spheres. A is then brought to rest by a second collision with B. Find the coefficient of restitution between B and the wall.

15. A small smooth sphere moves on a horizontal table and strikes an identical sphere lying at rest on the table at a distance d from a vertical wall, the impact being along the line of centres and perpendicular to the wall. Prove that the next impact between the spheres will take place at a distance $2de^2/(1 + e^2)$ from the wall, where e is the coefficient of restitution for all impacts involved. (L)

16. The coefficient of restitution between two small, smooth spheres A and B is e. The mass of A is m and of B is em. Show that, if A is at rest and B is projected with velocity u to collide directly with A, B is reduced to rest by the collision. If, instead, B is at rest and A is projected with the same speed to hit B directly, find the velocities of A and B after the collision. Show that the loss of kinetic energy is the same in each collision and is greatest when $e = \frac{1}{2}$. (SU)

17. Two small spheres of masses m and $2m$ are connected by a light inextensible string of length $2a$. When the string is taut and horizontal, its mid-point is fixed and the spheres are released from rest. The coefficient of restitution between the spheres is $\frac{1}{2}$. Show that the first impact brings the heavier sphere to rest, and the second impact brings the lighter sphere to rest. Find the velocity of each sphere immediately after the third impact. (L)

18. Three small smooth spheres A, B, C, of equal radius but of mass $m, 2m$ and m respectively, lie at rest and separated from one another on a smooth horizontal table in the order ABC with their centres in a straight line. Sphere A is projected with speed V directly towards sphere B. The coefficient of restitution at each collision is e, where $0 < e < 1$. Find, in terms of V and e, the speeds of the three spheres (a) after the first collision (between A and B), (b) after the second collision (between B and C). Hence show that there will be at least three collisions if $e^2 - 7e + 1 > 0$. (O&C)

Exercise 12.5 (miscellaneous)

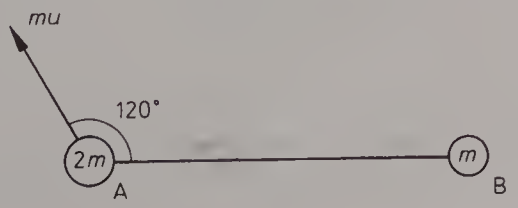
1. Two particles of masses $2m$ and $3m$ are moving towards each other with speeds $4u$ and $2u$ respectively. The direction of motion of the heavier particle is reversed by the impact and its speed after impact is u. Find (i) the magnitude of the impulse, (ii) the loss of energy. (C)

2. A particle of mass 4 kg moves under the action of a force $\mathbf{F}$ in the interval of time from $t = 0$ to $t = 2$. When $t = 0$ the velocity of the particle is $2\mathbf{i} - 3\mathbf{j}$. Given that $\mathbf{F} = 6t\mathbf{i} + 2\mathbf{j}$, find (a) the impulse of $\mathbf{F}$, (b) the velocity of the particle when $t = 2$, (c) the gain in kinetic energy.

3. A particle of mass m moves in a horizontal plane Oxy with speed v along the x-axis in the positive direction. It is subjected to a horizontal impulse $\mathbf{I}$ which turns its direction of motion through 30° in an anticlockwise sense and reduces its speed to $v/\sqrt{3}$. Find the vector $\mathbf{I}$. At the same instant an impulse $-\mathbf{I}$ is applied to a particle of mass $3m$ which is at rest. Find the magnitude and direction of the resultant velocity of this particle. (JMB)

4. A pump raises water from an underground reservoir through a height of 10 m and delivers it at a speed of $9\,\mathrm{m\,s^{-1}}$ through a circular pipe of internal diameter 20 cm. Taking 1 litre of water to have a mass of 1 kg and g to have the value $9{\cdot}8\,\mathrm{m\,s^{-2}}$ find (i) the mass of water raised per second, correct to 3 sig. fig.; (ii) the kinetic energy imparted to the water each second, correct to 3 sig. fig.; (iii) the effective power of the pump correct to 2 sig. fig.; (iv) the actual power of the pump if it is 70% efficient. If the water hits a vertical wall horizontally and does not rebound calculate the magnitude of the force exerted on the wall, correct to 2 sig. fig. (SU)

5. A light inextensible string AB has a particle of mass $2m$ attached at A and a particle of mass m attached at B. The particles are placed on a smooth horizontal table with the string taut. A horizontal impulse is applied to the particle at A of magnitude mu in a direction which makes an angle of 120° with AB. (i) Show that B



starts to move along BA with speed $u/6$. (ii) Find the components of the initial velocity of the particle at A along BA and perpendicular to BA and state the magnitude of the impulse of the tension in the string. (AEB 1977)

6. A composite particle of mass $m_1 + m_2$ is formed as a result of a head-on collision between two particles of masses m_1, m_2 travelling with speeds u_1, u_2 respectively. Find the speed of the composite particle in terms of m_1, m_2, u_1, u_2. Show that the collision leads to a loss of kinetic energy of amount

$$\frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right)(u_1 + u_2)^2. \qquad \text{(W)}$$

7. A gun of mass M fires a shell of mass m horizontally. The recoil of the gun is opposed by a constant force F which brings the gun to rest in t seconds. Show that the energy of the explosion is $\dfrac{F^2 t^2 (M + m)}{2Mm}$. (O)

8. A particle of mass m is dropped from rest from a point A which is at a height h above a horizontal surface. After hitting the surface the particle rebounds vertically and reaches a maximum height of $\frac{1}{2}h$ above the surface at the point B. Find expressions for the loss of energy and the impulse on the particle at the impact. At the instant of the particle's impact with the surface, a second particle, of mass $4m$, is dropped from rest at A. Show that the two particles collide at B. The two particles coalesce when they collide at B. When the combined particle hits the surface, one-half of its kinetic energy is lost in the impact. Show that, after the impact, the combined particle reaches a greatest height of $41h/100$ above the surface. (C)

9. Two beads, one of mass 3 g and the other of mass 2 g, are threaded on a smooth circular wire which is fixed with its plane vertical. The heavier bead is initially at rest at the lowest point L of the wire. The lighter bead is released from rest and is moving with speed $2\ \text{m s}^{-1}$ when it strikes the other bead. There is no loss of energy in the impact. Find the speed of each bead immediately after impact. In the subsequent motion each bead comes to rest instantaneously before reaching the highest point of the wire. Find, to the nearest millimetre, the height above L at which each bead comes first to rest. (C)

10. A pile-driver of mass 2000 kg is raised 1·6 m above a pile of mass 1500 kg and then dropped on to it. After impact the driver and the pile move together and the pile is driven 0·2 m into the ground. Find (i) the velocity of the pile-driver just before it hits the pile, (ii) the common velocity of pile and driver immediately after impact, (iii) the loss of energy on impact, stating the units in which it is measured, (iv) the mean resistance of the ground to penetration. (AEB 1980)

11. A particle A of mass $3m$ lies on a smooth horizontal rectangular table at a distance $5a$ from a point C at the edge of the table. The particle is attached to one end of a light inelastic string of length $6a$ which passes over a small smooth pulley at C.

To the other end of the string is attached a particle B of mass $2m$. The horizontal floor is at a depth $4a$ below C. The particle B is released from rest at C and motion takes place in a vertical plane perpendicular to the edge of the table. Show that the speed of the particles immediately after the string becomes taut is $\frac{2}{5}\sqrt{(2ga)}$. Find (i) the magnitude of the impulse which acts on the particle A, (ii) the loss of kinetic energy of the system due to the string becoming taut, (iii) the speed of the particle B just before it hits the floor. (C)

12. Two masses, one 3 kg and the other 2 kg, are connected by a light inextensible string which passes over a fixed smooth pulley and the system is released from rest. Find the acceleration of the system and show that the tension in the string is 23·52 N. After falling 2 metres the heavier mass strikes a fixed horizontal plane and is brought to rest. If the lighter mass does not reach the pulley, show that after 4/7 seconds the 3 kg mass is again in motion. Find the speed with which it begins to ascend and the impulsive tension in the string. (W)

13. Two particles A and B, each of mass m and connected by a light inextensible string of length $4a$, are resting on a smooth horizontal table with $AB = 2a$. A particle C of mass $3m$ is attached to the midpoint of the string and rests on the table so that $AC = BC$. The particles A and B are then simultaneously projected along the table with equal velocities perpendicular to AB and in the sense away from C. Show that, when the string tightens, the initial velocities of the particles A and B are each inclined at angle $\frac{1}{3}\pi$ with the line AB. (O)

14. A particle P of mass $2m$ is connected to a particle Q of mass $3m$ by means of a light inextensible string which passes over a small smooth pulley. The particles are released from rest with the string taut and its hanging parts vertical. (i) Calculate the acceleration of P and the tension in the string. At the instant when P is moving with velocity v it collides and coalesces with a stationary particle of mass $3m$ to form a particle P_1. (ii) Prove that the velocity of P_1 immediately after the collision is $5v/8$. (iii) Calculate the magnitude of the impulse of the tension in the string. (iv) Calculate the loss in kinetic energy of the whole system due to the collision.

The string breaks and particle Q hits a horizontal floor when moving with speed u and bounces to a height $u^2/(6g)$ above the floor. (v) Calculate the coefficient of restitution between Q and the floor. (AEB 1979)

15. A bullet of mass m travelling horizontally with speed u strikes perpendicularly a wooden block of mass M which is free to move horizontally in the line of motion of the bullet. The resistance to penetration is assumed uniform and the bullet comes to rest after penetrating a distance a into the block. Show that the ultimate common velocity of the system is $mu/(m + M)$ and that the total loss of kinetic energy during the penetration is $mMu^2/2(m + M)$. Deduce the value of the resistance and show that (i) the block moves a distance $am/(m + M)$ before the bullet comes to rest relative to the block, (ii) the penetration by the bullet ceases after a time $2a/u$. (W)

16. A particle of mass m is attached to one end of a light inextensible string of length a, the other end being attached to a fixed point O. Initially the string is taut and

makes an angle α ($\leqslant 90°$) with the downward vertical at O. The particle is released and, when it is at its lowest point A, coalesces with a stationary particle of mass m. The combined body next comes to rest with the string making an angle β with the downward vertical at O. Show that $1 - \cos\beta = \frac{1}{4}(1 - \cos\alpha)$.

The motion continues so that each time the body passes through A it coalesces with a stationary particle of mass m. If $\alpha = 90°$, (a) prove that the speed of the body just before passing through A for the fourth time is $\frac{1}{4}\sqrt{(2ag)}$, (b) find the angle (correct to the nearest degree) made by the string with the downward vertical at O when the body next comes to instantaneous rest after passing through A for the fourth time, (c) find the ratio of the tensions in the string just before and just after the body passes through A for the fourth time. (O&C)

17. Two equal particles are projected at the same instant from points A and B on horizontal ground, the first from A with speed u at an angle of elevation α and the second from B with speed v at an angle of elevation β. They collide directly when they are moving horizontally in opposite directions. Find v in terms of u, α and β, and show that $AB = \dfrac{u^2 \sin\alpha \sin(\alpha + \beta)}{g \sin\beta}$. The particles coalesce on impact. Find, in terms of u, α and β, the speed of the combined particle immediately after the collision. In the case when $\alpha = 30°$ and $\beta = 60°$ find AC, where C is the point where the combined particle returns to the ground. (JMB)

18. A ball is thrown vertically upwards from the floor with velocity V. It rebounds from the ceiling, which is at height h above the floor, and then rebounds from the floor. After this second rebound, it just reaches the ceiling again. If the coefficient of restitution between the ball and ceiling is e, and between the ball and floor is f, prove that

$$V^2 = \frac{2gh(1 - f^2 + e^2 f^2)}{e^2 f^2}. \qquad \text{(SU)}$$

19. A particle of mass m strikes a fixed smooth plane at speed u while moving in a direction inclined at an angle α to the plane, then rebounds at an angle β to the plane. Find expressions for (a) the coefficient of restitution between the particle and the plane, (b) the speed of the particle after impact, (c) the impulse exerted on the plane.

20. A particle of mass m, projected from a point A with speed u at an angle of elevation α, strikes a smooth vertical wall at a point B and after rebounding returns to A. Show that the total time taken to move from A to B and then back to A is $(2u \sin\alpha)/g$. Given that e is the coefficient of restitution between the particle and the wall, find expressions for (a) the loss of kinetic energy on impact, (b) the distance of A from the wall.

21. Two elastic spheres of masses m and M collide directly. Their velocities before impact are 12 m/s and 6 m/s respectively in the same direction, and after impact they are 5 m/s and 8 m/s respectively also in the same direction. Find the coefficient of restitution and also the ratio $m:M$. Find also the velocities of the spheres after

impact if they collide directly when travelling in opposite directions, the velocity of the heavier sphere being 4 m/s and that of the lighter sphere 14 m/s. (L)

22. Three small spheres A, B and C, of equal radii and masses m, $2m$ and $3m$ respectively, are placed with their centres in a straight line on a smooth horizontal table with B between A and C. The sphere A is given a velocity $5u$ in the direction $\overrightarrow{AB}$ and, as a result of the ensuing collision between A and B, B moves towards C with speed $3u$. Calculate (i) the magnitude and the sense of the velocity of A after the collision, (ii) the coefficient of restitution between A and B, (iii) the loss in kinetic energy due to the collision between A and B.

The sphere B strikes C and as a result C receives an impulse of $4mu$. Calculate (iv) the velocities of B and C after their collision, (v) the coefficient of restitution between B and C. (AEB 1978)

23. A small sphere A, of mass m moving with velocity $2u$ on a smooth horizontal plane, impinges directly on a small sphere B, of equal radius and mass $2m$ moving with velocity u in the same direction on the plane. Given that after impact B moves with velocity $3u/2$, calculate (i) the coefficient of restitution between A and B, (ii) the loss in kinetic energy due to the impact. The sphere B continues to move with velocity $3u/2$ until it hits a vertical wall from which it rebounds and is then brought to rest by a second impact with the approaching sphere A. Calculate (iii) the coefficient of restitution between B and the wall, (iv) the final speed of the sphere. (AEB 1980)

24. Particles A and B of mass m and $2m$ respectively lie at rest on a smooth horizontal plane. They are projected towards each other both with speed u. Show that, if the coefficient of restitution between the particles is $\frac{3}{4}$, their directions of motion are reversed on impact and find their speeds after impact. Find, also, the loss of kinetic energy during impact. If A subsequently hits a smooth vertical plane normally, show that A and B will collide again only if the coefficient of restitution between A and the vertical plane is greater than $1/8$. (SU)

25. Two particles A, B of masses M, m respectively are attached to a fixed point O by light inextensible strings of equal length. The particle A is held so that its string is taut and horizontal. It is then released so that it strikes the particle B which was hanging in equilibrium. Prove that if OB just reaches the horizontal then $Me = m$, where e is the coefficient of restitution between the particles. (O)

26. Two particles, A of mass $2m$ and B of mass m, moving on a smooth horizontal table in opposite directions with speeds $5u$ and $3u$ respectively, collide directly. Find their velocities after the collision in terms of u and the coefficient of restitution e. Show that the magnitude of the impulse exerted by B on A is $16mu(1 + e)/3$. Find the value of e for which the speed of B after the collision is $3u$. Moving at this speed B subsequently collides with a stationary particle C of mass km, and thereafter remains attached to C. Find the velocity of the combined particle and find the range of values of k for which a third collision will occur. (JMB)

27. Identical beads P and Q are threaded on a smooth circular hoop fixed in a horizontal plane. Initially Q is at rest at a point A of the hoop when P collides with it. Denoting the coefficient of restitution between P and Q by e, find the values of e for which the next collision takes place at A. (JMB)

13 Further particle dynamics

13.1 Acceleration as a function of displacement

Let us consider a particle moving along a straight line so that at time t its displacement from a fixed point on the line is s. The velocity v and the acceleration a of the particle at time t are given by:

$$v = \frac{ds}{dt} \quad \text{and} \quad a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

Suppose now that the acceleration a is regarded as a function of the displacement s, then, using the chain rule,

$$a = \frac{dv}{dt} = \frac{dv}{ds} \times \frac{ds}{dt} = \frac{dv}{ds} \times v = v\frac{dv}{ds}$$

$\therefore$ if $a = f(s)$, then $$v\frac{dv}{ds} = f(s),$$

giving $$\int v\,dv = \int f(s)\,ds$$

$\therefore$ $$\tfrac{1}{2}v^2 = \int f(s)\,ds$$

i.e. $$v^2 = 2\int f(s)\,ds$$

Thus if the acceleration a is a function of the displacement s, the relationship $a = v\dfrac{dv}{ds}$ may be used to obtain an expression for v^2 in terms of s.

Example 1 A particle of mass m moves in a straight line under the action of a force directed towards a fixed point O of the line. The magnitude of the force is $3ms^2$, where

s is the displacement of the particle from O at time t. Given that the particle starts from rest at a distance $2a$ from O, find the speed of the particle as it reaches O.

If v is the velocity of the particle at time t, then its acceleration is $v\dfrac{dv}{ds}$ in the direction of the displacement s. Since the force of magnitude $3ms^2$ acts in the opposite direction,

by Newton's second law, $\quad mv\dfrac{dv}{ds} = -3ms^2$

$$\therefore \qquad v\frac{dv}{ds} = -3s^2$$

$$\therefore \qquad \int v\,dv = -\int 3s^2\,ds$$

$$\tfrac{1}{2}v^2 = -s^3 + c$$

But $v = 0$ when $s = 2a$,

$\therefore \qquad 0 = -8a^3 + c \quad$ i.e. $\quad c = 8a^3$

Thus $\qquad v^2 = 2(8a^3 - s^3)$,

and when $s = 0$, $\qquad v^2 = 16a^3$.

Hence when the particle reaches O its speed is $4a^{3/2}$.

Example 2 Assuming that the earth is a sphere of radius R, the acceleration due to gravity of an object at a distance x from the centre of the earth is inversely proportional to x^2 when above the earth's surface and equal to g on the surface. Given that an object projected vertically upwards from the earth's surface with speed u rises to a maximum height R above the surface, find u in terms of g and R.

The acceleration of the object is inversely proportional to the square of the displacement x and in the opposite direction,

$$\therefore \qquad v\frac{dv}{dx} = -\frac{k}{x^2},$$

where v is the velocity and k is constant.
Since the acceleration is $-g$ when $x = R$,

$$-g = -\frac{k}{R^2} \quad \text{giving} \quad k = gR^2.$$

$$\therefore \qquad v\frac{dv}{dx} = -\frac{gR^2}{x^2}$$

$$\therefore \qquad \int v\,dv = -\int \frac{gR^2}{x^2}dx$$

$$\tfrac{1}{2}v^2 = \frac{gR^2}{x} + c$$

Given that $v = 0$ when $x = 2R$,

$$0 = \tfrac{1}{2}gR + c \quad \text{i.e.} \quad c = -\tfrac{1}{2}gR$$

$$\therefore \quad v^2 = \frac{2gR^2}{x} - gR$$

Since $v = u$ when $x = R$, $u^2 = \dfrac{2gR^2}{R} - gR = gR$

Hence $u = \sqrt{(gR)}$.

Exercise 13.1

In questions 1 to 4 a particle moves in a straight line so that after t seconds its displacement from a fixed point on the line is s metres and its velocity is $v\,\mathrm{m\,s^{-1}}$.

1. Given that the acceleration of the particle is $1/(1 + t)\,\mathrm{m\,s^{-2}}$ and that the particle starts from rest, find its speed after 2 seconds.

2. Given that the particle is subject to a retardation of $5e^{-t}\,\mathrm{m\,s^{-2}}$ and that its initial speed is $4\,\mathrm{m\,s^{-1}}$, find the time at which the particle first comes to rest.

3. Given that the acceleration of the particle is $2s\,\mathrm{m\,s^{-2}}$ and that $v = 2$ when $s = 0$, find the speed of the particle when $s = 4$.

4. Given that the particle is subject to a retardation of $2\sin^2 s\,\mathrm{m\,s^{-2}}$ and that $v = 1$ when $s = \pi/4$, find, correct to 3 significant figures, the speed of the particle when $s = \pi/3$.

5. A particle of mass m moving along a straight line is subject to a force $4mx$ acting towards a fixed point O on the line, where x is the displacement of the particle from O. Given that the particle starts from rest at a distance a from O, find the speed of the particle as it passes through O.

6. The acceleration due to gravity of an object at a distance x from the centre of the earth is inversely proportional to x^2 when above the earth's surface and equal to g on the surface. The earth is assumed to be a sphere of radius R. If an object falls from rest at a height $2R$ above the earth's surface, find its speed when it reaches a height R above the surface.

7. A particle of mass m is moving in a straight line under the action of a force $4m/x^3$ directed towards a fixed point O of the line, x being the displacement of the particle from O. Given that initially the particle is at a distance h from O, moving away from O at a speed of $2/h$, find expressions for (a) dx/dt in terms of x, and (b) x in terms of t and h.

8. A particle of mass m is moving in a straight line under the action of a force of magnitude $\frac{1}{4}e^{2x}$, where x is the displacement of the particle from a fixed

point O on the line. Initially the particle is at a point A travelling towards O at speed u. Given that the force on the particle acts in the direction of $\vec{OA}$ and that $OA = \ln(2u)$, find an expression for dx/dt in terms of x. Hence show that $x = \ln\left(\frac{2u}{1+ut}\right)$.

13.2 Acceleration as a function of velocity

We consider again a particle moving in a straight line so that at time t its displacement from a fixed point on the line is s and its velocity is v. If the acceleration of the particle is a function of the velocity, $f(v)$ say, then we may write:

$$(1)\ \frac{dv}{dt} = f(v), \quad \text{or} \quad (2)\ v\frac{dv}{ds} = f(v).$$

Equation (1) is used when a relationship between velocity and time is required. However, equation (2) may be used to obtain a relationship between velocity and displacement.

Example 1 A particle of mass m is projected vertically upwards from a point O with speed u and after time t its velocity is v. Given that the particle moves under gravity in a medium which exerts a resistance kv per unit mass, where k is a positive constant, find the time taken by the particle to reach its greatest height.

As the particle rises to its highest point the forces acting on it are its weight mg and the resistance mkv, both directed vertically downwards.

Thus, by Newton's second law, $\quad -mg - mkv = m\frac{dv}{dt}$

$$\therefore \qquad -(g + kv) = \frac{dv}{dt}$$

$$1 = -\frac{1}{(g + kv)}\frac{dv}{dt}$$

$$\therefore \qquad \int dt = -\int \frac{1}{g + kv}\,dv$$

$$t = -\frac{1}{k}\ln(g + kv) + c$$

Since $v = u$ when $t = 0$, $\quad c = \frac{1}{k}\ln(g + ku)$

$$\therefore \qquad t = -\frac{1}{k}\ln(g + kv) + \frac{1}{k}\ln(g + ku) = \frac{1}{k}\ln\left(\frac{g + ku}{g + kv}\right)$$

$$\therefore \quad \text{when } v = 0,\ t = \frac{1}{k}\ln\left(\frac{g + ku}{g}\right)$$

Hence the time taken by the particle to reach its greatest height is $\frac{1}{k}\ln\left(1 + \frac{ku}{g}\right)$.

Example 2 A particle of mass m moves in a straight line under the action of a retarding force $mv^4/2k^4$, where k is a positive constant and v is the velocity of the particle. Initially the particle is at a distance k^2 from a fixed point A, travelling away from A at speed k. Find the speed of the particle when its displacement from A is x. Find also the time which elapses before the particle reaches the point B, where $AB = 4k^2$.

Writing the acceleration of the particle in the form $v\,dv/dx$ and using Newton's second law we have:

$$-\frac{mv^4}{2k^4} = mv\frac{dv}{dx}$$

$$\therefore \quad \frac{1}{k^4} = -\frac{2}{v^3}\frac{dv}{dx}$$

$$\therefore \quad \int \frac{1}{k^4}\,dx = \int\left(-\frac{2}{v^3}\right)dv$$

Thus
$$\frac{x}{k^4} = \frac{1}{v^2} + c$$

Since $v = k$ when $x = k^2$, $\quad \frac{1}{k^2} = \frac{1}{k^2} + c \quad$ i.e. $\quad c = 0$

$$\therefore \quad \frac{x}{k^4} = \frac{1}{v^2}$$

$$v^2 = \frac{k^4}{x}$$

Hence when the displacement of the particle from A is x, its speed is $k^2/\sqrt{x}$.
$\therefore$ if x is the displacement from A at time t,

$$\frac{k^2}{\sqrt{x}} = \frac{dx}{dt}$$

$$\therefore \quad \int k^2\,dt = \int x^{1/2}\,dx$$

$$k^2t = \tfrac{2}{3}x^{3/2} + C$$

Since $x = k^2$ when $t = 0$, $\quad C = -\tfrac{2}{3}k^3$

$$\therefore \quad k^2t = \tfrac{2}{3}(x^{3/2} - k^3)$$

Thus when $x = 4k^2$, $\quad k^2t = \tfrac{2}{3}(8k^3 - k^3) = 14k^3/3$

Hence the time which elapses before the particle reaches B is $14k/3$.

Example 3 A car of mass m moving on a straight horizontal road is subject to a constant resistance R. The engine is working at a constant rate kR. Given that in time t the car accelerates to a speed v from rest, where $v < k$, find an expression for t in terms of m, R, k and v. Find also an expression for the distance x travelled by the car in this time.

If the tractive force exerted by the engine is F when the car is travelling at speed v, then its rate of working $= kR = Fv$,

$$\therefore \qquad F = kR/v$$

Using Newton's second law, the equation of motion is

$$\frac{kR}{v} - R = m\frac{dv}{dt}$$

$$\therefore \qquad R\left(\frac{k-v}{v}\right) = m\frac{dv}{dt} \tag{1}$$

$$\therefore \qquad \int R\,dt = \int m\left(\frac{v}{k-v}\right)dv$$

$$= \int m\left(\frac{k}{k-v} - 1\right)dv$$

$$\therefore \qquad Rt = m\{-k\ln(k-v) - v\} + c$$

Since $v = 0$ when $t = 0$, $\quad c = mk\ln k$

$$\therefore \qquad Rt = m\{-k\ln(k-v) - v + k\ln k\}$$

Hence $$t = \frac{m}{R}\left\{k\ln\left(\frac{k}{k-v}\right) - v\right\}$$

Substituting $\dfrac{dv}{dt} = v\dfrac{dv}{dx}$ in equation (1),

$$R\left(\frac{k-v}{v}\right) = mv\frac{dv}{dx}$$

$$\therefore \qquad \int R\,dx = \int m\left(\frac{v^2}{k-v}\right)dv$$

$$\begin{array}{r} -v-k \\ -v+k\,\overline{)\,v^2} \\ \underline{v^2 - kv} \\ kv \\ \underline{kv - k^2} \\ k^2 \end{array}$$

By long division

$$\frac{v^2}{k-v} = -v - k + \frac{k^2}{k-v}$$

$$\therefore \qquad \int R\,dx = \int m\left\{\frac{k^2}{k-v} - v - k\right\}dv$$

$$\therefore \qquad Rx = m\{-k^2\ln(k-v) - \tfrac{1}{2}v^2 - kv\} + C$$

Since $v = 0$ when $x = 0$, $\quad C = mk^2\ln k$

$$\therefore \qquad Rx = m\{-k^2\ln(k-v) - \tfrac{1}{2}v^2 - kv + k^2\ln k\}$$

Hence $$x = \frac{m}{R}\left\{k^2\ln\left(\frac{k}{k-v}\right) - \tfrac{1}{2}v^2 - kv\right\}.$$

In certain types of problem it is quicker to use definite integrals rather then indefinite integrals. For instance, if in Example 3 we had been asked to find the time which elapses as the car accelerates from speed $\frac{1}{3}k$ to speed $\frac{2}{3}k$, we could have deduced from equation (1) that:

$$\int_{t_1}^{t_2} R\,dt = \int_{k/3}^{2k/3} m\left(\frac{v}{k-v}\right)dv$$

where $t = t_1$ when $v = \frac{1}{3}k$ and $t = t_2$ when $v = \frac{2}{3}k$

$$\therefore \quad R\int_{t_1}^{t_2} dt = m\int_{k/3}^{2k/3}\left(\frac{k}{k-v} - 1\right)dv$$

$$\therefore \quad R\Big[t\Big]_{t_1}^{t_2} = m\Big[-k\ln(k-v) - v\Big]_{k/3}^{2k/3}$$

$$\therefore \quad R(t_2 - t_1) = m(-k\ln\tfrac{1}{3}k - \tfrac{2}{3}k) - m(-k\ln\tfrac{2}{3}k - \tfrac{1}{3}k)$$

$$= mk(\ln\tfrac{2}{3}k - \ln\tfrac{1}{3}k + \tfrac{1}{3} - \tfrac{2}{3})$$

$$= mk(\ln 2 - \tfrac{1}{3})$$

Hence the time taken is $\dfrac{mk}{R}(\ln 2 - \frac{1}{3})$.

Exercise 13.2

1. A particle of unit mass, moving in a straight line with speed v at time t, is subject to a retarding force of magnitude $(3 + 2v)$. Given that initially $v = 66$, find the value of t when $v = 6$.

2. A particle travelling in a straight line is subject to a retardation of $(1 + v^2)$, where v is its speed. If the initial speed of the particle is u, find an expression for the distance it travels before coming to rest.

3. A particle is moving in a straight line so that when the displacement of the particle from a fixed point on the line is x, its velocity is v. Given that the acceleration of the particle is $k(1 - v)$, where k is constant and that $x = 0$ when $v = 0$, find an expression for x in terms of v.

4. A particle is projected horizontally from a point O on a horizontal plane with an initial velocity $2a/b$ where a and b are positive constants. The particle is subject to a retardation $a + bv$ where v is the velocity at time t from the commencement of the motion. Show that $bv = a(3e^{-bt} - 1)$ and, using this result or otherwise, find the distance gone by the particle when it comes to instantaneous rest. (O&C)

5. The retardation of a particle moving in a straight line with speed v is proportional to v^3. The initial speed of the particle is u and s is the displacement of the particle from its initial position at time t. Given that $v = \frac{1}{2}u$ when $s = 2/u$, find an expression for v in terms of u and s. Hence find an expression for t in terms of u and s.

6. Show that the acceleration of an object moving along a straight line may be written as $v\,dv/ds$.

A vehicle of mass 2500 kg moving on a straight course is subject to a single resisting force in the line of motion of magnitude kv newtons, where v metres per second is the velocity and k is constant. At 100 km/h this force is 2000 N. The vehicle is slowed down from 100 km/h to 50 km/h. Find (i) the distance travelled, (ii) the time taken. (AEB 1980)

7. A particle falls from rest and, at any moment, it experiences a resistance of size kmv, where m is the mass of the particle, v is its velocity and k is a constant.

(i) Show that, from the equation of motion we can deduce

$$\int \frac{dv}{g - kv} = \int dt.$$

(ii) Using this equation, show that the velocity at any time is

$$\frac{g}{k}(1 - e^{-kt}).$$

(iii) Show that, no matter for how long the particle falls, its velocity never exceeds a certain value. If this value is V, show that k is g/V. (SU)

8. A particle of mass m moving under gravity is subject to a resistance of magnitude mkv^2, where v is its speed and k is constant. Initially the particle is travelling vertically downwards at speed u, where $u^2 < g/k$. Find the speed of the particle when it has fallen a distance s.

9. A train, when braking, is subject to a retardation of $\left(1 + \frac{v}{100}\right)\text{m s}^{-2}$, where v is its speed at any time. The brakes are applied when it is travelling at $20\,\text{m s}^{-1}$.

(i) Show that it takes just over 18 seconds to come to rest.

(ii) Show that it will travel approximately 177 m after the brakes are applied.
[It may prove useful to know that $\frac{v}{v + 100}$ can be written as $1 - \frac{100}{v + 100}$.]

(iii) If it then accelerates with acceleration $\left(1 + \frac{v^2}{100}\right)\text{m s}^{-2}$, find, correct to 3 significant figures, how long it takes to attain a speed of $20\,\text{m s}^{-1}$. (SU)

10. A particle of mass m moves, in the direction of x increasing, along the positive x-axis under the action of two forces. These are (i) a force toward the origin O of magnitude kv, and (ii) a force away from O of magnitude kU^2/v, where k and U are positive constants and v is the speed of the particle at time t. Write down the equation of motion of the particle in the form of a differential equation in v and t, and show that the time taken for the particle to accelerate from speed $\frac{1}{4}U$ to speed $\frac{1}{2}U$ is $\frac{m}{2k}\ln\frac{5}{4}$.

Write the equation of motion in the form of a differential equation in v and x, and hence show that the distance travelled by the particle while it accelerates from speed $\frac{1}{4}U$ to speed $\frac{1}{2}U$ is given by

$$\frac{m}{k}\int_{U/4}^{U/2}\left(\frac{U^2}{U^2-v^2}-1\right)dv.$$

Show that this distance is $\frac{mU}{4k}\left(2\ln\frac{9}{5}-1\right)$. (C)

11. A particle of mass m moves in a straight line under the action of a retarding force of magnitude $k(4+v^2)$, where v is the speed of the particle at time t and k is constant. If the initial speed of the particle is u, find an expression of the time taken by the particle to come to rest. Find also an expression, in terms of m, k and u, for the distance travelled in this time.

12. An engine of mass m kilograms, driven by a constant tractive force of P newtons, experiences a resistance which is k times its speed in metres/second. Show that, when moving horizontally from rest, its speed t seconds after the start is $\frac{P}{k}(1-e^{-kt/m})$ metres/second. Hence deduce that it will have travelled mP/ek^2 metres in m/k seconds. What power is being developed by the engine when the time is m/k seconds after the start? (W)

13. A train is pulled along a level track by an engine which exerts a constant pull P newtons at all speeds and the total resistance to motion varies as the square of the speed $v\,\mathrm{m\,s^{-1}}$. The mass of the engine and train combined is 161 700 kg, the maximum speed attained is $120\,\mathrm{km\,h^{-1}}$ and the power then developed is 1078 kW. Show that (a) $P = 32\,340$, (b) $5\times10^4 v\frac{dv}{ds}=10^4-9v^2$, where s metres is the distance travelled. Hence show that the distance travelled from rest while attaining a speed of $20\,\mathrm{m\,s^{-1}}$ is about 1240 metres. (C)

14. A train of mass M moves on a straight horizontal track. At speeds less than V the resultant force on the train is constant and equal to P; at speeds not less than V the rate of working of the force is constant and equal to PV. Show that a speed $v(>V)$ is attained from rest in a time $M(V^2+v^2)/2PV$ and that the distance travelled in this time is $M(V^3+2v^3)/6PV$. (W)

15. A car of mass m moving on a level road is subject to a constant resistance R. The engine is working at a constant rate H. Find the time taken for the speed of the car to increase from u to $2u$, where $2Ru < H$. Find also, in terms of m, R, H and u, the distance travelled in this time.

16. A car of mass m moving on a level road is subject to a resisting force kv, where v is the speed of the car at time t and k is constant. When the engine of the car is working at full power S, the car can travel at a steady speed U. Find the

time taken by the car to accelerate under full power from rest to speed V, where $V < U$. Find also the distance travelled in this time. Give both answers in terms of m, S, U and V.

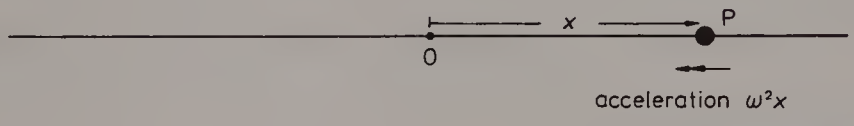
13.3 Simple harmonic motion

Any motion which satisfies a differential equation of the form

$$\ddot{x} = -\omega^2 x$$

where ω is constant, is described as *simple harmonic motion* or S.H.M.

For a particle moving in a straight line, the variable x is the displacement of the particle from a fixed point on the line. However, x may also be a displacement measured along a curved path. In the case of circular motion or the rotation of a rigid body x could represent an angle.



The diagram shows a particle P executing simple harmonic motion along a straight line. If x is the displacement of P from a fixed point O and $\ddot{x} = -\omega^2 x$, then the magnitude of the acceleration of P is a constant multiple of the distance OP. Since when $x > 0, \ddot{x} < 0$ and when $x < 0, \ddot{x} > 0$, the acceleration is always directed towards O. This leads to an alternative definition of simple harmonic motion.

When a particle moves in a straight line with acceleration always directed towards a fixed point of the line and proportional to its distance from that point, the particle is said to be executing simple harmonic motion.

The differential equation $\ddot{x} = -\omega^2 x$, i.e. $\dfrac{d^2x}{dt^2} = -\omega^2 x$, can be solved by expressing the acceleration of the particle in the form $v\,dv/dx$, where v is the velocity at time t, to give:

$$v\frac{dv}{dx} = -\omega^2 x$$

$$\therefore \quad \int 2v\,dv = -\int 2\omega^2 x\,dx$$

$$v^2 = -\omega^2 x^2 + c$$

If $v = 0$ when $x = a$, we find that

$$v^2 = \omega^2(a^2 - x^2) \tag{1}$$

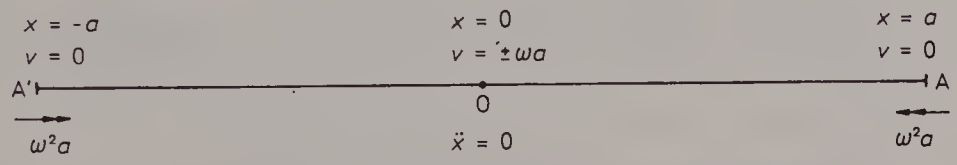
It can be shown that this leads to the general solution:

$$x = a\sin(\omega t + \varepsilon) \tag{2}$$

or the equivalent form:

$$x = A\cos\omega t + B\sin\omega t \tag{3}$$

From equations (1) and (2) we deduce that the particle is oscillating between points A and A', which lie on opposite sides of O at a distance a from O. The point O is called the *centre* or *central point* of the oscillation and a is the *amplitude.*



From equation (1) we see that the particle is momentarily at rest when it reaches A or A', and achieves its maximum speed ωa when passing through O.

Since $x = A\cos\omega t + B\sin\omega t$
and $v = -A\omega\sin\omega t + B\omega\cos\omega t,$

when ωt is increased to $\omega t + 2\pi$ the same values of x and v are obtained. This means that at times t and $t + 2\pi/\omega$ the particle passes through the same point with the same velocity. Hence $2\pi/\omega$ is the time taken to perform a complete oscillation and is called the *period* of the motion. In particular, $2\pi/\omega$ is the time taken by the particle to move from A to A' and back again to A.

The *frequency*, i.e. the number of complete oscillations per unit time, is $\omega/2\pi$. Summarising the properties of S.H.M. we have:

$\ddot{x} = -\omega^2 x,$
$v^2 = \omega^2(a^2 - x^2)$, where a is the amplitude,
$x = a\sin(\omega t + \varepsilon)$, or
$x = A\cos\omega t + B\sin\omega t,$
$T = 2\pi/\omega$, where T is the period of oscillations.

When solving problems involving S.H.M. standard formulae may be quoted without proof. It is usually best to begin be determining a and ω.

Example 1 A particle is performing simple harmonic oscillations between points A and A' which are 6 m apart. When the particle is at a distance $\sqrt{5}$ m from the mid-point O of AA' its speed is 10 m s^{-1}. Find (a) the period of the motion, (b) the maximum speed of the particle, (c) its maximum acceleration.

Let the equation of motion of the particle be

$$\ddot{x} = -\omega^2 x,$$

then

$$v^2 = \omega^2(a^2 - x^2),$$

where x m is the displacement of the particle from O, v m s^{-1} is its speed and a m is the amplitude of the motion.
But $AA' = 6$ m $= 2a$ m, which gives $a = 3$

$$\therefore \quad v^2 = \omega^2(9 - x^2)$$

Since $v = 10$ when $x = \sqrt{5}$, $100 = \omega^2(9 - 5)$ $\quad \therefore \quad \omega = 5$

(a) The period of the motion $= \dfrac{2\pi}{\omega}$ seconds $= \dfrac{2\pi}{5}$ seconds.

(b) The speed of the particle is greatest when $x = 0$
$\therefore$ the maximum speed $= \omega a$ m s^{-1} $= 15$ m s^{-1}.

(c) The acceleration of the particle is greatest when the magnitude of $\omega^2 x$ is greatest,
$\therefore$ the maximum acceleration $= \omega^2 a$ m s^{-2} $= 75$ m s^{-2}.

In S.H.M. problems involving time we need to find suitable expressions for x and v in terms of t. The forms of these expressions depend on the starting point chosen for the motion. One way of approaching such problems is to write:

$$x = A\cos\omega t + B\sin\omega t, \tag{3}$$
$$v = -A\omega\sin\omega t + B\omega\cos\omega t, \tag{4}$$

then to use any given conditions to determine A and B. We first apply this method to two important special cases.

Case I: $x = a$ when $t = 0$.

Substituting in (3): $a = A$
Since $v = 0$ when $x = a$, from (4): $B = 0$

$\therefore$ $\boxed{x = a \quad \text{when} \quad t = 0 \;\Rightarrow\; x = a\cos\omega t.}$

Case II: $x = 0$, $v > 0$ when $t = 0$.

Substituting in (3): $0 = A$
$\therefore$ $x = B\sin\omega t$
Since the amplitude of the motion is a and $v > 0$ when $t = 0$, we must have $B = a$.

$\therefore$ $\boxed{x = 0, v > 0 \quad \text{when} \quad t = 0 \;\Rightarrow\; x = a\sin\omega t.}$

Example 2 A particle moving in a straight line performs simple harmonic oscillations about a point O with amplitude 2 m and maximum speed 1 m s^{-1}. If P

and Q are the two points on the line which lie at a distance $\sqrt{2}$ m from O, find the time taken by the particle to move directly from P to Q.

Let the equation of motion be $\ddot{x} = -\omega^2 x$, then, assuming that $x = a$ when $t = 0$, $x = a\cos\omega t$.
Since the amplitude of the motion is 2 m, $a = 2$.
The maximum speed of the particle $= \omega a \text{ m s}^{-1} = 1 \text{ m s}^{-1}$

$\therefore \quad \omega = \frac{1}{2}$
Thus $x = 2\cos\frac{1}{2}t$.

Let P and Q be the points at which $x = \sqrt{2}$ and $x = -\sqrt{2}$ respectively.

For $x = \sqrt{2}$: $\sqrt{2} = 2\cos\frac{1}{2}t$
i.e. $\cos\frac{1}{2}t = 1/\sqrt{2}$

$\therefore$ the particle first passes through P when $\frac{1}{2}t = \frac{1}{4}\pi$ i.e. when $t = \frac{1}{2}\pi$.

For $x = -\sqrt{2}$: $-\sqrt{2} = 2\cos\frac{1}{2}t$
i.e. $\cos\frac{1}{2}t = -1/\sqrt{2}$

$\therefore$ the particle first passes through Q when $\frac{1}{2}t = \frac{3}{4}\pi$ i.e. when $t = \frac{3}{2}\pi$.
Hence the particle moves directly from P to Q in a time of π seconds.

Example 3 A particle is performing simple harmonic motion about a point O with amplitude 5 m and period $\frac{1}{2}\pi$ seconds. If P is the point at which the speed of the particle is 10 m s^{-1}, find the time taken by the particle to move directly from O to P.

Let the equation of motion of the particle be $\ddot{x} = -\omega^2 x$, and let $x = a\sin\omega t$, so that the particle is at O when $t = 0$.
Since the amplitude of the motion is 5 m, $a = 5$.
The period of oscillation $= \dfrac{2\pi}{\omega}\text{ s} = \dfrac{\pi}{2}\text{ s}, \qquad \therefore \quad \omega = 4.$

Thus $x = 5\sin 4t$ and $v = 20\cos 4t$.

Substituting $v = 10$: $10 = 20\cos 4t$
$\therefore$ $\cos 4t = \frac{1}{2}$

$\therefore$ the particle first passes through P when $4t = \pi/3$. Hence the particle moves directly from O to P in a time of $\pi/12$ seconds.

Example 4 A particle moves on a straight line through a fixed point O so that at time t seconds its displacement from O is x metres and its equation of motion is $\ddot{x} = -9x$. Given that $x = 5$ and $\dot{x} = -6$ when $t = \pi/6$, find the position and speed of the particle when $t = 2\pi/3$.

The general solution of the equation of motion is

$$x = A\cos 3t + B\sin 3t$$
$$\therefore \quad \dot{x} = -3A\sin 3t + 3B\cos 3t$$

Since $x = 5$ and $\dot{x} = -6$ when $t = \pi/6$,

$$5 = A\cos\tfrac{1}{2}\pi + B\sin\tfrac{1}{2}\pi$$

i.e. $5 = B$

and $-6 = -3A\sin\tfrac{1}{2}\pi + 3B\cos\tfrac{1}{2}\pi$

i.e. $-6 = -3A$

$\therefore$ $A = 2$ and $B = 5$

Thus $x = 2\cos 3t + 5\sin 3t$,

$\dot{x} = -6\sin 3t + 15\cos 3t$.

$\therefore$ when $t = 2\pi/3$, $x = 2\cos 2\pi + 5\sin 2\pi = 2$

and $\dot{x} = -6\sin 2\pi + 15\cos 2\pi = 15$

i.e. the particle is 2 m from O moving at $15\,\mathrm{m\,s^{-1}}$.

Exercise 13.3

[Standard formulae may be quoted without proof.]

1. A particle moves in a straight line with simple harmonic motion of amplitude 2·5 m. If the period of oscillation is π seconds, find the maximum speed of the particle and its maximum acceleration.

2. A particle moves in a straight line with simple harmonic motion about the point O as centre. The maximum speed of the particle is $6\,\mathrm{m\,s^{-1}}$ and its maximum acceleration is $18\,\mathrm{m\,s^{-2}}$. Find (a) the amplitude of the motion, (b) the period of the motion, (c) the speed of the particle when it is 1 m from O.

3. A particle of mass 3 kg moves in a straight line with simple harmonic motion between two points A and A' which are 15 m apart. Given that when the particle is at a distance of 3 m from A its speed is $2\,\mathrm{m\,s^{-1}}$, find the period of the motion. Find also the greatest force exerted on the particle during the motion.

4. A mass of 10 kg moves with simple harmonic motion. When it is 2 m from the centre of the oscillation, the velocity and acceleration of the body are $12\,\mathrm{m\,s^{-1}}$ and $162\,\mathrm{m\,s^{-2}}$ respectively. Calculate (i) the number of oscillations per minute; (ii) the amplitude of the motion; (iii) the force being applied to the body when it is at the extremities of its motion. (SU)

5. A particle P moves in a straight line so that its acceleration is always directed towards a point O in its path and is of magnitude proportional to the distance OP. When P is at the point A, where $OA = 1$ m, its speed is $3\sqrt{3}$ m/s and when P is at the point B, where $OB = \sqrt{3}$ m, its speed is 3 m/s. Calculate the maximum speed attained by P and the maximum value of OP. Show that P takes $\pi/18$ seconds to move directly from A to B. Find, in m/s correct to 2 significant figures, the speed of P one second after it passes O. (L)

6. A particle, A, is performing simple harmonic oscillations about a point O with amplitude 2 m and period 12π s. Find the least time from the instant when A passes through O until the instant when (i) its displacement is 1 m, (ii) its velocity is half that at O, (iii) its kinetic energy is half that at O. (AEB 1978)

7. A particle P is describing simple harmonic motion in the horizontal line $ADCB$, where $AD = DC = \frac{1}{2}CB$. The speed of P as it passes through C is 5 m/s and P is instantaneously at rest at A and B. Given that P performs 3 complete oscillations per second, calculate (i) the distance AB, (ii) the speed of P as it passes through D, (iii) the distance of P from C at an instant when the acceleration of P is 18π m/s^2, (iv) the time taken by P to go directly from D to A. (AEB 1978)

8. A particle P moves in a straight line with simple harmonic motion of period 2 seconds and maximum speed 4 m/s. Find the speed of P when it is at the point A which is $2/\pi$ metres from O, the centre of the path. Find also the time taken by P to move directly from O to A. When P is passing through O it strikes and adheres to a stationary particle which is free to move. If each particle is of mass 2 kg, find the kinetic energy lost in the collision. (L)

9. A particle moves in a straight line so that at time t its displacement from a fixed point on the line is x and its equation of motion is $\ddot{x} = -4x$. Given that $x = 3$ and $\dot{x} = -6$ when $t = \pi/4$, find (a) x in terms of t, (b) the values of x and $\dot{x}$ when $t = 3\pi/4$, (c) the least positive value of t for which $x = 0$.

10. A particle moves in a straight line so that at time t its displacement from a fixed point on the line is x and its equation of motion is $\ddot{x} = -16x$. Given that $x = 3$ and $\dot{x} = 4\sqrt{3}$ when $t = 0$, find (a) x in terms of t, (b) the least positive value of t for which $\dot{x} = 0$, (c) the amplitude of the oscillation.

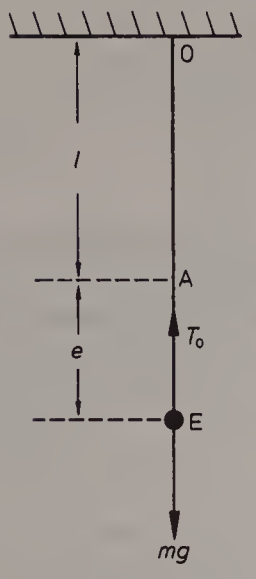
11. A particle P moves round a circle, diameter AB, with constant angular speed ω. A second particle Q moves along AB so that PQ is always perpendicular to AB, i.e. Q is the projection of P on AB. Show that the motion of Q is simple harmonic.

13.4 Forces producing simple harmonic motion

To show that a given set of forces acting on a particle produce simple harmonic motion we must prove that the equation of motion of the particle can be expressed in the form $\ddot{x} = -\omega^2 x$.

Since the tension in an elastic string is proportional to the extension in the string, it may be possible for a particle attached to an elastic string or a spring to perform simple harmonic motion.

Let us suppose that a light elastic string of natural length l and modulus of elasticity λ has one end fastened to a fixed point O. A particle of mass m is



attached to the other end. Let e be the extension in the string and T_0 the tension when the particle hangs in equilibrium at the point E.

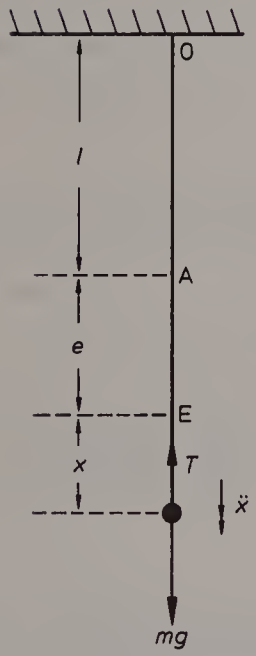
By Hooke's law, $T_0 = \dfrac{\lambda e}{l}$

Since the particle is in equilibrium,

$$T_0 - mg = 0$$

$$\therefore \quad \frac{\lambda e}{l} = mg$$

Consider now the forces on the particle when its displacement from E is x vertically downwards.



By Hooke's law, $T = \dfrac{\lambda(x+e)}{l}$

where T is the tension in the string. Applying Newton's second law vertically downwards:

$$mg - T = m\ddot{x}$$

$$\therefore \quad \frac{\lambda e}{l} - \frac{\lambda(x+e)}{l} = m\ddot{x}$$

$$\therefore \quad -\frac{\lambda x}{l} = m\ddot{x}$$

Hence the equation of motion of the particle is $\ddot{x} = -\dfrac{\lambda}{lm}x$. Since this equation is of the form $\ddot{x} = -\omega^2 x$, where $\omega = \sqrt{(\lambda/lm)}$, the particle can perform simple harmonic motion with centre E and period $2\pi\sqrt{(lm/\lambda)}$. The amplitude a of the motion is the maximum distance below E reached by the particle.

Provided that $a < e$ the string will remain taut throughout the motion and complete simple harmonic oscillations will be possible. However, if $a > e$ the particle will perform simple harmonic motion when it is below A, but move freely under gravity when the string becomes slack.

[Note that in the case of a particle suspended from a spring of natural length l, complete S.H.M. involving both extension and compression of the spring is theoretically possible when $a > e$.]

Example 1 A light elastic string of natural length l and modulus mg has one end fastened to a fixed point O. The other end of the string is attached to a particle of

mass m which hangs in equilibrium at a point E. Find the distance OE. The particle is now pulled down to a point A at a distance $5l/2$ vertically below O. If at time $t = 0$ the particle is released from rest at A, show that the subsequent motion is simple harmonic and find the speed of the particle as it passes through E. Find also the time at which the particle first passes through the point B which lies at a distance $7l/4$ vertically below O.

When the particle is at E let the tension in the string be T_0 and the extension e.

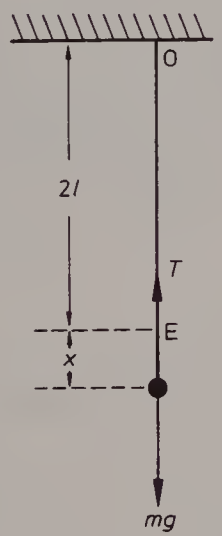
By Hooke's law: $T_0 = \dfrac{mg \times e}{l}$

But since the particle is in equilibrium, $T_0 = mg$

$$\therefore \quad \frac{mg \times e}{l} = mg$$

$$e = l$$

Hence $OE = l + e = 2l$.



Let T be the tension in the string when the displacement of the particle from E is x vertically downwards.

By Hooke's law: $T = \dfrac{mg(x + l)}{l}$

Applying Newton's second law vertically downwards:

$$mg - T = m\ddot{x}$$

$$\therefore \quad mg - \frac{mg(x + l)}{l} = m\ddot{x}$$

$$\therefore \quad \ddot{x} = -\frac{g}{l}x$$

Since this equation is of the form $\ddot{x} = -\omega^2 x$, where $\omega = \sqrt{(g/l)}$, the particle performs simple harmonic motion with centre E.
The speed v of the particle is given by the formula

$$v^2 = \omega^2(a^2 - x^2)$$

When the particle is at A, $x = \frac{1}{2}l$ and $v = 0$ $\quad \therefore \quad a = \frac{1}{2}l$

Hence $$v^2 = \frac{g}{l}(\tfrac{1}{4}l^2 - x^2).$$

$$\therefore \quad \text{when } x = 0, v^2 = \frac{g}{l} \times \frac{l^2}{4} = \frac{gl}{4}$$

Hence as the particle passes through E its speed is $\frac{1}{2}\sqrt{(gl)}$.

Since $x = \frac{1}{2}l$ and $v = 0$ when $t = 0$,

$$x = \tfrac{1}{2}l\cos\omega t, \quad \text{where} \quad \omega = \sqrt{(g/l)}.$$

When $x = -\frac{1}{4}l$, $\quad -\frac{1}{4}l = \frac{1}{2}l\cos\omega t$

i.e. $\quad \cos\omega t = -\frac{1}{2}$

$\therefore$ the particle first passes through B when $\omega t = 2\pi/3$

i.e. when $t = \dfrac{2\pi}{3}\sqrt{\left(\dfrac{l}{g}\right)}$.

We include the following example as a reminder that many problems involving particles attached to elastic strings can be solved using energy considerations rather than the theory of simple harmonic motion.

Example 2 An elastic string of natural length l and modulus $4mg$ has one end fastened to a fixed point O. The other end of the string is attached to a particle of mass m. If the particle is released from rest at O, find the distance it falls before coming instantaneously to rest at a point A.

Let x be the extension in the string when the particle reaches A. The kinetic energy of the particle is zero when it is released from O and when it comes to rest at A.
Since $OA = l + x$, the loss in gravitational potential energy is $mg(l + x)$.
The energy stored in the elastic string as it is stretched to the length $(l + x)$ is $4mgx^2/2l$ i.e. $2mgx^2/l$.
Thus, using the principle of conservation of mechanical energy,

$$\frac{2mgx^2}{l} = mg(l + x)$$

$\therefore \quad 2x^2 = l(l + x)$

$$2x^2 - lx - l^2 = 0$$

$$(x - l)(2x + l) = 0$$

Since $x > 0$, we must have $x = l$.
Hence the particle falls a distance $2l$ before coming to rest.

In the next example we consider oscillations in a horizontal plane.

Example 3 A particle P of mass m lies on a smooth horizontal table and is attached to two fixed points A, B on the table by two elastic strings each of natural length l. The strings AP, PB have moduli $2mg$ and mg respectively and $AB = 5l$. If E is the equilibrium position of the particle, find AE. Given that the particle is released from rest at the point C on AB such that $AC = l$, show that it performs simple harmonic motion, stating the period and the amplitude of the oscillations.

Let T be the tension in both strings when the particle is in equilibrium at E.
If $AE = l + e$, then the extensions in the strings AP and PB are e and $3l - e$ respectively.

By Hooke's law for AP: $T = \dfrac{2mge}{l}$

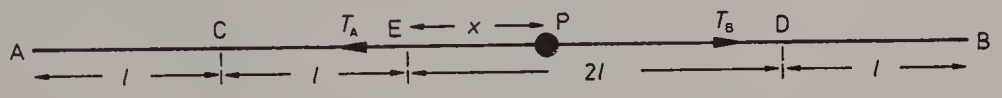
By Hooke's law for BP: $T = \dfrac{mg(3l - e)}{l}$

$$\therefore \quad \frac{2mge}{l} = \frac{mg(3l - e)}{l}$$

$$2e = 3l - e$$

$$\therefore \quad e = l$$

Hence $AE = l + e = 2l$.



Suppose now that the displacement of the particle P from E is x, as shown in the diagram.
Assuming that both strings are taut, i.e. that P lies between the points C and D, the extension in AP is $(l + x)$ and the extension in PB is $(2l - x)$.
Let T_A and T_B be the tensions in the strings AP and PB respectively, then applying Hooke's law,

$$T_A = \frac{2mg(l + x)}{l}, \quad T_B = \frac{mg(2l - x)}{l}$$

Using Newton's second law in the direction AB,

$$T_B - T_A = m\ddot{x}$$

$$\therefore \quad \frac{mg(2l - x)}{l} - \frac{2mg(l + x)}{l} = m\ddot{x}$$

$$-\frac{3mgx}{l} = m\ddot{x}$$

$$\therefore \quad \ddot{x} = -\frac{3g}{l}x$$

Thus the equation of motion of the particle is of the form $\ddot{x} = -\omega^2 x$, where $\omega = \sqrt{(3g/l)}$. Hence when both strings are taut the particle performs simple harmonic motion with centre E and period $2\pi\sqrt{(l/3g)}$.
Since the particle starts from rest at C, where $CE = l$, the amplitude of the simple harmonic motion is l.
The diagram shows that complete oscillations of this amplitude are possible with both strings taut. Thus when the particle is released at C it performs simple harmonic motion of period $2\pi\sqrt{(l/3g)}$ and amplitude l.

These examples illustrate the fact that in simple harmonic motion the centre of oscillation is an equilibrium position. If a displacement x from this equilibrium position is considered, then the equation of motion is obtained in the standard form

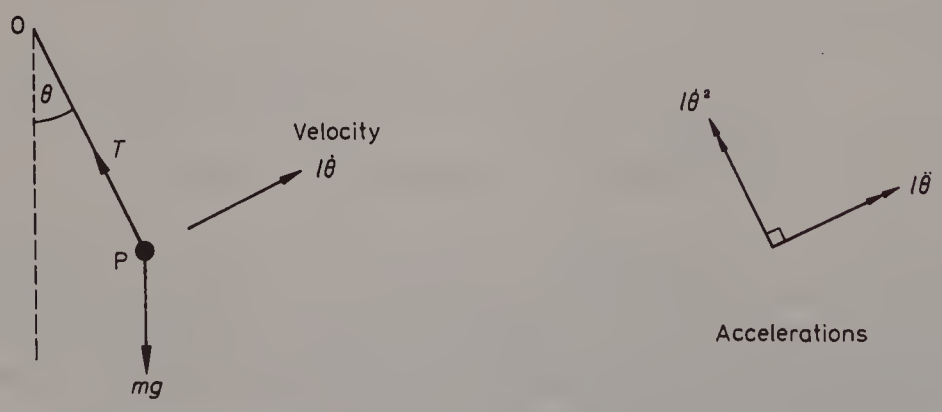
$$\ddot{x} = -\omega^2 x.$$

However, if a displacement from a point which is not an equilibrium position is used, then the simple harmonic motion equation takes the more general form

$$\ddot{x} + \omega^2 x = \text{constant}.$$

Finally, we show that the motion of a *simple pendulum* is approximately simple harmonic. A simple pendulum consists of a particle, known as the "bob", suspended from a fixed point by a light inextensible string. When the particle swings in a vertical plane with the string taut, the bob moves along a circular arc.

It was shown in Chapter 8 that when a particle moves in a circle with variable speed v, then the acceleration of the particle has components v^2/r towards the centre of the circle and dv/dt in the direction of motion. These results can now be used to find the equation of motion of a simple pendulum.



Let the bob of the pendulum be a particle P of mass m and let the length of the string be l. At the instant when the string makes an angle θ with the downward vertical, P is moving perpendicular to the string with velocity $l\dot{\theta}$. Hence the acceleration of P has components $l\dot{\theta}^2$ and $l\ddot{\theta}$ along and perpendicular to the string.

Applying Newton's second law perpendicular to the string,

$$mg \sin\theta = -ml\ddot{\theta}$$

Hence
$$\ddot{\theta} = -\frac{g}{l}\sin\theta$$

However, if θ is small, $\sin\theta \approx \theta$

$\therefore$
$$\ddot{\theta} \approx -\frac{g}{l}\theta$$

Thus small oscillations of a simple pendulum are approximately simple harmonic. The period of complete oscillations is $2\pi\sqrt{\frac{l}{g}}$.

A pendulum which beats seconds is called a *seconds pendulum*. Such a pendulum swings through its equilibrium position once every second and therefore each complete oscillation takes 2 seconds. Hence the length of a seconds pendulum is found by writing:

$$2\pi\sqrt{\frac{l}{g}} = 2.$$

Exercise 13.4

1. A particle of mass m is attached to one end of a light elastic string of length a and modulus $\frac{3}{2}mg$. The other end of the string is attached to a fixed point O and the particle hangs in equilibrium under gravity at the point E. Find the distance OE. If the particle is a *further* distance x below E show that the resultant force acting on the particle is proportional to x. The particle is pulled down to the point at a distance a below E and released from rest. Show that, in the subsequent motion and while the string is taut, the particle executes Simple Harmonic Motion and that its distance below E at time t after being released is

$$a\cos\left\{\left(\frac{3g}{2a}\right)^{1/2}t\right\}. \qquad \text{(O\&C)}$$

2. A light elastic string of natural length l has one end fastened to a fixed point O. The other end of the string is attached to a particle of mass m. When the particle hangs in equilibrium the length of the string is $\frac{7}{4}l$. The particle is displaced from equilibrium so that it moves vertically with the string taut. Show that the motion is simple harmonic with period $\pi\sqrt{(3l/g)}$.

At time $t = 0$ the particle is released from rest at a point A at a distance $\frac{3}{2}l$ vertically below O. Find (i) the depth below O of the lowest point L of the motion, (ii) the time taken to move from A to L, (iii) the depth below O of the particle at time $t = \frac{1}{3}\pi\sqrt{\frac{3l}{g}}$. (C)

3. A particle of mass m is attached to one end of a light elastic string of natural length l and modulus $2mg$. The other end of the string is fixed at a point A. The

particle rests on a support B vertically below A, with $AB = 5l/4$. Find the tension in the string and the reaction exerted on the particle by the support B. The support B is suddenly removed. Show that the particle will execute simple harmonic motion and find (i) the depth below A of the centre of oscillation, (ii) the period of the motion. (JMB)

4. A light elastic spring AB of natural length l has the end A attached to a fixed point and hangs vertically under gravity with a particle of mass m attached at B. The system is set in motion and performs small vertical oscillations. Show that the motion is simple harmonic. The particle of mass m at B is replaced by a particle of mass αm, and it is found that the period of oscillation is doubled. Find the value of α. (C)

5. One end of an elastic string of modulus mg and natural length a is attached to a fixed point O. To the other end A are attached two particles P and Q, P having mass $2m$ and Q having mass m. The particles hang down in equilibrium under gravity. If Q falls off, show that P subsequently performs simple harmonic motion and state the period and amplitude of this motion. If on the other hand P falls off, find the distance from O of the highest point reached by Q. (O&C)

6. An elastic string of natural length $2a$ and modulus λ has its ends attached to two points A, B on a smooth horizontal table. The distance AB is $4a$ and C is the mid-point of AB. A particle of mass m is attached to the mid-point of the string. The particle is released from rest at D, the mid-point of CB. Denoting by x the displacement of the particle from C, show that the equation of motion of the particle is

$$\frac{d^2x}{dt^2} + \frac{2\lambda}{ma}x = 0.$$

Find the maximum speed of the particle and show that the time taken for the particle to move from D directly to the mid-point of CD is $\frac{\pi}{3}\left(\frac{ma}{2\lambda}\right)^{1/2}$ (JMB)

7. A particle P of mass m is attached by two light elastic strings to two fixed points A and B on a smooth horizontal table. Each string is of natural length l and $AB = 4l$. If the strings AP and PB have moduli of elasticity λ and 3λ respectively, find the length of AP when the particle is in equilibrium. If the particle is given a small displacement from the equilibrium position along the line AB, show that while both strings are taut the particle executes simple harmonic motion. Find the period of the oscillations. Given that the string PB just becomes slack in the ensuing motion, find the maximum speed of the particle.

8. Two fixed points A and B on a smooth horizontal table are at a distance $5l$ apart. A particle of mass m lies between A and B. It is attached to A by means of a light elastic string of modulus mg and natural length l and to B by means of a light elastic string of modulus $2mg$ and natural length $2l$. If O is the point at which the particle would rest in equilibrium, find the distance OA. The particle is projected towards O

from the mid-point of AB with speed u. Show that while both strings remain taut the particle performs simple harmonic motion. Find the period and the amplitude of this motion. Show that the particle will perform complete simple harmonic oscillations if $2u^2 \leqslant 3gl$.

9. A particle of mass m moving on a smooth horizontal plane is attached to a point O of the plane by an elastic string of natural length l and modulus λ. Initially the particle is at the point A which lies at a distance l from O and is moving away from O at speed u. If the particle first comes to rest at the point B, use energy considerations to find the distance AB. Show that when the particle is moving in the line AB its equation of motion is of the form $\ddot{x} = -\omega^2 x$, where x is its displacement from A. Find the time taken by the particle to move directly from A to B. Find also the further time which elapses before the particle next comes to rest.

10. A particle of mass m is attached to two elastic strings each of natural length a. The first string has modulus $3mg$ and its other end is attached to a fixed point A. The second string has modulus $6mg$ and its other end is attached to a fixed point B which is at a distance $3a$ vertically below A. Show that the particle can rest in equilibrium at a point O between A and B with both strings taut. Find the distance y of O from A. The particle is in motion in the line AB with both strings taut. At time t, the particle is at P where $OP = x$. Prove that $\dfrac{d^2x}{dt^2} = -\dfrac{9g}{a}x$.

The particle is projected vertically upwards from the point at a distance $2a$ below A with speed u. Find the maximum value of u for which the upper string does not become slack in the subsequent motion. For this maximum value of u find the time taken for the particle to travel from O to its highest point. (O&C)

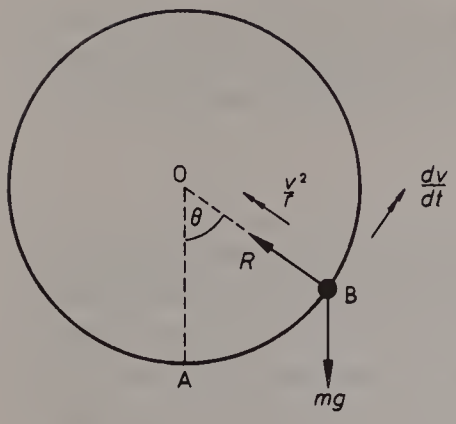
11. Find, to the nearest millimetre, the length of a seconds pendulum, (a) at Greenwich, where $g \approx 9{\cdot}81$, (b) at the equator, where $g \approx 9{\cdot}78$, (c) at the North Pole, where $g \approx 9{\cdot}83$, (d) on the surface of the moon, where $g \approx 1{\cdot}62$.

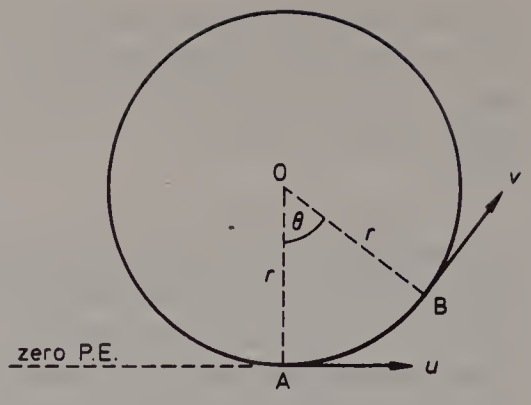
12. A musician wishes to use a simple pendulum as a makeshift metronome. Regarding a complete oscillation as two "beats", find, to the nearest centimetre, the length of pendulum required to produce (a) 50 beats per minute, (b) 80 beats per minute, (c) 100 beats per minute, (d) 160 beats per minute. [Take g as $9{\cdot}8\,\mathrm{m\,s^{-2}}$.]

13.5 Motion in a vertical circle

It was established in Chapter 8 that the acceleration of a particle moving in a circle, radius r, with variable speed v has components v^2/r towards the centre of the circle and dv/dt in the direction of motion.

When dealing with motion in a vertical circle we can obtain equations of motion by applying Newton's second law in these two directions. However, it may sometimes be more convenient to use an energy equation.





We consider first a small bead of mass m sliding on a smooth circular wire, of centre O and radius r, fixed in a vertical plane. Let u be the speed of the bead as it passes through the lowest point A of the wire and let v be the speed of the bead when it is at a point B such that $\angle AOB = \theta$. Since the wire is smooth the reaction R of the wire on the bead is perpendicular to the wire. Using Newton's second law radially and tangentially,

$$R - mg\cos\theta = m\frac{v^2}{r}$$

$$mg\sin\theta = -m\frac{dv}{dt}$$

The reaction R is perpendicular to the direction of motion and therefore does no work. This means that we can apply the principle of conservation of mechanical energy:

$$\text{(K.E. + P.E.) at } A = \text{(K.E. + P.E.) at } B$$

$$\therefore \quad \tfrac{1}{2}mu^2 = \tfrac{1}{2}mv^2 + mgr(1 - \cos\theta) \qquad (1)$$

As θ increases the speed of the bead decreases. If the bead comes to rest before it reaches the highest point of the wire it will slide back down the wire and perform oscillations about A. However, if the speed of the bead is greater than zero for all values of θ, it will execute complete circles.

From (1): $\quad v^2 = u^2 - 2gr(1 - \cos\theta)$

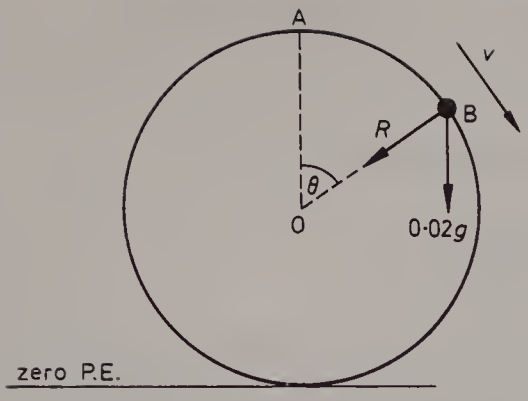
$\therefore$ if $\quad v^2 > 0 \quad$ for all $\quad \theta$

then $\quad u^2 > 2gr(1 - \cos\theta) \quad$ for all $\quad \theta$

Since the greatest value taken by $(1 - \cos\theta)$ is 2, the condition for performing complete circles is

$$u^2 > 4gr.$$

Example 1 A small ring of mass 0·02 kg is threaded on a smooth circular wire of centre O fixed in a vertical plane. The ring is slightly displaced from rest at the highest point A of the wire. Find the magnitude and direction of the reaction of the wire on the ring when the radius to the ring makes an angle of 45° with (a) the upward vertical, (b) the downward vertical.



Let v m s^{-1} be the speed of the ring when it is at a point B such that $\angle AOB = \theta$. Let the reaction of the wire on the ring be R N towards O and let the radius of the wire be r m.

Using the principle of conservation of mechanical energy,

$$\text{(K.E. + P.E.) at } A = \text{(K.E. + P.E.) at } B$$

$$\therefore \quad 0 + 0{\cdot}02g \times 2r = \tfrac{1}{2} \times 0{\cdot}02 \times v^2 + 0{\cdot}02g \times r(1 + \cos\theta)$$

$$\therefore \quad 4gr = v^2 + 2gr(1 + \cos\theta)$$

Thus $$v^2 = 2gr(1 - \cos\theta).$$

Applying Newton's second law towards O,

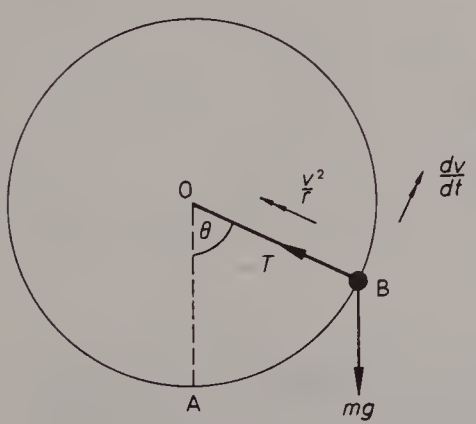
$$R + 0{\cdot}02g\cos\theta = 0{\cdot}02 \times \frac{v^2}{r}$$

$$\therefore \quad R + 0{\cdot}02g\cos\theta = 0{\cdot}02 \times 2g(1 - \cos\theta)$$

Thus $$R = 0{\cdot}02g(2 - 3\cos\theta).$$

(a) When $\theta = 45°$, $R = 0{\cdot}02 \times 9{\cdot}8 \times (2 - 3\cos 45°) \approx -0{\cdot}024$.
Hence when the radius to the ring makes an angle of 45° with the upward vertical the reaction of the wire is 0·024 N *away from O*.

(b) When $\theta = 135°$, $R = 0{\cdot}02 \times 9{\cdot}8 \times (2 - 3\cos 135°) \approx 0{\cdot}808$.
Hence when the radius to the ring makes an angle of 45° with the downward vertical the reaction of the wire is 0·808 N *towards O*.



Suppose now that a particle of mass m, suspended from a fixed point O by a light inextensible string of length r, moves in a vertical circle. Let u be the speed of the particle as it passes through its lowest point A and let v be the speed of the particle when it is at a point B such that $\angle AOB = \theta$.

Applying Newton's second law towards O,

$$T - mg\cos\theta = m\frac{v^2}{r} \qquad (1)$$

As in the case of a bead sliding on a circular wire, it can also be shown that

$$v^2 = u^2 - 2gr(1 - \cos\theta) \tag{2}$$

$$\therefore \quad T = \frac{m}{r}\{u^2 - 2gr(1 - \cos\theta)\} + mg\cos\theta$$

$$\text{i.e.} \quad T = \frac{mu^2}{r} - 2mg + 3mg\cos\theta \tag{3}$$

Thus when the particle passes through A its speed v and the tension T in the string take their maximum values. As θ increases, both v and T decrease, but the exact nature of the subsequent motion depends on the magnitude of u. We now consider in more detail the following possibilities:

(i) v reaches zero before T,
(ii) T reaches zero before v,
(iii) both v and T are positive for all values of θ.

Case (i): The particle comes to rest with the string taut.

Substituting $v = 0$ in (1) and (2), then rearranging:

$$T = mg\cos\theta \tag{4}$$
$$2gr - u^2 = 2gr\cos\theta \tag{5}$$

Using (4): $T > 0$ when $v = 0$

$\Leftrightarrow$ $\cos\theta > 0$ when $v = 0$

Hence the particle comes to rest at a point in the lower half of the circle.

Using (5): $\cos\theta > 0$ when $v = 0$

$\Leftrightarrow$ $2gr - u^2 > 0$

$\Leftrightarrow$ $2gr > u^2$

Hence the particle performs oscillations below the level of O with the string taut if and only if $u^2 < 2gr$.

Case (ii): The string becomes slack while the particle is moving.

Substituting $T = 0$ in (1) and (3), then rearranging:

$$v^2 = -gr\cos\theta \tag{6}$$
$$2gr - u^2 = 3gr\cos\theta \tag{7}$$

Using (6): $v^2 > 0$ when $T = 0$

$\Leftrightarrow$ $-1 < \cos\theta < 0$ when $T = 0$

Hence the string can become slack only at a point in the upper half of the circle.

Using (7): $-1 < \cos\theta < 0$ when $T = 0$

$\Leftrightarrow$ $-3gr < 3gr\cos\theta < 0$ when $T = 0$

$\Leftrightarrow$ $-3gr < 2gr - u^2 < 0$

$\Leftrightarrow$ $0 < u^2 - 2gr < 3gr$

$\Leftrightarrow$ $2gr < u^2 < 5gr$

Hence the string becomes slack at a point above the level of O if and only if $2gr < u^2 < 5gr$. The particle will then leave its circular path and move freely under gravity until the string is again taut.

Case (iii): *The particle performs complete circles*

Using (2): $v > 0$ for all θ

$\Leftrightarrow$ $u^2 > gr(2 - 2\cos\theta)$ for all θ

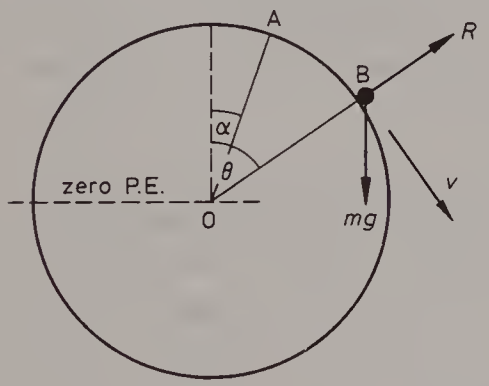
Using (3): $T > 0$ for all θ

$\Leftrightarrow$ $u^2 > gr(2 - 3\cos\theta)$ for all θ

Since $\cos\theta$ takes values from -1 to $+1$, both conditions are satisfied if and only if $u^2 > 5gr$. Hence the particle performs complete circles if and only if $u^2 > 5gr$.

Lastly we illustrate, by means of a worked example, a further way in which motion in a vertical circle may arise.

Example 2 A particle of mass m is released from rest at a point A on the outer surface of a smooth fixed sphere of centre O and radius r. Given that OA makes an angle α with the upward vertical, find an expression for the speed at which the particle is travelling when it leaves the surface of the sphere.



Let v be the speed of the particle when it reaches a point B such that OB makes an angle θ with the upward vertical. Since the sphere is smooth the reaction R of the sphere on the particle acts in the direction OB.

Since the reaction does no work mechanical energy is conserved during the motion,

$\therefore$ (K.E. + P.E.) at A = (K.E. + P.E.) at B

$\therefore$ $0 + mg \times r\cos\alpha = \frac{1}{2}mv^2 + mg \times r\cos\theta.$

i.e. $$2gr\cos\alpha = v^2 + 2gr\cos\theta \tag{1}$$

Applying Newton's second law towards O,

$$mg\cos\theta - R = m\frac{v^2}{r} \tag{2}$$

The particle will leave the surface of the sphere when R vanishes.
Putting $R = 0$ in equation (2) gives $gr\cos\theta = v^2$

$\therefore$ from (1): $2gr\cos\alpha = v^2 + 2v^2$

i.e. $v^2 = \frac{2}{3}gr\cos\alpha$

Hence the speed at which the particle leaves the surface is $\sqrt{(\frac{2}{3}gr\cos\alpha)}$.

Exercise 13.5

1. A particle P of mass 2 kg is suspended from a fixed point O by a light inextensible string of length 2·5 m and hangs in equilibrium at a point A. The particle is projected from A with horizontal speed 5 m s^{-1}. Given that when $\angle AOP = \theta$, the speed of the particle is v m s^{-1} and the tension in the string is T N, find expressions for v^2 and T in terms of θ. Hence find the values of v and T when $\theta = 40°$. Find also to the nearest degree, the greatest value attained by θ in the subsequent motion.

2. A small bead of mass 0·5 kg moves on a smooth circular wire of radius 2 m which is fixed in a vertical plane. When the radius to the bead makes an angle of 60° with the downward vertical the reaction between the bead and the wire is of magnitude 12 N. Find the speed of the bead at this instant and the magnitude of the reaction when the bead passes through the lowest point of the wire.

3. A particle of mass m is suspended from a fixed point O by a light inextensible string of length a. When the particle is in its lowest position it is given a horizontal speed of $\sqrt{(3ag)}$. If v is the speed of the particle when the string makes an angle θ with the upward vertical, find $\cos\theta$ and an expression for v^2 at the instant when the string becomes slack.

4. A particle of mass m is placed inside a smooth hollow sphere of radius a. The particle is set in motion at the lowest point of the sphere with speed $\sqrt{(\lambda ag)}$. Show that the particle will perform complete circles if $\lambda > 5$.

5. A small ring P of mass m is threaded on a smooth circular wire, of centre O and radius r, fixed in a vertical plane. Initially the ring is slightly displaced from the highest point of the wire. Find the magnitude of the force exerted by the ring on the wire when OP makes an angle θ with the upward vertical. Find the range of values of $\cos\theta$ for which this force acts towards O.

6. A particle of mass m slides down the smooth outside surface of a fixed sphere of radius a. At the top of the sphere its velocity is horizontal and of magnitude u. If θ is the angle that the radius to the particle makes with the upward vertical, show that the reaction between the particle and the sphere is

$$mg(3\cos\theta - 2) - mu^2/a.$$

Show that the particle, when just displaced from rest at the top, leaves the surface when $\theta = \cos^{-1}\frac{2}{3}$, and find its speed at that instant. (W)

7. A light rod OP of length a is free to rotate in a vertical plane about a fixed point O, and a mass m is attached to it at P. Denote by θ the angle between OP and the downward vertical. When $\theta = 0$, P is moving horizontally with speed u. Find the tension in the rod as a function of θ, and show that the value of the function is positive throughout the motion if $u^2 > 5ag$. Find the least value of the tension when $u^2 = 6ag$. Show that, if $u^2 < 2ag$, P remains below the level of O and deduce that, in this case also, the value obtained for the tension is positive throughout the motion. (JMB)

8. A particle P is projected horizontally with speed u from the lowest point A of the smooth inside surface of a fixed hollow sphere of internal radius a.

(i) In the case when $u^2 = ga$ show that P does not leave the surface of the sphere. Show also that, when P has moved halfway along its path from A towards the point at which it first comes to rest, its speed is $\sqrt{\{ga(\sqrt{3} - 1)\}}$.

(ii) Find u^2 in terms of ga in the case when P leaves the surface at a height $3a/2$ above A, and find, in terms of a and g, the speed of P as it leaves the surface. (JMB)

9. A particle of mass m, attached to a fixed point O by an inelastic string of length l, is allowed to fall from a point in the horizontal through O at a distance $l\cos\alpha$ from O. Find the speed of the particle when it first begins to move in a circle. In the subsequent motion, show that, when the string first makes an angle θ ($< \frac{1}{2}\pi - \alpha$) with the downward vertical, the speed v of the particle is $\sqrt{\{2gl(\cos\theta - \sin^3\alpha)\}}$ and the tension in the string is $3mv^2/2l + mg\sin^3\alpha$. Will the tension vanish during the subsequent motion? (W)

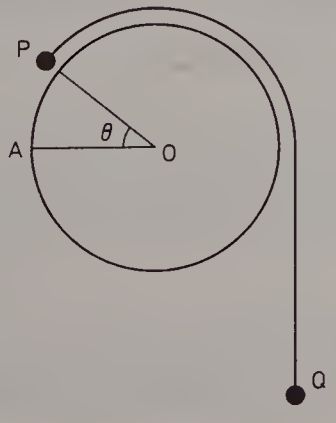
10. One end of a light inextensible string of length a is attached to a particle of mass m. The other end is attached to a fixed point O which is at a height $5a/2$ above the horizontal ground. Initially the string is taut and horizontal. The particle is then projected vertically downward with velocity $(2ag)^{\frac{1}{2}}$. When the string has turned through an angle θ ($< \pi$) find the velocity v of the particle and show that the tension in the string is $mg(2 + 3\sin\theta)$.

(a) If the string can withstand a tension of at least $5mg$, prove that the string will not break.

(b) If the string can withstand a tension of at most $7mg/2$, find the values of θ and v when the string breaks. In this case find the time to reach the vertical through O. Hence show that the particle strikes the ground at the point vertically below O. (O&C)

11. Two small smooth pegs O and C are fixed at the same level and $OC = a$. A light inextensible string of length $3a$ has one end attached to O and a particle P of mass m hanging at its other end vertically below O. The particle is projected with speed u parallel to OC. Find the speed of P when OP first makes an acute angle θ with the downward vertical and show that the tension in the string is then $mg(3\cos\theta - 2) + mu^2/3a$. Show that the string will reach a horizontal position if $u^2 \geqslant 6ga$. Given that $u^2 > 6ga$, find the tension when, after the string has struck the peg C, the moving portion CP makes an angle ϕ above the horizontal, and show that P will complete a semicircle with centre C if $u^2 \geqslant 12ga$. (JMB)

12.



A particle P of mass m is at rest at a point A on the smooth outer surface of a fixed sphere of centre O and radius a, OA being horizontal. The particle is attached to one end of a light inextensible string which is taut and passes over the sphere, its other end carrying a particle Q of mass $2m$ which hangs freely. The string lies in the vertical plane containing OA. The system is released from rest,

and after time t the angle POA is θ, as shown in the diagram. Show that, while P remains in contact with the sphere,

$$3a\left(\frac{d\theta}{dt}\right)^2 = 2g(2\theta - \sin\theta).$$

Find the tension in the string and the reaction of the sphere on P in terms of m, g and θ. (JMB)

Exercise 13.6 (miscellaneous)

1. A particle is in motion along a straight line. When at a distance x from a fixed point of the line its velocity is v. Show that its acceleration is $v\,dv/dx$.

The gravitational force per unit mass at a distance $x(> a)$ from the earth's centre is ga^2/x^2, where a is the radius of the earth, and g is constant. A particle is projected vertically upwards from the earth's surface at A with speed u. Show that if $u^2 < 2ag$, the particle rises to a maximum height $au^2/(2ag - u^2)$ above A. What happens if $u^2 > 2ag$? (O&C)

2. A man pushes a loaded cart of total mass m along a straight horizontal path against a constant resisting force λm. The cart starts from rest, and the man exerts a force, in the direction of the motion, of magnitude $Ke^{-\alpha x}$, where x is the distance moved, α is a positive constant and K is a constant greater than λm. Write down a differential equation, involving x and the speed v, for the motion of the cart (while it continues to move), and hence obtain v in terms of x and the given constants. (JMB)

3. A particle moves in a straight line in such a manner that when the particle is at the point P, at a distance x from some fixed point O on the line, the only force acting on the particle is in the direction **OP** and has magnitude $m\mu\left\{\frac{a}{x^3} - \frac{1}{x^2}\right\}$, where m is the mass of the particle and μ a positive constant.

If the particle is projected from the point $x = a$ with speed V in the direction **OP** find the set of values to which V must belong if the particle reaches the point $x = 2a$ but does not reach the point $x = 4a$. (C)

4. A particle moves along the x-axis so that its displacement x from the origin O at time t satisfies the differential equation $\frac{d^2x}{dt^2} + 9x - 18 = 0$. Its speed when $t = \pi/12$ is $21/\sqrt{2}$ in the direction of the positive x-axis and its speed when $t = \pi/6$ is 15 in the *opposite* direction. Find x at any time t. What is the greatest distance from O reached by the particle and what is its greatest speed? If the motion starts at $t = 0$, find the time at which the particle first passes through O. [The general solution of the differential equation may be quoted.] (O&C)

5. A particle is projected vertically upwards under (constant) gravity g from a point A on the earth's surface with an initial speed k. At time t after projection, the

particle is at height x above A and its speed is v; the air resistance is gv/k per unit mass. Show that, before the particle reaches its highest point, $\frac{dv}{dt} = -\frac{g}{k}(v + k)$, and $v = k(2e^{-gt/k} - 1)$. Hence find the corresponding value of x as a function of t. (O&C)

6. A bullet of mass 40 g is fired horizontally with velocity 800 m/s into a fixed block of material which exerts a constant resistance, R newtons, which stops it in 0·5 m. Show that $R = 2{\cdot}56 \times 10^4$ and find the time taken. An identical bullet is fired at the same velocity into a second fixed block which offers a resistance which varies as the square root of the speed of the bullet. The initial value of this resistance is $2{\cdot}56 \times 10^4$ N. Write down the equation of motion for the bullet and find the time taken to stop it. Explain why the bullet could never be stopped by a material which offered a resistance proportional to the velocity. (AEB 1979)

7. With the usual notation, prove that $\frac{dv}{dt} = v\frac{dv}{ds}$. A particle P of mass m moves in a straight line and starts from a point O with velocity u. When $OP = x$, where $x \geqslant 0$, the velocity v of P is given by $v = u + \frac{x}{T}$, where T is a positive constant. Show that, at any instance, the force acting on P is proportional to v. Given that the velocity of P at the point A is $3u$, calculate (i) the distance OA in terms of u and T, (ii) the time, in terms of T, taken by P to move from O to A, (iii) the work done, in terms of m and u, in moving P from O to A. (AEB 1980)

8. A point P is travelling in the positive direction on the x-axis with acceleration proportional to the square of its speed v. At time $t = 0$ it passes through the origin with speed gT and with acceleration g. Show that $dv/dx = v/(gT^2)$, and hence obtain an expression for v in terms of x, g and T. Prove that, at time t,

$$x = gT^2 \ln\left(\frac{T}{T - t}\right).$$

Sketch the graph of x against t for $0 \leqslant t < T$. (L)

9. A particle is projected vertically upwards with speed u from a point O. The air resistance is proportional to the fourth power of the velocity and is initially equal to the weight of the particle. If v is the speed of the particle when it is at a height z above O, show that $\frac{d}{dz}(\frac{1}{2}v^2) + g\left(1 + \frac{v^4}{u^4}\right) = 0$.
Solve this equation by writing w for v^2 and so prove that the particle is instantaneously at rest when its height above O is $\pi u^2/8g$. (W)

10. A body of mass m falls from rest under gravity through a resisting medium. The resistance is kv^2 per unit mass where k is constant and v is the velocity when the body has fallen a distance x.

(i) Given that u is the limiting value of the velocity, i.e. the velocity for which there would be zero acceleration, establish the differential equation $v\dfrac{dv}{dx} = k(u^2 - v^2)$.

(ii) By solving this differential equation obtain an expression for v in terms of x. (JMB)

11. A locomotive of mass M works at a constant rate H. Its motion is resisted by a force Mkv, where v is the speed at time t and k is constant. If $c^2 = H/(Mk)$, show that the equation of motion of the locomotive can be written $v\dfrac{dv}{dt} = k(c^2 - v^2)$.

Prove that the locomotive will increase its speed from $\frac{1}{3}c$ to $\frac{1}{2}c$ in time $\{1/(2k)\}\ln(32/27)$, and in this interval of time will travel a distance $\dfrac{c}{2k}\ln(1{\cdot}5) - \dfrac{c}{6k}$. (O&C)

12. A car of mass M is driven by an engine working at constant power and the resistance to motion is proportional to the square of the speed. If R is the resistance at full speed V on a horizontal road, prove that the distance covered as the speed increases from v_1 to v_2 is $\dfrac{MV^2}{3R}\log_e\left(\dfrac{V^3 - v_1^3}{V^3 - v_2^3}\right)$. (O)

13. (a) A particle travelling in a straight line is subject to a retardation of $(v^2 + 25)\,\mathrm{m\,s^{-2}}$, where $v\,\mathrm{m\,s^{-1}}$ is its speed at t seconds. If its initial speed is $12\,\mathrm{m\,s^{-1}}$, show that it travels a distance of $\log_e(13/5)$ metres before coming to rest.

(b) A particle moving with SHM oscillates between points A and B which are 10 m apart. When it is 1 m from A, its speed is $16\,\mathrm{m\,s^{-1}}$. Find (i) the periodic time; (ii) the maximum speed it attains; (iii) the maximum acceleration; (iv) the shortest time it takes to travel from A to C, where AC is $\frac{1}{4}AB$. (SU)

14. A particle moves along a straight line with simple harmonic motion of amplitude 0·5 m and of period 4 seconds. Find the maximum speed in m/s and the maximum acceleration in $\mathrm{m/s^2}$ of the particle. Find also in m/s the speed of the particle when it is 0·25 m from its central position. (Answers may be left in terms of π.) Find also the least time which elapses between two instants when the speed of the particle is half its maximum speed. (L)

15. A particle of mass m moves in simple harmonic motion about O in a straight line, under the action of a restoring force of magnitude proportional to the distance from O. At time $t = 0$ the speed of the particle is zero. After 1 second the speed of the particle is $2\,\mathrm{m\,s^{-1}}$, after a further second the speed is $2\sqrt{3}\,\mathrm{m\,s^{-1}}$ and subsequently the particle passes through O for the first time. Show that the speed of particle when it passes through O is $4\,\mathrm{m\,s^{-1}}$ and find (i) the period and amplitude of the motion, (ii) the time at which the speed of the particle is equal to $2\,\mathrm{m\,s^{-1}}$ after it first passes through O.

A stationary particle of mass m is placed at O and the moving particle collides and coalesces with this particle. The combined particle moves under the action of the

same restoring force as before. Find the period of the subsequent motion and the speed when the displacement is $12/\pi$ m from O. (C)

16. At time t the displacement from the origin of a particle moving along the x-axis under the action of a single force is given by $x = a\cos(\omega t + \alpha)$ where a, ω and α are constants. Find the acceleration of the particle in terms of its displacement from the origin and describe the force acting on the particle. What is the periodic time of this motion?

Two particles, each moving in simple harmonic motion, pass through their centres of oscillation at time $t = 0$. They are next at their greatest distances from their centres at times $t = 2$ s and $t = 3$ s respectively, having been at the same distance from their respective centres at time $t = 1$ s. Show that the amplitudes of their motions are in the ratio $1:\sqrt{2}$. When will the particles next be at their centres, simultaneously? When will the initial conditions next be repeated? (W)

17. A particle moves on a straight line through a fixed point O so that at time t its displacement from O is x and its equation of motion is $\ddot{x} = -\omega^2 x$. If when $t = 0$, $x = d$ and $\dot{x} = u$, show that $x = d\cos\omega t + (u/\omega)\sin\omega t$. Given that the first three times at which the particle passes through the point $x = d$ are $t = 0, t = 2$ and $t = 8$, find d in terms of u.

18. A rough horizontal platform moves horizontally in simple harmonic motion. A particle P of mass 2 kg rests on the platform without slipping and is at O, the centre of oscillations when the motion starts; the initial speed of P is 4 m s^{-1}. The period of the S.H.M. is 8 s. Find (i) the amplitude of the motion correct to 3 sig. fig., (ii) the distance OP, 1 s after the motion starts, correct to 3 sig. fig., (iii) an expression for the time taken for the particle to be first displaced a distance 1 m from O. Do not evaluate the expression.

Draw a diagram showing the forces acting on P with its acceleration and obtain the maximum frictional force acting on P. Hence give the range of values of the coefficient of friction in order that P should not slip. (SU)

19. The end A of a light elastic string AB, obeying Hooke's law and of natural length 0·5 m, is fixed. When a particle of mass 2 kg is attached to the string at B and hangs freely under gravity, the extension of the string in the equilibrium position is 0·075 m. Calculate, in newtons, the modulus of elasticity of the string. The particle is now pulled down vertically a further 0·1 m and released. Show that, until the string becomes slack, the motion of the particle is simple harmonic. Show that the time that elapses before the particle first passes through the equilibrium position is $(\pi\sqrt{3})/40$ s and find the speed of the particle when it is in this position. [Take g as 10 m/s^2]. (L)

20. One end of a light elastic string of natural length l and modulus λ is attached to a fixed point A, and a small mass m, attached to the other end, hangs in equilibrium at B. A small mass $2m$ is made to adhere to the under side of the first mass, and the combined mass is released from rest at B. Show that the combined mass oscillates

about a point C which is at a depth $l + \dfrac{3mgl}{\lambda}$ below A, and that the equation of motion is

$$\frac{d^2x}{dt^2} = -\frac{\lambda}{3ml}x,$$

where x is the depth below C of the combined mass at time t. Show that the speed of the combined mass when $x = 0$ is $2g\sqrt{(ml/3\lambda)}$. Find the force of adhesion between the masses in terms of m, g, λ and x. (JMB)

21. A particle P of mass m lies on a smooth horizontal table and is attached to two fixed points A, B on the table by two light elastic strings, each obeying Hooke's law and of natural lengths $2l, 3l$ and of moduli $2mg, mg$, respectively. If $AB = 7l$, show that, when P is in equilibrium, $AP = 5l/2$. The particle P is held at rest at the point C in the line AB, where $AC = 3l$ and C lies between A and B, and is then released. Show that the motion of P is simple harmonic of period $\pi\sqrt{(3l/g)}$. Find the maximum speed of P during the motion. (O&C)

22. One end of a light elastic string, of natural length $2a$ and modulus $4mg$, is attached to a fixed point A of a smooth horizontal table. A particle of mass m is attached to the other end of the string. The particle is released from rest from a point C of the table, where $AC = 3a$. Show that the particle reaches C again after time $(4 + \pi)\sqrt{(2a/g)}$. (L)

23. A scale pan of mass m is suspended from a fixed point by an elastic spring, of modulus $2mg$ and natural length l. When the scale pan is hanging at rest, a particle of mass $\frac{1}{2}m$ falls vertically on to the scale pan, striking it with speed $\sqrt{(lg)}$. Find the amplitude of the oscillations which the scale pan starts to make if the particle rebounds, the coefficient of restitution between the particle and the scale pan being $\frac{1}{4}$. (L)

24. A particle rests on the *outside* of a fixed smooth vertical hoop, of radius a, at its highest point A, and a second particle rests on the *inside* of the hoop at its lowest point B. The particle at A is just disturbed from rest and the particle at B is given a horizontal velocity v such that both particles leave the hoop at the same point P. Show that (i) the radius OP is inclined to the vertical at an angle $\cos^{-1}\frac{2}{3}$, (ii) $v = 2\sqrt{(ag)}$. (W)

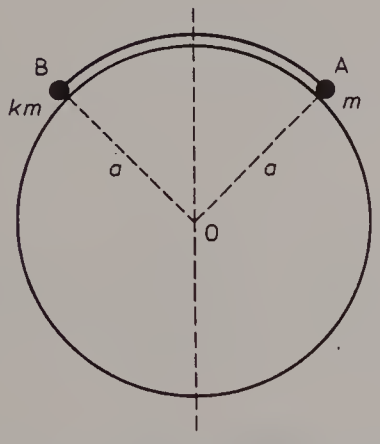
25. A smooth wire in the form of a circle of centre O and radius a is fixed in a vertical plane. Two beads, P of mass $2m$ and Q of mass m, are threaded on to the wire. Initially Q is at rest at A, the lowest point of the wire, and P is held at a point B on the wire, where the angle AOB is $\frac{1}{3}\pi$.

The particle P is then released. Find the speed of P when the angle AOP is θ and P has not yet reached A, and find the reaction then exerted on P by the wire. Show that, when P reaches A, its speed is $\sqrt{(ga)}$. Show that, when the particles collide, the impulse of the reaction between them is of magnitude $\frac{2}{3}m(1 + e)\sqrt{(ga)}$, where e is the

coefficient of restitution. In the case when $e = 1/5$ find the height above A which Q subsequently reaches before coming instantaneously to rest. (JMB)

26. A particle is at rest at the lowest point A inside a smooth hollow sphere of radius a. It is projected horizontally with velocity $\sqrt{(7ga/2)}$. Show that it loses contact with the sphere at a height $3a/2$ above A and will next strike the sphere at A. (C)

27.



Two particles A and B of masses m and km respectively, where $k < 1$, are joined by a light inextensible string of length $\frac{1}{2}\pi a$. The string is taut and the particles are held, at the same horizontal level, in contact with a fixed smooth cylinder of radius a. The axis of the cylinder is horizontal and the string is in a vertical plane perpendicular to the axis of the cylinder. The system is released from rest. If θ is the angle made with the vertical by the radius OA of the cylinder when the particle A is about to lose contact with the cylinder, show that $(3 + k)\cos\theta + 2k\sin\theta = (1 + k)\sqrt{2}$. (C)

28. Two small equal beads, threaded on a vertical smooth circular wire of radius a, are connected by a light inextensible string of length a. One bead is held at the highest point of the wire with the string taut and is then released. Find the speed of the beads at the instant the string first becomes slack. (O)

29. A particle of mass m is projected horizontally with velocity $\mathbf{u}$ from a point O in a medium producing a resisting force $-mk\dot{\mathbf{r}}$ where k is a positive constant and $\mathbf{r}$ is the position vector of the particle relative to O. The acceleration due to gravity is $\mathbf{g}$. Write down the equation of motion (involving $\ddot{\mathbf{r}}$) and show that at time t

(i) $\dfrac{d}{dt}(\mathbf{r}e^{kt}) = (\mathbf{u} + \mathbf{g}t)e^{kt}$,

(ii) $\mathbf{r} = \dfrac{1}{k}(1 - e^{-kt})\mathbf{u} + \dfrac{1}{k^2}(e^{-kt} + kt - 1)\mathbf{g}$.

If k is so small that k^2 may be neglected compared with k, show that approximately $\mathbf{r} = \mathbf{u}t(1 - \frac{1}{2}kt) + \mathbf{g}t^2(\frac{1}{2} - \frac{1}{6}kt)$. (JMB)

14 Probability

14.1 Permutations and combinations

A *permutation* is an arrangement of objects chosen from a given set. In this section we discuss ways of finding the total number of permutations of a set of objects under various conditions. For example, consider the number of ways in which we could arrange four guests in a row of four numbered chairs at a concert.

Any one of the 4 people may sit in the first chair. When the first person is seated, there are 3 people to be considered for the second place. This means that for each of the 4 ways of choosing the first person, there are 3 ways of choosing the second person to be seated. Similarly, in each of these $4 \times 3 = 12$ cases, we are left with 2 ways of choosing the occupant of the third chair. When 3 people are seated, the fourth chair must be filled by the 1 remaining person. Hence the total number of ways of seating the 4 people is $4.3.2.1$, i.e. 24.

Using the letters A, B, C, D to represent the 4 people, these 24 arrangements are as follows:

A B C D	B A C D	C A B D	D A B C
A B D C	B A D C	C A D B	D A C B
A C B D	B C A D	C B A D	D B A C
A C D B	B C D A	C B D A	D B C A
A D B C	B D A C	C D A B	D C A B
A D C B	B D C A	C D B A	D C B A

Since products of the type $4.3.2.1$ frequently arise in this type of work, the *factorial* notation is used. We write $4.3.2.1 = 4!$ which is read 'four factorial' or 'factorial four'. Thus for any positive integer n, $n! = n(n-1)(n-2)\ldots 3.2.1$.

The result of the above seating problem can now be generalised.

> The number of permutations or arrangements of n different objects is $n!$.

Example 1 In how many different ways can 4 people chosen from a set of 6 be seated in a row of four chairs?

There are 6 possible choices for the first chair, then 5 choices for the second, 4 for the third and 3 for the fourth. Hence the total number of ways of seating 4 people out of a set of six is 6.5.4.3, i.e. 360.
[In factorial notation this number is 6!/2!]

In general, the number of permutations or arrangements of r different objects chosen from a set of n objects, written ${}_nP_r$, is given by ${}_nP_r = \dfrac{n!}{(n-r)!}$.

When $r = n$ this expression becomes ${}_nP_n = n!/0!$, but as shown above ${}_nP_n = n!$. Hence, it is convenient to define 0! to be 1.

In some problems the order in which objects are arranged is not important. The number of *combinations* of r different objects out of a set of n is the number of different selections, irrespective of order, and is denoted by ${}_nC_r$, or sometimes $\binom{n}{r}$.

Let us suppose that a set of 3 cards is to be dealt from a pack of 52 playing cards. The number of permutations of 3 cards chosen from $52 = 52!/49!$ $= 52.51.50$. However, since any set of 3 cards can be arranged in 3! different ways, each set of 3 cards will appear 6 times in the list of all possible arrangements. For instance, the set containing cards X, Y and Z will appear as XYZ, XZY, YXZ, YZX, ZXY and ZYX. Hence the total number of ways of selecting 3 cards $= \dfrac{52!}{3!49!} = \dfrac{52.51.50}{3.2.1} = 22\,100.$

In general, the number of ways of selecting r objects from n unlike objects $= {}_nC_r = \dfrac{n!}{r!(n-r)!}$.

Example 2 A committee of 5 is to be formed from 12 men and 8 women. In how many ways can the committee be chosen so that there are 3 men and 2 women on it?

Number of ways of choosing 3 men from 12

$$= {}_{12}C_3 = \frac{12!}{3!9!} = \frac{12.11.10}{3.2.1} = 220.$$

Number of ways of choosing 2 women from 8

$$= {}_8C_2 = \frac{8!}{2!6!} = \frac{8.7}{2.1} = 28.$$

$\therefore$ the total number of ways of forming the committee

$$= {}_{12}C_3 \times {}_8C_2 = 220 \times 28 = 6160.$$

Exercise 14.1

1. Find the number of ways of arranging 6 different books on a shelf.

2. Find the number of arrangements of 8 items on a shopping list.

3. A lady has 8 house plants. In how many ways can she arrange 6 of them in a line on a window sill?

4. In a competition 6 household products chosen from 10 are to be listed in order of preference. In how many ways can this be done?

5. There are 25 entrants in a gymnastics competition. In how many different ways can the gold, silver and bronze medals be awarded?

6. In how many different ways can 8 books be arranged on a shelf if 2 particular books must be placed next to each other?

7. Three girls and 4 boys are to sit in a row of 7 chairs. If the girls wish to sit in adjacent chairs, how many different arrangements are possible?

8. How many numbers less than 3500 can be formed using one or more of the digits 1, 3, 5, 7, if (a) no digit can be repeated, (b) repetitions are allowed?

9. Verify that ${}_nC_{n-r} = {}_nC_r$ for the cases
(a) $n = 8, r = 5$; (b) $n = 10, r = 2$.

10. In how many ways can
(a) 4 photographs be chosen from 10 proofs,
(b) 3 representatives be chosen from 20 students,
(c) a hand of 5 cards be dealt from a set of 13,
(d) 11 players be selected from 12 cricketers?

11. Nine people are to go on a journey in cars which can take 2, 3 and 4 passengers respectively. In how many different ways can the party travel, assuming that the seating arrangements inside the cars are not important?

12. In how many ways can a set of 12 unlike objects be divided into (a) 2 sets of 6, (b) 3 sets of 4, (c) 6 sets of 2?

13. A team of 5 students, including a captain and a reserve, is to be selected for a general knowledge contest. In how many ways can the team be chosen from a short list of 12?

14. Find the number of different selections of two letters which can be made from the letters of the word PROBABILITY. How many of these selections do not contain a vowel?

15. Find the number of ways in which 8 books can be distributed to 2 boys, if each boy is to receive at least 2 books.

16. A committee of 6 is to be formed from 13 men and 7 women. In how many ways can the committee be selected given that (a) it must consist of 4 men and 2 women, (b) it must have at least one member of each sex?

14.2 The language of sets

A *set* is a collection of objects. These objects are the *elements* of the set. A set may be defined either by listing the elements or by describing their properties, e.g. $A = \{2,4,7\}$, B is the set of all triangles, $C = \{\ldots, -5, -3, -1, 1, 3, \ldots\}$ or {odd integers}. Using the colon : to mean 'such that', we may also write $P = \{p : p \text{ is a prime number}\}$, which is read '$P$ is the set of all elements p such that p is a prime number'. Whatever form is chosen it must always be possible to decide from the definition whether or not an object belongs to the set.

The statement $x \in S$ means 'x is an element of S' or 'x belongs to S'. Similarly $x \notin S$ means 'x is not an element of the set S'. Thus, for the sets defined above, $2 \in A$ but $2 \notin C$ and $9 \in C$ but $9 \notin P$.

Two sets are *equal* if they contain exactly the same elements, e.g. $\{16, 25, 36, 49, 64, 81\} = \{36, 81, 16, 25, 64, 49\} = \{\text{two digit squares}\}$. The way in which the elements are listed or described is not important.

Sometimes all the elements of one set are contained in another. If every element of a set A also belongs to a set B, then A is a *subset* of B, written $A \subseteq B$. If A is a subset of B and there is at least one element of B which is not an element of A, then A is a *proper subset* of B and we write $A \subset B$. For instance, if $X = \{a,c\}$, $Y = \{a,c,e\}$ and $Z = \{a, b, c, d\}$, then $X \subset Y$ and $X \subset Z$ but $Y \not\subset Z$. We may also write $Y \supset X$ and $Z \supset X$ using the symbol $\supset$ to mean 'contains as a proper subset'.

Three important properties of the set inclusion relation follow from the definition.

(1) Any set A is a subset of itself, i.e. $A \subseteq A$.

(2) $A \subseteq B$ and $B \subseteq C \Rightarrow A \subseteq C$.

(3) $A \subseteq B$ and $B \subseteq A \Leftrightarrow A = B$.

[Note that if p and q represent statements, then $p \Rightarrow q$ means 'if p then q' or 'p implies q'. The symbol $\Leftarrow$ means 'is implied by'. We write $p \Leftrightarrow q$ meaning 'p implies and is implied by q' when the statements p and q are equivalent.]

All the objects under consideration in any mathematical discussion form a set called the *universal set* (or *universe of discourse*), often denoted by $\mathscr{E}$. For instance, in a geometrical problem the universal set may be a set of points, a set of lines or a set of plane figures. Once a universal set $\mathscr{E}$ has been defined any other set considered must be one of its subsets. Thus for any set A, $A \subseteq \mathscr{E}$.

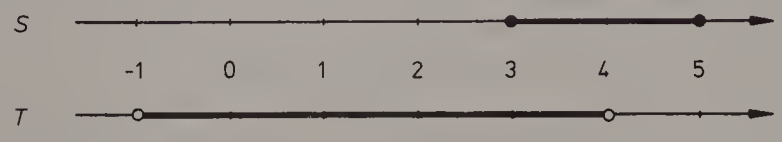
The set with no elements is called the *null* or *empty set*, written $\varnothing$ or sometimes $\{\ \}$. The empty set $\varnothing$ is considered to be a subset of any set A, i.e. $\varnothing \subseteq A$.

The *complement* of a set A is the set containing all the elements of $\mathscr{E}$ which are not elements of A and is written A'. For instance, if $\mathscr{E} = \{1,2,3,4,5\}$ and $A = \{1,3,5\}$ then $A' = \{2,4\}$. It follows from the definition that for any set A, $(A')' = A$. We also find that $\varnothing' = \mathscr{E}$ and $\mathscr{E}' = \varnothing$.

The *union* of sets A and B, written $A \cup B$, is the set containing all elements which belong to A or B (or both). For instance, if $A = \{1,2,3,4\}$ and $B = \{1,3,5\}$, then $A \cup B = \{1,2,3,4,5\}$.

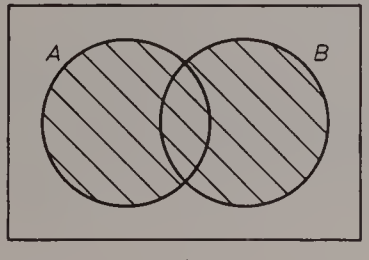
The *intersection* of sets A and B, written $A \cap B$, is the set containing all elements which belong to both A and B. For instance, if $A = \{2,4,6,8\}$ and $B = \{1,2,3,4\}$, then $A \cap B = \{2,4\}$.

Example 1 If $S = \{x \in \mathbb{R} : 3 \leqslant x \leqslant 5\}$ and $T = \{x \in \mathbb{R} : -1 < x < 4\}$, find $S \cup T$ and $S \cap T$.

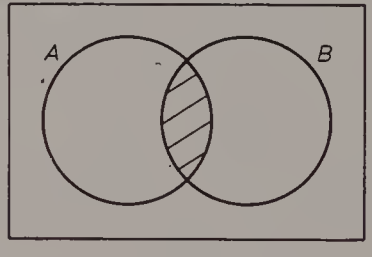


Representing S and T on the real number line we find that $S \cup T = \{x \in \mathbb{R} : -1 < x \leqslant 5\}$ and $S \cap T = \{x \in \mathbb{R} : 3 \leqslant x < 4\}$.

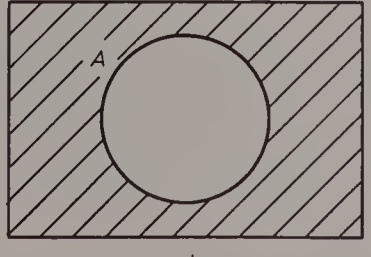
Venn† *diagrams* are used to represent relationships between sets. The universal set is shown as a rectangle and other sets as regions, often circles, within this rectangle.



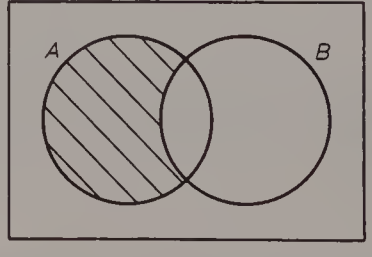
$A \cup B$



$A \cap B$



A'



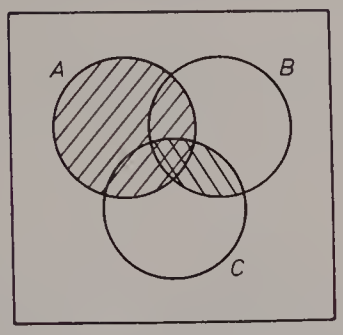
$A \cap B'$

Venn diagrams can be used to demonstrate properties of union and intersection, but such verifications are not regarded as rigorous proofs of the results.

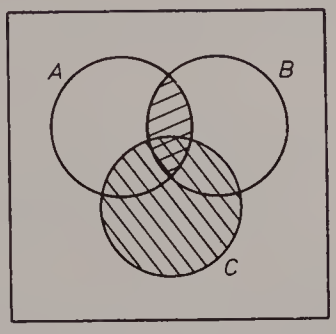
† *Venn, John* (1834–1923) English logician. He developed the theories of probability and mathematical logic. In his *Symbolic Logic* (1881) he improved the diagrammatic methods of earlier writers.

Example 2 Verify using Venn diagrams, the associative law

$$A \cap (B \cap C) = (A \cap B) \cap C.$$

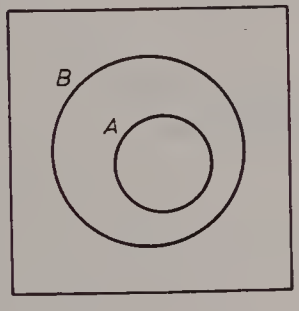


(1) $A \cap (B \cap C)$

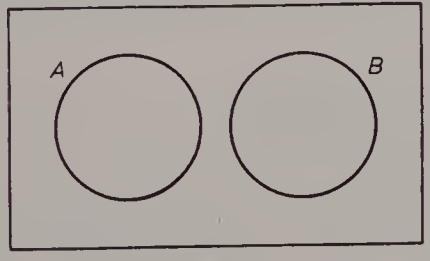


(2) $(A \cap B) \cap C$

In (1) the sets A and $(B \cap C)$ are shaded in different ways, so that the region shaded twice represents $A \cap (B \cap C)$. Similarly in (2) the region shaded twice represents $(A \cap B) \cap C$. Hence, in these diagrams, $A \cap (B \cap C) = (A \cap B) \cap C$.



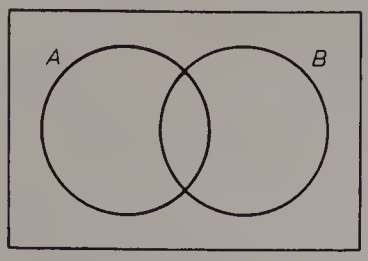
This diagram illustrates the fact that A is a subset of B, i.e. $A \subset B$. We see that in this case $A \cup B = B$ and $A \cap B = A$.



Here the sets A and B have no common elements, i.e. $A \cap B = \varnothing$. Such sets are said to be *disjoint*.

A', the complement of A, contains no elements of A,
$\therefore$ A and A' are disjoint and $A \cap A' = \varnothing$.
Since every element of the universal set $\mathscr{E}$ belongs to either A or A', for any set A, $A \cup A' = \mathscr{E}$.

Venn diagrams can also be used in problems concerning the numbers of elements in sets. The number of elements in a finite set A is called the *order* or *cardinal number* of the set and is denoted by $n(A)$.



By considering the diagram we see that in the sum $n(A)+n(B)$, the elements of $A \cap B$ have been counted twice.

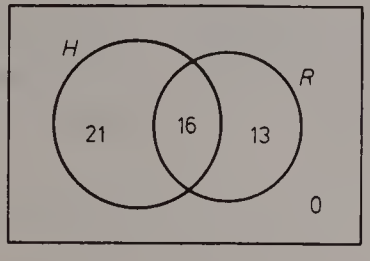
Hence $\boxed{n(A \cup B) = n(A)+n(B)-n(A \cap B).}$

Example 3 On a shelf of 50 books, there are 37 historical novels and 29 romances, but no other types of book. What conclusion may be drawn from this?

Let H and R be the sets of historical novels and romances respectively, then $n(H) = 37$, $n(R) = 29$ and $n(H \cup R) = 50$.

$$n(H \cup R) = n(H)+n(R)-n(H \cap R)$$
$$50 = 37+29-n(H \cap R)$$
$$\therefore \quad n(H \cap R) = 16.$$

It follows that there are 16 historical romances on the shelf.



The diagram shows one way of illustrating this result.

Example 4 In a survey of 100 housewives it is found that 65 use brand X and 55 use brand Y washing powder. What can be said about the numbers of housewives using both X and Y.

Let X and Y be the sets of housewives using those brands, then

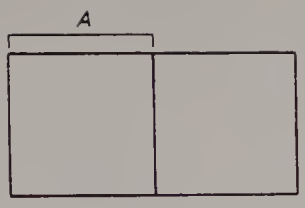
$$n(X \cup Y) = n(X)+n(Y)-n(X \cap Y)$$
$$= 65+55-n(X \cap Y)$$
$$= 120-n(X \cap Y).$$

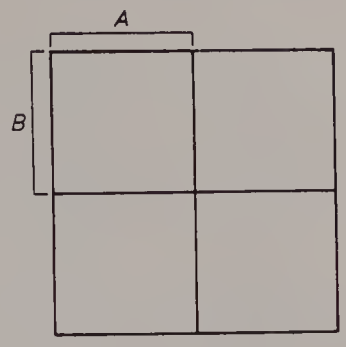
As there may be housewives who use neither X nor Y, we may assume only that $n(X \cup Y) \leqslant 100$,

$\therefore \quad n(X \cap Y) \geqslant 20$, i.e. at least 20 housewives use both X and Y.

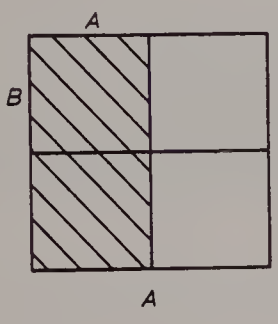
Karnaugh† *maps* provide a useful alternative to Venn diagrams when dealing with intersecting sets. Karnaugh maps for one set A and for two sets A and B are given below.

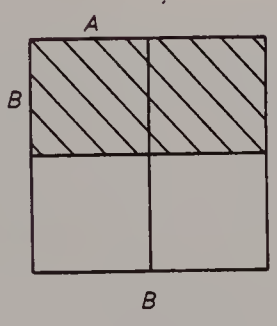
† Karnaugh maps representing sets are developments of diagrams used to simplify logic circuits in computers and electronic control systems, as described by *M. Karnaugh* in his article *The Map Method for Synthesis of Combinational Logic Circuits* (1953).

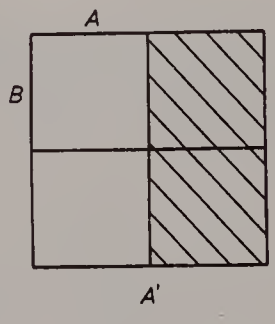


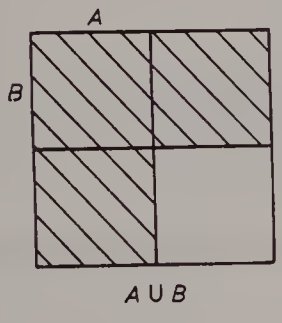


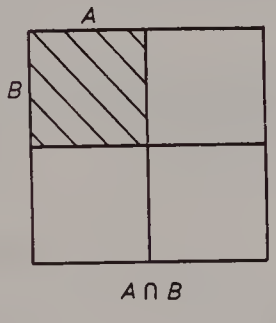
Any set can be represented on a Karnaugh map by shading the appropriate region.

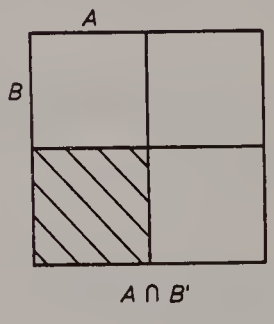


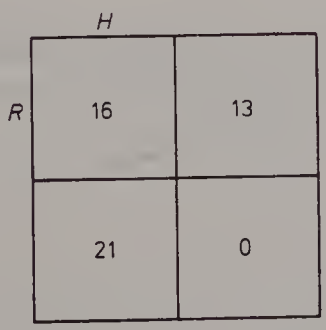






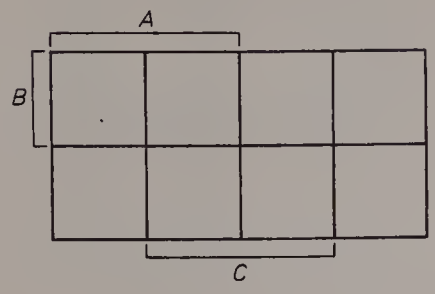


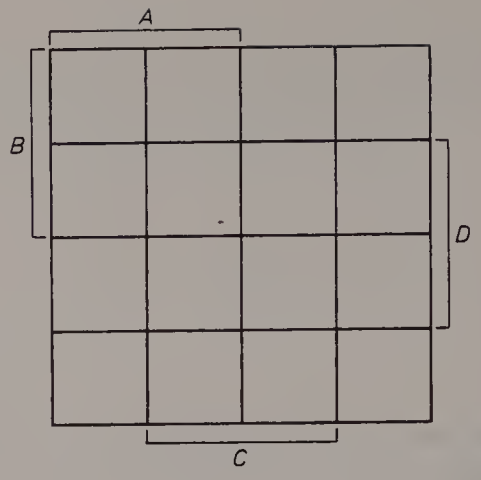




A Karnaugh map could have been used to illustrate the result of Example 3.

The Karnaugh system can be extended to larger numbers of sets. The maps for 3 sets and 4 sets are given here.





Exercise 14.2

1. If $A = \{x, y, z\}$, $B = \{x, p, q, r\}$, $C = \{x, y, p\}$, list the following sets:
(a) $A \cap B$, (b) $B \cup C$, (c) $(A \cup B) \cup C$,
(d) $(A \cup C) \cap B$, (e) $(A \cap C) \cap B$, (f) $B \cup (A \cap C)$.

2. Find $P \cup Q$ and $P \cap Q$ in each of the following cases
(a) $P = \{x \in \mathbb{R} : -2 < x < 4\}$, $Q = \{x \in \mathbb{R} : 0 < x < 7\}$,
(b) $P = \{x \in \mathbb{R} : 2 \leqslant x \leqslant 9\}$, $Q = \{x \in \mathbb{R} : x \leqslant 5\}$,
(c) $P = \{x \in \mathbb{R} : |x| < 10\}$, $Q = \{x \in \mathbb{R} : -4 < x < 3\}$,
(d) $P = \{x \in \mathbb{R} : -3 < x < 1\}$, $Q = \{x \in \mathbb{R} : 1 \leqslant x \leqslant 6\}$.

3. If $\mathscr{E} = \{1, 2, 3, 4, 5\}$, $A = \{1, 2, 3\}$, $B = \{5\}$, $C = \{3, 4\}$, list the following sets:
(a) $A \cap B$, (b) $C \cap A'$, (c) $(B \cup C) \cap \mathscr{E}$,
(d) $B' \cap C'$, (e) $A \cup \varnothing$, (f) $A' \cap (B \cup C)$.

4. If $\mathscr{E} = \{a, b, c, d\}$, $X = \{a, b\}$, $Y = \{b, c\}$, find expressions for the following sets using the symbols X, Y, X', Y', $\cup$ and $\cap$.
(a) $\{c, d\}$, (b) $\{b\}$, (c) $\{d\}$,
(d) $\{a, b, d\}$, (e) $\{a, c, d\}$, (f) $\{a, c\}$.

5. In each of the following cases draw a Venn diagram showing the universal set $\mathscr{E}$ and three sets A, B and C, then shade the region representing the given set.
(a) $A \cap B \cap C$, (b) $A' \cap B$, (c) $B' \cup C$,
(d) $(A \cup B) \cap C'$, (e) $A \cup (B \cap C')$, (f) $A' \cap B' \cap C'$.

6. Use Venn diagrams to verify the following:
(a) $A \cap (A \cup B) = A$, (b) $(A \cap B)' = A' \cup B'$,
(c) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

7. Draw a Venn diagram to show the relationships between the following sets: $\mathscr{E} = \{\text{all triangles}\}$, $P = \{\text{isosceles triangles}\}$, $Q = \{\text{equilateral triangles}\}$, $R = \{\text{right-angled triangles}\}$, $S = \{\text{acute angled triangles}\}$.

8. Draw a Venn diagram to show the following sets and their non-empty intersections: $\mathscr{E} = \{\text{positive integers}\}$, $A = \{\text{even numbers}\}$, $B = \{\text{multiples of 3}\}$, $C = \{\text{multiples of 9}\}$, $D = \{\text{multiples of 12}\}$.

9. Simplify the following expressions, using Venn diagrams if necessary.
(a) $A \cup (A \cap B)$, (b) $A \cap (A' \cap B)$,
(c) $(A \cap B) \cup (B' \cap A)$, (d) $(A \cup B') \cap [A \cup (B \cap C)]$.

10. In a survey of 100 children, it is found that 47 have at least one brother, 58 have at least one sister and that 32 children have both. How many of the children have no brothers and sisters?

11. Find an expression for $n(A \cup B \cup C)$ in terms of the numbers of elements in the sets A, B, C and their intersections.

12. Out of 50 people who enrol for evening classes, 30 enrol for Mathematics, 15 for Further Mathematics, 22 for Physics and 8 enrol for none of these subjects. Denoting the sets of students who enrol for Mathematics, Further Mathematics and Physics by M, F and P respectively, what conclusions may be drawn about the values of $n(M \cap F)$, $n(M \cap P)$ and $n(M \cap F \cap P)$?

13. In each of the following cases draw a Karnaugh map for three sets A, B and C, then shade the region representing the given set.
(a) $A \cap B \cap C$, (b) $A \cap C'$, (c) $B' \cup C$,
(d) $A' \cap B \cap C'$, (e) $A \cup (B' \cap C)$, (f) $(A' \cup B') \cap C'$.

14. Using Karnaugh maps or otherwise, simplify the following expressions.
(a) $A \cap (A \cup B')$, (b) $(B \cap C) \cup (B \cap C')$,
(c) $(A \cap B \cap C) \cup (A \cap B' \cap C)$, (d) $(A \cup B \cup C) \cap (A' \cup B \cup C)$.

14.3 Elementary theory of probability

In mathematics probability is the numerical value assigned to the likelihood that a particular event will take place. For instance, if we throw an unbiased die, we have equal chances of scoring any of the numbers 1, 2, 3, 4, 5 and 6. Since there is one chance in six of throwing a 3 the probability of the event occurring is said to be 1/6. Similarly when tossing a coin the probability that it lands heads is considered to be 1/2. This does not, of course, mean that in any two tosses we expect the coin to fall heads once, only that in a long series of tosses the number of heads will be approximately half the total number of throws.

To arrive at a formal definition, consider the set S of all possible *outcomes* of an experiment or *trial*. A set of this kind is sometimes called a *possibility space* or *sample space*. Any *event* A can be represented by the subset A of S, which contains all the outcomes in which the event occurs. If S contains a finite number

of *equally likely* outcomes, then the probability $P(A)$ that the event A will occur is given by

$$P(A) = \frac{n(A)}{n(S)}, \quad \text{i.e.} \quad \frac{\text{no. of favourable outcomes}}{\text{no. of possible outcomes}}.$$

If the event A is impossible, then $A = \varnothing$ and $P(A) = 0$. However, if the event A is certain to occur, then $A = S$ and $P(A) = 1$. Otherwise $P(A)$ will take some value between 0 and 1.

[When probabilities are calculated from experimental data, a definition such as the following may be used.

$$\text{Estimated probability} = \frac{\text{no. of successes}}{\text{no. of trials}}.$$

However, this value for the probability could be unreliable unless large numbers of observations are used.]

Example 1 What is the probability of drawing an ace at random from a pack of cards?

Since there are 4 aces in a pack of 52 cards, the probability of drawing an ace is 4/52, i.e. 1/13.

Example 2 If a letter is chosen at random from the word FACETIOUS, what is the probability that it is a vowel?

Since there are 5 vowels out of a total of 9 letters, the probability of choosing a vowel is 5/9.

Returning to the problem of throwing an unbiased die, the probability of throwing a 3 is 1/6, but the probability of not throwing a 3 is 5/6. We notice that the sum of these probabilities is 1. More generally, in a set S of equally likely possible outcomes, if A' denotes the subset of outcomes in which the event A does not occur, then A and A' are called *complementary* events and

$$P(A') = \frac{n(A')}{n(S)} = \frac{n(S) - n(A)}{n(S)} = 1 - \frac{n(A)}{n(S)}$$

$$\therefore \quad \boxed{P(A') = 1 - P(A).}$$

Hence, it seems reasonable to assume that in any sample space, if p is the probability that an event occurs and q is the probability that the event does not occur, then $p + q = 1$.

[Note that the event A' is sometimes referred to as 'not A' and may also be written $\bar{A}$.]

For experiments in which a trial is a throw of two or more dice, we can construct the sample space from the set of outcomes for a single die, $S = \{1, 2, 3, 4, 5, 6\}$. For instance, in the two-dice case, the possible outcomes can be expressed as ordered pairs such as $(1, 3)$, $(4, 6)$ and $(6, 4)$ formed from the elements of S.

A *sample space diagram* can be used to find probabilities as shown in the next example.

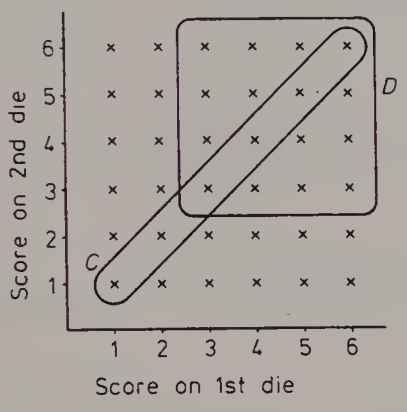
Example 3 If two dice are thrown together, what is the probability of the following events?

A: scoring a total of 2

B: scoring a total of 3

C: the same score on both dice

D: 3 or more on each die.



The diagram shows that when two dice are thrown, there are 36 possible outcomes. A score of 2 is obtained only when the score on each die is 1

$$\therefore \quad P(A) = \frac{1}{36}.$$

There are two outcomes $(1, 2)$ and $(2; 1)$ which produce a score of 3

$$\therefore \quad P(B) = \frac{2}{36} = \frac{1}{18}.$$

The set of outcomes in which both dice show the same score has 6 elements, as indicated in the diagram

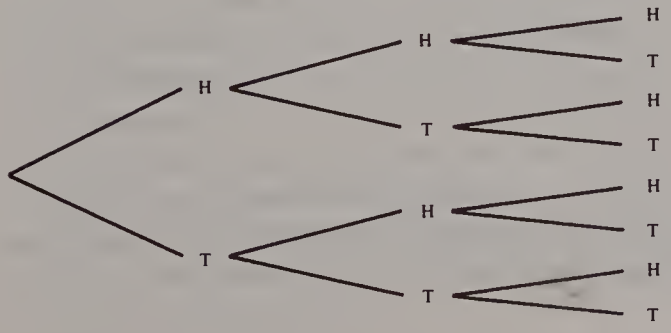
$$\therefore \quad P(C) = \frac{6}{36} = \frac{1}{6}.$$

The diagram also shows that there are 16 outcomes in which both dice show 3 or more

$$\therefore \quad P(D) = \frac{16}{36} = \frac{4}{9}.$$

Example 4 If a coin is spun three times, what is the probability that it lands heads once and tails twice?

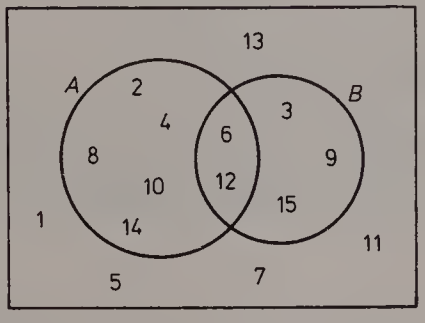
[In this case the elements of the sample space may be represented by the paths along a simple *tree diagram*.]



Since the given event occurs in 3 out of 8 equally likely outcomes, i.e. HTT, THT, TTH, the required probability is 3/8.

Let us now consider two different events A and B which may occur in an experiment. The set containing the outcomes in which A or B (or both) occur is the set $A \cup B$, so the probability of event A or event B (or both) is written $P(A \cup B)$. Similarly, the probability that both A and B occur is $P(A \cap B)$.

Example 5 An integer is chosen at random from the set $\{1, 2, 3, \ldots, 14, 15\}$. If A is the event of choosing an even number and B is the event of choosing a multiple of 3, find $P(A \cap B)$, $P(A \cup B)$ and $P(A' \cap B)$.



Using the possibility space diagram,

$$P(A \cap B) = \frac{2}{15}$$

$$P(A \cup B) = \frac{10}{15} = \frac{2}{3}$$

$$P(A' \cap B) = \frac{3}{15} = \frac{1}{5}$$

Some problems can be solved using the theory of permutations and combinations.

Example 6 If 4 cards are selected at random from a pack of 52, what is the probability that exactly 3 of them are diamonds?

The number of ways of selecting 3 diamonds from 13 $= {}_{13}C_3$.
The number of ways of selecting 1 card from the 39 which are not diamonds $= 39$.
$\therefore$ the number of ways of selecting 4 cards, 3 of which are diamonds $= 39 \times {}_{13}C_3$.
The number of ways of selecting any 4 cards from 52 $= {}_{52}C_4$.
Hence the probability that exactly 3 cards out of 4 are diamonds

$$= 39 \times {}_{13}C_3 \div {}_{52}C_4 = 39 \times \frac{13!}{3!10!} \times \frac{4!48!}{52!} \approx 0{\cdot}0412.$$

For a trial in which the possible outcomes are not equally likely, it is more difficult to arrive at a precise definition of the probability of an event. However, the subsets of a sample space S are regarded as events whatever the exact nature of the set S. In all cases $P(A)$, the probability of an event A, is a number or a weighting associated with the subset A. We can consider P to be a *probability mapping* which maps any subset A onto a real number p, where $0 \leqslant p \leqslant 1$. In any experiment the set of events (i.e. the set of subsets of the sample space) to be considered together with the appropriate probability mapping may be referred to as the *probability space.*

Exercise 14.3

In questions 1 to 4 find the probability of events A, B and C.

1. A die is thrown. A: scoring a four, B: scoring an odd number, C: scoring four or less.

2. A card is drawn from a pack. A: drawing a red card, B: drawing a seven, C: drawing a king, a queen or a jack.

3. Three coins are tossed. A: three heads, B: at least two tails, C: at least one of each.

4. It rained on exactly two days last week. A: it rained on Monday and Tuesday, B: it rained on two consecutive days, C: it rained on neither Monday nor Tuesday.

5. For the events in question 1, find
(a) $P(A \cup B)$, (b) $P(A \cap B)$, (c) $P(B \cup C)$, (d) $P(B \cap C')$.

6. For the events in question 2, find
(a) $P(A \cup B)$, (b) $P(A' \cap B)$, (c) $P(B \cap C')$, (d) $P(A \cap C')$.

7. An integer is chosen at random from the first 200 positive integers. Find the probability that it is (a) not divisible by 5, (b) a perfect square, (c) divisible by both 5 and 2, (d) divisible by neither 2 nor 7.

8. Two unbiased dice are thrown. Find the probability that the product of the scores is (a) odd, (b) a multiple of 3, (c) a multiple of 12.

9. A domino is drawn from a standard set of 28. Find the probability that the sum of its spots is (a) 2, (b) 8, (c) odd, (d) even.

10. In a cafeteria 80% of the customers order chips and 60% order peas. If 20% of those ordering peas do not want chips, find the probability that a customer chosen at random orders chips but not peas.

11. The points A, B, C, D and E are the vertices of a regular pentagon. All possible lines joining pairs of these points are drawn. If two of these lines are chosen at random, that is the probability that their point of intersection is (a) inside the pentagon, (b) one of the points A, B, C, D, E.

12. A box contains 4 discs numbered 1, 2, 2, 3. A disc is drawn from the box then replaced and a second disc is drawn. With the aid of a sample space diagram or otherwise, find the probability that (a) the total score is 6, (b) the total score is 4, (c) the numbers drawn are different, (d) the difference between the two numbers drawn is less than two.

13. A shopper with a keen eye for a bargain has bought a carton of assorted tins without labels. Assuming that the tins came from a large warehouse containing equal numbers of tins of soup, peaches and rice pudding, find the probability that (a) the first tin opened contains soup, (b) the first tin does not contain soup but the second does, (c) none of the first three tins contains soup. [You may find tree diagrams helpful.]

14. Two cards are selected at random without replacement from a set of five numbered 2, 3, 4, 4, 5. With the aid of a diagram or otherwise, find the probability that the numbers on the cards (a) are both even, (b) have a difference of 2, (c) have a sum of 7, (d) have a sum of 8.

15. A bag contains 3 blue beads and 2 red beads. A second bag contains 1 blue bead and 3 red beads. If one bead is drawn from each bag, find the probability that the beads are (a) both blue, (b) one blue and one red.

16. Write down in factorial form the number of different arrangements of the letters of the word EQUILIBRIUM. One of these arrangements is chosen at random. Find the probability that (a) the first two letters of the arrangement are consonants, (b) all the vowels are together.

17. There are 8 dusters in a drawer, 5 orange and 3 pink. If two of the dusters are chosen at random simultaneously, what is the probability that there will be one of each colour.

18. Four balls, two red, one blue and one white, are placed in a bag. The balls are then drawn at random from the bag one at a time and not replaced. Find the probability that (a) the first ball drawn is red, (b) the last ball drawn is white, (c) the red balls are drawn consecutively.

19. A hand of three cards is dealt from a well shuffled pack of 52. Find the probability that the hand contains (a) exactly one ace, (b) three cards of the same suit, (c) no two cards of the same suit.

20. On a plate of 12 assorted cakes, 3 are doughnuts. If 3 cakes are selected at random, find the probabilities that 0, 1, 2, 3 doughnuts are chosen.

21. A bag contains 25 clothes pegs, 15 plastic pegs and 10 wooden pegs. If 4 pegs are taken from the bag at random, find the probability that (a) all 4 are plastic, (b) 2 are plastic and 2 are wooden.

22. Four books are selected at random from a shelf containing 3 cookery books, 5 novels and 2 biographies. Find the probability that the four books (a) are all novels, (b) are 2 novels and 2 cookery books, (c) include at least one biography, (d) include at least one of each type.

14.4 Sum and product laws

If A and B are two different events which may occur in a sample space S, the set theory result

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

discussed in §14.2 can be used to find the probability that A or B (or both) will occur.

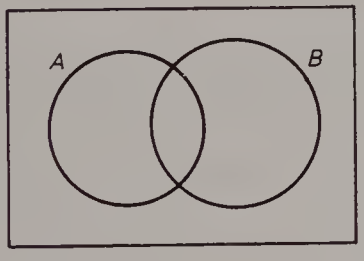
$$P(A \cup B) = \frac{n(A \cup B)}{n(S)} = \frac{n(A) + n(B) - n(A \cap B)}{n(S)}$$

$$= \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} - \frac{n(A \cap B)}{n(S)}$$

$$\therefore \quad \boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B).}$$

This is the *addition law* of probability.

Example 1 A card is drawn at random from a pack. A is the event of drawing an ace and B is the event of drawing a diamond. Find $P(A)$, $P(B)$, $P(A \cap B)$ and deduce $P(A \cup B)$.



$$P(A) = \frac{4}{52} = \frac{1}{13}, \quad P(B) = \frac{13}{52} = \frac{1}{4}.$$

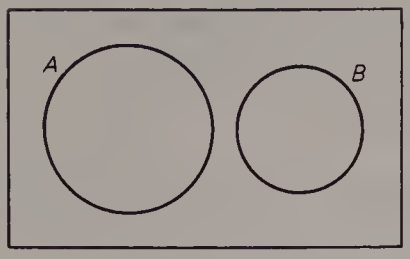
$P(A \cap B)$ is the probability of drawing the ace of diamonds, which is 1/52.

Hence, using the addition law,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}.$$

Events A and B are said to be *mutually exclusive* if they cannot occur at the same time, i.e. if the sets A and B are disjoint. This means that $A \cap B = \varnothing$ and $P(A \cap B) = 0$, so that for mutally exclusive events the addition law reduces to $P(A \cup B) = P(A) + P(B)$. In particular, if $B = A'$, then $P(A \cup B) = P(A \cup A') = P(S) = 1$, giving the result $1 = P(A) + P(A')$, i.e. $P(A') = 1 - P(A)$, which was obtained in the previous section.

Example 2 A disc is drawn at random from a box containing 12 discs of which 6 are red and 2 are blue. If event A is the choice of a red disc and event B is the choice of a blue disc, find $P(A)$, $P(B)$ and deduce $P(A \cup B)$.



$$P(A) = \frac{6}{12} = \frac{1}{2},\ P(B) = \frac{2}{12} = \frac{1}{6}$$

Since a disc cannot be both red and blue, the events A and B are mutually exclusive,

$$\therefore\ P(A \cup B) = P(A) + P(B)$$

$$= \frac{6}{12} + \frac{2}{12} = \frac{8}{12} = \frac{2}{3}.$$

A set of events is said to be *exhaustive* if at least one of them is certain to occur in any trial, e.g. if the events A and B are exhaustive, then $P(A \cup B) = 1$ and the addition law becomes $P(A)+P(B)-P(A \cap B) = 1$. This will reduce to $P(A)+P(B) = 1$, if A and B are also mutually exclusive.

In general, for events $A_1, A_2, \ldots, A_n$ which are both mutually exclusive and exhaustive

$$P(A_1)+P(A_2)+\ldots+P(A_n) = 1.$$

Two events A and B are said to be *independent* if the occurrence of event A has no effect on the probability of event B. However, if the probability of event B changes when A is known to have occurred then A and B are *dependent* upon one another.

Let us consider a playing card drawn at random from a pack and events A, B and C defined as follows: A: drawing a king, B: drawing a club, C: drawing a black king. The probability of drawing a club from the complete pack is 1/4. If the card drawn is known to be a king, i.e. if event A has occurred, then the probability that the card is a club remains 1/4. Hence events A and B are independent. Since $P(A) = 1/13$, $P(B) = 1/4$ and $P(A \cap B) = 1/52$, we find that $P(A \cap B) = P(A).P(B)$. However, if the card drawn is known to be a black king, i.e. if event C has occurred, then the probability that the card is a club is now 1/2 rather than 1/4. Hence events B and C are not independent. Since $P(B) = 1/4$, $P(C) = 1/26$ and $P(B \cap C) = 1/52$, in this case $P(B \cap C) \neq P(B).P(C)$.

This example illustrates the fact that

> two events A and B are independent if and only if
>
> $$P(A \cap B) = P(A).P(B)$$

This is the simplest form of the *multiplication* or *product law*. [The extension to events which are not independent is dealt with in the next section.]

When two dice are thrown, the scores obtained are independent of each other, so the product law may be applied. This provides an alternative approach to Example 3, §14.3.

The probability of throwing 1 on a single die is $\frac{1}{6}$, so the probability of throwing two 1's, i.e. of scoring a total of 2, is $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$.

The probabilities of the outcomes (1,2) and (2,1) are both equal to $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$. Hence the probability of scoring a total of 3 is $\frac{1}{36} + \frac{1}{36} = \frac{2}{36} = \frac{1}{18}$.

For any score on the 1st die, the probability of the 2nd score being the same is $\frac{1}{6}$. Thus the probability of throwing the same score on both dice is $\frac{1}{6}$.

Since the probability of scoring 3 or more on a single die is $\frac{2}{3}$, the probability of throwing 3 or more on each die is $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$.

Example 3 A boy must throw a 6 on a single die to start a game. Find the probability that he succeeds at his third attempt.

The probability of not throwing a 6 in his first turn $= \frac{5}{6}$.

The probability of not throwing a 6 in his second turn $= \frac{5}{6}$.

The probability of throwing a 6 in his third turn $= \frac{1}{6}$.

Hence the probability that he succeeds at his third attempt

$$= \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} = \frac{25}{216}.$$

Example 4 A card is drawn from a pack and then replaced, the suit being noted. Find the probability that in a sequence of four such trials exactly two diamonds are drawn.

The number of orders in which two diamonds and two other cards may be drawn in the four trials is $\frac{4!}{2!2!}$, i.e. 6.

In each of these 6 cases, the probability of drawing two diamonds and two other cards in the appropriate order is $\left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^2$.

Hence the probability that two diamonds are drawn

$$= 6 \times \frac{3.3}{4.4.4.4} = \frac{27}{128}.$$

Exercise 14.4

1. If A and B are random events such that $P(A) = 5/12$, $P(B) = 1/3$ and $P(A \cap B) = 1/4$, find $P(A \cup B)$.

2. In a certain school the probability that a pupil takes part in the annual musical evening is 0·25. The probability that a pupil helps with the school play is 0·1. If the probability of doing both is 0·02, find the probability that a pupil is involved in at least one of these activities.

3. A butcher observes that the probability that a certain customer buys beef is $\frac{1}{2}$ and the probability that she buys pork is $\frac{1}{3}$. If the probability that this customer buys neither pork nor beef is $\frac{1}{4}$, find the probability that she buys both beef and pork.

In questions 4 and 5 for the given events A, B and C
(i) write down any pairs of events which are (a) mutually exclusive, (b) exhaustive;
(ii) find (a) $P(A \cap B)$, (b) $P(A \cup C)$, (c) $P(B \cap C)$,
(d) $P(A' \cup C)$, (e) $P(B \cap C')$, (f) $P(A' \cup B')$.

4. A card is drawn at random from a pack of 52. A: drawing a black card, B: drawing a red card, C: drawing a heart.

5. Two unbiased dice are thrown and the total score recorded. A: a score less than 8, B: a score divisible by 5, C: a score greater than 4.

6. The events A and B are mutually exclusive. The events A, B and C are exhaustive. If $P(B) = 0{\cdot}6$ and $P(C) = 0{\cdot}3$, find the range of possible values of $P(A')$.

In questions 7 and 8, for the given events A, B and C, find $P(A)$, $P(B)$, $P(C)$, $P(A \cap B)$, $P(B \cap C)$, $P(A \cap C)$. Write down any pairs of events which are independent.

7. A card is drawn at random from a pack of 52. A: drawing a black card, B: drawing the jack of hearts, C: drawing a court card (i.e. jack, queen or king).

8. A disc is drawn at random from a bag containing discs numbered 1 to 48. A: drawing a multiple of 3, B: drawing a multiple of 4, C: drawing a multiple of 6.

9. If A and B are two independent events such that $P(A) = 1/4$ and $P(B) = 3/5$, find
(a) $P(A \cap B)$, (b) $P(A \cup B)$, (c) $P(A \cap B')$, (d) $P(A' \cup B')$.

10. A and B are two independent events, such that $P(A \cap B) = 1/3$ and $P(A \cup B) = 9/10$. Given that $P(A) > P(B)$, find $P(A)$ and $P(B)$.

11. If A, B and C are independent random events such that $P(A) = 2/5$, $P(B) = 1/3$, $P(C) = 3/4$, calculate $P(A \cup B \cup C)$ and $P(A' \cup B' \cup C')$.

12. A box contains 5 red balls and 3 blue balls. A second box contains 2 red balls and 3 blue balls. Find the probability that (a) if a ball is drawn at random from each box both will be blue, (b) if two balls are drawn at random from each box all four will be red.

13. A die is biased so that the probability of throwing a six is $\frac{1}{4}$. If the die is thrown twice find the probability of (a) two sixes, (b) at least one six.

14. A man estimates that the probability that he will be early for work is 1/3, the probability that he will be on time is 1/2 and the probability that he will be late is 1/6. Find the probability that in a particular 3-day period he will (a) arrive early every day, (b) never arrive on time, (c) be late at least once.

15. Three unbiased dice are thrown. Find the probability that (a) the score on each die is the same, (b) the total score is 5, (c) the total score is even.

16. Alan and Bob are playing a game which ends when either player is two points ahead of his opponent. If the probability that Alan wins any particular point is $\frac{2}{3}$, find the probabilities that the game ends when the number of points played is (a) 2, (b) 3, (c) 4.

17. Tom and Peter throw a single die in turn. The first player to throw a six wins. If Tom throws first, find the probability that Peter wins on his first, second or third turn. [Give your answer to 3 significant figures.]

18. Cards are drawn at random, with replacement, from a pack of 52. Find the probability that (a) the first two cards drawn are spades, (b) two out of the first three cards drawn are spades, (c) the third card drawn is a club, (d) the fourth card is the first black card to be drawn.

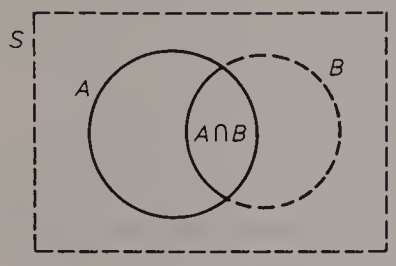
19. A certain committee has 6 members. The probability that any member will attend a particular meeting is 4/5. Find the probability that the chairman and at least 4 of the 5 other members will attend.

20. An unbiased coin is tossed 5 times. Find the probabilities of (a) 4 heads, (b) 3 heads, (c) at least 2 heads.

21. A bag contains 3 red discs, 4 blue discs and 5 green discs. A trial consists of selecting a disc at random, noting its colour then replacing it. In three such trials, what is the probability of selecting (a) three red discs, (b) two blue discs, (c) one green and two blue discs?

14.5 Conditional probability

The probability of an event B, given that an event A has occurred, is called the *conditional probability* of B upon A, or the probability of B given A, and is written $P(B|A)$. This means that $P(B|A)$ is the probability that B occurs, considering A as sample space.



Since the subset of A in which B occurs is $A \cap B$,

$$P(B|A) = \frac{n(A \cap B)}{n(A)} = \frac{n(A \cap B)}{n(S)} \Big/ \frac{n(A)}{n(S)}$$

$$\therefore \quad \boxed{P(B|A) = \frac{P(A \cap B)}{P(A)}.}$$

The general statement of the product law is obtained by rearranging this result:

$$P(A \cap B) = P(A).P(B|A).$$

Similarly

$$P(A \cap B) = P(B).P(A|B).$$

In the case of independent events the probability of B is unaffected by the occurrence of A, so $P(B|A) = P(B)$

$$\therefore \quad P(A \cap B) = P(A).P(B).$$

Example 1 Two cards are drawn in succession from a pack. What is the probability that they are both diamonds if (a) the first card is replaced before the second is drawn, (b) the first card is not replaced?

(a) Since both cards are drawn from a full pack, in each case the probability of drawing a diamond is $\frac{13}{52}$, i.e. $\frac{1}{4}$

$\therefore$ with replacement the probability of two diamonds is

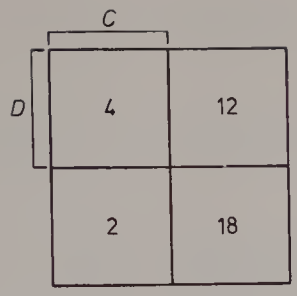
$$\frac{1}{4} \times \frac{1}{4} = \frac{1}{16}.$$

(b) The probability that the second card is a diamond given that a diamond has already been drawn is $\frac{12}{51} = \frac{4}{17}$,

$\therefore$ without replacement, the probability of two diamonds is

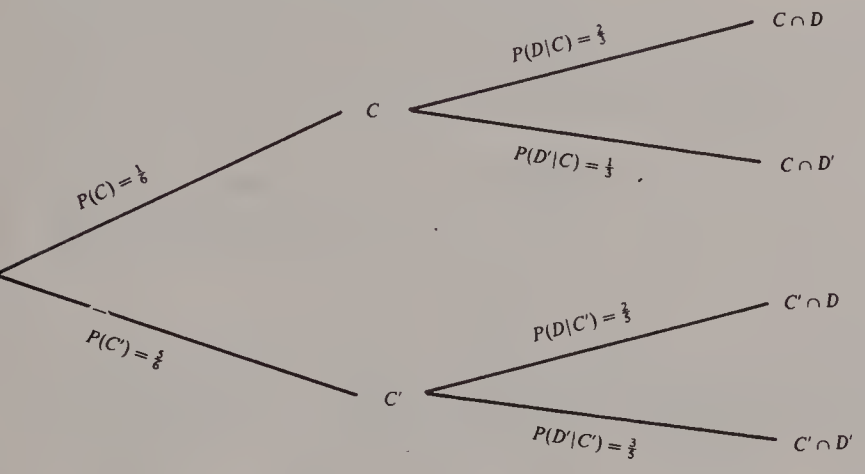
$$\frac{1}{4} \times \frac{4}{17} = \frac{1}{17}.$$

Consider now a trial in which two dice are thrown, where event C is throwing the same score on both dice and event D is throwing 3 or more on each of the dice.



The 36 possible outcomes in the sample space for this trial can be represented in a Karnaugh map for events C and D.

Relationships between probabilities associated with such a trial are often displayed in a *probability tree diagram.*



$$P(C \cap D) = \frac{1}{6} \cdot \frac{2}{3} = \frac{1}{9}$$

$$P(C \cap D') = \frac{1}{6} \cdot \frac{1}{3} = \frac{1}{18}$$

$$P(C' \cap D) = \frac{5}{6} \cdot \frac{2}{5} = \frac{1}{3}$$

$$P(C' \cap D') = \frac{5}{6} \cdot \frac{3}{5} = \frac{1}{2}$$

Probabilities can also be recorded in a *contingency table* as shown below. The diagram on the left shows which probabilities should be inserted in each part of the table. The completed table is on the right.

	D	D'	
C	$P(C \cap D)$	$P(C \cap D')$	$P(C)$
C'	$P(C' \cap D)$	$P(C' \cap D')$	$P(C')$
	$P(D)$	$P(D')$	

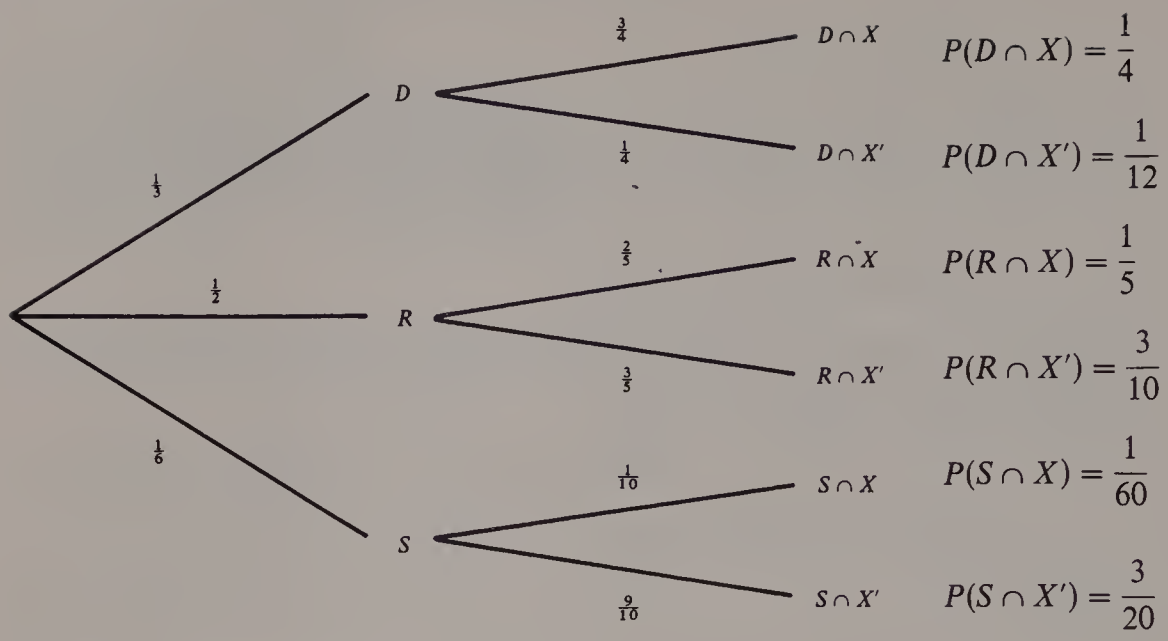
	D	D'	
C	$\frac{1}{9}$	$\frac{1}{18}$	$\frac{1}{6}$
C'	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{5}{6}$
	$\frac{4}{9}$	$\frac{5}{9}$	1

Both the tree diagram and the contingency table demonstrate that

$$P(C \cap D) + P(C \cap D') + P(C' \cap D) + P(C' \cap D') = P(C) + P(C') = 1.$$

Example 2 A motorist plans a journey and event X is arrival at his destination in less than three hours. He estimates the probabilities of dry weather D, rain R or snow S to be $\frac{1}{3}$, $\frac{1}{2}$ and $\frac{1}{6}$ respectively. The probabilities of event X in these conditions are $\frac{3}{4}$, $\frac{2}{5}$ and $\frac{1}{10}$ respectively. What is the probability that (a) the motorist completes his journey in under 3 hours, (b) if he fails to arrive in less than 3 hours, there was a fall of snow?

(a) On the assumption that events D, R and S are mutually exclusive, the following tree diagram can be constructed.



$$P(X) = P(D \cap X) + P(R \cap X) + P(S \cap X) = \frac{1}{4} + \frac{1}{5} + \frac{1}{60} = \frac{7}{15}$$

$\therefore$ the probability that the journey is completed in less than 3 hours is 7/15.

(b) $P(S|X') = \dfrac{P(S \cap X')}{P(X')} = \dfrac{P(S \cap X')}{1 - P(X)} = \dfrac{\frac{3}{20}}{1 - \frac{7}{15}} = \dfrac{3}{20} \cdot \dfrac{15}{8} = \dfrac{9}{32}.$

$\therefore$ given that the motorist failed to arrive in less than three hours, the probability that it snowed is 9/32.

	D	R	S
X	1/4	1/5	1/60
X′	1/12	3/10	3/20

It is sometimes helpful to display probabilities on a Karnaugh map. We give here the probability map for Example 2. This shows clearly that

$$P(X) = \frac{1}{4} + \frac{1}{5} + \frac{1}{60} = \frac{7}{15}.$$

When finding $P(S|X')$ we consider X' as sample space. Thus, using the lower half of the diagram only,

$$P(S|X') = \frac{3}{20} \Big/ \left(\frac{1}{12} + \frac{3}{10} + \frac{3}{20}\right) = \frac{3}{20} \Big/ \frac{32}{60} = \frac{9}{32}.$$

Exercise 14.5

Questions 1 to 5 concern a trial in which a domino is drawn from a complete set of 28. Event A: it has at least one six on it. Event B: it is a double, i.e. has two identical numbers on it. Event C: the sum of the numbers is six or less.

1. Find $P(A)$, $P(B)$, $P(C)$.

2. Find $P(B|A)$, $P(C|A)$, $P(A \cap B)$, $P(A \cap C)$.

3. Find $P(B|C)$, $P(C|B)$, $P(B|C')$, $P(C|B')$.

4. Find $P(B \cap C)$, $P(B \cap C')$, $P(B' \cap C)$.

5. Find $P(A \cap C')$, $P(A' \cap B')$, $P(A' \cap C')$.

Questions 6 to 10 Repeat questions 1 to 5 for a trial in which a card is drawn from a normal pack. Event A: drawing a king or a queen. Event B: drawing a black card. Event C: drawing a heart.

11. A bag contains 4 black discs and 2 white ones. If discs are drawn from the bag one at a time, find the probability that 2 black discs followed by 2 white discs are drawn (a) with replacement, (b) without replacement.

12. A bag contains 10 cards bearing the letters of the word STATISTICS. Find the probability that the first four letters drawn at random from the bag are S, I, T, S in that order (a) with replacement, (b) without replacement.

13. The probability that it will be foggy on a November morning is $\frac{1}{3}$. The probability that Mr. Jones will be late for work when it is foggy is $\frac{1}{2}$. The probability that he will be late if it is not foggy is $\frac{1}{8}$. If on a particular November morning Mr. Jones is late, find the probability that it is foggy.

14. On a day on the summer holidays the probability that Jane will go swimming is 0·6. The probability that both Jane and Karen will go swimming is 0·45. If Jane is seen in the swimming pool what is the probability that Karen also swims that day?

15. The probability that a light bulb has a life of more than 1000 hours is $\frac{1}{2}$ and the probability of a life of more than 2000 hours is $\frac{1}{10}$. What is the probability that a bulb that has already been in use for 1000 hours will last a further 1000 hours?

16. A football star is having injury problems. When he is playing the probability that his team will win is $\frac{3}{4}$, but otherwise it is only $\frac{1}{2}$. The probability that the player will be fit this Saturday is $\frac{1}{3}$. Find the probability that his team will win the match.

17. The probability that it will be sunny tomorrow is 0·25. The probability that Mrs. Brown will go shopping tomorrow is 0·2. The probability that both these events occur is 0·15. Find the probability that (a) neither event occurs, (b) Mrs. Brown goes shopping, given that it is sunny, (c) it is sunny given that Mrs. Brown goes shopping.

18. A bag contains 5 black beads and 3 white beads. A second bag contains 3 black beads and 5 white beads. A bead is drawn at random from the first bag and placed in the second bag. A bead is then drawn at random from the second bag

and placed in the first. Find the probability that each bag now contains (a) 4 black and 4 white beads, (b) the same numbers of each colour as it did initially.

19. A box A contains 3 red balls and 4 black balls. A box B contains 3 red balls and 2 black balls. One box is selected at random and then from that box one ball is selected at random. Find (a) the probability that the ball is red, (b) the probability that the ball came from box A, given that it is red.

20. A member of a slimming club claims that the probability that she will have a chocolate biscuit with her morning coffee is $\frac{1}{6}$ and that on the days that she resists the temptation, the probability that she exceeds her daily calorie allowance is $\frac{1}{5}$. If the probability that she consumes too many calories on any day is $\frac{1}{4}$, find the probability that she does so on a day when she has had a chocolate biscuit. Given that yesterday she kept within her calorie allowance, find the probability that she ate a chocolate biscuit.

Exercise 14.6 (miscellaneous)

1. How many even five-digit numbers can be formed using the digits 0, 1, 2, 3, 4, 5, 6 (excluding numbers which start with 0) if (a) no digit may be repeated, (b) repetition is allowed?

2. Find the number of different ways in which the letters of the word ISOSCELES can be arranged. How many of these arrangements (a) begin and end with E, (b) begin with S, (c) end with a vowel?

3. A man has found 6 records that he likes in a shop, but cannot decide which of them to buy. Assuming that he buys at least one, find the number of different selections he could make.

4. Six lines are drawn, no two of which are parallel. If no more than two of the lines pass through any one point, find the number of triangles formed.

5. In how many ways can a group of 9 people attending a conference be split into three sets of 3 people for discussion? Later the same people form new sets of 3. In how many ways can this be done if no two people remain together?

6. Find the number of ways in which 5 books can be distributed between three people A, B and C if the books are (a) all different, (b) indistinguishable.

7. Draw a Venn diagram to show the following sets and their non-empty intersections: $\mathscr{E} = \{\text{all quadrilaterals}\}$, $T = \{\text{trapeziums}\}$, $R = \{\text{rectangles}\}$, $P = \{\text{parallelograms}\}$, $K = \{\text{kites, i.e. two pairs of adjacent sides equal}\}$. Describe the elements of the sets $K \cap R$ and $K \cap P$.

8. Decide whether the following statements are true or false for sets A, B and C, giving brief reasons for your answers.

(a) $A \cap B = A \cup B \Leftrightarrow A = B$,
(b) $A \cup B = A \cup C \Leftrightarrow B = C$,
(c) $A \cap B = A \cap C \Leftrightarrow A = B \cap C$,
(d) $A \cap (B' \cup C') = \varnothing \Leftrightarrow A \subset (B \cup C)$.

9. In a certain school half the pupils under 16 years of age are girls, two thirds of the girls in the school are under 16 and a quarter of the pupils are boys of 16 and over. If there are 540 girls in the school, find the total number of pupils.

10. A drawer contains 4 different pairs of socks. Find the probability that (a) if 2 socks are selected at random they will form a pair, (b) if 4 socks are selected at random they will form two pairs.

11. (a) Two dice are thrown together, and the scores added. What is the probability that (i) the total score exceeds 8, (ii) the total score is 9, or the individual scores differ by 1, or both?
(b) A bag contains 3 red balls and 4 black ones. 3 balls are picked out, one at a time and not replaced. What is the probability that there will be 2 red and 1 black in the sample?
(c) A committee of 4 is to be chosen from 6 men and 5 women. One particular man and one particular woman refuse to serve if the other person is on the committee. How many different committees may be formed? (SU)

12. An analysis of the other subjects taken by A-level Mathematics candidates in a certain year showed that 20% of them took Further Mathematics, 50% took Physics and 5% took both Further Mathematics and Physics. A candidate is chosen at random from those who took A-level Mathematics. (i) Calculate the probability that the chosen candidate took neither Further Mathematics nor Physics. (ii) Given that the chosen candidate took at least one of Further Mathematics and Physics, calculate the probability that the candidate took Further Mathematics. (JMB)

13. Two cards are drawn simultaneously from a pack of 52. What is the probability that both are spades if one card is known to be (a) black, (b) red, (c) a spade, (d) a king?

14. An experiment consists of tossing two fair coins one after the other. Let X be the set of events in which the first coin is a head, Y be the set of events in which the second coin is a head and Z be the set of events in which both coins are heads or both coins are tails. Calculate the following probabilities: $p(X)$, $p(Y)$, $p(Z)$, $p(X \cap Z)$, $p(X \cap Y \cap Z)$. (AEB 1977)

15. Three cards are drawn at random without replacement from a pack of ten cards which are numbered from 1 to 10, respectively. Calculate (i) the probability that the numbers drawn consist of two even numbers and one odd number, (ii) the probability that at least one of the numbers drawn is a perfect square greater than 1, (iii) the probability that the smallest number drawn is the 5. (JMB)

16. A disc of diameter 5 cm is tossed at random on to a large sheet of paper ruled with parallel lines 8 cm apart. Assuming the lines to be of negligible thickness, find the probability that the disc lands on a line. If the paper is now ruled with 8 cm squares, find the probability that (a) the disc lands on a line, (b) the disc lands on the intersection of two lines.

17. A pack of 52 cards contains 4 suits each of 13 cards. If 13 cards are taken at random from the pack what is the probability that exactly 10 of them are spades? [You may take $\binom{52}{13} = 6{\cdot}35 \times 10^{11}$.] (O&C)

18. A bag contains 5 red, 4 orange and 3 yellow sweets. One after another, three children select and eat one sweet each. When the bag contains n sweets, the probability of any one child choosing any particular sweet is $1/n$. What are the probabilities that (a) they all choose red sweets, (b) at least one orange sweet is chosen, (c) each chooses a different colour, (d) all choose the same colour? [Answers to this question may be left as fractions in their lowest terms.] (O&C)

19. (a) Two independent events are such that there is a probability of $\frac{1}{6}$ that they will both occur and a probability of $\frac{1}{3}$ that neither will occur. Calculate their individual probabilities of occurring.
(b) A fair cubical die has three of its faces coloured red, two coloured blue and one coloured white. If the die is thrown six times calculate the probabilities that (i) a red face will be uppermost at least once, (ii) a red face will be uppermost exactly three times, (iii) each colour will be uppermost exactly twice. (W)

20. Four cards are drawn at random from a pack, one at a time with replacement. Find the probability that (a) no heart is drawn, (b) four hearts are drawn, (c) two hearts and two diamonds are drawn (in any order), (d) one card from each suit is drawn. (L)

21. Two events A and B are such that $P(A) = 0{\cdot}4$ and $P(A \cup B) = 0{\cdot}7$. (i) Find the value of $P(A' \cap B)$. (ii) Find the value of $P(B)$ if A and B are mutually exclusive. (iii) Find the value of $P(B)$ if A and B are independent. (JMB)

22. The following are three of the classical problems in probability.
(a) Compare the probability of a total of 9 with the probability of a total of 10 when three fair dice are tossed once (Galileo and Duke of Tuscany).
(b) Compare the probability of at least one six in 4 tosses of a fair die with the probability of at least one double-six in 24 tosses of two fair dice (Chevalier de Méré).
(c) Compare the probability of at least one six when 6 dice are rolled with the probability of at least two sixes when 12 dice are rolled (Pepys to Newton).
Solve each of these problems. (AEB 1978)

23. (a) From an ordinary pack of 52 cards two are dealt face downwards on a table. What is the probability that (i) the first card dealt is a heart, (ii) the second card dealt is a heart, (iii) both cards are hearts, (iv) at least one card is a heart?

(b) Bag A contains 3 white counters and 2 black counters whilst bag B contains 2 white and 3 black. One counter is removed from bag A and placed in bag B without its colour being seen. What is the probability that a counter removed from bag B will be white?
(c) A box of 24 eggs is known to contain 4 old and 20 new. If 3 eggs are picked at random determine the probability that (i) 2 are new and the other old, (ii) they are all new. (SU)

24. A hand of 5 cards is drawn from a pack of 52 playing cards. Find the probability of drawing (a) 5 cards of the same suit, (b) a pair plus a triple (e.g. 2 sevens and 3 aces). If a hand consists of 2 aces, a king, a jack and a 4 and a player keeps the pair of aces but discards the other 3 cards and draws 3 more cards from the remaining 47, find the probability of obtaining a pair plus a triple. (L)

25. If A, B and C are exhaustive events, such that A cannot occur at the same time as either B or C, show that
(a) $P(A)+P(B)+P(C) = 1+P(B \cap C)$, (b) $P(A') = P(B \cup C)$.

26. Two cards are drawn without replacement from a pack of playing cards. Using a tree diagram, or otherwise, calculate the probability (a) that both cards are aces, (b) that one (and only one) card is an ace, (c) that the two cards are of different suits. Given that at least one ace is drawn, find the probability that the two cards are of different suits. (L)

27. (a) Two events A and B are such that $P(A) = \frac{1}{3}$ and $P(B) = \frac{1}{2}$. If A' denotes the complement of A, calculate $P(A' \cap B)$ in each of the cases when (i) $P(A \cap B) = \frac{1}{8}$, (ii) A and B are mutually exclusive, (iii) A is a subset of B.
(b) A Scottish court may give any one of the three verdicts: 'guilty', 'not guilty' and 'not proven'. Of all the cases tried by the court, 70% of the verdicts are 'guilty', 20% are 'not guilty' and 10% are 'not proven'. Suppose that when the court's verdict is 'guilty', 'not guilty' and 'not proven', the probabilities that the accused is really innocent are 0·05, 0·95 and 0·25, respectively. Calculate the probability that an innocent person will be found 'guilty' by the court. (W)

28. Suppose that letters sent by first and second class post have probabilities of being delivered a given number of days after posting according to the following table (weekends are ignored).

Days to delivery	1	2	3
1st class	0·9	0·1	0
2nd class	0·1	0·6	0·3

The secretary of a committee posts a letter to a committee member who replies immediately using the same class of post. What is the probability that four or more days are taken from the secretary posting the letter to receiving the reply if (a) first class, (b) second class post is used? The secretary sends out four letters and each member replies immediately by the same class of post. Assuming the letters move independently, what is the probability that the secretary receives (a)

all the replies within three days using first class post, (b) at least two replies within three days using second class post? (O)

29. In a game, three cubical dice are thrown by a player who attempts to throw the same number on all three. What is the chance of the player (a) throwing the same number on all three, (b) throwing the same number on just two? If the first throw results in just two dice showing the same number, then the third is thrown again. If no two dice show the same number, then all are thrown again. The player then comes to the end of his turn. What is the chance of the player succeeding in throwing three identical numbers in a complete turn? What is the chance that all the numbers are different at the end of a turn? (O&C)

30. An experiment is performed with a die and two packs of cards. The die is thrown, and if it shows 1, 2, 3 or 4 a card is drawn at random from the first pack, which contains the usual 52 cards; if the score on the die is 5 or 6 a card is drawn from the second pack, which contains only 39 cards, all the clubs having been removed. X denotes the event 'The first pack with 52 cards is used', and Y denotes the event 'The card drawn is a diamond'. Calculate the probabilities (i) $P(X)$, (ii) $P(X \cap Y)$, (iii) $P(Y)$, (iv) $P(Y|X)$, (v) $P(X|Y)$. (C)

31. Each of two boxes contains ten discs. In one box four of the discs are red, two are white and four are blue; in the other box two are red, three are white and five are blue. One of these two boxes is chosen at random and three discs are drawn at random from it without replacement. Calculate (i) the probability that one disc of each colour will be drawn, (ii) the probability that no white disc will be drawn, (iii) the most probable number of white discs that will be drawn. Given that three blue discs were drawn, calculate the conditional probability that they came from the box that contained four blue discs. (W)

32. Write down an equation, in terms of probabilities, corresponding to each of the statements (i) the events A and B are independent, (ii) the events A and C are mutually exclusive. The events A, B and C are such that A and B are independent and A and C are mutually exclusive. Given that $P(A) = 0{\cdot}4$, $P(B) = 0{\cdot}2$, $P(C) = 0{\cdot}3$, $P(B \cap C) = 0{\cdot}1$, calculate $P(A \cup B)$, $P(C|B)$, $P(B|A \cup C)$. Also calculate the probability that one and only one of the events B, C will occur. (W)

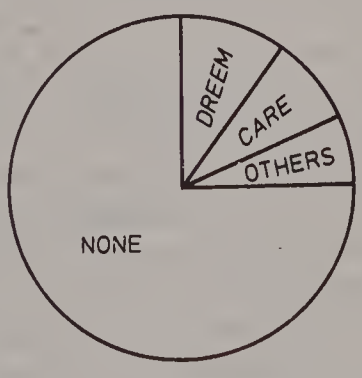
33. A pack of sixteen playing cards consists of the ace, king, queen and jack of each of the four suits, spades, hearts, diamonds and clubs. A man is dealt four cards at random from the pack. (i) Calculate the probability that he has been dealt exactly two aces. (ii) The man is asked whether he has been dealt any aces. He truthfully replies that he has. Calculate the probability that he has been dealt exactly two aces. (iii) By accident the man displays one of his four cards, which is seen to be the ace of spades. Calculate the probability that he has been dealt exactly two aces. Give your answers correct to two significant figures. (C)

15 Elementary statistics

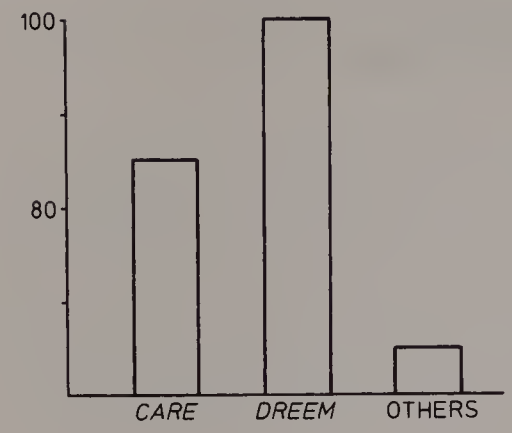
15.1 Graphical representation of data

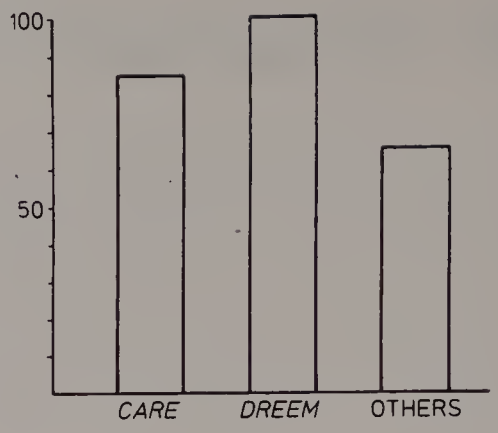
Statistics may be described as a subject concerned with the analysis of data obtained from such sources as experiments and surveys. In elementary work this will involve making tables, charts and graphs to present the data as clearly as possible. In more advanced work various numerical measures of the properties of collections of data are studied. These may be used to compare sets of observations, to draw conclusions from information gathered and to make predictions about future events. An important aspect of the subject is measuring the significance of such conclusions and predictions.

We first consider three common ways of presenting information in graphical form. Below are two *pie charts* showing the results of an investigation into which brands of fabric softener housewives use. The charts both show the relative proportions of the different brands used, but the first is quite misleading as it gives no indication that the majority of housewives use no softener at all.

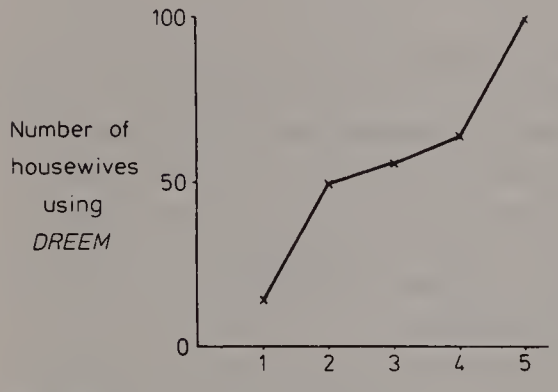


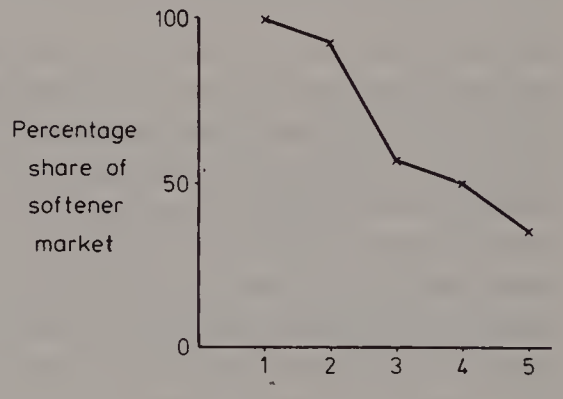
The same results can also be illustrated in the form of a *bar chart* or *block graph.* In both the given charts the vertical scale represents the number of housewives using the products. However, the first diagram exaggerates the lead of 'DREEM' in the market by starting the vertical axis at 60 instead of 0.





The following *line graphs* both represent the results of 5 successive surveys. This type of graph is used to illustrate a trend or pattern in results. However, in this case either graph on its own could be misleading because although sales of 'DREEM' are rising, its share of the market in fabric softeners is falling.





It is clear from these examples that care must be taken when making or interpreting graphs and charts, but it is also important to question the validity of the original data. In this survey a *sample* of 1000 housewives were questioned, but this is only a small proportion of the total *population* of housewives. The results could have been distorted because most of the people chosen belonged to the same income group or lived in the same area. The most reliable results are usually obtained by taking a *random sample*, in which each member of the population has an equal chance of being selected.

Another factor which may affect this type of investigation is the framing of the questions asked. For example, one person might give a different answer to each of the following four questions:

(1) Which product gives best value for money?
(2) Which product do you buy regularly?
(3) Which product did you buy this week?
(4) Which product do you prefer?

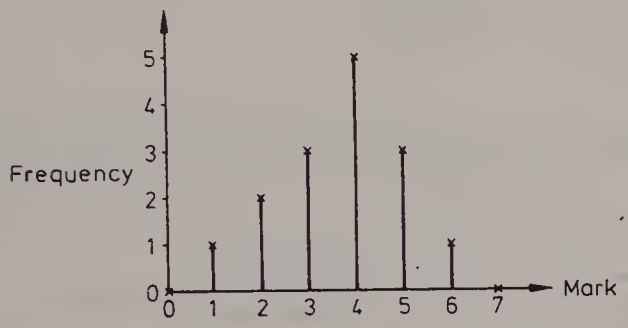
A survey based on only one of these questions could give misleading results.

Having considered some of the ways in which statistical information can be misunderstood or even misused, we now examine some more accurate ways of analysing and presenting results.

Example A In a short test given to a set of 15 students the marks were as follows:

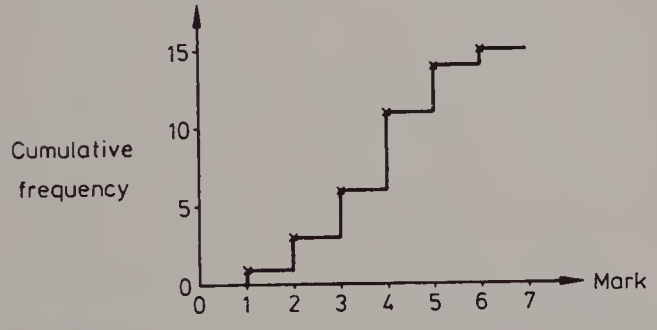
2 4 1 3 5 4 4 3 5 4 2 6 3 5 4

Since the test mark, x, of any particular student can take only certain values in the range 1 to 6, it is called a *discrete variable.* For a variable of this type it is helpful to make a table called a *frequency distribution* showing the frequency with which each value occurs. The corresponding diagram is called a *frequency chart.*



Mark	Frequency
1	1
2	2
3	3
4	5
5	3
6	1

When it is necessary to examine sums of frequencies, such as the number of students who scored 4 marks or less, a *cumulative frequency distribution* may be used. For a discrete variable x the cumulative frequency is the frequency of values *less than or equal to* x. As shown in the *cumulative frequency chart*, for a discrete variable, the cumulative frequency is a *step function.*



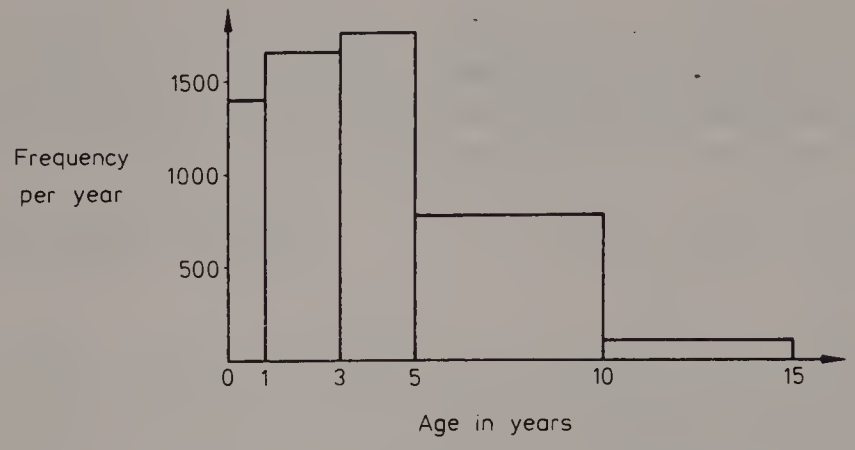
Mark	Cumulative frequency
1	1
2	3
3	6
4	11
5	14
6	15

Example B The table below shows the numbers of children known to have contracted whooping-cough in 1965.

Age	No. of cases (to the nearest 100)
Under 1 year	1400
1–2 years	3300
3–4 years	3500
5–9 years	3900
10–14 years	500

Since the age of a child can take any real value between 0 and 15 years, it is called a *continuous variable.* Because the variable is continuous and because the table gives frequencies for age groups of different sizes, a simple frequency chart

similar to the one used in Example A could be misleading. To allow for the various age ranges we draw a *histogram*, in which the *areas* (not the heights) of the columns are proportional to the frequencies.

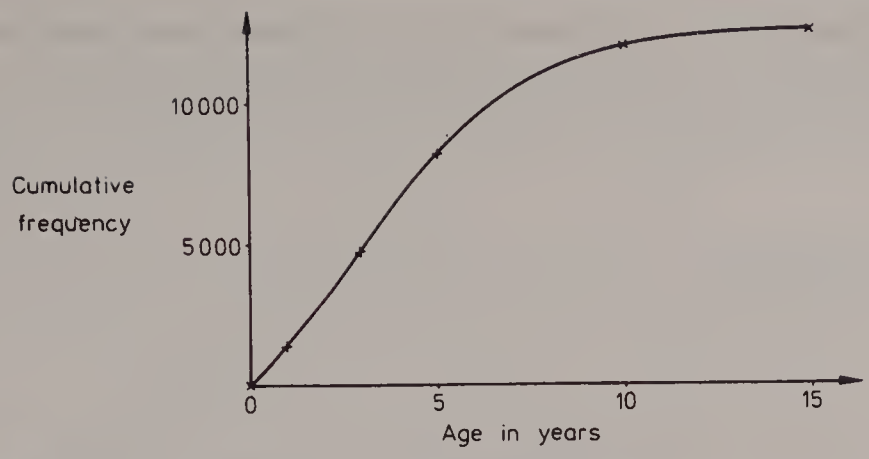


This means that the height of each column represents $\frac{\text{frequency}}{\text{column width}}$ (sometimes called *frequency density*). In this case the vertical scale may be taken to represent the number of cases of whooping-cough per one year age group.

The *frequency polygon* shown here is obtained from the histogram by joining the mid-point of the tops of the columns. As several frequency polygons can be drawn in the same diagram, they are often used for comparing distributions.

For a continuous variable x, it is convenient to regard the cumulative frequency as the frequency of values *less than* x. In this example the cumulative frequency does not move in a series of jumps, but increases gradually throughout the range 0 to 15 years. Thus the values in the cumulative frequency table are used to draw a curve called the *cumulative frequency graph* or *ogive*.

Age	Cumulative frequency
Under 1 year	1400
Under 3 years	4700
Under 5 years	8200
Under 10 years	12 100
Under 15 years	12 600



Example C The table below shows the number of staff meals issued per day in a college refectory over a period of 40 days.

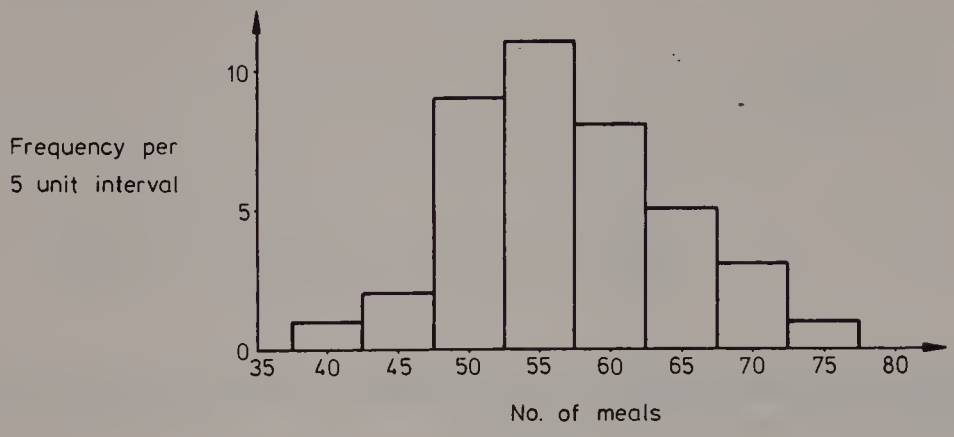
56	50	63	56	58	53	52	71	59	52
45	55	42	62	62	60	57	48	63	68
55	65	50	59	54	54	46	51	73	64
53	50	61	51	58	52	57	67	53	69

In this example the variable is again discrete. However, a frequency table of the type used in Example A would have little meaning as there is a range of 31 from the lowest value 42 to the highest 73. Any pattern in the distribution of observations will be shown more clearly by grouping the numbers. The range could be divided into intervals 40–50, 50–60, 60–70 and 70–80. One difficulty about this choice arises when we come to observations such as 50 which appear in two classes. This could be avoided by choosing intervals such as 40–49, 50–59, ... [In certain circumstances, it may be better to regard 50 as $\frac{1}{2}$ in the 40–50 class and $\frac{1}{2}$ in the 50–60 class.] The other main difficulty is that this grouping produces only 4 classes and will not illustrate the distribution in any detail. From this point of view, the intervals 40–44, 45–49, ... would be better. However, the mid-points of intervals are often used in calculations, so it is convenient if these are round numbers. Thus a reasonable choice in this example is the grouping 38–42, 43–47, ... into intervals of length 5 units with mid-points 40, 45, ... The *class limits* are the lowest and highest observations which could occur in each class, namely 38 and 42, 43 and 47, ... However, the *class boundaries* or *end-points* are taken, in this case, to be 37·5, 42·5, ...

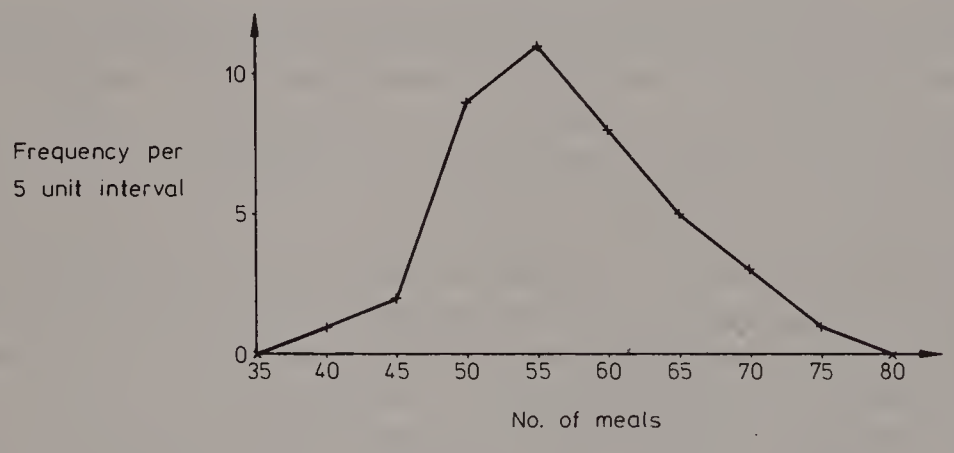
We may now draw up the frequency distribution for the grouped data using *tally marks* as shown below.

No. of meals		Frequency
38–42	I	1
43–47	II	2
48–52	𝍸 I	9
53–57	𝍸 𝍸 I	11
58–62	𝍸 III	8
63–67	𝍸	5
68–72	III	3
73–77	I	1

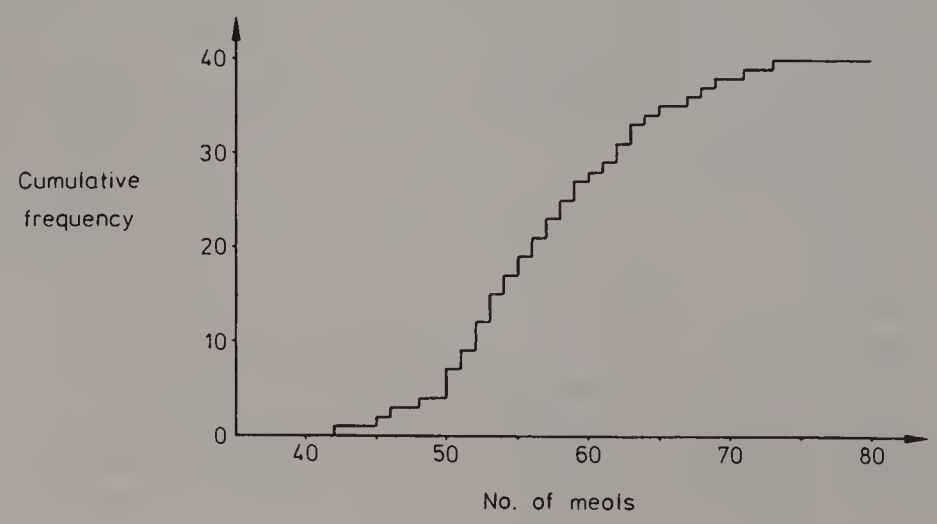
In the histogram of this distribution, since the columns are of equal width, their heights as well as their areas will be proportional to the frequencies.



When constructing a frequency polygon it is possible to plot zero frequencies at each end of the diagram to produce a closed polygon.

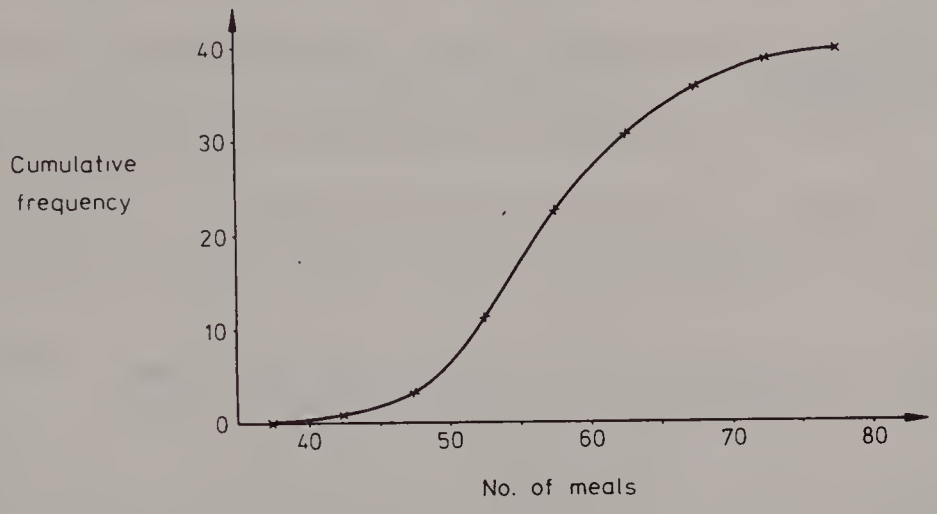


As in Example A the cumulative frequency chart constructed from the original data is a step diagram.

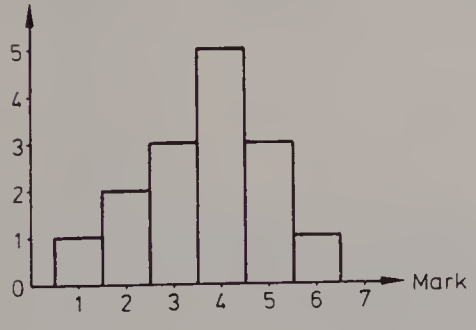


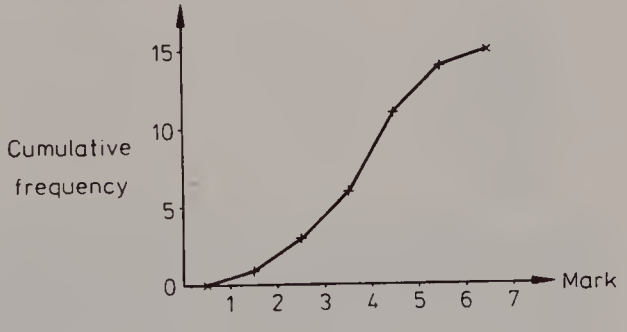
However, a cumulative frequency graph drawn using the grouped frequency distribution will provide a sufficiently good approximation for many purposes. To construct this graph we treat the number of meals as a continuous variable plotting the cumulative frequencies against the corresponding upper class boundaries.

No. of meals less than:	37·5	42·5	47·5	52·5	57·5	62·5	67·5	72·5	77·5
Cumulative frequency	0	1	3	12	23	31	36	39	40



In a similar way we could produce a histogram and a cumulative frequency polygon for the distribution in Example A by regarding it as a grouped distribution with class boundaries 0·5, 1·5, 2·5, ... However, such artificially produced diagrams have very little true meaning.





Exercise 15.1

[The data in questions 3 to 8 will be referred to in later exercises.]

1. Conduct a survey on a suitable topic of your own choice, such as the reading or television viewing habits of a group of students. Consider carefully (a) the questions to be used, (b) the way in which the data is to be recorded, (c) the graphical representation of the data collected. How could such a survey be conducted and the data presented in order to produce distorted or misleading results?

2. The table below shows the sales in two successive years of a particular type of car.

	2-door super	4-door super	4-door de-luxe	estate
1st year	8100	10 800	18 000	3600
2nd year	15 000	21 000	25 000	11 000

(a) Draw two pie charts whose areas allow the total sales for the two years to be compared.
(b) Draw a comparative bar chart, i.e. one in which there are two bars for each model of the car.
(c) Comment briefly on the advantages of the two ways of representing the data.

3. In an experiment the following 20 values of a variable x were recorded.

5	6	2	4	4	5	7	5	5	6
4	7	6	5	3	5	6	4	6	5

(a) Given that x is a discrete variable which can take only integral values, draw a frequency distribution chart and a cumulative frequency chart.
(b) Given that x is a continuous variable and that the above values are rounded to the nearest integer, draw a histogram and a cumulative frequency polygon.
(c) Given that x is a continuous variable and that the integral value n is recorded when $n \leqslant x < n+1$, draw a histogram and a cumulative frequency polygon.

4. The table below gives the number of dresses sold each day in a small shop over a 5-week period.

	Monday	Tuesday	Wednesday	Friday	Saturday
Week 1	2	2	4	4	9
Week 2	0	4	7	3	7
Week 3	1	0	3	1	6
Week 4	3	3	2	2	8
Week 5	1	3	5	3	8

Draw a frequency distribution chart and a cumulative frequency chart for the distribution. Comment briefly on other ways of representing the data graphically in order to show any patterns in the distribution more clearly.

5. In an investigation the following 50 values of a variable x were recorded:

6	11	12	8	9	8	9	4	2	7
9	7	4	11	7	5	8	7	6	4
9	6	3	4	6	7	5	6	6	9
4	9	7	1	4	7	7	4	10	9
5	8	10	7	4	9	5	7	9	11

Decide whether it would be more appropriate to represent the cumulative frequency distribution by a step diagram or an ogive given that (a) x is a discrete

variable which takes only integral values, (b) x is a continuous variable with values rounded to the nearest integer. Draw the appropriate graph in each case.

6. The length to the nearest minute of a man's journey to work over a period of 40 days are as follows:

57	56	33	56	49	56	44	56	45	58
62	57	42	68	49	44	59	63	50	58
53	58	50	56	60	59	45	51	65	53
50	38	65	43	51	63	56	61	67	61

Construct a suitable grouped frequency table, stating the class boundaries that you have used. Draw the corresponding histogram and cumulative frequency curve. Estimate from your graph the percentage of days on which the journey takes at least 45 minutes.

7. In a particular year 80 children enter a certain large infants' school. The ages of the children on the day they first attend the school are given in the table below.

Age in years	$4\frac{1}{4}$–	$4\frac{1}{2}$–	$4\frac{3}{4}$–	5–	$5\frac{1}{4}$–	$5\frac{1}{2}$–	6–	7–	$7\frac{3}{4}$–
Frequency	0	10	35	18	7	3	5	2	0

Draw a histogram, a frequency polygon and a cumulative frequency polygon to represent this distribution. Can you account for the entries in the table for children of $5\frac{1}{2}$ years and over? Estimate the probability that a child in the catchment area of this school is under 5 years old when he first attends school, stating any assumptions you have made.

8. The grouped frequency distribution of the test marks obtained by 150 students is given below.

Class limits	1–5	6–10	11–15	16–20	21–25
Frequency	0	4	12	22	30
Class limits	26–30	31–35	36–40	41–45	Over 45
Frequency	31	24	14	8	5

Draw a histogram to represent the data given that (a) the test was marked out of 50, (b) the test was marked out of 60, (c) the top mark was 54. Construct a cumulative frequency polygon in case (b). Give two reasons why this diagram represents an approximation to the true cumulative frequency distribution. Explain why a cumulative frequency curve may give a better approximation.

15.2 Mode, median and mean

The mode, median and mean of a distribution are all known as measures of *location* (or *central tendency*). For a collection of numerical data, they all indicate the general level or location of the observations by giving an 'average' or central value.

The *mode* is the observation which occurs most frequently. In some cases there are two or more modes.

The *median* is the middle observation when the observations are listed in order of magnitude. If the number of items is even, the average of the middle two is used.

The *mean* of a set of observations is the average or arithmetic mean. For a set of n observations $x_1, x_2, \ldots, x_n$, the mean $\bar{x} = \frac{1}{n}(x_1 + x_2 + \ldots + x_n)$. Using the Σ notation, this becomes $\bar{x} = \frac{1}{n}\sum_{r=1}^{n} x_r$ or simply $\bar{x} = \frac{1}{n}\sum x$.

Of these measures of location, the mean is the one used most frequently, although it can be distorted by extreme values at the ends of a distribution. The median has the advantage that it is not affected in this way.

Example 1 Find the mode, median and mean of the following sets of observations (a) 6, 7, 7, 3, 8, 5, 3, 9; (b) 18, 13, 14, 13, 11, 12, 13, 12, 11.

(a) The distribution has two modes, 3 and 7.

Arranged in order of magnitude the observations are 3, 3, 5, 6, 7, 7, 8, 9. Thus the median is $\frac{1}{2}(6+7) = 6{\cdot}5$. The mean is $\frac{1}{8}(6+7+7+3+8+5+3+9) = 6$.

(b) The mode of this distribution is 13.

Arranged in order of magnitude the observations are 11, 11, 12, 12, 13, 13, 13, 14, 18. Thus the median is 13.

The mean is $\frac{1}{9}(2.11 + 2.12 + 3.13 + 14 + 18) = 13$.

[Note that arranging a set of observations in order of magnitude is called forming an *array*.]

We now return to Examples A, B and C discussed in the previous section.

Example A From both the frequency distribution and the frequency chart, it is clear that the mode is 4.

Since there are 15 marks, the median is the 8th mark in order of magnitude. Hence, using the cumulative frequency distribution or the corresponding chart, we find that the median is 4.

When calculating the mean, the total of the marks can be found using the frequency distribution.

Mark (x)	Frequency (f)	fx
1	1	1
2	2	4
3	3	9
4	5	20
5	3	15
6	1	6
	15	55

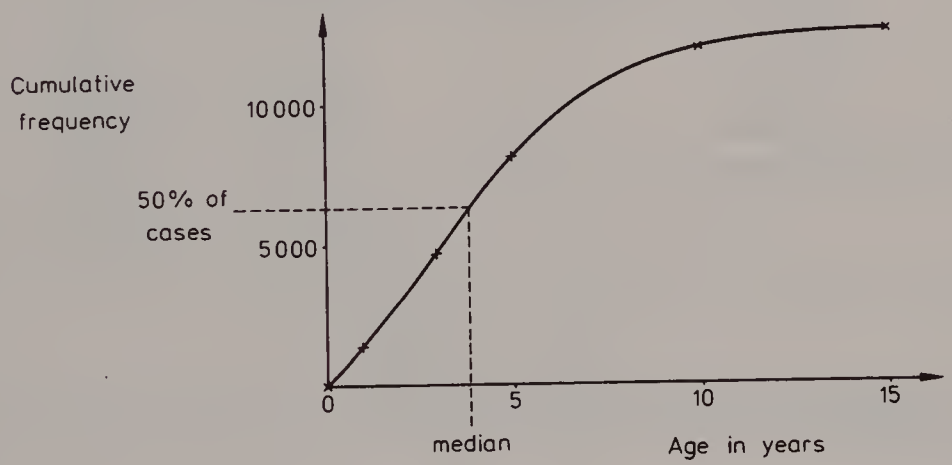
The formula for the mean can now be expressed as

$$\bar{x} = \frac{1}{n}\sum fx \quad \text{where} \quad n = \sum f.$$

Hence the mean mark $= \frac{55}{15} \approx 3{\cdot}67.$

Example B For grouped data it is not possible to find a true mode, as the values of individual observations are not known. Instead the term *modal class* may be used to describe the interval corresponding to the highest column in the histogram. In this case the modal class is the 3–4 year age group. [Note that this is not the group with the highest number of cases of whooping-cough.]

The median is a value of the variable which has equal numbers of observations above and below it. For grouped data an approximate value of the median can be found either graphically or by calculation. In this example the median is the age below which 6300 whooping-cough cases lie. Thus the median can be estimated from the ogive by reading off the age corresponding to a cumulative frequency of 6300 cases, which is approximately 4 years.

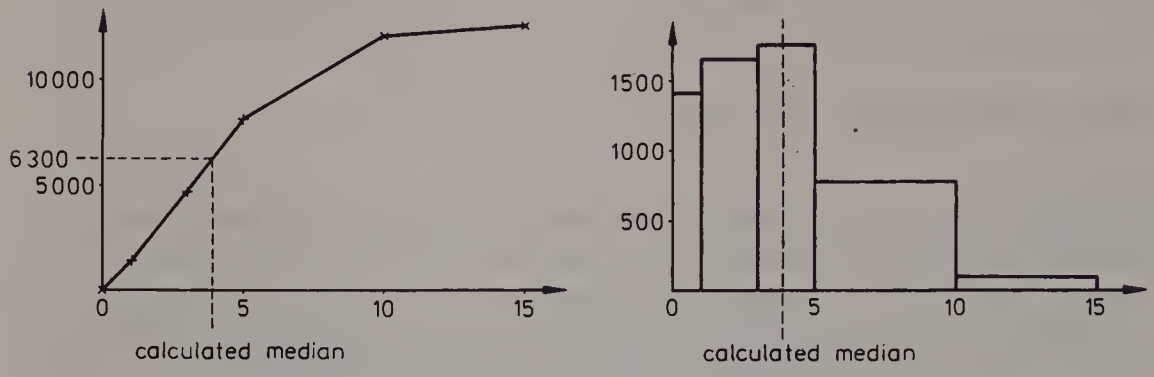


To estimate the median of a grouped frequency distribution by calculation, we assume that the observations in any class are evenly spaced, i.e. that the known points on the ogive can be joined with straight lines. This way of estimating intermediate values of a function is called *linear interpolation.* In this example there are 4700 cases under 3 years old and 8200 under 5 years old, so the median age lies in the 3–4 year class which contains 3500 cases. Since there are a total of 6300 cases below the median, 1600 of the cases in the 3–4 year class must lie below the median and 1900 above it. Thus we assume that the median divides this 2 year interval in the ratio 1600 : 1900

$$\therefore \quad \text{the median} = 3 + \left(\frac{1600}{3500} \times 2\right) \text{years} \approx 3{\cdot}9 \text{ years}.$$

Finding the median by calculation is equivalent to using a cumulative frequency polygon rather than a curve. This calculated value can also be regarded as the

value of the variable at which a vertical line divides the histogram into two equal areas.



To find an approximate value for the mean of a grouped frequency distribution, we again assume that observations are evenly spaced. It then follows that the mean of the observations in any one class lies at the mid-point of the class interval. Hence if there are f observations in a class with mid-value x, then the sum of these observations is taken to be fx.

Class	Mid-value (x)	Frequency (f)	fx
Under 1 year	0·5	1400	700
1–2 years	2	3300	6600
3–4 years	4	3500	14 000
5–9 years	7·5	3900	29 250
10–14 years	12·5	500	6250
		$n = 12\,600$	56 800

$$\bar{x} = \frac{1}{n}\sum fx = \frac{56\,800}{12\,600} \approx 4{\cdot}5$$

Hence the mean age is approximately 4·5 years.

Example C From the original data this distribution is found to have modes 50, 52 and 53. As these values occur only three times each, this information is probably of little value. The fact that the 53–57 group is the modal class may be more useful.

There are 40 observations, so the median is the average of the 20th and 21st, which is 56. An estimate of the median value can be obtained from the grouped frequency distribution, either graphically or by calculation, using the methods given in Example B. To estimate the median graphically we again use the ogive and obtain an approximate value of 56. To estimate the median by calculation we note, from the cumulative frequency table in §15.1, that 12 values of the variable are less than 52·5 and 23 values are less than 57·5. Since the median lies between the 20th and 21st value, it must be in the 5 unit interval from 52·5 to 57·5, with 8 values below it and 3 above. Assuming that these 11 observations are evenly spaced, the approximate value obtained for the median is $52{\cdot}5 + \left(\frac{8}{11} \times 5\right) \approx 56{\cdot}1$.

If it is necessary to find the mean from the original data without the aid of a calculator, it helps to use the deviations of the observations from an arbitrary origin. Taking 55 as origin the deviations are:

1	−5	8	1	3	−2	−3	16	4	−3
−10	0	−13	7	7	5	2	−7	8	13
0	10	−5	4	−1	−1	−9	−4	18	9
−2	−5	6	−4	3	−3	2	12	−2	14

Cancelling equal and opposite numbers, the sum of the deviations reduces to

$$16+8+18+6+12+14 = 74$$

$$\therefore \quad \text{the mean} = 55 + \frac{74}{40} \approx 56{\cdot}9.$$

In this case a good approximation to the value of the mean can be obtained from the grouped data. This estimated value is found to be $56\frac{7}{8}$.

Exercise 15.2

1. Find the mode, median and mean of the following sets of integers
(a) 3, 4, 2, 3, 1, 3, 5, 7, 3, 4,
(b) 24, 38, 8, 13, 25, 3, 29,
(c) 21, 19, 20, 20, 22, 20, 21, 22, 21.

2. If an additional integer 20 is included in each of the lists given in question 1, find the new values of the mode, median and mean in each case.

In questions 3 to 8 use the data given in the corresponding questions in Exercise 15.1.

3. Find the mode, median and mean of the distribution, distinguishing where necessary between cases (a), (b) and (c).

4. Find the mode, median and mean of the distribution. Comment briefly on the usefulness of these measures in this context.

5. Use a frequency distribution table to find the mean of this distribution. Comment on the significance of your answer in cases (a) and (b).

6. (i) State the modal class of the grouped frequency distribution obtained in Exercise 15.1 and use this distribution to estimate the mean of the original data. Use the cumulative frequency curve to obtain a value for the median.
(ii) Form an array of the 40 items of data and write down the mode and the median of the distribution. Use a working origin of 50 to find the mean.
(iii) Comment on your answers to (i) and (ii).

7. (i) Calculate the approximate values of the median and mean of this distribution.

(ii) Assuming that children under $5\frac{1}{2}$ years of age enter a reception class, calculate the median age and the mean age of such children.
(ii) Discuss the significance of your answers.

8. Use linear interpolation to find the median of this distribution. Estimate the value of the mean in each of the cases (a), (b) and (c).

9. Construct a flow diagram for a procedure to find the mean of a set of numbers $x_1, x_2, \ldots, x_r$ which occur with frequencies $f_1, f_2, \ldots, f_r$ using a pocket calculator without built-in statistics functions. [Note that unless it is possible to calculate $\sum f$ and $\sum fx$ simultaneously, it will be necessary to input the numbers $f_1, f_2, \ldots, f_r$ twice.]

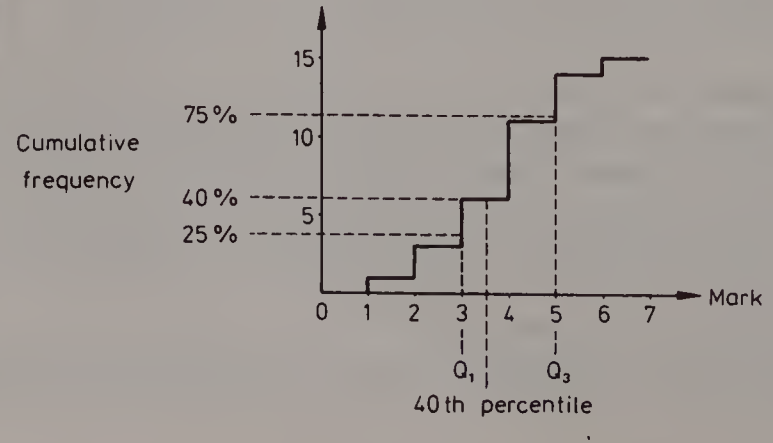
15.3 Measures of spread

Consider two sets of numbers (i) 1, 13, 15, 20, 21 and (ii) 12, 13, 14, 14, 17. Both have mean 14, but the first is more widely spread than the second. Thus, in addition to measures of location it is useful to have measures of *spread*, sometimes called *dispersion* or *variation*. The simplest of these is the *range*, which is the difference between the highest and the lowest items of data. In the above examples the ranges are (i) 20, (ii) 5. The main disadvantage of the range is that it is often determined by two freak values at the ends of a distribution.

Other measures of spread are obtained using quantities called *percentiles*. Theoretically, the percentiles divide a distribution into 100 equal parts. For instance, the 40th percentile is a value which has 40% of the observations in a distribution on or below it and 60% on or above it. Clearly the 50% percentile is the median. Other percentiles are found by similar methods to those used to find the median in the previous section. The interval between any two percentiles can be used as a measure of spread. Particularly useful in this way are the 25th percentile, referred to as the *first* or *lower quartile* Q_1, and the 75th percentile, the *third* or *upper quartile* Q_3. The *semi-interquartile range* is half the interval between the lower and upper quartiles, i.e. $\frac{1}{2}(Q_3 - Q_1)$. This is a more useful measure than the range because it is not affected by a few untypical values.

We return again to the examples discussed earlier in the chapter.

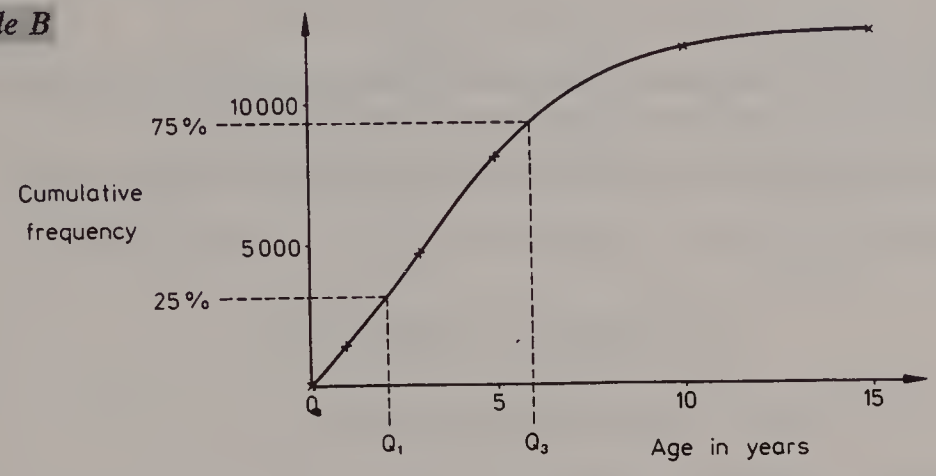
Example A



Using the cumulative frequency chart we find that the 40th percentile lies between 3 and 4. We take its value to be 3·5. The lower quartile is 3 and the upper quartile is 5

$\therefore$ the semi-interquartile range is $\frac{1}{2}(5-3)$, i.e. 1.

Example B



There are a total of 12600 observations. Hence the lower quartile is the age below which 3150 of the cases occur. From the graph this is approximately 2·1 years. Similarly the upper quartile corresponds to a cumulative frequency of 9450 and is approximately 5·9 years. Hence the semi-interquartile range is $\frac{1}{2}(5{\cdot}9-2{\cdot}1)$ years, i.e. 1·9 years.

By calculation the lower quartile is estimated, using the cumulative frequency table, to be

$$1+\left(\frac{\{3150-1400\}}{3300}\times 2\right)\text{years} \approx 2{\cdot}1 \text{ years}.$$

The upper quartile is $5+\left(\frac{\{9450-8200\}}{3900}\times 5\right)$ years $\approx 6{\cdot}6$ years.

Hence the calculated estimate of the semi-interquartile range is $\frac{1}{2}(6{\cdot}6-2{\cdot}1)$ years, i.e. 2·25 years.

[It appears likely that the whooping-cough cases are not spread evenly throughout the 5–9 year age group and that more cases occur in the lower half of the class. Thus it is probable that the estimate of 1·9 years obtained graphically is more accurate than the calculated value of 2·25 years.]

Further measures of spread are obtained by considering the set of *deviations* of the values of a variable x from the mean $\bar{x}$. For a set of n observations the sum of these deviations

$$=\sum(x-\bar{x})=\sum x-n\bar{x}=\sum x-n\left(\frac{1}{n}\sum x\right)=0.$$

Thus the mean deviation is always zero and cannot be used as a measure of spread. However, the *mean absolute deviation*, i.e. $\frac{1}{n}\sum|x-\bar{x}|$, is sometimes useful.

The main disadvantage when using this quantity is that moduli are difficult to manipulate algebraically.

The most frequently used measures of dispersion are the *variance*, which is the mean squared deviation, $\text{var}(x) = \frac{1}{n}\sum(x-\bar{x})^2$ and the *standard deviation*, which is the square root of the variance $\sqrt{\left\{\frac{1}{n}\sum(x-\bar{x})^2\right\}}$.

[Note that the variance is measured in square units, whereas the units of the standard deviation are the same as those of the original data.]

For examples (i) and (ii) considered at the beginning of the section, we have

(i) $\text{variance} = \frac{1}{5}\{(1-14)^2+(13-14)^2+(15-14)^2+(20-14)^2+(21-14)^2\}$

$$= \frac{1}{5}\{169+1+1+36+49\} = 51{\cdot}2$$

$\therefore$ the variance is 51·2 and the standard deviation is $\sqrt{(51{\cdot}2)} \approx 7{\cdot}2$.

(ii) $\text{variance} = \frac{1}{5}\{(-2)^2+(-1)^2+0^2+0^2+3^2\} = 2{\cdot}8$

$\therefore$ the variance is 2·8 and the standard deviation is $\sqrt{(2{\cdot}8)} \approx 1{\cdot}7$.

To avoid cumbersome calculations with a mean which is not a whole number, we now rearrange the formula for the variance.

$$\text{var}(x) = \frac{1}{n}\sum(x-\bar{x})^2 = \frac{1}{n}\sum(x^2-2x\bar{x}+\bar{x}^2)$$

$$= \frac{1}{n}\sum x^2 - \frac{1}{n}\sum 2x\bar{x} + \frac{1}{n}\sum \bar{x}^2$$

$$= \frac{1}{n}\sum x^2 - 2\bar{x}.\frac{1}{n}\sum x + \frac{1}{n}.n\bar{x}^2$$

$$= \frac{1}{n}\sum x^2 - 2\bar{x}^2 + \bar{x}^2$$

Hence the variance,

$$\boxed{\text{var}(x) = \frac{1}{n}\sum x^2 - \bar{x}^2.}$$

When working from a frequency table in which a value x has frequency f we may write:

$$\boxed{\text{var}(x) = \frac{1}{n}\sum f(x-\bar{x})^2 = \frac{1}{n}\sum fx^2 - \bar{x}^2.}$$

In Example A, assuming that a calculator with built-in statistical data programs is not available, the variance and standard deviation can be obtained in the following way.

Mark (x)	Frequency (f)	fx	fx^2
1	1	1	1
2	2	4	8
3	3	9	27
4	5	20	80
5	3	15	75
6	1	6	36
	15	55	227

$$n = \sum f = 15, \qquad \bar{x} = \frac{1}{n}\sum fx = \frac{55}{15}$$

$$\therefore \text{ var}(x) = \frac{1}{n}\sum fx^2 - \bar{x}^2 = \frac{1}{15} \times 227 - \left(\frac{55}{15}\right)^2.$$

Hence the variance is 1·69 and the standard deviation is 1·30 (both to 3 s.f.).

The variance of a grouped frequency distribution can be estimated using the formula $\text{var}(x) = \frac{1}{n}\sum fx^2 - \bar{x}^2$ by taking the values of x to be the mid-values in each class.

In Example B the class intervals are quite long, so the estimated variance is likely to differ noticeably from the true value. However, in Example C the class intervals are shorter, so the grouped frequency distribution should give a better value for the variance.

From the original data the variance is found to be approximately 50·7 (measured in 'square meals') and the standard deviation is 7·12 meals. Using the grouped data the estimated variance is 57·1 and the standard deviation 7·56.

Exercise 15.3

1. Find the median, the quartiles and the semi-interquartile range for the following sets of integers.
(a) 8, 3, 5, 7, 11, 9, 7, 6, 7, 5, 8, 11,
(b) 28, 18, 32, 27, 42, 23, 49, 28, 33, 30,
(c) 2, −4, 10, 0, −5, 11, 6, −4, 11.

2. Find the mean, the variance and the standard deviation of the sets of integers given in question 1.

In questions 3 to 8 use the data given in the corresponding questions in Exercise 15.1.

3. Use the appropriate cumulative frequency diagram to find the following in cases (a), (b) and (c):
(i) the 40th and the 70th percentile,
(ii) the semi-interquartile range.

4. Find the variance and standard deviation of the distribution of sales.

5. (i) Use the appropriate cumulative frequency graph to estimate the lower quartile, median and upper quartile of this distribution in cases (a) and (b).
(ii) Use the frequency distribution table to find the standard deviation of the distribution.

6. Find the lower and upper quartiles of this distribution using (i) the original data, (ii) the grouped data and linear interpolation, (iii) an ogive based on the grouped data. Comment on any differences between your answers.

7. (i) Calculate approximate values, to the nearest month, of the 30th and the 65th percentile.
(ii) Estimate the variance of the distribution, stating the units in which your answer is measured.

8. The 150 test marks are to be graded as follows: the top 15% of students, grade I; the next 30%, grade II; the next 35%, grade III; the bottom 20%, grade IV. Use a cumulative frequency polygon to suggest suitable class limits for each grade.

9. Construct a flow diagram for a procedure to find the standard deviation of a set of numbers $x_1, x_2, \ldots, x_r$ which occur with frequencies $f_1, f_2, \ldots, f_r$ using a pocket calculator without built-in statistical functions. [Assume that $n = \sum f$ and $m = \frac{1}{n}\sum fx$ have already been calculated and may be read as data.]

15.4 Change of variable

Suppose we have a distribution of n values of a variable x with mean $\bar{x}$ and standard deviation σ_x. Consider a new variable y, such that $y = ax + b$, where a and b are constants. Let the distribution of the n values of y have mean $\bar{y}$ and standard deviation σ_y, then

$$\bar{y} = \frac{1}{n}\sum y = \frac{1}{n}\sum (ax+b) = \frac{1}{n}\sum ax + \frac{1}{n}\sum b$$

$$= a.\frac{1}{n}\sum x + \frac{1}{n}.nb = a\bar{x} + b$$

$$\text{var}(y) = \frac{1}{n}\sum(y-\bar{y})^2 = \frac{1}{n}\sum\{(ax+b)-(a\bar{x}+b)\}^2$$

$$= \frac{1}{n}\sum(ax-a\bar{x})^2 = a^2 . \frac{1}{n}\sum(x-\bar{x})^2 = a^2\,\text{var}(x).$$

Hence, if $y = ax+b$, then $\bar{y} = a\bar{x}+b$ and $\text{var}(y) = a^2\,\text{var}(x)$.

Example 1 During an industrial dispute a trade union negotiated a pay increase for its members of £8 a week plus 10%, so that a man earning £x per week received an increase of £$(8+x/10)$ per week. For the original wage distribution the mean was £120 per week and the standard deviation £20 per week. Find the new mean and standard deviation.

Let £y be the new weekly wage of a man originally earning £x per week, then

$$y = x+8+\frac{x}{10} = \frac{11}{10}x+8$$

$$\therefore \quad \bar{y} = \frac{11}{10}\bar{x}+8 = \left(\frac{11}{10}\times 120\right)+8 = 140.$$

$$\text{New s.d.} = \frac{11}{10}\times \text{original s.d.} = \frac{11}{10}\times 20 = 22.$$

Hence the new mean weekly wage is £140 and the standard deviation is £22.

A change of variable is sometimes used to simplify the calculation of the mean and standard deviation of a distribution. For instance, if a set of values of a variable x appear to have a mean approximately equal to x_0, then we can take this *assumed mean* as a working origin. If we let $y = x-x_0$, then $\bar{x} = x_0+\bar{y}$ and $\text{var}(x) = \text{var}(y)$.

Example 2 The weights in kilogrammes of the players selected for a certain Rugby Union football team are 76, 79, 85, 86, 76, 75, 86, 105, 88, 104, 102, 103, 89, 92, 95. Find the mean and standard deviation of the distribution.

If x kg is the weight of a member of the team, let $y = x-90$, then the 15 values of y for the team are -14, -11, -5, -4, -14, -15, -4, 15, -2, 14, 12, 13, -1, 2, 5.

Cancelling equal and opposite values of y,

$$\bar{y} = \frac{1}{15}(-5-4) = -\frac{9}{15} = -0{\cdot}6.$$

$$\sum y^2 = (-14)^2+(-11)^2+\ldots+2^2+5^2 = 1563$$

$$\therefore \quad \text{var}(y) = \frac{1}{15}\sum y^2-\bar{y}^2 = \frac{1563}{15}-(0{\cdot}6)^2 = 103{\cdot}84$$

$$\therefore \quad \bar{x} = 90+\bar{y} = 89{\cdot}4 \quad \text{and} \quad \text{var}(x) = \text{var}(y) = 103{\cdot}84$$

Hence the mean weight is 89·4 kg and the standard deviation is 10·2 kg.

When using a calculator a change of variable may be used to improve accuracy. For instance, if the values of a variable are given to 5 digits or more, squares of at least 9 digits will be produced in an accurate calculation of the variance. A change to a variable whose values have fewer digits would reduce the truncation and rounding errors generated by a calculator of limited capacity.

In certain situations a change of variable of the form $y = \frac{1}{c}(x - x_0)$ is useful.

If $y = \frac{1}{c}(x - x_0)$ then $x = x_0 + cy$

$\therefore \quad \bar{x} = x_0 + c\bar{y}$ and $\text{var}(x) = c^2 \text{var}(y)$.

Applying a transformation of this type to a distribution may be referred to as '*coding*' the data. We illustrate the method using the grouped frequency distribution obtained in Example C, §15.1. We take an assumed mean of 55 as the value of x_0 and the class width 5 as the value of c, so that $y = \frac{1}{5}(x - 55)$.

Class	Mid-value (x)	y	Frequency (f)	fy	fy^2
38–42	40	−3	1	−3	9
43–47	45	−2	2	−4	8
48–52	50	−1	9	−9	9
53–57	55	0	11	0	0
58–62	60	1	8	8	8
63–67	65	2	5	10	20
68–72	70	3	3	9	27
73–77	75	4	1	4	16
			40	15	97

$$n = \sum f = 40, \quad \sum fy = 15, \quad \sum fy^2 = 97$$

$$\bar{y} = \frac{1}{n}\sum fy = \frac{15}{40} = \frac{3}{8}$$

$$\therefore \quad \bar{x} = 55 + 5.\frac{3}{8} \approx 56{\cdot}9$$

$$\text{var}(y) = \frac{1}{n}\sum fy^2 - \bar{y}^2 = \frac{97}{40} - \left(\frac{3}{8}\right)^2$$

$$\therefore \quad \text{s.d. of } x \text{ values} = 5\sigma_y = 5\sqrt{\left(\frac{97}{40} - \frac{9}{64}\right)} \approx 7{\cdot}56.$$

Hence the mean of the distribution is 56·9 and the standard deviation is 7·56.

Exercise 15.4

1. Find the mean, variance and standard deviation of the numbers

1, 3, 3, 4, 6, 6, 7, 8, 8, 9.

Deduce the mean, variance and standard deviation of the following sets of numbers:
(a) 4, 6, 6, 7, 9, 9, 10, 11, 11, 12,
(b) 4, 12, 12, 16, 24, 24, 28, 32, 32, 36,
(c) 4·1, 4·3, 4·3, 4·4, 4·6, 4·6, 4·7, 4·8, 4·8, 4·9.

2. The mean and standard deviation of a set of mid-day temperatures for the month of June are 23°C and 5°C respectively. Find the mean and standard deviation for the same distribution measured in degrees Fahrenheit.

3. Use a frequency table and an assumed mean of 5 to calculate the mean, variance and standard deviation of the distribution of the discrete variable x given in question 3. Exercise 15.1. Deduce the mean, variance and standard deviation of the variable in case (c).

4. (i) Use an appropriate change of variable to estimate the standard deviation of the data given in question 6, Exercise 15.1 from the grouped frequency table.
(ii) With the aid of a calculator, find the standard deviation of the original distribution.
(iii) Comment on the difference between your answers.

5. Use an appropriate change of variable to find the mean and standard deviation of the distribution given in question 8, Exercise 15.1 assuming that the test is marked out of 60. The marks are to be scaled using the formula $z = ax+b$, where x is the original mark and z the new mark. If the mean and standard deviation of the scaled marks are to be approximately 50 and 15 respectively, find suitable values of a and b correct to 2 significant figures.

6. The variable x_1 represents the mark of a student in an examination. The distribution of marks for a certain group of students has mean $\bar{x}_1$ and standard deviation σ_1. Given that the marks are to be scaled so that the values of the new mark x_2 have mean $\bar{x}_2$ and standard deviation σ_2, prove that

$$x_2 = \bar{x}_2 + \frac{\sigma_2}{\sigma_1}(x_1 - \bar{x}_1).$$

Use this formula to standardise the marks of students A, B and C as given in the table below, taking $\bar{x}_2 = 50$ and $\sigma_2 = 15$. Hence find a combined mark for each student.

Subject	Mean	Standard deviation	A	B	C
Mathematics	60	20	90	96	70
English	45	10	52	46	70

Exercise 15.5 (miscellaneous)

1. The following table gives an analysis by numbers of employees of the size of UK factories of less than 1000 employees manufacturing clothing and footwear.

Number of employees	11–19	20–24	25–99	100–199	200–499	500–999	Total
Number of factories	1500	800	2300	700	400	100	5800

Calculate as accurately as the data allow the mean and median of this distribution, showing your working. If 90% of the factories have less than N employees, estimate N. (O & C)

2. An inspection of 34 aircraft assemblies revealed a number of missing rivets as shown in the following table.

Number of rivets missing	0–2	3–5	6–8	9–11	12–14	15–17	18–20	21–23
Frequency	4	9	11	6	2	1	0	1

Draw a cumulative frequency curve. Use this curve to estimate the median and the quartiles of the distribution. (O & C)

3. The distribution of the times taken when a certain task was performed by each of a large number of people was such that its twentieth percentile was 25 minutes, its fortieth percentile was 50 minutes, its sixtieth percentile was 64 minutes and its eightieth percentile was 74 minutes. Use linear interpolation to estimate (i) the median of the distribution, (ii) the upper quartile of the distribution, (iii) the percentage of persons who performed the task in 40 minutes or less. (JMB)

4. The weights of a number of school children were measured and the mean weight was found to be 54·1 kg, with standard deviation 3·2 kg. Subsequently it was discovered the weighing machine was reading 0·9 kg under the true weight. How should the mean and standard deviation be revised? Give a clear indication of your reasoning. (O & C)

5. A frequency distribution has values $x_1, x_2, \ldots, x_n$ with frequencies $f_1, f_2, \ldots, f_n$ respectively. It has mean $\bar{x}$ and standard deviation σ. Find the mean and standard deviation of the distribution which has values $d_1, d_2, \ldots, d_n$ with frequencies $f_1, f_2, \ldots, f_n$ respectively, where $d_r = \dfrac{x_r - \bar{x}}{\sigma}$. (AEB 1977)

6. 100 pupils were tested to determine their intelligence quotient (I.Q.), and the results were as follows:

I.Q.	45–	55–	65–	75–	85–	95–	105–	115–	125–134
No. of pupils	1	1	2	6	21	29	24	12	4

All I.Q.s are given to the nearest integer.
(i) Calculate the mean, and the standard deviation.
(ii) Draw a cumulative frequency graph, and estimate how many pupils have I.Q.s within 1 s.d. on either side of the mean. (SU)

7. The haemoglobin levels were measured in a sample of 50 people and the results were as follows, each being correct to 1 place of decimals.

13·5	15·6	16·3	12·3	13·1	14·2	12·4	11·3	14·0	14·6
13·6	14·8	12·7	10·9	11·0	11·4	15·0	10·1	15·4	11·3
10·7	14·6	13·5	15·1	12·1	12·0	14·2	11·4	15·0	13·3
13·2	9·1	16·9	14·2	15·0	13·6	14·8	11·4	14·8	15·7
13·5	13·5	12·9	13·8	13·7	16·2	11·6	13·8	14·2	10·7

(a) Group the data into eight classes, 9·0–9·9, 10·0–10·9, . . . , 16·0–16·9.
(b) What are the smallest and largest possible measurements which could be included in the class 9·0–9·9?
(c) Draw a histogram of the grouped data and use it to estimate the median value of the sample, showing your working.
(d) Find the true median of the sample. (SU)

8. A small firm has a total labour force of 267 of whom 250 are craftsmen and the rest managers. The distribution of the weekly pay of all employees is as follows, where, for example, 22·5– means that employees in this group earn at least £22.50, but less than £25.00.

Craftsmen Wages in £	No. of employees	Managers Wages in £	No. of employees
20.0–	29	50.0–	3
22.5–	33	52.5–	9
25.0–	35	55.0–	5
27.5–	40		
30.0–	51		
32.5–	37		
35.0–	25		

Display this information on a block diagram. Calculate (using a working zero of £31.25, or otherwise) the mean pay of (i) the craftsmen, (ii) the managers, (iii) all 267 employees. Comment on these means as they describe the situation in the firm and discuss other measures, such as the median or mode which might be used instead or as well. (SU)

9. The following table gives the maximum temperature in 60 cities on a certain day. Readings are to the nearest degree.

72	61	43	54	54	48	59	55	61	50	55	48
30	66	41	55	48	57	61	48	46	61	30	50
66	73	54	48	66	61	45	57	48	70	68	43
52	50	46	64	46	50	50	50	48	37	45	53
64	50	39	32	66	68	41	70	48	73	39	43

Gather these measurements into classes at intervals of 5°, starting at 29·5°. From the information in this form construct a cumulative frequency table and draw a graph to display the cumulative frequency. Use this to find the median and semi-interquartile range. For the grouped distribution calculate the mean and standard deviation, explaining what effect the grouping is likely to have had on each.

(AEB 1978)

10. On 1 January 1975 the heights of ten boys, measured in centimetres, were 172·1, 168·6, 174·1, 174·6, 176·1, 167·6, 174·1, 170·1, 172·6, 171·1. Find the mean and standard deviation of these heights. If on 1 January 1976 each boy was 2·5 cm taller, calculate the mean height and variance on that date. If the heights had been measured in inches, find the variance of the heights of the ten boys stating the units in which your answer is given. [Take 2·54 cm = 1 inch.] (L)

11. For a finite population of size N, the mean of a measurement X is μ_X and the variance σ_X^2 is given by $\sigma_X^2 = \frac{1}{N}\sum(X-\mu_X)^2$. If $Y = aX+b$, where a and b are fixed constants, show that $(aN\sigma_X)^2 = N\sum Y^2 - (\sum Y)^2$.

For the finite population given in the table below, calculate the average earnings and the standard deviation of the earnings.

Earnings to nearest £	18–22	23–27	28–32	33–37	38–42
Number of workers	2	2	2	4	5
Earnings to nearest £	43–47	48–52	53–57	58–62	63–67
Number of workers	7	4	7	9	8

(L)

12. The set of positive integers $x_1, x_2, \ldots, x_9$ has a mean of 5 and $\sum_{i=1}^{9}(x_i-5)^2 = 36$. A tenth integer (x_{10}) is added to the set, and the mean and variance of the set of ten integers are denoted by $\bar{x}$ and Var (x) respectively.

(a) Calculate the values of $\bar{x}$ and Var (x) if $x_{10} = 5$.
(b) Calculate the values of $\bar{x}$ and Var (x) if $x_{10} = 9$.
(c) If $0 < x_{10} < 5$ state two inequalities satisfied by $\bar{x}$.
(d) Without doing long calculations, suggest which values for x_{10} would result in Var (x) being greater than 4.

[In this question you should take Var $(x) = \frac{1}{10}\sum_{i=1}^{10}(x_i-\bar{x})^2$.] (L)

13. (i) A distribution, A, contains N_1 observations and has mean and standard deviation μ_1 and σ_1 respectively. A second distribution, B, of N_2 observations has mean and standard deviation μ_2 and σ_2 respectively. The standard deviation of the combined distribution of N_1+N_2 observations is σ. Show that

$$(N_1+N_2)\sigma^2 = N_1\sigma_1^2+N_2\sigma_2^2+\frac{N_1N_2}{N_1+N_2}(\mu_1-\mu_2)^2.$$

(ii) Two teachers A and B each marked 100 exam papers with the following results

Mark	35	36	37	38	39	40	41	42	43	44	45
No. marked by A	4	10	14	18	12	12	10	8	4	5	3
No. marked by B	8	15	16	20	18	5	3	5	2	2	6

The mean mark for A is 39·23 with variance 6·4371. The mean mark for B is 38·51. Calculate the variance of B's marks. It is decided to add 0·72 to each of B's marks to make the mean for his papers the same as for A's. Calculate the mean and variance of the combined set of 200 papers (with B's marks adjusted).

(AEB 1975)

14. The following table gives the frequency distribution of 100 observed values of a random variable x.

x =	1	2	3	4	5	6
Frequency =	12	13	30	20	18	7

(a) Suppose x is the score obtained on a throw of a certain die.
(i) Calculate the mean and the variance of the observed scores.
(ii) Use the data to estimate the probability of throwing an even score with this die. Hence estimate the probability that six throws of the die will result in three even and three odd scores.

(b) Suppose, instead, that x is a continuous random variable and that in compiling the above frequency table the observed values were recorded to the nearest integer. Use linear interpolation to estimate the median of the observed values and the proportion of the observed values which exceeded 4·8. (JMB)

16 Probability distributions

16.1 Discrete probability distributions

The theory of probability is used to set up mathematical models of random experiments or trials. In its simplest form a *probability model* of an experiment consists of a description of the sample space (i.e. the set of possible outcomes) together with estimates of the probabilities associated with the outcomes. Let us consider, for instance, a trial in which a single unbiased die is thrown and the score noted. This can be described as a trial with a sample space containing 6 outcomes each with probability 1/6. We can use this model to make predictions about the behaviour of dice in experiments involving large numbers of such trials. The theory can be tested by comparing these predictions with the observed results of the experiments.

Since the score obtained in any particular throw of a single die cannot be predicted in advance, the trial is said to be *random* and the score obtained is called a *random variable.* In general, the values of a random variable are real numbers assigned to the elements of a sample space. This leads to the following formal definition.

> A random variable X is a function which assigns a real number x to each element s of a sample space S.

A random variable such as the height of students in a class, which may take any real value within a given interval, is said to be *continuous.* A random variable such as the score on a die, which takes only certain specified values, is described as *discrete.*

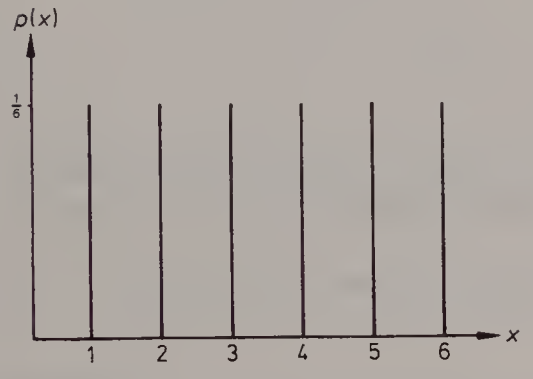
As we saw in Chapter 14, any subset A of a sample space S is called an event and the probability that event A occurs is denoted by $P(A)$. Thus the probability that the random variable X takes the value x may be written $P(X = x)$. We can now define a probability function on the set of values taken by X.

The probability function p of the discrete random variable X is defined by $p(x) = P(X = x)$, where $0 \leqslant p(x) \leqslant 1$ for all values of x.

The set of values $x_1, x_2, x_3, \ldots$ of a discrete random variable X, together with the corresponding values $p_1, p_2, p_3, \ldots$ of the probability function p, form a *discrete probability distribution.*

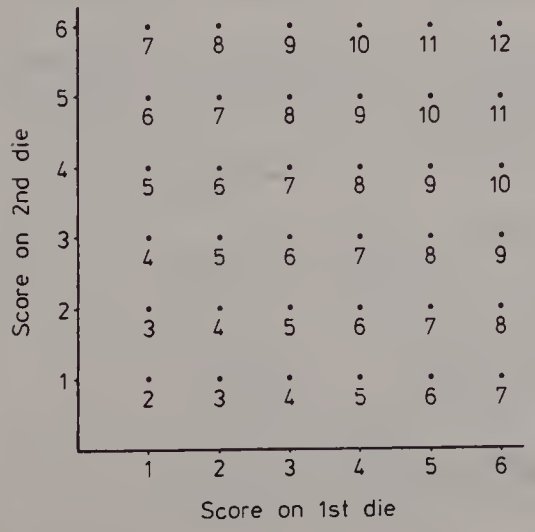
Since the sum of the probabilities of all possible outcomes of any experiment must be 1, all discrete probability distributions have the following property.

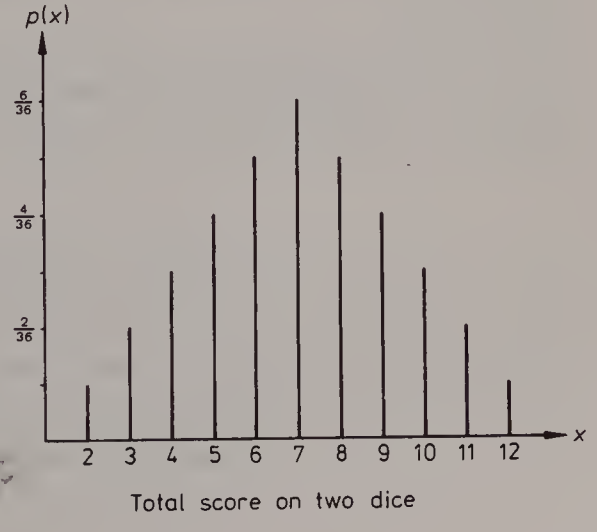
If a discrete random variable X takes the n values $x_1, x_2, \ldots x_n$ with probabilities $p_1, p_2, \ldots, p_n$ respectively, then $\sum_{r=1}^{n} p_r = 1$.



If X represents the score obtained when a single die is thrown, then $p(x) = 1/6$ for $x = 1, 2, 3, 4, 5$ and 6. A distribution such as this, in which all values of the random variable are equally likely, is said to be *rectangular* or *uniform.*

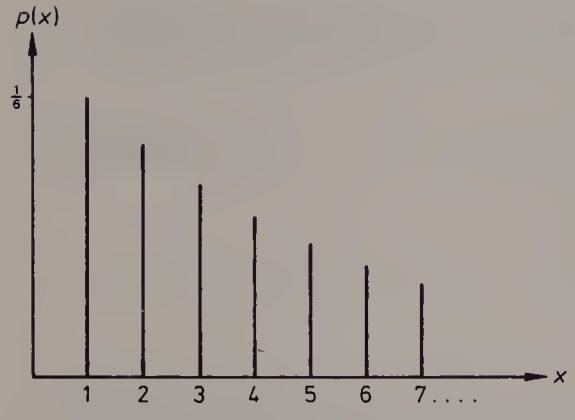
For a trial in which two dice are thrown and the total score noted the sample space S can be regarded as a set of 36 ordered pairs such as (1, 3), (4, 6) and (6, 4). The total score obtained in any throw is a discrete random variable defined on S and takes the values 2, 3, 4, ..., 12. The following diagrams show the sample space and corresponding probability distribution for this two dice trial.





A further type of discrete probability distribution is obtained when a die is thrown repeatedly and the random variable X is the number of throws needed to obtain a six. We then find that

$$p(1) = \frac{1}{6}, p(2) = \frac{5}{6} \times \frac{1}{6}, p(3) = \left(\frac{5}{6}\right)^2 \times \frac{1}{6}, p(4) = \left(\frac{5}{6}\right)^3 \times \frac{1}{6}, \ldots$$



Clearly X may take any one of the infinite set of values $1, 2, 3, 4, \ldots$ Since the corresponding probabilities form a geometric progression, this distribution is described as *geometric*.

In order to discuss the properties of probability distributions in more detail we need to define quantities similar to the measures of location and spread used to describe frequency distributions.

For a frequency distribution in which a set of n observations $x_1, x_2, \ldots, x_n$ occur with frequencies $f_1, f_2, \ldots, f_n$,

$$\text{the mean } \bar{x} = \frac{1}{n}\sum f_r x_r, \text{ where } n = \sum f_r,$$

$$\text{the variance} = \frac{1}{n}\sum f_r (x_r - \bar{x})^2 = \frac{1}{n}\sum f_r x_r^2 - \bar{x}^2$$

If n is fairly large the relative frequency f_r/n is approximately equal to the probability $p(x_r)$. Thus we obtain suitable definitions of the mean and variance of a probability distribution by replacing f_r/n by $p(x_r)$ in the formulae used for frequency distributions.

The mean of a probability distribution is called the *expectation* or *expected value* of the random variable X and is denoted by $E(X)$ or μ. The *variance* is denoted by $\text{Var}(X)$ or σ^2. These quantities are defined as follows:

$$\boxed{\begin{aligned} E(X) = \mu &= \sum x_r p(x_r) \\ \text{Var}(X) = \sigma^2 &= \sum (x_r - \mu)^2 p(x_r) \end{aligned}}$$

The expression for $\text{Var}(X)$ can be rearranged to give:

$$\boxed{\text{Var}(X) = \sigma^2 = \sum x_r^2 p(x_r) - \mu^2}$$

For the two dice trial discussed earlier:

$$E(X) = 2.\tfrac{1}{36} + 3.\tfrac{2}{36} + \ldots + 12.\tfrac{1}{36} = 7$$
$$\mathrm{Var}(X) = 2^2.\tfrac{1}{36} + 3^2.\tfrac{2}{36} + \ldots + 12^2.\tfrac{1}{36} - 7^2 = \tfrac{35}{6} \approx 5{\cdot}83.$$

Example 1 A box contains 10 coloured discs of which 2 are red. A man pays 10 pence to play a game in which discs are drawn one at a time from the box, without replacement, until a red disc is drawn. He will receive 25 pence if the first disc drawn is red, 20 pence if the second disc is red, 5 pence if the third disc is red, but nothing otherwise. Find the man's expected profit or loss in any game.

If the random variable X is the amount in pence received in any game, then X takes the values 25, 20, 5 and 0.

$p(25) = \frac{2}{10} = \frac{1}{5}$, $p(20) = \frac{8}{10} \times \frac{2}{9} = \frac{8}{45}$ and $p(5) = \frac{8}{10} \times \frac{7}{9} \times \frac{2}{8} = \frac{7}{45}$.

$\therefore$ the expected value of X

$$= 25 \times p(25) + 20 \times p(20) + 5 \times p(5) + 0 \times p(0)$$
$$= 25 \times \tfrac{1}{5} + 20 \times \tfrac{8}{45} + 5 \times \tfrac{7}{45} + 0 = 9\tfrac{1}{3}$$

Since the man pays 10 pence per game, his expected loss in pence is $\frac{2}{3}$.

In the next example we make use of the following results.

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1), \quad \sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1), \quad \sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2.$$

Example 2 The probability distribution of a random variable X is given by

$$P(X = r) = k(n - r) \quad \text{for} \quad r = 1, 2, 3, \ldots, n.$$

Find k in terms of n. Find also the mean and variance of X.

Since X is a discrete random variable, $\sum_1^n p(r) = 1$.

But $\sum_1^n p(r) = \sum_1^n k(n-r) = k\sum_1^n n - k\sum_1^n r$

$$= kn^2 - k.\tfrac{1}{2}n(n+1) = \tfrac{1}{2}kn(n-1)$$

Hence $k = 2/n(n-1)$.

$$E(X) = \sum_1^n rp(r) = \sum_1^n kr(n-r)$$

$$= kn\sum_1^n r - k\sum_1^n r^2$$

$$= kn.\frac{1}{2}n(n+1) - k.\frac{1}{6}n(n+1)(2n+1)$$

$$= \frac{2}{n(n-1)}.\frac{n(n+1)}{6}\{3n - (2n+1)\} = \frac{1}{3}(n+1)$$

$$\text{Var}(X) = \sum_1^n r^2 p(r) - \mu^2 = \sum_1^n kr^2(n-r) - \frac{1}{9}(n+1)^2$$

$$= kn\sum_1^n r^2 - k\sum_1^n r^3 - \frac{1}{9}(n+1)^2$$

$$= kn.\frac{1}{6}n(n+1)(2n+1) - k.\frac{1}{4}n^2(n+1)^2 - \frac{1}{9}(n+1)^2$$

$$= \frac{2}{n(n-1)}.\frac{n^2(n+1)}{12}\{2(2n+1) - 3(n+1)\} - \frac{1}{9}(n+1)^2$$

$$= \frac{1}{6}n(n+1) - \frac{1}{9}(n+1)^2$$

$$= \frac{1}{18}(n+1)\{3n - 2(n+1)\} = \frac{1}{18}(n+1)(n-2)$$

Thus the mean and variance of X are $\frac{1}{3}(n+1)$ and $\frac{1}{18}(n+1)(n-2)$ respectively.

Exercise 16.1

1. A student has a week-end job in a warehouse. The probability that he will be given 6 hours work is 1/6, the probability of 9 hours work is 1/3 and the probability of 12 hours work is 1/2. Find the expected number of hours the student will work.

2. A man pays x pence to play a game in which two unbiased dice are rolled. If the total score on the dice is an even number n, then the man receives n pence. Otherwise he receives nothing. Calculate the value of x which makes the game fair.

3. A bag contains one 50p coin, three 20p coins, seven 10p coins and several 5p coins. Given that when one coin is selected at random the expectation is 10p, find the number of 5p coins. Find also the expectation when two coins are selected at random without replacement.

4. Three discs are drawn at random, without replacement from a bag containing three black discs and five red discs. Find (a) the most probable number of black discs drawn, (b) the expected number of black discs drawn.

5. A bag X contains 3 red balls and 6 white balls and a second bag Y contains 5 red balls and 4 white balls. A ball is chosen at random from bag X and placed in bag Y. A ball is then chosen at random from bag Y and placed in bag X. If R is the number of red balls in bag X after these operations, find the probability distribution of R. Hence find the expected value of R.

6. A simple 'fruit machine' has three wheels on which there are pictures of fruit. When 10p is paid into the machine, the wheels spin and then come to rest independently, each showing one fruit. The probabilities that the three wheels show

a lemon are 0·4, 0·2 and 0·1 respectively. The machine pays out 10p for 1 lemon, 30p for 2 lemons and £1·60 for 3 lemons. Find the expected profit or loss per play.

7. In a certain gambling game a player nominates an integer x from 1 to 6 inclusive and he then throws three fair cubical dice. Calculate the probabilities that the number of x's thrown will be 0, 1, 2 and 3. The player pays 5 pence per play of the game and he receives 48 pence if the number of x's thrown is three, 15 pence if the number of x's thrown is two, 5 pence if only one x is thrown and nothing otherwise. Calculate the player's expected gain or loss per play of the game. (JMB)

8. In a certain snooker match the first man to win 3 frames wins the match. The winner of each frame receives £20 and the winner of the match receives an additional £40. Find the expected cost of the prizes in a match between players A and B if the probability that A wins a frame against B is (a) 0·5, (b) 0·2.

9. A player throws a die whose faces are numbered 1 to 6 inclusive. If the player obtains a six he throws the die a second time, and in this case his score is the sum of 6 and the second number; otherwise his score is the number obtained. The player has no more than two throws. Let X be the random variable denoting the player's score. Write down the probability distribution of X, and determine the mean of X. Show that the probability that the sum of two successive scores is 8 or more is 17/36. Determine the probability that the first of two successive scores is 7 or more, given that their sum is 8 or more. (C)

10. At a horse show there is a jumping competition in which each horse attempts to clear 4 successive fences, each fence being more difficult than the previous one. A particular horse has probability $(5 - i)/5$ of clearing the ith fence $(i = 1, 2, 3, 4)$, independent of its performance over the other fences. (i) Find the probability that the horse clears all four fences. (ii) Find the probability that the horse does not clear any fence. (iii) Determine the expected value of the number of fences that the horse clears. (C)

11. A disc is drawn from a bag containing 10 discs numbered $0, 1, 2, \ldots 9$. The random variable X is defined as the square of the number drawn. Find $E(X)$ and $\mathrm{Var}(X)$.

12. A box contains 6 cards all of different colours. A card is drawn at random from the box and its colour noted. If three cards are drawn, with replacement, find the mean and variance of the number of different colours obtained.

13. Let X denote the number of heads obtained when a fair coin is tossed three times. Find the mean and the variance of X. A pack of six cards consists of three cards labelled H and the other three labelled T. Three cards are drawn at random without replacement from the pack. Let Y denote the number of H's obtained. Calculate the probabilities that Y takes the values 0, 1, 2 and 3 respectively. Show that the mean of Y is equal to the mean of X and express the variance of Y as a percentage of the variance of X. (JMB)

14. In a certain field, each puffball which is growing in one year gives rise to a number, X, of new puffballs in the following year. None of the original puffballs is present in the following year. The probability distribution of the random variable X is as follows:

$$P(X = 0) = P(X = 2) = 0{\cdot}3, P(X = 1) = 0{\cdot}4.$$

Find the probability distribution of Y, the number of puffballs resulting from there being two puffballs in the previous year, and show that the variance of Y is 1·2. Hence, or otherwise, determine the probability distribution of the number, Z, of puffballs present in year 3, given that there was a single puffball present in year 1. Find also the mean and variance of Z. (C)

15. A cubical die is biased in such a way that the probability of scoring $n (n = 1 \text{ to } 6)$ is proportional to n. Determine the mean value and variance of the score obtained in a single throw. What would be the mean and variance if the score showing were doubled? (O&C)

16. A random variable R takes values $1, 2, \ldots, n$ with equal probabilities. Determine the expectation μ of R and show that the variance σ^2 is given by $12\sigma^2 = n^2 - 1$. Find $P(|R - \mu| > \sigma)$ in the case $n = 100$. (C)

17. The probability distribution of a random variable R is defined by $P(R = r) = kr$ for $r = 1, 2, \ldots, n$. Find the value of k. Hence find the mean and variance of the distribution.

18. Ten identically shaped discs are in a bag; two of them are black, the rest white. Discs are drawn at random from the bag in turn and NOT replaced. Let X be the number of discs drawn up to AND INCLUDING the first black one. List the values of X and the associated theoretical probabilities. Calculate the mean value of X and its standard deviation. What is the most likely value of X? If instead each disc is replaced before the next is drawn, construct a similar list of values and probabilities and point out the chief differences between the two lists. (SU)

19. (a) Expand $(1 - x)^{-2}$ in a series of ascending powers of x, giving the first three terms and the term in x^{r-1}.
    (b) A boy is throwing stones at a tin can. The probability of hitting the can with any particular stone is p. Find the probability that the boy first hits the can with the rth stone. If the random variable R is the number of throws needed to hit the can, find the expected value of R in terms of p.

20. A faulty gas-lighter works successfully, on average, once in three attempts. When using it one goes on trying until it does work and the gas lights. What is the probability that the gas lights (i) first time, (ii) second time, (iii) at the fifth attempt? What is the probability that it will not have worked after five attempts? The sellers of the lighter promise that if one is returned, they will pay 25p for each time it fails to work before the gas is lit. What is one's expectation in taking up this offer? What is the probability of getting one's money back (or more) if the original price was £1? If

four people take up this offer, what is the probability that at least two get their money back (or more)? (AEB 1978)

21. An ordinary unbiased cubical die is to be thrown repeatedly. Show that the probability that a score of 6 will occur for the first time on the nth throw with the first $(n-1)$ throws resulting in one odd score and $(n-2)$ even scores (other than 6) is equal to $(n-1)(\frac{1}{4})(\frac{1}{3})^{n-1}$ for $n \geqslant 2$. If the die is to be thrown as many times as it is necessary for a score of 6 to occur, find, using the above result, or otherwise, the probability that only one of the throws will result in an odd score.

[Hint: For $|x| < 1$, $\sum_{r=0}^{\infty} (r+1)x^r = (1-x)^{-2}$] (JMB)

16.2 The binomial distribution

A *Bernouilli*† *trial* is one in which there are two possible outcomes, usually referred to as success and failure. If the probability of success is p and the probability of failure is q, then $p + q = 1$.

If a random variable X represents the number of successes in a sequence of n independent repetitions of a Bernouilli trial, then the resulting probability distribution is called a *binomial distribution.*

Consider, for example, a trial in which an unbiased die is rolled once and success is defined as obtaining a six. In one trial the probability of success is 1/6 and the probability of failure is 5/6. Suppose now that the random variable X is the number of sixes obtained in a sequence of 5 rolls of the die.

Clearly $$P(X = 0) = (\tfrac{5}{6})^5$$

The number of arrangements of a sequence of 4 failures and 1 success = 5.
The probability of each such arrangement $= (\frac{5}{6})^4(\frac{1}{6})$

$$\therefore \quad P(X = 1) = 5(\tfrac{5}{6})^4(\tfrac{1}{6})$$

The number of arrangements of a sequence of 3 failures and 2 successes $= \frac{5!}{3!2!} = 10$.
The probability of each such arrangement $= (\frac{5}{6})^3(\frac{1}{6})^2$

$$\therefore \quad P(X = 2) = 10(\tfrac{5}{6})^3(\tfrac{1}{6})^2$$

Using similar arguments it follows that

$$P(X = 3) = 10(\tfrac{5}{6})^2(\tfrac{1}{6})^3, P(X = 4) = 5(\tfrac{5}{6})(\tfrac{1}{6})^4, P(X = 5) = (\tfrac{1}{6})^5.$$

We see that the probabilities of 0, 1, 2, 3, 4, 5 sixes in a sequence of 5 rolls of a die are given by the terms in the binomial expansion of $(\frac{5}{6} + \frac{1}{6})^5$.

To obtain a more general result let us assume that the probability of success in a certain trial is p and that the probability of failure is q, where $q = 1 - p$.

†*Bernouilli, Jacques* (1654–1705) Swiss mathematician: one of the members of a large family who were all gifted mathematically. His *Ars Conjectandi* of 1713 contains a considerable quantity of work on probability.

The number of arrangements of a sequence of $(n-r)$ failures and r successes is $\dfrac{n!}{(n-r)!r!}$, written $\binom{n}{r}$ or ${}_nC_r$.
The probability of each such arrangement $= q^{n-r}p^r$.
Hence the probability of r successes in n independent trials is $\binom{n}{r}q^{n-r}p^r$.
Thus the probabilities that $r = 0, 1, 2, \ldots, n$ are the terms in the binomial expansion of $(q+p)^n$. This distribution is described as the binomial distribution with *parameters* n and p.

A random variable X is said to have a binomial distribution $B(n, p)$, if

$$P(X = r) = \binom{n}{r}q^{n-r}p^r \quad \text{for} \quad r = 0, 1, 2, \ldots, n,$$

where $0 < p < 1$ and $q = 1 - p$.

This binomial distribution arises when X represents the number of successes in n *independent* trials, each trial having the *constant* probability p of success.

Example 1 A doctor estimates that his treatment of a particular illness is successful in $66\frac{2}{3}\%$ of cases. Find the probability that he will treat successfully exactly 4 out of the 6 patients currently in his care. Find also the probability of success in at least 2 out of the 6 cases.

If X is the number of successes, then X has a binomial distribution with parameters $n = 6$ and $p = \frac{2}{3}$.

$$P(X = 4) = \binom{6}{4}\left(\tfrac{1}{3}\right)^2\left(\tfrac{2}{3}\right)^4 = \frac{6!}{2!4!} \times \frac{2^4}{3^6} = \tfrac{80}{243} \approx 0{\cdot}329$$

$\therefore$ the probability of success in exactly 4 out of 6 cases is 0·329.

$P(X = 0) = \left(\tfrac{1}{3}\right)^6 = \tfrac{1}{729}$, $P(X = 1) = \binom{6}{1}\left(\tfrac{1}{3}\right)^5\left(\tfrac{2}{3}\right) = \tfrac{4}{243}$,

$\therefore$ the probability of at least 2 successes

$= 1 - \tfrac{1}{729} - \tfrac{4}{243} = \tfrac{716}{729} \approx 0{\cdot}982.$

If a random variable X has the binomial distribution $B(n, p)$, then

$$\sum_{r=0}^{n} P(X = r) = \sum_{r=0}^{n}\binom{n}{r}q^{n-r}p^r = (q+p)^n = 1.$$

Thus we have verified that for the distribution $B(n, p)$,

$$\sum_{r=0}^{n} p_r = 1, \quad \text{where} \quad p_r = P(X = r).$$

We now obtain the mean and variance of X.

$$E(X) = \sum_{r=0}^{n} r\binom{n}{r} q^{n-r}p^r$$

$$= \sum_{r=0}^{n} \frac{r(n!)}{(n-r)!r!} q^{n-r}p^r$$

$$= \sum_{r=1}^{n} \frac{n!}{(n-r)!(r-1)!} q^{n-r}p^r$$

$$= np \sum_{r=1}^{n} \frac{(n-1)!}{(n-r)!(r-1)!} q^{n-r}p^{r-1} = np(q+p)^{n-1} = np.$$

$$\text{Var}(X) = \sum_{r=0}^{n} r^2\binom{n}{r} q^{n-r}p^r - \{E(X)\}^2$$

Writing $r^2 = r(r-1) + r$, we obtain:

$$\text{Var}(X) = \sum_{r=0}^{n} \frac{r(r-1)n!}{(n-r)!r!} q^{n-r}p^r + \sum_{r=0}^{n} \frac{r(n!)}{(n-r)!r!} q^{n-r}p^r - n^2p^2$$

$$= \sum_{r=2}^{n} \frac{n!}{(n-r)!(r-2)!} q^{n-r}p^r + np - n^2p^2$$

$$= n(n-1)p^2 \sum_{r=2}^{n} \frac{(n-2)!}{(n-r)!(r-2)!} q^{n-r}p^{r-2} + np - n^2p^2$$

$$= n(n-1)p^2(q+p)^{n-2} + np - n^2p^2$$
$$= n(n-1)p^2 + np - n^2p^2$$
$$= np - np^2 = np(1-p) = npq$$

$\therefore$ the mean and variance of the binomial distribution $B(n, p)$ are given by

$$\mu = np, \quad \sigma^2 = npq = np(1-p).$$

Example 2 The components produced by a particular machine are tested by taking samples containing 5 components and noting the number of rejects in each. The given table shows the results for 100 samples

Number of rejects (x)	0	1	2	3	4	5
Frequency (f)	10	27	31	20	9	3

(a) Calculate the mean and variance of the number of rejects.
(b) Estimate the probability p that a component selected at random is a reject.

(c) By assuming that the number of rejects in a sample of 5 components has a binomial distribution with parameters $n = 5$ and p, write down the theoretical mean and variance.

(d) Obtain the expected frequency distribution for 100 samples.

(a) $\sum f = 10 + 27 + 31 + 20 + 9 + 3 = 100$

$\sum fx = 0 \times 10 + 1 \times 27 + 2 \times 31 + 3 \times 20 + 4 \times 9 + 5 \times 3 = 200$

$\sum fx^2 = 0^2 \times 10 + 1^2 \times 27 + 2^2 \times 31 + 3^2 \times 20 + 4^2 \times 9 + 5^2 \times 3 = 550$

$$\therefore \quad \text{the mean } \bar{x} = \frac{\sum fx}{\sum f} = \frac{200}{100} = 2,$$

$$\text{the variance} = \frac{\sum fx^2}{\sum f} - \bar{x}^2 = \frac{550}{100} - 2^2 = 1{\cdot}5.$$

(b) Total number of rejects $= \sum fx = 200$

Total number of components $= 5 \times \sum f = 500$

$\therefore$ the probability p that any component is a reject $= \frac{200}{500} = \frac{2}{5}$.

(c) The mean and variance of the binomial distribution with parameters $n = 5$ and $p = 2/5$ are respectively:

$$\mu = np = 2, \qquad \sigma^2 = np(1 - p) = 1{\cdot}2.$$

(d) $$P(\text{no rejects}) = \left(\frac{3}{5}\right)^5 \approx 0{\cdot}08,$$

$$P(1 \text{ reject}) = \binom{5}{1}\left(\frac{3}{5}\right)^4\left(\frac{2}{5}\right) \approx 0{\cdot}26,$$

$$P(2 \text{ rejects}) = \binom{5}{2}\left(\frac{3}{5}\right)^3\left(\frac{2}{5}\right)^2 \approx 0{\cdot}35,$$

$$P(3 \text{ rejects}) = \binom{5}{3}\left(\frac{3}{5}\right)^2\left(\frac{2}{5}\right)^3 \approx 0{\cdot}23,$$

$$P(4 \text{ rejects}) = \binom{5}{4}\left(\frac{3}{5}\right)\left(\frac{2}{5}\right)^4 \approx 0{\cdot}08,$$

$$P(5 \text{ rejects}) = \left(\frac{2}{5}\right)^5 \approx 0{\cdot}01.$$

Multiplying these probabilities by 100 gives the expected frequencies in 100 samples (correct to the nearest integer).

Number of rejects	0	1	2	3	4	5
Expected frequency	8	26	35	23	8	1

[Note that because of rounding errors the sum of these expected frequencies is 101. This does not invalidate the results.]

We now consider an alternative way of finding the mean and variance of certain discrete probability distributions including the binomial distribution.

The *probability generating function* (p.g.f.) for the random variable X, which takes the values $0, 1, 2, \ldots, r, \ldots$ with probabilities $p_0, p_1, p_2, \ldots, p_r, \ldots$ is given by

$$G(t) = p_0 + p_1 t + p_2 t^2 + \ldots + p_r t^r + \ldots$$

Thus by using a "dummy" variable t, we are able to write down a single expression $G(t)$ representing a complete probability distribution. We note that since $\sum p_r = 1$, it follows that $G(1) = 1$.

For the binomial distribution with parameters n and p,

$$G(t) = q^n + \binom{n}{1} q^{n-1} pt + \binom{n}{2} q^{n-2} p^2 t^2 + \ldots + \binom{n}{r} q^{n-r} p^r t^r + \ldots$$

$\therefore$ the p.g.f. of the binomial distribution $B(n, p)$ is

$$G(t) = (q + pt)^n, \quad \text{where} \quad q = 1 - p.$$

If $G(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3 + \ldots + p_r t^r + \ldots$ then, differentiating with respect to t,

$$\begin{aligned} G'(t) &= p_1 + 2p_2 t + 3p_3 t^2 + \ldots + rp_r t^{r-1} + \ldots \\ G''(t) &= 2p_2 + 3.2p_3 t + \ldots + r(r-1)p_r t^{r-2} + \ldots \\ \therefore \quad G'(1) &= p_1 + 2p_2 + 3p_3 + \ldots + rp_r + \ldots = \sum rp_r \\ G''(1) &= 2p_2 + 3.2p_3 + \ldots + r(r-1)p_r + \ldots \\ &= \sum r(r-1)p_r = \sum r^2 p_r - \sum rp_r \end{aligned}$$

Hence $E(X) = \sum rp_r = G'(1)$

and $\text{Var}(X) = \sum r^2 p_r - [E(X)]^2 = G''(1) + G'(1) - [G'(1)]^2$

i.e. for a distribution with p.g.f. $G(t)$

$$\mu = G'(1), \quad \sigma^2 = G''(1) + G'(1) - [G'(1)]^2.$$

For the binomial distribution $B(n, p)$:

$$\begin{aligned} G(t) &= (q + pt)^n \\ G'(t) &= np(q + pt)^{n-1} \qquad \therefore \quad G'(1) = np \\ G''(t) &= n(n-1)p^2(q + pt)^{n-2} \qquad \therefore \quad G''(1) = n(n-1)p^2 \end{aligned}$$

Hence $\mu = np$, $\sigma^2 = n(n-1)p^2 + np - n^2p^2 = np(1-p)$.

These results are the same as those obtained earlier by elementary methods.

Exercise 16.2

1. An unbiased die is thrown 9 times. Find the probability of (a) 2 sixes, (b) at least 2 sixes. Find also the mean and variance of the number of sixes thrown.

2. In a large batch of plastic mouldings 25% have faults. If a random sample of 10 mouldings is inspected, find the probability that the sample contains

(a) 3 faulty mouldings, (b) no more than 2 faulty mouldings. Find also the mean and variance of the number of faulty mouldings.

3. The probability that a certain darts player hits the "treble 20" with one dart is 0·3. Find the probability that the player scores at least one treble 20 with 3 darts. If the probability of scoring at least one treble 20 with n darts is greater than 0·9, find the least possible value of n.

4. One in eight of the torches assembled by a machine are imperfect. Eight torches are chosen at random. Find the probability that (a) one is imperfect, (b) at least 6 are imperfect. The probability that a box of n torches contains none that are imperfect is less than 0·1. Find the range of possible values of n.

5. Given that the binomial distribution $B(n, p)$ has mean 9·6 and standard deviation 2·4, find n and p. Hence obtain the probability generating function of the distribution.

6. Find the probability generating function of a binomial distribution whose mean is 1·8 and whose variance is 0·99. If the random variable X has this binomial distribution, (i) state the greatest possible value that X may take (with non-zero probability), (ii) write down a numerical expression for the probability that a randomly observed value of X will be equal to 3. (JMB)

7. A lake contains 1000 fish of species A, 2000 fish of species B and no other fish. Ten fish are taken from the lake at random. Explain why the probabilities of obtaining various numbers of fish of species A in the ten can be very well approximated by a binomial probability distribution, and hence estimate to three significant figures the probability of obtaining 4 fish of species A and six of species B. Estimate also to three significant figures the probability that in a catch of 5 fish altogether there are more fish of species A than there are of species B. (C)

8. A manufacturing company sends out invoices to its customers allowing 5% discount to those who settle their accounts within 15 days. In the past, 40% of the customers took advantage of the discount terms. On a particular day the company sent out 10 invoices. Examine critically the appropriateness of the binomial distribution model for this situation, discussing (a) the requirement of constant probability from trial to trial, (b) the independence of trials. Assuming the model to be appropriate, calculate the probability that, of the 10 invoices, less than 2 are settled within 15 days. For days when 10 invoices are sent out, calculate the mean and variance of the distribution of the number of invoices settled within 15 days. (L)

9. The numbers of males in a sample of 810 litters of 4 cats are recorded in the table below:

No. of males	0	1	2	3	4
No. of litters	156	326	242	74	12

Calculate the mean and variance of the number of males in this sample of litters. Obtain a corresponding table of the above form for a binomial distribution with $p = \frac{1}{3}$.

10. Six dice were thrown together 1000 times. At each throw, the number of ones obtained was noted and the frequency is shown in the table below:

No. of 1's in a single throw	0	1	2	3	4	5	6
No. of throws	370	380	150	90	9	1	0

Calculate the mean and variance of the number of ones obtained in the experiment. Produce a corresponding table of the expected frequencies, to the nearest integer in each case, based on the assumption that the six dice are unbiased and that the 1000 throws are independent. (L)

11. (a) Berg and Korner have a long rivalry in tennis: in the last 25 games, Berg has won 15 times. They start a new series of 8 games. Assuming that the binomial model may be applied, what is the probability that Berg will win at least 6 of these?

(b) Prove that the mean of the binomial distribution is np. A set of 100 pods, each containing 4 peas, was examined to see how many of the peas were good. The following were the results:

No. of good peas in pod	0	1	2	3	4
No. of pods	7	20	35	30	8

(i) What was the probability of getting a good pea? (ii) Calculate the theoretical frequencies of 0, 1, 2, 3, 4 good peas, using the associated theoretical binomial distribution. (SU)

12.

X	frequency
0	36
1	37
2	20
3	5
4	2

The number of wrong numbers (X) obtained in making a set of ten telephone calls is recorded in the given table which records the results obtained in making 100 sets of 10 telephone calls. Assuming that wrong numbers occur randomly, suggest a suitable statistical model to correspond to the making of 10 telephone calls and recording the number of wrong numbers obtained. Using this model estimate the percentage of sets of ten telephone calls in which 2 or more wrong numbers are obtained. What will this percentage be if the chance of getting a wrong number when making a call is doubled? (L)

13. (i) Find the probability of obtaining exactly two sixes in six throws of a single die [You may assume that $(5/6)^5 = 0{\cdot}402$.]

(ii) If two dice are to be thrown together and the sum of their scores is recorded, find the probability that (a) the sum in one throw will be eleven, (b) a score of

eleven will occur exactly twice in six throws [assume that $17^4/18^5 = 0{\cdot}044$], (c) in one throw the score will be less than eight, given that exactly one die shows a four. (L)

14. A crossword puzzle is published in *The Times* each day of the week, except Sunday. A man is able to complete, on average, 8 out of 10 of the crossword puzzles. (i) Find the expected value and the standard deviation of the number of completed crosswords in a given week. (ii) Show that the probability that he will complete at least 5 in a given week is 0·655 (to 3 significant figures). (iii) Given that he completes the puzzle on Monday, find, to three significant figures, the probability that he will complete at least 4 in the rest of the week. (iv) Find, to three significant figures, the probability that, in a period of four weeks, he completes 4 or less in only one of the four weeks. (C)

15. 4 girls and 3 boys plan to meet together on the following Saturday. The probability that each boy will be present is $\frac{2}{3}$ independently of the other boys. Find the probabilities that (a) 0, (b) 1, (c) 2, (d) 3 boys will be present. The probability that each girl will be present is $\frac{1}{2}$ independently of the other girls and of the boys. (e) Find the probability that the number of girls present will equal the number of boys. (f) Find the probability that both sexes will be present. (g) Afterwards it was reported that the gathering had included at least one boy and at least one girl. What is the probability that there were equal numbers of boys and girls in the light of this additional information? (O&C)

16. Each trial of a binomial experiment has a constant probability p of yielding a success. (i) Given that 7 successes occurred in 15 independent trials show that the conditional probability that the first and last trials were successes is 1/5. (ii) Find the value of p given that in 20 independent trials the probability of exactly 4 successes is twice that of exactly 6 successes. (JMB)

17. The random variable X has the binomial distribution defined by

$$P(X = r) = \binom{n}{r}(1 - p)^{n-r}p^r, \quad r = 0, 1, 2, \ldots, n.$$

Find an expression for $P(X = r)/P(X = r - 1)$. Hence deduce the most likely values of X when (a) $n = 20$, $p = 0{\cdot}4$, (b) $n = 14$, $p = 0{\cdot}2$.

18. Cards are drawn at random and with replacement from an ordinary pack of playing cards until three spades have been drawn. Find the probability that the number of draws required is (a) exactly six, (b) at least six. Find an expression for P_n, the probability that exactly n cards have to be drawn to get three spades ($n \geqslant 3$). Hence, or otherwise, find the most likely value (or values) of n. (L)

19. A random variable X has p.g.f. $G(t) = 1/(2 - t)^2$. Find the expressions for (a) $P(X = r)$, (b) $E(X)$, (c) $\mathrm{Var}(X)$.

20. A random variable X is defined as the number of throws of an unbiased die needed to obtain a six. Find the p.g.f. for X and hence determine $E(X)$ and $\mathrm{Var}(X)$.

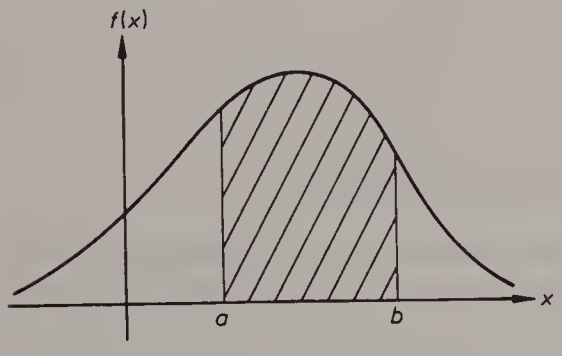
16.3 Continuous probability distributions

If X is a continuous random variable which may take any real value within some given interval, $0 \leqslant x \leqslant 10$ say, then it is difficult to give a meaning to probabilities such as $P(X = 2)$, $P(X = 7{\cdot}32054)$, $P(X = 17/39)$. If $P(X = x)$ were non-zero for all values of x, the sum of all such probabilities would be infinite. This means that we must regard the value of $P(X = x)$ as zero. Thus, in order to set up a suitable probability model we clearly need to define a new type of function.

The *probability density function* (p.d.f.) of the continuous random variable X is a function $f(x)$ such that

$$P(a \leqslant X \leqslant b) = \int_a^b f(x)\,dx.$$

[Since $P(X = a) = P(X = b) = 0$, the values of $P(a < X < b)$, $P(a \leqslant X < b)$ and $P(a < X \leqslant b)$ are also given by this integral.]



It follows that we can now think of probabilities as areas under the graph of $f(x)$. As probabilities are never negative, $f(x) \geqslant 0$ for all values of x. Since the sum of the probabilities of all possible outcomes of an experiment is 1, the total area under the graph must be 1.

i.e. $$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

For a discrete random variable X, $E(X) = \mu = \sum xp(x)$ and $\mathrm{Var}(X) = \sigma^2 = \sum(x - \mu)^2 p(x) = \sum x^2 p(x) - \mu^2$. To obtain similar formulae for the mean and variance of a continuous random variable X, let us consider the probability that the value of X lies between x and $x + \delta x$. Provided that δx is small, this probability is given approximately by $f(x)\delta x$. Thus we let

$$E(X) = \lim_{\delta x \to 0} \sum x f(x)\,\delta x = \int_{-\infty}^{\infty} x f(x)\,dx.$$

Applying similar arguments to the variance, we have:

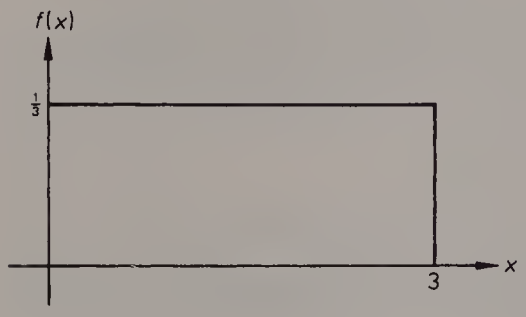
For a continuous random variable X,

$$E(X) = \mu = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$\mathrm{Var}(X) = \sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)\,dx = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2.$$

One of the simplest types of continuous probability distribution is the *rectangular* or *uniform* distribution. A continuous random variable X is said to be *uniformly distributed* over the interval a to b if $f(x)$ has a constant non-zero value for $a \leqslant x \leqslant b$ and is zero otherwise.

Example 1 Find the mean and variance of the continuous random variable X which is uniformly distributed over the interval from 0 to 3.



Since the total area under the graph of the probability density function $f(x)$ must be 1,

$f(x) = \frac{1}{3}$ for $0 \leqslant x \leqslant 3$

$f(x) = 0$ otherwise.

$$\therefore \quad E(X) = \int_0^3 x \times \frac{1}{3}\,dx = \left[\frac{1}{6}x^2\right]_0^3 = \frac{3}{2}$$

$$\text{Var}(X) = \int_0^3 x^2 \times \frac{1}{3}\,dx - \left(\frac{3}{2}\right)^2 = \left[\frac{1}{9}x^3\right]_0^3 - \frac{9}{4} = 3 - \frac{9}{4} = \frac{3}{4}$$

Hence the mean of the distribution is $1\frac{1}{2}$ and the variance is $\frac{3}{4}$.

A *mode* or *modal value* of a continuous random variable X with probability density function $f(x)$ is a value of x for which $f(x)$ takes a maximum value. Thus the modes of a distribution are the x-coordinates of the maximum points on the graph of $f(x)$.

The *median* of a continuous random variable X with probability density function $f(x)$ is the value m such that $\int_{-\infty}^{m} f(x)\,dx = 0{\cdot}5$.

Example 2 A continuous random variable X has probability density function $f(x) = kx(9 - x^2)$ for $0 \leqslant x \leqslant 3$ and $f(x) = 0$ otherwise. Find (a) the value of k, (b) the mean, (c) the mode, (d) the median.

(a) $$\int_0^3 f(x)\,dx = k\int_0^3 (9x - x^3)\,dx = k\left[\frac{9x^2}{2} - \frac{x^4}{4}\right]_0^3 = \frac{81k}{4}$$

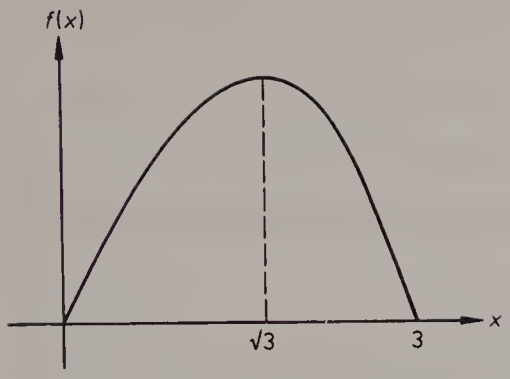
Since the total area under the graph of $f(x)$ must be 1, $k = 4/81$.

(b) $$E(X) = \int_0^3 xf(x)\,dx = \frac{4}{81}\int_0^3 (9x^2 - x^4)\,dx$$

$$= \frac{4}{81}\left[3x^3 - \frac{x^5}{5}\right]_0^3 = 4\left(1 - \frac{3}{5}\right) = \frac{8}{5}$$

Hence the mean of the distribution is 1·6.

(c)



$f(x) = \frac{4}{81}(9x - x^3)$,

$f'(x) = \frac{4}{81}(9 - 3x^2) = \frac{4}{27}(3 - x^2)$

$\therefore$ in the interval $0 \leqslant x \leqslant 3$, $f'(x) = 0$ when $x = \sqrt{3}$.
The sketch shows that this gives a maximum value of $f(x)$.

Hence the mode of the distribution is 1·73 (to 3 s.f.)

(d) If m is the median, $\int_0^m f(x)\,dx = \frac{1}{2}$

$$\therefore \quad \frac{4}{81}\int_0^m (9x - x^3)\,dx = \frac{4}{81}\left[\frac{9x^2}{2} - \frac{x^4}{4}\right]_0^m = \frac{1}{81}(18m^2 - m^4) = \frac{1}{2}$$

$$36m^2 - 2m^4 = 81$$
$$2m^4 - 36m^2 + 81 = 0$$

$$\therefore \quad m^2 = \frac{36 \pm \sqrt{(36^2 - 8.81)}}{4} = \frac{36 \pm 18\sqrt{(4-2)}}{4} = 9 \pm \frac{9}{2}\sqrt{2}$$

Since $0 < m < 3$, we must have $m^2 = 9 - \frac{9}{2}\sqrt{2}$

$$\therefore \quad m = \sqrt{\left\{9 - \frac{9}{2}\sqrt{2}\right\}} \approx 1{\cdot}62$$

Hence the median of the distribution is 1·62 (to 3 s.f.).

Another function used when modelling continuous probability distributions is defined as follows:

The *cumulative distribution function* (c.d.f.) of the continuous random variable X is the function $F(x)$ such that

$$F(x) = P(X \leqslant x).$$

If X has probability density function $f(x)$, then

$$F(t) = P(X \leqslant t) = \int_{-\infty}^{t} f(x)\,dx.$$

It follows that: $F'(x) = f(x)$.

Example 3 A random variable X has cumulative distribution function

$$F(x) = \begin{cases} 0 & \text{for } x \leqslant 0 \\ \frac{1}{6}x^2 & \text{for } 0 \leqslant x \leqslant 2 \\ -2 + 2x - \frac{1}{3}x^2 & \text{for } 2 \leqslant x \leqslant 3 \\ 1 & \text{for } x \geqslant 3. \end{cases}$$

Find (a) $P(X \leqslant 1{\cdot}5)$, (b) the median, (c) the probability density function $f(x)$.

(a) $P(X \leqslant 1{\cdot}5) = F(1{\cdot}5) = \frac{1}{6}(1{\cdot}5)^2 = 0.375$.

(b) The median is the value m such that $F(m) = \frac{1}{2}$.

Since $F(0) = 0$ and $F(2) = \frac{2}{3}$, m lies in the interval from 0 to 2 and thus $F(m) = \frac{1}{6}m^2$.

$\therefore$ $\frac{1}{6}m^2 = \frac{1}{2}$ which gives $m = \sqrt{3}$.

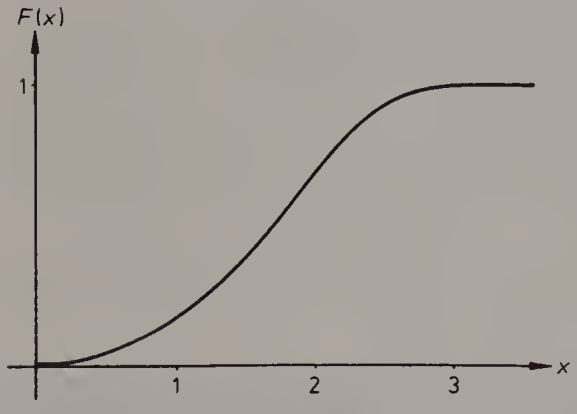
Hence the median of the distribution is 1·73 (to 3 s.f.).

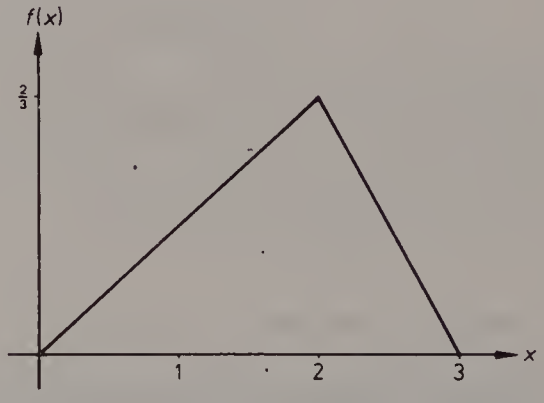
(c) The probability density function $f(x) = F'(x)$.

$\therefore$ for $0 \leqslant x \leqslant 2$, $f(x) = \dfrac{d}{dx}(\frac{1}{6}x^2) = \frac{1}{3}x$;

for $2 \leqslant x \leqslant 3$, $f(x) = \dfrac{d}{dx}(-2 + 2x - \frac{1}{3}x^2) = 2 - \frac{2}{3}x$;

otherwise, $f(x) = 0$.





Example 4 A random variable X has probability density function $f(x) = \frac{1}{2}x$ for $0 \leqslant x \leqslant 2$ and $f(x) = 0$ otherwise. Find the cumulative distribution function $F(x)$ of the variable X. Hence find the cumulative distribution function $G(y)$ of the variable $Y = X^2$. Deduce the probability density function $g(y)$.

For $0 \leqslant t \leqslant 2$, $F(t) = \displaystyle\int_0^t \tfrac{1}{2}x\,dx = \left[\tfrac{1}{4}x^2\right]_0^t = \tfrac{1}{4}t^2$.

$\therefore$ the cumulative distribution function of X is given by

$F(x) = 0\,(x \leqslant 0)$, $F(x) = \frac{1}{4}x^2\,(0 \leqslant x \leqslant 2)$, $F(x) = 1\,(x \geqslant 2)$.

$G(y) = P(Y \leqslant y) = P(X^2 \leqslant y) = P(X \leqslant \sqrt{y}) = F(\sqrt{y})$

$\therefore$ the cumulative distribution function of Y is given by

$G'(y) = 0 (y \leqslant 0)$, $G(y) = \frac{1}{4}y (0 \leqslant y \leqslant 4)$, $G(y) = 1 (y \geqslant 4)$.

Since $g(y) = G'(y)$, we find that $g(y) = \frac{1}{4} (0 \leqslant y \leqslant 4)$ and $g(y) = 0$ otherwise.

[We see that Y is uniformly distributed over the interval from 0 to 4.]

Exercise 16.3

1. Find the mean and variance of the continuous random variable X which is uniformly distributed over the interval
(a) from 0 to 1, (b) from 1 to 5,
(c) from 0 to k, (d) from a to b, where $b > a$.

2. A continuous random variable X has probability density function

$$f(x) = k(3x^2 - x^3) \quad \text{for} \quad 0 \leqslant x \leqslant 3 \quad \text{and} \quad f(x) = 0 \text{ otherwise.}$$

Find the value of k. Find also the mean, mode and standard deviation of X.

3. (a) A continuous variable x is distributed at random between the values 2 and 3 and has a probability density function of $6/x^2$. Find the median value of x.
(b) A continuous random variable X takes values between 0 and 1, with a probability density function of $Ax(1 - x)^3$. Find the value of A, and the mean and standard deviation of X. (SU)

4. A continuous variable X is distributed at random between two values, $x = 0$ and $x = 2$, and has a probability density function of $ax^2 + bx$. The mean is 1·25. (i) Show that $b = \frac{3}{4}$, and find the value of a. (ii) Find the variance of X. (iii) Verify that the median value of X is approximately 1·3. (iv) Find the mode. (SU)

5. The random variable X has a probability density function given by

$$P(x) = \begin{cases} kx(1 - x^2) & (0 \leqslant x \leqslant 1), \\ 0 & \text{elsewhere,} \end{cases}$$

k being a constant. Find the value of k and find also the mean and variance of this distribution. Find the median of the distribution. (O&C)

6. The continuous random variable X has probability density function given by

$$f(x) = \begin{cases} k(1 + x^2) & \text{for} \quad -1 \leqslant x \leqslant 1, \\ 0 & \text{otherwise,} \end{cases}$$

where k is a constant. Find the value of k, and determine $E(X)$ and $\text{Var}(X)$. A is the event $X > \frac{1}{2}$; B is the event $X > \frac{3}{4}$. Find (i) $P(B)$, (ii) $P(B|A)$. (C)

7. Petrol is delivered to a garage every Monday morning. At this garage the weekly demand for petrol, in thousands of units, is a continuous random variable X distributed with a probability density function of the form

$$f(x) = \begin{cases} ax^2(b - x) & 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise.} \end{cases}$$

(i) Given that the mean weekly demand is 600 units, determine the values of a and b. (ii) If the storage tanks at this garage are filled to their total capacity of 900 units every Monday morning, what is the probability that in any given week the garage will be unable to meet the demand for petrol? (AEB 1978)

8. A continuous random variable X has the probability density function f defined by

$$\begin{aligned} f(x) &= \tfrac{1}{3}cx, & 0 \leqslant x < 3, \\ f(x) &= c, & 3 \leqslant x \leqslant 4, \\ f(x) &= 0, & \text{otherwise,} \end{aligned}$$

where c is a positive constant. Find (i) the value of c, (ii) the mean of X, (iii) the value, a, for there to be a probability of 0·85 that a randomly observed value of X will exceed a. (JMB)

9. A continuous random variable X takes values between 0 and 1 only and its probability density function f is given by $f(x) = kx^2(1 - x)$. Prove that $k = 12$, and find $E(X)$. The cumulative distribution function F is such that $F(x) = P(X \leqslant x)$. Express $F(x)$ in terms of x, and sketch the graph of $y = F(x)$. Verify by calculation that the median of X lies between 0·60 and 0·62. (C)

10. The random variable X has the probability density function

$$f(x) = ae^{-ax} (0 < x < \infty).$$

Show that the cumulative distribution function of X is $F(x) = 1 - e^{-ax}$. The above distribution, with parameter $a = 0{\cdot}8$, is proposed as a model for the length of life, in years, of a species of bird. Find the expected frequencies, of a total of 50 birds, that would fall in the class intervals (years) 0−, 1−, 2−, 3−, 4−, 5 and over. (O)

11. A random variable X takes values in the interval $0 < x < 3$ and has probability density function $f(x)$, where

$$f(x) = \begin{cases} ax & (0 < x \leqslant 1), \\ \frac{1}{2}a(3 - x) & (1 < x < 3). \end{cases}$$

(i) Find the value of a. (ii) Find the expected value and the standard deviation of X. (iii) Find the (cumulative) distribution function $F(x)$. (iv) Find $P(|X - 1| < \frac{1}{2})$. (C)

12. The continuous random variable X has probability density function $f(x)$ defined by

$$f(x) = \begin{cases} c/x^4 & (x < -1) \\ c(2 - x^2) & (-1 \leqslant x \leqslant 1) \\ c/x^4 & (x > 1). \end{cases}$$

(i) Show that $c = \frac{1}{4}$. (ii) Sketch the graph of $f(x)$. (iii) Determine the cumulative distribution function $F(x)$. (iv) Determine the expected value of X and the variance of X. (C)

13. The continuous random variable X has (cumulative) distribution function given by

$$F(x) = \begin{cases} (1 + x)/8 & (-1 \leqslant x \leqslant 0), \\ (1 + 3x)/8 & (0 \leqslant x \leqslant 2), \\ (5 + x)/8 & (2 \leqslant x \leqslant 3), \end{cases}$$

with $F(x) = 0$ for $x < -1$, and $F(x) = 1$ for $x > 3$.
(i) Sketch the graph of the probability density function $f(x)$.
(ii) Determine the expectation of X and the variance of X.
(iii) Determine $P(3 \leqslant 2X \leqslant 5)$. (C)

14. The length X of an offcut of wooden planking is a random variable which can take any value up to 0·5 m. It is known that the probability of the length being not more than x metres $(0 \leqslant x \leqslant 0{\cdot}5)$ is equal to kx. Determine (i) the value of k, (ii) the probability density function of X, (iii) the expected value of X, (iv) the standard deviation of X (correct to 3 s.f.). (C)

15. The (cumulative) distribution function for a continuous random variable X is given by

$$F(x) = \begin{cases} 0 & \text{for} \quad x < -2 \\ k(4x - \frac{1}{3}x^3 + \frac{16}{3}) & \text{for} \quad -2 \leqslant x \leqslant 2, \\ 1 & \text{for} \quad x > 2. \end{cases}$$

Find (i) the value of k, (ii) $P(-1 < X < 1)$, (iii) the probability density function for X, (iv) the mean and variance of X. (C)

16. A random variable X has probability density function $f(x) = \frac{2}{3}x$ for $1 \leqslant x \leqslant 2$ and $f(x) = 0$ otherwise. Find the cumulative distribution function $F(x)$ of the variable X. Hence find the cumulative distribution function $G(y)$ of the variable $Y = X^{1/2}$. Deduce the probability density function $g(y)$.

17. X is a continuous random variable which is uniformly distributed in the interval $-1 \leqslant x \leqslant 1$. A second random variable Y is given by $Y = X^2$. Show that, if $0 < y < 1$, then $P(Y \leqslant y) = \sqrt{y}$, and deduce the probability density function of Y. Two independent observations of Y are made, and Z denotes the larger of the two observations. By considering the probability that both values of Y are

less than or equal to z, where $0 \leqslant z \leqslant 1$, or otherwise, prove that z is uniformly distributed. Determine the probability density function of W, the smaller of the two values of Y. (C)

18. Write down the probability density function and the cumulative distribution function of a random variable which is uniformly distributed over the interval $[a, b]$.

A person drives to work, a distance of 12 miles. The time he takes, in minutes, to cover the journey is uniformly distributed over the interval $[20, 30]$. Let V denote the car's average speed in miles per hour on such a journey. Find an expression for $P(V \leqslant v)$, where v is an arbitrary value between 24 and 36. Hence, or otherwise, find (i) the probability density function of V, (ii) the probability that on such a journey the car's average speed will be in excess of 30 m.p.h., (iii) the median average speed for the journey. (W)

19. A continuous random variable X has probability density function defined by

$$f(x) = 2x/a^2 \qquad 0 \leqslant x \leqslant a$$
$$f(x) = 0, \qquad \text{otherwise.}$$

(a) Find the variance of X. (b) Find the cumulative distribution function of X. (c) If $Y = 1 - (X/a)$, find (i) $P(Y > X)$, (ii) the probability density function of Y. (W)

20. A random variable X has probability density function $f(x) = |x|$ for $-1 \leqslant x \leqslant 1$ and $f(x) = 0$ otherwise. Find the probability density function $g(y)$ of the random variable $Y = X^2$.

16.4 The normal distribution

The *normal distribution* is the most important continuous probability distribution in theoretical statistics. It is found to provide a good probability model for the distributions of many variables such as the heights of people and other measurements of biological characteristics. One of the early uses of the normal distribution was as a model for the distribution of errors in experimental observations.

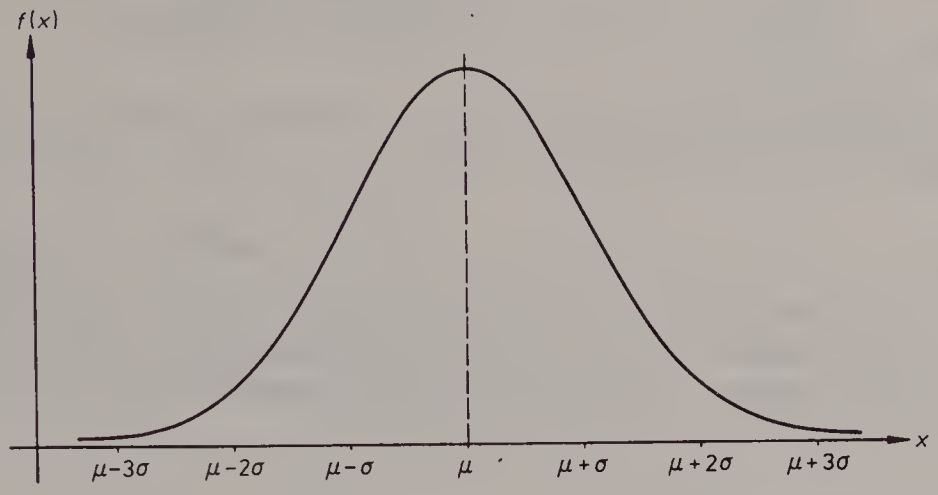
A continuous random variable X is said to be *normally distributed* with mean μ and variance σ^2 if it has probability density function.

$$f(x) = \frac{1}{\sigma\sqrt{(2\pi)}} e^{-(x-\mu)^2/2\sigma^2} \quad (\sigma > 0)$$

This distribution is denoted by $N(\mu, \sigma^2)$.

We cannot prove by elementary methods that this is a valid definition of a probability distribution. Clearly $f(x) \geqslant 0$ for all values of x, but for now we must

assume that the total area under the graph of $f(x)$ is 1 and that the distribution has mean μ and variance σ^2.



The graph of $f(x)$ is a bell-shaped curve called the *normal curve*. We see that
(1) the curve is symmetrical about the line $x = \mu$;
(2) the x-axis is an asymptote to the curve;
(3) when $x = \mu$, $f(x)$ takes its maximum value $1/\sigma\sqrt{(2\pi)}$ (i.e. $0{\cdot}4/\sigma$ approximately).
As described in the previous section, the probabilities associated with a continuous random variable X are represented by areas under the graph of the p.d.f. $f(x)$. Thus for the distribution $N(\mu, \sigma^2)$,

$$P(X \leqslant a) = \int_{-\infty}^{a} f(x)\,dx = \int_{-\infty}^{a} \frac{1}{\sigma\sqrt{(2\pi)}} e^{-(x-\mu)^2/2\sigma^2}\,dx.$$

This integral can be simplified by substituting

$$z = \frac{x-\mu}{\sigma} \quad \text{and} \quad dz = \frac{1}{\sigma}dx.$$

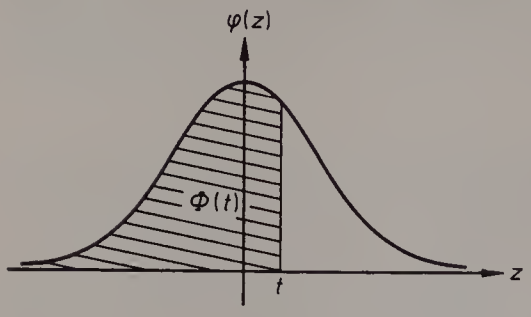
Writing $c = \dfrac{a-\mu}{\sigma}$, we have

$$P(X \leqslant a) = P(Z \leqslant c) = \int_{-\infty}^{c} \frac{1}{\sqrt{(2\pi)}} e^{-z^2/2}\,dz.$$

Hence the new random variable $Z = (X - \mu)/\sigma$ has probability density function $\phi(z) = \dfrac{1}{\sqrt{(2\pi)}} e^{-z^2/2}$. We deduce that Z is normally distributed with mean 0 and variance 1. We therefore call Z a *standardised* or *standard* normal variable. To summarise:

> If a normal variable X has the distribution $N(\mu, \sigma^2)$ then the random variable $Z = (X - \mu)/\sigma$ has the *standard normal distribution* $N(0, 1)$.

Using this change of variable any problem involving a normal distribution can be restated in terms of the standard distribution $N(0, 1)$.



The (cumulative) *distribution function* for the standard normal variable Z is denoted by $\Phi(z)$,
where $\Phi(z) = P(Z \leqslant z)$,

$$\therefore \quad \Phi(t) = \int_{-\infty}^{t} \phi(z)\, dz,$$

where $\phi(z) = \dfrac{1}{\sqrt{(2\pi)}} e^{-z^2/2}$.

Since these integrals are difficult to evaluate, tables of values have been compiled. Table 1, which appears towards the end of this book, gives values of $\Phi(z)$ for $0 \leqslant z \leqslant 4$. For example, $P(Z \leqslant 2{\cdot}18) = \Phi(2{\cdot}18) = 0{\cdot}98537$. If intermediate values of $\Phi(z)$ are required linear interpolation can be used.

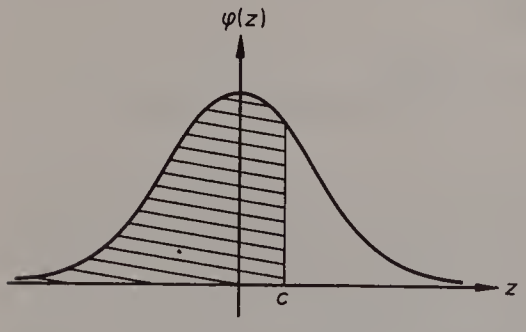
Example 1 Use Table 1 to estimate the value of $\Phi(0{\cdot}6354)$.

$\Phi(0{\cdot}64) - \Phi(0{\cdot}63) = 0{\cdot}7389 - 0{\cdot}7357 = 0{\cdot}0032$

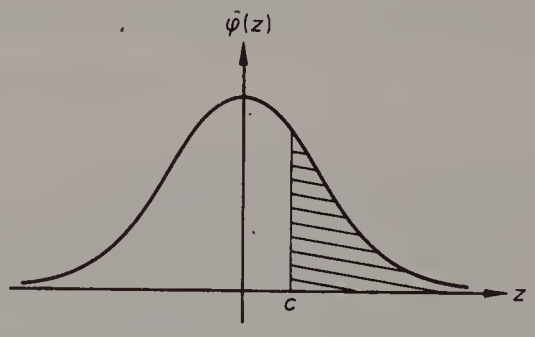
$\therefore$ by linear interpolation,

$$\Phi(0{\cdot}6354) = 0{\cdot}7357 + \tfrac{54}{100} \times 0{\cdot}0032 \approx 0{\cdot}7374.$$

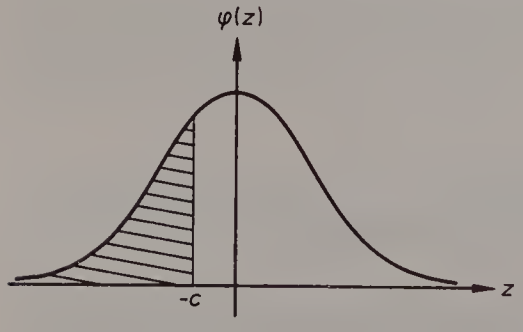
The following diagrams show how the symmetry of the normal distribution can be used to obtain $\Phi(z)$ for negative values of z.



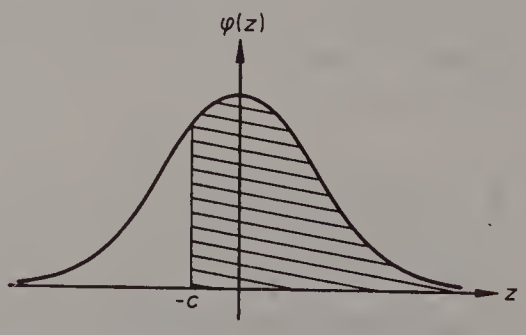
$P(Z \leqslant c) = \Phi(c)$



$P(Z \geqslant c) = 1 - \Phi(c)$



$P(Z \leqslant -c) = 1 - \Phi(c)$



$P(Z \geqslant -c) = \Phi(c)$

We may apply these results as follows:

$$P(Z \leqslant -0.72) = 1 - \Phi(0{\cdot}72) = 1 - 0{\cdot}7642 = 0.2358$$
$$P(-1{\cdot}5 \leqslant Z \leqslant -0{\cdot}5) = P(Z \geqslant -1{\cdot}5) - P(Z \geqslant -0.5)$$
$$= \Phi(1{\cdot}5) - \Phi(0{\cdot}5) = 0{\cdot}9332 - 0{\cdot}6915 = 0{\cdot}2417$$

In the next example we use our knowledge of the standard normal distribution to find probabilities associated with a different normal distribution.

Example 2 Given that a continuous random variable X is normally distributed with mean 20 and variance 16, find $P(15 \leqslant X \leqslant 24)$.

X has mean $\mu = 20$ and standard deviation $\sigma = 4$,
$\therefore$ the corresponding standard normal variable is

$$Z = (X - \mu)/\sigma = (X - 20)/4$$

$$P(X \leqslant 24) = P\left(Z \leqslant \frac{24 - 20}{4}\right) = P(Z \leqslant 1) = \Phi(1) = 0{\cdot}8413$$

$$P(X \leqslant 15) = P\left(Z \leqslant \frac{15 - 20}{4}\right) = P(Z \leqslant -1{\cdot}25) = 1 - \Phi(1{\cdot}25) = 0{\cdot}1056$$

$\therefore$ $P(15 \leqslant X \leqslant 24) = 0{\cdot}8413 - 0{\cdot}1056 = 0{\cdot}7357.$

We now demonstrate a different way of using values of the normal distribution function.

Example 3 Given that $\Phi(t) = 0{\cdot}81$, use Table 1 to estimate the value of t.

$\Phi(0{\cdot}87) = 0{\cdot}8078$, $\Phi(0{\cdot}88) = 0{\cdot}8106$,
$\therefore$ if $\Phi(t) = 0{\cdot}81$, then $0{\cdot}87 < t < 0{\cdot}88$.
$\Phi(t) - \Phi(0{\cdot}87) = 0{\cdot}0022$, $\Phi(0{\cdot}88) - \Phi(0{\cdot}87) = 0{\cdot}0028$,
$\therefore$ by linear interpolation, $t = 0{\cdot}87 + \frac{22}{28} \times 0.01 \approx 0.878$.

For certain values of $\Phi(z)$, the values of z can be obtained directly from Table 2, which gives *percentage points* of the normal distribution $N(0, 1)$. For instance, given that $\Phi(t) = 0{\cdot}8$, we know that 20% of the distribution lies "to the right" of the value $z = t$. Thus, using Table 2, $t = 0{\cdot}842$.

Example 4 A random variable X is normally distributed with mean μ and variance σ^2. Given that $P(X < 8) = 0{\cdot}05$ and $P(X > 15) = 0{\cdot}1$, find μ and σ^2.

Letting $Z = (X - \mu)/\sigma$ and using tables,
$0{\cdot}05 = P(Z > 1{\cdot}645) = P(Z < -1{\cdot}645)$
and $0{\cdot}1 = P(Z > 1{\cdot}282)$.

$$\therefore \quad -1{\cdot}645 = \frac{8 - \mu}{\sigma} \quad \text{i.e.} \quad 1{\cdot}645\sigma = \mu - 8 \tag{1}$$

$$1{\cdot}282 = \frac{15 - \mu}{\sigma} \quad \text{i.e.} \quad 1{\cdot}282\sigma = 15 - \mu \tag{2}$$

Adding (1) and (2): $2{\cdot}927\sigma = 7$

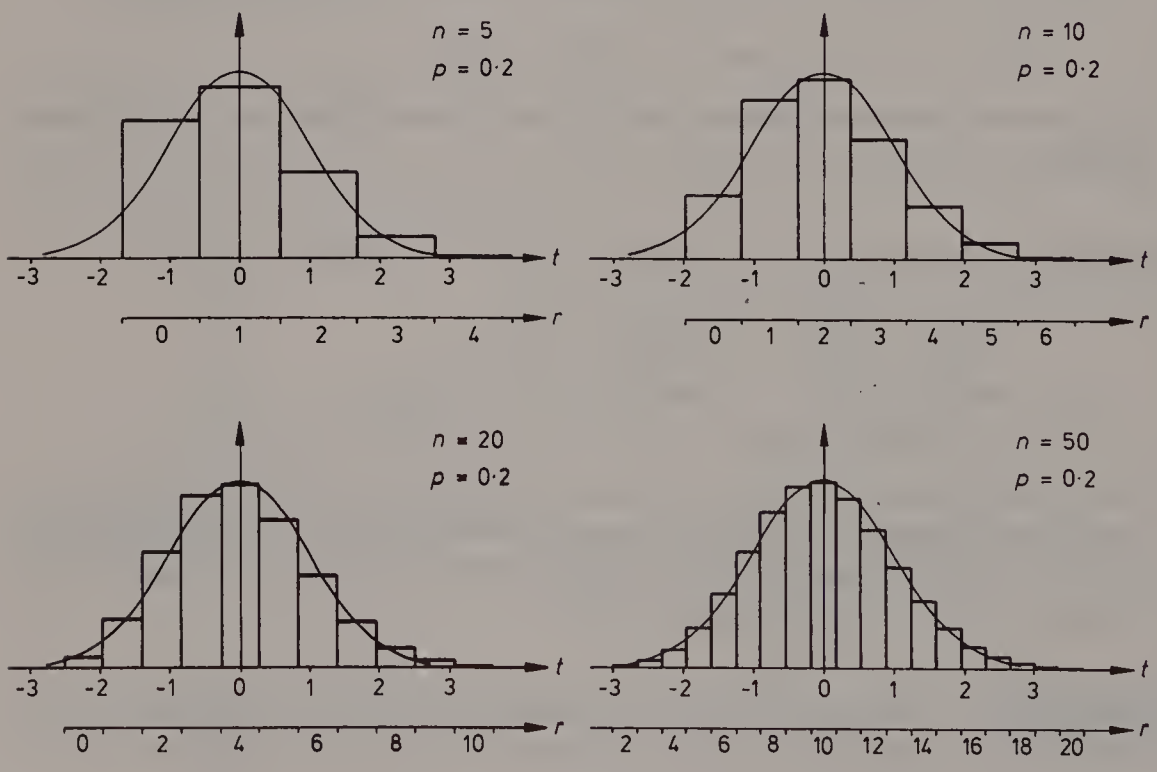
$$\therefore \quad \sigma = \frac{7}{2{\cdot}927} \quad \text{and} \quad \sigma^2 \approx 5{\cdot}72$$

Substituting in (1): $$\mu = 8 + \frac{1{\cdot}645 \times 7}{2{\cdot}927} \approx 11{\cdot}9$$

The normal distribution arises theoretically as a limit of the binomial distribution, as described in the following theorem:

> If R is a random variable with distribution $B(n, p)$, then the distribution of the variable $T = (R - \mu)/\sigma$, where $\mu = np$ and $\sigma = \sqrt{(npq)}$, tends to $N(0, 1)$ as n tends to infinity.

The proof of this statement is rather difficult, but we can demonstrate the limiting process involved by means of an example. The given set of histograms shows the distribution of the standardised binomial variable T as the value of n increases, but p remains fixed.



Transforming back to variables with mean $\mu = np$ and variance $\sigma^2 = npq$ we draw the following important conclusion.

> If R is a random variable with distribution $B(n, p)$, then for large n, the distribution of R can be approximated by the normal distribution $N(np, npq)$, where $q = 1 - p$.

This approximation will be close enough for most practical purposes if both np and nq are greater than 5. However, more accurate results will be obtained if both np and nq are greater than 20.

Before considering a numerical example we note that when using the distribution of a continuous variable X as an approximation to the distribution of a discrete variable R, it is necessary to make a *continuity correction*. For instance, if R takes only the integral values $0, 1, 2, \ldots$, then these are assumed to correspond to the intervals $X \leqslant 0{\cdot}5$, $0{\cdot}5 \leqslant X \leqslant 1{\cdot}5$, $1{\cdot}5 \leqslant X \leqslant 2{\cdot}5, \ldots$.

Let us now suppose that R has a binomial distribution with $n = 50$, $p = 0{\cdot}2$, then R will have mean $\mu = np = 10$ and variance $\sigma^2 = npq = 8$. Thus the corresponding normal variable X also has mean 10 and variance 8. Using $N(10, 8)$ as an approximation to $B(50, 0{\cdot}2)$:

$$\begin{aligned} P(12 \leqslant R \leqslant 15) &= P(11{\cdot}5 \leqslant X \leqslant 15{\cdot}5) \\ &= P\left(\frac{11{\cdot}5 - 10}{\sqrt{8}} \leqslant \frac{X - 10}{\sqrt{8}} \leqslant \frac{15{\cdot}5 - 10}{\sqrt{8}}\right) \\ &= P(0{\cdot}5303 \leqslant Z \leqslant 1{\cdot}9445) \\ &= \Phi(1{\cdot}9445) - \Phi(0{\cdot}5303) \\ &= 0{\cdot}9741 - 0{\cdot}7020 = 0{\cdot}2721 \end{aligned}$$

$$\therefore \quad P(12 \leqslant R \leqslant 15) \approx 0{\cdot}27.$$

This turns out to be a fairly good approximation to the true value, which is 0·26 (to 2 d.p.). Even better results are obtained when the value of p is closer to 0·5.

Exercise 16.4

1. Given that X is normally distributed with mean 0 and variance 1, use Table 1 to find

(a) $P(X \leqslant 0{\cdot}3)$, (b) $P(X \geqslant 1{\cdot}2)$,
(c) $P(X > -2)$, (d) $P(X \leqslant -2{\cdot}23)$,
(e) $P(0{\cdot}8 \leqslant X \leqslant 1{\cdot}8)$, (f) $P(-1{\cdot}1 < X < 0{\cdot}5)$.

2. Given that Z is a standardised normal variable, use linear interpolation to find

(a) $P(Z \leqslant 1{\cdot}537)$, (b) $P(Z < -0.862)$,
(c) $P(Z > -1{\cdot}9035)$, (d) $P(Z \geqslant 2{\cdot}0067)$,
(e) $P(|Z| \geqslant 1{\cdot}263)$, (f) $P(|Z + 1| < 0{\cdot}616)$.

3. Given that X is normally distributed with mean 5 and standard deviation 1, find

(a) $P(X < 6{\cdot}7)$, (b) $P(X \leqslant 4{\cdot}2)$, (c) $P(3{\cdot}7 < X < 5{\cdot}7)$.

4. Given that X is normally distributed with mean 0·2 and standard deviation 0·01, find

(a) $P(X \geqslant 0{\cdot}205)$, (b) $P(X > 0{\cdot}198)$, (c) $P(0{\cdot}199 \leqslant X \leqslant 0{\cdot}201)$.

5. Given that X is normally distributed with mean 12 and variance 1·5, find
(a) $P(X \leqslant 14)$, (b) $P(X > 9{\cdot}6)$, (c) $P(9 \leqslant X \leqslant 10)$.

6. Use Table 1 to estimate the value of t given that
(a) $\Phi(t) = 0{\cdot}67$, (b) $\Phi(t) = 0{\cdot}72$, (c) $\Phi(t) = 0{\cdot}19$.

7. Use Table 2 to find the value of t such that
(a) $\Phi(t) = 0{\cdot}7$, (b) $\Phi(t) = 0{\cdot}99$, (c) $\Phi(t) = 0{\cdot}9995$.

8. The weights of a large number of cats are normally distributed with mean 1·55 kg and standard deviation 0·2 kg. Find the percentage of cats with weights (a) less than or equal to 1 kg, (b) between 1·2 and 1·3 kg, (c) between 1·5 and 1·75 kg, (d) greater than or equal to 2 kg. (L)

9. The weights of oranges are distributed normally with a mean of 96 g and standard deviation 25 g. For selling purposes, oranges are divided into three grades, I, II and III. Grade I oranges weigh more than 126 g, grade III oranges weigh less than 60 g. All remaining oranges are classed grade II. Calculate the percentage of oranges in each grade. Grade I, II and III oranges sell respectively for 12p, 8p and 4p each. Calculate the expected receipts from the sale of 80 000 oranges. (L)

10. Tests made on 2 types of electric light bulb show the following: Type *A*, lifetime distributed normally with an average life of 1150 hours and a standard deviation of 30 hours. Type *B*, long-life bulb, average lifetime of 1900 hours, with standard deviation of 50 hours. (i) What percentage of bulbs of type *A* could be expected to have a life of more than 1200 hours? (ii) What percentage of type *B* would you expect to last longer than 1800 hours? (iii) What lifetime limit would you estimate would contain the central 80% of the production of type *A*? (SU)

11. Batteries for a transistor radio have a mean life under normal usage of 160 hours, with a standard deviation of 30 hours. Assuming the battery life follows a normal distribution, (a) calculate the percentage of batteries which have a life between 150 hours and 180 hours; (b) calculate the range, symmetrical about the mean, within which 75% of the battery lives lie; (c) if a radio takes four of these batteries and requires all of them to be working, calculate the probability that the radio will run for at least 135 hours. (O&C)

12. Adult men in a certain country have heights which are normally distributed with a mean of 1·81 m and a standard deviation of 0.05 m.
(a) Estimate the number of men in a random sample of 100 men whose heights, when measured to the nearest 0·01 m, exceed 1·90 m.
(b) Find the probability that a man selected at random will have a height, when measured to the nearest 0·01 m, less than 1·72 m.
(c) Find the height which is exceeded by 90% of the men. (L)

13. Find the mean and variance of the normally distributed variable X in each of the following cases:
(a) $P(X > 3) = 0{\cdot}5$, $P(3 < X < 5) = 0{\cdot}1$.
(b) $P(X > 20) = P(X < 10) = 0{\cdot}05$.
(c) $P(X < 60) = 0{\cdot}01$, $P(X > 90) = 0{\cdot}3$.
(d) $P(X > 10) = 0{\cdot}1$, $P(9 < X < 10) = 0{\cdot}2$.

14. A machine is producing curtain rails whose lengths are normally distributed with mean 2·4 m. If 5% of the rails are longer than 2·42 m, find the standard deviation of the distribution. If the machine is adjusted so that 95% of the rails have lengths between 2·38 m and 2·42 m, find the new standard deviation of the distribution.

15. In an examination 30% of the candidates fail and 10% achieve distinction. Last year the pass mark (out of 200) was 84 and the minimum mark required for a distinction was 154. Assuming that the marks of the candidates were normally distributed, estimate the mean mark and the standard deviation. (O&C)

16. A random variable R has the binomial distribution $B(n, p)$. In each of the following cases find both the probability of r successes and the normal approximation to it.
(a) $n = 10, p = \frac{1}{2}, r = 6$,
(b) $n = 20, p = \frac{1}{2}, r = 12$,
(c) $n = 50, p = \frac{1}{2}, r = 30$,
(d) $n = 20, p = \frac{1}{4}, r = 4$,
(e) $n = 50, p = \frac{1}{4}, r = 10$,
(f) $n = 100, p = \frac{1}{4}, r = 20$.

17. Estimate the probability of obtaining more than 60 heads in 100 tosses of an unbiased coin.

18. In a large consignment of peaches 20% of the fruit are bruised. Estimate the probability of fewer than 10 bruised peaches in a box of 60.

19. Tutorial groups, each of 9 students, are picked at random from a large population of which two-thirds are men. (i) What is the probability that a group contains men only? (ii) Calculate the probability that the number of women in a group is either 1, 2, 3, 4 or 5. (iii) Compare this with the probability that a normal variable with the same mean and variance as the distribution appropriate to (ii) should lie between 0·5 and 5·5. (AEB 1979)

20. In a particular community the probability of a man having brown eyes is 2/5. If a random sample of 6 men is to be taken, find the probability that (a) exactly 3 men, (b) at least 3 men will have brown eyes. A second random sample of 100 men is to be taken. Use the normal distribution as an approximation to the binomial distribution, to estimate to 3 decimal places the probabilities that (a) at least half the men have brown eyes and (b) the number of men with brown eyes is between 30 and 45 inclusive. (L)

21. It is known that 30% of an apple crop has been attacked by insects. A random sample of 150 is selected from the crop. Assuming that the distribution of the

number of damaged apples in random samples of 150 may be approximated to by a normal distribution, estimate the probability that (a) more than half the sample is damaged, (b) less than 10% is damaged, (c) the number of damaged apples in the sample lies between 35 and 50, inclusive. (L)

22. In Urbania selection for the Royal Flying Corps (RFC) is by means of an aptitude test based on a week's intensive military training. It is known that the scores of potential recruits on this test follow a normal distribution with mean 45 and standard deviation 10. (i) What is the probability that a randomly chosen recruit will score between 40 and 60? (ii) What percentage of the recruits is expected to score more than 30? (iii) In a particular year 100 recruits take the test. Assuming that the pass mark is 50, calculate the probability that less than 35 recruits qualify for the RFC. (AEB 1978)

16.5 The Poisson distribution

The *Poisson*† *distribution* was originally derived as a limit of the binomial distribution. It is found to provide a useful model for distributions in which the variable is the number of occurrences of a particular event in a given interval of time or space. Some examples of variables which may have a Poisson distribution are: the number of customers entering a shop in an hour, the number of claims received by an insurance company in a day, the number of flaws in a given length of copper wire, the number of earthworms in a given area of land.

To demonstrate the way in which the Poisson distribution arises as a limit of the binomial distribution, we consider the number of telephone calls received in an hour by a small business. Assuming that the average number of calls received per hour is 4, we can construct a simple binomial model by dividing each hour into 12 five-minute "trials" and taking the probability of a call in any five-minute interval to be 1/3. However, since the firm may receive more than one call in a five-minute period, this model is not entirely satisfactory. It would be better to regard an hour as 60 one-minute trials, taking the probability of a call in any one-minute interval to be 1/15. Remembering the speed with which a modern switchboard can process calls, we should perhaps reduce the time interval to 10 seconds or even to 2 seconds. The given table shows that we are producing a sequence of binomial distributions with increasing numbers of trials but constant mean.

Interval	n	p	Mean	Variance
5 minutes	12	$\frac{1}{3}$	4	$4 \times \frac{2}{3} \approx 2{\cdot}67$
1 minute	60	$\frac{1}{15}$	4	$4 \times \frac{14}{15} \approx 3{\cdot}73$
10 seconds	360	$\frac{1}{90}$	4	$4 \times \frac{89}{90} \approx 3{\cdot}96$
2 seconds	1800	$\frac{1}{450}$	4	$4 \times \frac{449}{450} \approx 3{\cdot}99$

The limit of this sequence is the Poisson distribution with mean and variance 4.

†*Poisson, Siméon Denis* (1781–1840) French mathematician. His main contributions were to the theory of electricity and magnetism, but in 1837 he published *Recherches sur la probabilité des jugements*, dealing with probability distributions.

We obtain the general form of the Poisson distribution by considering the limit of the binomial distribution $B(n, p)$ as n tends to infinity in such a way that the mean $\mu = np$ remains constant.

$$P(X = r) = \binom{n}{r} p^r (1-p)^{n-r}$$

$$= \frac{n(n-1)\dots(n-r+1)}{r!}\left(\frac{\mu}{n}\right)^r\left(1-\frac{\mu}{n}\right)^{n-r}$$

$$= \frac{n}{n}\left(\frac{n-1}{n}\right)\dots\left(\frac{n-r+1}{n}\right)\frac{\mu^r}{r!}\left(1-\frac{\mu}{n}\right)^n\left(1-\frac{\mu}{n}\right)^{-r}$$

$$= \left(1-\frac{1}{n}\right)\dots\left(1-\frac{r-1}{n}\right)\frac{\mu^r}{r!}\left(1-\frac{\mu}{n}\right)^n \Big/ \left(1-\frac{\mu}{n}\right)^r$$

$$\therefore \quad \lim_{n\to\infty} P(X = r) = \frac{\mu^r}{r!}\lim_{n\to\infty}\left(1-\frac{\mu}{n}\right)^n$$

It can be shown that

$$\lim_{n\to\infty}\left(1-\frac{\mu}{n}\right)^n = e^{-\mu}$$

Thus, as $n \to \infty$, $P(X = r) \to \frac{\mu^r}{r!}e^{-\mu}$.

This result leads to the following definition:

A discrete random variable X is said to have a Poisson distribution with parameter μ, where $\mu > 0$, if

$$P(X = r) = \frac{\mu^r}{r!}e^{-\mu} \quad \text{for} \quad r = 0, 1, 2, \dots$$

Considering again the limit of the distribution $B(n, p)$ as $n \to \infty$, we see that if the mean $\mu = np$ remains constant, then $p \to 0$ and the variance $\sigma^2 = np(1-p)$ tends to μ. This suggests that the Poisson distribution has mean μ and variance μ. These results can proved either directly or by using the appropriate probability generating function. For instance:

$$E(X) = \sum_{r=0}^{\infty} r\left(\frac{\mu^r}{r!}e^{-\mu}\right) = \mu e^{-\mu}\sum_{r=1}^{\infty}\frac{\mu^{r-1}}{(r-1)!}$$

$$= \mu e^{-\mu}\left(1 + \mu + \frac{\mu^2}{2!} + \frac{\mu^3}{3!} + \dots\right) = \mu e^{-\mu} \times e^{\mu} = \mu.$$

[For other proofs see Exercise 16.5 questions 12 and 13.]

It follows from this discussion that the Poisson distribution is a suitable model for the distribution of the number of events occurring in a specified interval of continuous time or space, provided that

(a) the events occur at random and independently, but may not occur simultaneously;
(b) the expected number of events in any interval is proportional to the size of the interval.

Example 1 A small business receives an average of 4 telephone calls per hour. Use the appropriate Poisson distribution to find the probability that (a) 7 calls are received in a two hour period, (b) up to 3 calls are received in an interval of 30 minutes.

(a) The expected number of calls in 2 hours is 8.
Thus if X is the number of calls received in 2 hours, X has a Poisson distribution with parameter 8

$$\therefore \quad P(X = 7) = \frac{8^7}{7!}e^{-8} \approx 0{\cdot}140.$$

Hence the probability that 7 calls are received in a two hour period is 0·140.

(b) The expected number of calls in 30 minutes is 2.
Thus if Y is the number of calls received in 30 minutes, Y has a Poisson distribution with parameter 2

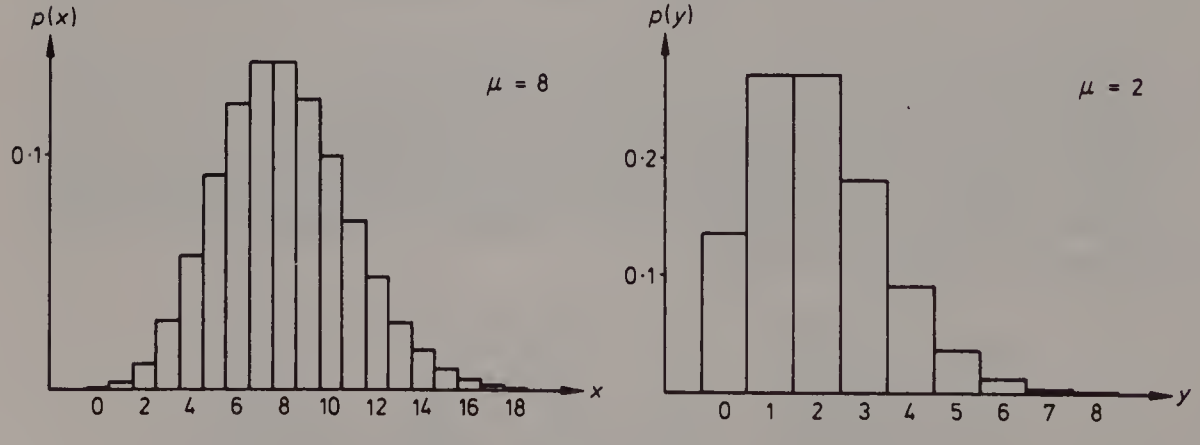
$$\therefore \quad P(Y \leqslant 3) = P(Y = 0) + P(Y = 1) + P(Y = 2) + P(Y = 3)$$

$$= e^{-2} + 2e^{-2} + \frac{2^2}{2!}e^{-2} + \frac{2^3}{3!}e^{-2}$$

$$= (1 + 2 + 2 + \tfrac{4}{3})e^{-2} \approx 0{\cdot}857.$$

Hence the probability that up to 3 calls are received in 30 minutes is 0·857.

The following diagrams show the Poisson distributions used in Example 1.



We have seen that the Poisson distribution with parameter $\mu = np$ is a limit of the binomial distribution $B(n, p)$. This means that under certain conditions a Poisson distribution can be used as an approximation to a binomial distribution with the same mean. In practice a Poisson approximation gives fairly accurate results when $n > 50$, $p < 0{\cdot}1$ and $np \leqslant 5$. It may also produce acceptable approximations to binomial distributions which do not satisfy these conditions. However, for a

binomial distribution such that both np and $n(1-p)$ are greater than 5, a normal approximation is usually more appropriate.

It can be shown that as μ tends to infinity, the distribution of the standardised Poisson variable $(X-\mu)/\sqrt{\mu}$ approaches the standard normal distribution $N(0, 1)$. It follows that for large μ, the normal distribution $N(\mu, \mu)$ can be used as an approximation to the Poisson distribution with parameter μ. Fairly good approximations are obtained for $\mu > 5$, but for accurate results it is necessary to have $\mu > 20$.

Exercise 16.5

1. In a school laboratory an average of 8 test-tubes are broken in 20 working days. Find the probability that (a) 5 test-tubes will be broken in the next 10 days; (b) fewer than 2 test-tubes will be broken in the next 3 days.

2. The number of heavy vehicles driving past a certain historic building in an hour is assumed to have a Poisson distribution with mean 3. Find the probability that (a) in the next hour more than 3 heavy vehicles will drive by; (b) in the next two hours fewer than 5 heavy vehicles will drive by.

3. A textile manufacturer estimates that there is an average of 2 faults in every 100 metres of fabric he produces. Find the probability that there is at least 1 fault in a 10-metre length of fabric. Write down an expression for the probability that there is no fault in an n-metre length of fabric. What is the greatest integer n such that this probability is greater than 0·9?

4. The number of plants of a certain species found in one square metre of woodland is assumed to have a Poisson distribution with mean 0·003. Find the probability of observing at least 2 specimens in an area of 400 m^2. A botanist is more than 99% certain that he will find at least one specimen in an area of n m^2. Find the least possible value of the integer n.

In questions 5 to 8 draw up a table showing the values, correct to 3 d.p., of the probabilities $p(0), p(1), p(2), \ldots$ for (a) the binomial distribution $B(n, p)$; (b) the Poisson distribution with parameter $\mu = np$; (c) the normal approximation to (a); (d) the normal approximation to (b). Comment briefly on your results.

5. $n = 20, p = 0{\cdot}5$.

6. $n = 20, p = 0{\cdot}1$.

7. $n = 50, p = 0{\cdot}1$.

8. $n = 100, p = 0{\cdot}08$.

9. It is found that on average 0·5% of the watches produced by a certain factory are faulty. Estimate the probability that (a) in a batch of 200 watches there are no faulty watches; (b) in a batch of 500 watches more than 2 are faulty.

10. On average 1 out of 50 patients treated with a certain drug suffer side effects. Estimate the probability that (a) if 80 patients take the drug then exactly 2 will have side effects, (b) if 200 patients take the drug then at least 4 will have side effects.

11. In a certain experiment the probability of success in any trial is 0·03. Use appropriate approximations to find the probabilities of (a) at least 2 successes in 50 trials, (b) at least 4 successes in 100 trials, (c) at least 20 successes in 500 trials.

12. Use the identity $r^2 \equiv r(r-1) + r$ to show that

$$\sum_{r=0}^{\infty} \frac{r^2 x^r}{r!} = x^2 e^x + x e^x.$$

Hence show that the Poisson distribution with parameter μ has variance μ.

13. Show that the probability generating function for the Poisson distribution with parameter μ is given by $G(t) = e^{\mu(t-1)}$. Hence show that the distribution has mean μ and variance μ.

14. The number of road accidents at a certain traffic roundabout has been found to follow a Poisson distribution with a mean of 0·8 accidents per week. Calculate, correct to 2 significant figures, the probabilities that (i) there will be at least 2 accidents in a particular week, (ii) there will be exactly 3 accidents in a particular three-week period. After road works have been carried out, the traffic is observed for a period of 8 weeks and in that time only 1 accident occurs. Calculate the probability of at most 1 accident in 8 weeks if the mean is still 0·8 accidents per week. Comment briefly on this result. (C)

15. (a) I recorded the number of phone calls I received over a period of 150 days:

No. of calls	0	1	2	3	4
No. of days	51	54	36	6	3

(i) Find the average number of calls per day. (ii) Calculate the frequencies of the comparable Poisson distribution.

(b) A firm selling electrical components packs them in boxes of 60. On average 2% of the components are faulty. What is the chance of getting more than 2 defective components in a box? (Use the Poisson distribution.) (SU)

16. An aircraft has 116 seats. The airline has found, from long experience, that on average 2·5% of people with tickets for a particular flight do not arrive for that flight. If the airline sells 120 seats for a particular flight, determine, using a suitable approximation, the probability that more than 116 people arrive for that flight. Determine also the probability that there are empty seats on the flight. (C)

17. A number of different types of fungi are distributed at random in a field. Eighty per cent of these fungi are mushrooms, and the remainder are toadstools. Five per cent of the toadstools are poisonous. A man who cannot distinguish between mushrooms and toadstools, wanders across the field and picks a total of 100 fungi. Determine, correct to 2 significant figures, using approximations, the probability that the man has picked (i) at least 20 toadstools, (ii) exactly two poisonous toadstools. (C)

18. A new telephone directory is to be published, and before publication takes place the entries have to be checked for misprints and any necessary corrections made. Previous experience suggests that, on average, 0·1% of the entries will require correction, and that entries requiring correction will be randomly distributed. The directory will contain 240 000 entries altogether, printed on 800 pages with 300 entries per page. There are two methods of making corrections: for method A the costs are £C per page containing one entry requiring correction and £3C per page containing two or more entries requiring correction; for method B the costs are £2C per page containing one or more entries requiring correction. Calculate the expected costs of correction for each method. (C)

19. In a certain office the number of reams of typing paper used per month has a Poisson distribution with mean 25. Estimate the probability that more than 25 reams of paper will be used in the next month. Given that it is more than 95% certain that there is enough paper in the stationery cupboard to last a month, find the minimum number of reams of paper in the cupboard.

20. Show that the variance of a Poisson distribution is equal to the mean. The number of bacteria in 1 ml of inoculum has a Poisson distribution with mean 2·0. If at least 3 bacteria are needed for a dose to be infective, find the probability that a dose of 1 ml will cause infection. Find approximate limits, symmetrical about the mean, between which lies 95% of the distribution of the number of bacteria in 100 ml of inoculum. (O&C)

16.6 The algebra of expectation and variance

The expectation $E(X)$ and the variance $\mathrm{Var}(X)$ of a random variable X were defined earlier in the chapter.

$$E(X) = \mu = \begin{cases} \sum x_r p(x_r), & \text{for discrete } X \\ \displaystyle\int_{-\infty}^{\infty} x f(x)\,dx, & \text{for continuous } X \end{cases}$$

$$\mathrm{Var}(X) = \sigma^2 = \begin{cases} \sum (x_r - \mu)^2 p(x_r), & \text{for discrete } X \\ \displaystyle\int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx, & \text{for continuous } X \end{cases}$$

If a new random variable $g(X)$ is defined in terms of X by a formula such as $g(X) = 2X + 3$ or $g(X) = X^2$, then the expectation of $g(X)$ is given by:

$$E[g(X)] = \begin{cases} \sum g(x_r) p(x_r), & \text{for discrete } X \\ \displaystyle\int_{-\infty}^{\infty} g(x) f(x)\,dx & \text{for continuous } X \end{cases}$$

If X is a discrete variable which takes the values $x_1, x_2, \ldots$ then we can show that this formula for $E[g(X)]$ is consistent with the definition of $E(X)$ by letting $Y = g(X)$.

(i) If Y takes distinct values $y_1 = g(x_1)$, $y_2 = g(x_2), \ldots$, then

$$P(Y = y_r) = P\big(g(X) = g(x_r)\big) = P(X = x_r) = p(x_r),$$

$$\therefore \quad E[g(X)] = E(Y) = \sum_{\text{all } r} y_r P(Y = y_r) = \sum_{\text{all } r} g(x_r)p(x_r).$$

(ii) If the values $g(x_1), g(x_2), \ldots$ are not all distinct then we have relationships such as $y_k = g(x_i) = g(x_j) = \ldots$. In these cases

$$P(Y = y_k) = P(X = x_i) + P(X = x_j) + \ldots = p(x_i) + p(x_j) + \ldots$$

$$\therefore \quad y_k P(Y = y_k) = y_k p(x_i) + y_k p(x_j) + \ldots$$
$$= g(x_i)p(x_i) + g(x_j)p(x_j) + \ldots$$

$$\therefore \quad E[g(X)] = E(Y) = \sum_{\text{all } k} y_k P(Y = y_k) = \sum_{\text{all } r} g(x_r)p(x_r).$$

Thus the formula for $E[g(X)]$ is valid for discrete X. We will assume that it is also valid for continuous X.

Example 1 A discrete random variable X takes the values $0, 1, 2, 3$ and $p(0) = 0{\cdot}1$, $p(1) = p(3) = 0{\cdot}2$, $p(2) = 0{\cdot}5$. Find $E(X^2 - X)$.

$$\begin{aligned} E(X^2 - X) &= \sum (x^2 - x)p(x) \\ &= (0^2 - 0)0{\cdot}1 + (1^2 - 1)0{\cdot}2 + (2^2 - 2)0{\cdot}5 + (3^2 - 3)0{\cdot}2 \\ &= 0 + 0 + 2 \times 0{\cdot}5 + 6 \times 0{\cdot}2 = 2{\cdot}2. \end{aligned}$$

Example 2 Find $E(X^3)$ for the continuous random variable X which has probability density function $f(x) = 2(1 - x)$ for $0 \leqslant x \leqslant 1$ and $f(x) = 0$ otherwise.

$$E(X^3) = \int_0^1 x^3 f(x)\,dx = \int_0^1 x^3 . 2(1 - x)\,dx$$

$$= 2\int_0^1 (x^3 - x^4)\,dx = 2\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1 = 2\left(\frac{1}{4} - \frac{1}{5}\right) = 0{\cdot}1.$$

Two useful properties of expectation follow from the definition of $E[g(X)]$.

> If X is a random variable, discrete or continuous,
> (i) $E(aX + b) = aE(X) + b$, for constants a and b.
> (ii) $E[g(X) + h(X)] = E[g(X)] + E[h(X)]$.

We give here the proof of (i) for discrete X. The remaining proofs are left as an exercise.

$$\begin{aligned} E(aX + b) &= \sum (ax_r + b)p(x_r) \\ &= \sum \{ax_r p(x_r) + bp(x_r)\} = a\sum x_r p(x_r) + b\sum p(x_r) \end{aligned}$$

But $\quad \sum x_r p(x_r) = E(X) \quad$ and $\quad \sum p(x_r) = 1$

Hence $\quad E(aX + b) = aE(X) + b.$

It is now possible to express the variance of a random variable in terms of expected values:

$$\text{Var}(X) = E[(X - \mu)^2], \quad \text{where} \quad \mu = E(X)$$

Thus, using results (i) and (ii):

$$\text{Var}(X) = E(X^2 - 2\mu X + \mu^2) = E(X^2) - 2\mu E(X) + \mu^2 = E(X^2) - \mu^2$$

i.e.

$$\boxed{\text{Var}(X) = E(X^2) - [E(X)]^2.}$$

Hence if a and b are constants,

$$\begin{aligned}\text{Var}(aX + b) &= E[(aX + b)^2] - [E(aX + b)]^2 \\ &= E(a^2X^2 + 2abX + b^2) - [aE(X) + b]^2 \\ &= a^2E(X^2) + 2abE(X) + b^2 - a^2[E(X)]^2 - 2abE(X) - b^2 \\ &= a^2E(X^2) - a^2[E(X)]^2 = a^2\{E(X^2) - [E(X)]^2\}\end{aligned}$$

i.e.

$$\boxed{\text{Var}(aX + b) = a^2\,\text{Var}(X)}$$

To develop the theory of expectation further we must now consider the distribution of two discrete random variables X and Y with probability functions $p_X(x)$ and $p_Y(y)$ respectively. Let X take the values $x_1, x_2, \ldots, x_i, \ldots$ and let Y take the values $y_1, y_2, \ldots, y_j, \ldots$

The *joint probability distribution* of X and Y is the set of all probabilities $p(x_i, y_j)$, where $p(x_i, y_j) = P(X = x_i, Y = y_j)$.

For example if the possible values of X are 0, 1, 2 and the values of Y are 0, 1, 2, 3, then the joint distribution will be a set of 12 probabilities, which can be displayed in a two-way table as follows:

X \ Y	0	1	2	3	Total
0	$p(0,0)$	$p(0,1)$	$p(0,2)$	$p(0,3)$	$p_X(0)$
1	$p(1,0)$	$p(1,1)$	$p(1,2)$	$p(1,3)$	$p_X(1)$
2	$p(2,0)$	$p(2,1)$	$p(2,2)$	$p(2,3)$	$p_X(2)$
Total	$p_Y(0)$	$p_Y(1)$	$p_Y(2)$	$p_Y(3)$	1

As indicated in the table, the row totals give the values of $p_X(x)$. For instance

$$p(0,0) + p(0,1) + p(0,2) + p(0,3) = p_X(0).$$

Similarly the column totals are the values of $p_Y(y)$. Since these probabilities appear

in the margins, they are called the *marginal distributions* of X and Y. Using this table we could obtain the probability distribution of a random variable such as $X + Y$. We see that $P(X + Y = 0) = p(0,0)$, $P(X + Y = 1) = p(1,0) + p(0,1)$, and so on. However, it is possible to obtain general formulae for the expected values of $X + Y$ and other functions of X and Y.

Taking into account all ordered pairs (x_i, y_j):

$$\begin{aligned} E(X + Y) &= \sum_i \sum_j (x_i + y_j) p(x_i, y_j) \\ &= \sum_i \left[\sum_j x_i p(x_i, y_j) \right] + \sum_j \left[\sum_i y_j p(x_i, y_j) \right] \\ &= \sum_i \left[x_i \sum_j p(x_i, y_j) \right] + \sum_j \left[y_j \sum_i p(x_i, y_j) \right] \\ &= \sum_i x_i p_X(x_i) + \sum_i y_j p_Y(y_j) = E(X) + E(Y). \end{aligned}$$

Similar arguments can be used to extend this result to any *linear combination* of random variables.

> If X and Y are random variables and a and b are constants, then
>
> $$E(aX + bY) = aE(X) + bE(Y).$$
>
> If $X_1, X_2, \ldots, X_n$ are random variables, then
>
> $$E(X_1 + X_2 + \ldots + X_n) = E(X_1) + E(X_2) + \ldots + E(X_n).$$

[These results are valid for both discrete and continuous variables, but in the continuous case the proofs require integration methods not covered in this course.]

If random variables X and Y are *independent*, then

$$P(X = x_i, Y = y_j) = P(X = x_i) \times P(Y = y_j)$$

i.e.

$$p(x_i, y_j) = p_X(x_i) p_Y(y_j)$$

$$\begin{aligned} E(XY) &= \sum_i \sum_j x_i\, y_j p(x_i, y_j) \\ &= \sum_i \left[\sum_j x_i y_j p_X(x_i) p_Y(y_j) \right] \\ &= \sum_i \left[x_i p_X(x_i) \sum_j y_j p_Y(y_j) \right] \\ &= \left[\sum_i x_i p_X(x_i) \right] \left[\sum_j y_j p_Y(y_j) \right] = E(X)E(Y). \end{aligned}$$

$\therefore$

> If X and Y are independent random variables, then
>
> $$E(XY) = E(X)E(Y).$$

We are now ready to consider the variance of a linear combination of random variables. In particular:

$$\begin{aligned}\mathrm{Var}(X+Y) &= E[(X+Y)^2] - [E(X+Y)]^2 \\ &= E(X^2 + 2XY + Y^2) - [E(X) + E(Y)]^2 \\ &= E(X^2) + 2E(XY) + E(Y^2) - [E(X)]^2 - 2E(X)E(Y) - [E(Y)]^2 \\ &= \{E(X^2) - [E(X)]^2\} + 2\{E(XY) - E(X)E(Y)\} \\ &\quad + \{E(Y^2) - [E(Y)]^2\}\end{aligned}$$

The quantity $E(XY) - E(X)E(Y)$ is called the *covariance* of X and Y, and is denoted by $\mathrm{Cov}(X, Y)$.
Thus, in general:

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + 2\,\mathrm{Cov}(X, Y) + \mathrm{Var}(Y).$$

However, if the variables X and Y are independent,

$$\mathrm{Cov}(X, Y) = E(XY) - E(X)E(Y) = 0$$

$$\therefore \quad \mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y).$$

As before, this basic result can be extended:

If X and Y are independent random variables and a and b are constants, then

$$\mathrm{Var}(aX + bY) = a^2\,\mathrm{Var}(X) + b^2\,\mathrm{Var}(Y).$$

Two important special cases are given by:

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X-Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$$

If $X_1, X_2, \ldots, X_n$ are independent random variables,

$$\mathrm{Var}(X_1 + X_2 + \ldots + X_n) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \ldots + \mathrm{Var}(X_n).$$

Example 3 In the ice-cream dessert served in a motorway cafeteria, the masses of the portions of ice-cream are distributed about a mean of 80 g with standard deviation 6 g. The fruit topping has mean mass 20 g and standard deviation 4 g. If the cost of 100 g of ice-cream is 5 p and the cost of 100 g of topping is 10 p, find the mean and standard deviation of the cost of a dessert.

Let the random variables X and Y represent the masses in grammes of portions of ice-cream and topping respectively. If Z is the total cost in pence of the dessert, then

$$Z = \tfrac{5}{100}X + \tfrac{10}{100}Y = \tfrac{1}{20}X + \tfrac{1}{10}Y$$

$$\therefore \quad E(Z) = \tfrac{1}{20}E(X) + \tfrac{1}{10}E(Y) = \tfrac{80}{20} + \tfrac{20}{10} = 6,$$

$$\text{Var}(Z) = (\tfrac{1}{20})^2 \text{Var}(X) + (\tfrac{1}{10})^2 \text{Var}(Y)$$
$$= \frac{6^2}{20^2} + \frac{4^2}{10^2} = \frac{9+16}{100} = \frac{1}{4}$$

Hence the cost of a dessert has mean 6 p and standard deviation $\frac{1}{2}$ p.

We now state without proof two important theorems concerning linear combinations of normal random variables.

If X and Y are independent normal random variables with distributions $N(\mu_x, \sigma_x^2)$ and $N(\mu_y, \sigma_y^2)$ respectively, then $Z = aX + bY$, where a and b are constants, is also a normal random variable and has mean $a\mu_x + b\mu_y$, variance $a^2\sigma_x^2 + b^2\sigma_y^2$.

Sums and differences of independent normal variables are of particular interest:

$Z = X + Y$ has the distribution $N(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)$,
$Z = X - Y$ has the distribution $N(\mu_x - \mu_y, \sigma_x^2 + \sigma_y^2)$.

If $X_1, X_2, \ldots, X_n$ are independent observations of a random variable with distribution $N(\mu, \sigma^2)$, then $Z = X_1 + X_2 + \ldots + X_n$ is normally distributed with mean $n\mu$, variance $n\sigma^2$.

Example 4 Brand X beefburgers have masses normally distributed about a mean of 65 g with variance 15 g^2. The masses of Brand Y beefburgers are normally distributed with mean 100 g, variance 25 g^2. Find the probability that the mass of 3 Brand X beefburgers is greater than the mass of 2 Brand Y beefburgers.

[It is tempting to think that what we need to find here is $P(3X - 2Y > 0)$. However, in the expression $3X - 2Y$, $3X$ represents 3 times the mass of one beefburger, not the mass of three beefburgers.]

Let $D = (X_1 + X_2 + X_3) - (Y_1 + Y_2)$, where X_1, X_2, X_3 represent the masses (in grammes) of 3 Brand X beefburgers and Y_1, Y_2 represent the masses of 2 Brand Y beefburgers, then

$$E(D) = E(X_1) + E(X_2) + E(X_3) - E(Y_1) - E(Y_2)$$
$$= 3 \times 65 - 2 \times 100 = -5$$
$$\text{Var}(D) = \text{Var}(X_1) + \text{Var}(X_2) + \text{Var}(X_3) + \text{Var}(Y_1) + \text{Var}(Y_2)$$
$$= 3 \times 15 + 2 \times 25 = 95$$

Letting Z be the standardised normal variable $(D + 5)/\sqrt{95}$,

$$P(D > 0) = P(Z > 5/\sqrt{95})$$
$$= P(Z > 0{\cdot}5130)$$
$$= 1 - \Phi(0{\cdot}5130) = 1 - 0{\cdot}6961 = 0{\cdot}3039$$

$\therefore$ the required probability is approximately 0·304.

Two further results are sometimes useful.

If independent random variables X and Y have distributions $B(m, p)$ and $B(n, p)$, then $X + Y$ has distribution $B(m + n, p)$.

If independent random variables X and Y have Poisson distributions with means λ and μ, then $X + Y$ has a Poisson distribution with mean $\lambda + \mu$.

Methods of proof are indicated in Exercise 16.6 questions 23 and 25.

Exercise 16.6

1. A discrete random variable X takes the values 1, 2, 3, 4, 5 with probabilities $p(1) = p(2) = 0{\cdot}1, p(3) = 0{\cdot}15, p(4) = 0{\cdot}4, p(5) = 0{\cdot}25$. Find $E(X)$, $E(X^2)$ and $E(1/X)$. Hence find $E(2X^2 - 1)$ and $E[(X + 1)^2/X]$.

2. A discrete random variable X takes the values $-1, 0, 1, 2$ with probabilities $p(-1) = 0{\cdot}2, p(0) = 0{\cdot}3, p(1) = 0{\cdot}4, p(2) = 0{\cdot}1$. Find $E(X)$, $E(X^2)$ and $E(X^4)$. Hence find $\mathrm{Var}(X)$ and $\mathrm{Var}(X^2)$.

3. A continuous random variable X has probability density function

$$f(x) = \tfrac{1}{2}(3 - x) \text{ for } 1 \leqslant x \leqslant 3 \text{ and } f(x) = 0 \text{ otherwise.}$$

Find $E(X)$, $E(X^2)$ and $E(X^3)$. Hence find $E(3X^3 + 2X)$ and $\mathrm{Var}(X)$.

4. A continuous random variable X has probability density function

$$f(x) = 6x(1 - x) \text{ for } 0 \leqslant x \leqslant 1 \text{ and } f(x) = 0 \text{ otherwise.}$$

Find $E(X)$, $E(X^2)$, and $E(\sqrt{X})$. Hence find $E[X(X - 1)]$ and $\mathrm{Var}(\sqrt{X})$.

5. A random variable X takes the values 5, 8, 4, 7 with equal probability and an independent variable Y takes the values 4, 7, 1 with equal probability. Find the mean and variance of the following:

(a) X (b) Y, (c) $4X + 3$, (d) $5 - 2Y$,
(e) $X + Y$, (f) $X - Y$, (g) $3X - Y$, (h) $2X + 3Y$.

6. Independent random variables X and Y are such that $E(X) = 10$, $\mathrm{Var}(X) = 9$ and $E(Y) = 7$, $\mathrm{Var}(Y) = 1$. Find (a) the mean and variance of $5X - 3Y + 2$, (b) the values of $E(X^2)$, $E(Y^2)$ and $E(X^2 + Y^2)$.

7. A factory produces nuts with masses distributed about a mean of 5 g with a standard deviation of 0·2 g. These are screwed on to bolts whose masses have mean

20 g and standard deviation 0·1 g. If each bolt is fitted with two nuts, find the mean and standard deviation of the resulting total mass.

8. In a can containing baked beans and 5 small sausages, the mass of each sausage is distributed about a mean of 20 g with standard deviation 2 g. The mean mass of the baked beans is 100 g with standard deviation 5 g. Find the mean mass of the total contents of a can and the standard deviation.

9.

X \ Y	0	1	2
1	0·06	0·12	0·02
2	0·15	0·3	0·05
3	0·09	0·18	0·03

The given table shows the joint probability distribution of two discrete random variables X and Y.

(a) Use the marginal distributions of X and Y to find $E(X)$, $\mathrm{Var}(X)$ and $E(Y)$, $\mathrm{Var}(Y)$.

(b) Are there any values of x and y for which

$$P(X = x, Y = y) \neq P(X = x) \times P(Y = y)?$$

Use your answer to determine whether or not X and Y are independent.

(c) Find the distribution of XY i.e. $P(XY = 0)$, $P(XY = 1)$, $P(XY = 2), \ldots$ Hence determine whether or not $E(XY) = E(X)E(Y)$.

(d) Find the distribution of $X + Y$. Hence determine whether or not

$$\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y).$$

10. Repeat question 9 for the following joint distributions.

(i)

X \ Y	0	1	2	3
0	0·03	0·1	0.05	0·02
1	0·05	0	0·25	0·1
2	0·02	0·2	0·1	0·08

(ii)

X \ Y	1	2	3
0	0·05	0·2	0·05
1	0·2	0	0·2
2	0·05	0·2	0·05

11. A random variable X takes the values 0 and 2 each with probability 0·5. Find $E(X)$, $E(X^2)$, $E(X^3)$, $E(X^4)$. Use the formula $\mathrm{Var}(Y) = E(Y^2) - [E(Y)]^2$ to verify that (a) if $Y = 3X - 5$, $\mathrm{Var}(Y) = 9\,\mathrm{Var}(X)$,

(b) if $Y = X^2 + X$, $\mathrm{Var}(Y) \neq \mathrm{Var}(X^2) + \mathrm{Var}(X)$.

12. The daily demand, x kilogrammes, for fresh fish at a certain fishmonger's shop is a continuous random variable with probability density function f, where f is given by $f(x) = (30 - x)/450$, $0 \leqslant x \leqslant 30$, $f(x) = 0$, otherwise. The fishmonger receives a delivery of 20 kg of fresh fish each morning. He makes a profit of 40 pence on each kilogramme of fish sold on the day the fish is delivered; fish left unsold on the day it is

delivered is sold to a fertiliser manufacturer at a loss of 5 pence per kg. Show that on a day when x is less than 20, the fishmonger's profit is $(45x - 100)$ pence and that on a day when x is 20 or more, the fishmonger's profit is £8. Hence determine the fishmonger's expected daily profit from the sale of fresh fish. (JMB)

13 A sample of four mice is to be chosen at random from a litter of ten mice, of which six are male and four are female. Let X denote the number of male mice in the sample. Calculate the values of $P(X = r)$ for $r = 0, 1, 2, 3, 4$. Hence find the mean and the standard deviation of X. Deduce the mean and the standard deviation of the number of female mice in the sample. Independently of the above, a sample of four mice is to be chosen at random from another litter of ten mice, of which four are male and six are female. Calculate the mean and the standard deviation of the total number of male mice in the combined sample of eight mice. (W)

14. The joint probability distribution of two discrete random variables X and Y is displayed in the following table, the entries in the body of the table being $P(X = x, Y = y)$.

		x		
		0	1	2
y	0	0·1	0·1	0
	1	0·1	0·2	0·1
	2	0·1	0·2	0·1

(i) Find $E(X)$ and $E(Y)$. (ii) Determine the distribution of $Z = 2X - Y$ and verify that $E(Z) = 2E(X) - E(Y)$. (iii)Let (X_1, Y_1) and (X_2, Y_2) denote two independent observations of (X, Y). By using the distribution of Z, or otherwise, find the probability that

$$2(X_1 - X_2) = Y_1 - Y_2.$$ (W)

15. On a piece-work operation a company pays a bonus to any employee who processes in excess of 300 kg of raw material in a day. The daily amounts processed by two employees A and B are independent and normally distributed, the mean and standard deviation of the amounts processed by A being 291 kg and 9 kg, respectively, and of the amounts processed by B being 282 kg and 12 kg, respectively. Calculate, to two significant figures in each case, (i) the proportion of days on which both A and B will earn the bonus, (ii) the probability that the combined daily amount processed by A and B will exceed 600 kg, (iii) the proportion of days on which the amount processed by B will exceed that processed by A. (W)

16. The diameters of axles supplied by a factory have a mean value of 19·92 mm and a standard deviation of 0·05 mm. The inside diameters of bearings supplied by another factory have a mean of 20·04 mm and a standard deviation of 0·03 mm. What is the mean and standard deviation of the random variable defined to be the diameter of a bearing less the diameter of an axle? Assuming that both dimensions are normally distributed, what percentage of axles and bearings taken at random will not fit? (O&C)

17. The time of arrival of a bus at a bus stop varies in a normal distribution with a mean of 09.00 a.m. and a standard deviation of 2 minutes. Independently a second bus departs from this stop at a time which varies in a normal distribution with a mean of 09.01 a.m. (i.e. one minute past 09.00 a.m.) and a standard deviation of 1 minute. Find the probability that (a) the first bus arrives before the second bus leaves; (b) this happens on 5 given consecutive days. (O&C)

18. The weight of a large loaf of bread is a normal variable with mean 420 g and standard deviation 30 g. The weight of a small loaf of bread is a normal variable with mean 220 g and standard deviation 10 g. (i) Find the probability that 5 large loaves weigh more than 10 small loaves. (ii) Find the probability that the total weight of 5 large loaves and 10 small loaves lies between 4·25 kg and 4·4 kg. (C)

19. The mass of a cheese biscuit has a normal distribution with mean 6 g and standard deviation 0·2 g. Determine the probability that (i) a collection of twenty-five cheese biscuits has a mass of more than 149 g, (ii) a collection of thirty cheese biscuits has a mass of less than 180 g, (iii) twenty-five times the mass of a cheese biscuit is less than 149 g. The mass of a ginger biscuit has a normal distribution with mean 10 g and standard deviation 0·3 g. Determine the probability that a collection of seven cheese biscuits has a mass greater than a collection of four ginger biscuits. (C)

20. Four sprinters A, B, C and D each run 100 m. The times (in seconds) that they take may be regarded as independent observations from a normal distribution with mean 14 and standard deviation 0·2. The athlete E runs 400 m. The time (in seconds) that this athlete takes may be regarded as an observation from a normal distribution with mean 58 and standard deviation 1·0 and is independent of the times of the sprinters. Determine the probability that (i) the time taken by E is less than 3 seconds greater than the sum of the times taken by the four sprinters, (ii) the time taken by E is less than 4 times as great as the time taken by A. For the four sprinters A, B, C and D, find the probability that A is the fastest sprinter of the four. Find also the probability that C and D (in either order) are the two slowest sprinters. (C)

21. P is a fixed point on the circumference of a circle, centre O, of radius r and Q is a point on the circumference such that $\angle POQ = \theta$, where θ is a random variable with a rectangular distribution in $[0, 2\pi]$. Find the mean and median values for the length of the shorter arc PQ. The length of the chord PQ is X. Find $E(X)$ and show that $\mathrm{Var}(X) = 2r^2(1 - 8/\pi^2)$. Find also $P(X > r\sqrt{3})$. (C)

22. X and Y are independent random variables with means and variances μ_x, σ_x^2 and μ_y, σ_y^2 respectively, and you may assume that for any such independent variables $E(XY) = E(X)E(Y)$. Find expressions for (i) the expectation of the area of a square of side X, (ii) the expectation and variance of the area of a rectangle of sides X and Y, (iii) the expectation of the volume of a cuboid with a square base of side X and with height Y. (C)

23. Two independent random variables X and Y have Poisson distributions with means λ and μ respectively. Obtain an expression for $P(X = r, Y = n - r)$. Hence express $P(X + Y = n)$ as the sum of a series. Deduce that $X + Y$ has a Poisson distribution with mean $\lambda + \mu$.

24. Telephone calls reach a secretary independently and at random, internal ones at a mean rate of 2 in any 5 minute period, and external ones at a mean rate of 1 in any 5 minute period. Calculate the probability that there will be more than 2 calls in any period of 2 minutes. (O&C)

25. (a) Verify that if a random variable X has probability generating function $G_X(t)$, then $G_X(t) = E(t^X)$.
(b) Two independent random variables X and Y have distributions $B(m, p)$ and $B(n, p)$ respectively. Write down expressions for $E(t^X)$ and $E(t^Y)$. Assuming that t^X and t^Y can be regarded as independent random variables, write down $E(t^X \times t^Y)$ and hence find the p.g.f. of the variable $Z = X + Y$. Deduce that Z has the distribution $B(m + n, p)$.

Exercise 16.7 (miscellaneous)

1. A random variable R takes the integer values $1, 2, \ldots, n$ each with probability $1/n$. Find the mean and variance of R. A pack of 15 cards bearing the numbers 1 to 15 is shuffled. Find the probability that the number on the top card is larger than that on the bottom card, giving reasons for your answer. If the sum of these two numbers is S, find (a) the probability that $S \leqslant 4$, (b) the expected value of S. (O&C)

2. The number X of radios sold per week by a certain shop is a binomially distributed random variable with $n = 10$ and $p = 0{\cdot}3$. The number Y of television sets sold per week by the same shop may be 0, 1 or 2 with $P(Y = 0) = 0{\cdot}6$, $P(Y = 1) = 0.3$, $P(Y = 2) = 0{\cdot}1$. The shopkeeper makes a profit of £2 on each radio sold and a profit of £20 on each television set sold. Find the shopkeeper's expected weekly profit from the sale of radios and television sets. (JMB)

3. (a) In a certain manufacturing process, it is known that approximately 10% of the items produced are defective. A quality control scheme is set up, by selecting 20 items out of a large batch, and rejecting the whole batch if 3 or more are defective. Find the probability that the batch is rejected.
(b) Two boys, John and David, play a game with a die. The die will be thrown 4 times. David will give John £x if there is an odd number of sixes, otherwise John will give David £1. If the game is to be a fair one to both John and David, find the value of x. (SU)

4. The probability of there being X unusable matches in a full box of Surelite matches is given by $P(X = 0) = 8k$, $P(X = 1) = 5k$, $P(X = 2) = P(X = 3) = k$,

$P(X \geqslant 4) = 0$. Determine the constant k and the expectation and variance of X. Two full boxes of Surelite matches are chosen at random and the total number Y of unusable matches is determined. Calculate $P(Y > 4)$, and state the values of the expectation and variance of Y. (C)

5. Fergus Lightfingers, an educated thief, has broken into a house and has come upon a large safe. Lying upon the table is a bunch of k similar keys, only one of which will open the safe. He considers two possible strategies: (i) randomly select keys, one at a time *without replacement*, until successful; (ii) randomly select keys, one at a time *with replacement*, until successful. Let N, a random variable, be the number of keys including the successful one tried by Fergus in order to open the safe. Derive the probability distribution of N for both strategies. Calculate the expected value of N in each case. Use your statistical judgment in order to select the strategy that Fergus should use. (You may assume that $\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}$ for $|x| < 1$.) (AEB 1978)

6. A batsman's performance is described by the following probability distribution. For each ball that he receives he has probability $\frac{1}{2}$ of scoring no runs, probability 1/5 of scoring one run and otherwise he is equally likely to score two runs, four runs, or to be bowled out. In the last case he receives no further balls.

(i) Determine the probability that the batsman's total score is 0.
(ii) Given that the batsman is not bowled by a particular ball, calculate the expectation of his score from that ball.
(iii) Calculate the expectation of the number of balls that the batsman receives.
(iv) Deduce the expectation of the batsman's total score. (C)

7. In a series of N independent trials the probability of success in any trial is constant and equal to p. Show that the probability of r successes is given by

$$P(r) = \binom{N}{r} p^r (1-p)^{N-r}.$$

Find $P(3)$ if $N = 5$ and $p = 0{\cdot}2$. Write a flow diagram for a program to print values of $P(r)$ if $N = 25$ and $p = 0{\cdot}4$ for $r = 0(1)25$. (L)

8. In a certain factory 70% of the microprocessors produced are found to be imperfect and have to be discarded. Find the probability that, in a randomly chosen sample of 5 microprocessors produced, the number of perfect specimens will be (i) 0, (ii) 1, (iii) 2, (iv) 3, (v) 4, (vi) 5. Verify that the probability of more than 3 perfect specimens is less than 1/30. If a sample of 200 is taken from the same set of microprocessors, find the probability that there are at least 50 perfect specimens. (AEB 1980)

9. Derive the mean and variance of the binomial distribution. Mass production of miniature hearing aids is a particularly difficult process and so the quality of these products is monitored carefully. Samples of size six are selected regularly and tested for correct operation. The number of defectives in each sample is

recorded. During one particular week 140 samples are taken and the distribution of the number of defectives per sample is given in the following table.

Number of defectives per samples (x)	0	1	2	3	4	5	6
Number of samples with x defectives (f)	27	36	39	22	10	4	2

Find the frequencies of the number of defectives per sample given by a binomial distribution having the same mean and total as the observed distribution.

(AEB 1978)

10. Show that the probability generating function for the binomial distribution arising from n independent trials, in each of which the probability of a success is p, is $(1 - p + pt)^n$. Show that the probability generating function for the number of trials needed to obtain a success is $pt/(1 - t + pt)$, and find the mean and variance of this distribution.

When a gun first fires at a target the probability of a hit is 1/5. Firing continues until the first hit, after which the gun is resighted and the probability of a hit becomes 2/5. Find the probability generating function for the number of shots required to obtain 2 hits and hence find the probability that fewer than 5 shots will be needed.

(O&C)

11. A random variable X takes values in the intervals 0 to 1 and 2 to 3, and has probability density function given by

$$f(x) = \begin{cases} 1 - x & (0 \leqslant x \leqslant 1), \\ x - a & (2 \leqslant x \leqslant 3), \\ 0 & \text{elsewhere.} \end{cases}$$

Determine (i) the value of a, (ii) the expectation, μ, of X, (iii) the variance of X, (iv) the (cumulative) distribution function $F(x)$, (v) the probability that $|X - \mu|$ exceeds 1.

(C)

12. The random variable X is the distance, in metres, that an inexperienced tight-rope walker has moved along a given tight-rope before falling off. It is given that

$$P(X > x) = 1 - x^3/64, \quad 0 \leqslant x \leqslant 4.$$

(i) Show that $E(X) = 3$. (ii) Find the standard deviation, σ, of X. (iii) Show that $P(|X - 3| < \sigma) = \frac{69}{80}\sqrt{\frac{3}{5}}$. (C)

13. Find the mean and the variance of the distribution whose probability density function f is given by

$$\begin{aligned} f(x) &= x, && 0 \leqslant x < 1, \\ f(x) &= 2 - x, && 1 \leqslant x \leqslant 2, \\ f(x) &= 0, && \text{otherwise.} \end{aligned}$$

A rectangle is constructed with adjacent sides of lengths x cm and $(2 - x)$ cm where x is a random value from the above distribution. Find the probability that the

area of the rectangle will exceed 0·75 cm^2. Also find the expected value of the area of the rectangle. (JMB)

14. The operational lifetime in hundreds of hours of a battery-operated minicalculator may be regarded as a continuous random variable having probability density function

$$f(x) = cx(10 - x), \qquad 5 \leqslant x \leqslant 10,$$
$$f(x) = 0, \qquad \text{otherwise.}$$

(a) Find the value of c and of the expected operational lifetime of such a minicalculator.

(b) The purchase price of such a minicalculator is £20 and its running cost (for batteries) amounts to 20 pence per hundred hours operation. Thus, the overall average cost in pence per hundred hours operation of a minicalculator whose operational lifetime is X hundred hours is given by $Y = 20 + (2000/X)$. (i) Evaluate $E(Y)$, the expected overall average cost per hundred hours. (ii) Find the probability that the overall average cost per hundred hours will exceed £2.70. (W)

15. A random variable X has (cumulative) distribution function

$$F(x) = \begin{cases} 0 & (x \leqslant 0) \\ \frac{1}{4}x^2 & (0 \leqslant x \leqslant 1) \\ \alpha x + \beta & (1 \leqslant x \leqslant 2) \\ \frac{1}{4}(5 - x)(x - 1) & (2 \leqslant x \leqslant 3) \\ 1 & (3 \leqslant x) \end{cases}$$

Find (i) the values of the constants α and β, (ii) the probability that $1{\cdot}5 \leqslant X \leqslant 2{\cdot}5$, (iii) the probability density function $f(x)$. Sketch the graph of $f(x)$, and hence, or otherwise, deduce the mean of X. Determine also the variance of X. (C)

16. A failure time, X, has density function $\lambda e^{-\lambda x}$ for $x \geqslant 0$, $\lambda > 0$. Find (a) the distribution function; (b) the mean and variance; (c) the median; (d) the time below which it is 95% certain that there will be no failure. (O)

17. Henri de Lade regularly travels from his home in the suburbs to his office in Paris. He always tries to catch the same train, the 08.05 from his local station. He walks to the station from his home in such a way that his arrival times form a normal distribution with mean 08.00 hours and standard deviation 6 minutes. (i) Assuming that his train always leaves on time, what is the probability that on any given day Henri misses his train? (ii) If Henri visits his office in this way 5 days each week and if his arrival times at the station each day are independent, what is the probability that he misses his train once and only once in a given week? (AEB 1980)

18. The score, S, gained by an expert rifleman with a single shot, is a random variable with the following probability distribution:

$$P(S = 8) = 0{\cdot}01, \quad P(S = 9) = 0{\cdot}29, \quad P(S = 10) = 0{\cdot}70.$$

(i) Use a normal approximation to determine the probability that the rifleman obtains six or more scores of 8 in a series of 900 independent shots. (ii) Find the expectation and variance of S. (iii) Use a normal approximation to determine the probability that the rifleman scores less than 96 with ten independent shots. (C)

19. The prices of houses on an estate agent's books are approximately normally distributed with a mean price of £18 000. Given that 90% of the houses are priced at less than £30 000, find the standard deviation of the house prices and the percentage of the houses with a price of less than £10 000. Find the mean price and the standard deviation of the prices when the price of each house is increased by (a) £2000, (b) 10%. (L)

20. The weights of grade A oranges are normally distributed with mean 200 g and standard deviation 12 g. Determine, correct to 2 significant figures, the probability that (i) a grade A orange weighs more than 190 g but less than 210 g, (ii) a sample of 4 grade A oranges weighs more than 820 g. The weights of grade B oranges are normally distributed with mean 175 g and standard deviation 9 g. Determine, correct to 2 significant figures, the probability that (iii) a grade B orange weighs less than a grade A orange, (iv) a sample of 8 grade B oranges weighs more than a sample of 7 grade A oranges. (C)

21. Mass-produced right circular cylindrical pipes have internal diameters that are normally distributed with a mean of 10 cm and a standard deviation of 0·4 cm. (i) Find the probability that a randomly chosen pipe will have an internal diameter greater than 10·3 cm. (ii) Find the expected number of pipes in a random sample of 100 pipes that have internal diameters in the range from 9·7 cm to 10·3 cm. (iii) Find the expected value of the internal cross-sectional area of a randomly chosen pipe; give your answer correct to three significant figures. (iv) Mass-produced pistons have diameters that are normally distributed with a mean of 9·9 cm and a standard deviation of 0·3 cm. Find the probability that a randomly selected piston will have a diameter less than the internal diameter of a randomly selected pipe. (W)

22. Two gauges are used to test the thickness of manufactured metal sheets. Over a long period it is found that 1·25% of the sheets will pass through the smaller gauge of 1·4 mm while 95·4% will pass through the larger gauge of 1·6 mm. Assuming that the distribution of sheet thickness is normal, find its mean and standard deviation. At the next stage of manufacture 2 sheets are clamped together to make a thick plate. Find the mean and the standard deviation of the thickness of these plates. If a plate will pass through a gauge of 3·2 mm but not through one of 2·8 mm, find the probability that both the sheets of which it is composed will pass through the 1·6 mm gauge but not through the 1·4 mm gauge. Find also the size of a gauge which will allow 99·5% of these plates to pass through. (O&C)

23. Next May, an ornithologist intends to trap one male cuckoo and one female cuckoo. The mass M of the male cuckoo may be regarded as being a normal random variable with mean 116 g and standard deviation 16 g. The mass F of the female cuckoo may be regarded as being independent of M and as being a normal random

variable with mean 106 g and standard deviation 12 g. Determine (i) the probability that the mass of the two birds together will be more than 230 g. (ii) the probability that the mass of the male will be more than the mass of the female. By considering $X = 9M - 16F$, or otherwise, determine the probability that the mass of the female will be less than nine-sixteenths of that of the male. Suppose that one of the two trapped birds escapes. Assuming that the remaining bird will be equally likely to be the male or the female, determine the probability that its mass will be more than 118 g. (C)

24. (a) The number of accidents notified in a factory per day over a period of 200 days gave rise to the following table:

No. of accidents	0	1	2	3	4	5
No. of days	127	54	14	3	1	1

(i) Calculate the mean number of accidents per day. (ii) Assuming that this situation can be represented by a suitable Poisson distribution, calculate the corresponding frequencies.

(b) Of items produced by a machine, approximately 3% are defective, and these occur at random. What is the probability that, in a sample of 144 items, there will be at least two which are defective? (SU)

25. Define a Poisson distribution giving the general probability formula; state the relationship between its mean and variance. The distribution of the number of vehicles observed passing under a motorway bridge in 100 successive intervals of 12 seconds, at a time when the traffic flow is 720 vehicles per hour, may be considered to be Poissonian. During how many of the intervals would one expect (i) no cars to have passed? (ii) 2 cars to have passed, (iii) more than 2 cars to have passed? Find the smallest value of N such that the probability of at most N cars in an interval is greater than 0·999. (AEB 1978)

26. Each play of a one-armed bandit machine has probability 0·04 of yielding a win. It may be assumed that this probability applies independently for each play of the machine. (a) Find the probability that in ten plays of the machine there will be (i) at least two wins, (ii) exactly two wins. (b) For a sequence of n plays of the machine write down an expression for the probability that there will be no win. Hence find the smallest number of plays of the machine for there to be a probability of at least 0·95 of winning at least once. (c) Use an appropriate method to find an approximate value for the probability that there will be six or fewer wins in 80 plays of the machine. (W)

27. Gnat larvae are distributed at random in pond water so that the number of larvae contained in a random sample of 10 cm^3 of pond water may be regarded as a random variable having a Poisson distribution with mean 0·2. Ten independent random samples, each of 10 cm^3, of pond water, are taken by a zoologist. Determine (correct to three significant figures) (i) the probability that none of the samples contain larvae, (ii) the probability that one sample contains a single larva and the remainder contain no larvae, (iii) the probability that one sample contains two or

more larvae and the remainder contain no larvae, (iv) the expectation of the total number of larvae contained in the ten samples, (v) the expectation of the number of samples containing no larvae. (C)

28. A prospector for gold examines exactly 800 pans of material every month. The contents of each pan may be assumed to be independent of the others with the probability of a pan containing gold being 0·005 for each pan. The prospector has a good month when 4 or more of the pans that he examines contain gold. Show that, to three significant figures, the probability that a randomly chosen month is a good month is 0·567. Determine the probability that (i) a randomly chosen period of four months contains more than two good months, (ii) a randomly chosen period of 24 months contains more than 12 good months. (C)

29. The number of organic particles suspended in a volume V cm^3 of a certain liquid follows a Poisson distribution with mean $0{\cdot}1V$. (a) Find the probabilities that a sample of 1 cm^3 of the liquid will contain (i) at least one organic particle, (ii) exactly one organic particle. (b) Use an appropriate approximate procedure to find the probability that a sample of 1000 cm^3 of the liquid will contain at least 90 organic particles. (c) The liquid is sold in vials, each vial containing 10 cm^3 of the liquid. The vials are dispatched for sale in boxes, each box containing 100 vials. Find the probability that a vial will contain at least one organic particle. Hence find the mean and the standard deviation of the number of vials per box of 100 vials that contain at least one organic particle. (W)

30. When a number is rounded off to its nearest integer value, the rounding off error X may be regarded as a random variable which is uniformly distributed over the interval $(-\frac{1}{2}, \frac{1}{2})$. Find the variance of X. If n numbers are rounded off to their nearest integer values and then summed, the total error in the sum is given by $Y = X_1 + X_2 + \ldots + X_n$, where $X_1, X_2, \ldots, X_n$ are the individual errors in the n numbers, which may be assumed to be independent and uniformly distributed over $(-\frac{1}{2}, \frac{1}{2})$. Write down the mean and the variance of Y in terms of n. Suppose n is large enough to justify approximating the distribution of Y by a normal distribution. Show that there is a probability in excess of 0·9 that Y will be numerically less than $\frac{1}{2}\sqrt{n}$. Find the largest integer value of n for which there is a probability of at least 0·6 that Y will be numerically less than 1·0. (W)

31. The joint probability function of two discrete random variables X and Y is given by $P(X = r, Y = s) = c|r + s|$ for $r = -1, 0, 1$ and $s = -2, -1, 0, 1, 2$,
$P(X = r, Y = s) = 0$, otherwise.
(a) Show that (i) $c = 0{\cdot}05$, (ii) $\mathrm{Var}(Y) = 2 + \mathrm{Var}(X)$.
(b) Evaluate $\mathrm{Var}(Y - X)$. (W)

32. A continuous random variable X is distributed with probability density function $f(x) = 3x^{-4}$, $x \geqslant 1$, $f(x) = 0$, otherwise. Find (i) the cumulative distribution function of X, (ii) the mean and the variance of X. The random variable Y is defined by $Y = X^{-1}$. By first finding the cumulative distribution function of Y, or otherwise, determine the probability density function of Y. Verify that $E(X)E(Y)$

and $E(XY)$ are not equal and explain why they have different values in this particular example. (W)

33. A continuous random variable X has probability density function

$$f(x) = 4x(1 - x^2), \qquad 0 \leqslant x \leqslant 1,$$
$$f(x) = 0, \qquad \text{otherwise.}$$

(i) Verify that $f(x)$ satisfies the requirements to be a probability density function. (ii) Find the mean value of X. (iii) Find the cumulative distribution function of X. (iv) If X denotes the area of a square of side Y find the probability density function of Y and use it to verify that $E(Y^2) = E(X)$. (W)

34. The probability density function of a continuous random variable X is given by

$$f(x) = \tfrac{2}{3}x, \qquad 1 \leqslant x \leqslant 2,$$
$$f(x) = 0, \qquad \text{otherwise.}$$

(a) Find (i) $E(X^2)$, (ii) $E(X^4)$, (iii) the cumulative distribution function of X.
(b) Suppose X is the length of the side of a square. (i) Find the mean and the variance of the area of the square. (ii) Show that the area of the square has a uniform distribution over a certain interval which should be specified. (W)

Formulae for reference

Series

Sum of A.P. $= \frac{1}{2}n\{2a + (n-1)d\}$, sum of G.P. $= \dfrac{a(1-r^n)}{1-r}$

For $|r| < 1$, sum to infinity of G.P. $= a/(1-r)$

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + b^n,$$

where n is a positive integer.

$$\sum_1^n r = \tfrac{1}{2}n(n+1); \quad \sum_1^n r^2 = \tfrac{1}{6}n(n+1)(2n+1); \quad \sum_1^n r^3 = \tfrac{1}{4}n^2(n+1)^2$$

Trigonometry

$$\cos^2\theta + \sin^2\theta = 1; \quad 1 + \tan^2\theta = \sec^2\theta; \quad \cot^2\theta + 1 = \operatorname{cosec}^2\theta.$$

$$\sin(A+B) = \sin A\cos B + \cos A\sin B \qquad \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$
$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$
$$\cos(A+B) = \cos A\cos B - \sin A\sin B \qquad \tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$
$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$

$$\text{Area of } \triangle ABC = \tfrac{1}{2}bc\sin A$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R, \quad a^2 = b^2 + c^2 - 2bc\cos A$$

Length of circular arc $= r\theta$, area of sector $= \frac{1}{2}r^2\theta$ (θ in radians)

Calculus

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}, \qquad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

y	$\sin x$	$\cos x$	$\tan x$	$\cot x$	$\sec x$	$\operatorname{cosec} x$
dy/dx	$\cos x$	$-\sin x$	$\sec^2 x$	$-\operatorname{cosec}^2 x$	$\sec x \tan x$	$-\operatorname{cosec} x \cot x$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int e^{kx}dx = \frac{1}{k}e^{kx} + c, \qquad \int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + c,$$

$$\int \frac{dx}{\sqrt{(a^2 - x^2)}} = \sin^{-1}\frac{x}{a} + c, \qquad \int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c.$$

Mechanics

Straight line motion with constant acceleration

$$v = u + at; \quad s = ut + \tfrac{1}{2}at^2; \quad s = \tfrac{1}{2}(u+v)t; \quad v^2 = u^2 + 2as.$$

Hooke's law: $T = \dfrac{\lambda}{l}x$; energy stored in string $= \dfrac{\lambda x^2}{2l}$.

Uniform body	Length, area or volume	Centre of mass
Triangular lamina, height h	$\frac{1}{2}h \times$ base	$\frac{1}{3}h$ from base
Circular arc, radius r, angle at centre 2α	$2r\alpha$	$\dfrac{r\sin\alpha}{\alpha}$ from centre
Sector of circle, radius r, angle at centre 2α	$r^2\alpha$	$\dfrac{2r\sin\alpha}{3\alpha}$ from centre
Solid cone or pyramid, height h	$\frac{1}{3}h \times$ base area	$\frac{1}{4}h$ from base
Solid hemisphere, radius r	$\frac{2}{3}\pi r^3$	$\frac{3}{8}r$ from centre
Hemispherical shell, radius r	$2\pi r^2$	$\frac{1}{2}r$ from centre

Probability

$$P(A \cup B) = P(A) + P(B) - P(A \cap B), \quad P(A \cap B) = P(A) \times P(B \mid A)$$

$$E(X) = \mu = \begin{cases} \sum x_r p(x_r) & X \text{ discrete} \\ \int_{-\infty}^{\infty} x f(x)\,dx & X \text{ continuous} \end{cases}$$

$$\mathrm{Var}(X) = \sigma^2 = \begin{cases} \sum (x_r - \mu)^2 p(x_r) = \sum x_r^2 p(x_r) - \mu^2, & X \text{ discrete} \\ \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2, & X \text{ continuous} \end{cases}$$

Binomial distribution B(n, p):

$$P(X = r) = \binom{n}{r} p^r (1-p)^{n-r} \quad (r = 0, 1, 2, \ldots, n)$$

$$E(X) = np, \quad \mathrm{Var}(X) = np(1-p)$$

Poisson distribution, parameter μ:

$$P(X = r) = \frac{\mu^r}{r!} e^{-\mu} \quad (r = 0, 1, 2, \ldots)$$

$$E(X) = \mu, \quad \mathrm{Var}(X) = \mu$$

Properties of expectation and variance:

$$E[g(X)] = \begin{cases} \sum g(x_r) p(x_r) & X \text{ discrete} \\ \int_{-\infty}^{\infty} g(x) f(x)\,dx & X \text{ continuous} \end{cases}$$

$$E(aX + b) = aE(X) + b, \qquad E(aX + bY) = aE(X) + bE(Y)$$

$$\mathrm{Var}(X) = E(X^2) - [E(X)]^2, \quad \mathrm{Var}(aX + b) = a^2 \mathrm{Var}(X)$$

For independent random variables $E(XY) = E(X)E(Y)$ and
$\mathrm{Var}(aX + bY) = a^2 \mathrm{Var}(X) + b^2 \mathrm{Var}(Y)$

$$\mathrm{Cov}(X, Y) = E[\{X - E(X)\}\{Y - E(Y)\}] = E(XY) - E(X)E(Y)$$

Statistics

Mean, $\bar{x} = \frac{1}{n}\sum x$

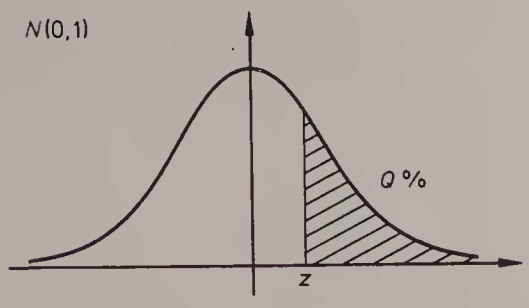
Variance, $\mathrm{var}(x) = \frac{1}{n}\sum (x - \bar{x})^2 = \frac{1}{n}\sum x^2 - \bar{x}^2$

Table 1 The normal distribution function $\Phi(z)$

z	·00	·01	·02	·03	·04	·05	·06	·07	·08	·09
0·0	·5000	·5040	·5080	·5120	·5160	·5199	·5239	·5279	·5319	·5359
01	·5398	·5438	·5478	·5517	·5557	·5596	·5636	·5675	·5714	·5753
0·2	·5793	·5832	·5871	·5910	·5948	·5987	·6026	·6064	·6103	·6141
0·3	·6179	·6217	·6255	·6293	·6331	·6368	·6406	·6443	·6480	·6517
0·4	·6554	·6591	·6628	·6664	·6700	·6736	·6772	·6808	·6844	·6879
0·5	·6915	·6950	·6985	·7019	·7054	·7088	·7123	·7157	·7190	·7224
0·6	·7257	·7291	·7324	·7357	·7389	·7422	·7454	·7486	·7517	·7549
0·7	·7580	·7611	·7642	·7673	·7704	·7734	·7764	·7794	·7823	·7852
0·8	·7881	·7910	·7939	·7967	·7995	·8023	·8051	·8078	·8106	·8133
0·9	·8159	·8186	·8212	·8238	·8264	·8289	·8315	·8340	·8365	·8389
1·0	·8413	·8438	·8461	·8485	·8508	·8531	·8554	·8577	·8599	·8621
1·1	·8643	·8665	·8686	·8708	·8729	·8749	·8770	·8790	·8810	·8830
1·2	·8849	·8869	·8888	·8907	·8925	·8944	·8962	·8980	·8997	·9015
1·3	·9032	·9049	·9066	·9082	·9099	·9115	·9131	·9147	·9162	·9177
1·4	·9192	·9207	·9222	·9236	·9251	·9265	·9279	·9292	·9306	·9319
1·5	·9332	·9345	·9357	·9370	·9382	·9394	·9406	·9418	·9429	·9441
1·6	·9452	·9463	·9474	·9484	·9495	·9505	·9515	·9525	·9535	·9545
1·7	·9554	·9564	·9573	·9582	·9591	·9599	·9608	·9616	·9625	·9633
1·8	·9641	·9649	·9656	·9664	·9671	·9678	·9686	·9693	·9699	·9706
1·9	·9713	·9719	·9726	·9732	·9738	·9744	·9750	·9756	·9761	·9767
2·0	·97725	·97778	·97831	·97882	·97932	·97982	·98030	·98077	·98124	·98169
2·1	·98214	·98257	·98300	·98341	·98382	·98422	·98461	·98500	·98537	·98574
2·2	·98610	·98645	·98679	·98713	·98745	·98778	·98809	·98840	·98870	·98899
2·3	·98928	·98956	·98983	·99010	·99036	·99061	·99086	·99111	·99134	·99158
2·4	·99180	·99202	·99224	·99245	·99266	·99286	·99305	·99324	·99343	·99361
2·5	·99379	·99396	·99413	·99430	·99446	·99461	·99477	·99492	·99506	·99520
2·6	·99534	·99547	·99560	·99573	·99585	·99598	·99609	·99621	·99632	·99643
2·7	·99653	·99664	·99674	·99683	·99693	·99702	·99711	·99720	·99728	·99736
2·8	·99744	·99752	·99760	·99767	·99774	·99781	·99788	·99795	·99801	·99807
2·9	·99813	·99819	·99825	·99831	·99836	·99841	·99846	·99851	·99856	·99861
3·0	·99865	·99869	·99874	·99878	·99882	·99886	·99889	·99893	·99896	·99900

z	3·1	3·2	3·3	3·4	3·5	3·6	3·7	3·8	3·9	4·0
Φ	·99903	·99931	·99952	·99966	·99977	·99984	·99989	·99993	·99995	·99997

Table 2 Percentage points of the normal distribution



The shaded area represents $Q\%$ of the total distribution.

$$Q(z) = 100P(Z \geqslant z) = 100\{1 - \Phi(z)\}$$

Q	z	Q	z	Q	z
50	0·000	1·5	2·170	0·2	2·878
40	0·253	1·0	2·326	0·15	2·968
30	0·524	0·9	2·366	0·1	3·090
20	0·842	0·8	2·409	0·05	3·291
10	1·282	0·7	2·457	0·01	3·719
5	1·645	0·6	2·512	0·005	3·891
4	1·751	0·5	2·576	0·001	4·265
3	1·881	0·4	2·652	0·0005	4·417
2·5	1·960	0·3	2·748	0·0001	4·753
2	2·054	0·25	2·807	0·00005	4·892

Answers

Exercise 1.1 (p. 4)

1. (a) **PS**, (b) **AE**, (c) **AB**, (d) **0**, (e) **MR**, (f) **PS**.
2. (a) false, (b) true, (c) true, (d) true, (e) false, (f) true.
3. (a) true, (b) false, (c) true, (d) true, (e) false, (f) false.
4. (a) 3 km E, (b) 10 km N, (c) 1 km N 30° E, (d) 1 km S.
5. (a) $2\sqrt{2}$ km NE, (b) $2\sqrt{2}$ km NE, (c) $2\sqrt{2}$ km SE, (d) $2\sqrt{2}$ km NW.
6. (a) 1 km S 30° E, (b) $\sqrt{3}$ km S, (c) $\sqrt{3}$ km S 60° E, (d) $\sqrt{3}$ km N 60° E.
7. $\mathbf{p}+\mathbf{q}$, $\mathbf{q}-\mathbf{p}$, $\frac{1}{2}(\mathbf{p}+\mathbf{q})$, $\frac{1}{2}(\mathbf{q}-\mathbf{p})$.
8. $-\mathbf{a}$, $\mathbf{a}-\mathbf{b}$, $2\mathbf{b}$, $2\mathbf{b}-\mathbf{a}$.
9. $2\mathbf{a}+\mathbf{b}$, $\mathbf{a}+\mathbf{b}$.

Exercise 1.2 (pp. 7–8)

1. (a) $2\mathbf{b}$, (b) $-3\mathbf{a}$, (c) $2\mathbf{b}-6\mathbf{a}$, (d) $\mathbf{b}-\mathbf{a}$, (e) $\mathbf{a}$, (f) $2\mathbf{b}-\mathbf{a}$.
2. (a) $\mathbf{z}$, (b) $\mathbf{x}$, (c) $-\mathbf{x}$, (d) $\mathbf{z}$, (e) $-\mathbf{x}+\mathbf{z}$, (f) $\mathbf{y}-\mathbf{z}$.
3. (a) $-\mathbf{x}+\mathbf{y}$, (b) $\mathbf{x}+\mathbf{z}$, (c) $-\mathbf{x}+\mathbf{y}-\mathbf{z}$, (d) $2\mathbf{x}-\mathbf{y}$, (e) $2\mathbf{x}-\mathbf{y}+2\mathbf{z}$, (f) $-\mathbf{y}+2\mathbf{z}$.
4. (a) $5\mathbf{i}+3\mathbf{j}$, (b) $-\mathbf{i}+6\mathbf{j}$, (c) $-4\mathbf{i}+2\mathbf{j}$, (d) $2\mathbf{i}-\mathbf{j}$, (e) $3\mathbf{i}+4\mathbf{j}$, (f) $\mathbf{i}+5\mathbf{j}$.
5. (a) $-4\mathbf{i}+5\mathbf{j}+2\mathbf{k}$, (b) $3\mathbf{j}+2\mathbf{k}$, (c) $\mathbf{i}-\mathbf{j}+\mathbf{k}$, (d) $\mathbf{i}+2\mathbf{j}+3\mathbf{k}$.
6. (a) 13, (b) $\sqrt{5}$, (c) 10, (d) 3, (e) 7, (f) $5\sqrt{3}$.
7. (a) $\frac{8}{9}\mathbf{i}-\frac{4}{9}\mathbf{j}-\frac{1}{9}\mathbf{k}$, (b) $\frac{8}{9}\mathbf{i}-\frac{4}{9}\mathbf{j}-\frac{1}{9}\mathbf{k}$, (c) $-\frac{8}{9}\mathbf{i}+\frac{4}{9}\mathbf{j}+\frac{1}{9}\mathbf{k}$.
8. (a) 5, (b) 3, (c) 6.
9. (a) $\sqrt{29}$, (b) $\sqrt{29}$, (c) 5.
10. (a) $\frac{1}{\sqrt{2}}\mathbf{i}-\frac{1}{\sqrt{2}}\mathbf{j}$, (b) $-\frac{7}{9}\mathbf{i}+\frac{4}{9}\mathbf{j}+\frac{4}{9}\mathbf{k}$, (c) $\mathbf{i}$.
11. $\mathbf{b}-\mathbf{a}$, $\mathbf{b}-2\mathbf{a}$; $2\mathbf{b}-2\mathbf{a}$, $2\mathbf{b}-3\mathbf{a}$, $\mathbf{b}-2\mathbf{a}$.
12. $\mathbf{a}-\mathbf{b}$, $\mathbf{c}-\mathbf{b}$, $\mathbf{a}-2\mathbf{b}+\mathbf{c}$; $2\mathbf{a}-3\mathbf{b}+2\mathbf{c}$.
13. (a) 6, $-\frac{1}{3}\mathbf{i}+\frac{2}{3}\mathbf{j}+\frac{2}{3}\mathbf{k}$, $2\mathbf{i}-3\mathbf{j}+\mathbf{k}$,
    (b) 22, $\frac{6}{11}\mathbf{i}-\frac{9}{11}\mathbf{j}-\frac{2}{11}\mathbf{k}$, $-\mathbf{i}+4\mathbf{j}-2\mathbf{k}$,

(c) $\sqrt{10}, \frac{3}{\sqrt{10}}\mathbf{j} - \frac{1}{\sqrt{10}}\mathbf{k}, \mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{5}{2}\mathbf{k}$,

(d) $3\sqrt{6}, -\frac{2}{\sqrt{6}}\mathbf{i} - \frac{1}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}, 2\mathbf{i} + \frac{5}{2}\mathbf{j} - \frac{1}{2}\mathbf{k}$.

14. 50.

Exercise 1.3 (pp. 8–9)

1. $\mathbf{a}+\mathbf{b}+\mathbf{c}, \mathbf{b}+\mathbf{c}-\mathbf{a}, \mathbf{a}+\mathbf{c}-\mathbf{b}, \mathbf{a}+\mathbf{b}-\mathbf{c}$.
2. (a) $\mathbf{b}-\mathbf{a}$, (b) $3\mathbf{b}-2\mathbf{a}$, (c) $3\mathbf{b}-\mathbf{c}$, (d) $\mathbf{a}-\frac{3}{2}\mathbf{b}$, (e) $-\mathbf{a}-\frac{1}{2}\mathbf{b}$,
   (f) $\mathbf{a}+\frac{3}{2}\mathbf{b}$.
3. (a) $2\sqrt{6}, 5\sqrt{2}, 20$, (b) $\frac{2}{3}\mathbf{i}+\frac{1}{3}\mathbf{j}-\frac{2}{3}\mathbf{k}$.
4. (a) $6\sqrt{3}$, (b) $\frac{1}{9}\sqrt{3}\mathbf{i} - \frac{5}{9}\sqrt{3}\mathbf{j} + \frac{1}{9}\sqrt{3}\mathbf{k}$, (c) $-2\mathbf{j}+3\mathbf{k}$.
5. 1, 4.
7. (a) $-\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$, (b) $\frac{3}{2}\sqrt{29}$.
8. $\mathbf{a}+k(2\mathbf{b}-\mathbf{a})$; 3, $6\mathbf{b}-2\mathbf{a}$.
9. $\frac{1}{3}\mathbf{a}+\frac{1}{3}\mathbf{b}$.
10. (4, −7, 8): 54 sq. units.

Exercise 2.3 (pp. 17–18)

1. (a) 13, 22·6°, (b) 8·06, 60·3°, (c) 9·85, 75·3°,
   (d) 14·5, 33°, (e) 14·1, 37·3°, (f) 8·86, 17·9°.
2. 133·5°.
3. 14·3 N at 81·4° to **G**.
4. 3·81 N, 15·7 N.
5. 9·43 N, N 12° W.
6. 13·0 N, S $37\frac{1}{2}$° E.
7. 16·8 N, N 57·6° E.
8. (a) 5, N 53·1° E, (b) 7, E, (c) 6·71, S 63·4° E, (d) 7·07, SW.
9. (a) 6, (b) 17, (c) 9·95.
10. 15·3 N.
11. $\sqrt{5}$.
12. $4\sqrt{10}, 6\sqrt{5}$.

Exercise 2.4 (pp. 21–22)

1. (a) 15·3 N, 12·9 N, (b) 1·46 N, 6·85 N, (c) 7·05 N, 9·71 N.
2. (a) $-W, N\cos\alpha, T\sin(\alpha+\beta)$, (b) $-W\sin\alpha, 0, T\cos\beta$,
   (c) $W\cos\alpha, -N, -T\sin\beta$.
3. 11·5 N, N 15·5° W.
4. 29·3 N, N 15·8° W.
5. 344·7°.
6. 15·6, 76·6°.
7. 15·2 N, 66·8°.
8. 4·03 N.
9. (a) 48 N, 40 N, 46·2 N, (b) 14 N, 30 N, 19·2 N, (c) 134·2 N,
   (d) 20·6 N, 157·4°.

Exercise 2.5 (pp. 27–28)

1. 31·4 N, S 84·6° W.
2. 12·9, 6·39.
3. $5\sqrt{3}$.
4. 6, 8.
5. (a) $20\sqrt{3}$, (b) 20, (c) $10\sqrt{3}$.
6. $3\frac{1}{2}$ N, $12\frac{1}{2}$ N.
7. 1·87 N, 4·41 N.
8. 37·1 N, 19·7 N.
9. (a) $5W$, $5\sqrt{3}W$, (b) $10W/\sqrt{3}$, $10W/\sqrt{3}$.
10. 4·14 N, 43°.
11. $8W$. $6W$.
12. $10\sqrt{3}$ N, $20\sqrt{3}$ N at 30° to vertical.
13. (a) $\sqrt{5}W$, $2\sqrt{2}W$, (b) 63·4°, 45°.
14. $2\sqrt{2}W$, $\sqrt{5}W$, 26·6°.
15. (a) $5{\cdot}80W$, $4{\cdot}24W$, $5{\cdot}02W$, (b) $7{\cdot}07W$, $5{\cdot}39W$, $5{\cdot}39W$.

Exercise 2.6 (pp. 28–30)

1. 69·6°.
2. $\sqrt{10}$, 51·3°.
3. $0 \leqslant \theta \leqslant 36{\cdot}9°$.
4. $\sqrt{65}$, N 60·3° E.
6. $2\sqrt{5}$ N, $4\sqrt{2}$ N.
7. $5\sqrt{3}$ N.
8. 154·9°.
9. 6·32 N, S 71·6° W.
10. 15·4 N, S 37·6° W.
11. $2\sqrt{5}W$.
12. 12·9 N, 45·2 N.
13. 46·8°, 11·5 N.
15. (a) 25/8 N, (b) 2·62 N.
16. $W/\sqrt{3}$.
19. 16·6 N, 6·50 m.

Exercise 3.1 (pp. 34–35)

1. 13, $25\mathbf{i}+6\mathbf{j}$.
2. $-2\mathbf{i}+4\mathbf{j}$.
3. $24\mathbf{i}+3\mathbf{j}+12\mathbf{k}$; $8\mathbf{i}+\mathbf{j}+4\mathbf{k}$, 11.
4. 85 m.
5. $5\mathbf{i}+8\mathbf{j}$; 7.
6. $-3\mathbf{i}+6\mathbf{j}$.
7. (a) 13, N 67·4° E, (b) $2\sqrt{10}$, N 71·6° E, (c) $4\sqrt{5}$, S 63·4° E, (d) $\sqrt{97}$, S 24° W.
8. (a) 13·6 m s^{-1}, N 45·2° E, (b) 59·7 km/h, S 74·4° E, (c) 16·4 m s^{-1}, N 6·6° W, (d) 98·0 km/h, S 48·8° E.
9. (a) 4·18 m s^{-1}, (b) 30 m.
10. 14·6 km/h, N 87° E.

Exercise 3.2 (pp. 37–38)

1. 9·43 km/h from S 58° W.
2. 41·6 km/h, S 70·1° E.
3. 15, 126·9°.
4. $\mathbf{i}-4\mathbf{j}$, $4\mathbf{i}+5\mathbf{j}$.
5. 33·1 km/h, S 65° W.
6. $20\sqrt{3}$ km/h, N 60° W.
7. 10·7 km/h from N 37·5° W.
8. 90 km/h.
9. N 73·7° W; 2 min.
10. 2·25 m s^{-1}; at 63·3° to the bank.
11. N 47·3° W; 1 h 30 min.
12. 76·8 km/h; S 64·8° W.

Exercise 3.3 (pp. 41–42)

1. 151·8°, 1 h 47 min.
2. $4\sqrt{3}$ km/h, N 11·4° E, 6 km/h.
3. 28·8 km/h on S 33·7° W; 8·32 km.
4. 5·57 m s^{-1} on 321·1°, 115 s.
5. 25 km/h on S 36·9° E; 60 m.
6. 576 km/h, N 51·3° E; 31·2 km; 10.04, N 38·7° W.
7. $500\sqrt{2}$ km/h, SE; 25·9 km.
8. (ii) $3\mathbf{i} + 4\mathbf{j}$.
9. 3, 500 s; $1000\sqrt{10}$ m.
10. N 73·1° E; 2·9 km.

Exercise 3.4 (pp. 43–45)

1. 203°; 157°; 10·4 knots E.
2. $6\sqrt{3}$ knots, 120°.
3. (i) due north, 100 s, 583 m, (ii) N 36·9° W, 125 s, 500 m.
4. 95·2 s; 92·3 s; 125 s.
5. 032·8°, 237·2°; 64·4 min.
6. 65 km h^{-1} from N 32·6° E; N 5·8° E, S 54·2° W.
7. 97 min, 69 min, 137 min.
8. 14.36, N 45·2° W; 13.42.
9. N; 26·8 min.
10. (a) 3 m s^{-1}, (b) $\mathbf{i} + 3\mathbf{j}$; $\sqrt{181}$ m s^{-1}.
11. (i) 50 km h^{-1} towards N 73·7° W, (ii) 112 m, (iii) 31·36 m, 107·52 m.
12. 42 km/h, S 8·2° E; 13.06$\frac{1}{4}$, 8·66 km; 13.25.
13. N 41·4° E, 45 min; N 48·2° E, 7·45 km.
14. 18·0 km/h; 13.12; 4·16 km/h.

Exercise 4.1 (pp. 49–50)

1. (a) $9t^2 - 5$, $18t$, (b) $4t + 3$, 4.
2. (a) $6t^2 + 7$, $2t^3 + 7t$, (b) $20 - 10t$, $5 + 20t - 5t^2$,
   (c) $8t^3 - 12t^2 + 4$, $2t^4 - 4t^3 + 4t - 2$.
3. 0, 3; 4.
4. 24; 32 m.
5. 36 m, 9 m s^{-1}.
6. 1 s, 9 s; 16 m s^{-1}.
7. 29, 52 m.
8. -12 m s^{-1}, 16 m; 48 m.
9. (a) $2\frac{1}{2}$, (b) 5 m.
10. (a) -15 m s^{-1}, -4 m s^{-2}, (b) 0 s, 5 s, (c) 72 m.
11. (a) $9t - 3t^2$, $\frac{9}{2}t^2 - t^3$, (b) 27 m.
12. (a) 0 s, 6 s, (b) 32 m, (c) 64 m, (d) 36 m s^{-1}.
13. (a) 1 s, 4 s, (b) 10 m, (c) 18$\frac{1}{2}$ m, (d) 7 m s^{-1}.
14. (a) 0 s, 2$\frac{1}{4}$ s, 6 s, (b) 60$\frac{3}{4}$ m, (c) 171$\frac{1}{2}$ m, (d) 90 m s^{-1}.
15. 5$\frac{1}{3}$, 10, 14$\frac{2}{3}$; 6 m s^{-1}.
16. (a) 27 m, (b) 16 m s^{-1}, (c) 12 m s^{-2}.

Exercise 4.2 (pp. 54–56)

1. 11, 24.
2. 12, 2.
3. 16, −3.
4. −7, −4.
5. $\frac{1}{2}$, 10.
6. 0, 24.
7. 10, 10 or 15, −10.
8. 1, 60.
9. $-2\frac{1}{2}$, 2.
10. 0, $1\frac{1}{2}$.
11. $\frac{1}{4}$ km.
12. 1/48.
13. $31\frac{1}{4}$ m, 5 s.
14. 14 m s^{-1}, 8 m.
15. 5 m s^{-1}, 10 s.
16. 34 m, 20 m s^{-1}.
17. $2\frac{1}{2}$ m s^{-2}, 1 s.
18. 40 m s^{-1}, 80 m.
19. 1750 m; 48, 100, 32, 45.
20. 30, 28·8 km.
21. 12 min.
22. 2, 10·05.
23. 20.
24. 1 min, 900 m.
25. 3 s, 15 m.
26. 20 m.

Exercise 4.3 (pp. 60–62)

1. $1\frac{1}{4}$ s; $2\frac{1}{2}$ s, 53·0 m.
2. 60 m; 35·8 m s^{-1} at 33·2° to horizontal.
3. 50 m; $y = 2x - x^2/50$.
4. 5, 2·5; 30·8 at 54·2° to the horizontal.
5. 20 m s^{-1}, 19·6 m s^{-1}; 44·1 m.
6. 11·2 m s^{-1}; 3·2 m.
8. 4·5 km; 26·6°
9. $y = \frac{5}{3}x - \frac{49x^2}{90u^2}$; 2·1, 13·5 m.
10. 14, 36·9°.
11. 28; $15° < \alpha < 75°$.
12. $6u^2/g$; $2u/g$, $4u/g$.
13. $u/(g \sin \alpha)$.
14. $2V \sin \theta(U + V \cos \theta)/g$.
15. $12V^2/13g$; $2V/(g\sqrt{13})$.
16. $\sqrt{(V^2 - 2gh)}$, $2\sqrt{(V^2 \sin^2 \theta - 2gh)}/g$; 60°.
18. 63·4°, 26·6°; 4 : 5.
19. $81\frac{2}{3}$; 69·3 m s^{-1}.
20. $2u_2/g$.

Exercise 4.4 (p. 64)

1. (a) $3\mathbf{i} + 6t^2\mathbf{k}$, $12t\mathbf{k}$, (b) $2t\mathbf{i} + 4t^3\mathbf{j} - 6\mathbf{k}$, $2\mathbf{i} + \mathbf{12}t^2\mathbf{j}$.
2. −3 m s^{-1}, 13 m s^{-1}.
3. $3(t-2)\mathbf{i} + 2(t^2-1)\mathbf{j} + t(1-t^2)\mathbf{k}$.
4. $(3t^2+1)\mathbf{i} + \mathbf{j} - 4t\mathbf{k}$, $(t^3+t)\mathbf{i} + (t+3)\mathbf{j} - 2t^2\mathbf{k}$.
5. $3\mathbf{i} + (2 - 10t)\mathbf{j}$, $3t\mathbf{i} + (2t - 5t^2)\mathbf{j}$.
6. $31\frac{1}{4}$ m, 200 m, 5 s.
7. $10\mathbf{i} + (10\sqrt{3} - 10t)\mathbf{j}$, $10t\mathbf{i} + (10\sqrt{3}t - 5t^2)\mathbf{j}$.
8. 48 m, 24 m; 32·0 m s^{-1}.

Exercise 4.5 (pp. 65–67)

1. $25\frac{11}{27}$ m, 20 m s^{-1}; $t < 2\frac{1}{6}$.
2. $343\frac{3}{4}$ m; 5 s; 49 m s^{-2}; 108 m s^{-1}.

3. (a) 7, (b) $2t(35+3t-2t^2)$, (c) $0 < t < 5$.
6. (i) $b+\frac{1}{2}ft^2-ut$, (ii) $b+(V-u)t-V^2/2f$;
   $b-u^2/2f$; car overtakes bus after time $(2fb-V^2)/2f(u-V)$.
7. (a) U/g, (b) $2U/g$.
8. 18·3 m, 61·6 m, 3·83 s.
9. (i) 40 m, (ii) 10 m, (iii) 20 m s^{-1} at 45° to horizontal,
   (iv) $2\sqrt{2}$ s; $(20+5\sqrt{2}, 8\frac{3}{4})$, $(20-5\sqrt{2}, 8\frac{3}{4})$.
10. 40, 45°; 15°.
11. $D^2\tan^2\theta-20D\tan\theta+20h+D^2=0$.
15. $\frac{3}{5}\mathbf{i}-\frac{4}{5}\mathbf{k}$; 2 s, $2\mathbf{j}+16\mathbf{k}$.
16. 5, 25.

Exercise 5.1 (pp. 71–72)

1. 48 N.
2. 0·2 m s^{-2}.
3. 12 kg.
4. 40 N.
5. $2\mathbf{i}-5\mathbf{j}$.
6. $12\mathbf{i}+4\mathbf{j}$.
7. 6·4 m s^{-2} at 17·4° to the 9 N force.
8. 84·9°.
9. 6·25 m s^{-2}, S 53·1° E.
10. 2·05 m s^{-2}, N 61·6° W.
11. 1·8 s.
12. 15 N.
13. 10 kg.
14. 10 m s^{-1}.
15. 8, 540 m.
16. 20 m s^{-1}; $6\frac{2}{3}$, 10.

Exercise 5.2 (pp. 74–75)

1. (a) 98 N, (b) 14·7 N, (c) 2·45 N.
2. 6·2 m s^{-2}.
3. 62·4 N.
4. (a) 440 N, (b) 392 N, (c) 360 N.
5. (a) decelerating at 1·2 m s^{-2}, (b) accelerating at 0·8 m s^{-2},
   (c) moving with uniform speed.
6. 33·1 N, 280 N.
7. 1·71 m s^{-2}, 87 N.
8. 5 N, 11·5°.
9. 8·49 m s^{-2}, 24·5 N.
10. 2·44 s.
11. (a) 8·28 N, (b) 14·3 N.
12. 2·12 m s^{-2}, 55·4 N.

Exercise 5.3 (pp. 79–81)

1. (a) 3 m s^{-2}, (b) 350 N.
2. (a) 240 N, 80 N, (b) 2640 N, 880 N.
3. 6000 N, 1200 N, 600 N; 0·25 m s^{-2}, 7200 N, 3600 N.
4. 4·2 m s^{-2}, 28 N.
5. 4·9 m s^{-2}, 44·1 N.
6. $\frac{M-m}{M+m}g, \frac{2Mm}{M+m}g$.
7. $24mg/5$.
8. 2·45 m s^{-2}, 8·82 N.
9. 0·392 m; $mg/\sqrt{2}$ N $= 6{\cdot}93m$ N at 45° to vertical.
10. 3·92 m s^{-2}, 17·64 N; 30·6 N at 30° to vertical.

11. $m \sin \alpha > M$.
12. 0·5 s.
13. 3·75 m.
14. 0·98 m s^{-2}, 34·3 N; 1·4 m s^{-1}; $1\frac{1}{6}$ m.

Exercise 5.4 (pp. 83–84)

1. 7·84 m s^{-2}, upwards; 17·64 N.
2. 2·8 m s^{-2}, downwards; 28 N.
3. 6·2 m s^{-2} upwards, 5·8 m s^{-2} downwards; 32 N, 16 N.
4. 1·96 m s^{-2} down, 5·88 m s^{-2} up; 31·4 N, 15·7 N.
5. 1·51 m s^{-2} down, 2·26 m s^{-2} down; 41·5 N, 15·1 N.
6. 1·89 m s^{-2}.
7. $49 \sin \alpha/(4+\sin^2 \alpha)$ m s^{-1}.
8. 0·8 m s^{-2} vertically, 3·2 m s^{-2} down plane.

Exercise 5.5 (pp. 84–87)

1. 3·70 kg.
2. 24·1, 242·5°.
3. 1250, 40 s.
4. $10\mathbf{i}+20\mathbf{j}$, $92\mathbf{i}+10\mathbf{j}$.
5. 9·16 N.
6. (a) 791 N, (b) 651 N.
7. 2 m s^{-1}; 1980 N, 1960 N, 1920 N.
8. (a) 784 N, 0°, (b) 809 N, 14·3°.
9. $\sqrt{(2gx \sin \alpha)}$.
10. 12·5 m.
11. (a) 2, (b) 56·4°
12. 1071 N, 315 N.
13. 3·15 m.
14. (a) $2\frac{1}{3}$ m s^{-1}, (b) $1\frac{13}{15}$ m s^{-1}.
15. 1·70 m s^{-2}, 83·4 N at 25° to the vertical.
16. 47·04 N, 23·52 N; 2·06 m.
17. 30°; $mg\sqrt{3}$ at 30° to vertical; $\frac{3}{2}mg$, $\frac{1}{4}g$, $\frac{3\sqrt{3}}{2}mg$ at 30° to vertical.
18. $\frac{2}{3}mg$.
19. 48 N, 24 N.
20. 525 N, 11·5°.

Exercise 6.1 (p. 90)

1. (a) 10 N, (b) 0, (c) 5 N.
2. (a) 5 N down plane, (b) 0, (c) 5 N up plane.
3. (a) 3, (b) $12\sqrt{2}/5 = 3{\cdot}39$, (c) $4\sqrt{2} = 5{\cdot}66$.
4. $\tan 25° = 0{\cdot}466$.
5. (a) 3·98, (b) 2·45.
6. 0·733 m s^{-2}.
7. 5·75 m s^{-2}.
8. 3·20 m s^{-2}.
9. 0·147.
10. 1·34 m s^{-2}.

Exercise 6.2 (pp. 93–94)

1. 9·8 N, $\frac{1}{4}$.
2. 1·5 s.
3. (a) 6·21 m s^{-2}, (b) 2·79 m s^{-2}.
4. 7·38.

5. (a) $\tan\alpha$, (b) $(g\sin\alpha - a)/g\cos\alpha$.
6. 23·04 m s^{-2}, 61·0 N.
7. 36·3 N, 0·48.
8. $\mu < 1$, 5/13.
9. 4·2 m s^{-2}, 22·4 N.
10. $\frac{2}{3}$; 34·6 N at 45° to horizontal.
11. 1·94 m s^{-2} up the plane, 39·3 N.
12. (a) 1·86 m s^{-2} up the plane, (b) 1·25 m s^{-2} down the plane.

Exercise 6.3 (pp. 95–96)

1. 14°.
2. (a) 12·5 N, (b) 5·47 N.
3. (a) $\mu \geqslant \frac{1}{3}$, (b) $\mu \geqslant \frac{1}{5}\sqrt{3}$, (c) $\mu \geqslant \frac{1}{7}\sqrt{3}$.
5. $\frac{1}{6}\sqrt{3}$.
6. $2{\cdot}87 \leqslant P \leqslant 4{\cdot}79$.
7. (a) 16, (b) $24\frac{8}{13}$.
8. $M \geqslant 3m$.

Exercise 6.4 (p. 98)

1. (a) 6, (b) 5·66, (c) 9·74.
2. 12·5 N, 36·9°.
3. 1·94 N at 14° to horizontal.
4. $W\sin\lambda$ at λ to horizontal.
5. (a) 1·20 N at 18·4° to plane, i.e. at 11·6° above horizontal,
   (b) 4·49 N at 18·4° to plane, i.e. at 48·4° above horizontal.
6. (a) $W\sin(\lambda - \alpha)$ at λ to plane. (b) $W\sin(\lambda + \alpha)$ at λ to plane.

Exercise 6.5 (pp. 101–102)

1. 5 N.
2. 9 N.
3. 2·4 N.
4. 1·2 m.
5. 26·6°, $a(3 + \sqrt{5})$.
6. 6·2 m s^{-2}.
7. 1·8 m.
8. $2{\cdot}5W/8$.
9. 4·8 N in PA, 2 N in PB; 0·9 m.
10. 20.
11. 18 N; 4·5 cm, 7·5 cm, 12 cm.
12. 1·25 m, 1·75 m.

Exercise 6.6 (pp. 102–106)

1. (a) 140 N, (b) 135 N.
2. $\frac{1}{2}W$.
3. (a) 15·7, (b) 7·84; 3·92 m s^{-2}.
4. 65, $\frac{1}{2}$.
5. (i) $\dfrac{\mu W}{1+2\mu^2}$, (ii) $\dfrac{3\mu W}{\sqrt{(1+4\mu^2)}}$.
8. 99·0, 33·0 N.
9. (a) 4·5 m s^{-2}; 42 N, 24 N, (b) 2 m s^{-2}; 42 N, 24 N.

11. (a) $\mu < \frac{1}{8}$, (b) $\mu < \frac{1}{8}$. 12. (ii) $\frac{1}{8}g(2-\sqrt{3}), \frac{1}{8}Mg(6+\sqrt{3})$.
14. 14°, $2{\cdot}60\,\mathrm{m\,s^{-2}}$.
15. (a) 0·09 N at 1·3° to horizontal,
    (b) 1·45 N at 21·3° to horizontal.
16. $\lambda(2\sec\theta - 1)$, $W + \lambda(2 - \cos\theta)$, $\lambda(2\tan\theta - \sin\theta)$.
17. (i) $5mg$, $4\sqrt{2}mg$, (ii) 53·1°, 45°.

Exercise 7.1 (pp. 109–110)

1. 0, 0, Ps, $-Fs$.
2. $-Wx$, Px.
3. $Ws\sin\alpha$, 0; Fs.
4. Ts, 0; Fs, $Ws\sin\alpha$.
5. 105 J, 15 m.
6. −9000 J, 450 N.
7. 45 kN.
8. 770 N.
9. 62·5 J.
10. 300 J; 5/8.
11. 200 J; 45.
12. 270 J, 180 J; $6\,\mathrm{m\,s^{-1}}$.

Exercise 7.2 (pp. 113–114)

1. $10{\cdot}5\,\mathrm{m\,s^{-1}}$.
2. (a) 21·6 m above A, (b) 40 m below A.
3. $v^2/2g$.
4. 4·05 m.
5. $5\,\mathrm{m\,s^{-1}}$.
6. $14\,\mathrm{m\,s^{-1}}$.
7. $2{\cdot}8\,\mathrm{m\,s^{-1}}$.
8. 214 J.
9. 147 J; 7 m.
10. $(M+m)v^2/2(M-m)g$.
11. $1{\cdot}4\,\mathrm{m\,s^{-1}}$.
12. 10 m.
13. 8 m.
14. (a) $3{\cdot}5\,\mathrm{m\,s^{-1}}$, (b) 2·6 m.
15. 19·6 J; $\frac{1}{4}$.

Exercise 7.3 (pp. 117–118)

1. (a) 0·2 J, (b) 5 J.
2. (a) 0·25 J, (b) 9 J.
3. 2·25 J.
4. 15 N.
5. 7·5 J.
6. $12k$, $3ka$.
7. 0·56 m.
8. $2\,\mathrm{m\,s^{-1}}$.
9. $\frac{1}{2}\sqrt{(ag)}$.
10. 4 m.
11. $7\,\mathrm{m\,s^{-1}}$.
12. 60 cm; $1{\cdot}4\,\mathrm{m\,s^{-1}}$.
13. 2·5 N.
14. 1·2 m; (a) $\frac{7}{5}\sqrt{3}$, (b) 0·7 m.

Exercise 7.4 (pp. 121–122)

1. 392 W.
2. 40 s.
3. 400 W.
4. 500 kg.
5. 28·8 kW.
6. 6·8 kW.
7. 335 W.
8. 72 kN.
9. 1·2 kW; $0{\cdot}1\,\mathrm{m\,s^{-2}}$.

10. (a) 12 kW, (b) 14·7 kW, (c) 3 kW.
11. 21 kW.
12. 30 m s^{-1}.
13. (a) $33\frac{1}{3}$ kW, (b) 52 kW, (c) $14\frac{2}{3}$ kW.
14. $6\frac{5}{6}$ kN, $13\frac{1}{3}$ m s^{-1}.
15. (a) 96 km/h, (b) 36 km/h; 1/6 m/s^2.
16. 0·98 m s^{-2}.
17. 2, 1·4 m/s^2.
18. 20 kW; (i) 6·93 kW, (ii) 0·201 m s^{-2}.

Exercise 7.5 (pp. 122–127)

1. 1/8.
2. 30/49.
3. 10 N.
4. 53·1°.
5. 68·6 J, 7/12.
6. 2·4 m.
7. 2·8 m s^{-1}.
8. 7 m s^{-1}, 2·4 m.
9. 3, 2·57 m s^{-1}.
10. $\frac{3}{2}\sqrt{(ga)}$, $2a$.
11. 30 mg/19.
13. (i) $2\sqrt{(2gb)}$, (ii) $\sqrt{(10gb)}$, (iii) 26·6°, (iv) $4b$; $-\frac{1}{2}c$.
14. (i) $2b-l$, (ii) b^2/l.
15. $4{\cdot}5\times10^4$ N, 450 kW; 24 m s^{-1}.
16. 1800 N, $804\frac{1}{6}$ N.
17. 320 kW.
18. 113 kg, 22·3 kW.
19. (i) $19\frac{7}{11}$ km h^{-1}, (ii) 0·53 m s^{-2}; 60.
20. 500 N, 20 m/s.
21. (a) $(1000H-MVg\sin\alpha)/MV$, (b) $(1000H+MVg\sin\alpha)/MV$; $1000H/3MVg$, $3V/4$.
22. 400 kW, 15 m/s, 20 000 N.
23. 0·395 m s^{-2}.
24. (i) 0·25 m/s, (ii) 22·5 m/s; $233\frac{1}{3}$ m, $15\frac{5}{9}$ s.
25. 1200 N; 1/40; 10·5 kW.
26. $R+mg\sin\alpha+2mU^2\sin\alpha/h$, $2U(R+mg\sin\alpha+2mU^2\sin\alpha/h)$, $mgh+2mU^2+Rh\operatorname{cosec}\alpha$.

Exercise 8.1 (pp. 130–131)

1. $\frac{3\pi}{2}$ rad s^{-1}, 0·42 m s^{-1}.
2. 4 rad s^{-1}.
3. $7{\cdot}3\times10^{-5}$ rad s^{-1}, 470 m s^{-1}.
4. 2×10^{-7} rad s^{-1}, 30 km/s.
5. 57 cm.
6. (a) $\frac{1}{1+t^2}$, (b) $\frac{6}{9+4t^2}$, (c) $\frac{2t}{1+t^4}$.
7. (a) $r(\cos\omega t\mathbf{i}+\sin\omega t\mathbf{j})$, (b) $r(-\sin\omega t\mathbf{i}+\cos\omega t\mathbf{j})$.
8. (a) 2 rad s^{-1}, 6 m s^{-1}, (b) 1 rad s^{-1}, 5 m s^{-1}, (c) -2π rad s^{-1}, 2π m s^{-1}, (d) k rad s^{-1}, ak m s^{-1}.
9. $a(\cos\omega t\mathbf{i}+\sin\omega t\mathbf{j})$, $a(-\sin\omega t\mathbf{i}+\cos\omega t\mathbf{j})$, $a\omega(\sin\omega t-\cos\omega t)\mathbf{i}-a\omega(\cos\omega t+\sin\omega t)\mathbf{j}$.

Exercise 8.2 (pp. 133–134)

1. (a) $9\,\mathrm{m\,s^{-2}}$, (b) $4\,\mathrm{m\,s^{-2}}$, (c) $20\,\mathrm{m\,s^{-2}}$, (d) $63{\cdot}2\,\mathrm{m\,s^{-2}}$.
2. $22\frac{1}{2}$ N.
3. 12 N.
4. 375 N, 49 N.
5. $6\,\mathrm{m\,s^{-1}}$.
6. 0·4.
7. 0·7 m.
8. $1{\cdot}75\,\mathrm{rad\,s^{-1}}$.

Exercise 8.3 (pp. 137–139)

1. 49 N, $2{\cdot}8\,\mathrm{m\,s^{-1}}$.
2. 127·4 N, 3 m.
3. $mg\,\mathrm{cosec}\,\theta$; $rg\cot\theta$.
4. 2·45 m.
5. $3{\cdot}15\,\mathrm{m\,s^{-1}}$.
6. $\sqrt{(3ag)}$.
7. $5m(8u^2+9ag)/72a$ in AP, $5m(8u^2-9ag)/72a$ in BP.
8. $17\frac{1}{2}\,\mathrm{m\,s^{-1}}$.
9. 17·7°.
10. $35\,\mathrm{m\,s^{-1}}$, $5\sqrt{7}\,\mathrm{m\,s^{-1}}$.
11. $7\sqrt{5}\,\mathrm{m\,s^{-1}}$; 12·7°.

Exercise 8.4 (pp. 139–142)

1. $10\pi/9\,\mathrm{rad\,s^{-1}}$, $0{\cdot}42\,\mathrm{m\,s^{-1}}$.
2. $25/\pi$, 50.
3. $-6/(5t^2-6t+9)$.
4. $6\,\mathrm{m\,s^{-1}}$ S 30° W.
5. $5/36 \leqslant \mu < 5/9$.
6. 60 N, 75 N.
7. (i) 60°, (ii) $2mg$, (iii) $\pi\sqrt{(2a/g)}$ s.
8. 60°.
9. (i) $\sqrt{5}$, (ii) 1/6 N.
10. $\frac{1}{2}ma(2\omega^2+\Omega^2)$, $\frac{1}{2}ma(\Omega^2-2\omega^2)$.
11. $\sqrt{\{g(a^2-h^2)/h\}}$.
12. $\frac{2}{3}mg\sqrt{3}$, $m(3u^2-gl\sqrt{3})/3l$.
13. $m(v^2\cos\alpha+gl\sin^2\alpha)/l\sin\alpha$, $m(v^2-gl\cos\alpha)/l$.
14. 36·3°; $97{\cdot}8\,\mathrm{m\,s^{-1}}$.
15. $mg(\sqrt{3}-1)$.
16. $\sqrt{(2rg/11)}$ to $\sqrt{(2rg)}$.
17. $\sqrt{(45ag/8)}$.

Exercise 9.1 (pp. 147–149)

1. -42 N m, 14 N m, -21 N m, 0 N m.
2. 42 N m, -30 N m, -42 N m, 30 N m, 0 N m.
3. $(4+\sqrt{3})$ N m, $3\sqrt{3}$ N m, $(2+5\sqrt{3})$ N m.
4. (a) 6 N, 2 m, (b) 8 N, 3 m.
5. (a) 1 N, 10 m, (b) 1 N, 15 m.
6. 50 N m.
7. $F = G = 4$.
8. (a) $5\sqrt{2}$ N, 4 m, (b) 15 N, 3 m.
9. (a) $4\sqrt{2}$ N, 5 m, (b) 6·5 N, 12 m.
10. $6\sqrt{2}$ N along BD.
11. $20\sqrt{3}$ N m, $10\sqrt{3}$ N m, 0 N m, 0 N m, $10\sqrt{3}$ N m, $20\sqrt{3}$ N m; 10 N along CD

Exercise 9.2 (pp. 154–156)

1. 4 N at 90° to AB, 4 m.
2. 2 N at 90° to AB, 5 m.
3. $6\sqrt{3}$ N at 60° to AB, 1 m.
4. 6 N at 60° to AB, 2 m.
5. (a) $5P$ at 53·1° to AB, (b) $6a$.
6. $3x - 7y + 7 = 0$.

7. $2x - 5y + 10 = 0$. 8. *a*, *b*. 9. *c*. 10. *a*, *c*. 11. *b*, *c*.
12. *c*. 13. *c*. 14. *a*, *c*. 15. *b*, *c*. 16. *c*. 17. *a*.
18. *b*. 19. *a*, *c*. 20. 6 N m clockwise; $y = x + 1$.
21. (a) 5, 13, (b) 2·5, 19·5. 22. $7\sqrt{5}$ N, 1 m; 28 N m in sense *CBA*.
23. (a) 10; 28 N along *AD*, (b) 18 N, $14\sqrt{3}$ N m in sense *ABC*.
24. (a) $25P$, $24y = 7x$, (b) (i) $S = -18P$, $T = 15P$, (ii) $72Pa$, (iii) clockwise.

Exercise 9.3 (pp. 162–164)

1. 18 cm from *A*. 2. 1·05 m from *P*. 3. 5 cm. 4. 3 cm, 4 cm.
5. (2, 1). 6. $(3, -\frac{1}{2})$. 7. $3\mathbf{i} + \mathbf{j}$. 8. $2\mathbf{i} + 2{\cdot}5\mathbf{j}$. 9. (3, 0).
10. (0·4, 1·6). 11. 10 cm, 3 cm. 12. 10 cm. 13. $1\frac{2}{3}$ cm from *AB*.
14. $6\frac{1}{3}$ cm from *AD*, $4\frac{4}{5}$ cm from *AB*. 15. 11 cm from both *AB* and *AF*.
16. $3\frac{1}{2}$ cm from *AE*, 4 cm from *AB*. 17. 4 cm from *AD*, 3·5 cm from *AB*.
18. $7a/5$ above base, $16a/11$ above base.

Exercise 9.4 (pp. 170–171)

3. $\dfrac{2a(14 + 3\pi)}{3(8 + \pi)}$. 4. $\dfrac{2a(26 - 3\pi)}{3(16 - \pi)}$. 5. $\dfrac{28r}{9\pi}$.

6. $\dfrac{27\sqrt{3}a}{3\sqrt{3} + 4\pi}$. 7. $\dfrac{11h}{28}$.

8. (a) $\dfrac{7\sqrt{3}r}{8\pi + 3\sqrt{3}}$, (b) $\dfrac{7\sqrt{3}r}{4\pi - 3\sqrt{3}}$. 9. $28\pi a^3/3$, $12a/7$. 11. $3r$.

Exercise 9.5 (pp. 175–176)

1. 6, 103/15; (9/7, 103/70). 2. 153/4, 414/5; (17/8, 23/5).
3. ln 2, 7/48; (2 ln 2, 7/24). 4. 1/4, 2/5; (3/8, 3/5). 5. (20/7, 0).
6. (3/5, 0). 7. $(0, \pi/8)$. 8. (0, 7/3). 9. $\left(\dfrac{1}{e-1}, \dfrac{e+1}{4}\right)$.
10. $(1, \pi/8)$. 11. (2, 0). 12. (2/3, 0). 13. (2 ln 2, 0).
14. $\left(\dfrac{e^2 - 3}{2(e^2 - 1)}, 0\right)$. 15. (9/7, 0). 16. (0, 14/5).
17. $2\ln 2 - 1$, $2\ln 2 - \frac{3}{4}$, $2(\ln 2)^2 - 4\ln 2 + 2$; (1·65, 0·24).
18. 1/6; (1/2, 2/5).

Exercise 9.6 (pp. 176–179)

1. (a) $13P$, $12/5$, (b) $3a$; $12Pa$, sense ABC. 2. $\lambda = 5/2$; $5a/2$; $5a/7$.
3. (c) (i) $3\sqrt{2}$ along AC, $\sqrt{2}$ along BD; $7a$;
   (ii) 7 along AD, $4\sqrt{2}$ along CA, $6\sqrt{2}$ along BD.
4. (a) $4\sqrt{3}P, \frac{\sqrt{3}}{3}P; \frac{7\sqrt{3}}{3}P$; (b) $\frac{5\sqrt{3}}{3}P, \frac{\sqrt{3}}{3}P; \frac{7\sqrt{3}}{3}Pa$ in sense ABC.
5. $p = 11, q = 14$; (a) $2\sqrt{19}$ N, (b) $12\sqrt{3}$ N m.
6. (i) $m = 2, n = 6$, (iii) $x = 3a\sqrt{3}$; $\sqrt{31}$, $5Pa\sqrt{3}$.
7. $(7a/18, 4a/9)$.
8. 4·8 cm from OA, 8·8 cm from OC.
9. $\frac{11}{24}\pi a^3, \frac{5}{24}\pi a^3$; $\frac{21}{88}a, \frac{27}{40}a$ from centre of hemisphere.
11. $\frac{3a}{8}$. 12. $\frac{32}{15}\sqrt{2}; \frac{8\pi}{5}$. 13. 16/3; 8π; $(0, 20/3)$.
14. (0·864, 0·333). 15. 4/3, $104\pi/15$; (15/13, 0). 16. $3\pi a^2$, $(\pi a, 5a/6)$.

Exercise 10.1 (pp. 186–188)

1. 360 N, 2·5 m. 2. 41 N at A, 33 N at B. 3. 1·6 m.
4. 0·6 m, 20 N. 5. (a) 1200 N (b) 3 m. 6. 0·4 m.
7. $5W/8$ at A, $3W/8$ at B. 8. 5. 9. $5a/3$, $5a/2$; $\tan^{-1} 10/21 \approx 25{\cdot}5°$.
10. (a, a); 1/2. 11. $2/(\pi + 2)$. 12. $1/\pi$.
13. $\tan^{-1} 3/4 \approx 36{\cdot}9°$; $3aW$. 14. $\tan^{-1} 11/13 \approx 40{\cdot}2°$; $26aW/9$.
15. $\sqrt{2}$. 16. $2W/3$.
17. $h + \left(\frac{3a^2 - h^2}{8a + 4h}\right)$ from vertex of cone; (a) 17°, (b) $h = a\sqrt{3}$.
18. 30°; $1/\sqrt{3}$. 19. 36·9°. 20. (a) 53·1°, (b) 26·6°.

Exercise 10.2 (pp. 195–197)

1. (a) 10, (b) $10\sqrt{5}$ N at 26·6° to vertical.
2. $\frac{\sqrt{3}}{2}mg$ in AC, $\frac{1}{2}mg$ in BC; $\frac{1}{2}a$. 3. 2 N at A, $2\sqrt{17}$ N at C.
4. 30 N at 60° to vertical. 5. 45 cm, 16 N.
6. (a) $20\sqrt{3}$ N, 90°, (b) $15\sqrt{5}$ N, 63·4°. 7. 5/6; 5W/12, 13W/12.
8. (a) $W\sqrt{3}$ at both A and C, (b) $2a$.
9. (a) 15 N, (b) 9 N at 53·1° to vertical. 10. $5W$, 75 cm. 11. $\frac{1}{2}$; 30 cm.
12. 5/24. 13. (a) $\frac{1}{2}mg \cot\alpha$, (b) $\frac{1}{2}\cot\alpha$.
14. 45°, $\frac{1}{2}W$; half way up, all the way. 15. (a) 30°, (b) $1/\sqrt{3}$.
17. 1/3.

Exercise 10.3 (pp. 202–205)

1. (a) 24 N, (b) $12\sqrt{3}$ N, 13 N.
2. (a) 60 N, (b) 48·2 N at 41·6° to horizontal.
3. (a) 15, (b) 47·4 N at 71·6° to horizontal.
4. $4\sqrt{3}$ N perpendicular to rod; $2\sqrt{3}$ N, 10 N.
5. (a) 14 N, 30 N (b) 1·92 m. 6. (a) 4/7, (b) $\tan^{-1}\frac{2}{3} \approx 33{\cdot}7°$.
7. 5 N, 1/2. 8. 121 N, 17·5 N; 7/48. 9. $\frac{1}{2}\tan\alpha, 2W\tan\alpha$.
10. $3W/10, W/10$; 0·8. 11. 1/5. 12. $8W/15$; $24W/75, 43W/75$.
13. $(5\sqrt{3})/18$; $(W\sqrt{3})/2$. 14. $5a/3$. 15. $\tan^{-1}(4/3)$; 40, 900 N.
16. 10, 6/17. 17. $W/4$, 1/3. 18. $11W/4$. 19. 5 m.

Exercise 10.4 (pp. 209–211)

1. (a) $5\sqrt{10}$ N, $5\sqrt{5}$ N (b) 21·8°, 45°. 2. $\frac{1}{2}W\sqrt{5}$, 63·4°.
3. $\frac{1}{3}W\sqrt{3}, \frac{1}{2}W$.
4. (ii) $W\sqrt{(1+\mu^2)}$ at $\tan^{-1}\mu$ to vertical, (iii) $\mu W = \frac{1}{2}W\cot\theta$.
5. (a) 7/10, (b) $5W/2, 3W/4$. 6. C; 67·4°. 7. C will slip.
8. (a) $9W/5, 12W/5$, (b) 3·2°. 9. $\sqrt{3}/2$. 10. $2-\sqrt{3}$.
11. $\frac{3}{4}W, \frac{1}{4}W\tan\alpha$; $1/\sqrt{3}$.
13. (i) $\frac{9w}{25}$, (ii) $W+\frac{27w}{125}$, (iii) $\frac{36w}{125}$.

Exercise 10.5 (pp. 213–215)

1. AB: $\sqrt{3}$ kN tension; BC: $4\sqrt{3}$ kN tension; CD: $2\sqrt{3}$ kN thrust; AD: $2\sqrt{3}$ kN thrust; BD: $2\sqrt{3}$ kN tension.
2. AB and BC: 5 kN thrust; BD: no stress; CD: 4 kN tension; AD: 3 kN tension.
3. AB: 2 kN thrust; BC: $2\sqrt{2}$ kN tension; CD: 4 kN tension; AC: $2\sqrt{2}$ kN thrust.
4. AB: 8 kN thrust; BC: $4\sqrt{3}$ kN thrust; CD: $2\sqrt{3}$ kN tension; BD: 4 kN tension.
5. AB: $10\sqrt{2}$ kN thrust; BC: $5\sqrt{2}$ kN thrust; CD: 5 kN tension; DE: $5\sqrt{2}$ kN tension; AE: 10 kN tension; BD: 5 kN thrust; BE: 5 kN tension.
6. AB: 4 kN tension; BC: 3 kN thrust; CD: 3 kN thrust; AD: 4 kN tension; AC: 15 kN tension.
7. AB: $4\sqrt{3}$ kN tension; BC: $2\sqrt{3}$ kN tension; CD: $4\sqrt{3}$ kN thrust; DE: $3\sqrt{3}$ kN thrust; AE: $8\sqrt{3}$ kN thrust; BD: $2\sqrt{3}$ kN tension; BE: $2\sqrt{3}$ kN thrust.
8. AB and BC: 6 kN tension; CD and AE: $6\sqrt{3}$ kN thrust; BD and BE: $2\sqrt{3}$ kN tension; DE: $4\sqrt{3}$ kN thrust.
9. $P = \frac{1}{3}W\sqrt{2}$; AB: $\frac{2}{3}W$ thrust; BC: $\frac{1}{3}W\sqrt{2}$ tension; CD: $\frac{1}{3}W$ thrust; AC: $\frac{2}{3}W\sqrt{2}$ tension; AD: $\frac{2}{3}W$ thrust.
10. (ii) AB: 390 N; BC: 650 N; CD: 150 N; AD: 250 N; AC: $120\sqrt{5}$ N (iii) AC and CD.

Exercise 10.6 (pp. 215–222)

2. $4/11$; $7:2$. 4. $5a/2$, $5a/2$; $\tan^{-1}(1/6) \approx 9{\cdot}5°$; $W/6$.
7. (a) $12\frac{1}{2}$ N; $7\frac{1}{2}$ N in the direction CA, (b) $6\frac{1}{4}$ N; $6\frac{1}{4}$ N at $53{\cdot}1°$ to CA produced; 8 N acting 0·625 m from A.
8. $\frac{1}{2}W\sqrt{5}$; $7l/12$.
9. (a) 15 N, (b) 8/3 m, (c) 42·7 N at $73{\cdot}7°$ to the horizontal.
10. (ii) $\frac{1}{4}\sqrt{3}$, (iii) $24c/7$. 11. $3/46$. 12. $3/8$, $5l/3$.
14. $\frac{3}{2}mg\cot\alpha$; $mg\cot\alpha$. 15. $\cos^{-1}(2m/M)$.
16. $W(1-\sin 2\theta)$; $18{\cdot}4°$.
17. $AX = \frac{2}{3}a$, $13W$ at $67{\cdot}4°$ to XA; $CY = 11a/12$.
18. (i) $25W/37$, $20W/37$, (iii) $13/40$.
20. $(8a/15, 3a/16)$; $128/45 \approx 2{\cdot}84$.
21. $h + \left(\dfrac{6l^2 - h^2}{4h + 12l}\right)$ from vertex of cone. 22. $\sin^{-1}\left(\dfrac{k+3}{3k-24}\right)$.
23. (i) $35\sqrt{2}$ N, (ii) $5\sqrt{2}$ N.
24. Force at the ground. 25. $\frac{1}{4}Wl\sqrt{3}$.
26. $\frac{1}{2}W/5$; $\frac{1}{2}W$, 0. 27. Slipping at A.
28. (i) $\frac{1}{2}W$, (ii) $\frac{1}{2}W$, W, (iv) $\frac{1}{2}$.
29. $2W\sqrt{3}$ along BA; $\frac{1}{3}W\sqrt{3}$, $3W/2$ at A. $\frac{1}{3}W\sqrt{3}$, $\frac{1}{2}W$ at C.
30. $\frac{2}{3}W\sqrt{3}$ between cylinders, $\frac{1}{3}W\sqrt{3}$ at both walls. 31. $1\sqrt{3}$.
32. (i) $35W/4$, (ii) 0, $5W$ at A; $7W$, $5W/4$ at B, (iii) $21aW/2$.
33. (i) 2400 N, (ii) 2600 N at $67{\cdot}4°$ to vertical,
(iii) AB: 2400 N; BC: 700 N; CD: $700\sqrt{2}$ N; DE: 1400 N; BD: $700\sqrt{2}$ N; BE: $1000\sqrt{2}$ N; rods AB, BC, BD in compression.
34. (i) 14, (ii) AB: $4/\sqrt{3}$ kN tension; BC: $8/\sqrt{3}$ kN thrust; CD: $8/\sqrt{3}$ kN thrust; DE: $16/\sqrt{3}$ kN thrust; AE: $2/\sqrt{3}$ kN thrust; BD: $4/\sqrt{3}$ kN tension; BE: $4\sqrt{3}$ kN thrust, (iii) 2 kN vertically downwards.

Exercise 11.1 (pp. 227–228)

1. $6\mathbf{i} - 8\mathbf{j}$, $4\mathbf{i} + 3\mathbf{j}$; $10\mathbf{i} - 5\mathbf{j}$.
2. $3\sqrt{2}$ N, $5\sqrt{2}$ N. 3. 27 N.
4. (a) $\mathbf{r} = (1+\lambda)\mathbf{i} + (1+2\lambda)\mathbf{j}$, (b) $\mathbf{r} = -2\lambda\mathbf{i} + (5\lambda+2)\mathbf{j} + (3\lambda-1)\mathbf{k}$,
(c) $\mathbf{r} = 3\mathbf{i} - 2\lambda\mathbf{j} + 3\lambda\mathbf{k}$.
5. $\mathbf{r} = (2+3\lambda)\mathbf{i} + (\lambda-1)\mathbf{j} - 2\lambda\mathbf{k}$, $\mathbf{r} = (9+\mu)\mathbf{i} - \mu\mathbf{j} + (3\mu-1)\mathbf{k}$; $8\mathbf{i} + \mathbf{j} - 4\mathbf{k}$; $\mathbf{r} = (8+4t)\mathbf{i} + \mathbf{j} + (t-4)\mathbf{k}$.
6. 11 N, $\mathbf{r} = (3+7\lambda)\mathbf{i} + (5+6\lambda)\mathbf{j} + 6\lambda\mathbf{k}$.
7. $3x + y = 26$.
8. $3\sqrt{13}$ N, $\frac{2}{3}$; $2x - 3y + 2 = 0$.
9. (a) 15**YX**, where $BX : XC = 3:2$, $AY : YB = 2:1$,
(b) 24**XY**, where $AX : XC = 1:3$, $AY : YB = 1:5$,
(c) 6**YX**, where $AX : XC = 2:1$ and $AY = YB$,
(d) 21**XY**, where $AX : XC = 1:2$, $AY : YB = 3:4$.

Exercise 11.2 (pp. 231–232)

1. $\mathbf{r}_A = (3 + 5t)\mathbf{i} + (4 + 2t)\mathbf{j}$, $\mathbf{r}_B = (5 + 2t)\mathbf{i} + (8 + t)\mathbf{j}$; $\mathbf{AB} = (2 - 3t)\mathbf{i} + (4 - t)\mathbf{j}$; $\sqrt{10}$ m.
2. $4\mathbf{i} - 6\mathbf{j}$; $(4t - 2)\mathbf{i} + (3 - 6t)\mathbf{j}$; $t = \frac{1}{2}$.
3. $(2 - 5t)\mathbf{i} + (4t - 18)\mathbf{j}$; $2\sqrt{41}$ km; $16\mathbf{i} + 19\mathbf{j}$, $8\mathbf{i} + 9\mathbf{j}$.
4. $8\cos\omega t\,\mathbf{i} + 8\sin\omega t\,\mathbf{j}$. 5. $\mathbf{i} - 2\mathbf{j}$, 3 m; 6 m s^{-1}.
6. (a) $\pi/4$, (b) $5\pi/4$. 7. (a) 1/3, (b) 3.
8. (a) $8t\mathbf{i} + (9t^2 - 2)\mathbf{j}$, $m(8\mathbf{i} + 18t\mathbf{j})$, (b) $-6t^2\mathbf{j} + 2t\mathbf{k}$, $m(-12t\mathbf{j} + 2\mathbf{k})$, (c) $(2 - 2t)\mathbf{i} + (6\cos 2t)\mathbf{j}$, $m\{-2\mathbf{i} - (12\sin 2t)\mathbf{j}\}$, (d) $e^{-t}(1 - t)\mathbf{i} - 2\mathbf{j}$, $me^{-t}(t - 2)\mathbf{i}$.
9. (a) $\sqrt{19}$ m s^{-1}, (b) $20\mathbf{i}$. 10. $12\mathbf{i} + 30\mathbf{j}$, $18\mathbf{i} + 45\mathbf{j}$.
11. 40 m s^{-1}, $18\mathbf{i} - 35\mathbf{j}$. 12. $5\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$.
13. $2\mathbf{i} + 6\mathbf{j}$; $9\sqrt{10}$ m.
14. $(16t - 5t^2)\mathbf{j}$, $12t\mathbf{i} + (25 - 5t^2)\mathbf{j}$; 15 m.

Exercise 11.3 (pp. 238–240)

1. 31·6°, 5 m s^{-1}. 2. $\sqrt{41}$ N $\approx$ 6·40 N, 50·6°.
3. (a) 2 N, (b) -3 N, (c) $\sqrt{2}$ N, (d) $-5/3$ N.
4. (a) $8\mathbf{i} - 6\mathbf{k}$, $3\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$, (b) $-10\mathbf{j} + 24\mathbf{k}$, $5\mathbf{i} - 12\mathbf{j} - 5\mathbf{k}$, (c) $-6\mathbf{i} + 18\mathbf{j} - 27\mathbf{k}$, $24\mathbf{i} + 17\mathbf{j} + 6\mathbf{k}$, (d) $\frac{2}{3}(4\mathbf{i} - 8\mathbf{j} + \mathbf{k})$, $\frac{1}{3}(\mathbf{i} + 4\mathbf{j} + 28\mathbf{k})$.
5. (a) 2 J, (b) 0 J, (c) 21 J, (d) -11 J.
6. (a) $8m$ J, (b) $24m$ J, (c) $360m$ J, (d) 0 J.
7. (a) $4m\mathbf{i}$, $4t\mathbf{i} - 5\mathbf{k}$; $16mt$, (b) $6m(t\mathbf{j} - \mathbf{k})$, $3t^2\mathbf{j} - 6t\mathbf{k}$; $18mt(t^2 + 2)$, (c) $-me^{-t}\mathbf{j}$, $\mathbf{i} + e^{-t}\mathbf{j}$; $-me^{-2t}$. (d) $-m(\sin t\mathbf{j} + 3\cos t\mathbf{k})$, $\cos t\mathbf{j} - 3\sin t\mathbf{k}$; $4m\sin 2t$.
8. 14 J; 8 m s^{-1}. 9. -114 J; 6; -96 W, -18 W.
10. $t^2\mathbf{i} + (1 - 3t)\mathbf{j} + \mathbf{k}$; 20 J; 31 W.
11. $(4t + 6)\mathbf{i} - 2\sin t\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$, $t^3 + 32t + 48 + 4\sin 2t$. 12. 24 J.
13. $10t^4 + 6t^2 + 2t$; 544 J. 14. $\int_1^2 \mathbf{F}.(\mathbf{i} - 2\lambda\mathbf{j})\,d\lambda$; 12.

Exercise 11.4 (pp. 240–242)

1. (i) 26·7 N in the direction of the vector $3\mathbf{i} + 13\mathbf{j}$; $-36/65$, (ii) $6\mathbf{i} + 26\mathbf{j}$.
2. $(2T^4 - 2)\mathbf{i} + (3T^2 + 3)\mathbf{j}$; $94\mathbf{i} + 30\mathbf{j}$.
3. $6\mathbf{i} + 8\mathbf{j}$, 6; $6\mathbf{i} + 8\mathbf{j} + 6(\mathbf{i}\cos(\pi t/12) + \mathbf{j}\sin(\pi t/12))$; $-\pi^2/24$, $-\pi^2\sqrt{3}/24$; $(9 - \sqrt{3}\pi)\mathbf{i} + (8 + 3\sqrt{3} + \pi)\mathbf{j}$. 4. $1/(1 + \cos\theta)$; towards O.
5. $2/\sqrt{13}$. 6. (i) $(10 - 2t)\mathbf{i} + (5 + 4t)\mathbf{j}$, (ii) 14.00 hrs, (iii) $4\sqrt{5}$ km.
7. $(260 - 35t)\mathbf{i} - 5\sqrt{3}t\mathbf{j}$; 62 km, 19.00 hrs.
8. (a) $\mathbf{v} - \mathbf{u}$, (b) $\mathbf{p} + t(\mathbf{v} - \mathbf{u})$; $\left\{p^2 - \dfrac{[\mathbf{p}.(\mathbf{u} - \mathbf{v})]^2}{|\mathbf{u} - \mathbf{v}|^2}\right\}^{1/2}$

9. (i) $\mathbf{F}.\mathbf{v}$, (ii) $\int_0^T \mathbf{F}.\mathbf{v}\,dt$; $-4a\sin 2t\mathbf{i} + 2a\cos 2t\mathbf{j}$,
$-8ma\cos 2t\mathbf{i} - 4ma\sin 2t\mathbf{j}$; $12ma^2\sin 4t$; $6ma^2$, $\frac{1}{4}\pi$.
10. $\frac{1}{2}ma^2(2 - 2\cos\theta)\dot{\theta}^2 + mga(1 + \cos\theta) = 2mga$.

Exercise 12.1 (pp. 248–250)

1. (a) $8\mathbf{i} - 28\mathbf{j}$, (b) $2\mathbf{i} - \frac{2}{3}\mathbf{j} + \mathbf{k}$, (c) 65, (d) 4.
2. (a) $10\mathbf{i} + 6\mathbf{j}$, (b) $2\mathbf{i} - 52\mathbf{j} + 8\mathbf{k}$, (c) $3\mathbf{j}$.
3. (a) 480 N s, (b) 30 000 N s, (c) 12 N s.
4. (a) $12\mathbf{j}$, (b) $-2\mathbf{i} + 6\mathbf{k}$, (c) $-60\mathbf{k}$. 5. 54 N s. 6. 4 N s.
7. 600 N. 8. (a) 16·8 m s^{-1}, (b) 7·2 m s^{-1}. 9. 7·5 s.
10. $12\mathbf{i} + 15\mathbf{j}$; $2\mathbf{i} + 2{\cdot}5\mathbf{j}$. 11. $8\mathbf{i} + 4\mathbf{j}$; 10 J
12. (a) 14 m s^{-1}, 240 J gain, (b) 2 m s^{-1}, 48 J loss, (c) 10 m s^{-1}, 96 J gain.
13. 2·5 N s. 14. $20\sqrt{3}$ N s. 15. 600 N. 16. 12 m s^{-1}.
17. 2 N at 53·1° to horizontal
18. (a) 2 m s^{-1}, $6\sqrt{3}$ N s, (b) $(4\sqrt{3})/3$ m s^{-1}, $4\sqrt{6}$ N s.
19. 1·5 m s^{-1} along AB, 5·22 m s^{-1} at 73·3° to AB; 4·5 N s.
20. (a) $2\sqrt{2}$ m s^{-1}, $2\sqrt{2}$ m s^{-1}, 4 m s^{-1}, (b) 2 m s^{-1}, $2\sqrt{3}$ m s^{-1}, 4 m s^{-1}.

Exercise 12.2 (pp. 252–253)

1. 5 m s^{-1}. 2. 720 m s^{-1}. 3. 7·5 m s^{-1} towards A; 78·75 J.
4. 2·5 m s^{-1}, 2512·5 J. 5. 5 m s^{-1}. 6. 7 m s^{-1}. 7. $2\mathbf{i} - 5\mathbf{j}$.
8. $5\mathbf{i} + \frac{5}{2}\mathbf{j}$. 9. 28 J. 10. $3u$, $2u$. 11. 6 m s^{-1}, 26 kJ.
12. 3·5 m s^{-1}, 2000 N. 13. 0·6 m s^{-1}, 720 N s. 14. 30.
15. (a) 14 m s^{-1}, (b) 23·8 m s^{-1}. 16. $u/2g\sin\alpha$.

Exercise 12.3 (pp. 258–260)

1. 4800 N, 5 s. 2. 9000 N, 1/600 s. 3. 2 m s^{-1}, 25 m. 4. 1.
5. (a) 70 N s, (b) 77 N s. 6. 10/7 s, $14\sqrt{2}$ m s^{-1}.
7. 4·2 m s^{-1}, 78 400 N. 8. 90 160 N, 1/7 s. 9. 1·25 m.
10. 469 m s^{-1}. 11. h, $\frac{1}{4}h$. 12. 0·1 m. 13. 2·1 m s^{-1}, 6·3 N s.
14. 7·35 m s^{-1}, $2\frac{1}{4}$ s. 15. 3 s, 4·2 m s^{-1}. 16. (a) $\frac{7}{3}h$, (b) $\frac{2}{3}m\sqrt{(\frac{2}{3}gh)}$.
17. 2·8 m s^{-1}, 20·6 J. 18. (i) $\frac{3}{4}mgh$, (ii) $9m^2gh/4M$.
19. (ii) $\frac{12}{5}m\sqrt{(gc)}$, (iii) $\frac{33}{25}c$, (iv) $\frac{12}{5}mgc$.
20. 40 m s^{-1}, 80 J; 1200 J; $(d + 14)$ cm; 1, 8000 N.

Exercise 12.4 (pp. 265–267)

1. (a) 2·5 m, (b) 14·7 J, (c) 4·2 N s. 2. $\frac{3}{4}$, 4·05 m. 3. $\frac{2}{3}$.
4. $8\sqrt{5}$ m s^{-1}, 60 N s. 5. $1/\sqrt{3}$; 45°. 6. 60°.
7. $4\frac{1}{6}$ m s^{-1} at 53·1° to horizontal. 8. $2u$, $4u$; $18mu^2$. 9. $\frac{1}{2}$.

10. (a) $1\,\mathrm{m\,s^{-1}}$, $4\,\mathrm{m\,s^{-1}}$, same direction.
(b) $\frac{1}{2}\,\mathrm{m\,s^{-1}}$, $5\frac{1}{2}\,\mathrm{m\,s^{-1}}$, opposite directions.
11. $\frac{1}{4}$, 2:3. 12. $5\,\mathrm{m\,s^{-1}}$, $-4\,\mathrm{m\,s^{-1}}$, $16\,\mathrm{m\,s^{-1}}$; yes.
13. $u/4$, $3u/16$, $9u/16$; yes. 14. 2/9. 16. $u(1-e)$, u.
17. $\frac{1}{2}\sqrt{(2ag)}$, $\frac{1}{4}\sqrt{(2ag)}$ in same direction.
18. (a) $\frac{1}{3}V(1-2e), \frac{1}{3}V(1+e), 0$ (b) $\frac{1}{3}V(1-2e), \frac{1}{9}V(1+e)(2-e), \frac{2}{9}V(1+e)^2$.

Exercise 12.5 (pp. 267–272)

1. (i) $9mu$, (ii) $81mu^2/4$. 2. (a) $12\mathbf{i}+4\mathbf{j}$, (b) $5\mathbf{i}-2\mathbf{j}$, (c) 32 J.

3. $-\frac{mv}{2}\mathbf{i}+\frac{mv}{2\sqrt{3}}\mathbf{j}$; $\frac{v}{3\sqrt{3}}$ at 30° to the x-axis in a clockwise sense.

4. (i) 283 kg, (ii) 11 500 J, (iii) 39 kW, (iv) 56 kW; 2500 N.
5. (ii) $u/6$, $(u\sqrt{3})/4$; $mu/6$. 6. $|m_1u_1-m_2u_2|/(m_1+m_2)$.
8. $\frac{1}{2}mgh$, $m(1+\sqrt{2})\sqrt{(gh)}$. 9. $1{\cdot}6\,\mathrm{m\,s^{-1}}$, $0{\cdot}4\,\mathrm{m\,s^{-1}}$; 131 mm, 8 mm.
10. (i) $5{\cdot}6\,\mathrm{m\,s^{-1}}$, (ii) $3{\cdot}2\,\mathrm{m\,s^{-1}}$, (iii) 13 440 J, (iv) 123·9 kN.
11. (i) $\frac{6}{5}m\sqrt{(2ga)}$, (ii) $\frac{6}{5}mga$, (iii) $\frac{2}{5}\sqrt{(17ga)}$.
12. $1{\cdot}96\,\mathrm{m\,s^{-2}}$; $1{\cdot}12\,\mathrm{m\,s^{-1}}$, 3·36 N s.
14. (i) $g/5$, $12mg/5$, (iii) $9mv/8$, (iv) $15mv^2/16$, (v) $1/\sqrt{3}$.
15. $mMu^2/2a(m+M)$. 16. (b) 16°, (c) 5:6.

17. $\frac{u\sin\alpha}{\sin\beta}$; $\frac{1}{2}u\left|\frac{\sin(\alpha-\beta)}{\sin\beta}\right|$; $\frac{u^2}{g\sqrt{3}}$.

19. (a) $\cot\alpha\tan\beta$, (b) $u\cos\alpha\sec\beta$, (c) $mu\sin(\alpha+\beta)\sec\beta$.
20. (a) $\frac{1}{2}mu^2(1-e^2)\cos^2\alpha$, (b) $u^2e\sin 2\alpha/g(1+e)$.
21. $\frac{1}{2}$; 2:7; $2\,\mathrm{m\,s^{-1}}$, $7\,\mathrm{m\,s^{-1}}$ in opposite directions.
22. (i) u along $\overrightarrow{BA}$, (ii) 4/5, (iii) $3mu^2$,
(iv) u, $4u/3$ along $\overrightarrow{BC}$, (v) 1/9.
23. (i) $\frac{1}{2}$, (ii) $\frac{1}{4}mu^2$, (iii) $\frac{2}{3}$, (iv) u. 24. $4u/3$, $u/6$; $7mu^2/12$.
26. $\frac{1}{3}(7-8e)u$, $\frac{1}{3}(7+16e)u$ in direction $\overrightarrow{AB}$; 1/8;
$3u/(1+k)$, $k>\frac{1}{2}$.

27. $e=1,\frac{1}{3},\frac{1}{5},\frac{1}{7},\ldots,\frac{1}{1+2n},\ldots$

Exercise 13.1 (pp. 275–276)

1. $\ln 3\,\mathrm{m\,s^{-1}}$. 2. $\ln 5$ seconds. 3. $6\,\mathrm{m\,s^{-1}}$. 4. $0{\cdot}585\,\mathrm{m\,s^{-1}}$.
5. $2a$. 6. $\sqrt{(gR/3)}$. 7. (a) $2/x$, (b) $\sqrt{(4t+h^2)}$. 8. $-\frac{1}{2}e^x$.

Exercise 13.2 (pp. 279–282)

1. $\ln 3$. 2. $\frac{1}{2}\ln(1+u^2)$. 3. $-\frac{1}{k}\{v+\ln(1-v)\}$.

4. $\frac{a}{b^2}(2 - \ln 3)$. 5. $\frac{2u}{2 + su}; \frac{s^2}{4} + \frac{s}{u}$. 6. (i) 482 m, (ii) 24·1 s.

8. $\left[\frac{g}{k}(1 - e^{-2ks}) + u^2 e^{-2ks}\right]^{1/2}$. 9. (iii) 11·1 s.

10. $\frac{dv}{dt} = \frac{k(U^2 - v^2)}{mv}; v\frac{dv}{dx} = \frac{k(U^2 - v^2)}{mv}$.

11. $\frac{m}{2k}\tan^{-1}(\frac{1}{2}u); \frac{m}{2k}ln(1 + \frac{1}{4}u^2)$. 12. $\frac{P^2}{k}\left(1 - \frac{1}{e}\right)$.

15. $\frac{m}{R^2}\left\{H\ln\left(\frac{H - Ru}{H - 2Ru}\right) - Ru\right\}$;

$\frac{m}{R^3}\left\{H^2\ln\left(\frac{H - Ru}{H - 2Ru}\right) - HRu - \frac{3}{2}R^2u^2\right\}$.

16. $\frac{mU^2}{2S}\ln\left(\frac{U^2}{U^2 - V^2}\right); \frac{mU^2}{2S}\left\{U\ln\left(\frac{U + V}{U - V}\right) - 2V\right\}$.

Exercise 13.3 (pp. 286–287)

1. $5\,m\,s^{-1}, 10\,m\,s^{-2}$. 2. (a) 2 m, (b) $\frac{2}{3}\pi$ s, (c) $3\sqrt{3}\,m\,s^{-1}$.
3. 6π s; 2·5 N. 4. (i) $270/\pi$, (ii) $\frac{2}{3}\sqrt{13}$ m, (iii) $540\sqrt{13}$ N.
5. $6\,m\,s^{-1}$, 2 m; $5{\cdot}9\,m\,s^{-1}$. 6. (i) π s, (ii) 2π s, (iii) $3\pi/2$ s.

7. (i) $\frac{5}{3\pi}$ m, (ii) $\frac{5}{2}\sqrt{3}\,m\,s^{-1}$, (iii) $\frac{1}{2\pi}$ m, (iv) $\frac{1}{18}$ s.

8. $2\sqrt{3}\,m\,s^{-1}$, 1/6 s; 8 J.
9. (a) $x = 3\cos 2t + 3\sin 2t$, (b) $-3, 6$, (c) $3\pi/8$.
10. (a) $x = 3\cos 4t + \sqrt{3}\sin 4t$, (b) $\pi/24$, (c) $2\sqrt{3}$.

Exercise 13.4 (pp. 293–295)

1. $5a/3$. 2. (i) $2l$, (ii) $\frac{1}{2}\pi\sqrt{(3l/g)}$, (iii) $15l/8$.
3. $\frac{1}{2}mg, \frac{1}{2}mg$; (i) $3l/2$, (ii) $\pi(2l/g)^{1/2}$. 4. 4.
5. $2\pi\sqrt{(2a/g)}$, a; $\frac{1}{2}a$ above 0. 6. $\sqrt{(2a\lambda/m)}$.

7. $5l/2, \pi(lm/\lambda), \sqrt{(\lambda l/m)}$. 8. $2l; \pi\sqrt{\left(\frac{2l}{g}\right)}, \left(\frac{2u^2l + gl^2}{4g}\right)^{1/2}$.

9. $u\sqrt{\left(\frac{lm}{\lambda}\right)}; \frac{\pi}{2}\sqrt{\left(\frac{lm}{\lambda}\right)}; \pi\sqrt{\left(\frac{lm}{\lambda}\right)} + \frac{2l}{u}$. 10. $\frac{16}{9}a; \sqrt{(5ag)}; \frac{\pi}{6}\sqrt{\left(\frac{a}{g}\right)}$.

11. (a) 994 mm, (b) 991 mm, (c) 996 mm, (d) 164 mm.
12. (a) 143 cm, (b) 56 cm, (c) 36 cm, (d) 14 cm.

Exercise 13.5 (pp. 300–302)

1. $49\cos\theta - 24$, $58{\cdot}8\cos\theta - 19{\cdot}2$; 3·68, 25·8; 61°.
2. $6{\cdot}18\,\mathrm{m\,s^{-1}}$, 19·35 N. 3. $\frac{1}{3}$, $\frac{1}{3}ag$.
5. $mg|3\cos\theta - 2|$; $\frac{2}{3} < \cos\theta < 1$. 6. $\sqrt{(2ag/3)}$.
7. $\dfrac{mu^2}{a} - 2mg + 3mg\cos\theta$; mg. 8. (ii) $\frac{7}{2}ag$, $\sqrt{(\frac{1}{2}ag)}$.
9. $\cos\alpha\sqrt{(2gl\sin\alpha)}$; no. 10. $\{2ag(1+\sin\theta)\}^{1/2}$; (b) $\pi/6$, $(3ag)^{1/2}$; $(a/g)^{1/2}$.
11. $\{u^2 - 6ag(1-\cos\theta)\}^{1/2}$; $\dfrac{mu^2}{2a} - 3mg(1+\sin\phi)$.
12. $\frac{2}{3}mg(1+\cos\theta)$, $\frac{1}{3}mg(5\sin\theta - 4\theta)$.

Exercise 13.6 (pp. 302–307)

1. Particle escapes from the earth.
2. $mv\dfrac{dv}{dx} = Ke^{-\alpha x} - \lambda m$; $\left\{\dfrac{2K}{\alpha m}\left(1 - e^{-\alpha x}\right) - 2\lambda x\right\}^{1/2}$.
3. $\frac{1}{2}(\mu/a)^{1/2} \leqslant V \leqslant \frac{3}{4}(\mu/a)^{1/2}$.
4. $5\cos 3t + 12\sin 3t + 2$; 15, 39; $t \approx 0{\cdot}967$. 5. $\dfrac{2k^2}{g}\left(1 - e^{-gt/k}\right) - kt$.
6. $\frac{1}{800}$ s; $\dfrac{dv}{dt} = -16\times 10^3(2v)^{1/2}$, $\frac{1}{400}$ s; $v = Ae^{-kt}$, A and k positive constants.
7. (i) $2uT$, (ii) $T\ln 3$, (iii) $4mu^2$. 8. $v = gTe^{-x/gT^2}$
9. $v^2 = u^2\tan\left(\dfrac{\pi}{4} - \dfrac{2gz}{u^2}\right)$. 10. (ii) $u\sqrt{(1 - e^{-2kx})}$.
13. (b) (i) $3\pi/8$ s, (ii) $26\frac{2}{3}\,\mathrm{m\,s^{-1}}$, (iii) $142\frac{2}{9}\,\mathrm{m\,s^{-2}}$, (iv) $\pi/16$ s.
14. $\dfrac{\pi}{4}$ m/s, $\dfrac{\pi^2}{8}$ m/s^2; $\dfrac{\pi\sqrt{3}}{8}$ m/s, $\frac{2}{3}$ s.
15. (i) 12 s, $24/\pi$ m, (ii) $t = 5$; $12\sqrt{2}$ s, $\sqrt{2}\,\mathrm{m\,s^{-1}}$.
16. $\ddot{x} = -\omega^2 x$, $2\pi/\omega$; $t = 12$ s, $t = 24$ s. 17. $4u/\pi$.
18. (i) 5·09 m, (ii) 3·60 m, (iii) $\dfrac{4}{\pi}\sin^{-1}\left(\dfrac{\pi}{16}\right)$; 2π N, $\mu \geqslant \pi/g$.
19. $\frac{400}{3}$ N; $\dfrac{2}{\sqrt{3}}$ m/s. 20. $2mg + \dfrac{2\lambda}{3l}x$. 21. $\sqrt{(\frac{1}{3}gl)}$.
23. $\dfrac{5\sqrt{2}}{24}l$. 25. $\{ag(2\cos\theta - 1)\}^{1/2}$, $2mg(3\cos\theta - 1)$; $8a/25$.
28. $\sqrt{(3ga/2)}$. 29. $\ddot{\mathbf{r}} = \mathbf{g} - k\dot{\mathbf{r}}$.

Exercise 14.1 (pp. 310–311)

1. 720.
2. 40 320.
3. 20 160.
4. 151 200.
5. 13 800.
6. 10 080.
7. 720.
8. (a) 48. (b) 180.
10. (a) 210, (b) 1140, (c) 1287, (d) 12.
11. 1260.
12. (a) 462, (b) 5775, (c) 10 395.
13. 15 840
14. 38, 16.
15. 238.
16. (a) 15 015, (b) 37 037.

Exercise 14.2 (pp. 316–317)

1. (a) $\{x\}$, (b) $\{x, y, p, q, r\}$, (c) $\{x, y, z, p, q, r\}$, (d) $\{x, p\}$, (e) $\{x\}$, (f) $\{x, y, p, q, r\}$.
2. (a) $\{x \in \mathbb{R} : -2 < x < 7\}$, $\{x \in \mathbb{R} : 0 < x < 4\}$,
   (b) $\{x \in \mathbb{R} : x \leqslant 9\}$, $\{x \in \mathbb{R} : 2 \leqslant x \leqslant 5\}$,
   (c) $\{x \in \mathbb{R} : |x| < 10\}$, $\{x \in \mathbb{R} : -4 < x < 3\}$,
   (d) $\{x \in \mathbb{R} : -3 < x \leqslant 6\}$, $\varnothing$.
3. (a) $\varnothing$, (b) $\{4\}$, (c) $\{3, 4, 5\}$, (d) $\{1, 2\}$, (e) $\{1, 2, 3\}$, (f) $\{4, 5\}$.
4. (a) X', (b) $X \cap Y$, (c) $X' \cap Y'$, (d) $X \cup Y'$, (e) $X' \cup Y'$, (f) $(X \cup Y) \cap (X' \cup Y')$ or $(X \cap Y') \cup (X' \cap Y)$.
9. (a) A, (b) $\varnothing$, (c) A, (d) A.
10. 27.
11. $n(A)+n(B)+n(C)-n(A \cap B)-n(B \cap C)-n(C \cap A)+n(A \cap B \cap C)$.
12. $n(M \cap F) \geqslant 3$, $n(M \cap P) \geqslant 10$, $n(M \cap F \cap P) \leqslant 12$.
14. (a) A, (b) B, (c) $A \cap C$, (d) $B \cup C$.

Exercise 14.3 (pp. 321–322)

1. $\frac{1}{6}, \frac{1}{2}, \frac{2}{3}$.
2. $\frac{1}{2}, \frac{1}{13}, \frac{3}{13}$.
3. $\frac{1}{8}, \frac{1}{2}, \frac{3}{4}$.
4. $\frac{1}{21}, \frac{2}{7}, \frac{10}{21}$.
5. (a) $\frac{2}{3}$, (b) 0, (c) $\frac{5}{6}$, (d) $\frac{1}{6}$.
6. (a) $\frac{7}{13}$, (b) $\frac{1}{26}$, (c) $\frac{1}{13}$, (d) $\frac{5}{13}$.
7. (a) 0·8, (b) 0·07, (c) 0·1, (d) 0·43.
8. (a) $\frac{1}{4}$, (b) $\frac{5}{9}$, (c) $\frac{7}{36}$.
9. (a) $\frac{1}{14}$, (b) $\frac{3}{28}$, (c) $\frac{3}{7}$, (d) $\frac{4}{7}$.
10. 0·32.
11. (a) $\frac{1}{9}$, (b) $\frac{2}{3}$.
12. (a) $\frac{1}{16}$, (b) $\frac{3}{8}$, (c) $\frac{5}{8}$, (d) $\frac{7}{8}$.
13. (a) $\frac{1}{3}$, (b) $\frac{2}{9}$, (c) $\frac{8}{27}$.

14. (a) $\frac{3}{10}$, (b) $\frac{3}{10}$, (c) $\frac{3}{10}$, (d) $\frac{1}{5}$.

15. (a) $\frac{3}{20}$, (b) $\frac{11}{20}$.

16. $\frac{11!}{3!2!}$; (a) $\frac{2}{11}$, (b) $\frac{1}{77}$.

17. $\frac{15}{28}$.

18. (a) $\frac{1}{2}$, (b) $\frac{1}{4}$, (c) $\frac{1}{2}$.

19. (a) 0·204, (b) 0·052, (c) 0·398.

20. $\frac{21}{55}, \frac{27}{55}, \frac{27}{220}, \frac{1}{220}$.

21. (a) 0·108, (b) 0·374.

22. (a) $\frac{1}{42}$, (b) $\frac{1}{7}$, (c) $\frac{2}{3}$, (d) $\frac{1}{2}$.

Exercise 14.4 (pp. 326–327)

1. $\frac{1}{2}$.

2. 0·33.

3. $\frac{1}{12}$.

4. (i) (a) A and B, A and C, (b) A and B;
   (ii) (a) 0, (b) $\frac{3}{4}$, (c) $\frac{1}{4}$, (d) $\frac{1}{2}$, (e) $\frac{1}{4}$, (f) 1.

5. (i) (a) none, (b) A and C;
   (ii) (a) $\frac{1}{9}$, (b) 1, (c) $\frac{7}{36}$, (d) $\frac{5}{6}$, (e) 0, (f) $\frac{8}{9}$.

6. $0{\cdot}6 \leqslant P(A') \leqslant 0{\cdot}9$.

7. $\frac{1}{2}, \frac{1}{52}, \frac{3}{13}, 0, \frac{1}{52}, \frac{3}{26}$; A and C.

8. $\frac{1}{3}, \frac{1}{4}, \frac{1}{6}, \frac{1}{12}, \frac{1}{12}, \frac{1}{6}$; A and B.

9. (a) $\frac{3}{20}$, (b) $\frac{7}{10}$, (c) $\frac{1}{10}$, (d) $\frac{17}{20}$.

10. $\frac{5}{6}, \frac{2}{5}$.

11. $\frac{9}{10}, \frac{9}{10}$.

12. (a) $\frac{9}{40}$, (b) $\frac{1}{28}$.

13. (a) $\frac{1}{16}$, (b) $\frac{7}{16}$.

14. (a) $\frac{1}{27}$, (b) $\frac{1}{8}$, (c) $\frac{91}{216}$.

15. (a) $\frac{1}{36}$, (b) $\frac{1}{36}$, (c) $\frac{1}{2}$.

16. (a) $\frac{5}{9}$, (b) 0, (c) $\frac{20}{81}$.

17. 0·302.

18. (a) $\frac{1}{16}$, (b) $\frac{9}{64}$, (c) $\frac{1}{4}$, (d) $\frac{1}{16}$.

19. 0·590.

20. (a) $\frac{5}{32}$, (b) $\frac{5}{16}$, (c) $\frac{13}{16}$.

21. (a) $\frac{1}{64}$, (b) $\frac{2}{9}$, (c) $\frac{5}{36}$.

Exercise 14.5 (pp. 330–332)

1. $\frac{1}{4}, \frac{1}{4}, \frac{4}{7}$.

2. $\frac{1}{7}, \frac{1}{7}, \frac{1}{28}, \frac{1}{28}$.

3. $\frac{1}{4}, \frac{4}{7}, \frac{1}{4}, \frac{4}{7}$.

4. $\frac{1}{7}, \frac{3}{28}, \frac{3}{7}$.

5. $\frac{3}{14}, \frac{15}{28}, \frac{3}{14}$.

6. $\frac{2}{13}, \frac{1}{2}, \frac{1}{4}$.

7. $\frac{1}{2}, \frac{1}{4}, \frac{1}{13}, \frac{1}{26}$.

8. $0, 0, \frac{2}{3}, \frac{1}{2}$.

9. $0, \frac{1}{2}, \frac{1}{4}$.

10. $\frac{3}{26}, \frac{11}{26}, \frac{33}{52}$.

11. (a) $\frac{4}{81}$, (b) $\frac{1}{15}$.

12. (a) $\frac{27}{5000}$, (b) $\frac{1}{140}$.

13. $\frac{2}{3}$.

14. $\frac{3}{4}$.

15. $\frac{1}{5}$.

16. $\frac{7}{12}$.

17. (a) 0·7, (b) 0·6, (c) 0·75.

18. (a) $\frac{25}{72}$, (b) $\frac{19}{36}$.

19. (a) $\frac{18}{35}$, (b) $\frac{5}{12}$.

20. $\frac{1}{2}, \frac{1}{9}$.

Exercise 14.6 (pp. 332–336)

1. (a) 1260, (b) 8232.
2. 30 240; (a) 840, (b) 10 080, (c) 13 440.
3. 63.
4. 20.
5. 280, 36.
6. (a) 243, (b) 21.
7. $K \cap R = \{\text{squares}\}$; $K \cap P = \{\text{rhombuses}\}$.
8. (a) true, (b) false, (c) false, (d) false.
9. 1200.
10. (a) $\frac{1}{7}$, (b) $\frac{3}{35}$.
11. (a) (i) $\frac{5}{18}$, (ii) $\frac{1}{3}$, (b) $\frac{12}{35}$, (c) 294.
12. (i) $\frac{7}{20}$, (ii) $\frac{4}{13}$.
13. (a) $\frac{6}{77}$, (b) 0, (c) $\frac{2}{15}$, (d) $\frac{2}{33}$.
14. $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}$.
15. (i) $\frac{5}{12}$, (ii) $\frac{8}{15}$, (iii) $\frac{1}{12}$.

16. $\frac{5}{8}$; (a) $\frac{55}{64}$, (b) $\frac{25\pi}{256}$. 17. $4{\cdot}12 \times 10^{-6}$.

18. (a) $\frac{1}{22}$, (b) $\frac{41}{55}$, (c) $\frac{3}{11}$, (d) $\frac{3}{44}$.

19. (a) $\frac{1}{3}, \frac{1}{2}$, (b) (i) $\frac{63}{64}$, (ii) $\frac{5}{16}$, (iii) $\frac{5}{72}$.

20. (a) $\frac{81}{256}$, (b) $\frac{1}{256}$, (c) $\frac{3}{128}$, (d) $\frac{3}{32}$.

21. (i) 0·3, (ii) 0·3, (iii) 0·5.

22. (a) $P(9) = \frac{25}{216}$, $P(10) = \frac{27}{216}$,

(b) $P(\text{at least one six}) = 1 - \left(\frac{5}{6}\right)^4 = 0{\cdot}518$;

$P(\text{at least one double six}) = 1 - \left(\frac{35}{36}\right)^{24} = 0{\cdot}491$.

(c) $P(\text{at least one six}) = 1 - \left(\frac{5}{6}\right)^6 = 0{\cdot}665$;

$P(\text{at least two sixes}) = 1 - \left(\frac{5}{6}\right)^{12} - 2\left(\frac{5}{6}\right)^{11} = 0{\cdot}619$.

23. (a) (i) $\frac{1}{4}$, (ii) $\frac{1}{4}$, (iii) $\frac{1}{17}$, (iv) $\frac{15}{34}$; (b) $\frac{13}{30}$;

(c) (i) $\frac{95}{253}$, (ii) $\frac{285}{506}$.

24. (a) $\frac{33}{16\,660}$, (b) $\frac{6}{4165}$; $\frac{11}{1081}$.

26. (a) $\frac{1}{221}$, (b) $\frac{32}{221}$, (c) $\frac{13}{17}$; $\frac{25}{33}$.

27. (a) (i) $\frac{3}{8}$, (ii) $\frac{1}{2}$, (iii) $\frac{1}{6}$; (b) 0·14.

28. (a) 0·01, (b) 0·87; (a) 0·9606, (b) 0·0847.

29. (a) $\frac{1}{36}$, (b) $\frac{5}{12}$; $\frac{73}{648}, \frac{25}{81}$.

30. (i) $\frac{2}{3}$, (ii) $\frac{1}{6}$, (iii) $\frac{5}{18}$, (iv) $\frac{1}{4}$, (v) $\frac{3}{5}$.

31. (i) $\frac{31}{120}$, (ii) $\frac{91}{240}$, (iii) 1; $\frac{2}{7}$.

32. (i) $P(A \cap B) = P(A).P(B)$. (ii) $P(A \cap C) = 0$; $\frac{13}{25}, \frac{1}{2}, \frac{9}{35}; \frac{3}{10}$.

33. (i) 0·22, (ii) 0·30, (iii) 0·44.

Exercise 15.1 (pp. 343–345)

5. (a) step diagram, (b) ogive.

6. One possibility is 32·5, 37·5, 42·5, ..., 67·5, 72·5; 88%.
7. 9/14, excluding the 10 late entrants (possibly transfers from schools in other areas).

Exercise 15.2 (pp. 349–350)

1. (a) 3, 3, $3\frac{1}{2}$. (b) no mode, 24, 20, (c) 20 and 21, 21, $20\frac{2}{3}$.
2. (a) 3, 3, 5, (b) no mode, 22, 20, (c) 20, $20\frac{1}{2}$, $20\frac{3}{5}$.
3. (a) 5, 5, 5, (b) 5, $5\frac{1}{14}$, 5, (c) 5, $5\frac{4}{7}$, $5\frac{1}{2}$.
4. 3, 3, 3·64. 5. 6·84.
6. (i) $52{\cdot}5 \leqslant x < 57{\cdot}5$, $54\frac{3}{8}$, 55, (ii) 56, 56, 54·175.
7. (i) 4·96, 5·14, (ii) 4·93, 4·95.
8. 26·6; (a) 27, (b) 27·2, (c) 27·1.

Exercise 15.3 (pp. 353–354)

1. (a) 7; $5\frac{1}{2}$, $8\frac{1}{2}$; $1\frac{1}{2}$, (b) 29; 27, 33; 3, (c) 2; −4, 10; 7.
2. (a) 7·25, 5·19, 2·28, (b) 31, 71·8, 8·47, (c) 3, 39·8, 6·31.
3. (a) (i) 5, 6, (ii) 1; (b) (i) 4·8, 5·7, (ii) 0·8; (c) (i) 5·3, 6·2, (ii) 0·8.
4. 6·31, 2·51.
5. (i) (a) 5, 7, 9, (b) 4·9, 6·9, 8·8; (ii) 2·45.
6. (i) 49·5, 59·5, (ii) 48·9, 60·3, (iii) 49, 60.
7. (i) 4 yrs 10 months, 5 yrs 1 month, (ii) 46·5 months^2.
8. I: 38 −, II: 28 − 37, III: 19 − 27, IV: 0 − 18.

Exercise 15.4 (p. 357)

1. 5·5, 6·25, 2·5; (a) 8·5, 6·25, 2·5, (b) 22, 100, 10, (c) 4·55, 0·0625, 0·25.
2. 73·4°F, 9°F. 3. 5, 1·5, 1·22; 5·5, 1·5, 1·22.
4. (i) 7·84, (ii) 7·98. 5. 27·2, 9·69; 1·5, 9·2.
6. Mathematics: $72\frac{1}{2}$, 77, $57\frac{1}{2}$; English: $60\frac{1}{2}$, $51\frac{1}{2}$, $87\frac{1}{2}$; Combined: 133, $128\frac{1}{2}$, 145.

Exercise 15.5 (pp. 358–361)

1. 86·6, 44·1; 188·1. 2. 6·5; 4·2, 9·0.
3. (i) 57 min, (ii) 71·5 min, (iii) 32%.
4. mean 55 kg, s.d. 3·2 kg. 5. 0, 1.
6. (i) 100·7, 14·5, (ii) 72.
7. (b) 8·95, 9·95, (c) 13·53, (d) 13·55.
8. (i) £28·87, (ii) £54·04, (iii) £30·47.
9. 52°, 7·5°; 53·2°, 10·7°.
10. 172·1 cm, 2·608 cm; 174·6 cm, 6·8 cm^2; 1·054 in^2.
11. £49, £12·85.
12. (a) 5, 3·6, (b) 5·4, 5·04, (c) $4{\cdot}5 < \bar{x} < 5{\cdot}0$, (d) $x_{10} \geqslant 8$ or $x_{10} \leqslant 2$.

13. (ii) 6·8299; 39·23, 6·6335.
14. (a) (i) 3·4, 2, (ii) 0·4, 0·276; (b) $3\frac{1}{3}$, 19·6%.

Exercise 16.1 (pp. 366–369)

1. 10. 2. $3\frac{1}{2}$. 3. 14; 20 p. 4. (a) 1, (b) 9/8.
5. $p(2) = 2/15$, $p(3) = 8/15$, $p(4) = 1/3$; 3·2. 6. 0·8 p loss.
7. $\frac{125}{216}, \frac{75}{216}, \frac{15}{216}, \frac{1}{216}$; 2 p loss. 8. (a) £122.50, (b) £112·672.
9. $p(x) = 1/6, x = 1, 2, 3, 4, 5$; $p(6) = 0$; $p(x) = 1/36, x = 7, 8, 9, 10, 11, 12$; $4\frac{1}{12}$; 6/17.
10. (i) 24/625, (ii) 24/625, (iii) 2. 11. 28·5, 721·05. 12. 2·53, 0·305.
13. $1\frac{1}{2}, \frac{3}{4}, \frac{1}{20}, \frac{9}{20}, \frac{9}{20}, \frac{1}{20}$; 60%.
14.

r	0	1	2	3	4
$P(Y = r)$	0·09	0·24	0·34	0·24	0·09

r	0	1	2	3	4
$P(Z = r)$	0·447	0·232	0·222	0·072	0·027

$E(Z) = 1$, $\mathrm{Var}(Z) = 1{\cdot}2$.

15. $4\frac{1}{3}, 2\frac{2}{9}; 8\frac{2}{3}, 8\frac{8}{9}$. 16. $\frac{1}{2}(n + 1)$, 0·42.
17. $2/n(n + 1)$; $\frac{1}{3}(2n + 1)$, $\frac{1}{18}(n - 1)(n + 2)$.
18. $x = 1, 2, 3, \ldots, 9$; $p(x) = \frac{9}{45}, \frac{8}{45}, \frac{7}{45}, \ldots, \frac{1}{45}$; $\frac{11}{3}, \frac{2}{3}\sqrt{11}$; 1;

$x = 1, 2, 3, \ldots$; $p(x) = \dfrac{10 - x}{45}$.

19. (a) $1 + 2x + 3x^2 + \ldots + rx^{r-1} + \ldots$, (b) $p(1 - p)^{r-1}$; $1/p$.
20. (i) $\frac{1}{3}$, (ii) $\frac{2}{9}$, (iii) $\frac{16}{243}; \frac{32}{243}$; 50 p; $\frac{16}{81}$; 0·177. 21. 3/16.

Exercise 16.2 (pp. 373–376)

1. (a) 0·279, (b) 0·457; 1·5, 1·25. 2. (a) 0·250, (b) 0·526; 2·5, 1·875.
3. 0·657; 7. 4. (a) 0·393, (*b*) 0·000 085 2; $n \geqslant 18$.
5. 24, 0·4; $(0{\cdot}6 + 0{\cdot}4t)^{24}$.
6. $(0{\cdot}55 + 0{\cdot}45t)^4$; (i) 4, (ii) $4(0{\cdot}55)(0{\cdot}45)^3$. 7. 0·228, 0·210.
8. 0·0464; 4, 2·4.
9. $\frac{4}{3}, \frac{356}{405}$;

No. of males	0	1	2	3	4
Expected no. of litters	160	320	240	80	10

10. 0·991, 0·977;

No. of ones	0	1	2	3	4	5	6
Expected no. of throws	335	402	201	54	8	1	0

11. (a) 0·315, (b) (i) 0·53, (ii) 5, 22, 37, 28, 8. 12. 26·4%, 62·4%.
13. (i) 0·201; (ii) (a) 0·0556, (b) 0·0367, (c) 0·6.
14. (i) 4·8, 0·98, (iii) 0·737, (iv) 0·388.
15. (a) $\frac{1}{27}$, (b) $\frac{2}{9}$, (c) $\frac{4}{9}$, (d) $\frac{8}{27}$, (e) 0·299, (f) 0·903, (g) 0·328.
16. (ii) 1/5. 17. $\dfrac{(n-r+1)p}{r(1-p)}$; (a) 8, (b) 2, 3.
18. (a) $\dfrac{135}{2048}$; (b) $\dfrac{459}{512}$; $\binom{n-1}{2}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^{n-3}$; 8, 9.
19. (a) $(r+1)/2^{r+2}$, (b) 2, (c) 4. 20. $t/(6-5t)$; 6, 30.

Exercise 16.3 (pp. 381–384)

1. (a) 1/2, 1/12, (b) 3, 4/3, (c) $k/2$, $k^2/12$, (d) $(a+b)/2$, $(b-a)^2/12$.
2. 4/27; 9/5, 2, 3/5. 3. (a) 2·4, (b) 20; 1/3, 0·178.
4. (i) $-3/16$, (ii) 19/80, (iv) 2. 5. 4, 8/15, 11/225; 0·541.
6. 3/8, 0, 2/5; (i) 85/512, (ii) 85/152. 7. (i) 12, 1, (ii) 0·0523.
8. (i) 2/5, (ii) 13/5, (iii) 3/2.
9. 0·6; $F(x) = 0$ for $x \leqslant 0$, $F(x) = 4x^3 - 3x^4$ for $0 \leqslant x \leqslant 1$, $F(x) = 1$ for $x \geqslant 1$.
10. 28, 12, 6, 2, 1, 1.
11. (i) 2/3, (ii) 4/3, $(\sqrt{14})/6$, (iii) $F(x) = 0$ for $x \leqslant 0$, $F(x) = \frac{1}{3}x^2$ for $0 \leqslant x \leqslant 1$, $F(x) = \frac{1}{6}(-3 + 6x - x^2)$ for $1 \leqslant x \leqslant 3$, $F(x) = 1$ for $x \geqslant 3$, (iv) 13/24.
12. (iii) $F(x) = -\frac{1}{12}x^3$ for $x \leqslant -1$, $F(x) = \frac{1}{2} + \frac{1}{2}x - \frac{1}{12}x^3$ for $-1 \leqslant x \leqslant 1$, $F(x) = 1 - \frac{1}{12}x^3$ for $x \geqslant 1$, (iv) $E(X) = 0$, $\mathrm{Var}(X) = 11/15$.
13. (ii) 1, 5/6, (iii) 1/4.
14. (i) $k = 2$, (ii) $f(x) = 2$ for $0 \leqslant x \leqslant 0{\cdot}5$, $f(x) = 0$ otherwise, (iii) 0·25, (iv) 0·144.
15. (i) 3/32, (ii) 11/16, (iii) $f(x) = 3(4 - x^2)/32$ for $-2 \leqslant x \leqslant 2$, $f(x) = 0$ otherwise, (iv) 0, 4/5.
16. $F(x) = 0$ for $x \leqslant 1$, $F(x) = \frac{1}{3}(x^2 - 1)$ for $1 \leqslant x \leqslant 2$, $F(x) = 1$ for $x \geqslant 2$; $G(y) = 0$ for $y \leqslant 1$, $G(y) = \frac{1}{3}(y^4 - 1)$ for $1 \leqslant y \leqslant \sqrt{2}$, $G(y) = 1$ for $y \geqslant \sqrt{2}$; $g(y) = 4y^3/3$ for $1 \leqslant y \leqslant \sqrt{2}$, $g(y) = 0$ otherwise.
17. $1/(2\sqrt{y})$; $(1/\sqrt{w}) - 1$.
18. $f(x) = 1/(b-a)$ for $a \leqslant x \leqslant b$, $f(x) = 0$ otherwise; $F(x) = 0$ for $x \leqslant a$, $F(x) = (x-a)/(b-a)$ for $a \leqslant x \leqslant b$, $F(x) = 1$ for $x \geqslant b$; $P(V \leqslant v) = 3 - 72/v$; (i) $f(v) = 72/v^2$, (ii) 0·4, (iii) 28·8 m.p.h.
19. (a) $a^2/18$, (b) $F(x) = 0$ for $x \leqslant a$, $F(x) = x^2/a^2$ for $0 \leqslant x \leqslant a$, $F(x) = 1$ for $x \geqslant a$, (c) (i) $1/(a+1)^2$, (ii) $2(1-y)$ for $0 \leqslant y \leqslant 1$, 0 otherwise.
20. $g(y) = 1$ for $0 \leqslant y \leqslant 1$, $g(y) = 0$ otherwise.

Exercise 16.4 (pp. 389–392)

1. (a) 0·6179, (b) 0·1151, (c) 0·97725, (d) 0·01287, (e) 0·1760, (f) 0·5558.
2. (a) 0·9378, (b) 0·1944, (c) 0·9715, (d) 0·02239, (e) 0·2065, (f) 0·2976.
3. (a) 0·9554, (b) 0·2119, (c) 0·6612.
4. (a) 0·3085, (b) 0·5793, (c) 0·0796.
5. (a) 0·9487, (b) 0·9750, (c) 0·04412
6. (a) 0·44, (b) 0·583, (c) −0·878.
7. (a) 0·524, (b) 2·326, (c) 3·291.
8. (a) 0·3%, (b) 6·6%, (c) 44%, (d) 1·2%.
9. 11·51%, 81%, 7·49%; £6528.64.
10. (i) 4·8%, (ii) 97·7%; 1111·5 to 1188·5.
11. (a) 37·82%, (b) 125·5 hrs to 194·5 hrs, (c) 0·405.
12. (a) 4 men, (b) 0·0446, (c) 1·746 m.
13. (a) 3, 62·3, (b) 15, 9·24, (c) 84·5, 111, (d) 8·31, 1·74.
14. 0·0122, 0·0102. 15. 104, 38·8.
16. (a) 0·205, 0·205, (b) 0·120, 0·119, (c) 0·0419, 0·0416, (d) 0·190, 0·179, (e) 0·0985, 0·0932, (f) 0·0493, 0·0474.
17. 0·0287. 18. 0·210. 19. (i) 0·026, (ii) 0·932, (iii) 0·923.
20. (a) 4320/15625; (b) 7120/15625; (a) 0·026, (b) 0·853.
21. (a) Negligible, (b) negligible, (c) 0·806.
22. (i) 0·648, (ii) 92·65%, (iii) 0·654.

Exercise 16.5 (pp. 395–397)

1. (a) 0·156, (b) 0·663. 2. (a) 0·353, (b) 0·285.
3. 0·181; $e^{-0.02n}$; 5. 4. 0·337; 1536.

5.

r	0	1	2	3	4	5	6	7	8	9	10
(a)				0·001	0·005	0·015	0·037	0·074	0·120	0·160	0·176
(b)			0·002	0·008	0·019	0·038	0·063	0·090	0·113	0·125	0·125
(c)				0·001	0·005	0·015	0·037	0·073	0·119	0·160	0·177
(d)	0·001	0·002	0·005	0·011	0·021	0·036	0·057	0·080	0·103	0·120	0·126

r	11	12	13	14	15	16	17	18	19	20	21
(a)	0·160	0·120	0·074	0·037	0·015	0·005	0·001				
(b)	0·114	0·095	0·073	0·052	0·035	0·022	0·013	0·007	0·004	0·002	0·001
(c)	0·160	0·119	0·073	0·037	0·015	0·005	0·001				
(d)	0·120	0·103	0·080	0·057	0·036	0·021	0·011	0·005	0·002	0·001	

6.

r	0	1	2	3	4	5	6	7	8	9
(a)	0·122	0·270	0·285	0·190	0·090	0·032	0·009	0·002		
(b)	0·135	0·271	0·271	0·180	0·090	0·036	0·012	0·003	0·001	
(c)	0·132	0·223	0·291	0·223	0·101	0·027	0·004			
(d)	0·144	0·217	0·276	0·217	0·106	0·032	0·006	0·001		

7.

r	0	1	2	3	4	5	6	7	8	9
(a)	0·005	0·029	0·078	0·139	0·181	0·185	0·154	0·108	0·064	0·033
(b)	0·007	0·034	0·084	0·140	0·175	0·175	0·146	0·104	0·065	0·036
(c)	0·017	0·033	0·070	0·121	0·167	0·186	0·167	0·121	0·070	0·033
(d)	0·022	0·037	0·073	0·119	0·160	0·177	0·160	0·119	0·073	0·037

r	10	11	12	13
(a)	0·015	0·006	0·002	0·001
(b)	0·018	0·008	0·003	0·001
(c)	0·012	0·004	0·001	
(d)	0·015	0·005	0·001	

8.

r	0	1	2	3	4	5	6	7	8	9
(a)		0·002	0·009	0·025	0·054	0·090	0·123	0·144	0·146	0·129
(b)		0·003	0·011	0·029	0·057	0·092	0·122	0·140	0·140	0·124
(c)	0·003	0·005	0·013	0·027	0·050	0·080	0·112	0·137	0·146	0·137
(d)	0·004	0·007	0·015	0·030	0·052	0·080	0·110	0·132	0·140	0·132

r	10	11	12	13	14	15	16	17	18
(a)	0·102	0·073	0·047	0·028	0·015	0·007	0·003	0·001	0·001
(b)	0·099	0·072	0·048	0·030	0·017	0·009	0·005	0·002	0·001
(c)	0·112	0·080	0·050	0·027	0·013	0·005	0·002	0·001	
(d)	0·110	0·080	0·052	0·030	0·015	0·007	0·003	0·001	

9. (a) 0·368, (b)0·456. 10. (a) 0·258, (b) 0·567.
11. (a) Poisson approximation: 0·442,
(b) Poisson approximation: 0·353,
(c) Normal approximation: 0·119.
14. (i) 0·19, (ii) 0·21, (iii) 0·012.
15. (a) (i) 1·04, (ii) 53, 55, 29, 10, 3; (b) 0·121.
16. 0·647, 0·185. 17. (i) 0·55, (ii) 0·18.
18. £266C, £415C. 19. 0·460; 33.
20. 0·323; 172 to 228.

Exercise 16.6 (pp. 403–407)

1. 3·6, 14·5, 0·35, 28, 5·95. 2. 0·4, 1, 2·2; 0·84, 1·2.
3. $1\frac{2}{3}$, 3, $5\frac{4}{5}$; $20\frac{11}{15}$, $\frac{2}{9}$.
4. $\frac{1}{2}$, $\frac{3}{10}$, $\frac{24}{35}$; $-\frac{1}{5}$, $\frac{73}{2450} \approx 0{\cdot}0298$.
5. (a) 6, 2·5, (b) 4, 6, (c) 27, 40, (d) -3, 24,
(e) 10, 8·5, (f) 2, 8·5, (g) 14, 28·5, (h) 24, 64.
6. (a) 31, 234, (b) 109, 50, 159. 7. 30 g, 0·3 g. 8. 200 g, 6·71 g.
9. (a) 2·1, 0·49; 0·8, 0·36, (b) no; independent,

(c)

xy	0	1	2	3	4	6
$p(xy)$	0·3	0·12	0·32	0·18	0·05	0·03

; yes,

(d)

$x + y$	1	2	3	4	5
$p(x + y)$	0·06	0·27	0·41	0·23	0·03

; yes.

10. (i) (a) 1·2, 0·56; 1·7, 0·81, (b) all values; not independent,

(c)

xy	0	1	2	3	4	6
$p(xy)$	0·27	0	0·45	0·1	0·1	0·08

; no,

(d)

$x + y$	0	1	2	3	4	5
$p(x + y)$	0·03	0·15	0·07	0·47	0·2	0·08

; no.

(ii) (a) 1, 0·6; 2, 0·6, (b) all values, not independent,

(c)

xy	0	1	2	3	4	6
$p(xy)$	0·3	0·2	0·05	0·2	0·2	0·05

; yes,

(d)

$x + y$	1	2	3	4	5
$p(x + y)$	0·05	0·4	0·1	0·4	0·05

; yes.

11. 1, 2, 4, 8. 12. £3.33.
13. $\frac{1}{210}, \frac{4}{35}, \frac{3}{7}, \frac{8}{21}, \frac{1}{14}$; 2·4, 0·8; 1·6, 0·8; 4, 1·13.
14. (i) 0·9, 1·2,

(ii)

z	-2	-1	0	1	2	3	4
$p(z)$	0·1	0·1	0·3	0·2	0·2	0·1	0

,

(iii) 0·2.

15. (i) 0·011, (ii) 0·036, (iii) 0·27. 16. 0·12 mm, 0·0583 mm; 1·98%
17. (a) 0·673, (b) 0·138. 18. (i) 0·089, (ii) 0·661.
19. (i) 0·841, (ii) 0·500, (iii) 0·421; 0·994.
20. (i) 0·823, (ii) 0·059; 1/4, 1/6. 21. $\frac{1}{2}r\pi, \frac{1}{2}r\pi$; $4r/\pi$; 1/3.
22. (i) $\mu_x^2 + \sigma_x^2$; (ii) $\mu_x\mu_y$; $\mu_x^2\sigma_y^2 + \mu_y^2\sigma_x^2 + \sigma_x^2\sigma_y^2$; (iii) $\mu_y(\mu_x^2 + \sigma_x^2)$.
23. $\dfrac{\lambda^r\mu^{n-r}}{r!(n-r)!}e^{-(\lambda+\mu)}$; $\displaystyle\sum_{r=0}^{n}\frac{\lambda^r\mu^{n-r}}{r!(n-r)!}e^{-(\lambda+\mu)}$. 24. 0·121.
25. (b) $(1 - p + pt)^m$, $(1 - p + pt)^n$; $(1 - p + pt)^{m+n}$.

Exercise 16.7 (pp. 407–414)

1. $(n + 1)/2$, $(n^2 - 1)/12$; 1/2; (a) 2/105, (b) 16.

2. £16. 3. (a) 0·323, (b) $£1\frac{32}{65} \approx £1.49$.
4. 1/15; 2/3, 34/45; 1/75; 4/3, 68/45.

5. (i) $P(N = r) = \frac{1}{k}(1 \leqslant r \leqslant k)$, $E(N) = \frac{1}{2}(k + 1)$,

(ii) $P(N = r) = \frac{1}{k}\left(\frac{k-1}{k}\right)^{r-1}$ $(r \geqslant 1)$, $E(N) = k$; first strategy.

6. (i) 1/5, (ii) 8/9, (iii) 10, (iv) 8. 7. 0·0512.
8. (i) 0·168, (ii) 0·360, (iii) 0·309, (iv) 0·132, (v) 0·028, (vi) 0·002; 0·947.

9.

No. of defectives	0	1	2	3	4	5	6
Expected frequency	16	42	45	26	8	1	0

10. $\frac{1}{p}, \frac{1-p}{p^2}; \frac{2t^2}{(5-3t)(5-4t)}; \frac{194}{625} = 0{\cdot}3104$.

11. (i) 2, (ii) $1\frac{1}{2}$, (iii) $1\frac{5}{12}$,
(iv) $F(x) = 0$ for $x \leqslant 0$, $F(x) = x - \frac{1}{2}x^2$ for $0 \leqslant x \leqslant 1$, $F(x) = \frac{1}{2}$ for $1 \leqslant x \leqslant 2$,
$F(x) = \frac{1}{2}x^2 - 2x + \frac{5}{2}$ for $2 \leqslant x \leqslant 3$, $F(x) = 1$ for $x \geqslant 3$, (v) $\frac{3}{4}$.
12. (ii) $\sqrt{(3/5)}$. 13. 1, 1/6; 3/4; 5/6.
14. (a) 3/250, $6\frac{7}{8}$, (b) (1) £3.20, (ii) 0·792.
15. (i) 1/2, −1/4, (ii) 7/16,
(iii) $f(x) = \frac{1}{2}x$ for $0 \leqslant x \leqslant 1$, $f(x) = \frac{1}{2}$ for $1 \leqslant x \leqslant 2$,
$f(x) = \frac{3}{2} - \frac{1}{2}x$ for $2 \leqslant x \leqslant 3$, $f(x) = 0$ otherwise; 3/2; 5/12.

16. (a) $1 - e^{-\lambda x}$, (b) $\frac{1}{\lambda}, \frac{1}{\lambda^2}$, (c) $\frac{1}{\lambda}\ln 2$ (d) $\frac{1}{\lambda}\ln\left(\frac{20}{19}\right)$.

17. (i) 0·202, (ii) 0·410.
18. (i) 0·880, (ii) 9·69, 0·2339, (iii) 0·180.
19. £9360, 19·6%; (a) £20 000, £9360; (b) £19 800, £10 296.
20. (i) 0·60, (ii) 0·20, (iii) 0·95, (iv) 0·50.
21. (i) 0·227, (ii) 54·7, (iii) 78·7 cm^2, (iv) 0·579.
22. 1·514 mm, 0·051 mm; 3·028 mm, 0·072 mm; 0·895; 3·214 mm.
23. (i) 0·345, (ii) 0·692; 0·0033; 0·304.

24. (a) (i) 0·5, (ii)

No. of accidents	0	1	2	3	4	5
Expected no. of days	121	61	15	3	0	0

(b) Binomial model: 0·932, Poisson approximation: 0·929.

25. (i) 9, (ii) 26, (iii) 43; 8.
26. (a) (i) 0·0581, (ii) 0·0519; (b) $(0{\cdot}96)^n$, 74; (c) 0·955.
27. (i) 0·135, (ii) 0·271, (iii) 0·0290, (iv) 2, (v) 8·19.
28. (i) 0·419, (ii) 0·676.
29. (a) (i) 0·0952, (ii) 0·0905; (b) 0·853; (c) 0·632; 63·2, 4·82.
30. 1/12; 0, $n/12$; 16. 31. (b) 2·2.

32. (i) $F(x) = 0$ for $x \leqslant 1$, $F(x) = 1 - x^{-3}$ for $x \geqslant 1$;
(ii) 3/2, 3/4; p.d.f. of Y: $g(y) = 3y^2$ for $0 \leqslant y \leqslant 1$, $g(y) = 0$ otherwise.

33. (ii) 8/15,
(iii) $F(x) = 0$ for $x \leqslant 0$, $F(x) = x^2(2 - x^2)$ for $0 \leqslant x \leqslant 1$, $F(x) = 1$ for $x \geqslant 1$;
(iv) $g(y) = 8y^3(1 - y^4)$ for $0 \leqslant y \leqslant 1$, $g(y) = 0$ otherwise.

34. (a) (i) 5/2, (ii) 7,
(iii) $F(x) = 0$ for $x \leqslant 1$, $F(x) = \frac{1}{3}(x^2 - 1)$ for $1 \leqslant x \leqslant 2$, $F(x) = 1$ for $x \geqslant 2$;
(b) (i) 5/2, 3/4, (ii) interval from 1 to 4.

Index

Acceleration, 46, 69
  as a function of displacement, 273
  as a function of velocity, 276
  as a vector, 31, 62, 229
  constant, 50, 63
  due to gravity, 53, 63, 67
Accelerations, related, 81
Addition formulae, 415
Addition law of probability, 323
Amplitude (of S.H.M.), 283
Angle between two vectors, 235
Angle of friction, 97
Angular acceleration, 128
Angular velocity, 128
Arc, length of circular, 415
Area of a sector, 415
Area of a triangle, 415
Arithmetic progression, 415
Arrangements, 308
Array of statistical data, 346
Assumed mean, 355
Average speed, 31
Average velocity, 31

Banked roads, 136
Bernouilli, Jacques, 369
Bernouilli trial, 369
Binomial distribution, 369, 417
  normal approximation, 388
  Poisson approximation, 394
  probability generating function, 373
  sum of binomial variables, 403
Binomial theorem, 415

Cardinal number, 313
Cartesian components, 5, 16
Cartesian coordinates in three dimensions, 6
Central tendency, measures of, 345
Centre of gravity, 157, 162
  composite bodies, 160
  derivation of standard results, 164
  set of particles, 158
  triangular lamina, 157
Centre of mass, 162
  standard results, 416
Centroid, 162
  plane area, 172, 174
  volume of revolution, 174
Change of variable (statistics), 354
Circular motion, 128, 132, 229, 292, 295
Class boundaries, 341
Class limits, 341
Closest approach, 40
Coding of statistical data, 356
Coefficient of friction, 88
Combinations, 309
Complementary event, 318
Complement of a set, 312
Component of a force, 18
Components, Cartesian, 5, 16
Component vectors, 5, 233
Composite bodies, 160, 182
Conditional probability, 328
Conical pendulum, 134
Connected particles, 76
Conservation of energy, 111, 116, 255
Conservation of linear momentum, 250, 263
Conservative force, 112
Constant acceleration formulae, 51, 63, 416
Contingency table, 329
Continuity correction, 389
Continuous probability distribution, 377
Continuous variable, 339, 362
Coplanar forces, 149
  see also Equilibrium conditions

Couple, 145, 147
Covariance, 401, 417
Cumulative distribution function, 379
  standard normal variable, 386, 418
Cumulative frequency, 339, 340, 343

Density function, *see* Probability density function
Differentiation, standard formulae, 416
Differentiation of position vectors, 62
Direct impact, 261, 263
Direction vector, 223
Discrete probability distribution, 363
Discrete variable, 339, 362
Disjoint sets, 313
Dispersion, measures of, 350
Displacement, as a vector, 2, 31
  relative, 38
  using a definite integral, 48
Displacement–time graph, 46
Distance between two points, 6
Distribution function, (cumulative), 379
  standard normal variable, 386, 418
Dot product, 233

Elastic impact, 260
Elastic potential energy, 116
Elastic string, 99
  energy stored in, 117, 416
  in S.H.M., 287
Empty set, 311
Energy, 108, 110, 116
  conservation of, 111, 116
Energy equations, 237, 254, 255
Equilibrium, 11
  limiting, 94
  of a particle, 22, 94
Equilibrium conditions, 180, 197, 201
  three forces, 188, 190
  two forces, 181
Equivalent statements, 311
Equivalent systems of forces, 151
Event, 317
Expectation, continuous variable, 377, 397, 417
  discrete variable, 364, 397, 417
  function of a random variable, 397, 417
  linear combination of random variables, 400, 417
  linear function of a random variable, 398, 417
  product of independent random variables, 400, 417
  sum of random variables, 400
  using probability generating function, 373
Expectation algebra, 397, 417
Expected frequency, 372
Expected value, 364

Factorial notation, 308
Finite series, 415
First moment of area, 172
Forces, 11
  polygon of, 23
  triangle of, 24
Frameworks, 211
Frequency, 339, 340
  density, 340
  distribution, 364
Frequency of S.H.M., 283
Friction, 12, 88, 112
  angle of, 97
  coefficient of, 88
  cone of, 97
  laws of, 88
  limiting, 88

Geometric distribution, 364
Geometric progression, 415
Graphical representation of data, 337
Gravity, force of, 12
  acceleration due to, 53, 63, 67
Grouping of data, 341

Histogram, 340
Hooke, Robert, 99
Hooke's law, 99, 416

Impact, 245, 250
  direct, 261, 263
  indirect, oblique, 261, 262
Implication signs, 311
Impulse–momentum equation, 244, 254
Impulse of a force, 243
Impulsive tension, 246
Independent events, 324
Independent random variables, 400
Indirect impact, 261, 262
Inelastic impact, 261
Inextensible string, 11
Integration, standard forms, 416
Interception, 38
Intersection of sets, 312

Joint probability distribution, 399, 417
Joule, unit of work, 108

Karnaugh map, 314, 329
Kinetic energy, 108, 110

Lami, Bernard, 25

Lami's theorem, 25, 191
Lamina, 11
Least force problems, 95, 97
Length of circular arc, 415
Light frameworks, 211
Light rods, 211
Light string, 11, 12
Limiting equilibrium, 94
Limiting friction, 88
Linear combinations, of normal variables, 402
of random variables, 400, 401, 417
Linear interpolation, 347, 386
Line of action of resultant, 143, 149
parallel forces, 144, 149
vector equation, 224
Location, measures of, 345

Magnitude of a vector, 1, 5
Marginal distribution, 112
Mass, 12
Mathematical model, 10
Mean, 346
binomial distribution, 371, 417
continuous probability distribution, 377, 397
discrete probability distribution, 364, 397
frequency distribution, 364
grouped data, 348
Poisson distribution, 393, 417
using probability generating function, 373
see also Expectation
Mean absolute deviation, 351
Mechanical energy, 110
conservation of, 111, 116
Median of a distribution, 346, 378
grouped data, 347
Mid-point, position vector of, 6
Modal class, 347
Mode, 346, 378
grouped data, 347
Models, mathematical, 10
Modulus of elasticity, 99
Moment, of a couple, 147
of a force, 145
of an area, 172
Moments, principle of, 147
Momentum, 69, 244
conservation of, 250, 263
continuous change in, 245
Mutually exclusive events, 323

Natural length, 99
Newton, Sir Isaac, 68
Newton, unit of force, 13, 69
Newton's experimental law, 261
Newton's law of restitution, 261, 263
Newton's laws of motion, 68, 70
Newton's second law, vector form, 230
Non-conservative force, 112
Normal approximation,
to binomial distribution, 388
to Poisson distribution, 395
Normal curve, 385
Normal distribution, 384
as a limit of the binomial distribution, 388
linear combination of normal variables, 402
sum and difference of normal variables, 402
table of percentage points, 419
Normal distribution function, 386
table of values, 418
Normal reaction, 12, 88
Null set, 311
Null vector, 1

Oblique impact, 261, 262
Ogive, 340
Oscillations, simple harmonic, 282, 287

Parallel forces, 144, 149, 180
Parallelogram law, 3, 14
Particle, 11
Pendulum, 292
Percentage points of the normal distribution, 387, 419
Percentiles, 350
Period of S.H.M., 283
Permutations, 308
Perpendicular vectors, 233, 235
Poisson, Siméon Denis, 392
Poisson approximation to the binomial distribution, 394
Poisson distribution, 392, 417
as a limit of the binomial distribution, 392
normal approximation, 395
probability generating function, 396
sum of Poisson variables, 403
Polygon of forces, 23
Position vector, 6
differentiation of, 62
Possibility space, 317
Potential energy, 110
elastic, 116
gravitational, 110
Power, 118
vector approach, 238

Probability, 317
  conditional, 328
Probability mapping, 320
Probability space, 320
Probability density function, 377
  normal variable, 384
  standard normal variable, 385
Probability distributions, 417
  continuous, 377
  discrete, 363
  joint, 399
  marginal, 400
Probability function, 363
Probability generating function, 373, 396
Probability model, 362
Product law (probability), 324, 328
Progressions, arithmetic and geometric, 415
Projectile, 56
  range, 58
  time of flight, 58
  trajectory, 57
  vector equations, 63
Projection of a vector, 233
Proper subset, 311
Pulley systems, 81

Quartiles, 350

Random sample, 338
Random trial, 362
Random variables, 362
  independent, 400
  see also Expectation, Variance
Range of a distribution, 350
Ratio theorem, 226
Reaction, normal, 12, 87
  total, 12, 87, 96
Rectangular distribution, 363, 378
Relative displacement, 38
Relative frequency, 364
Relative motion, 13, 228
Relative velocity, 35, 228
Resolution of, forces, 18
  vectors, 5
  velocities, 33
Resolved part, 19
Restitution, coefficient of, 261
  Newton's law of, 261, 263
Resultant, displacement, 2
  force, 14, 151
  line of action, 143, 149, 224
  using ratio theorem, 226
  velocities and accelerations, 33
Rigid body, 11
Rough surface, 12
  equilibrium on, 94
  motion on, 91

Sample space, 317, 362
  diagram, 319
Scalar product, 233
Scalar quantity, 1
Scaling of marks, 357
Seconds pendulum, 293
Sector, area of, 415
Selections, 309
Semi-interquartile range, 350
Series, 415
Sets, 311
Simple harmonic motion, 282
  forces producing S.H.M., 287
  standard formulae, 283
Simple pendulum, 292
SI units, 13
Smoothly hinged rods, 206
Smooth surface, 11, 12, 180
Speed, 31
Spread, measures of, 350
Springs, 99, 287
Standard deviation, 352
Standardisation of marks, 357
Standard normal distribution, 385
Statistical tables, 418
Step function, 339
Straight line, vector equation, 223
Straight line motion, 46, 50
Subset, 311
Sum and product laws, 323, 324, 328
Systems, of coplanar forces, 149
  of forces in vector form, 225
  of rigid bodies, 205

Tables, statistical, 418
Tension, 12
  in a light rod, 212
Three-dimensional coordinate system, 6
Thrust in a light rod, 212
Toppling and sliding, 183, 185
Torque, 145
Total reaction, 12, 88, 96
Trajectory of a projectile, 56
Tree diagram, 319, 329
Triangle, area of, 415
Triangle law, 3, 14
Triangle of forces, 24
Trigonometric formulae, 415

Uniform acceleration, 51, 63
Uniform distribution, 363, 378
Uniform motion in a circle, 128

Union of sets, 312
Units, SI, 13
Unit vector, 5
Universal set, 311

Variables, continuous, 339
  discrete, 339
Variance, 352, 364, 417
  binomial distribution, 371
  continuous probability distribution, 377, 397
  discrete probability distribution, 364, 397
  for grouped data, 353
  in terms of expectation, 399
  linear combination of random variables, 401
  linear function of a random variable, 399
  Poisson distribution, 393
  sum and differences of random variables, 401
Variation, measures of, 350
Vector equation of a line, 223
Vectors, 1
  addition, 3
  angle between, 235
  differentiation of, 62
  polygon, 15, 33
  position, 6
  scalar product, 233
  subtraction, 3
  triangle, 14, 35
  unit, 5
Velocity, 46
  as a vector, 31, 62, 228
  relative, 35, 228
Velocity–time graph, 47
  for constant acceleration, 51
Venn, John, 312
Venn diagrams, 312
Vertical circle, motion in a, 295

Watt, unit of power, 118
Weight, 12, 72, 156
Work done, by a constant force, 107
  by a variable force, 115
  in extending an elastic string, 115
  using vector methods, 235
Work–energy equations, 108, 109, 112, 113, 237, 254

RICHMOND UPON THAMES
COLLEGE

Advanced Mathematics: An Applied Course provides a complete course in applied mathematics for A level courses which include both pure and applied mathematics. It will also be particularly useful for students studying mechanics as part of a course in engineering.

Topics are presented in a logical sequence and are arranged in short chapters to allow flexibility. The text contains many worked examples and numerous exercises providing both routine practice and opportunities to develop understanding.

Advanced Mathematics: A Pure Course (ISBN 0 7135 2821 4) provides the companion course in pure mathematics.

Advanced Mathematics: Book 1 (ISBN 0 7135 1272 5) and *Advanced Mathematics: Book 2* (ISBN 0 7135 1322 5) integrate the pure and applied chapters from these two books to form a *combined* course in pure mathematics, statistics, and mechanics which includes chapters on numerical methods and probability.

ISBN 0-7135-2820-6

9 780713 528206

UNWIN HYMAN

A COMBINED COURSE IN
PURE MATHEMATICS, STATISTICS
& MECHANICS

ADVANCED MATHEMATICS

BOOK 2

PERKINS & PERKINS

ADVANCED MATHEMATICS

BOOK 2

Martin and Patricia Perkins

Bell & Hyman
London

First published in 1983 by
Bell & Hyman Limited
Denmark House
37–39 Queen Elizabeth Street
London SE1 2QB

© Martin and Patricia Perkins 1983

All rights reserved. No part of this publication may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photocopying, recording or otherwise, without the prior permission of Bell & Hyman Limited.

British Library Cataloguing in Publication Data

Perkins, Martin
Advanced mathematics.
Book 2.
1. Mathematics—1961-
I. Title II. Perkins, Pat
510 QA39.2

ISBN 0-7135-1322-5

Typeset by Polyglot Pte Ltd, Singapore
Printed in Great Britain at the
University Press, Cambridge

Contents

Preface vii

List of symbols viii

1 Mathematical reasoning 1
- 1.1 Deductive and inductive processes 1
- 1.2 Implication and equivalence 4
- 1.3 Types of proof 7
- 1.4 Proof by induction 9

2 Some functions and their properties 17
- 2.1 Symmetry, continuity and differentiability 17
- 2.2 Some graphs with related equations 22
- 2.3 Inverse functions 26
- 2.4 Polynomials; remainders and factors 30
- 2.5 Graphs of rational functions 34
- 2.6 Partial fractions 39

3 Exponential and logarithmic functions 49
- 3.1 Exponential functions 49
- 3.2 Logarithmic functions 54
- 3.3 $\ln x$ defined in integral form 58
- 3.4 Integration using logarithms 61
- 3.5 Introduction to hyperbolic functions 65

4 Methods of integration 69
- 4.1 Standard forms 69
- 4.2 Use of partial fractions 73
- 4.3 Integration by substitution 77
- 4.4 Integration by parts 83
- 4.5 Use of inverse trigonometric functions 87
- 4.6 Properties of the definite integral 90

5 Coplanar forces; centre of gravity 99
5.1 Parallel forces, moments and couples 99
5.2 Systems of coplanar forces 107
5.3 Centre of gravity; centre of mass 112
5.4 Standard results by integration 120
5.5 Centroids of areas and volumes 128

6 Equilibrium of rigid bodies 136
6.1 Parallel forces in equilibrium 136
6.2 Three force problems 144
6.3 General conditions for equilibrium 153
6.4 Rigid body systems 161
6.5 Light frameworks 167

7 Vectors and scalar product 179
7.1 Properties of vectors 179
7.2 Equations of lines and planes 183
7.3 Forces and their lines of action 189
7.4 Use of vectors in kinematics and dynamics 193
7.5 Scalar product 198
7.6 Further lines and planes 203
7.7 The scalar product in mechanics 210

8 Impulse and momentum 221
8.1 The impulse of a force 221
8.2 Conservation of momentum 228
8.3 Work, energy and momentum 231
8.4 Newton's law of restitution 238

9 Series and expansions 251
9.1 Summation of finite series 251
9.2 Maclaurin's theorem 256
9.3 The binomial series 258
9.4 Exponential and logarithmic series 264
9.5 The expansions of $\sin x$ and $\cos x$ 269

10 Some numerical methods 273
10.1 Numerical integration 273
10.2 Approximate solutions to equations 281
10.3 The Newton-Raphson method 287
10.4 Other iterative processes 291

11 Differential equations 303
11.1 Differential equations and their solution curves 303
11.2 Elementary methods of solution 307
11.3 Equations with separable variables 313
11.4 Formulation of differential equations 318
11.5 The equations $d^2y/dx^2 \pm n^2y = \text{constant}$ 321

12 Further particle dynamics 332
12.1 Acceleration as a function of displacement 332
12.2 Acceleration as a function of velocity 335
12.3 Simple harmonic motion 341
12.4 Forces producing simple harmonic motion 346
12.5 Motion in a vertical circle 354

13 Coordinate geometry 367
13.1 Angles, distances and areas 367
13.2 Further work on circles 372
13.3 Loci using parametric forms 376
13.4 Conic sections 382

14 Complex numbers 391
14.1 Algebraic form 391
14.2 Roots of polynomial equations 395
14.3 Geometrical representation 399
14.4 Modulus-argument form 403
14.5 Loci in the Argand diagram 409
14.6 De Moivre's theorem for an integral index 417

15 Probability distributions 427
15.1 Discrete probability distributions 427
15.2 The binomial distribution 434
15.3 Continuous probability distributions 442
15.4 The normal distribution 449
15.5 The Poisson distribution 457
15.6 The algebra of expectation and variance 462

16 Samples and significance 480
16.1 Sampling distributions 480
16.2 Unbiased estimators 486
16.3 Confidence intervals 491
16.4 Introduction to hypothesis testing 496

17 Correlation and regression 509
17.1 Bivariate distributions 509
17.2 Linear correlation 512
17.3 Rank correlation 518
17.4 Regression lines 523

18 Matrices and determinants 533
18.1 Linear transformations in three dimensions 533
18.2 3×3 matrices and determinants 540
18.3 Inverse matrices: adjoint method 546
18.4 Sets of linear equations 548
18.5 Inverse matrices: row reduction method 555

19 Binary operations and groups 564
19.1 Definition of a group 564
19.2 The integers modulo n 571
19.3 Symmetries and permutations 573
19.4 Properties of groups 578
19.5 Isomorphisms 584

Formulae for reference 594

Statistical tables 600

Answers 602

Index 658

Preface

This is the second of two books designed to cover most Advanced Level single subject Mathematics syllabuses. The two volumes together provide a thorough treatment of the Pure Mathematics and Mechanics required at this level, together with a choice of further options including probability and statistics, numerical methods, matrices and group theory. We hope that the inclusion of these additional topics will enable students and teachers to plan a course of study to suit individual syllabus requirements. In allowing this opportunity for choice we have covered various topics which appear on Further Mathematics syllabuses. This means that Book 2 could be used as a starting point for some of this more advanced work.

We are most grateful for permission to use questions from past G.C.E. Advanced Level examinations. These are acknowledged as follows:

University of London University Entrance and Schools Examination Council (L)
The Associated Examining Board (AEB)
Joint Matriculation Board (JMB)
University of Cambridge Local Examinations Syndicate (C)
Oxford and Cambridge Schools Examination Board (O&C)
Oxford Delegacy of Local Examinations (O)
Southern Universities' Joint Board for School Examinations (SU)
Welsh Joint Education Committee (W)

We must again thank the staff of Messrs. Bell and Hyman, especially Caroline Paines and Antonia Murphy, for both their enthusiasm and their patience. We have also been helped and encouraged by many friends and colleagues.

M.L.P.
P.P.

List of symbols

$\mathbb{Z}$	set of integers
$\mathbb{Z}^+$	set of positive integers
$\mathbb{Z}_n$	set of integers modulo n, $\{0, 1, 2, \ldots, n-1\}$
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	set of real numbers
$\mathbb{R}^+$	set of positive real numbers
$\mathbb{R}_0^+$	set of positive real numbers and zero
$\mathbb{R}^2$	set of ordered pairs of real numbers
$\mathbb{C}$	set of complex numbers
$[a, b]$	the closed interval $\{x \in \mathbb{R}: a \leqslant x \leqslant b\}$
(a, b)	the open interval $\{x \in \mathbb{R}: a < x < b\}$
$p \Rightarrow q$	p implies q; if p then q
$p \Leftarrow q$	p is implied by q; if q then p
$p \Leftrightarrow q$	p is equivalent to q
$\sim p$	not p
$\sum_{r=1}^{n} f(r)$	$f(1) + f(2) + \ldots + f(n)$
$\binom{n}{r}$	the binomial coefficient: ${}_nC_r = \dfrac{n!}{r!(n-r)!}$ for $n \in \mathbb{Z}^+$, $\dfrac{n(n-1)\ldots(n-r+1)}{r!}$ for $n \in \mathbb{Q}$
$\lvert x \rvert$	the modulus of x
$[x]$	the integral part of x
e^x, $\exp x$	exponential function of x
$\log_a x$	logarithm to the base a of x
$\ln x$	the natural logarithm of x, $\log_e x$
$\lg x$	the common logarithm of x, $\log_{10} x$

$\sin^{-1}$	inverse function of sin with range $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$
$\cos^{-1}$	inverse function of cos with range $[0, \pi]$
$\tan^{-1}$	inverse function of tan with range $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$
δx	small increment of x
$f'(x), f''(x), f'''(x)$	first, second and third derivatives of $f(x)$ with respect to x
$f^{(r)}(x)$	rth derivative of $f(x)$ with respect to x for $r > 3$
$\dot{x}, \ddot{x}, \ldots$	first, second, ... derivatives of x with respect to time
i	square root of -1
z	complex number, $z = x + iy = r(\cos\theta + i\sin\theta)$
$\mathrm{Re}(z)$	real part of z, $\mathrm{Re}(z) = x = r\cos\theta$
$\mathrm{Im}(z)$	imaginary part of z, $\mathrm{Im}(z) = y = r\sin\theta$
z^*	complex conjugate of z, $z^* = x - iy$
$\lvert z\rvert$	modulus of z, $\lvert z\rvert = \sqrt{(x^2 + y^2)} = r$
$\arg z$	principal argument of z, $\arg z = \theta$, where $\sin\theta = x/r$, $\cos\theta = y/r$ and $-\pi < \theta \leqslant \pi$
Δ	determinant of a matrix
$\det \mathbf{A}$	determinant of the matrix $\mathbf{A}$
$\mathbf{A}^{-1}$	inverse of the matrix $\mathbf{A}$
$\mathbf{A}^T$	transpose of the matrix $\mathbf{A}$
$\mathrm{adj}\,\mathbf{A}$	adjoint of the matrix $\mathbf{A}$
$(G, *)$	group G with binary operation $*$
$+_n$, $\times_n$	addition and multiplication modulo n
$\cong$	is isomorphic to
$\mathbf{PQ}$	vector represented by line segment $\vec{PQ}$
$\lvert\mathbf{PQ}\rvert$, PQ	magnitude of vector $\mathbf{PQ}$
$\lvert\mathbf{a}\rvert$, a	magnitude of vector $\mathbf{a}$
$\hat{\mathbf{a}}$	unit vector in the direction of $\mathbf{a}$
$\mathbf{i}, \mathbf{j}, \mathbf{k}$	unit vectors in the directions of the x-, y- and z-axes
$\mathbf{a} . \mathbf{b}$	scalar product of $\mathbf{a}$ and $\mathbf{b}$
g	acceleration due to gravity (taken as $9{\cdot}8\ \mathrm{m\ s^{-2}}$ unless otherwise stated)
$P(A)$	probability of event A
$P(A \mid B)$	probability of event A given event B
$X, Y, R, \ldots$	random variables
$x, y, r, \ldots$	values of the random variables $X, Y, R, \ldots$
$p(x)$	value of the probability function $P(X = x)$ of the discrete random variable X
$p_1, p_2, \ldots$	probabilities of the values $x_1, x_2, \ldots$ of the discrete random variable X
$f(x), g(x), \ldots$	value of the probability density function of the continuous random variable X
$F(x), G(x), \ldots$	value of the (cumulative) distribution function $P(X \leqslant x)$ of the continuous random variable X

$E(X), \mu$	expectation (or mean) of the random variable X
$\text{Var}(X), \sigma^2$	variance of the random variable X
$G(t)$	probability generating function for a random variable which takes the values $0, 1, 2, \ldots$
$B(n, p)$	binomial distribution, parameters n and p
$N(\mu, \sigma^2)$	normal distribution, mean μ and variance σ^2
Z	the standardised normal variable with distribution $N(0, 1)$
ϕ	probability density function for $N(0, 1)$
Φ	cumulative distribution function for $N(0, 1)$
$x_1, x_2, \ldots$	observations
$f_1, f_2, \ldots$	frequencies with which the observations $x_1, x_2, \ldots$ occur
n	sample size
$\bar{x}$	sample mean
$\text{var}(x)$	sample variance, $\text{var}(x) = \frac{1}{n}\sum(x - \bar{x})^2$
s^2	unbiased estimate of population variance from a sample, $s^2 = \frac{1}{n-1}\sum(x - \bar{x})^2$
$\bar{X}$	random variable representing the sample mean
V	random variable representing the sample variance
μ	population mean
σ^2	population variance
σ	population standard deviation
H_0	null hypothesis
H_1	alternative hypothesis
$\text{cov}(x, y)$	covariance of observations $(x_1, y_1), (x_2, y_2), \ldots$
$\text{Cov}(X, Y)$	covariance of random variables X and Y
r	sample (product moment) correlation coefficient
ρ	population (product moment) correlation coefficient
r_S	Spearman's rank correlation coefficient
r_K	Kendall's rank correlation coefficient
α, β	population parameters for a regression line
a, b	least squares estimates of α, β

1 Mathematical reasoning

1.1 Deductive and inductive processes

Pure mathematics is the study of pattern and structure in systems involving sets of objects such as numbers or points in space. The theory of the subject is developed by logical argument based on precise definitions of the terms used. In general, the type of reasoning used in scientific work is different from that used in mathematics. A scientist observes some natural phenomenon, then uses the data he has collected to formulate laws governing the behaviour of the objects concerned. If results obtained from later experiments also satisfy these laws, they may be accepted by other scientists and used as the basis of further work. This process, by which particular items of data lead to a general theory, is called *inductive* reasoning. Scientific laws obtained in this way cannot be regarded as absolute truths. New evidence may be found which shows that the theory is untenable or that it needs modification.

A mathematician may use inductive processes to produce *conjectures* which he then attempts to prove. One such conjecture is that there are no positive integers a, b and c such that $a^n + b^n = c^n$, where n is any positive integer greater than 2. It is believed that the mathematician Fermat† may have proved this, but he left no record of his method of proof. The conjecture has since been proved for certain values of n, but the general statement has yet to be proved or disproved. Another famous mathematical conjecture, known as the "four colour problem", has only recently been proved. This is the statement that any geographical map (or division of a plane into regions) can be coloured using only four different colours, so that no two countries (or regions) with a common boundary have the same colour.

To *prove* a mathematical statement, we must establish its truth by logical reasoning. A mathematical *proof* is a chain of reasoning in which a statement or theorem is shown to follow from previously proved statements or from initial assumptions called premises, axioms or postulates. This process, in which one

†*Fermat, Pierre de* (1601–1665) French mathematician. He made important contributions to probability theory and the theory of numbers. He wrote his famous conjecture in the margin of his copy of the *Arithmetica* written by the Greek mathematician Diophantus in about A.D. 275.

statement follows logically from another, is called *deductive* reasoning or *deduction.* We now give a simple example to illustrate the use of the process.

Example 1 What can be deduced from the following three statements?

(a) John always enjoys a boiled egg for breakfast.
(b) There was no bacon left for breakfast today.
(c) If there is no bacon, John boils an egg for breakfast.

If we form these statements into a chain, taking (b) then (c) then (a), we may deduce that John enjoyed his breakfast today.

The next example shows that an apparently logical argument may be invalid.

Example 2 Decide whether the conclusion drawn from the three given statements is valid.

(a) There is a robin in Mrs. Brown's garden.
(b) No other bird in Mrs. Brown's garden has a red breast.
(c) Mrs. Brown can see a bird with a red breast.
Conclusion: Mrs. Brown can see a robin.

The argument is invalid for various reasons. Robins are not the only birds with red breasts. Mrs. Brown may not be watching the birds in her own garden. She may, for instance, be looking at a bullfinch in a neighbour's garden.

We now demonstrate an invalid mathematical argument.

Consider the equation:	$x + \frac{1}{x} = 1$	(1)
Multiply (1) by x,	$x^2 + 1 = x$	(2)
Multiply (2) by x,	$x^3 + x = x^2$	(3)
Rearranging (3),	$x^3 = x^2 - x$	
Rearranging (2),	$x^2 - x = -1$	
$\therefore$	$x^3 = -1$	(4)
Hence	$x = -1$	

Clearly $x = -1$ does not satisfy the original equation. The flaw in the argument becomes apparent when we consider the steps in more detail.

Rearranging (2),	$x^2 - x + 1 = 0$
Rearranging (3),	$x(x^2 - x + 1) = 0$
Adding,	$(x + 1)(x^2 - x + 1) = 0$
i.e.	$x^3 + 1 = 0$

We can now see that combining equations (2) and (3) to produce equation (4) results in the introduction of the factor $(x + 1)$. It is this extra factor which leads to the invalid solution $x = -1$.

[It can be shown that the equation $x + \frac{1}{x} = 1$ has no real roots.]

The way in which deductive reasoning is used in various types of mathematical proof is discussed later in the chapter. However, although deduction forms the basis of mathematical reasoning, in such fields as problem solving and original research less reliable processes such as intuition or "trial and error" may be involved.

In applied mathematics both inductive and deductive processes are used. An applied mathematician states in mathematical form laws obtained from experimental data. He may be interested in such things as the performance of an aircraft, the growth of a population of bacteria or the current in an electrical circuit. Using the laws governing the system under consideration, he sets up a mathematical model, often in the form of a set of equations. Predictions about the behaviour of the system can now be made using this model. In general, setting up a mathematical model is an inductive process, but using the model to make predictions involves mainly deductive reasoning.

Exercise 1.1

1. Decide what conclusion can be deduced from the following sets of statements.

(a) Most people disapprove of immorality.
Stealing is immoral.
The Knave of Hearts stole some tarts.

(b) N is an even number greater than 2.
A prime number has only itself and 1 as factors.
Any even number is divisible by 2.

(c) Boycott was the vice-captain of the England cricket team.
It is impossible to play cricket with a broken arm.
If the captain cannot play, the vice-captain takes over.
The England cricket captain broke his arm.

2. Decide whether the conclusions drawn from the following sets of statements are valid. Give reasons for your answer in each case.

(a) Swallow-tail butterflies feed on Milk Parsley.
Milk Parsley grows in the Norfolk fens.
Conclusion: There are swallow-tail butterflies in Norfolk.

(b) Mathematicians are highly intelligent.
People who are not absent-minded never wear odd socks.
Highly intelligent people are thought to be absent-minded.
Conclusion: Mathematicians are thought to wear odd socks.

(c) Opposite angles of a parallelogram are equal.
A rhombus is a parallelogram whose diagonals cut at right angles.
$ABCD$ has two pairs of equal angles.
The diagonals of $ABCD$ cut at right angles.
Conclusion: $ABCD$ is a rhombus.

3. Find the flaw in the following argument:

Let	$x = y$
then	$x^2 = xy$
and	$x^2 - y^2 = xy - y^2$
Factorising,	$(x + y)(x - y) = y(x - y)$
$\therefore$	$x + y = y$
Since $x = y$,	$2x = x$
Hence	$2 = 1$

4. Read the short story given below, then consider the statements which follow it. In each case decide whether the statement is (a) true, (b) false or (c) could be either true or false.

A pedestrian had just stepped off the pavement when a vehicle, travelling at high speed, came round the corner. The car driver quickly applied his brakes and the man on the zebra crossing narrowly escaped serious injury. The police were called to the scene of the incident and later the guilty party was prosecuted for dangerous driving.

(1) At the beginning of the story a man stepped off the pavement.
(2) The car was travelling at high speed.
(3) The man on the zebra crossing was not injured.
(4) After the incident the police were called to the scene.
(5) A driver was found guilty of dangerous driving.
(6) A man escaped serious injury because the car driver braked quickly.
(7) A vehicle came round the corner just after a pedestrian had stepped off the pavement.
(8) A man was crossing the road at the time of the incident.
(9) The car stopped on or near the zebra crossing.
(10) The car stopped.
(11) The car driver was prosecuted for dangerous driving.
(12) The incident involved at least one driver.
(13) The zebra crossing was near a bend in the road.
(14) The events described in the story could not have taken place on a dual carriageway.
(15) None of the people referred to in the story could have been riding a bicycle.
(16) It is possible that the incident involved two vehicles.
(17) The story states that a vehicle being driven at high speed knocked down a man on a zebra crossing, injuring him only slightly.
(18) Three people are mentioned in the story, apart from the police.

1.2 Implication and equivalence

It is sometimes convenient to use letters and symbols to represent statements and relations between them.

For two statements denoted by p and q,

$p \Rightarrow q$ means "p implies q" or "if p, then q",

$p \Leftarrow q$ means "p is implied by q" or "if q, then p",

$p \Leftrightarrow q$ means "p implies q and q implies p" i.e. "the statements p and q are equivalent".

Example 1 Use the symbols $\Rightarrow$ and $\Leftrightarrow$ to connect the following statements:

a: $\triangle ABC$ is isosceles b: In $\triangle ABC$, $\angle B = \angle C$
c: $\triangle ABC$ is equilateral d: In $\triangle ABC$, $AB = AC$

We see that $b \Rightarrow a, b \Rightarrow d; c \Rightarrow a, c \Rightarrow b, c \Rightarrow d; d \Rightarrow a, d \Rightarrow b$.
Since $b \Rightarrow d$ and $d \Rightarrow b$, we have $b \Leftrightarrow d$
i.e. the statements b and d are equivalent.

The symbols $\not\Rightarrow$ "does not imply" and $\not\Leftrightarrow$ "is not equivalent to" may also be used.

Example 2 Use the symbols $\not\Rightarrow$ and $\not\Leftrightarrow$ to connect statements a, b, c and d.

By considering an isosceles triangle ABC in which $AB = BC$, we see that $a \not\Rightarrow b$ and $a \not\Leftarrow d$. We also find that $a \not\Leftarrow c, b \not\Leftarrow c$ and $d \not\Rightarrow c$. Hence $a \Leftrightarrow b$, $a \not\Leftrightarrow c, a \not\Leftrightarrow d, b \not\Leftrightarrow c, c \not\Leftrightarrow d$.

The *negation* of a statement p is denoted by $\sim p$ (or sometimes p') and read "not p". The negations of a, b, c and d in *Example 1* are as follows:

$\sim a$: $\triangle ABC$ is not isosceles $\sim b$: In $\triangle ABC$, $\angle B, \neq \angle C$
$\sim c$: $\triangle ABC$ is not equilateral $\sim d$: In $\triangle ABC$, $AB \neq AC$

Here are some of the connections between these new statements:

$$\sim a \Rightarrow \sim c, \quad \sim b \Leftrightarrow \sim d, \quad \sim d \Rightarrow \sim c, \quad \sim d \not\Rightarrow \sim a.$$

These examples illustrate some basic properties of negation and implication.

(1) If $p \Rightarrow q$, then $\sim q \Rightarrow \sim p$.
(2) If $p \Leftrightarrow q$, then $\sim p \Leftrightarrow \sim q$.

It is also important to notice that writing $p \not\Rightarrow q$ is *not* equivalent to writing $p \Rightarrow \sim q$. For instance in *Example 2*, the statement $a \not\Rightarrow d$ means: "if $\triangle ABC$ is isosceles, then it does not necessarily follow that $AB = AC$". However, writing $a \Rightarrow \sim d$ would mean: "if $\triangle ABC$ is isosceles, then $AB \neq AC$", which is clearly false.

Other phrases which are used in mathematical statements, such as "if and only if" or "necessary and sufficient condition", can be replaced by implication symbols. The table below shows some sets of equivalent statements.

$p \Rightarrow q$	p only if q	q is a necessary condition that p
$p \Leftarrow q$	p if q	q is a sufficient condition that p
$p \Leftrightarrow q$	p if and only if q	q is a necessary and sufficient condition that p

Consider now the following statements concerning a positive integer n.

a: n is the square of an even number
b: n is a multiple of 4
c: n is an even number

We see that $a \Rightarrow b \Rightarrow c$ and that, using the phrases in the given table:

the statement b is true if a is true, but b is true only if c is true;
the truth of c is a necessary condition for b to be true, but a is a sufficient condition for b to be true.

Clearly implication signs provide the most economical way of expressing connections between statements. The process of deduction is based on a *law of implication* which we can now state in the form:

$$\text{if} \quad a \Rightarrow b \quad \text{and} \quad b \Rightarrow c, \quad \text{then} \quad a \Rightarrow c.$$

A deductive proof can always be reduced to a series of these steps or closely related ones such as, "if $a \Rightarrow b$ and $a \Rightarrow c$, where b and c together imply d, then $a \Rightarrow d$."

Exercise 1.2

1. Use the symbols $\Rightarrow$ and $\Leftrightarrow$ to connect the statements p, q and r.

(a) p: $ABCD$ is a square
q: $ABCD$ has four equal sides
r: $ABCD$ is a parallelogram

(b) p: Ivor is Welsh
q: Ivor lives in Cardiff
r: Ivor is British

(c) p: $KLMN$ is a rectangle 3 cm by 4 cm
q: The area of rectangle $KLMN$ is 12 cm^2
r: $KLMN$ is a rectangle with diagonals 5 cm long

(d) p: The integer n is an even multiple of 5
q: The integer n has final digit 0
r: The integer n has a pair of prime factors which differ by 3.

2. Use the symbol $\not\Rightarrow$ to connect the statements given in question 1.

3. Use the symbols $\Rightarrow$ and $\Leftrightarrow$ to connect the negations $\sim p$, $\sim q$ and $\sim r$ of the statements given in question 1.

4. Write down all possible connections between the statements p, q, $\sim p$ and $\sim q$ using the symbols $\Rightarrow$ and $\Leftrightarrow$.

(a) p: In quadrilateral $ABCD$, $AC = BD$
q: $ABCD$ is a rectangle

(b) p: In $\triangle ABC$, $\angle A$ is obtuse
q: In $\triangle ABC$, $\angle B = 90°$

(c) p: For integers x and y, $x - y$ is odd
q: For integers x and y, $x^2 - y^2$ is divisible by 4

(d) p: C lies on the circle with AB as diameter
q: In $\triangle ABC$, $\angle C$ is not a right angle.

5. Using only the terms and symbols $p, q, \sim p, \sim q, \Rightarrow, \Leftrightarrow, \not\Rightarrow$ and $\not\Leftrightarrow$, write down a statement equivalent to each of the following:

(a) $\sim p \Rightarrow \sim q$, (b) $p \Leftrightarrow \sim q$, (c) $\sim p \not\Rightarrow q$,
(d) $p \not\Leftrightarrow q$, (e) $p \Rightarrow \sim q$, (f) $\sim p \Leftrightarrow q$,

6. Use the symbol $\Rightarrow$ to state the connection between the statements p and q in each of the following cases.

(a) q is true only if p is true,
(b) p is a necessary condition that q,
(c) q is a sufficient condition that p,
(d) p is true if q is true,
(e) p is a sufficient condition that q,
(f) p is true only if q is true.

7. Use the terms "if", "only if", "necessary condition" and "sufficient condition" to express the connection between the statements, "$\triangle ABC$ is equilateral" and "$AB = AC$" in four different ways.

1.3 Types of proof

We first give an example of an elementary deductive proof.

Example 1 Prove that $a \times 0 = 0$ for all real values of a.

[You may assume the following properties of real numbers:

(i) $a + (b + c) = (a + b) + c$ associative law for addition
(ii) $a(b + c) = ab + ac$ distributive law
(iii) $a + 0 = a = 0 + a$ for all real a
(iv) any real number a has an additive inverse $(-a)$, such that $a + (-a) = 0 = (-a) + a$.]

	$b + 0 = b$	property (iii)
$\Rightarrow$	$a(b + 0) = ab$	both sides multiplied by a
$\Rightarrow$	$ab + a \times 0 = ab$	distributive law
$\Rightarrow$	$-ab + (ab + a \times 0) = -ab + ab$	$-ab$ added to both sides
$\Rightarrow$	$(-ab + ab) + a \times 0 = -ab + ab$	associative law for $+$
$\Rightarrow$	$0 + a \times 0 = 0$	property (iv)
$\Rightarrow$	$a \times 0 = 0$	property (iii)

Another important type of proof is called *proof by contradiction* (or "reductio ad absurdum".) To prove a theorem by this method we assume that the theorem does not hold, then show that this assumption leads to a contradiction. In its simplest form this process is equivalent to proving a statement p true by showing $\sim p$ to be

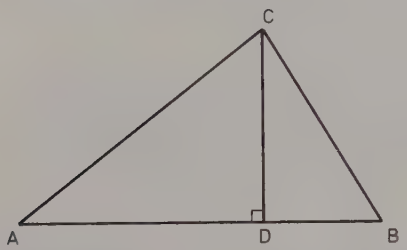
false. For instance, the standard proof of the statement "$\sqrt{2}$ is irrational" begins with the assumption that $\sqrt{2}$ is rational. In the example given here a statement of the form $p \Rightarrow q$ is proved by showing that $\sim q \Rightarrow \sim p$.

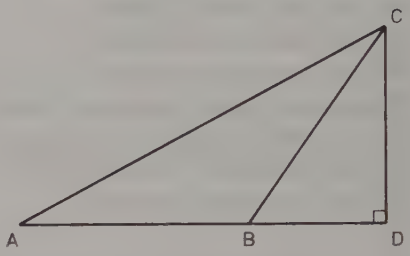
Example 2 Prove that in $\triangle ABC$, if $AB^2 + BC^2 = AC^2$, then $\angle ABC = 90°$.

[You may assume Pythagoras' theorem.]

Let us assume that in $\triangle ABC$, $\angle ABC \neq 90°$.

(1) $\angle ABC < 90°$ (2) $\angle ABC > 90°$





Case (1): By Pythagoras' theorem in $\triangle ADC$, $AC^2 = AD^2 + DC^2$

But $AD < AB$ and $DC < BC$, $\therefore$ $AC^2 < AB^2 + BC^2$.

Case (2): By Pythagoras' theorem in $\triangle ADC$, $AC^2 = AD^2 + DC^2$

$\therefore$ $AC^2 = (AB + BD)^2 + DC^2 = AB^2 + 2AB.BD + BD^2 + DC^2$

By Pythagoras' theorem in $\triangle BDC$, $BD^2 + DC^2 = BC^2$

$\therefore$ $AC^2 = AB^2 + 2AB.BD + BC^2 > AB^2 + BC^2$.

Thus, in both cases, if $AB^2 + BC^2 = AC^2$, the assumption that $\angle ABC \neq 90°$ leads to a contradiction.
Hence, if $AB^2 + BC^2 = AC^2$, then $\angle ABC = 90°$.

This result is, of course, the converse of Pythagoras' theorem. In general, the *converse* of the statement $p \Rightarrow q$ is the statement $q \Rightarrow p$.

As in the case of Fermat's famous conjecture, mentioned in §1.1, it is sometimes difficult to establish the truth of a general statement about such objects as real numbers or sets. However, a statement can be proved false by producing just one *counter-example.*

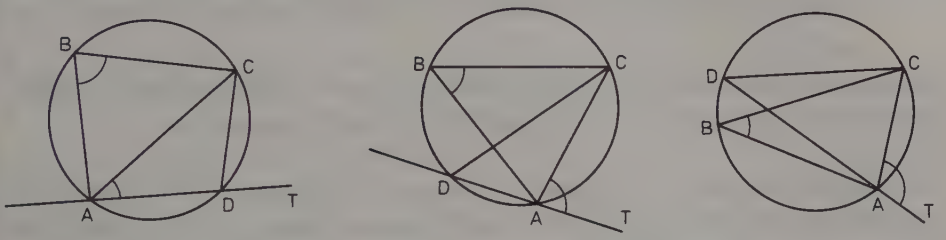
Example 3 Discuss the statement that if n is a positive integer, then $n^2 - n + 17$ is prime.

[To prove that this statement is true it would be necessary to show that it holds for all positive integral values of n. However, the statement can be proved false by finding one value of n, such that $n^2 - n + 17$ is not prime.]

If we test the statement by writing $n = 1, 2, 3, 4, 5 \ldots$, we obtain the prime numbers $17, 19, 23, 29, 37, \ldots$. This suggests that the statement could be true. However, if $n = 17$, $n^2 - n + 17 = 17^2$, which is not prime. Hence the statement is false.

Exercise 1.3

1. Using the given diagrams, show that if $\angle ABC = \angle CAT$, the assumption that AT cuts the circle again at a point D leads to a contradiction. What conclusion can be drawn from this?



2. Show that if $p_1, p_2, p_3, \ldots, p_n$ are prime numbers, then the number $p_1p_2p_3 \ldots p_n + 1$ is either a prime not already listed or a multiple of such a prime. Hence prove by contradiction that there are infinitely many prime numbers.

3. Write down the converses of the following statements. In each case decide whether (i) the statement, (ii) its converse are true or false.
(a) If a diameter of a circle bisects a chord, it is perpendicular to that chord.
(b) If $2x - 3 = 0$, then $2x^2 - x - 3 = 0$.
(c) If a pentagon is regular, it must be equilateral.
(d) If x is greater than y, then x^2 is greater than y^2.
(e) If a positive integer N is divisible by 9, then the sum of its digits is divisible by 9.
(f) If any set of six socks taken from a drawer contains a pair, then there must be five pairs of socks in the drawer.

In questions 4, 5 and 6 decide whether the given statements are true or false. If a statement is true, prove it; if it is false, give a counter-example.

4. (a) The square of any odd number is of the form $4k + 1$, where k is a non-negative integer.
(b) Any number of the form $4k + 1$ is the square of an odd number.

5. (a) If $0 < p < q$, then $\log p < \log q$.
(b) If $0 < p < q$, then $\int_0^p f(x)dx < \int_0^q f(x)dx$.

6. (a) If $n^3 - n$ is divisible by 4, then n must be divisible by 4.
(b) If n is a positive integer, then $n^3 - n$ is divisible by 3.

1.4 Proof by induction

The elementary deductive methods discussed so far are sometimes found to be inadequate when attempting to prove that a proposition, such as "the number $4^n + 5$ is divisible by 3", holds for all positive integral values of n. The difficulty is

that this general statements is equivalent to the infinite sequence of propositions "3 is a factor of $4^1 + 5, 4^2 + 5, 4^3 + 5, 4^4 + 5, \ldots$" However, propositions of this kind can often be proved using *mathematical induction.* Although the method is related to the type of inductive reasoning used in science, it is nevertheless a rigorous procedure in which conclusions are reached by sound deductive reasoning.

The principle of a proof by induction can be stated formally as follows:

> Given a proposition $P(n)$ involving a positive integer n, if (i) $P(k) \Rightarrow P(k+1)$ and (ii) $P(1)$ is true, then $P(n)$ is true for all positive integral values of n.

A useful mental picture of a proof by induction is created by considering the proposition that a man can climb a certain uniform staircase with n steps. The statement will be proved if we can show

(i) that the man is capable of climbing from any step to the one above and
(ii) that the man can reach the foot of the staircase.

Thus it may be helpful to think of a general proposition $P(n)$ as a staircase to be climbed. The steps of the staircase are the propositions $P(1), P(2), P(3), \ldots$. It is possible to proceed from one "step" to the next if we can prove that for any k, $P(k+1)$ may be deduced from $P(k)$. If, in addition, it is shown that $P(1)$ is true (i.e. that it is possible to set foot on the first step of the staircase), then it follows that $P(2)$ is true, then $P(3)$ is true and so on for all positive integral values of n.

Example 1 Prove that the number, $a_n = 4^n + 5$, is divisible by 3 for all positive integral values of n.

Consider the numbers $a_k = 4^k + 5$ and $a_{k+1} = 4^{k+1} + 5$.

$$a_{k+1} - a_k = (4^{k+1} + 5) - (4^k + 5)$$
$$= 4^{k+1} - 4^k = 4^k(4 - 1) = 4^k \times 3$$
$$\therefore \quad a_{k+1} = a_k + 4^k \times 3$$

Hence, if a_k is divisible by 3, so is a_{k+1}.
$\therefore$ if the proposition is true for $n = k$, it is also true for $n = k + 1$.
$a_1 = 4^1 + 5 = 9$, which is divisible by 3,
$\therefore$ the proposition is true for $n = 1$.
Hence, by induction, the number $4^n + 5$ must be divisible by 3 for all positive integral values of n.

When dealing with a finite series, it is sometimes possible to suggest a formula for the sum to n terms by examining the first few sums, $S_1, S_2, S_3, \ldots$. Such a result can often be proved by the method of induction.

Example 2 If S_n is the sum of the first n terms of the series

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \ldots + \frac{1}{n(n+1)} + \ldots,$$

find S_1, S_2 and S_3, then suggest a formula for S_n. Prove that the result is correct by the method of induction.

$$S_1 = \frac{1}{1 \times 2} = \frac{1}{2}, \qquad S_2 = S_1 + \frac{1}{2 \times 3} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3},$$

$$S_3 = S_2 + \frac{1}{3 \times 4} = \frac{2}{3} + \frac{1}{12} = \frac{3}{4}.$$

These results suggests that $S_n = \dfrac{n}{n+1}$

Assuming that this result is true for $n = k$, $S_k = \dfrac{k}{k+1}$

$$\therefore \quad S_{k+1} = S_k + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)}$$

$$= \frac{k^2 + 2k + 1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

$\therefore$ if the result holds for $n = k$, it also holds for $n = k + 1$.

When $n = 1$, $S_n = S_1 = \dfrac{1}{2}$ and $\dfrac{n}{n+1} = \dfrac{1}{1+1} = \dfrac{1}{2}$

$\therefore$ the result holds for $n = 1$.

Hence, by induction, it is true that $S_n = n/(n+1)$ for all positive integral values of n.

The method of induction can also be used to prove the binomial theorem:

$$(a+b)^n = a^n + {}_nC_1 a^{n-1}b + {}_nC_2 a^{n-2}b^2 + \ldots + {}_nC_r a^{n-r}b^r + \ldots + {}_nC_{n-1} ab^{n-1} + b^n.$$

Assuming that the theorem is true for $n = k$,

$$(a+b)^k = a^k + {}_kC_1 a^{k-1}b + {}_kC_2 a^{k-2}b^2 + \ldots + {}_kC_{r-1} a^{k-r+1}b^{r-1} + {}_kC_r a^{k-r}b^r + \ldots + b^k,$$

then, multiplying by $(a+b)$,

$$(a+b)^{k+1} = a^{k+1} + {}_kC_1 a^k b + {}_kC_2 a^{k-1}b^2 + \ldots + {}_kC_r a^{k-r+1}b^r + \ldots$$
$$+ a^k b + {}_kC_1 a^{k-1}b^2 + \ldots + {}_kC_{r-1} a^{k-r+1}b^r + \ldots + b^{k+1}.$$

But ${}_kC_1 + 1 = k + 1 = {}_{k+1}C_1$, and

$$
\begin{aligned}
{}_kC_r + {}_kC_{r-1} &= \frac{k!}{r!(k-r)!} + \frac{k!}{(r-1)!(k-\{r-1\})!} \\
&= \frac{k!}{r!(k-r+1)!}\{(k-r+1)+r\} \\
&= \frac{(k+1)!}{r!(k+1-r)!} = {}_{k+1}C_r
\end{aligned}
$$

$$
\therefore \quad (a+b)^{k+1} = a^{k+1} + {}_{k+1}C_1 a^k b + {}_{k+1}C_2 a^{k-1}b^2 + \ldots \\
\ldots + {}_{k+1}C_r a^{k+1-r}b^r + \ldots + b^{k+1}.
$$

$\therefore$ if the theorem is true for $n = k$ it is also true for $n = k + 1$.
Since $(a + b)^1 = a^1 + b^1$, the theorem is true for $n = 1$. Hence, by induction, the binomial theorem is true for all positive integral values of n.

[Note that the basic principle of induction can be extended in various ways. Suppose, for instance, that for some proposition $P(n)$, where n is a positive integer, it can be shown that $P(k) \Rightarrow P(k + 1)$, but that the proposition $P(1)$ is false or meaningless. Clearly $P(n)$ cannot be proved true for all positive integral values of n. However, provided that there is some positive integer m such that $P(m)$ is true, then by induction, $P(n)$ holds for all integral values of n greater than or equal to m.]

Exercise 1.4

In questions 1 to 8 prove, by induction, that the given statements are true for all positive integral values of n.

1. The sum of the first n terms of the series
$(1 \times 3) + (2 \times 4) + (3 \times 5) + \ldots + r(r + 2) + \ldots$ is $\frac{1}{6}n(n + 1)(2n + 7)$.

2. The sum of the first n terms of the series
$$\frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \frac{1}{5 \times 7} + \ldots + \frac{1}{(2r-1)(2r+1)} + \ldots \text{ is } \frac{n}{2n+1}.$$

3. $\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$.

4. $\sum_{r=1}^{n} \frac{r}{2^r} = 2 - \frac{n+2}{2^n}$.

5. $n^3 + 3n^2 - 10n$ is divisible by 3.

6. $3^{2n} - 1$ is a multiple of 8.

7. $7^n + 4^n + 1^n$ is divisible by 6.

8. If $\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$, then $\mathbf{M}^n = \begin{pmatrix} 1 & 0 \\ 3n & 1 \end{pmatrix}$.

9. Prove that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$ for all positive integers n. Hence find the following sums:
(a) $1^2 + 2^2 + 3^2 + \ldots + 20^2$,
(b) $21^2 + 22^2 + 23^2 + \ldots + 40^2$,
(c) $2^2 + 4^2 + 6^2 + \ldots + 40^2$,
(d) $1^2 + 3^2 + 5^2 + \ldots + 39^2$.

10. Prove that the sum to n terms of the series

$$\frac{1}{1 \times 2 \times 3} + \frac{1}{2 \times 3 \times 4} + \frac{1}{3 \times 4 \times 5} + \ldots \text{ is } \frac{1}{4} - \frac{1}{2(n+1)(n+2)}.$$

Hence find the sum to infinity of the series.

11. If $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}$, prove that $\mathbf{A}^n = \begin{pmatrix} 2n+1 & -n \\ 4n & 1-2n \end{pmatrix}$, where n is a positive integer. Given that the transformation represented by $\mathbf{A}$ is applied to the points of the x, y plane n times in succession, find the image of
(a) the point $(0, -1)$,
(b) the line $y = x$.

12. Given that $f(x) = x/(x+1)$, find $f^2(x)$ i.e. $f(f(x))$. Find also $f^3(x)$ and suggest a possible form for $f^n(x)$. Prove that your result is correct by mathematical induction.

Exercise 1.5 (*miscellaneous*)

In questions 1 to 6 explain why the given chain of reasoning is unsound.

1.
$$\begin{aligned} & -6x + 9 = 2x + 1 \\ \Rightarrow \quad & x^2 - 6x + 9 = x^2 + 2x + 1 \\ \Rightarrow \quad & (x-3)^2 = (x+1)^2 \\ \Rightarrow \quad & x - 3 = x + 1 \\ \Rightarrow \quad & -3 = 1 \end{aligned}$$

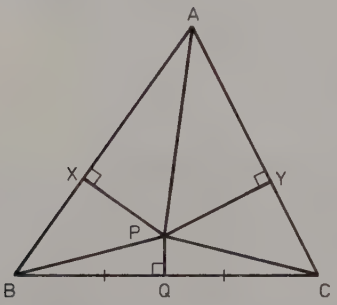
2.
$$\begin{aligned} & 2 > 1 \\ \Rightarrow \quad & 2\lg(0{\cdot}2) > \lg(0{\cdot}2) \\ \Rightarrow \quad & \lg(0{\cdot}2)^2 > \lg(0{\cdot}2) \\ \Rightarrow \quad & \lg(0{\cdot}04) > \lg(0{\cdot}2) \\ \Rightarrow \quad & 0{\cdot}04 > 0{\cdot}2 \end{aligned}$$

3.
$$\begin{aligned} & (2x+3)(3x+1) = (6x+5)(x+1) \\ \Rightarrow \quad & 6x^2 + 11x + 3 = 6x^2 + 11x + 5 \\ \Rightarrow \quad & 3 = 5 \end{aligned}$$

4.
$$\begin{aligned} & a + b > 0 \\ \Rightarrow \quad & a^2 + ab > 0 \\ \Rightarrow \quad & a^2 + 2ab + b^2 > ab + b^2 \\ \Rightarrow \quad & (a + b)^2 > (a + b)b \\ \Rightarrow \quad & a + b > b \end{aligned}$$

Hence if $a + b > 0$, then $a + b > b$.

5.



In $\triangle ABC$ the point P is constructed such that AP bisects $\angle A$ and PQ is the perpendicular bisector of BC. PX and PY are drawn perpendicular to AB and AC respectively.

In $\triangle$s APX and APY, $\angle AXP = \angle AYP$, $\angle PAX = \angle PAY$ and AP is common,
$\therefore$ $\triangle$s APX and APY are congruent.
Hence $PX = PY$ and $AX = AY$.
In $\triangle$s PXB and PYC, $\angle BXP = \angle CYP = 90°$, $PX = PY$ and $PB = PC$.
$\therefore$ $\triangle$s PXB and PYC are congruent and hence $XB = YC$.
Thus $AX + XB = AY + YC$ i.e. $AB = AC$.
Hence any triangle is isosceles.

6. To prove that $a_n = 7^n - 11$ is divisible by 6, where n is an integer greater than 1.

Proof: $a_{k+1} - a_k = (7^{k+1} - 11) - (7^k - 11)$
$= 7^{k+1} - 7^k = 7^k(7 - 1) = 7^k \times 6,$

$\therefore$ if a_k is divisible by 6, so is a_{k+1}.
Hence, by induction, a_n is divisible by 6 for all integral values of n greater than 1.

7. In proving a statement of the form $p \Rightarrow q$, it has been shown that p implies either r or s. What further steps are required to complete the proof?

8. In proving that the proposition $P(n)$ is true for all positive integral values of n, it has been shown that $P(k) \Rightarrow P(k + 2)$. What further steps are required to complete the proof?

9. Prove that if n is any positive integer,
$1 \times 2 \times 3 + 2 \times 3 \times 4 + \ldots + n(n + 1)(n + 2) = \frac{1}{4}n(n + 1)(n + 2)(n + 3)$.

10. Prove that $\dfrac{d}{dx}(x^n) = nx^{n-1}$ for all positive integral values of n.

11. Prove that $\displaystyle\sum_{r=2}^{n} \frac{1}{r^2 - 1} = \frac{3}{4} - \frac{2n + 1}{2n(n + 1)}$.

12. Given that $\mathbf{A} = \begin{pmatrix} 1+\cos\theta & \sin\theta \\ \sin\theta & 1-\cos\theta \end{pmatrix}$, prove that $\mathbf{A}^{n+1} = 2^n \times \mathbf{A}$, where n is a positive integer.

13. If $u_{n+1} = 2u_n + 1$ where n is a positive integer and $u_1 = 5$, prove by induction that $u_n = 3 \times 2^n - 1$.

14. If $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$, find $\mathbf{A}^2$, $\mathbf{A}^3$ and conjecture a form for $\mathbf{A}^n$ where n is any positive integer. Prove the truth of your conjecture by mathematical induction. (O & C)

15. Prove by induction that the sum of the first n terms of the series $\frac{1}{1\times2\times3} + \frac{4}{2\times3\times4} + \frac{7}{3\times4\times5} + \ldots$ is $\frac{n^2}{(n+1)(n+2)}$. Find the limit of this sum as n tends to infinity.

16. If $S_n = 1.n + 2(n-1) + 3(n-2) + \ldots + (n-1).2 + n.1$, where n is a positive integer, prove that $S_{n+1} - S_n = \frac{1}{2}(n+1)(n+2)$. By induction, or otherwise, prove that $S_n = \frac{1}{6}n(n+1)(n+2)$. (O)

17. Prove that, if n is a positive integer greater than one, $n^5 - n$ is a multiple of 5. Hence, or otherwise, prove that all square numbers are of the form $5r$ or $5r \pm 1$ where r is an integer. (JMB)

18. Let A, B and C be subsets of some universal set $\mathscr{E}$. For the three statements below, prove those that are true and, by taking $\mathscr{E}$ to be the set of all positive integers, provide counter-examples to demonstrate the falsity of the others:
(i) $(A \cap B = A \cap C) \Rightarrow (B = C)$, (ii) $(A \cap B = A \cup B) \Rightarrow (A = B)$,
(iii) $(A \cap B) \cup C = (A \cup C) \cap B$.

[In this question Venn diagrams may be used to examine the plausibility of a statement but will not be accepted as proofs of truth or falsity.] (C)

19. The real function f, defined for all $x \in \mathbb{R}$, is said to be multiplicative if, for all $x \in \mathbb{R}$, $y \in \mathbb{R}$, $f(xy) = f(x)f(y)$. [$\mathbb{R}$ denotes the set of real numbers.] Prove that if f is a multiplicative function then
(i) either $f(0) = 0$ or $f(x) \equiv 1$,
(ii) either $f(1) = 1$ or $f(x) \equiv 0$,
(iii) $f(x^n) = \{f(x)\}^n$ for all positive integers n.
Give an example of a non-constant multiplicative function. (C)

20. The matrix $\mathbf{X} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ has real non-zero elements; the matrix $\mathbf{P}$ is of the form $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ where $a \neq b$, and $\mathbf{M} = \begin{pmatrix} 4 & 2 \\ -3 & -1 \end{pmatrix}$. Given that $\mathbf{XM} = \mathbf{PX}$, show

that $\frac{q}{p} = \frac{4-a}{3} = \frac{2}{1+a}$ and find a possible matrix $\mathbf{P}$ and a possible corresponding matrix $\mathbf{X}$. Show that, if n is a positive integer, $\mathbf{M}^n = \mathbf{X}^{-1}\mathbf{P}^n\mathbf{X}$. Deduce the elements of $\mathbf{M}^n$ in terms of n. (JMB)

21. Prove that if n is a positive integer, $10^n - 1$ is divisible by 9. Hence prove that a necessary and sufficient condition for a positive integer to be divisible by 9 is that the sum of its digits is divisible by 9.

2 Some functions and their properties

2.1 Symmetry, continuity and differentiability

A *function* f with *domain* X and *codomain* Y is a rule which assigns to each element $x \in X$ exactly one *image* $y \in Y$. The set containing the images under f of all the elements of the domain is called the *range* (or image set). In this chapter we will be considering functions with domain the set of real numbers $\mathbb{R}$ (or some subset of $\mathbb{R}$), and codomain $\mathbb{R}$. The notation used to define such functions will take one of the following forms: $f(x) = x^2 + 1$, $y = x^2 + 1$ or $f: x \to x^2 + 1$. Unless otherwise stated, the domain of a function f is assumed to be the set of all real numbers for which an image is defined. For instance, the domain of the function $f(x) = x/(x^2 - 1)$ is taken to be the set $\mathbb{R} - \{-1, 1\}$ i.e. all real numbers except -1 and $+1$. We now examine some of the properties possessed by various functions defined in $\mathbb{R}$.

A function $f(x)$ is said to be *odd* if $f(-x) = -f(x)$. The graph of an odd function is symmetrical about the origin, i.e. if (a, b) lies on the graph so does $(-a, -b)$. The function $f(x)$ is said to be *even* if $f(-x) = f(x)$. The graph of an even function is symmetrical about the y-axis i.e. if (a, b) lies on the graph so does $(-a, b)$.

Example 1 Decide whether each of the functions f, g and h is odd, even or neither, given that

$$f(x) = x + 1/x,\ x \in \mathbb{R},\ x \neq 0,$$
$$g(x) = 2\cos x - x,\ x \in \mathbb{R},$$
$$h(x) = \sqrt{(4 - x^2)},\ x \in \mathbb{R},\ |x| \leqslant 2.$$

$f(-x) = (-x) + 1/(-x) = -x - 1/x = -f(x)$

$\therefore$ f is an odd function.

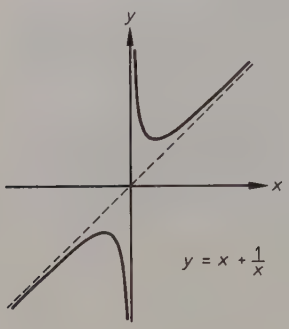
$g(-x) = 2\cos(-x) - (-x) = 2\cos x + x$

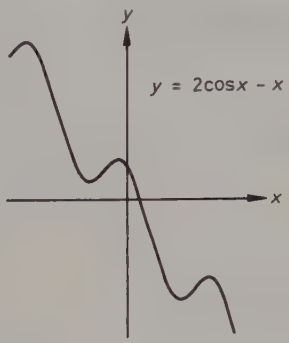
$\therefore$ g is neither odd nor even.

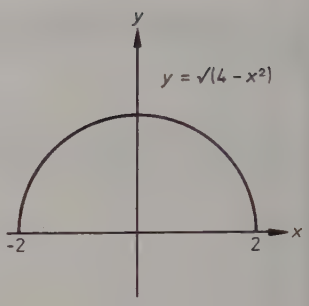
$h(-x) = \sqrt{(4 - \{-x\}^2)} = \sqrt{(4 - x^2)} = h(x)$

$\therefore$ h is an even function.

The sketches below show the graphs of f, g and h.



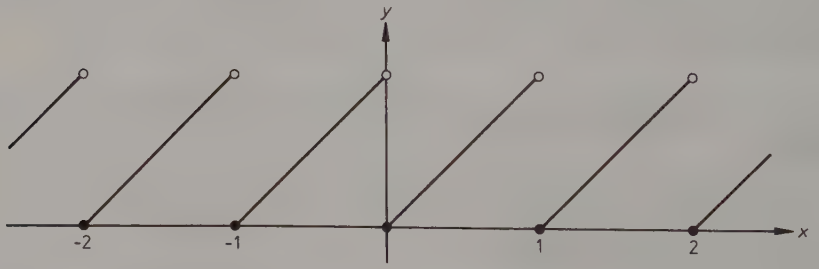




A function $f(x)$ is said to be *periodic* if there is a constant k such that $f(x + k) = f(x)$ for all values of x. The graph of the function forms a repeating pattern such that any translation parallel to the x-axis through k units leaves the pattern unchanged. The *period* of the function is the least positive constant a such that $f(x + a) = f(x)$ for all values of x. The trigonometric functions $\sin x$ and $\cos x$ are periodic and have period 2π. The function $\tan x$ is also periodic, but has period π.

Example 2 Sketch the graph of the function $f(x) = x - [x]$ and decide whether it is periodic. [Note that $[x]$ denotes the greatest integer less than or equal to x.]

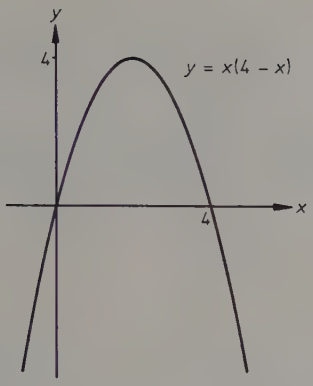
When x is an integer, $[x] = x$, $\therefore \quad f(x) = x - x = 0$
When $0 \leqslant x < 1$, $[x] = 0$, $\therefore \quad f(x) = x$
For all values of x, $f(x + 1) = (x + 1) - [x + 1]$
$= x + 1 - ([x] + 1)$
$= x - [x] = f(x)$

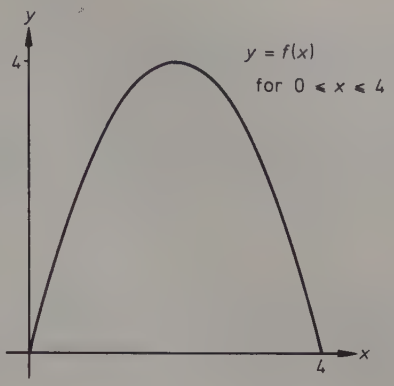


The sketch shows that $f(x)$ is periodic with period 1.
[Open and closed circles are used to show that, for instance, the value of $f(1)$ is 0 rather than 1.]

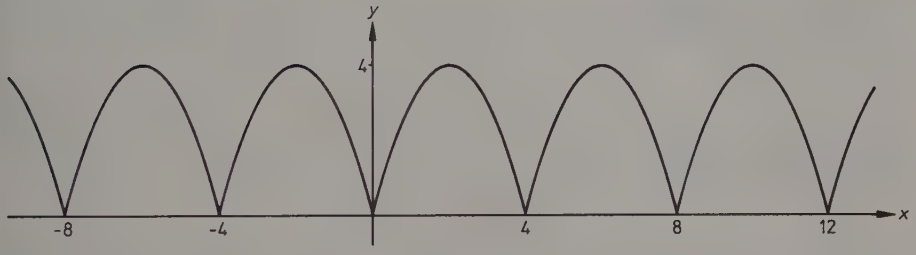
Example 3 Given that f is a periodic function such that for $0 \leqslant x \leqslant 4$, $f(x) = x(4 - x)$, sketch the graph $y = f(x)$ in each of the following cases,

(a) the period of f is 4,
(b) f is an odd function and has period 8.

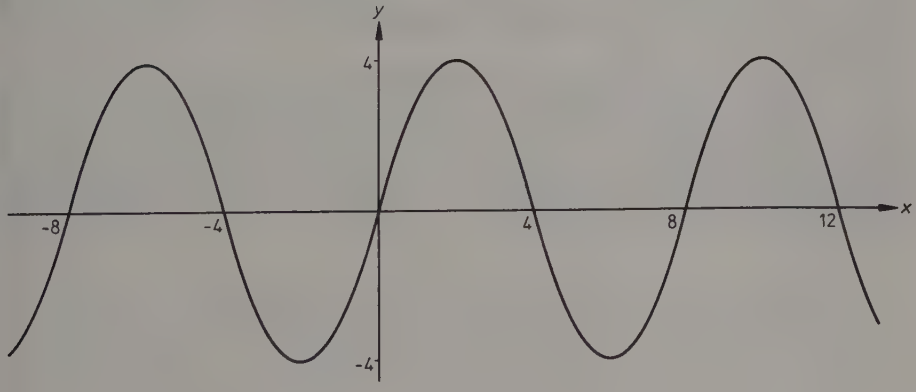




(a)



(b)

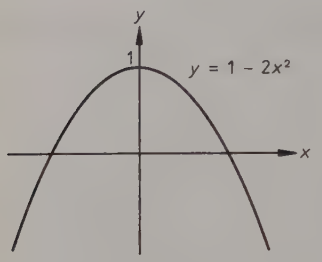


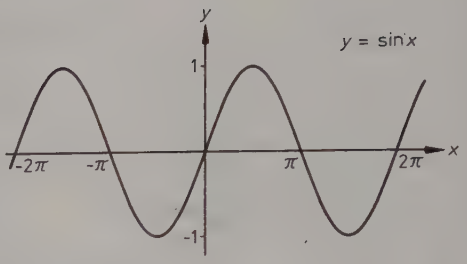
If f and g are functions defined on $\mathbb{R}$, then the *composite function* fg (also written $f \circ g$) is defined by $fg(x) = f(g(x))$.

Example 4 The functions f and g are defined by $f: x \to 1 - 2x^2$ and $g: x \to \sin x$. Sketch the graphs of the functions f, g and fg. State whether each of these functions is

(a) odd, even or neither,

(b) periodic, and if so give the period.

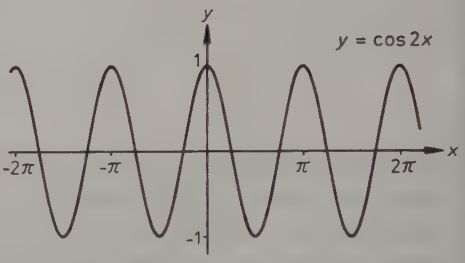




f is an even function but not periodic.
g is an odd function and periodic with period 2π.

$fg(x) = 1 - 2\sin^2 x = \cos 2x$

$\therefore$ fg is an even function and periodic with period π.

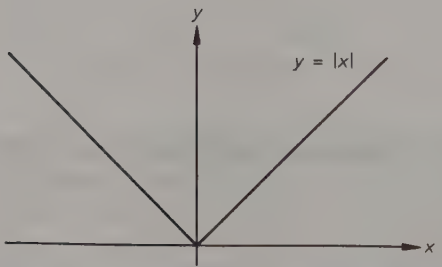


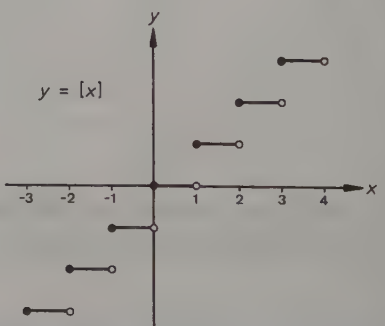
A function $f(x)$ is said to be *continuous* at $x = a$ if, when $x \to a$ from above and when $x \to a$ from below, $f(x) \to f(a)$

i.e. if $\lim_{x \to a} f(x) = f(a)$.

Less formally, $f(x)$ is continuous at $x = a$, if there is no break or *discontinuity* in its graph at that point. If the function $f(x)$ is continuous for all real values of x, it is called a *continuous function*. The graph of a continuous function is a single unbroken line or curve.

Many of the functions met in elementary mathematics are continuous e.g. $x^3 - x + 3$, $\sin x$, $|x|$. A function such as $\sqrt{(4 - x^2)}$, which is defined for $-2 \leqslant x \leqslant 2$, is continuous in that interval. However, other familiar functions have one or more discontinuities e.g. $1/x$ at $x = 0$; $\tan x$ at $x = \pm\frac{1}{2}\pi, \pm\frac{3}{2}\pi, \pm\frac{5}{2}\pi, \ldots$; $[x]$ at all integral values of x.





The *derivative* of a function $f(x)$ at $x = a$ is defined as

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

If this limit exists $f(x)$ is said to be *differentiable* at $x = a$. Given a function $f(x)$ the *derived function* $f'(x)$ can often be found using the standard rules for differentiation. However, any general formula for $f'(x)$ will be invalid where $f(x)$ has a discontinuity and possibly at certain other points. For instance, if $f(x) = 1/x$, the result $f'(x) = -1/x^2$ is meaningless when $x = 0$. The function $f(x) = [x]$ is not differentiable when x is an integer, but for non-integral values of x, $f'(x) = 0$. If we consider the function $f(x) = |x|$ near $x = 0$, we find that

for $h > 0$, $$\frac{f(0+h) - f(0)}{h} = \frac{h - 0}{h} = 1,$$

but for $h < 0$, $$\frac{f(0+h) - f(0)}{h} = \frac{-h - 0}{h} = -1.$$

Hence the expression $\dfrac{f(0+h) - f(0)}{h}$ approaches no single value as $h \to 0$. Thus although $|x|$ is a continuous function it is not differentiable at $x = 0$.

Exercise 2.1

In questions 1 to 4 decide whether the given functions are odd, even or neither.

1. (a) $x^4 - 6$, (b) $x^3 + 5$, (c) $x^3 + 5x$.

2. (a) $1 + \dfrac{1}{x^2}$, (b) $\dfrac{1}{1 + x^2}$, (c) $\dfrac{x}{1 + x^2}$.

3. (a) $1 - \cos x$, (b) $x^2 \sin x$, (c) $\sec x + \tan x$.

4. (a) $[x]$, (b) $[x] + \frac{1}{2}$, (c) $[x^2]$.

In questions 5 to 8 decide whether the given functions are periodic and if so state the period.

5. (a) $\sin 3x$, (b) $1 + \tan 2x$, (c) $\cos \frac{1}{4}(x + \pi)$.

6. (a) $x + \cos x$, (b) $\sin x + \cos x$, (c) $\sin x \cos x$.

7. (a) $\sin 2x - \sin 3x$, (b) $x \sin 3x$, (c) $\sin 3x \cos 2x$.

8. (a) $|\cos x|$, (b) $\sin |x|$, (c) $[\tan x]$.

In questions 9 to 12 sketch the graphs of the functions f, g, fg and gf. In each case state whether the function is odd, even or neither, give its range and, if periodic, its period.

9. $f(x) = x^2 - 1$, $g(x) = 2x + 1$.

10. $f(x) = \frac{1}{2}(\pi - x)$, $g(x) = \sin x$.

11. $f(x) = \cos x$, $g(x) = x^2$.

12. $f(x) = |x|$, $g(x) = x(x^2 - 3)$.

In questions 13 to 16 sketch the graphs of the given periodic functions.

13. $f(x) = x^2$ for $-2 < x \leqslant 2$, f has period 4.

14. $f(x) = |x|$ for $-1 < x \leqslant 1$, f has period 2.

15. $f(x) = x^3$ for $0 \leqslant x \leqslant 1$,
$f(x) = -x$ for $-1 < x < 0$, f has period 2.

16. $f(x) = x$ for $0 \leqslant x \leqslant k$,
$f(x) = k$ for $k < x < 2k$, f has period $2k$.

In questions 17 to 20 sketch the graphs of the given functions. State any values of x at which
(a) there is a discontinuity,
(b) the function is not differentiable.

17. $1/(x^2 - 1)$. 18. $|x^2 - 1|$. 19. $x + [x]$.

20. $f(x) = x$ for $|x| \leqslant 1$, $f(x) = x^2 - 2$ for $|x| > 1$.

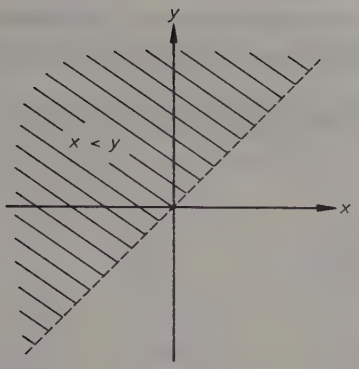
21. Given that f is a periodic function such that for $0 \leqslant x \leqslant 1$, $f(x) = x^2 - 2x$, sketch the graph of the function in each of the following cases
(a) f has period 1,
(b) f is an even function and has period 2,
(c) f is an odd function and has period 2.

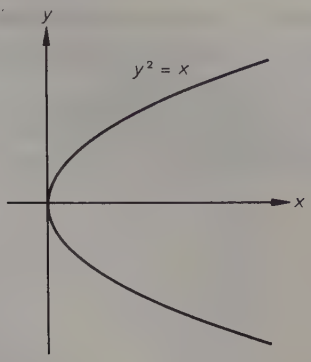
22. A function f is defined by $f(x) = x^2 + 3x$ for $x < 0$, $f(x) = kx$ for $x \geqslant 0$. Show that $f(x)$ is differentiable at $x = 0$ for only one value of k and state this value.

2.2 Some graphs with related equations

In this section we consider some graphs with equations of the form $y = f(x)$ together with graphs which have related equations such as $y = |f(x)|$, $y = 1/f(x)$, $y = \{f(x)\}^2$, $y^2 = f(x)$.

However, we first digress a little to note that not all graphs represent functions. There are many ways in which two real numbers x and y may be related, e.g. "$x < y$", "$y^2 = x$" or perhaps "$x - y$ is an integer". Each of these statements is said to define a *relation* on the set of real numbers. The graph of the relation is the set of points (x, y) such that x and y are related in the stated manner.



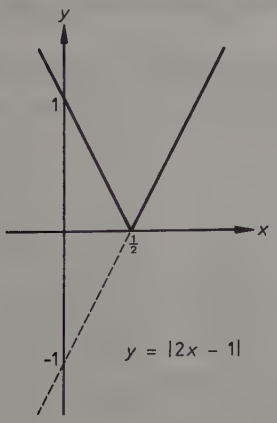


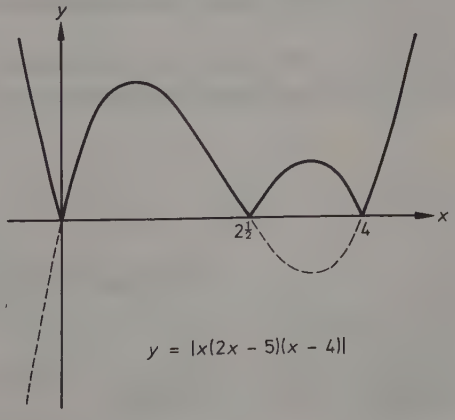
Thus, in general, an equation (or inequality) connecting x and y defines a relation on $\mathbb{R}$. Such an equation defines a function of x with domain $\mathbb{R}$ (or a subset of $\mathbb{R}$) only if there is at most one value of y corresponding to any given value of x. For instance the equation $y^2 = x$ specifies a relation, but not a function of the form $f: x \to y$, whereas the equation $y = \sqrt{x}$ defines a function with domain $\{x \in \mathbb{R}: x \geqslant 0\}$.

The equation $y = |f(x)|$

When $f(x) \geqslant 0$, the graph of $y = |f(x)|$ coincides with the graph of $y = f(x)$. When $f(x) < 0$, the graph of $y = |f(x)|$ is the reflection in the x-axis of the graph of $y = f(x)$.

[In the following sketches broken lines are used for the graph of $y = f(x)$ and unbroken lines for the graph of $y = |f(x)|$.]



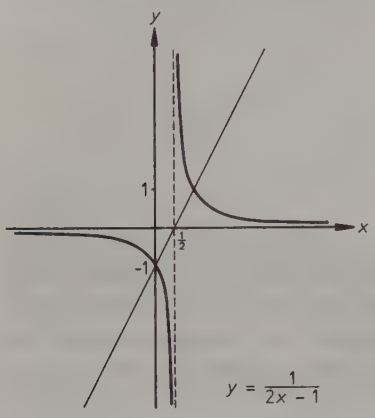


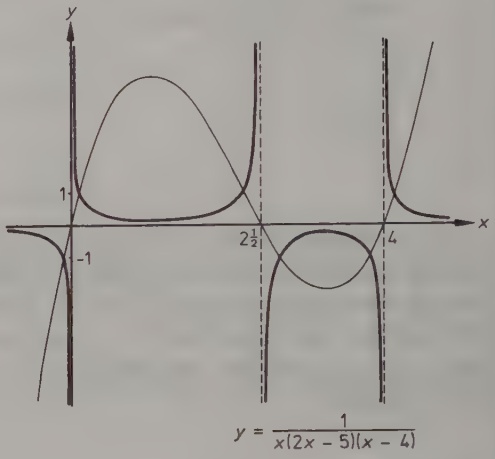
The equation $y = 1/f(x)$

If the graph of $y = f(x)$ meets the x-axis at $x = a$, then the line $x = a$ is an asymptote to the graph of $y = 1/f(x)$ and vice versa.

If there is a maximum (minimum) point on the graph of $y = f(x)$ when $x = k$, then provided that $f(k) \neq 0$, there is a minimum (maximum) point on the graph of $y = 1/f(x)$ when $x = k$.

The graphs of $y = f(x)$ and $y = 1/f(x)$ always lie on the same side of the x-axis and intersect when $f(x) = \pm 1$.





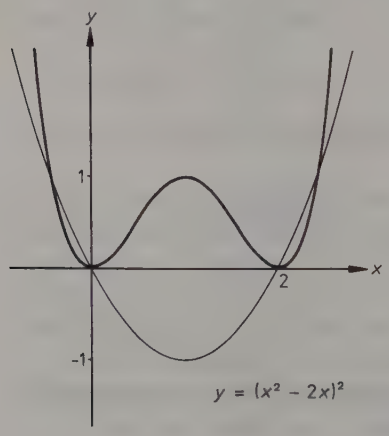
The equation $y = \{f(x)\}^2$

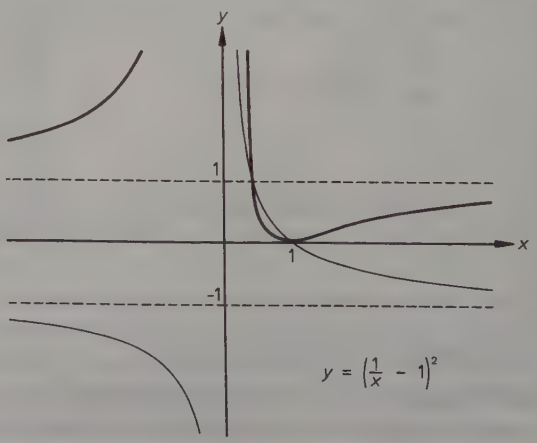
The graph of $y = \{f(x)\}^2$ lies entirely on or above the x-axis and cuts the graph of $y = f(x)$ when $f(x) = 0$ or 1.

$$y = \{f(x)\}^2 \Rightarrow \frac{dy}{dx} = 2f(x)f'(x)$$

Thus, since $dy/dx = 0$ when $f'(x) = 0$, there is a stationary point on the graph of $y = \{f(x)\}^2$ at any value of x for which there is a stationary point on the graph of $y = f(x)$.

Since $dy/dx = 0$ when $f(x) = 0$, any point at which the graph of $y = f(x)$ cuts the x-axis is a stationary point on the graph of $y = \{f(x)\}^2$.





The equation $y^2 = f(x)$

At values of x such that $f(x) < 0$, there are no points on the graph of $y^2 = f(x)$.

At values of x such that $f(x) > 0$, there are two possible values of y, namely $\pm\sqrt{\{f(x)\}}$. Hence the graph of $y^2 = f(x)$ is symmetrical about the x-axis.

It is also useful to note that if $0 < f(x) < 1$, then $\sqrt{\{f(x)\}} > f(x)$ and if $f(x) > 1$, then $\sqrt{\{f(x)\}} < f(x)$.

$$y^2 = f(x) \Rightarrow 2y\frac{dy}{dx} = f'(x)$$

$$\therefore \text{ if } f'(x) = 0, \text{ and } f(x) > 0, \text{ then } \frac{dy}{dx} = 0.$$

Thus, any stationary point above the x-axis on the curve $y = f(x)$ gives rise to a pair of stationary points on the curve $y^2 = f(x)$.

To determine how the graph of $y^2 = f(x)$ behaves near the x-axis, the expression for dy/dx must be considered further. For instance, suppose that $f(x) = x(x-3)^2$.

If $$y^2 = x(x-3)^2 = x^3 - 6x^2 + 9x$$

then differentiating with respect to x,

$$2y\frac{dy}{dx} = 3x^2 - 12x + 9 = 3(x-1)(x-3)$$

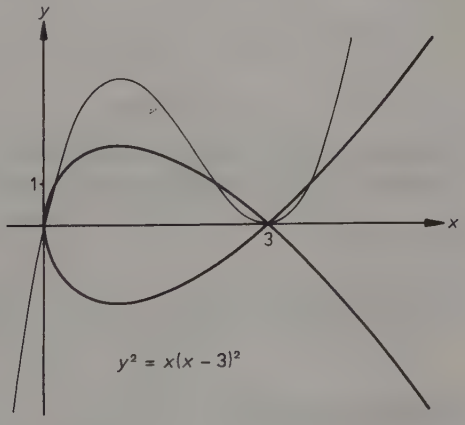
But $y = \pm(x-3)\sqrt{x}$

$\therefore$ provided that $x \neq 3$,

$$\frac{dy}{dx} = \pm\frac{3(x-1)}{2\sqrt{x}}$$

As $x \to 0$, $\dfrac{dy}{dx} \to \pm\infty$.

As $x \to 3$, $\dfrac{dy}{dx} \to \pm\sqrt{3}$.



Exercise 2.2

In questions 1 to 6 sketch the graphs with given equations

1. (a) $y = x^2$, (b) $y = \frac{1}{2}x^2$, (c) $y = x^2 - 4$, (d) $y = (x+2)^2$.

2. (a) $y = \sin x$, (b) $y = 2\sin x$, (c) $y = \sin 2x$, (d) $y = \sin(x + \frac{1}{2}\pi)$.

3. (a) $y = \dfrac{1}{x}$, (b) $y = \dfrac{4}{x}$, (c) $y = \dfrac{1}{x+1}$, (d) $y = \dfrac{x+2}{x+1}$.

4. (a) $y = |3 - 2x|$, (b) $y = |x^2 - 4|$, (c) $y = |x^3 - 3x^2|$.

5. (a) $y = |\cos x|$, (b) $y = |\cos x| - 1$, (c) $y = |2\cos x + 1|$.

6. (a) $y = \dfrac{1}{x-2}$, (b) $y = \dfrac{1}{2-x}$, (c) $y = \dfrac{1}{|x-2|}$.

In questions 7 to 16 sketch the given pair of curves in the same diagram. Label clearly any stationary points and asymptotes.

7. $y = x^2 + 2x$, $y = 1/(x^2 + 2x)$.

8. $y = x^2 + 2x + 1$, $y = 1/(x^2 + 2x + 1)$.

9. $y = x^2 + 2x + 2$, $y = 1/(x^2 + 2x + 2)$.

10. $y = x^3 + 2x^2 - 15x$, $y = 1/(x^3 - 2x^2 - 15x)$.

11. $y = (2x - 1)/(x + 2)$, $y = (x + 2)/(2x - 1)$.

12. $y = 2x - 3$, $y = (2x - 3)^2$.

13. $y = 4 - x^2$, $y = (4 - x^2)^2$.

14. $y = \sin x$, $y = \sin^2 x$.

15. $y = 2x/(1 - x)$, $y = 4x^2/(1 - x)^2$.

16. $y = 3x^2 - x^3$, $y^2 = 3x^2 - x^3$.

In questions 17 to 20 sketch the curves $y = f(x)$ and $y^2 = f(x)$ in the same diagram, showing clearly the behaviour of the curves near the x-axis.

17. $f(x) = x^3$.

18. $f(x) = x(x + 3)^2$.

19. $f(x) = (x + 1)(x - 2)^2$.

20. $f(x) = x^2(4 - x^2)$.

2.3 Inverse functions

Consider a function f with domain X and range Y. If every element of Y is the image of one and only one element of X, then f is said to be *one-one*. For any one-one function $f : X \to Y$ we define an *inverse function* $f^{-1} : Y \to X$ such that if $y = f(x)$ then $f^{-1}(y) = x$.

It follows from the definition that

$$f^{-1}f(x) = f^{-1}(y) = x \quad \text{and} \quad ff^{-1}(y) = f(x) = y$$

i.e. the composite functions $f^{-1}f$ and ff^{-1} are the identity mappings on the sets X and Y respectively.

Example 1 One-one functions f and g are defined on $\mathbb{R}$ by $f: x \to 3x - 1$ and $g: x \to \lg(1 - x)$, $x < 1$. Find expressions for f^{-1} and g^{-1}.

Let $y = f^{-1}(x)$ then $f(y) = x$

$\therefore \quad 3y - 1 = x$

$y = \frac{1}{3}(x + 1)$

Hence $f^{-1}: x \to \frac{1}{3}(x + 1)$.

Let $y = g^{-1}(x)$ then $g(y) = x$

$\therefore \quad \lg(1 - y) = x$

$1 - y = 10^x$

$y = 1 - 10^x$

Hence $g^{-1}: x \to 1 - 10^x$.

Example 2 A one-one function f is defined by $f: x \to 1/\sqrt{(x - 1)}$ where $x \in \mathbb{R}$, $x > 1$.
Find an expression for f^{-1}, stating its domain. Verify that $f^{-1}f(x) = x$ where $x \in \mathbb{R}$, $x > 1$.

$$\begin{aligned} y = f^{-1}(x) &\Rightarrow f(y) = x \\ &\Rightarrow 1/\sqrt{(y - 1)} = x \\ &\Rightarrow 1/(y - 1) = x^2 \\ &\Rightarrow 1 = x^2(y - 1) \\ &\Rightarrow y - 1 = 1/x^2 \\ &\Rightarrow y = 1 + 1/x^2 \end{aligned}$$

$\therefore \quad f^{-1}: x \to 1 + 1/x^2$.

The domain of f^{-1} is the same as the range of f
i.e. $\mathbb{R}^+$, the set of positive real numbers.

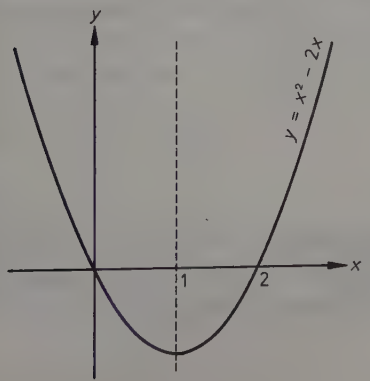
$f^{-1}f(x) = f^{-1}(u)$, where $u = f(x) = 1/\sqrt{(x - 1)}$

$\therefore \quad f^{-1}f(x) = 1 + 1/u^2 = 1 + (x - 1) = x$.

Example 3 A function f is defined on $\mathbb{R}$ by $f: x \to x^2 - 2x$. Explain why f^{-1} does not exist. Suggest a restricted domain for which f^{-1} does exist.

$f(0) = f(2) = 0$ i.e. 0 is the image under f of both 0 and 2,

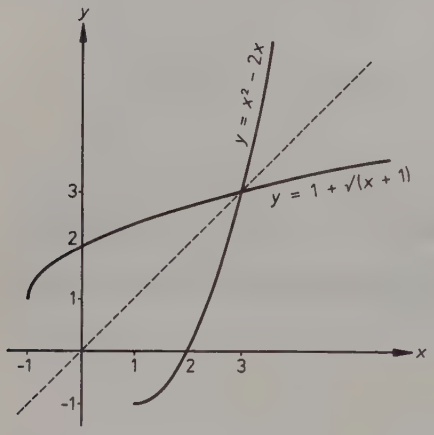
$\therefore$ the function f defined over $\mathbb{R}$ is not one-one and has no inverse.



The sketch of the graph $y = x^2 - 2x$ shows that for $x \geqslant 1$ there is only one value of x corresponding to each value of y, $\therefore$ over the domain $\{x \in \mathbb{R}: x \geqslant 1\}$ the function f is one-one and f^{-1} does exist. [Note that f^{-1} also exists for the restricted domain $\{x \in \mathbb{R}: x \leqslant 1\}$.]

In general, a function f has an inverse if and only if any line drawn parallel to the x-axis cuts the graph $y = f(x)$ exactly once. Clearly a function whose graph has one or more turning points does not have an inverse.

If a point (a, b) lies on the graph $y = f(x)$ then the point (b, a) lies on the graph $y = f^{-1}(x)$. Hence each graph is the reflection of the other in the line $y = x$.



The diagram illustrates this relationship for the function

$$f: x \to x^2 - 2x, \quad x \geqslant 1.$$

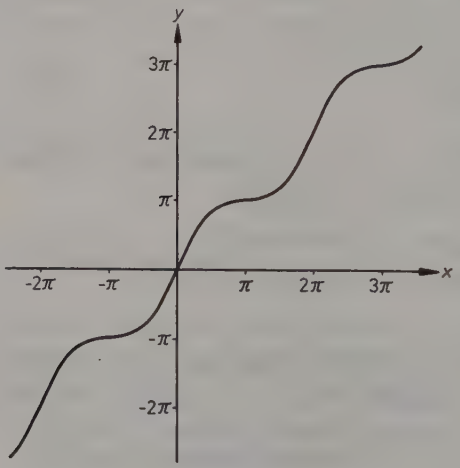
The equation of the graph of f^{-1} is $y^2 - 2y = x, y \geqslant 1$, which may be rearranged as

$$(y-1)^2 = x + 1, \ y \geqslant 1$$

i.e. $$y - 1 = \sqrt{(x+1)}$$

i.e. $$y = 1 + \sqrt{(x+1)}.$$

In some cases it is possible to show that a function f has an inverse, but difficult to find an expression for $f^{-1}(x)$. Consider, for example, the function $f: x \to x + \sin x$.



$$y = x + \sin x$$

$$\Rightarrow \quad \frac{dy}{dx} = 1 + \cos x$$

$$\Rightarrow \quad \frac{dy}{dx} \geqslant 0 \quad \text{for all} \quad x \in \mathbb{R}.$$

A sketch of the graph $y = x + \sin x$ shows that the function f must be one-one.

Hence the inverse f^{-1} exists and its graph has equation $y + \sin y = x$. However, it is impossible to rearrange this equation in the form $y = f^{-1}(x)$. Thus although f^{-1} exists, in practice it is difficult to find $f^{-1}(x)$ for many values of x.

Exercise 2.3

In questions 1 to 14 state whether or not the function f is one-one. If the function f is one-one, find an expression for f^{-1}, stating its domain. For any function that is not one-one, give two distinct values of x which have the same image.

1. $f:x \to 2x + 1$.
2. $f:x \to 5 - x$.
3. $f:x \to x(x + 2)$.
4. $f:x \to x^2 - 2, x \geqslant 0$.
5. $f:x \to 1 + \frac{1}{x}, x \neq 0$.
6. $f:x \to \frac{1}{9 - x}, x \neq 9$.
7. $f:x \to \sqrt{(3 - x)}, x \leqslant 3$.
8. $f:x \to \sqrt{(1 - x^2)}, -1 \leqslant x \leqslant 1$.
9. $f:x \to 3^x$.
10. $f:x \to \lg(x + 2), x > -2$.
11. $f:x \to \frac{1}{x^2 + 4}$.
12. $f:x \to \frac{2x}{x - 1}, x \neq 1$.
13. $f:x \to \cos 3x$.
14. $f:x \to \sin^{-1} x, -1 \leqslant x \leqslant 1$.

15. One-one functions f and g are defined on $\mathbb{R}$ by $f:x \to 5 - 4x$ and $g:x \to 1 - x^2, x \geqslant 0$. Given that h is the composite function fg, find expressions for $h(x)$ and $h^{-1}(x)$. Verify that $h^{-1} = g^{-1}f^{-1}$.

16. Functions f, g and h are defined on $\mathbb{R}$ by $f:x \to 3 - x$, $g:x \to 3x/(x - 3)$, $x \neq 3$ and $h:x \to \sqrt{(4 - x^2)}$, $0 \leqslant x \leqslant 2$. Show that each of the functions is its own inverse. Find the composite function gf and verify that its inverse is fg.

17. Find the inverse of each of the following functions defined on $\mathbb{R}$. Verify that, over the appropriate domains, $f^{-1}f(x) = x$ and $ff^{-1}(x) = x$.

(a) $f:x \to (x - 2)^2, x \geqslant 2$,
(b) $f:x \to 1 + \lg(x - 1), x > 1$,
(c) $f:x \to \frac{1 - x}{1 + x}, x \neq -1$,
(d) $f:x \to \frac{1}{\sqrt{(x^2 - 1)}}, x > 1$.

In questions 18 to 23 the function f has domain $\mathbb{R}^+$, the set of positive real numbers. In each case sketch the graph of f and explain why no inverse function exists. Find the maximum domain of the form $\{x \in \mathbb{R} : x \geqslant k\}$ for which f would have an inverse f^{-1} and state the domain of the inverse.

18. $f:x \to x^2 - 4x + 8$.
19. $f:x \to 6x - x^2$.
20. $f:x \to |x - 4|$.
21. $f:x \to 2x^3 - 3x^2$.
22. $f:x \to x + \frac{1}{x}$.
23. $f:x \to x - \sqrt{x}$.

24. Decide whether the following functions have inverses, giving brief reasons for your answers.

(a) $f: x \to x^3 + 3x^2, x \in \mathbb{R}$,
(b) $f: x \to x^3 + 3x, x \in \mathbb{R}$,
(c) $f: x \to 2x + \cos x, x \in \mathbb{R}$,
(d) $f: x \to x + 2\sin x, x \in \mathbb{R}$,
(e) $f: x \to |\lg x|, x \in \mathbb{R}^+$,
(f) $f: x \to |x| + 2x, x \in \mathbb{R}$.

25. Differentiable functions f and g are defined on $\mathbb{R}$ such that f is the inverse of g. Prove that, if $f(a) = b$, then $f'(a)g'(b) = 1$.

2.4 Polynomials; remainders and factors

If, when a polynomial $P(x)$ is divided by $(x - a)$, the quotient is $Q(x)$ and the remainder is R, then

$$P(x) \equiv (x - a)Q(x) + R.$$

This identity leads to two familiar results.

The remainder theorem: When $P(x)$ is divided by $(x - a)$ the remainder is $P(a)$.

The factor theorem: If $P(a) = 0$, then $(x - a)$ is a factor of $P(x)$.

This approach can be extended to the division of a polynomial $f(x)$ by a polynomial $g(x)$ of degree less than or equal to the degree of $f(x)$. If the division gives quotient $Q(x)$ and remainder $R(x)$ then

$$f(x) \equiv g(x)Q(x) + R(x)$$

where $R(x)$ is of lower degree than $g(x)$.

[In particular, if $g(x)$ is a quadratic function then $R(x)$ is of the form $Ax + B$.]

Example 1 A polynomial $P(x)$ is a multiple of $x - 3$ and the remainder when $P(x)$ is divided by $x + 3$ is 12. Find the remainder when $P(x)$ is divided by $x^2 - 9$.

When $P(x)$ is divided by $x^2 - 9$, let the quotient be $Q(x)$ and the remainder $Ax + B$, then

$$P(x) \equiv (x^2 - 9)Q(x) + Ax + B.$$

Since $x - 3$ is a factor of $P(x)$, by the factor theorem,

$$P(3) = 0 \qquad \therefore \quad 3A + B = 0 \tag{1}$$

Since when $P(x)$ is divided by $x + 3$ the remainder is 12, by the remainder theorem,

$$P(-3) = 12 \qquad \therefore \quad -3A + B = 12 \tag{2}$$

Using (1) and (2), $A = -2$, $B = 6$.

Hence the remainder when $P(x)$ is divided by $x^2 - 9$ is $-2x + 6$.

Suppose now that a polynomial $f(x)$ has a repeated factor $(x - a)$,

so that $f(x) \equiv (x - a)^2 g(x)$

Differentiating, $f'(x) = (x - a)^2 g'(x) + 2(x - a)g(x)$

$= (x - a)\{(x - a)g'(x) + 2g(x)\}$

Hence, if $f(x)$ has a repeated factor $(x - a)$, then $(x - a)$ is also a factor of $f'(x)$.

It can further be shown, by the method of question 21, Exercise 2.4, that $(x - a)^2$ is a factor of a polynomial $f(x)$ if and only if $f(a) = f'(a) = 0$.

Example 2 Given that the polynomial $f(x) = x^3 + 3x^2 - 9x + k$ has a repeated linear factor, find the possible values of k.

Differentiating, $f'(x) = 3x^2 + 6x - 9$

$= 3(x^2 + 2x - 3) = 3(x - 1)(x + 3)$

$\therefore$ the repeated factor of $f(x)$ is $(x - 1)$ or $(x + 3)$.
If $(x - 1)$ is a factor of $f(x)$, then $f(1) = 0$
i.e. $1 + 3 - 9 + k = 0$ giving $k = 5$.
If $(x + 3)$ is a factor of $f(x)$, then $f(-3) = 0$
i.e. $-27 + 27 + 27 + k = 0$ giving $k = -27$.
Thus the possible values of k are 5 and -27.

When dealing with polynomial equations it may be helpful to use the relationships between roots and coefficients. In the case of the cubic equation $ax^3 + bx^2 + cx + d = 0$ with roots α, β and γ,

$$\alpha + \beta + \gamma = -\frac{b}{a}, \quad \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}, \quad \alpha\beta\gamma = -\frac{d}{a}.$$

Similar results can be obtained for equations of higher degree.

Example 3 Find the roots of the equation $4x^3 + 12x^2 - 15x + 4 = 0$ given that it has a repeated root.

Let $f(x) = 4x^3 + 12x^2 - 15x + 4$
then $f'(x) = 12x^2 + 24x - 15$

$= 3(4x^2 + 8x - 5) = 3(2x - 1)(2x + 5)$

Any repeated root of the equation $f(x) = 0$ is also a root of the equation $f'(x) = 0$
$\therefore$ the repeated root must be either $-5/2$ or $1/2$.
[It is possible to decide which of these values is the required root by evaluating $f(-5/2)$ and $f(1/2)$. However, the method given below may be quicker.]

The product of the roots of $f(x) = 0$ is $-4/4$ i.e. -1.
Thus, if the repeated root is $-5/2$, the remaining root is $-1 \div (-5/2)^2$ i.e. $-4/25$.
If the repeated root is $1/2$, the remaining root is $-1 \div (1/2)^2$ i.e. -4.
Since the sum of the roots of $f(x) = 0$ is $-12/4$ i.e. -3, the roots must be $1/2$, $1/2$, -4.

Let us consider two polynomials $P(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n$ and

$$Q(x) = b_0x^n + b_1x^{n-1} + \ldots + b_n.$$

It can be shown that if $P(x) \equiv Q(x)$ i.e. if $P(x)$ and $Q(x)$ are equal for all values of x, then $a_0 = b_0, a_1 = b_1, \ldots, a_n = b_n$.

One way of proving this result uses the fact that if $P(x) \equiv Q(x)$, then $P'(x) \equiv Q'(x)$, $P''(x) \equiv Q''(x), \ldots, P^{(n)}(x) \equiv Q^{(n)}(x)$.

Substituting $x = 0$, we find that $P(0) = Q(0) \Rightarrow a_n = b_n$,

$P'(0) = Q'(0) \Rightarrow a_{n-1} = b_{n-1}, \ldots, P^{(n)}(0) = Q^{(n)}(0) \Rightarrow a_0 = b_0$.

We deduce that if two polynomials are identically equal, then corresponding coefficients are equal. Problems involving such identities are solved by *comparing coefficients*, by substituting suitable values of x, or by a combination of these methods.

Example 4 Given that $x^2 - 3x + 7 \equiv a(x-1)^2 + b(x-1) + c$, find the values of a, b and c.

Comparing coefficients of x^2: $1 = a$ $\quad\therefore\quad a = 1$

Putting $x = 1$: $1 - 3 + 7 = 0 + 0 + c$ $\quad\therefore\quad c = 5$

Putting $x = 0$: $7 = a - b + c$ $\quad\therefore\quad b = -1$

Hence $x^2 - 3x + 7 \equiv (x-1)^2 - (x-1) + 5$.

[Note that putting $x = 0$ is equivalent to comparing constant terms.]

Example 5 If $x^2 + 1$ is a factor of $3x^4 + x^3 - 4x^2 + px + q$, find the values of p and q.

Let $3x^4 + x^3 - 4x^2 + px + q \equiv (x^2 + 1)(ax^2 + bx + c)$.

Comparing coefficients of x^4: $3 = a$

x^3: $1 = b$

x^2: $-4 = a + c$

x: $p = b$

Comparing constant terms: $q = c$

Hence $p = 1$ and $q = -7$.

Exercise 2.4

1. Find the remainder when

(a) $x^3 - 4x^2 + 3x - 7$ is divided by $x - 4$,

(b) $x^4 + 2x^3 + 5x - 8$ is divided by $x + 3$,

(c) $2x^3 - 5x^2 + 3x + 1$ is divided by $2x + 1$,

(d) $x^3 - 2x^2 - 5$ is divided by $(x + 1)(x - 2)$,

(e) $x^4 + 3x^3 - 12x - 16$ is divided by $x^2 - 4$,

(f) $2x^4 - 3x^3 - 9x^2 - 7$ is divided by $x^2 - x - 6$.

2. When a polynomial $f(x)$ is divided by $(x - 2)$ the remainder is -2, and when it is divided by $(x - 3)$ the remainder is 3. Given that the remainder when $f(x)$ is divided by $(x - 2)(x - 3)$ is $px + q$, find the values of p and q.

3. When a polynomial is divided by $(x + 1)$ the remainder is 7, and when it is divided by $(x - 4)$ the remainder is -8. Find the remainder when the polynomial is divided by $(x + 1)(x - 4)$.

4. When the polynomial $x^3 + px^2 + qx + 7$ is divided by $x^2 - 3x + 2$, the remainder is $5x - 3$. Find the values of p and q.

5. When $P(x) = x^3 + ax^2 + bx + c$ is divided by $x^2 - 4$ the remainder is $2x + 11$. Given that $x + 1$ is a factor of $P(x)$, find the values of a, b and c.

6. Show, by putting $a = -b$, that $a + b$ is a factor of $(a + b + c)^3 - (a^3 + b^3 + c^3)$. Factorise the expression completely.

7. Show, by putting $x = y$, that $x - y$ is a factor of $(x - y)^3 + (y - z)^3 + (z - x)^3$. Factorise the expression completely.

8. Given that the polynomial $f(x) = 2x^3 - 3x^2 - 12x + k$ has a repeated linear factor, find the possible values of k and factorise $f(x)$ in each case.

9. Factorise the polynomial $3x^4 + 8x^3 + 16$ given that it has a repeated linear factor.

10. Find the values of a and b and factorise the polynomial

$$f(x) = x^3 + 4ax^2 + bx + 3a$$

given that it is divisible by $(x - 1)^2$.

11. Find the roots of the equation $x^3 - 6x^2 - 63x - 108 = 0$ given that it has a repeated root.

12. Given that $P(x) = 8x^3 - 12x^2 - 18x + k$, find the values of k such that the equation $P(x) = 0$ has a repeated root. Give the roots of the equation in each case.

13. The roots of the equation $2x^3 + 3x^2 + kx - 6 = 0$ are in arithmetic progression. By letting the roots be $\alpha - \lambda, \alpha, \alpha + \lambda$, or otherwise, find the value of k and solve the equation.

14. Given that the roots of the equation $3x^3 - 7x^2 + px + 24 = 0$ are in geometric progression, find the value of p and solve the equation.

15. If $ax^3 + 2x^2 + bx + 2b - c \equiv x^3 + 2x^2 + (c + 2)x + b + c$, find the values of a, b and c.

16. If $x^3 - 2x + 7 \equiv a(x-1)^3 + b(x-1)^2 + c(x-1) + d$, find the values of a, b, c and d.

17. Find constants a, b, c and d such that

$$n^3 \equiv an(n+1)(n+2) + bn(n+1) + cn + d.$$

18. Show that no constants a and b can be found such that

$$n^2 \equiv a(n+1)^2 + b(n+1).$$

19. Given that $x^4 - 6x^3 + 10x^2 + ax + b$ is a perfect square, find the values of a and b.

20. Given that $x^2 - x + 1$ is a factor of $2x^4 - 3x^3 + px^2 + qx - 3$, find the values of p and q.

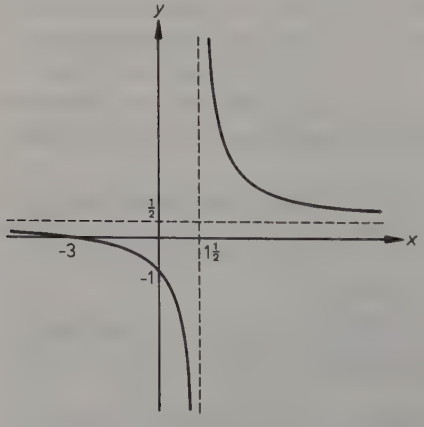
21. If $f(x)$ and $g(x)$ are polynomials and $f(x) \equiv (x-a)^2 g(x) + Ax + B$, find $f'(x)$ and hence find A and B in terms of $f(a)$ and $f'(a)$. Deduce that $(x-a)$ is a repeated factor of $f(x)$ if and only if $f(a) = f'(a) = 0$.

Find all real values of k for which the equation $x^3 - 3kx^2 + 2k + 2 = 0$ has repeated roots and, for each such k, solve the equation completely. (O & C)

2.5 Graphs of rational functions

A rational function is a function of the form $P(x)/Q(x)$, where $P(x)$ and $Q(x)$ are polynomials. The graph of this function cuts the x-axis where $P(x) = 0$ and has asymptotes parallel to the y-axis where $Q(x) = 0$.

One of the simplest types of rational function takes the form $(ax + b)/(cx + d)$. In general, the graph of this function is a curve called a *rectangular hyperbola* with asymptotes parallel to the x- and y-axes.



The diagram shows the graph of $y = (x+3)/(2x-3)$.
The curve cuts the axes at $(-3, 0)$ and $(0, -1)$.
As $x \to 1\frac{1}{2}$, $y \to \pm\infty$.

$$y = \frac{x+3}{2x-3} = \frac{1 + 3/x}{2 - 3/x}$$

$\therefore$ as $x \to \pm\infty$, $y \to \frac{1}{2}$.

Hence the lines $x = 1\frac{1}{2}$ and $y = \frac{1}{2}$ are asymptotes to the curve.

[If necessary, we can verify that such curves have no turning points by finding dy/dx.

In this case

$$\frac{dy}{dx} = \frac{(2x-3)1-(x+3)2}{(2x-3)^2} = -\frac{9}{(2x-3)^2}.$$

Thus the gradient of the curve is always negative.]

Another important set of rational functions are defined by expressions of the form $y = (ax+b)/(px^2+qx+r)$. When sketching the graphs of these functions it is helpful to find the range of values taken by y and also to note where changes in the sign of y occur.

We illustrate this approach by considering the graph of $y = 4x/(x^2+1)$.
The graph cuts the axes only at the origin $(0,0)$.
Since (x^2+1) is always positive, there can be no asymptotes parallel to the y-axis and the sign of y will always be the same as the sign of x.
As $x \to \infty, y \to 0$ from above and as $x \to -\infty, y \to 0$ from below,
$\therefore$ the x-axis is an asymptote to the curve.
To find the range of the function $y = 4x/(x^2+1)$, we rearrange the equation as a quadratic equation in x:

$$y(x^2+1) = 4x \quad \text{i.e.} \quad yx^2 - 4x + y = 0.$$

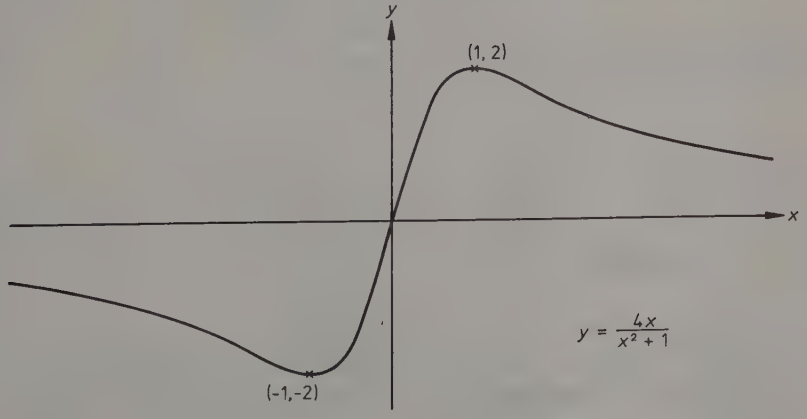
Since we are concerned only with values of y given by real values of x, the range of values taken by y is the set of values for which this quadratic equation has real roots.
The equation $ax^2+bx+c=0$ has real roots if $b^2-4ac \geqslant 0$,
$\therefore$ for real values of x, $(-4)^2 - 4y.y \geqslant 0$

$$16 - 4y^2 \geqslant 0$$

$$y^2 \leqslant 4$$

i.e. for real values of x, $-2 \leqslant y \leqslant 2$.

Thus all points of the graph lie on or between the lines $y = \pm 2$.



When $y = -2$ and when $y = 2$ the quadratic equation in x has equal roots, $x = -1$ and $x = 1$ respectively. Hence the line $y = -2$ is a tangent to the curve at the point $(-1,-2)$ and the line $y = 2$ is a tangent to the curve at the point $(1,2)$. It follows that $(-1,-2)$ must be a minimum point on the curve and $(1,2)$ a maximum point.

For every non-zero value of y in the interval $-2 < y < 2$ the quadratic equation in x has two distinct real roots. Hence there are no other turning points on the curve. [Note that a graph of similar form will be obtained for any function $(ax + b)/(px^2 + qx + r)$ if the denominator $(px^2 + qx + r)$ is never zero.]

Example 1 Sketch the curve $y = \dfrac{2x - 3}{x^2 + 2x - 3}$.

$x = 0 \Rightarrow y = 1$ $\quad\therefore$ the curve cuts the y-axis at the point $(0; 1)$.

$y = 0 \Leftrightarrow 2x - 3 = 0 \Leftrightarrow x = 1\frac{1}{2}$

$\therefore$ the curve cuts the x-axis at the point $(1\frac{1}{2}, 0)$.

$x^2 + 2x - 3 = 0 \Leftrightarrow (x + 3)(x - 1) = 0 \Leftrightarrow x = -3$ or $x = 1$

$\therefore$ the lines $x = -3$ and $x = 1$ are asymptotes to the curve.

As $x \to \pm\infty, y \to 0$ $\quad\therefore$ the x-axis is an asymptote to the curve.

The following table shows the changes in the sign of y.

	$x < -3$	$-3 < x < 1$	$1 < x < 1\frac{1}{2}$	$x > 1\frac{1}{2}$
$x + 3$	$-$	$+$	$+$	$+$
$x - 1$	$-$	$-$	$+$	$+$
$2x - 3$	$-$	$-$	$-$	$+$
y	$-$	$+$	$-$	$+$

Rearranging the equation as a quadratic in x:

$$y(x^2 + 2x - 3) = 2x - 3 \quad \text{i.e.} \quad yx^2 + (2y - 2)x - 3y + 3 = 0$$

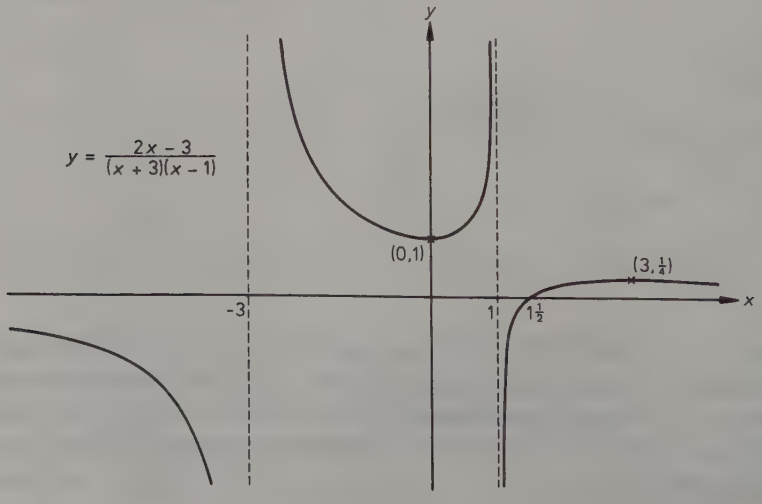
$\therefore$ for real values of x, $(2y - 2)^2 - 4y(-3y + 3) \geqslant 0$

$$4(y - 1)^2 + 12y(y - 1) \geqslant 0$$

$$4(y - 1)\{(y - 1) + 3y\} \geqslant 0$$

$$(y - 1)(4y - 1) \geqslant 0$$

$\therefore$ for real values of x, $y \leqslant \frac{1}{4}$ or $y \geqslant 1$.



Hence there are no points on the curve between the lines $y = \frac{1}{4}$ and $y = 1$.
There is a stationary point on the curve whenever the quadratic equation in x has repeated roots, i.e. when $y = \frac{1}{4}$, $x = 3$ and when $y = 1, x = 0$. Since y cannot take values between $\frac{1}{4}$ and 1, $(3, \frac{1}{4})$ is a maximum point and $(0, 1)$ a minimum point.
There is one other form that the graph of $y = (ax + b)/(px^2 + qx + r)$ can take.

Consider, for example, the curve $y = x/(x + 2)(x - 1)$.
This curve cuts the axes only at the origin $(0, 0)$.
The asymptotes to the curve are the lines $x = -2$, $x = 1$ and the x-axis.
Rearranging the equation as a quadratic in x:

$$y(x^2 + x - 2) = x \quad \text{i.e.} \quad yx^2 + (y - 1)x - 2y = 0.$$

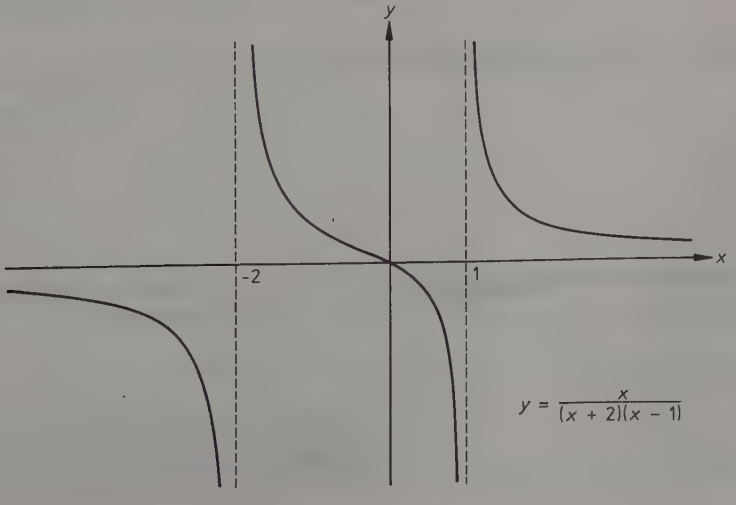
The nature of the roots of this quadratic equation depends on the sign of the expression

$$(y - 1)^2 - 4y(-2y) \quad \text{i.e.} \quad (y - 1)^2 + 8y^2.$$

Since this expression is always positive, the equation must have two real distinct roots for all non-zero values of y. Hence there are no stationary points on the curve and no restrictions on the values of y.

Again it is useful to determine the sign of y throughout the domain of the function.

	$x < -2$	$-2 < x < 0$	$0 < x < 1$	$x > 1$
$x + 2$	$-$	$+$	$+$	$+$
x	$-$	$-$	$+$	$+$
$x - 1$	$-$	$-$	$-$	$+$
y	$-$	$+$	$-$	$+$



[These methods can be extended to functions of the form $P(x)/Q(x)$ where both $P(x)$ and $Q(x)$ are quadratic functions.]

Exercise 2.5

In questions 1 to 6 sketch the given curve.

1. $y = \dfrac{1}{x-1}$ 2. $y = \dfrac{x}{x-1}$ 3. $y = \dfrac{x-1}{x}$

4. $y = \dfrac{3-x}{2+x}$ 5. $y = \dfrac{2x+1}{x-2}$ 6. $y = \dfrac{x}{3x+4}$

In questions 7 to 16 find the range of possible values of y. Sketch each curve showing clearly any asymptotes and turning points.

7. $y = \dfrac{5x-25}{x^2+3x-4}$ 8. $y = \dfrac{2x+1}{x^2+2}$

9. $y = \dfrac{x+1}{x(x+3)}$ 10. $y = \dfrac{x-1}{x(x+3)}$

11. $y = \dfrac{2-3x}{x^2-3x+2}$ 12. $y = \dfrac{15-10x}{4+x^2}$

13. $y = \dfrac{1-x}{1-x+x^2}$ 14. $y = \dfrac{x+6}{x^2+7x+10}$

15. $y = \dfrac{1}{x-1} + \dfrac{1}{4-x}$ 16. $y = \dfrac{1}{x-1} + \dfrac{1}{x-4}$

17. If a, b and c are all positive, sketch the curve $y = \dfrac{x-a}{(x-b)(x-c)}$ when

(i) $a < b < c$,
(ii) $b < a < c$,
(iii) $b < c < a$.

18. By considering the limiting value of y as $x \to \pm\infty$, show that the line $y = 1$ is an asymptote to the curve $y = (x^2+1)/(x^2-1)$. Sketch the curve.

19. Sketch the curves

(a) $y = \dfrac{3x^2}{x^2+x-2}$, (b) $y = \dfrac{3x^2}{x^2+x+2}$.

20. Sketch the curve $y = \dfrac{x^2 - 3x + 4}{x - 3}$ showing clearly any asymptotes and turning points.

2.6 Partial fractions

A sum of two fractions can be expressed as a single fraction by a method such as the following:

$$\frac{3}{3x-2} - \frac{x}{x^2+1} \equiv \frac{3(x^2+1) - x(3x-2)}{(3x-2)(x^2+1)} \equiv \frac{2x+3}{(3x-2)(x^2+1)}.$$

In this section we consider the reverse process by which a rational function is expressed as a sum of two or more simpler fractions called *partial fractions.*

We will deal first with *proper fractions* i.e. fractions in which the degree of the numerator is less than the degree of the denominator. The methods used in the examples which follow are based on the assumption that a proper fraction can be expressed as a sum of partial fractions which are also proper, such as

$$\frac{A}{x-a}, \quad \frac{Ax+B}{ax^2+bx+c}.$$

[A rigorous proof of this statement is beyond the scope of this course.]

Example 1 Express $\dfrac{2x-5}{(x+2)(x-1)}$ in partial fractions.

Let $$\frac{2x-5}{(x+2)(x-1)} \equiv \frac{A}{x+2} + \frac{B}{x-1}$$

then, multiplying both sides of the identity by $(x+2)(x-1)$,

$$2x - 5 \equiv A(x-1) + B(x+2)$$

Putting $x = 1$: $\quad 2 - 5 = 3B \quad \therefore \quad B = -1$

Putting $x = -2$: $\quad -4 - 5 = -3A \quad \therefore \quad A = 3$

$$\therefore \quad \frac{2x-5}{(x+2)(x-1)} \equiv \frac{3}{x+2} - \frac{1}{x-1}.$$

Example 2 Express $\dfrac{x+1}{2x^3 - 5x^2 + 2x}$ in partial fractions.

$$2x^3 - 5x^2 + 2x \equiv x(2x^2 - 5x + 2) \equiv x(x-2)(2x-1)$$

$$\therefore \quad \text{let} \quad \frac{x+1}{2x^3 - 5x^2 + 2x} \equiv \frac{A}{x} + \frac{B}{x-2} + \frac{C}{2x-1}$$

then $$x + 1 \equiv A(x-2)(2x-1) + Bx(2x-1) + Cx(x-2)$$

Putting $x = 0$: $1 = 2A$ $\quad\therefore\quad A = \frac{1}{2}$
Putting $x = 2$: $3 = 6B$ $\quad\therefore\quad B = \frac{1}{2}$
Putting $x = \frac{1}{2}$: $\frac{3}{2} = C \times \frac{1}{2} \times (-\frac{3}{2})$ $\quad\therefore\quad C = -2$

$$\therefore \quad \frac{x+1}{2x^3 - 5x^2 + 2x} \equiv \frac{1}{2x} + \frac{1}{2(x-2)} - \frac{2}{2x-1}.$$

So far we have determined unknown constants by substituting suitable values of x. In examples involving quadratic factors some constants can be obtained more quickly by comparing coefficients of powers of x.

Example 3 Express $\dfrac{4x^2}{(x-3)(x^2+3)}$ in partial fractions.

Let $$\frac{4x^2}{(x-3)(x^2+3)} \equiv \frac{A}{x-3} + \frac{Bx+C}{x^2+3}$$

then $$4x^2 \equiv A(x^2+3) + (Bx+C)(x-3)$$
Putting $x = 3$: $36 = 12A$ $\quad\therefore\quad A = 3$
Putting $x = 0$: $0 = 3A - 3C$ $\quad\therefore\quad C = 3$

Comparing coefficients of x^2:

$$4 = A + B \qquad \therefore \quad B = 1$$

$$\therefore \quad \frac{4x^2}{(x-3)(x^2+3)} \equiv \frac{3}{x-3} + \frac{x+3}{x^2+3}$$

In the next example we consider a fraction with the repeated factor $(x+1)^2$ in the denominator. Using the "proper fraction" rule, there will be a corresponding partial fraction of the form $(Ax+K)/(x+1)^2$. However, we obtained simpler partial fractions by writing $K = A + B$:

$$\frac{Ax+K}{(x+1)^2} \equiv \frac{Ax+A+B}{(x+1)^2} \equiv \frac{A(x+1)+B}{(x+1)^2} \equiv \frac{A}{x+1} + \frac{B}{(x+1)^2}.$$

Example 4 Express $\dfrac{x-5}{(x+1)^2(x-1)}$ in partial fractions.

Let $$\frac{x-5}{(x+1)^2(x-1)} \equiv \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1}$$

then $$x - 5 \equiv A(x+1)(x-1) + B(x-1) + C(x+1)^2$$
Putting $x = 1$: $-4 = 4C$ $\quad\therefore\quad C = -1$
Putting $x = -1$: $-6 = -2B$ $\quad\therefore\quad B = 3$

Comparing coefficients of x^2:

$$0 = A + C \qquad \therefore \quad A = 1$$

$$\therefore \quad \frac{x-5}{(x+1)^2(x-1)} \equiv \frac{1}{x+1} + \frac{3}{(x+1)^2} - \frac{1}{x-1}$$

A fraction in which the degrees of the numerator and denominator are equal can be expressed as the sum of a constant and a proper fraction,

e.g. $$\frac{x^2+1}{x^2-1} \equiv 1 + \frac{2}{x^2-1}.$$

We use this fact in the next example.

Example 5 Express $\dfrac{x^2+2}{(x-1)(2x+1)}$ in partial fractions.

Let $$\frac{x^2+2}{(x-1)(2x+1)} \equiv A + \frac{B}{x-1} + \frac{C}{2x+1}$$

then $$x^2+2 \equiv A(x-1)(2x+1) + B(2x+1) + C(x-1)$$

Putting $x = 1$: $\quad 3 = 3B \qquad \therefore \quad B = 1$

Putting $x = -\frac{1}{2}$: $\quad \frac{9}{4} = -\frac{3}{2}C \qquad \therefore \quad C = -\frac{3}{2}$

Comparing coefficients of x^2:

$$1 = 2A \qquad \therefore \quad A = \tfrac{1}{2}$$

$$\therefore \quad \frac{x^2+2}{(x-1)(2x+1)} \equiv \frac{1}{2} + \frac{1}{x-1} - \frac{3}{2(2x+1)}.$$

When dealing with fractions in which the degree of the numerator is greater than the degree of the denominator, we may either use long division to express the fraction as the sum of a polynomial and a proper fraction or extend the method of Example 5. If the degree of the numerator exceeds that of the denominator by n, then the expression in partial fractions must include a polynomial of degree n.

Example 6 Express $\dfrac{x^3-3x^2+1}{x^2-x-2}$ in partial fractions.

Method 1

$$\begin{array}{r} x-2 \\ x^2-x-2\overline{)\,x^3-3x^2\qquad +1} \\ x^3-\ x^2-2x \\ \hline -2x^2+2x+1 \\ -2x^2+2x+4 \\ \hline -3 \end{array}$$

$$\therefore \quad \frac{x^3-3x^2+1}{x^2-x-2} \equiv x - 2 - \frac{3}{x^2-x-2}$$

Let $$\frac{-3}{x^2-x-2} \equiv \frac{A}{x+1} + \frac{B}{x-2}$$

then $-3 \equiv A(x-2) + B(x+1)$

Putting $x = -1$: $-3 = -3A$ $\therefore A = 1$

Putting $x = 2$: $-3 = 3B$ $\therefore B = -1$

$$\therefore \quad \frac{x^3 - 3x^2 + 1}{x^2 - x - 2} \equiv x - 2 + \frac{1}{x+1} - \frac{1}{x-2}$$

Method 2

Let $$\frac{x^3 - 3x^2 + 1}{x^2 - x - 2} \equiv Ax + B + \frac{C}{x+1} + \frac{D}{x-2}$$

then $x^3 - 3x^2 + 1 \equiv (Ax + B)(x+1)(x-2) + C(x-2) + D(x+1)$

Putting $x = -1$: $-3 = -3C$ $\therefore C = 1$

Putting $x = 2$: $-3 = 3D$ $\therefore D = -1$

Putting $x = 0$: $1 = -2B - 2C + D$ $\therefore B = -2$

Comparing coefficients of x^3:

$$1 = A.$$

$$\therefore \quad \frac{x^3 - 3x^2 + 1}{x^2 - x - 2} \equiv x - 2 + \frac{1}{x+1} - \frac{1}{x-2}.$$

We now summarise the rules for expressing a rational function in partial fractions.

(1) To a linear factor $(x - a)$ in the denominator, there corresponds a fraction of the form $A/(x - a)$.

(2) To a quadratic factor $(ax^2 + bx + c)$, there corresponds a fraction of the form $(Ax + B)/(ax^2 + bx + c)$.

(3) To a repeated factor $(x - a)^n$, there correspond n fractions of the form $A_1/(x - a), A_2/(x - a)^2, \ldots, A_n/(x - a)^n$.

(4) To a repeated factor $(ax^2 + bx + c)^n$, there correspond n fractions

$$\frac{A_1x + B_1}{(ax^2 + bx + c)}, \frac{A_2x + B_2}{(ax^2 + bx + c)^2}, \ldots, \frac{A_nx + B_n}{(ax^2 + bx + c)^n}.$$

[Note that there are many ways of expressing a rational function as a sum of two or more fractions e.g. $\dfrac{1}{(x+1)(x+2)} \equiv \dfrac{x+1}{x+2} - \dfrac{x}{x+1} \equiv \dfrac{x^2}{x+1} - \dfrac{x^2 + x - 1}{x+2}$.

However, the simplest and most useful form is the expression in partial fractions according to rules (1) to (4) above.]

Exercise 2.6

Express in partial fractions

1. $\dfrac{1}{(x+1)(x+2)}$

2. $\dfrac{4 - x}{(x-1)(x+2)}$

3. $\dfrac{3x-10}{(x-2)(x-4)}$

4. $\dfrac{4x-9}{x(x-3)}$

5. $\dfrac{x+7}{(x-2)(x+1)}$

6. $\dfrac{5x+3}{(x+1)(2x+1)}$

7. $\dfrac{x}{x^2-1}$

8. $\dfrac{5}{6-x-x^2}$

9. $\dfrac{10x-1}{(2x+1)(4x-1)}$

10. $\dfrac{4-2x}{(2x-1)(2x+5)}$

11. $\dfrac{17x+11}{(x-2)(x+3)(x+1)}$

12. $\dfrac{5x^2-12x-5}{(x^2-1)(x-2)}$

13. $\dfrac{3x+1}{(x+1)(x^2+1)}$

14. $\dfrac{x+2}{(2x-1)(x^2+1)}$

15. $\dfrac{3x+1}{x(2x^2+1)}$

16. $\dfrac{x^2-10}{(x^2+3)(2x-1)}$

17. $\dfrac{x^2-13}{(x-1)^2(x+2)}$

18. $\dfrac{3x^2+7x+1}{x^3+2x^2+x}$

19. $\dfrac{2x^2-3x-2}{x^3-x^2}$

20. $\dfrac{x^2+23}{(x+1)^3(x-2)}$

21. $\dfrac{2x^2-5x-5}{(2x^2+5)(4x-5)}$

22. $\dfrac{x+2}{(x-2)(x^2-x+2)}$

23. $\dfrac{x^2+1}{x^2-1}$

24. $\dfrac{x^2}{x^2-x-2}$

25. $\dfrac{x(x-2)}{(3x-1)(x-1)}$

26. $\dfrac{x^3}{x^2-4}$

27. $\dfrac{x^2-x}{(x^2+3)(x^2+2)}$

28. $\dfrac{3x^3+2x^2+2x-3}{(x^2+2)(x+1)^2}$

29. $\dfrac{2x^4-2x^3+x}{(2x-1)^2(x-2)}$

30. $\dfrac{x^6-x^5-4x^2+x}{x^4+3x^2+2}$

31. Express $f(x)=\dfrac{4x+5}{(x+1)(2x+3)}$ in partial fractions. Hence find $f'(x)$ and $f''(x)$.

32. Find the coordinates of the points of inflexion on the curves

(a) $y = \dfrac{2x}{1-x^2}$, (b) $y = \dfrac{7(x+1)}{x(x-7)}$.

Exercise 2.7 (miscellaneous)

1. The functions f, g, h and k are defined by

$$\begin{aligned} &f: x \to \sin^2 3x, && x \in \mathbb{R}, \\ &g: x \to 1/(1-x), && x \in \mathbb{R},\ x \neq 1, \\ &h: x \to \sqrt{(1-x^2)}, && x \in \mathbb{R},\ |x| \leqslant 1, \\ &k: x \to \operatorname{cosec} \tfrac{1}{2}x, && x \in \mathbb{R},\ x \neq 2n\pi (n \in \mathbb{Z}). \end{aligned}$$

For each function, state whether it is even, odd or neither. State also whether or not it is periodic, giving the period where appropriate.

2. Sketch the graphs of the following functions. State any values of x at which
(a) there is a discontinuity,
(b) the function is not differentiable.

$$\begin{aligned} f(x) &= |x-2|, & g(x) &= [\tfrac{1}{2}x], \\ h(x) &= 1/(x^2 - x - 2), & k(x) &= |x^4 - x^3|. \end{aligned}$$

3. The functions f and g are defined by $f: x \to \sin 2x, x \in \mathbb{R}$, $g: x \to \cot x$, $x \in \mathbb{R}$, $x \neq k\pi (k \in \mathbb{Z})$. State the periods of f and g. Find the period of the function $f.g$. On separate axes, sketch the graph of f, g and $f.g$ for the interval $\{x: -\pi < x < \pi, x \neq 0\}$. Find the range of the function $f.g$. (JMB)

4. Determine whether each of the three functions, f, g, h defined in $\mathbb{R}$ is an even function, an odd function or neither, given that $f(x) = \pi/2 - \sin(x/2)$, $g(x) = \pi/2 - \cos(x/3)$, $h(x) = \cos(\sin x)$. Determine the period, if it exists, of each of the two functions $G(x) = g(x) - f(x)$, $H(x) = g(x) - h(x) + \sin[f(2x)]$. (L)

5. Find the ranges of the given functions f, g, h and k. State whether f^{-1}, g^{-1}, h^{-1}, k^{-1} exist, giving brief reasons for each answer.

$$\begin{aligned} &f: x \to x^2 + 4, && x \in \mathbb{R},\ 0 \leqslant x \leqslant 4; \\ &g: x \to x^2 - 4x, && x \in \mathbb{R},\ 0 \leqslant x \leqslant 4; \\ &h: x \to 3\cos x + 4\sin x, && x \in \mathbb{R}; \\ &k: x \to 1 - \lg(x+1), && x \in \mathbb{R},\ x \geqslant 0. \end{aligned}$$

6. Sketch the graph of the function f defined by $f(x) = 4x/(1+x^2)$, x real. Let g denote the function f restricted to the domain $[-a, a]$, so that $g(x) = 4x/(1+x^2)$, $-a \leqslant x \leqslant a$, where a is the largest positive number such that the function g is one-one. What is the value of a? Obtain an expression for the inverse function g^{-1}, and state the domain and range of g^{-1}. (W)

7. The function ϕ is defined by $\phi(x) = x^3 + 2x - 1$, and the inverse function ϕ^{-1} is denoted by ψ. Find the values of $\psi(2)$ and $\psi'(2)$. (W)

8. The domain of the function f is the set $D = \{x : x \in \mathbb{R}, -2 < x < 3\}$. The function $f : D \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 2x - 1 & \text{for } -2 < x \leqslant 1, \\ x^2 & \text{for } 1 < x \leqslant 2, \\ 10 - 3x & \text{for } 2 < x < 3. \end{cases}$$

Find the range of this function and sketch its graph. Explain why there is no inverse function to f. Suggest an interval such that f, restricted to this interval, will have an inverse function. Give an expression for the inverse function in this case. (L)

9. Explain what is meant by the statement "f is a function from a set A to a set B." A function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$\begin{cases} f(x) = \sin 2x, & 0 \leqslant x \leqslant \pi/4, \\ f(x) = 2 - 4x/\pi, & \pi/4 < x \leqslant \pi/2, \\ f \text{ is an odd function, } f \text{ is periodic with period } \pi. \end{cases}$$

Sketch the graph of $y = f(x)$ for $-\pi \leqslant x \leqslant \pi$. Find two different intervals, each of length $\pi/2$, such that f, restricted to either interval, will have an inverse. In each case, give an expression for the inverse function. (L)

10. (i) f_1 is a periodic odd function with a period 2π and f_2 is a periodic even function with a period 3π. Functions F and G are defined by $F(x) = f_2(x) . f_1(x)$, $G(x) = f_2[f_1(x)]$. State whether the functions F and G are
(a) odd, even or neither,
(b) periodic, and if periodic state a period.
Functions f_3 and f_4 are defined by $f_3(x) = x \cos x$, $f_4(x) = \cos(\sin 3x)$. State whether f_3 and f_4 are odd, even or neither and if periodic state a period.
(ii) A mapping $f : A \to \mathbb{R}$, where $A = \{x : x \in \mathbb{R} \text{ and } -\pi/4 \leqslant x \leqslant \pi/4\}$, is defined by $f(x) = \cos x + \sin x$. Find $f^{-1}(1)$. (L)

11. A function f is defined on $\mathbb{R}$ by $f : x \to |x + [x]|$ where $[x]$ indicates the greatest integer less than or equal to x, e.g. $[3] = 3$, $[2{\cdot}4] = 2$, $[-3{\cdot}6] = -4$. Sketch the graph of the function for $-3 \leqslant x \leqslant 3$. What is the range of f? Is the mapping one-one?
The function g is defined by $g : x \to |x + [x]|, x \in \mathbb{R}^+, x \notin \mathbb{Z}^+$. Find the rule and domain of the inverse function g^{-1}. (C)

12. Sketch the curve $y = |x^2 - 5x|$. Find the ranges of values of x for which $|x^2 - 5x| < 6$.

13. Sketch the curves $y = x^2(3 - x)$ and $y^2 = x^2(3 - x)$, showing clearly the behaviour of the curves near the x-axis.

14. Using a sketch graph, or otherwise, solve the inequality $\left|\frac{1}{1+2x}\right| < 1$.

15. Prove that the curves $y = \frac{x}{x+1}$ and $y = \left(\frac{x}{x+1}\right)^2$ intersect at only one point.

Sketch these curves in the same diagram. Label your sketch so that it is immediately clear which curve is which. (O&C)

16. Given that $f(x) \equiv 6x^2 + x - 12$, find the minimum value of $f(x)$ and the values of x for which $f(x) = 0$. Using the same axes, sketch the curves $y = f(x)$ and $y = 1/f(x)$, labelling each clearly. Deduce that there are four values of x for which $[f(x)]^2 = 1$. Find these values, each to two decimal places. (L)

17. The polynomial $P(x)$ leaves a remainder of 2 when divided by $(x - 1)$ and a remainder of 3 when divided by $(x - 2)$. The remainder when $P(x)$ is divided by $(x - 1)(x - 2)$ is $ax + b$. By writing $P(x) \equiv (x - 1)(x - 2)Q(x) + ax + b$, find the values of a and b. Given also that $P(x)$ is a cubic polynomial with coefficient of x^3 equal to unity, and that -1 is a root of the equation $P(x) = 0$, obtain $P(x)$. Show that the equation $P(x) = 0$ has no other real roots. (JMB)

18. Find the remainder when the polynomial $P(x) = x^3 - 5x + 2$ is divided by
(a) $x^2 + 1$,
(b) $(x - 1)^2$.

19. Using the remainder theorem, or otherwise, show that $x + a + b + c$ is a factor of

$$(x + a)(x + b)(x + c) + (b + c)(c + a)(a + b).$$

Hence, or otherwise, solve the equation

$$(x + 2)(x - 3)(x - 1) + 4 = 0. \quad \text{(C)}$$

20. Factorise completely the expressions
(a) $x(y^3 - z^3) + y(z^3 - x^3) + z(x^3 - y)^3$,
(b) $12abc + 4(a^3 + b^3 + c^3) - (a + b + c)^3$.

21. If $f(x)$ is a polynomial in x, show that when $f(x)$ is divided by $x - a$ the remainder is $f(a)$.
When $x^3 + ax^2 + bx + c$ is divided by $x + 3$ the remainder is -26 and when divided by $x^2 - x - 2$ the remainder is 14. Find the values of a, b and c.
(AEB 1978)

22. Show that if a polynomial $f(x)$ is divided by $(x - a)(x - b)$, where $a \neq b$, then the remainder is $\left\{\frac{f(a) - f(b)}{a - b}\right\}x + \left\{\frac{af(b) - bf(a)}{a - b}\right\}$.

23. Solve the equation $x^4 - 4x^3 - 20x^2 + 32 = 0$ given that it has a repeated root.

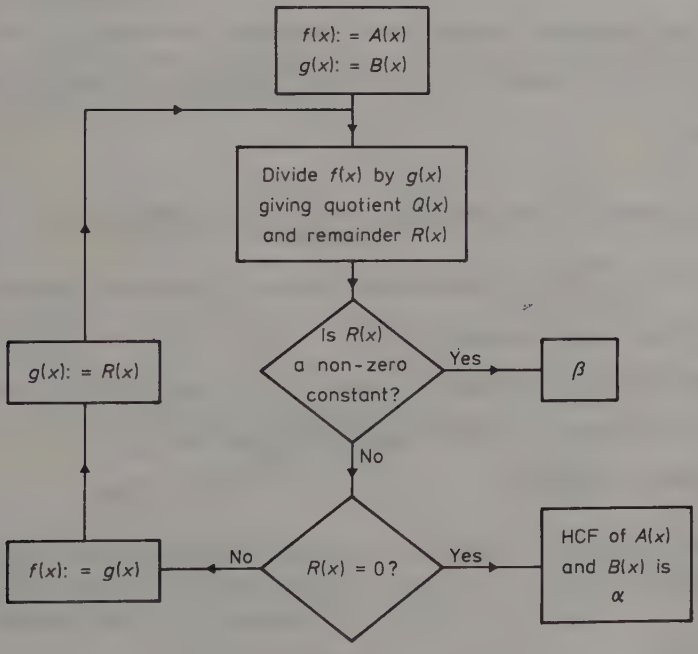
24. Given that the polynomial $f(x) = x^4 - 8x^3 + 10x^2 + p$ has a repeated linear factor, find the possible values of p.

25. The equation $x^3 - px^2 + qx - r = 0$ is such that its roots are in geometric progression. Show that
(a) one root is $\sqrt[3]{r}$,
(b) $p^3r = q^3$.

26. If the expression $2x^2 - xy - y^2 + 2x + ky - 4$ is the product of two linear factors, find the possible values of k.

27. (i) Given that $x + 1$ is a factor of $x^3 + 2x^2 + 3x + c$, find c.

(ii) In the algebraic operation of polynomial division, one polynomial $f(x)$ is divided by another polynomial $g(x)$ of order equal to or lower than that of $f(x)$. Polynomials $Q(x)$ and $R(x)$ are found such that $f(x) = g(x)Q(x) + R(x)$ and $R(x)$ is of lower order than $g(x)$. Carry out the process described in the flow chart when $A(x) = x^3 + 2x^2 + 3x + 2$ and $B(x) = x^2 - x - 2$.



State what should be written in the spaces α and β for the flow chart to be applicable to any two polynomials of suitable degree. (AEB 1980)

28. By considering the equation $yx^2 + (y - 1)x + (y - 1) = 0$, where $y \neq 0$, as a quadratic equation in x, find the range of values of y for which this equation has real roots. State the value of x which satisfies this equation when $y = 0$. Using these results, or otherwise, determine the range of the function $f: x \to \dfrac{x + 1}{x^2 + x + 1}$, where x is real. Sketch the graph of $f(x)$ giving the coordinates of any turning points. Explain why f has no inverse function. (JMB)

29. In each of the following cases find the range of the function f. Sketch the graph of f, showing clearly any asymptotes and turning points.

(a) $f(x) = \dfrac{x-1}{2x+1}$,

(b) $f(x) = \dfrac{4}{(1-x)(3+x)}$,

(c) $f(x) = \dfrac{3(x+1)}{x(x-3)}$,

(d) $f(x) = \dfrac{2x-1}{x^2-4}$.

30. Sketch each of the following curves, showing any stationary points and indicating clearly the form of the curve when $|x|$ becomes large.

(a) $y = x - \dfrac{1}{x+1}$.

(b) $y = \dfrac{x^2}{x-1}$,

(c) $y = \dfrac{x^2}{1+x^2}$,

(d) $y = \dfrac{x-1}{(x+1)^2}$.

31. State the maximum domain and determine the range of the function $f(x) = \dfrac{x^2+2x+3}{x^2-2x-3}$. Sketch the curve $y = \dfrac{x^2+2x+3}{x^2-2x-3}$ indicating clearly any asymptotes. (O)

32. (a) The function f is defined by $f(x) = \dfrac{x}{1+x^2}$. Show that $f'(x) = \dfrac{1-x^2}{(1+x^2)^2}$ and $f''(x) = \dfrac{-6x+2x^3}{(1+x^2)^3}$. Find the local maxima, minima and points of inflexion of f, and sketch the graph of f.

(b) The function g is defined by $g(x) = \dfrac{x^2}{1-x^2}$. Show that $g(x)$ cannot take values between -1 and 0. Sketch the graph of g. (W)

Express in partial fractions:

33. $\dfrac{3x+11}{(3x-2)(2x+3)}$

34. $\dfrac{3x-2}{(x-1)^2(2x-1)}$

35. $\dfrac{x}{x^3-x^2+x-1}$

36. $\dfrac{2}{x^3-2x^2-x+2}$

37. $\dfrac{5+x^2}{3+5x-2x^2}$

38. $\dfrac{3x}{1-x^3}$

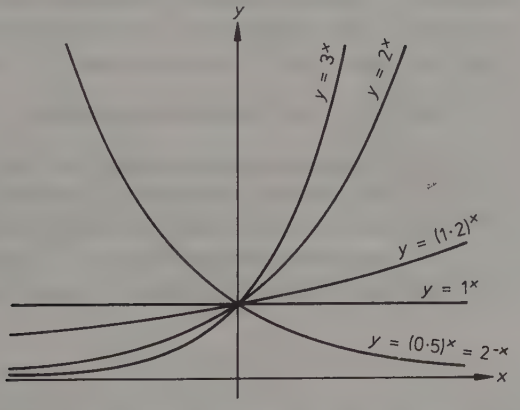
39. $\dfrac{x(x+1)(x-2)}{(x-1)(x+2)}$

40. $\dfrac{7x^2-2x+3}{(x^2-1)^2}$

3 Exponential and logarithmic functions

3.1 Exponential functions

Exponent is another word for index. A function such as 2^x, in which the index is variable, is called *an exponential function.*



The diagram shows the graphs of $y = a^x$ for various positive values of a. We see that each graph lies entirely above the x-axis and cuts the y-axis at the point $(0, 1)$. We now consider how to differentiate the function a^x.

Let $\quad f(x) = a^x, \quad$ then $\quad f(x + h) = a^{x+h}$

$\therefore \quad f(x + h) - f(x) = a^{x+h} - a^x = a^x(a^h - 1)$

$\therefore \quad f'(x) = \lim_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{a^x(a^h - 1)}{h} = a^x \lim_{h \to 0} \left(\frac{a^h - 1}{h} \right).$$

In particular, $f'(0) = a^0 \lim_{h \to 0} \left(\dfrac{a^h - 1}{h} \right) = \lim_{h \to 0} \left(\dfrac{a^h - 1}{h} \right)$. Since $f'(0)$ is the gradient

of the curve $y = a^x$ at the point (0, 1), it seems reasonable to assume that this limit exists and is equal to some constant k.
Thus, if $f(x) = a^x$, then $f'(x) = ka^x$, where the value of k depends on the value of a.
It follows that if $y = a^x$, then $\frac{dy}{dx} = ky$, i.e. the rate of change of y is proportional to y.

Because of this property the theory of exponential functions has many important applications. There are various scientific laws which state that the rate of change of a certain quantity is proportional to the quantity itself. We give here three simple examples of such laws.

(1) *Law of growth.* If a population of simple organisms such as bacteria is allowed to increase without restriction, then the rate of growth is proportional to the size of the population. If n is the number of organisms present at time t, then $\frac{dn}{dt} = kn$, where k is constant.

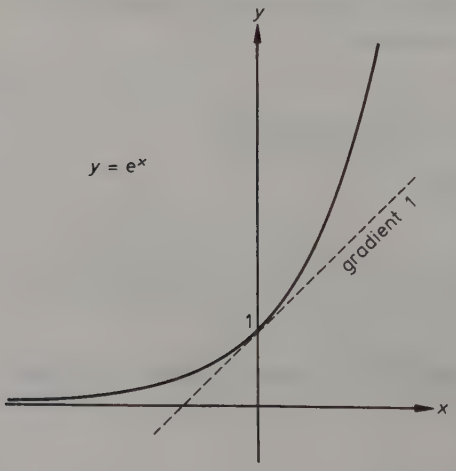
(2) *Law of radio-active decay.* The rate at which a radio-active substance decays is proportional to the quantity of the substance present. If a mass m is present at time t, then $\frac{dm}{dt} = -km$, where k is constant.

(3) *Newton's law of cooling.* If a body is placed in cooler surroundings then its temperature decreases at a rate proportional to the difference between the temperature of the body and that of its surroundings. If T is the temperature difference at time t, then $\frac{dT}{dt} = -kT$, where k is constant.

We arrive at a way of expressing such laws in exponential form by looking again at graphs of the form $y = a^x$. If, as before, the constant k represents the gradient of a particular curve at the point (0, 1), we see that as a increases, k also increases, taking negative values for $0 < a < 1$ and positive values for $a > 1$. The value of a for which $k = 1$ is denoted by e. This value is of special interest because when

$$y = e^x, \quad \frac{dy}{dx} = e^x = y.$$

It can be shown that e is an irrational number approximately equal to 2·718.



The function e^x is called *the exponential function* and is sometimes written $\exp x$. [In numerical work values of e^x are found using a scientific calculator or mathematical tables.]

The result $\frac{d}{dx}(e^x) = e^x$ can be extended as follows.

Let $y = e^u$, where $u = kx$

then $\frac{dy}{du} = e^u$ and $\frac{du}{dx} = k$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = ke^u.$$

Hence, if $y = e^{kx}$, then $\frac{dy}{dx} = ke^{kx} = ky$.

It is now possible to express the scientific laws mentioned earlier in exponential form:

(1) $n = n_0 e^{kt}$, (2) $m = m_0 e^{-kt}$, (3) $T = T_0 e^{-kt}$,

where n_0, m_0 and T_0 are the values of the variables when $t = 0$.

Some useful results to remember:

$$\frac{d}{dx}(e^x) = e^x, \qquad \int e^x\,dx = e^x + c$$

$$\frac{d}{dx}(e^{kx}) = ke^{kx}, \qquad \int e^{kx}\,dx = \frac{1}{k}e^{kx} + c.$$

Example 1 Differentiate e^{x^2+1} with respect to x.

Let $y = e^u$, where $u = x^2 + 1$

$$\frac{dy}{du} = e^u, \qquad \frac{du}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = e^u \times 2x = 2xe^{x^2+1}.$$

[Students thoroughly familiar with the differentiation of composite functions may write:

$$\frac{d}{dx}(e^{x^2+1}) = e^{x^2+1}\frac{d}{dx}(x^2 + 1) = 2xe^{x^2+1}.]$$

Example 2 Find $\int \sin x\, e^{\cos x}\,dx$.

$$\frac{d}{dx}(e^{\cos x}) = e^{\cos x}\frac{d}{dx}(\cos x) = -\sin x\, e^{\cos x}$$

$$\therefore \quad \int \sin x\, e^{\cos x}\,dx = -e^{\cos x} + c.$$

Example 3 Sketch the curve $y = x^2e^{-x}$.

$x = 0 \Leftrightarrow x^2e^{-x} = 0 \Leftrightarrow y = 0$

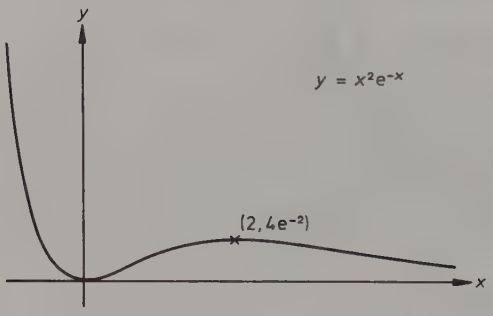
$\therefore$ the curve cuts the x- and y-axes at the origin.
Differentiating (using the product rule),

$$\frac{dy}{dx} = x^2(-e^{-x}) + 2xe^{-x} = (-x^2 + 2x)e^{-x} = x(2 - x)e^{-x}$$

$$\therefore \quad \frac{dy}{dx} = 0 \Leftrightarrow x(2 - x)e^{-x} = 0 \Leftrightarrow x = 0 \text{ or } x = 2.$$

	$x < 0$	$0 < x < 2$	$x > 2$
x	−	+	+
$2 - x$	+	+	−
e^{-x}	+	+	+
$\frac{dy}{dx}$	−	+	−

$\therefore$ when $x = 0$, there is a minimum point $(0, 0)$,
when $x = 2$, there is a maximum point $(2, 4e^{-2})$.



As $x \to -\infty$, $y \to \infty$.
As x increases through positive values, e^x increases at a much greater rate than x^2,
$\therefore$ as $x \to \infty$, $y \to 0$.

[The behaviour of the function x^ne^{-x} as $x \to \infty$ is also considered in *Exercise 9.4*, question 12.]

Exercise 3.1

In questions 1 to 7 differentiate the given functions with respect to x.

1. (a) e^{2x} (b) $5e^{-x}$ (c) e^{3x+5}

2. (a) e^{2x^3} (b) $e^{\sqrt{x}}$ (c) $e^{-1/x}$

3. (a) $e^{\sin x}$ (b) $e^{\cos 2x}$ (c) $e^{4\tan x}$

4. (a) xe^{x^2} (b) x^2e^{5x} (c) $e^{x\cos x}$

5. (a) $e^{2x}\cos 3x$ (b) $e^{-x^2}\sin x$ (c) $(x+1)^3 e^{x/2}$

6. (a) $\sqrt{(1-e^{4x})}$ (b) $e^{(e^x)}$ (c) $\sin(e^x)$

7. (a) $\frac{1}{2}(e^x - e^{-x})^2$ (b) $\dfrac{e^x}{1+e^{-x}}$ (c) $\dfrac{e^{\sin^2 x}}{e^{-\cos^2 x}}$

8. Find (a) $\int e^{3x}\,dx$, (b) $\int e^{-x}\,dx$, (c) $\int e^{x+4}\,dx$.

9. Evaluate, leaving your answers in terms of e,

(a) $\int_0^1 6e^{2x}\,dx$ (b) $\int_{-2}^0 e^{(-x/2)}\,dx$, (c) $\int_{-1}^1 \{3/e^{3(x-1)}\}\,dx$.

10. Find dy/dx in terms of x when y is defined implicitly by:
(a) $e^y = x$, (b) $xe^y = 1$, (c) $e^{x+y} = x^2$.

11. If $y = e^x \sin x$, show that $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = 0$.

12. Find the equation of the tangent to the curve $y = e^{3x-5}$ at the point where $x = 2$.

13. Find in terms of e the area enclosed by the curve $y = e^{1-x}$, the axes and the line $x = 1$.

14. By considering the graphs of $y = e^x$ and $y = e^{-x}$, or otherwise, make rough sketches of:
(a) $y = e^x + e^{-x}$, (b) $y = e^x - e^{-x}$, (c) $y = e^x - x$.

In questions 15 to 20 sketch the given curves. Give the coordinates of any turning points and show clearly the behaviour of the curve when $|x|$ is large.

15. $y = xe^{-x}$. 16. $y = (x+1)e^x$. 17. $y = xe^{-x^2/2}$.

18. $y = (x-1)^2 e^{2x}$. 19. $y = e^x/x$. 20. $y = e^{1-\cos x}$.

21. A particle moves in a straight line so that after t seconds its distance from a fixed point 0 is s metres, where $s = t^2 e^{2-t}$. Find the distance of the particle from 0 when it first comes to rest and its acceleration at that point.

22. Find the values of x for which the function $(x^2 - 2x - 1)e^{2x}$ has maximum or minimum values, distinguishing between them.

23. Find the values of x between 0 and 2π for which the function $e^x \cos x$ has maximum or minimum values, distinguishing between them.

24. Find the area of the finite region bounded by the curves $y = e^{2x}$, $y = e^{-x}$ and the line $x = 1$. If this region is rotated completely about the x-axis, find the volume of the solid formed. (AEB 1976)

3.2 Logarithmic functions

If a and x are positive real numbers such that $x = a^p$, then p is the logarithm of x to the base a,

i.e. $$x = a^p \Leftrightarrow \log_a x = p.$$

The laws of logarithms can be summarised as follows:

$$\log_a xy = \log_a x + \log_a y$$
$$\log_a x/y = \log_a x - \log_a y$$
$$\log_a x^n = n \log_a x$$
$$(\log_a b)(\log_b c) = \log_a c$$

The relationship between logarithms to different bases leads to two useful formulae:

$$\log_a x = \frac{1}{\log_x a}, \quad \log_a x = \frac{\log_b x}{\log_b a}.$$

Now that the exponential function e^x has been defined, we can introduce logarithms to the base e. These are called *natural logarithms* or *Napierian logarithms*†. The natural logarithm of x may be written $\log_e x$, but is more often denoted by $\ln x$.

Thus $$x = e^p \Leftrightarrow \ln x = p.$$

In numerical work values of $\ln x$ are found using a calculator or mathematical tables.

Example 1 Solve the equation $e^x - 2 - 3e^{-x} = 0$.

Letting $y = e^x$, $\quad e^x - 2 - 3e^{-x} = 0.$

$$\Rightarrow \quad y - 2 - 3/y = 0$$
$$\Rightarrow \quad y^2 - 2y - 3 = 0$$
$$\Rightarrow \quad (y - 3)(y + 1) = 0$$
$$\Rightarrow \quad y = 3 \quad \text{or} \quad y = -1$$
$$\Rightarrow \quad e^x = 3, \quad \text{since} \quad e^x > 0$$
$$\Rightarrow \quad x = \ln 3 \approx 1{\cdot}1$$

To investigate the properties of logarithmic functions we use our knowledge of exponential functions.

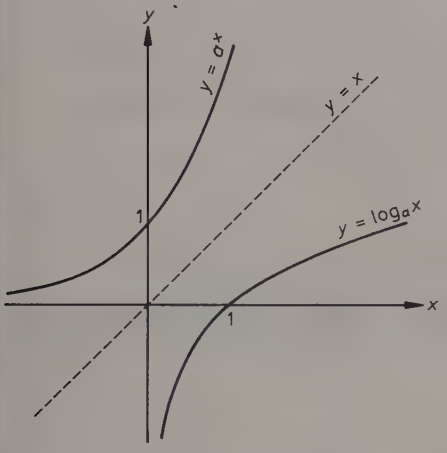
If $f(x) = a^x$, then $y = f^{-1}(x) \Rightarrow f(y) = x \Rightarrow a^y = x \Rightarrow y = \log_a x$.

†*Napier, John* (1550–1617) Scottish mathematician. He was the inventor of logarithms and spent 25 years producing tables. He also invented the calculating apparatus known as "Napier's Bones".

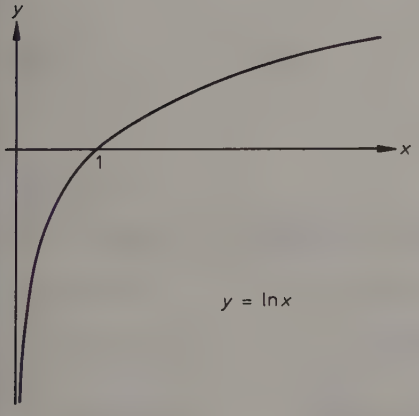
Hence the function $\log_a x$ is the inverse of the function a^x.

Thus $a^{\log_a x} = x$ and $\log_a(a^x) = x$.

It also follows that the graph of $y = \log_a x$ is the reflection in the line $y = x$ of the graph of $y = a^x$.



Since $\log_a x$ is defined only when $x > 0$, there are no points on the graph of $y = \log_a x$ for $x \leqslant 0$.
When $x = 1, y = \log_a 1 = 0$
$\therefore$ the graph of $y = \log_a x$ passes through the point $(1, 0)$ for all values of a.



If $y = \ln x$, then $e^y = x$.

Differentiating with respect to x,

$$\frac{d}{dx}(e^y) = 1$$

But $$\frac{d}{dx}(e^y) = \frac{d}{dy}(e^y)\frac{dy}{dx}$$

$$= e^y \frac{dy}{dx} = x\frac{dy}{dx}$$

$\therefore$ $$x\frac{dy}{dx} = 1 \quad \text{i.e.} \quad \frac{dy}{dx} = \frac{1}{x}.$$

Hence, for $x > 0$, $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$ and $\displaystyle\int \frac{1}{x}\,dx = \ln x + c.$

Example 2 Differentiate $\ln(x^2 + 1)$ with respect to x.

Let $y = \ln u$, where $u = x^2 + 1$

$$\frac{dy}{du} = \frac{1}{u} \qquad \frac{du}{dx} = 2x$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{u} \times 2x = \frac{2x}{x^2 + 1}.$$

Some problems may be simplified by using the laws of logarithms.

Example 3 If $y = \ln\left(\dfrac{2x+1}{1-3x}\right)$, find $\dfrac{dy}{dx}$.

$$y = \ln\left(\frac{2x+1}{1-3x}\right) = \ln(2x+1) - \ln(1-3x)$$

$$\therefore \quad \frac{dy}{dx} = \frac{2}{2x+1} - \frac{(-3)}{1-3x} = \frac{2(1-3x) + 3(2x+1)}{(2x+1)(1-3x)}$$

$$= \frac{5}{(2x+1)(1-3x)}$$

Some functions involving variable exponents or complicated products and quotients are differentiated using a process called *logarithmic differentiation*. The next two examples illustrate the method.

Example 4 Differentiate 2^x with respect to x.

Let $y = 2^x$, then $\ln y = \ln(2^x) = x \ln 2$.

Differentiating, $\dfrac{1}{y}\dfrac{dy}{dx} = \ln 2$

$$\therefore \quad \frac{dy}{dx} = y \ln 2 = 2^x \ln 2.$$

Example 5 If $y = \dfrac{x}{\sqrt{(x^2-2)}}$, find $\dfrac{dy}{dx}$.

$$\ln y = \ln x - \ln\sqrt{(x^2-2)} = \ln x - \tfrac{1}{2}\ln(x^2-2)$$

$$\therefore \quad \frac{1}{y}\frac{dy}{dx} = \frac{1}{x} - \frac{1}{2}\left(\frac{2x}{x^2-2}\right) = \frac{x^2 - 2 - x^2}{x(x^2-2)} = \frac{-2}{x(x^2-2)}$$

$$\therefore \quad \frac{dy}{dx} = y\left\{\frac{-2}{x(x^2-2)}\right\} = \frac{x}{\sqrt{(x^2-2)}}\left\{\frac{-2}{x(x^2-2)}\right\} = -\frac{2}{(x^2-2)^{3/2}}$$

Example 6 Sketch the curve $y = \dfrac{1}{x}\ln x$.

$\ln x$ is not defined for $x \leqslant 0$,

$\therefore$ the curve lies entirely to the right of the y-axis.

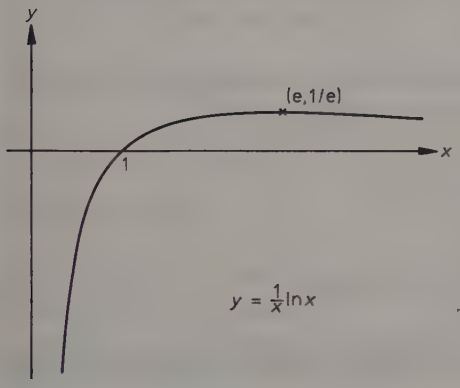
$$y = 0 \Rightarrow \frac{1}{x}\ln x = 0 \Rightarrow x = 1$$

$\therefore$ the curve cuts the x-axis at the point $(1, 0)$.
Differentiating (using the product rule),

$$\frac{dy}{dx} = \frac{1}{x} \times \frac{1}{x} + \left(-\frac{1}{x^2}\right) \ln x = \frac{1}{x^2}(1 - \ln x)$$

$$\therefore \quad \frac{dy}{dx} = 0 \Leftrightarrow 1 - \ln x = 0 \Leftrightarrow x = e$$

When $x < e$, $\frac{dy}{dx} > 0$ and when $x > e$, $\frac{dy}{dx} < 0$,
$\therefore$ the point $(e, 1/e)$ is a maximum point.



As $x \to 0, 1/x \to \infty$ and $\ln x \to -\infty$
$\therefore \quad y \to -\infty$.
When x is large, the rate of increase of $\ln x$ is very small
$\therefore$ as $x \to \infty$, $y \to 0$.

[The behaviour of the function $\frac{1}{x} \ln x$ as $x \to \infty$ is also considered in *Exercise 9.4*, question 12.]

Exercise 3.2

In questions 1 to 7 differentiate the given functions with respect to x.

1. (a) $\ln 3x$, (b) $\ln(x + 3)$, (c) $\ln(2x - 1)$.

2. (a) $\ln(x^3 + 4)$, (b) $\ln(\sin 2x)$, (c) $\ln(\sec x)$.

3. (a) $x^2 \ln x$, (b) $(\ln x)/x^2$, (c) $\ln(\ln x)$.

4. (a) $\ln(x^3)$, (b) $\ln\sqrt{(4x + 5)}$, (c) $x \ln(1/x)$.

5. (a) $\ln(\sec x + \tan x)$, (b) $\ln(x^2 + 4)^2$, (c) $\log_{10} x$.

6. (a) $\ln\left(\frac{x^2}{3x - 2}\right)$, (b) $\ln\sqrt{\left(\frac{x}{1 + x}\right)}$, (c) $\ln\frac{\cos x}{\sqrt{(1 - x^2)}}$.

7. (a) $e^{2 \ln x}$, (b) $\ln(e^{\tan x})$, (c) $\ln(3x^2 e^{-x})$.

8. Find the equation of the tangent to the curve $y = \ln(3x - 5)$ at the point where $x = 2$.

9. If $x = 1 + \ln t$ and $y = \ln(1 + t)$, find dy/dx in terms of t.

10. By considering the graph of $y = \ln x$, or otherwise, sketch the following:
(a) $y = \ln(-x)$, (b) $y = \ln(1/x)$, (c) $y = \ln(1 + x)$.

11. Given that $\ln 2 = p$ and $\ln 3 = q$, express the following in terms of p and q.
(a) $\ln 12e$, (b) $\log_3 2$, (c) $\log_6 e$, (d) 6.

12. Solve the following equations
(a) $e^{3t} + 2e^t = 3e^{2t}$, (b) $2e^x + 5 = 3e^{-x}$.

13. Sketch the following curves, giving the coordinates of any turning points.
(a) $y = |\ln x|$, (b) $y = 1/\ln x$, (c) $y = (\ln x)^2$.

14. Show that the curve $y = x - \ln x$ has one turning point. Show also that there are no points of inflexion on the curve. Sketch the curve.

15. If $y = \ln(x^2 - 5)$, show that $\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 = 2e^{-y}$.

16. If $y = \sin 2x \ln(\tan x)$, show that $\dfrac{d^2y}{dx^2} + 4y = 4\cot 2x$.

In questions 17 to 20 use logarithmic differentiation to find the derivatives of the given functions.

17. (a) 3^x, (b) 5^{2x}, (c) $10^{\sin x}$.

18. (a) x^x, (b) $x^{\ln x}$, (c) $(\ln x)^x$.

19. (a) $\dfrac{x^2 + 1}{(2x + 1)^2}$, (b) $\dfrac{x^3}{\sqrt{(1 - x)}}$, (c) $\dfrac{(x - 1)^2 e^{4x}}{(x + 1)^2}$

20. (a) $\sqrt{\left\{\dfrac{(x + 2)^3}{1 - 5x}\right\}}$, (b) $\sqrt[3]{\left\{\dfrac{x^2 - 2}{x^2 + 4}\right\}}$, (c) $\dfrac{\sqrt{(\sin 2x)}}{\cos^3 x}$.

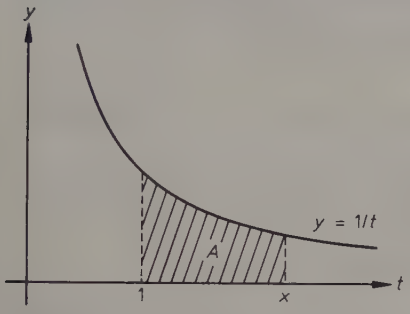
3.3 $\ln x$ defined in integral form

In this section we consider a more rigorous approach to the theory of exponential and logarithmic functions. In §3.1 we assumed with little justification that a^x is a continuous differentiable function defined for all real values of x. However, any elementary definition of a^x is valid only for rational values of x. Similarly, the laws of indices and the rule for differentiating x^n can be proved by elementary methods only for rational values of the exponents. We avoid these difficulties by using definitions of $\ln x$ and $\exp x$ which are valid for both rational and irrational values of x.

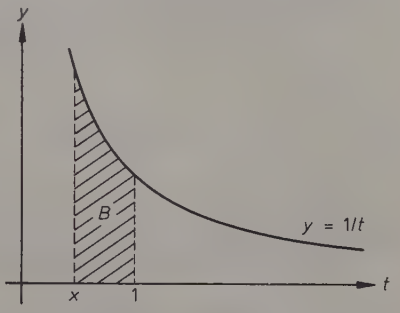
We define $\ln x$ as a definite integral by writing:

$$\ln x = \int_1^x \frac{1}{t}\,dt, \quad \text{where } x \text{ is real and positive.}$$

It follows that when $x = 1$, $\ln x = 0$. Other values of $\ln x$ are related to areas under the curve $y = 1/t$ as shown in the diagrams below.



When $x > 1$, $\ln x = A$.



When $0 < x < 1$, $\ln x = -B$.

Thus the domain of the function $\ln x$ is the set of positive real numbers and its range is the set of all real numbers.

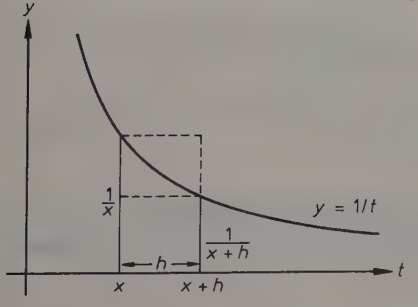
We can now define $\exp x$ for all real values of x by letting the function $\exp x$ be the inverse of the function $\ln x$, so that

$$y = \exp x \Leftrightarrow \ln y = x$$

To differentiate the function $\ln x$, let $f(x) = \ln x$, then

$$f(x+h) - f(x) = \ln(x+h) - \ln x$$

$$= \int_1^{x+h} \frac{1}{t}\,dt - \int_1^x \frac{1}{t}\,dt = \int_x^{x+h} \frac{1}{t}\,dt.$$



From the sketch we see that

$$h \times \frac{1}{x+h} < \int_x^{x+h} \frac{1}{t}\,dt < h \times \frac{1}{x}$$

$$\therefore \quad \frac{1}{x+h} < \frac{f(x+h) - f(x)}{h} < \frac{1}{x}$$

As $h \to 0$, $\dfrac{1}{x+h} \to \dfrac{1}{x}$,

$$\therefore \quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \frac{1}{x}$$

i.e.

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

To differentiate the function $\exp x$, let $y = \exp x$, then

$$\ln y = x \Rightarrow \frac{1}{y}\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = y$$

i.e.
$$\frac{d}{dx}(\exp x) = \exp x$$

We develop the theory further by considering $\ln(x^k)$, where k is any rational number ($k \in \mathbb{Q}$).

$$\frac{d}{dx}\{\ln(x^k)\} = \frac{1}{x^k} \times kx^{k-1} = \frac{k}{x}$$

$$\therefore \quad \frac{d}{dx}\{\ln(x^k) - k\ln x\} = \frac{k}{x} - k \times \frac{1}{x} = 0$$

$\therefore$ the value of $\{\ln(x^k) - k\ln x\}$ must be constant.
When $x = 1$, $\ln(x^k) - k\ln x = \ln 1 - k\ln 1 = 0$,
$\therefore$ the constant value of $\{\ln(x^k) - k\ln x\}$ is zero.

Thus for rational k, $\ln(x^k) = k\ln x$
and hence
$$x^k = \exp(k\ln x) \qquad (1)$$

The constant e is now defined as the real number such that $\ln e = 1$. Substituting $x = e$ in equation (1) we have

$$e^k = \exp(k\ln e) = \exp k, \quad \text{for rational } k.$$

Thus it seems reasonable to give a meaning to the expression e^x when x is irrational by letting $e^x = \exp x$ for all real x.
It follows from this definition that $\log_e x = \ln x$.
More generally, it is consistent with equation (1) to define a^x by writing

$$\boxed{a^x = \exp(x\ln a) \quad \text{for all real } x}$$

It is a consequence of this definition that

$$y = \log_a x \Rightarrow a^y = x \Rightarrow \exp(y\ln a) = x \Rightarrow y\ln a = \ln x$$

$$\therefore \quad \boxed{\log_a x = (\ln x)/(\ln a).}$$

We conclude by summarising the important points in this discussion.

We needed definitions of e^x and $a^x (a > 0)$ valid for irrational as well as rational values of x. We found that using the definitions

$$\ln x = \int_1^x \frac{1}{t}\,dt, \quad y = \exp x \Leftrightarrow \ln y = x,$$

it followed that, for rational values of k,

$$e^k = \exp k, \quad a^k = \exp(k\ln a).$$

Thus suitable definitions of e^x and a^x are

$$e^x = \exp x, \quad a^x = \exp(x \ln a).$$

We deduce from these definitions that

$$\log_e x = \ln x, \quad \log_a x = (\ln x)/(\ln a).$$

These results provide a firmer basis for the work of §3.1 and §3.2.

[Note that the theory of exponential and logarithmic functions defined in terms of $\exp x$ and $\ln x$ is extended in *Exercise 3.3* and in *Exercise 4.6*, question 11.]

Exercise 3.3

1. (a) Use the result $\frac{d}{dx}(\ln x) = \frac{1}{x}$ to prove that

$$\frac{d}{dx}(\ln xy - \ln x - \ln y) = 0.$$

   (b) Deduce that $\ln xy = \ln x + \ln y$.
   (c) By substituting $u = \ln x$, $v = \ln y$, show that

$$(\exp u)(\exp v) = \exp(u + v).$$

2. Use the methods of question 1 to show that
   (a) $\ln(x/y) = \ln x - \ln y$.
   (b) $(\exp u)/(\exp v) = \exp(u - v)$.

3. Use the definition $x^p = \exp(p \ln x)$ or the equivalent statement $\ln(x^p) = p \ln x$ to prove that, for all real values of p,

$$y = x^p \Rightarrow \frac{dy}{dx} = px^{p-1}.$$

3.4 Integration using logarithms

By differentiating logarithmic functions we can obtain various useful integration results.

For $x > 0$, $\frac{d}{dx}(\ln x) = \frac{1}{x}$,

$\therefore$ for $x > 0$, $$\int \frac{1}{x}\,dx = \ln x + c. \qquad (1)$$

For $x < 0$, $\frac{d}{dx}\{\ln(-x)\} = \frac{1}{(-x)} \times (-1) = \frac{1}{x}$

$\therefore$ for $x < 0$, $$\int \frac{1}{x}\,dx = \ln(-x) + c. \qquad (2)$$

For $ax + b > 0$, $\dfrac{d}{dx}\{\ln(ax + b)\} = \dfrac{1}{ax + b} \times a = \dfrac{a}{ax + b}$

$\therefore$ for $ax + b > 0$, $\displaystyle\int \frac{1}{ax + b}\,dx = \frac{1}{a}\ln(ax + b) + c.$

Example 1 Find $\displaystyle\int \frac{1}{3x - 2}\,dx$ when (a) $3x - 2 > 0$, (b) $3x - 2 < 0$.

(a) $\displaystyle\int \frac{1}{3x - 2}\,dx = \frac{1}{3}\ln(3x - 2) + c$, for $3x - 2 > 0$.

(b) $\displaystyle\int \frac{1}{3x - 2}\,dx = -\int \frac{1}{2 - 3x}\,dx = -\frac{1}{(-3)}\ln(2 - 3x) + c$

$\displaystyle = \frac{1}{3}\ln(2 - 3x) + c$, for $3x - 2 < 0$.

Example 2 Evaluate (a) $\displaystyle\int_5^9 \frac{1}{x - 3}\,dx$, (b) $\displaystyle\int_1^2 \frac{1}{x - 3}\,dx$.

(a) Within the limits of integration $x - 3 > 0$,

$\therefore$ $\displaystyle\int_5^9 \frac{1}{x - 3}\,dx = \Big[\ln(x - 3)\Big]_5^9 = \ln 6 - \ln 2 = \ln(6/2) = \ln 3.$

(b) Within the limits of integration $x - 3 < 0$,

$\therefore$ $\displaystyle\int_1^2 \frac{1}{x - 3}\,dx = -\int_1^2 \frac{1}{3 - x}\,dx = \Big[\ln(3 - x)\Big]_1^2 = \ln 1 - \ln 2 = -\ln 2.$

When there are no stated restrictions on the values of x, results (1) and (2) can be combined in the statement:

$$\int \frac{1}{x}\,dx = \ln|x| + c$$

Similarly

$$\boxed{\int \frac{1}{ax + b}\,dx = \frac{1}{a}\ln|ax + b| + c.}$$

We stress, however, that it should not be necessary to use modulus signs when evaluating definite integrals.

Example 3 Find $\displaystyle\int \frac{x - 1}{x + 1}\,dx$.

$$\int \frac{x - 1}{x + 1}\,dx = \int \left\{1 - \frac{2}{x + 1}\right\} dx = x - 2\ln|x + 1| + c.$$

Example 4 Find $\int_1^3 \frac{2x^2+3x}{2x-1}dx$.

$$\begin{array}{r} x+2 \\ 2x-1\overline{)2x^2+3x} \\ \underline{2x^2-x} \\ 4x \\ \underline{4x-2} \\ 2 \end{array}$$

By division $\dfrac{2x^2+3x}{2x-1} = x+2+\dfrac{2}{2x-1}$

Noting that within the limits of integration $2x-1>0$,

$$\int_1^3 \frac{2x^2+3x}{2x-1}dx = \int_1^3 \left\{x+2+\frac{2}{2x-1}\right\}dx$$

$$= \left[\tfrac{1}{2}x^2+2x+\ln(2x-1)\right]_1^3$$

$$= (\tfrac{9}{2}+6+\ln 5)-(\tfrac{1}{2}+2+\ln 1) = 8+\ln 5.$$

To obtain a general integration formula, let $y=\ln\{f(x)\}$, then

$$\frac{dy}{dx} = \frac{1}{f(x)}\times\frac{d}{dx}\{f(x)\} = \frac{f'(x)}{f(x)}$$

$\therefore$ for $f(x)>0$, $\int\frac{f'(x)}{f(x)}dx = \ln\{f(x)\}+c.$

When the sign of $f(x)$ is unknown we may write

$$\boxed{\int\frac{f'(x)}{f(x)}dx = \ln|f(x)|+c.}$$

Example 5 Find $\int\frac{x}{3x^2+5}dx$.

If $f(x)=3x^2+5$, then $f'(x)=6x$,
$\therefore$ noting that $3x^2+5>0$ for all values of x,

$$\int\frac{x}{3x^2+5}dx = \frac{1}{6}\int\frac{6x}{3x^2+5}dx = \frac{1}{6}\ln(3x^2+5)+c.$$

Example 6 Find $\int\tan x\,dx$.

If $f(x)=\cos x$, then $f'(x)=-\sin x$,

$$\therefore \int\tan x\,dx = -\int\frac{(-\sin x)}{\cos x}dx = -\ln|\cos x|+c$$

$$= \ln\frac{1}{|\cos x|}+c = \ln|\sec x|+c.$$

Exercise 3.4

Find the following indefinite integrals.

1. (a) $\int \frac{1}{2x+7}\,dx$ for $2x+7>0$, (b) $\int \frac{1}{5x-2}\,dx$ for $5x-2>0$.

2. (a) $\int \frac{1}{1-x}\,dx$ for $1-x>0$, (b) $\int \frac{1}{2x-1}\,dx$ for $2x-1<0$.

3. (a) $\int \frac{3}{3x-2}\,dx$ for $3x<2$, (b) $\int \frac{8}{3-4x}\,dx$ for $4x<3$.

4. (a) $\int \frac{1}{x+3}\,dx$, (b) $\int \frac{3}{3-x}\,dx$, (c) $\int \frac{2}{4x+5}\,dx$.

5. (a) $\int \frac{x}{x-1}\,dx$, (b) $\int \frac{x-2}{x-4}\,dx$, (c) $\int \frac{x+1}{2x+1}\,dx$.

6. (a) $\int \frac{x^2-1}{x}\,dx$, (b) $\int \frac{(1-x)^2}{1+x^2}\,dx$, (c) $\int \frac{1-x+x^2}{1-x}\,dx$.

7. (a) $\int \frac{x}{1-x^2}\,dx$, (b) $\int \frac{x^2}{x^3+8}\,dx$, (c) $\int \frac{x+1}{2x^2+4x-1}\,dx$.

8. (a) $\int \frac{\cos x}{1+\sin x}\,dx$, (b) $\int \cot x\,dx$, (c) $\int (1+\tan 2x)\,dx$.

9. (a) $\int \frac{e^x}{1+e^x}\,dx$, (b) $\int \frac{1}{x\ln x}\,dx$, (c) $\int \frac{1}{(1+\sqrt{x})\sqrt{x}}\,dx$.

Evaluate the following definite integrals.

10. (a) $\int_2^4 \frac{5}{x}\,dx$, (b) $\int_3^9 \frac{1}{x-1}\,dx$, (c) $\int_{-4}^0 \frac{1}{1-2x}\,dx$.

11. (a) $\int_0^{\pi/2} \tan\tfrac{1}{2}x\,dx$, (b) $\int_0^3 \frac{1}{x-4}\,dx$, (c) $\int_3^5 \frac{x}{x+1}\,dx$.

12. (a) $\int_0^{\pi/2} \tan\tfrac{1}{2}x\,dx$, (b) $\int_3^4 \frac{2x-3}{x^2-3x+1}\,dx$, (c) $\int_0^{\ln 2} \frac{e^{-x}}{1+e^{-x}}\,dx$.

13. Evaluate, giving answers correct to 3 significant figures.

(a) $\int_2^4 \frac{x+2}{x^2+4x-7}\,dx$, (b) $\int_{\pi/6}^{\pi/3} \frac{\sec^2 x}{\tan x}\,dx$.

14. Express $\dfrac{1}{2x-1} - \dfrac{2}{3-x}$ as a single fraction and hence evaluate, correct to 3 decimal places, $\displaystyle\int_1^2 \frac{1-x}{(2x-1)(3-x)}\,dx$.

15. Find the area enclosed by the curve $y = 1/(x-2)$ and the line $y = 7 - 2x$.

16. Find the volume of the solid formed by rotating about the x-axis the area bounded by the curve $y = \sqrt{x} + 1/\sqrt{x}$, the x-axis and the lines $x = 1$, $x = 3$.

3.5 Introduction to hyperbolic functions

Hyperbolic sine and cosine are defined as follows:

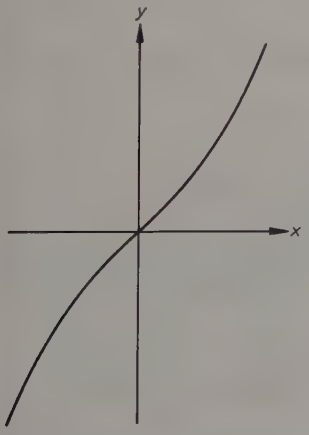
$$\sinh x = \tfrac{1}{2}(e^x - e^{-x}), \quad \cosh x = \tfrac{1}{2}(e^x + e^{-x})$$

The remaining hyperbolic functions can be expressed in terms of $\sinh x$ and $\cosh x$.

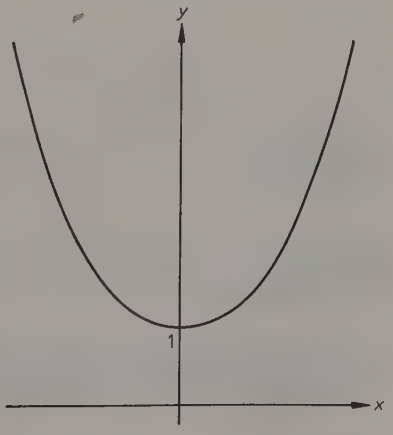
$$\tanh x = \frac{\sinh x}{\cosh x}, \quad \coth x = \frac{\cosh x}{\sinh x}, \quad \operatorname{sech} x = \frac{1}{\cosh x}, \quad \operatorname{cosech} x = \frac{1}{\sinh x}$$

[Note that cosh, sech, cosech, coth are pronounced as they are written, tanh is pronounced 'tanch' and sinh as 'shine' or sometimes 'sinch'.]

The graphs of $y = \sinh x$ and $y = \cosh x$ may be sketched from the graphs of $y = e^x$ and $y = e^{-x}$.



$y = \sinh x$



$y = \cosh x$

$$y = \sinh x = \tfrac{1}{2}(e^x - e^{-x}) \Rightarrow \frac{dy}{dx} = \tfrac{1}{2}(e^x + e^{-x}) = \cosh x$$

$$y = \cosh x = \tfrac{1}{2}(e^x + e^{-x}) \Rightarrow \frac{dy}{dx} = \tfrac{1}{2}(e^x - e^{-x}) = \sinh x$$

$$\therefore \quad \frac{d}{dx}(\sinh x) = \cosh x, \quad \frac{d}{dx}(\cosh x) = \sinh x.$$

We now establish one of the most useful identities connecting hyperbolic functions.

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \{\tfrac{1}{2}(e^x + e^{-x})\}^2 - \{\tfrac{1}{2}(e^x - e^{-x})\}^2 \\ &= \tfrac{1}{4}\{(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})\} \\ &= \tfrac{1}{4} \times 4 = 1\end{aligned}$$

i.e. $\cosh^2 x - \sinh^2 x = 1.$

Exercise 3.5

1. Differentiate with respect to x

(a) $\sinh 2x$, (b) $\cosh 3x$, (c) $\sinh^2 x$, (d) $\sqrt{(\cosh x)}$.

2. Find (a) $\int \sinh x\,dx$, (b) $\int \cosh 2x\,dx$, (c) $\int \tanh x\,dx$.

3. Differentiate with respect to x

(a) $\tanh x$, (b) $\coth x$, (c) $\operatorname{sech} x$, (d) $\operatorname{cosech} x$.

4. Use the definitions of $\sinh x$ and $\cosh x$ to prove

(a) $\sinh 2x = 2\sinh x \cosh x$, (b) $\cosh 2x = \cosh^2 x + \sinh^2 x$.

5. By substituting $y = e^x$, solve the equations

(a) $3\cosh x - 2\sinh x = 3$, (b) $5\cosh x + 7\sinh x = 1$.

Exercise 3.6 (*miscellaneous*)

In questions 1 to 3 differentiate the given functions.

1. (a) xe^{3x}, (b) $e^{2x}\ln 4x$, (c) $a^x (a > 0)$.

2. (a) $\sin(\ln x)$, (b) $(1 - x^2)\ln x$, (c) $\ln\left(\tan\frac{1}{2}x\right)$.

3. (a) $\ln\left(\frac{x-3}{3x-1}\right)$, (b) $\ln\left(\frac{1+\sin x}{1-\sin x}\right)$, (c) $\ln\{(x^3)\sqrt{(1-x^2)}\}$.

4. Sketch the given curves, giving the coordinates of any stationary points.

(a) $y = e^x + 4e^{-x}$, (b) $y = x^3 e^{-x}$.

5. Find the value of x for which the curve $y = x^{1/x} (x > 0)$ has a stationary point. State the nature of this point.

6. Show that if $y = e^{4x}\cos 3x$, then d^2y/dx^2 can be expressed in the form $25e^{4x}\cos(3x + \alpha)$. Give the value of $\tan\alpha$.

7. Sketch the curves

(a) $y = \dfrac{x+1}{x-1}$, (b) $y = \left|\dfrac{x+1}{x-1}\right|$, (c) $y = \ln\left|\dfrac{x+1}{x-1}\right|$.

8. Given that $x \ln x \to 0$ as $x \to 0$, sketch the curves

(a) $y = x \ln x$, (b) $y = \dfrac{1}{x} + \ln x$.

9. Find the coordinates of any turning points on the curve $y = e^{-x/2}(x^2 + x - 2)$. Sketch this curve and the curve $y = x^2 + x - 2$ on the same axes.

10. Evaluate (a) $\displaystyle\int_2^6 \frac{1}{2x-3}\,dx$, (b) $\displaystyle\int_0^5 \frac{x}{x^2+5}\,dx$, (c) $\displaystyle\int_{\pi/2}^{\pi} \cot\frac{1}{3}x\,dx$.

11. Find (a) $\displaystyle\int \frac{x}{3x+1}\,dx$, (b) $\displaystyle\int \frac{(x-2)(x+2)}{x^3}\,dx$, (c) $\displaystyle\int \frac{2x^2+1}{x^2+x+1}\,dx$

12. Simplify $\log_e\left[\dfrac{(1+x)e^{-2x}}{1-x}\right]^{1/2}$ and show that its derivative is $x^2/(1-x^2)$. Hence, or otherwise, evaluate dy/dx at $x = 0$ for the function $y = \left[\dfrac{(1+x)e^{-2x}}{1-x}\right]^{1/2}$.

(JMB)

13. If $\tan y = \log_e x^2$, show that $x\dfrac{dy}{dx} = 2\cos^2 y$. Hence show that

$$x^2\frac{d^2y}{dx^2} + 2(1 + 2\sin 2y)\cos^2 y = 0.$$

(AEB 1980)

14. If $y = (x - 0{\cdot}5)e^{2x}$, find dy/dx and hence or otherwise calculate, correct to one decimal place, the mean value of xe^{2x} in the interval $0 \leqslant x \leqslant 2{\cdot}5$. (AEB 1980)

15. The coordinates of a point on a curve are given in terms of a parameter θ by $x = a\{\ln(\cot\frac{1}{2}\theta) - \cos\theta\}$, $y = a\sin\theta$, where a is a positive constant and $0 < \theta \leqslant \frac{1}{2}\pi$. Show that, if $\theta \neq \frac{1}{2}\pi$, $\dfrac{dy}{dx} = -\tan\theta$.

The tangent at any point P of the curve meets the x-axis at T. Prove that the length of PT is equal to a. (C)

16. Sketch the region in the x-y plane within which all three of the following inequalities are satisfied:

(i) $y < x + 1$, (ii) $y > (x-1)^2$, (iii) $xy < 2$.

Determine the area of this region. (JMB)

17. Sketch the curve $y = 1 + 2e^{-x}$, showing clearly the behaviour of the curve as $x \to +\infty$. Find the area of the finite region enclosed by the curve and the lines $x = 0$, $x = 1$, $y = 1$. Find also the volume formed when this region is rotated completely about the line $y = 1$. (L)

18. Sketch the curve whose equation is $y + 3 = \dfrac{6}{x - 1}$. Find the coordinates of the points where the line $y + 3x = 9$ intersects the curve and show that the area of the region enclosed between the curve and the line is $\frac{3}{2}(3 - 4\log_e 2)$. Determine the equations of the two tangents to the curve which are parallel to the line. (JMB)

19. Sketch the curve $y = x/(1 + x^2)$, finding its turning points, showing that the origin is a point of inflexion, and indicating the behaviour of y when $|x|$ is large. Calculate the area of the region defined by the inequalities

$$0 \leqslant x \leqslant a, 0 \leqslant y \leqslant x/(1 + x^2),$$

and show that this area has the value 1 when $a = \sqrt{(e^2 - 1)}$. (L)

20. A curve has the equation $y = a/(b + e^{-cx})$, $(a \neq 0, b > 0, c > 0)$. Show that it has one point of inflexion, and that the value of y at the point of inflexion is half the limiting value y as $x \to \infty$. (O)

21. Sketch the curve $y = (x^2 - 7x + 10)/(x - 6)$, indicating clearly any vertical asymptotes and turning values. Determine the finite area bounded by the x-axis and that part of the curve between the points where it crosses the x-axis. (O)

22. Sketch the curve $y = \log_e(x - 2)$. The inner surface of a bowl is of the shape formed by rotating completely about the y-axis that part of the x-axis between $x = 0$ and $x = 3$ and that part of the curve $y = \log_e(x - 2)$ between $y = 0$ and $y = 2$. The bowl is placed with its axis vertical and water is poured in. Calculate the volume of water in the bowl when the bowl is filled to a depth $h(<2)$. If water is poured into the bowl at a rate of 50 cubic units per second, find the rate at which the water level is rising when the depth of the water is 1·5 units. (AEB 1979)

23. By means of a diagram, or otherwise, show that

$$\frac{1}{r+1} < \int_r^{r+1} \frac{1}{x}\,dx < \frac{1}{r}.$$

Deduce that the sum to N terms of the series $1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{r} + \ldots$ lies between $\log_e(1 + N)$ and $1 + \log_e N$, where $N > 1$. Show that the sum of one hundred million terms of this series lies between 18·4 and 19·5. (JMB)

4 Methods of integration

4.1 Standard forms

We first list some important integration results.

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c, \quad (n \neq -1)$$

$$\int \sin x\,dx = -\cos x + c, \qquad \int \cos x\,dx = \sin x + c$$

$$\int \sec^2 x\,dx = \tan x + c, \qquad \int \operatorname{cosec}^2 x\,dx = -\cot x + c$$

$$\int e^x\,dx = e^x + c, \qquad \int \frac{1}{x}\,dx = \ln x + c, \quad (x > 0)$$

When using these standard forms it is worth noting that if we can integrate a function $f(x)$, we should also be able to integrate $f(ax + b)$. For instance, remembering that $\int \cos x\,dx = \sin x + c$, to find $\int \cos(2x - 3)\,dx$ we try differentiating $\sin(2x - 3)$:

$$y = \sin u, \quad \text{where} \quad u = 2x - 3$$

$$\Rightarrow \frac{dy}{du} = \cos u \qquad \frac{du}{dx} = 2$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 2\cos u = 2\cos(2x - 3)$$

Hence $$\int \cos(2x - 3)\,dx = \tfrac{1}{2}\sin(2x - 3) + c.$$

Example 1 Find (a) $\int e^{x/2}\,dx$, (b) $\int (1-2x)^3\,dx$, (c) $\int \frac{1}{(3x+4)^4}\,dx$.

(a) $\frac{d}{dx}\{e^{x/2}\} = e^{x/2} \times \frac{1}{2} = \frac{1}{2}e^{x/2}$

$$\therefore \quad \int e^{x/2}\,dx = 2e^{x/2} + c.$$

(b) $\frac{d}{dx}\{(1-2x)^4\} = 4(1-2x)^3 \times (-2) = -8(1-2x)^3$

$$\therefore \quad \int (1-2x)^3\,dx = -\tfrac{1}{8}(1-2x)^4 + c.$$

(c) $\frac{d}{dx}\{(3x+4)^{-3}\} = -3(3x+4)^{-4} \times 3 = -\frac{9}{(3x+4)^4}$

$$\therefore \quad \int \frac{1}{(3x+4)^4}\,dx = -\frac{1}{9(3x+4)^3} + c.$$

Integrals involving trigonometric functions can often be found using basic trigonometric identities.

$$\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \tan x - x + c.$$

$$\int \cos^2 x\,dx = \int \tfrac{1}{2}(1 + \cos 2x)\,dx = \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x + c.$$

$$\int \sin 3x \cos 2x\,dx = \int \tfrac{1}{2}(\sin 5x + \sin x)\,dx = -\tfrac{1}{10}\cos 5x - \tfrac{1}{2}\cos x + c.$$

In the next two examples we consider integrals of the form

$$\int \{f(x)\}^n f'(x)\,dx.$$

Example 2 Find $\int \sin^2 x \cos x\,dx$ and hence $\int \cos^3 x\,dx$.

[We note that $f(x) = \sin x \Rightarrow f'(x) = \cos x$ and therefore try differentiating $\sin^3 x$.]

$$\frac{d}{dx}(\sin^3 x) = 3\sin^2 x \cos x$$

$$\therefore \quad \int \sin^2 x \cos x\,dx = \tfrac{1}{3}\sin^3 x + c.$$

Hence
$$\int \cos^3 x\,dx = \int (\cos^2 x)\cos x\,dx$$
$$= \int (1 - \sin^2 x)\cos x\,dx$$
$$= \int (\cos x - \sin^2 x \cos x)\,dx$$
$$= \sin x - \tfrac{1}{3}\sin^3 x + c.$$

Example 3 Find $\int 2x(1 + x^2)^4\,dx$.

[We note that $f(x) = 1 + x^2 \Rightarrow f'(x) = 2x$ and therefore try differentiating $(1 + x^2)^5$.]

$$\frac{d}{dx}\{(1 + x^2)^5\} = 5(1 + x^2)^4 \times 2x$$

$$\therefore \quad \int 2x(1 + x^2)^4\,dx = \tfrac{1}{5}(1 + x^2)^5 + c.$$

Two general results involving logarithms were derived in §3.4:

$$\int \frac{1}{ax + b}\,dx = \frac{1}{a}\ln(ax + b) + c, \quad \text{for} \quad ax + b > 0$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln\{f(x)\} + c, \quad \text{for} \quad f(x) > 0$$

or
$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c.$$

These were used to integrate functions such as:

$$\frac{1}{3x - 2}, \quad \frac{x - 1}{x + 1}, \quad \frac{2x^2 + 3x}{2x - 1}, \quad \frac{x}{3x^2 + 5} \quad \text{and} \quad \tan x.$$

Finally we note that when obtaining an indefinite integral it may be possible to produce several different answers.

(i) $\int (2x + 2)\,dx = x^2 + 2x + c$

or $\int (2x + 2)\,dx = \int 2(x + 1)\,dx = (x + 1)^2 + c.$

(ii) $\int 2\sin x \cos x\,dx = \sin^2 x + c$ or $\int 2\sin x \cos x\,dx = -\cos^2 x + c$

or $\int 2\sin x \cos x\,dx = \int \sin 2x\,dx = -\tfrac{1}{2}\cos 2x + c.$

However, apparently dissimilar results turn out to differ only by a constant. For example, in (ii)

$$\sin^2 x = -\cos^2 x + 1 = -\tfrac{1}{2}\cos 2x + \tfrac{1}{2}.$$

Exercise 4.1

Find the given indefinite integrals.

1. $\int \frac{x-3}{x^3}\,dx$
2. $\int (x^3-1)^2\,dx$
3. $\int (x+1)\sqrt{x}\,dx$
4. $\int \cos 3x\,dx$
5. $\int e^{4x-3}\,dx$
6. $\int \sin(1-2x)\,dx$
7. $\int (1-x)^4\,dx$
8. $\int \frac{1}{(x+4)^2}\,dx$
9. $\int \frac{1}{(2x-5)^3}\,dx$
10. $\int \frac{1}{2x-5}\,dx$
11. $\int \sec^2 5x\,dx$
12. $\int \cot \tfrac{1}{2}x\,dx$
13. $\int \sin^2 x\,dx$
14. $\int \sin x \cos 3x\,dx$
15. $\int \cot^2 x\,dx$
16. $\int \sin x \cos^2 x\,dx$
17. $\int \frac{\sin x}{\cos^2 x}\,dx$
18. $\int \sin^3 x\,dx$
19. $\int \frac{x+1}{x+2}\,dx$
20. $\int \frac{x^2}{x-1}\,dx$
21. $\int \frac{x^3-x}{x^2+5}\,dx$
22. $\int \frac{1}{\sqrt{(5x+3)}}\,dx$
23. $\int \frac{x^3}{1+x^4}\,dx$
24. $\int x^3(1+x^4)^3\,dx$

Evaluate the given definite integrals.

25. $\int_0^1 e^{1-x}\,dx$
26. $\int_{-2/3}^{1/3} (2+3x)^5\,dx$
27. $\int_0^{\pi/4} \cos^3 x \sin x\,dx$
28. $\int_0^{\pi/2} \sin 2x \sin 5x\,dx$
29. $\int_0^{\pi/12} \cos^2 3x\,dx$
30. $\int_0^{\pi/2} \sin^3 x \cos^2 x\,dx$
31. $\int_0^{\pi/3} \tan x \sec^2 x\,dx$
32. $\int_0^{\pi/4} \cos 3x \cos x\,dx$
33. $\int_0^{\pi/2} \tan^2\left(x - \frac{\pi}{4}\right)dx$
34. $\int_0^{\pi/2} \frac{\cos x}{1+\sin x}\,dx$
35. $\int_1^2 \frac{6}{(1-2x)^2}\,dx$
36. $\int_0^{1/2} \frac{x}{1-x^2}\,dx$

37. Show that $\int_0^{\pi/4} (1 + \tan x)^2 \, dx = 1 + \ln 2$.

38. Find $\frac{d}{dx}\{\ln(\sec x + \tan x)\}$ and hence evaluate $\int_0^{\pi/6} \sec x \, dx$.

39. By expressing $\cos x$ in terms of $\cos \frac{1}{2}x$, or otherwise, find

$$\int \frac{1}{1 + \cos x} dx.$$

40. Prove that $\sin^4 x = \frac{1}{8}(\cos 4x - 4\cos 2x + 3)$. Hence evaluate

$$\int_0^{\pi} \sin^4 x \, dx.$$

4.2 Use of partial fractions

Most rational functions of the form $P(x)/Q(x)$, where $P(x)$ and $Q(x)$ are polynomials, are more easily integrated when expressed in partial fractions.

As shown in §2.6, a linear factor of the form $(ax + b)$ in the denominator of a rational function leads to a partial fraction $A/(ax + b)$. Such fractions are integrated using the rule:

$$\int \frac{1}{ax + b} dx = \frac{1}{a}\ln(ax + b) + c, \quad \text{for} \quad ax + b > 0.$$

Example 1 Find $\int \frac{1}{(x - 2)(x + 1)} dx$ for $x > 2$.

Let $$\frac{1}{(x - 2)(x + 1)} \equiv \frac{A}{x - 2} + \frac{B}{x + 1}$$

then $$1 \equiv A(x + 1) + B(x - 2)$$

Putting $x = 2$: $$1 = 3A \quad \therefore \quad A = \tfrac{1}{3}$$

Putting $x = -1$: $$1 = -3B \quad \therefore \quad B = -\tfrac{1}{3}$$

$$\therefore \int \frac{1}{(x - 2)(x + 1)} dx = \int \left\{\frac{1}{3(x - 2)} - \frac{1}{3(x + 1)}\right\} dx$$

$$= \tfrac{1}{3}\ln(x - 2) - \tfrac{1}{3}\ln(x + 1) + c$$

$$= \tfrac{1}{3}\ln\left(\frac{x - 2}{x + 1}\right) + c, \quad \text{for} \quad x > 2.$$

Partial fractions are not needed when integrating rational functions of the form $f'(x)/f(x)$.

Example 2 Find $\int \frac{2x-1}{(x-2)(x+1)}dx$.

If $f(x) = (x-2)(x+1) = x^2 - x - 2$, then $f'(x) = 2x - 1$

$$\therefore \quad \int \frac{2x-1}{(x-2)(x+1)}dx = \ln|(x-2)(x+1)| + c.$$

A quadratic factor in the denominator of a rational function may give rise to a partial fraction of the form $f'(x)/f(x)$.

Example 3 Find $\int \frac{1-x^2}{x(x^2+1)}dx$, for $x > 0$.

Let $$\frac{1-x^2}{x(x^2+1)} \equiv \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

then $$1 - x^2 \equiv A(x^2+1) + (Bx+C)x$$

Putting $x = 0$ $\quad 1 = A \quad \therefore \quad A = 1$

Comparing coefficients, x^2: $\quad -1 = A + B \quad \therefore \quad B = -2$

x: $\quad 0 = C \quad \therefore \quad C = 0$

$$\therefore \quad \int \frac{1-x^2}{x(x^2+1)}dx = \int \left\{\frac{1}{x} - \frac{2x}{x^2+1}\right\}dx$$

$$= \ln x - \ln(x^2+1) + c$$

$$= \ln\left(\frac{x}{x^2+1}\right) + c, \quad \text{for} \quad x > 0.$$

Example 4 Evaluate $\int_0^1 \frac{1}{(x+1)(x^2+2x+2)}dx$.

Let $$\frac{1}{(x+1)(x^2+2x+2)} \equiv \frac{A}{x+1} + \frac{Bx+C}{x^2+2x+2}$$

then $$1 \equiv A(x^2+2x+2) + (Bx+C)(x+1)$$

Putting $x = -1$: $\quad 1 = A \quad \therefore \quad A = 1$

Putting $x = 0$: $\quad 1 = 2A + C \quad \therefore \quad C = -1$

Comparing coefficients of x^2: $\quad 0 = A + B \quad \therefore \quad B = -1$

$$\therefore \quad \int_0^1 \frac{1}{(x+1)(x^2+2x+2)}dx = \int_0^1 \left\{\frac{1}{x+1} - \frac{x+1}{x^2+2x+2}\right\}dx$$

$$= \int_0^1 \left\{\frac{1}{x+1} - \frac{1}{2}\left(\frac{2x+2}{x^2+2x+2}\right)\right\}dx$$

$$= \left[\ln(x+1) - \tfrac{1}{2}\ln(x^2+2x+2)\right]_0^1$$

$$= (\ln 2 - \tfrac{1}{2}\ln 5) - (\ln 1 - \tfrac{1}{2}\ln 2)$$

$$= \tfrac{3}{2}\ln 2 - \tfrac{1}{2}\ln 5$$

$$= \tfrac{1}{2}(3\ln 2 - \ln 5) = \tfrac{1}{2}\ln(8/5).$$

Example 5 Find $\int \frac{x-4}{(x-1)^2(2x+1)}\,dx$.

Let $$\frac{x-4}{(x-1)^2(2x+1)} \equiv \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(2x+1)}$$

then $$x - 4 \equiv A(x-1)(2x+1) + B(2x+1) + C(x-1)^2$$

Putting $x = 1$: $\quad -3 = 3B \quad \therefore \quad B = -1$

Putting $x = -\frac{1}{2}$: $\quad -\frac{9}{2} = \frac{9}{4}C \quad \therefore \quad C = -2$

Comparing coefficients of x^2: $\quad 0 = 2A + C \quad \therefore \quad A = 1$

$$\therefore \int \frac{x-4}{(x-1)^2(2x+1)}\,dx = \int \left\{ \frac{1}{x-1} - \frac{1}{(x-1)^2} - \frac{2}{2x+1} \right\} dx$$

$$= \ln|x-1| + \frac{1}{x-1} - \ln|2x+1| + c$$

$$= \frac{1}{x-1} + \ln\left|\frac{x-1}{2x+1}\right| + c.$$

Exercise 4.2

Find the following indefinite integrals.

1. $\int \frac{3}{(x-1)(x+2)}\,dx$
2. $\int \frac{1}{1-x^2}\,dx$
3. $\int \frac{1}{x(x-3)}\,dx$
4. $\int \frac{x}{x^2-4}\,dx$
5. $\int \frac{4x}{x^2-2x-3}\,dx$
6. $\int \frac{2x-5}{(x-2)(x-3)}\,dx$
7. $\int \frac{3x+5}{(x+1)(x+3)}\,dx$
8. $\int \frac{x-1}{x(2x-1)}\,dx$
9. $\int \frac{2x-11}{x^2-5x+4}\,dx$
10. $\int \frac{x^2}{1-x^2}\,dx$
11. $\int \frac{x-2}{x^2-4x-8}\,dx$
12. $\int \frac{x^3}{x^2-4}\,dx$

13. $\int \frac{2x^2 + 1}{(x + 1)^2(x - 2)}\,dx$

14. $\int \frac{x + 2}{x^2(x - 1)}\,dx$

15. $\int \frac{x}{(x + 3)^2}\,dx$

16. $\int \frac{x^2 + 2x - 1}{(x + 1)(x^2 + 1)}\,dx$

17. $\int \frac{x + 1}{(x^2 + 1)(x - 1)}\,dx$

18. $\int \frac{x}{(x^2 + 3)(x^2 + 5)}\,dx$

19. $\int \frac{4x^2 + x + 4}{(2x + 1)(x^2 + 2)}\,dx$

20. $\int \frac{x^3 - 3x - 4}{x^3 - 1}\,dx$

Evaluate the following definite integrals.

21. $\int_1^2 \frac{x}{(x + 1)(x + 2)}\,dx$

22. $\int_3^4 \frac{5}{x^2 + x - 6}\,dx$

23. $\int_4^5 \frac{2x}{x^2 - 4x + 3}\,dx$

24. $\int_0^{1/3} \frac{3x}{1 - x^2}\,dx$

25. $\int_0^{1/3} \frac{3 + x}{(1 - x)(1 + 3x)}\,dx$

26. $\int_2^3 \frac{1}{x(x - 1)}\,dx$

27. $\int_1^2 \frac{x - 3}{x(x + 1)(x + 3)}\,dx$

28. $\int_0^2 \frac{x^3 + 4x^2 + 3x - 2}{x^2 + 4x + 3}\,dx$

29. $\int_1^2 \frac{1}{x^2(x + 1)}\,dx$

30. $\int_3^4 \frac{x^2 + x - 3}{(x - 2)^2(x - 1)}\,dx$

31. $\int_3^4 \frac{x^3}{(x - 1)(x - 2)}\,dx$

32. $\int_4^5 \frac{2x^2 + 10x + 40}{(x - 3)^2(x + 1)}\,dx$

33. $\int_3^4 \frac{2x + 1}{(x - 2)(x^2 + 1)}\,dx$

34. $\int_1^2 \frac{2(x + 2)^2}{x^2(x^2 + 4)}\,dx$

35. $\int_0^1 \frac{(x - 2)^2}{x^3 + 1}\,dx$

36. $\int_2^3 \frac{x + 3}{(x - 1)(x^2 - x + 2)}\,dx$

37. Find $\int \frac{4}{(x-1)(x+3)}\,dx$ for (a) $x > 1$, (b) $-3 < x < 1$, (c) $x < -3$.

38. Find $\int \frac{x-4}{2x^2-x-3}\,dx$ for (a) $x > 1\frac{1}{2}$, (b) $x < 1\frac{1}{2}$.

4.3 Integration by substitution

The integration methods considered so far rely on the use of differentiation results in reverse. Integrating by substitution is a more systematic procedure, roughly equivalent to reversing the "chain rule" method for differentiating composite functions.

Suppose that as a result of applying the chain rule we have:

$$\frac{dy}{dx} = 10(2x+3)^4$$

We can work "backwards" to an expression for y as follows:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$\therefore$ substituting $u = 2x + 3$ gives $\frac{dy}{dx} = 10u^4$, $\frac{du}{dx} = 2$ and thus

$$10u^4 = \frac{dy}{du} \times 2 \quad \text{i.e.} \quad \frac{dy}{du} = 5u^4.$$

Hence $$y = u^5 + c = (2x+3)^5 + c.$$

The above method can be modified to produce a useful general result.

Let $$\frac{dy}{dx} = F(x), \quad \text{so that} \quad y = \int F(x)\,dx.$$

If, in terms of another variable u, $F(x) = G(u)$,

then $$\frac{dy}{du} = \frac{dy}{dx} \times \frac{dx}{du} = F(x)\frac{dx}{du} = G(u)\frac{dx}{du}$$

$\therefore$ $$y = \int G(u)\frac{dx}{du}\,du$$

i.e. if $F(x) = G(u)$, then $\int F(x)\,dx = \int G(u)\frac{dx}{du}\,du.$

Hence, when simplifying an integral $\int F(x)\,dx$ by a change to a new variable u, we

must express $F(x)$ in terms of u and replace dx by $\frac{dx}{du}du$.

[Note that in this context a statement such as $dx = \frac{dx}{du}du$ is often used as shorthand for "dx may be replaced by $\frac{dx}{du}du$".]

There are three different ways of expressing suitable substitutions.

(i) If $f(x) = u$, then $f'(x) = \frac{du}{dx}$,

and $$dx = \frac{dx}{du}du = \frac{1}{f'(x)}du$$

$\therefore$ we write $$f'(x)\,dx = du.$$

(ii) If $x = g(u)$, then $\frac{dx}{du} = g'(u)$,

$\therefore$ we write $$dx = g'(u)du.$$

(iii) If $f(x) = g(u)$, then $f'(x)\frac{dx}{du} = g'(u)$,

$\therefore$ we write $$f'(x)\,dx = g'(u)\,du.$$

Example 1 Find $\int 10(2x + 3)^4\,dx$.

Let $2x + 3 = u$, then $2dx = du$.

$$\therefore \int 10(2x + 3)^4\,dx = \int 5(2x + 3)^4 \times 2\,dx$$

$$= \int 5u^4 du$$

$$= u^5 + c = (2x + 3)^5 + c.$$

Example 2 Find $\int \sin^3 2x \cos 2x\,dx$.

Let $\sin 2x = u$, then $2\cos 2x\,dx = du$.

$$\therefore \int \sin^3 2x \cos 2x\,dx = \int \tfrac{1}{2}\sin^3 2x \times 2\cos 2x\,dx$$

$$= \int \tfrac{1}{2}u^3\,du$$

$$= \tfrac{1}{8}u^4 + c = \tfrac{1}{8}\sin^4 2x + c.$$

Example 3 Find $\int x\sqrt{(2x-1)}\,dx$.

Let $2x-1=u^2$, $\quad 2x=u^2+1$

then $2dx=2u\,du$, $\quad x=\frac{1}{2}(u^2+1)$

$\therefore\quad dx=u\,du$

$$\int x\sqrt{(2x-1)}\,dx=\int\tfrac{1}{2}(u^2+1)\times u\times u\,du$$

$$=\frac{1}{2}\int(u^4+u^2)\,du$$

$$=\tfrac{1}{2}(\tfrac{1}{5}u^5+\tfrac{1}{3}u^3)+c$$

$$=\tfrac{1}{30}u^3(3u^2+5)+c$$

$$\therefore\quad \int x\sqrt{(2x-1)}\,dx=\tfrac{1}{30}(2x-1)^{3/2}(3\{2x-1\}+5)+c$$

$$=\tfrac{1}{15}(2x-1)^{3/2}(3x+1)+c.$$

When the method of substitution is applied to a definite integral, the limits of integration must be changed as well as the variable.

Example 4 Evaluate $\int_0^1 4x(3x^2-1)^3\,dx$.

Let $3x^2-1=u$, $\quad x=1\Rightarrow u=2$

then $6x\,dx=du$ $\quad x=0\Rightarrow u=-1$

$$\int_0^1 4x(3x^2-1)^3\,dx=\int_0^1 \tfrac{2}{3}(3x^2-1)^3\times 6x\,dx$$

$$=\int_{-1}^{2}\tfrac{2}{3}u^3\,du$$

$$=\left[\tfrac{1}{6}u^4\right]_{-1}^{2}$$

$$=\tfrac{1}{6}\{2^4-(-1)^4\}=\tfrac{15}{6}=2\tfrac{1}{2}.$$

Example 5 Evaluate $\int_0^2\sqrt{(4-x^2)}\,dx$.

$$\int_0^2 \sqrt{(4-x^2)}\,dx$$

$$= \int_0^{\pi/2} 2\cos\theta \times 2\cos\theta\,d\theta$$

$$= \int_0^{\pi/2} 4\cos^2\theta\,d\theta$$

$$= \int_0^{\pi/2} (2 + 2\cos 2\theta)\,d\theta$$

$$= \Big[2\theta + \sin 2\theta\Big]_0^{\pi/2}$$

$$= \pi$$

Let $x = 2\sin\theta$
then $dx = 2\cos\theta\,d\theta$

$4 - x^2 = 4 - 4\sin^2\theta$
$= 4(-\sin^2\theta)$
$= 4\cos^2\theta$

$\therefore \sqrt{(4-x^2)} = 2\cos\theta$

$x = 2 \Rightarrow \theta = \frac{1}{2}\pi$
$x = 0 \Rightarrow \theta = 0$

When integrating trigonometric functions one useful substitution is $\tan\frac{1}{2}\theta = t$, where

$$\sin\theta = \frac{2t}{1+t^2}, \quad \cos\theta = \frac{1-t^2}{1+t^2}, \quad \tan\theta = \frac{2t}{1-t^2}$$

Example 6 Evaluate $\displaystyle\int_0^{\pi/2} \frac{4}{3+5\cos\theta}\,d\theta$.

$$\int_0^{\pi/2} \frac{4}{3+5\cos\theta}\,d\theta$$

$$= \int_0^1 \frac{4(1+t^2)}{8-2t^2} \times \frac{2}{1+t^2}\,dt$$

$$= \int_0^1 \frac{4}{4-t^2}\,dt$$

$$= \int_0^1 \left\{\frac{1}{2+t} + \frac{1}{2-t}\right\} dt$$

$$= \Big[\ln(2+t) - \ln(2-t)\Big]_0^1$$

$$= (\ln 3 - \ln 1) - (\ln 2 - \ln 2)$$

$$= \ln 3$$

Let $\tan\frac{1}{2}\theta = t$
then $\frac{1}{2}\sec^2\frac{1}{2}\theta\,d\theta = dt$
$\therefore (1 + \tan^2\frac{1}{2}\theta)\,d\theta = 2dt$
$\therefore d\theta = \dfrac{2}{1+t^2}\,dt$

$$3 + 5\cos\theta = 3 + 5\left(\frac{1-t^2}{1+t^2}\right)$$

$$= \frac{3(1+t^2) + 5(1-t^2)}{1+t^2}$$

$$= \frac{8-2t^2}{1+t^2}$$

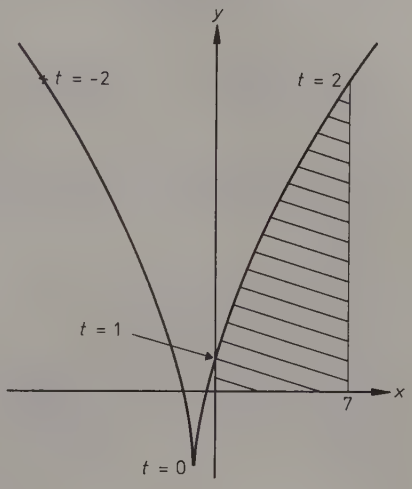
$\theta = \frac{1}{2}\pi \Rightarrow t = 1$
$\theta = 0 \Rightarrow t = 0$

A change of variable may be helpful when finding areas and volumes by integration. For instance, to evaluate $\displaystyle\int_a^b y\,dx$ it may be simpler to work in terms of y rather than x. If x and y are given in parametric form it will usually be best to use the parameter as variable in any integrations.

Example 7 Find the area bounded by the curve $x = t^3 - 1$, $y = 5t^2 - 4$, the x- and y-axes and the line $x = 7$.

As shown in the sketch below, values of x from 0 to 7 are given by values of t from 1 to 2.
Since $x = t^3 - 1$, we have $dx = 3t^2\,dt$.

$$\text{Required area} = \int_0^7 y\,dx$$
$$= \int_1^2 (5t^2 - 4) \times 3t^2\,dt$$
$$= \int_1^2 (15t^4 - 12t^2)\,dt$$
$$= \left[3t^5 - 4t^3\right]_1^2$$
$$= (96 - 32) - (3 - 4) = 65.$$



Exercise 4.3

Use the given substitutions to find the following indefinite integrals.

1. $\int (3x - 2)^3\,dx, 3x - 2 = u.$
2. $\int (1 - 2x)^4\,dx, 1 - 2x = u.$
3. $\int x(1 - x)^5\,dx, 1 - x = u.$
4. $\int 2x^2(x^3 - 1)^3\,dx, x^3 - 1 = u.$
5. $\int \tan^2 x \sec^2 x\,dx, \tan x = u.$
6. $\int \sin x \cos^5 x\,dx, \cos x = u.$
7. $\int x\sqrt{(x^2 + 4)}\,dx, x^2 + 4 = t^2.$
8. $\int x\sqrt{(1 - x)}\,dx, 1 - x = t^2.$
9. $\int \frac{x}{\sqrt{(2x + 1)}}\,dx, 2x + 1 = t^2.$
10. $\int \frac{e^x}{1 + e^x}\,dx, e^x = t.$
11. $\int \cos^3 \theta\,d\theta, \sin \theta = x.$
12. $\int \sin^3 \theta \cos^2 \theta\,d\theta, \cos \theta = x.$
13. $\int \frac{\sin \theta}{1 + \cos 2\theta}\,d\theta, \cos \theta = x.$
14. $\int \frac{\sec^2(\sqrt{x})}{\sqrt{x}}\,dx, \sqrt{x} = \theta.$

Use the given substitutions to find the following definite integrals.

15. $\int_4^5 (x-1)\sqrt{(x-4)}\,dx$, $x-4=u^2$.

16. $\int_0^1 x^2(1-x)^{1/2}\,dx$, $1-x=u^2$.

17. $\int_e^{e^2} \frac{1}{x\ln x}\,dx$, $\ln x = t$.

18. $\int_0^1 \frac{e^x}{(1+e^x)^2}\,dx$, $e^x = t$.

19. $\int_0^2 \frac{1}{\sqrt{(4-x^2)}}\,dx$, $x = 2\sin\theta$.

20. $\int_0^{\sqrt{3}} \frac{1}{9+x^2}\,dx$, $x = 3\tan\theta$.

21. $\int_0^{\pi/2} \sin 2x\sqrt{(\sin x)}\,dx$, $\sin x = u^2$.

22. $\int_0^{1/2} (1-x^2)^{1/2}\,dx$, $x = \sin u$.

23. $\int_0^{\pi/2} \frac{1}{5\cos\theta+4}\,d\theta$, $\tan\frac{1}{2}\theta = t$.

24. $\int_0^{\pi/4} \frac{1}{5\cos^2\theta-1}\,d\theta$, $\tan\theta = t$.

Find the following indefinite integrals.

25. $\int x(2x+3)^4\,dx$

26. $\int x^3(x^2-1)^3\,dx$

27. $\int \sqrt{(1+3x)}\,dx$

28. $\int \frac{x}{\sqrt{(3+x)}}\,dx$

29. $\int \frac{x^2}{(x^3-1)^2}\,dx$

30. $\int \frac{x+1}{(x-1)^3}\,dx$

31. $\int xe^{x^2}\,dx$

32. $\int \frac{1}{x\sqrt{(\ln x)}}\,dx$

33. $\int \frac{e^x}{4-e^{2x}}\,dx$

Find the following definite integrals

34. $\int_{-1}^2 x^2\sqrt{(x^3+1)}\,dx$

35. $\int_0^3 x\sqrt{(9-x^2)}\,dx$

36. $\int_0^3 \sqrt{(9-x^2)}\,dx$

37. $\int_1^2 \frac{x}{\sqrt{(4-x^2)}}\,dx$

38. $\int_1^2 \frac{x^2}{\sqrt{(4-x^2)}}\,dx$

39. $\int_0^{\ln 2} \frac{1}{1+e^x}\,dx$

40. Find the area between the x-axis and the curve $x = 2t^3 - 1$, $y = t^2 - 1$ for $1 \leqslant t \leqslant 2$.

41. Find the area of the finite region bounded by the curve $x = 5t^2$, $y = 2t^3$ and the line $x = 5$. Find also the volume of the solid formed when this region is rotated through π radians about the x-axis.

42. Find the area bounded by the x-axis and the curve $x = \theta - \sin\theta$, $y = 1 - \cos\theta$ for $0 \leqslant \theta \leqslant 2\pi$.

43. Using the substitution $\tan\frac{1}{2}x = t$, or otherwise, show that

$$\int_{\pi/3}^{\pi/2} \frac{1}{1 + \sin x + \cos x}\,dx = 0{\cdot}237$$

correct to 3 significant figures.

44. Using the substitution $x = 2\cos^2\theta + 5\sin^2\theta$, or otherwise, evaluate

$$\int_2^{7/2} \frac{1}{\sqrt{\{(x-2)(5-x)\}}}\,dx.$$

45. Using the substitution $\tan x = t$, or otherwise, show that

$$\int \frac{1}{1 - 2\sin^2 x}\,dx = \tfrac{1}{2}\ln\{\tan(x + \tfrac{1}{4}\pi)\} + c.$$

4.4 Integration by parts

A further general method of integration is obtained by considering the formula for differentiating a product. If u and v are functions of x, then

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrating,

$$uv = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx$$

Rearranging,

$$\boxed{\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx} \qquad (1)$$

This rule provides a way of expressing the integral of a product in terms of a second integral, which may be easier to find. The method is called *integration by parts.*

Example 1 Find $\int x\cos x\,dx$.

In rule (1), let $u = x$, $\dfrac{dv}{dx} = \cos x$,

$$\frac{du}{dx} = 1, \qquad v = \sin x,$$

$$\therefore \int x\cos x\,dx = x\sin x - \int(\sin x \times 1)\,dx$$

$$= x\sin x - \int \sin x\,dx = x\sin x + \cos x + c.$$

[Note that it is not necessary to introduce an arbitrary constant when integrating $\frac{dv}{dx}$ to give v, since rule (1) is valid for any function v such that $\frac{dv}{dx} = \cos x$.]

Example 2 Find $\int x \ln x \, dx$.

In rule (1), let $u = \ln x$, $\frac{dv}{dx} = x$,

$$\frac{du}{dx} = \frac{1}{x}, \qquad v = \tfrac{1}{2}x^2,$$

$$\therefore \int x \ln x \, dx = \tfrac{1}{2}x^2 \ln x - \int \left(\tfrac{1}{2}x^2 \times \frac{1}{x}\right) dx$$

$$= \tfrac{1}{2}x^2 \ln x - \int \tfrac{1}{2}x \, dx = \tfrac{1}{2}x^2 \ln x - \tfrac{1}{4}x^2 + c.$$

For definite integrals the rule for integration by parts becomes

$$\boxed{\int_a^b u \frac{dv}{dx} dx = \Big[uv\Big]_a^b - \int_a^b v \frac{du}{dx} dx} \qquad (2)$$

Example 3 Evaluate $\int_0^1 x^2 e^x \, dx$.

In rule (2), let $u = x^2$, $\frac{dv}{dx} = e^x$,

$$\frac{du}{dx} = 2x, \qquad v = e^x,$$

$$\therefore \int_0^1 x^2 e^x \, dx = \Big[x^2 e^x\Big]_0^1 - \int_0^1 (e^x \times 2x) \, dx$$

$$= e - 2\int_0^1 x e^x \, dx.$$

In rule (2), let $u = x$, $\frac{dv}{dx} = e^x$,

$$\frac{du}{dx} = 1, \qquad v = e^x,$$

$$\therefore \int_0^1 x e^x \, dx = \Big[x e^x\Big]_0^1 - \int_0^1 e^x \, dx$$

$$= e - \Big[e^x\Big]_0^1 = e - (e - 1) = 1.$$

Hence $\int_0^1 x^2 e^x \, dx = e - 2\int_0^1 xe^x \, dx = e - 2.$

Integration by parts is also used to integrate functions which are not products.

Example 4 Find $\int \ln x \, dx$.

In rule (1), let $u = \ln x, \quad \frac{dv}{dx} = 1,$

$$\frac{du}{dx} = \frac{1}{x}, \qquad v = x,$$

$$\therefore \quad \int \ln x \, dx = x \ln x - \int \left(x \times \frac{1}{x} \right) dx$$

$$= x \ln x - \int dx = x \ln x - x + c.$$

The next example illustrates a further way in which integration by parts may be used.

Example 5 Find $\int e^x \cos x \, dx$.

Integrating by parts,

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx \qquad \text{(i)}$$

Integrating by parts,

$$\int e^x \sin x \, dx = e^x(-\cos x) - \int (-\cos x) e^x \, dx$$

$$= -e^x \cos x + \int e^x \cos x \, dx$$

Substituting this expression for $\int e^x \sin x \, dx$ in (i),

$$\int e^x \cos x \, dx = e^x \sin x - \left(-e^x \cos x + \int e^x \cos x \, dx \right)$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

$$\therefore \quad 2\int e^x \cos x \, dx = e^x \sin x + e^x \cos x + k$$

Hence $\int e^x \cos x \, dx = \frac{1}{2} e^x (\sin x + \cos x) + c.$

Exercise 4.4

Find the following indefinite integrals.

1. $\int x \sin x \, dx$
2. $\int x \cos \frac{1}{2}x \, dx$
3. $\int xe^{-x} \, dx$
4. $\int x \ln 2x \, dx$
5. $\int xe^{2x} \, dx$
6. $\int x \sec^2 x \, dx$
7. $\int x \sin x \cos x \, dx$
8. $\int x \tan^2 x \, dx$
9. $\int x \cos^2 x \, dx$
10. $\int x^2 \cos x \, dx$
11. $\int x^3 e^x dx$
12. $\int x^3 \ln x \, dx$
13. $\int x(\ln x)^2 \, dx$
14. $\int \ln(x - 1) \, dx$
15. $\int e^{-x} \sin x \, dx$

Find the following definite integrals.

16. $\int_0^{\pi} x \cos x \, dx$
17. $\int_{-1}^{1} xe^x \, dx$
18. $\int_0^{\pi/3} x \sin 3x \, dx$
19. $\int_1^e \ln x \, dx$
20. $\int_0^{\pi} x \sin^2 x \, dx$
21. $\int_1^2 x^2 \ln x \, dx$
22. $\int_2^3 x^2 e^{-x} \, dx$
23. $\int_1^e \frac{1}{x^3} \ln x \, dx$
24. $\int_0^{\pi/2} x^2 \sin x \, dx$
25. $\int_0^{\pi} e^x \sin x \, dx$
26. $\int_1^2 x^2 e^{2x} \, dx$
27. $\int_e^{e^2} \ln(\sqrt{x}) \, dx$

28. Find $\int x(x + 1)^4 \, dx$ using (a) integration by parts, (b) the substitution $x + 1 = t$.

29. Find, using integration by parts or otherwise,

(a) $\int x\sqrt{(x - 1)} \, dx$, (b) $\int \frac{x}{(x + 2)^3} dx$.

30. Use integration by parts to show that

(a) $2\int \cos^2 x \, dx = \cos x \sin x + x + c$,

(b) $3\int \cos^3 x \, dx = \cos^2 x \sin x + 2 \sin x + c$.

4.5 Use of inverse trigonometric functions

The inverse trigonometric functions $\sin^{-1}x$ and $\tan^{-1}x$ are defined as follows:

$$y = \sin^{-1}x \Rightarrow \sin y = x \quad \text{and} \quad -\tfrac{1}{2}\pi \leqslant y \leqslant \tfrac{1}{2}\pi$$
$$y = \tan^{-1}x \Rightarrow \tan y = x \quad \text{and} \quad -\tfrac{1}{2}\pi < y < \tfrac{1}{2}\pi$$

In Book 1 two important integration results were obtained by differentiating $\sin^{-1}\frac{x}{a}$ and $\tan^{-1}\frac{x}{a}$.

$$\int \frac{1}{\sqrt{(a^2 - x^2)}}\,dx = \sin^{-1}\frac{x}{a} + c, \quad \int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c.$$

[These integrals can also be found by substituting $x = a\sin\theta$ and $x = a\tan\theta$ respectively.]

Example 1 Evaluate (a) $\int_0^1 \frac{1}{\sqrt{(4 - x^2)}}\,dx$, (b) $\int_0^2 \frac{1}{4 + x^2}\,dx$.

(a) $$\int_0^1 \frac{1}{\sqrt{(4 - x^2)}}\,dx = \left[\sin^{-1}\frac{x}{2}\right]_0^1 = \sin^{-1}\frac{1}{2} - \sin^{-1}0 = \frac{\pi}{6}.$$

(b) $$\int_0^2 \frac{1}{4 + x^2}\,dx = \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^2 = \frac{1}{2}\tan^{-1}1 - \frac{1}{2}\tan^{-1}0 = \frac{\pi}{8}.$$

Example 2 Find (a) $\int \frac{1}{\sqrt{(9 - 4x^2)}}\,dx$, (b) $\int \frac{1}{25 + 16x^2}\,dx$.

(a) Let $2x = u$, then $4x^2 = u^2$ and $2dx = du$,

$$\therefore \int \frac{1}{\sqrt{(9 - 4x^2)}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{(9 - u^2)}}\,du$$
$$= \frac{1}{2}\sin^{-1}\frac{u}{3} + c.$$
$$= \frac{1}{2}\sin^{-1}\frac{2x}{3} + c.$$

(b) Let $4x = u$, then $16x^2 = u^2$ and $4dx = du$,

$$\therefore \int \frac{1}{25 + 16x^2}\,dx = \frac{1}{4}\int \frac{1}{25 + u^2}\,du$$
$$= \frac{1}{4} \times \frac{1}{5}\tan^{-1}\frac{u}{5} + c$$
$$= \frac{1}{20}\tan^{-1}\frac{4x}{5} + c.$$

The method of Example 2 can be extended to integrals of the form

$$\int \frac{1}{\sqrt{(ax^2+bx+c)}}dx, \quad \int \frac{1}{ax^2+bx+c}dx.$$

Example 3 Find $\int \frac{1}{x^2-2x+2}dx$.

$x^2-2x+2 = (x^2-2x+1)+1 = (x-1)^2+1$

$$\therefore \int \frac{1}{x^2-2x+2}dx = \int \frac{1}{(x-1)^2+1}dx$$

$$= \int \frac{1}{u^2+1}du \qquad \text{letting } x-1=u,\ dx=du$$

$$= \tan^{-1}u + c$$

$$= \tan^{-1}(x-1) + c.$$

The remaining examples illustrate further uses of inverse trigonometric functions.

Example 4 Find $\int \frac{x+3}{x^2+9}dx$.

$$\int \frac{x+3}{x^2+9}dx = \int \left\{\frac{x}{x^2+9} + \frac{3}{x^2+9}\right\}dx = \frac{1}{2}\int \frac{2x}{x^2+9}dx + 3\int \frac{1}{x^2+9}dx$$

$$= \frac{1}{2}\ln(x^2+9) + \tan^{-1}\frac{x}{3} + c.$$

Example 5 Evaluate $\int_0^1 \frac{3-x}{(x+1)(x^2+1)}dx$.

Let $\frac{3-x}{(x+1)(x^2+1)} \equiv \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$

then $3-x \equiv A(x^2+1) + (Bx+C)(x+1)$

Putting $x=-1$: $4 = 2A$ $\quad\therefore\quad A = 2$

Putting $x=0$: $3 = A + C$ $\quad\therefore\quad C = 1$

Comparing coefficients of x^2:

$0 = A + B$ $\quad\therefore\quad B = -2$

$$\therefore \int_0^1 \frac{3-x}{(x+1)(x^2+1)}dx = \int_0^1 \left\{\frac{2}{x+1} + \frac{(-2x+1)}{x^2+1}\right\}dx$$

$$= \int_0^1 \left\{\frac{2}{x+1} - \frac{2x}{x^2+1} + \frac{1}{x^2+1}\right\}dx$$

$$= \left[2\ln(x+1) - \ln(x^2+1) + \tan^{-1}x\right]_0^1$$

$$= (2\ln 2 - \ln 2 + \tan^{-1}1) - (2\ln 1 - \ln 1 + \tan^{-1}0)$$

$$= \ln 2 + \tfrac{1}{4}\pi$$

Example 6 Find $\int \frac{2x^2}{\sqrt{(1-x^2)}}dx$.

$$\int \frac{2x^2}{\sqrt{(1-x^2)}}dx$$

$$= \int \frac{2\sin^2\theta}{\cos\theta} \times \cos\theta\, d\theta$$

$$= \int (1-\cos 2\theta)\, d\theta$$

$$= \theta - \tfrac{1}{2}\sin 2\theta + c$$

$$= \theta - \sin\theta\cos\theta + c$$

$$= \sin^{-1}x - x\sqrt{(1-x^2)} + c.$$

Let $x = \sin\theta$
then $dx = \cos\theta\, d\theta$
$1 - x^2 = 1 - \sin^2\theta$
$= \cos^2\theta$
$\therefore \sqrt{(1-x^2)} = \cos\theta$

Exercise 4.5

Find the following indefinite integrals.

1. $\int \frac{1}{\sqrt{(16-x^2)}}dx$
2. $\int \frac{1}{x^2+25}dx$
3. $\int \frac{1}{\sqrt{(1-4x^2)}}dx$
4. $\int \frac{x+1}{x^2+1}dx$
5. $\int \frac{1}{\sqrt{(4-25x^2)}}dx$
6. $\int \frac{1}{4+9x^2}dx$
7. $\int \frac{x^2-x}{(x+1)(x^2+1)}dx$
8. $\int \frac{4}{(1-x)(1+x^2)}dx$
9. $\int \frac{6(x-3)}{x(x^2+9)}dx$
10. $\int \frac{5x^2+4x-20}{(x+2)(x^2+4)}dx$
11. $\int \frac{1}{x^2+2x+5}dx$
12. $\int \frac{1}{\sqrt{(5+4x-x^2)}}dx$

Evaluate the following definite integrals.

13. $\int_0^{\sqrt{3}} \frac{1}{x^2+9}dx$
14. $\int_0^5 \frac{1}{\sqrt{(25-x^2)}}dx$
15. $\int_0^{1/2} \frac{1}{4x^2+1}dx$
16. $\int_{-2}^{2} \frac{x+2}{x^2+4}dx$
17. $\int_0^1 \frac{4}{(x+1)(x^2+1)}dx$
18. $\int_3^4 \frac{1}{x^2-6x+10}dx$
19. $\int_{-1/2}^{1/2} \frac{1}{\sqrt{(3-4x-4x^2)}}dx$

Use appropriate substitutions to find the following indefinite integrals.

29. $\int \frac{e^x + e^{2x}}{1 + e^{2x}} dx$

21. $\int \frac{1}{e^x + e^{-x}} dx$

22. $\int \sqrt{(1 - x^2)}\, dx$

23. $\int \frac{1 - x}{\sqrt{(1 - x^2)}} dx$

Use the given substitutions to find the following indefinite integrals.

24. $\int \frac{2x}{x^2 - 2x + 10} dx$, $x = u + 1$.

25. $\int \frac{x}{x^4 + 2x^2 + 2} dx$, $x^2 = u - 1$.

26. $\int \frac{1}{x\sqrt{(x^2 - 1)}} dx$, $x = \frac{1}{u}$.

27. Evaluate correct to 3 significant figures

(a) $\int_1^3 \frac{4 - 3x}{(2x - 1)(x^2 + 1)} dx$,

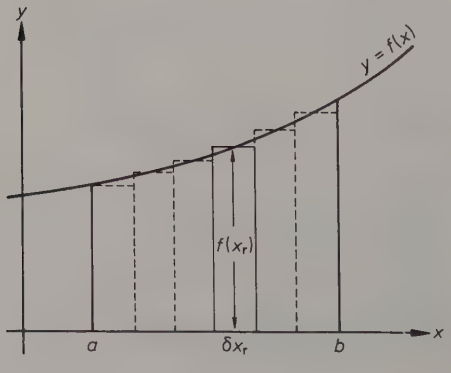
(b) $\int_0^1 \frac{7x + 2}{(x + 1)^2(x^2 + 4)} dx$

28. Use integration by parts to show that

$$\int \sin^{-1} x\, dx = x \sin^{-1} x + \sqrt{(1 - x^2)} + c.$$

4.6 Properties of the definite integral

The definite integral is defined formally as the limit of a sum by considering a function $f(x)$ in the interval $a \leqslant x \leqslant b$.



Suppose that this interval is divided into n subintervals of lengths $\delta x_1, \delta x_2, \ldots, \delta x_n$. If n is increased in such a way that the values of $\delta x_1, \delta x_2, \ldots, \delta x_n$ tend to zero, then

$$\int_a^b f(x)\, dx = \lim_{n \to \infty} \left\{ \sum_{r=1}^{n} f(x_r)\, \delta x_r \right\},$$

where x_r lies in the interval of length δx_r.

This definition may be expressed more simply as

$$\int_a^b y\,dx = \lim_{\delta x \to 0} \sum y\,\delta x.$$

In general the value of this integral represents the area bounded by the curve $y = f(x)$, the x-axis, the lines $x = a$ and $x = b$. We must remember, however, that areas above the x-axis are positive and areas below the x-axis are negative.

The most important properties of the definite integral are as follows:

$$\int_a^b f(x)\,dx + \int_a^b g(x)\,dx = \int_a^b \{f(x) + g(x)\}\,dx.$$

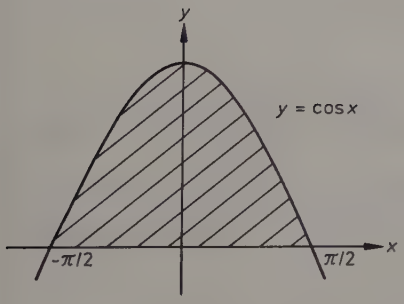
$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx, \quad \text{where } k \text{ is a constant.}$$

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx,$$

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx.$$

If $f(x)$ is an *even function* then the curve $y = f(x)$ will be symmetrical about the y-axis. Hence the area under the curve between $x = -a$ and $x = 0$ is equal to the area between $x = 0$ and $x = a$.

$\therefore$ if $f(x)$ is an even function $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx.$



For instance, if $f(x) = \cos x$,

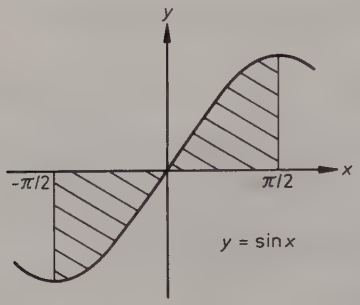
$f(-x) = \cos(-x) = \cos x = f(x)$

$$\therefore \int_{-\pi/2}^{\pi/2} \cos x\,dx = 2\int_0^{\pi/2} \cos x\,dx$$

$$= 2\Big[\sin x\Big]_0^{\pi/2} = 2.$$

If $f(x)$ is an *odd function*, the areas bounded by the curve $y = f(x)$ and the x-axis between $x = -a$ and $x = 0$ and between $x = 0$ and $x = a$ are numerically equal but opposite in sign,

$\therefore$ if $f(x)$ is an odd function $\displaystyle\int_{-a}^{a} f(x)\,dx = 0.$



For instance, if $f(x) = \sin x$,
$f(-x) = \sin(-x) = -\sin x = -f(x)$

$$\therefore \quad \int_{-\pi/2}^{\pi/2} \sin x \, dx = 0.$$

The symmetries of the sine function can also be used when integrating functions of the form $xf(\sin x)$. We illustrate the method by a simple example.

Example 1 Evaluate $I = \int_0^{\pi} x \sin x \, dx$.

$$I = -\int_{\pi}^{0} (\pi - u)\sin(\pi - u)\, du$$

Let $x = \pi - u$
then $dx = -du$

$x = \pi \Rightarrow u = 0$
$x = 0 \Rightarrow u = \pi$

$$= \int_0^{\pi} (\pi - u)\sin u \, du$$

$$= \int_0^{\pi} \pi \sin u \, du - \int_0^{\pi} u \sin u \, du$$

$$= \pi \int_0^{\pi} \sin u \, du - I$$

$$\therefore \quad I = \tfrac{1}{2}\pi \int_0^{\pi} \sin u \, du = \tfrac{1}{2}\pi \Big[-\cos u \Big]_0^{\pi} = \tfrac{1}{2}\pi\{1 - (-1)\} = \pi.$$

Finally we consider definite integrals with *infinite limits*. One type of infinite integral is defined as follows:

$$\int_0^{\infty} f(x)\, dx = \lim_{t \to \infty} \int_0^{t} f(x)\, dx,$$

provided that this limit exists.

Example 2 Evaluate $\int_0^{\infty} e^{-x} dx$.

$$\int_0^{t} e^{-x} dx = \Big[-e^{-x} \Big]_0^{t} = -e^{-t} + 1$$

$$\therefore \quad \int_0^{\infty} e^{-x} dx = \lim_{t \to \infty} (-e^{-t} + 1) = 1.$$

Exercise 4.6

1. Evaluate the following integrals using the properties of odd and even functions.

(a) $\int_{-1}^{1} x^4\,dx$, (b) $\int_{-3}^{3} x(x^2-1)\,dx$, (c) $\int_{-\pi}^{\pi} \sin^2 x\,dx$,

(d) $\int_{-\pi/3}^{\pi/3} \frac{1}{1+\cos 2x}\,dx$, (e) $\int_{-4}^{4} \frac{x}{x^2-25}\,dx$, (f) $\int_{-1}^{1} \frac{x\cos x}{1+\tan^2 x}\,dx$.

2. Sketch the curve $y=(x^2-1)(x-3)$. Hence evaluate $\int_{-7}^{9} (x^2-1)(x-3)\,dx$.

3. Using sketch graphs, or otherwise, evaluate

(a) $\int_{-2}^{2} [x]\,dx$, (b) $\int_{-2}^{2} |x|\,dx$, (c) $\int_{-2}^{2} (x+|x|)\,dx$.

4. Evaluate (a) $\int_{0}^{3} (x-2)\,dx$, (b) $\left|\int_{0}^{3} (x-2)\,dx\right|$, (c) $\int_{0}^{3} |x-2|\,dx$.

5. Evaluate (a) $\int_{0}^{3} x(x-2)\,dx$, (b) $\int_{0}^{3} |x(x-2)|\,dx$.

6. Prove that $\int_{0}^{a} f(x)\,dx = \int_{0}^{a} f(a-x)\,dx$. Hence, or otherwise, show that

(a) $\int_{0}^{\pi} \frac{x\sin x}{4-\cos^2 x}\,dx = \frac{1}{4}\pi \ln 3$, (b) $\int_{0}^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \frac{1}{4}\pi^2$.

7. Evaluate $\int_{0}^{1} e^x\,dx$. Hence show that

$$\lim_{n\to\infty} \frac{1}{n}\left\{e^{1/n} + e^{2/n} + e^{3/n} + \ldots + e\right\} = e-1.$$

8. Evaluate $\int_{0}^{1} \frac{x\,dx}{\sqrt{(1+x)}}$. By considering integration as the limit of a sum, or otherwise, show that

$$\lim_{n\to\infty} \frac{1}{n\sqrt{n}}\left\{\frac{1}{\sqrt{(n+1)}} + \frac{2}{\sqrt{(n+2)}} + \ldots + \frac{n}{\sqrt{(2n)}}\right\} = \frac{2}{3}(2-\sqrt{2}). \quad \text{(JMB)}$$

9. Explain what is wrong with the following mathematical arguments and evaluate the integrals I and J correctly:

(i) Let $I = \int_{0}^{2} \frac{1}{x^2-2x+2}\,dx$. Then putting $x-1=\cot\theta$ we have

$$I = \int_{\cot^{-1}(1)}^{\cot^{-1}(-1)} \frac{-\operatorname{cosec}^2\theta}{\cot^2\theta+1}\,d\theta = -\int_{-\pi/4}^{\pi/4} d\theta = -\tfrac{1}{2}\pi.$$

(ii) Let $J = \int_0^{\pi} e^{|\sin x|}|\cos x|\,dx$. Then putting $u = |\sin x|$ we have $du = |\cos x|\,dx$ and so $J = \int_0^0 e^u\,du = 0$. (C)

10. Evaluate (a) $\int_0^{\infty} e^{-x/2}\,dx$, (b) $\int_1^{\infty} \frac{1}{x^2}\,dx$, (c) $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$.

11. Use appropriate substitutions to show that

(i) $\int_1^{a^k} \frac{1}{t}\,dt = k\int_1^{a} \frac{1}{u}\,du$, (ii) $\int_a^{ab} \frac{1}{t}\,dt = \int_1^{b} \frac{1}{u}\,du$.

Defining $\ln x$ as $\int_1^x \frac{1}{t}\,dt$, deduce that

(iii) $\ln(a^k) = k\ln a$, (iv) $\ln(ab) = \ln a + \ln b$.

Exercise 4.7 (*miscellaneous*)

In questions 1 to 15 evaluate the given definite integrals.

1. $\int_0^4 \frac{x}{2x+1}\,dx$ 2. $\int_{-1/2}^0 \frac{x}{(x+1)^2}\,dx$ 3. $\int_1^2 \frac{1}{4x^2-1}\,dx$

4. $\int_{-1}^4 \frac{1}{\sqrt{(3x+4)}}\,dx$ 5. $\int_2^3 \frac{x^2+1}{x(x-1)}\,dx$

6. $\int_0^2 (4-x^2)\sqrt{x}\,dx$ 7. $\int_0^2 x\sqrt{(4-x^2)}\,dx$

8. $\int_0^{\pi} (\sin x + \cos x)^2\,dx$ 9. $\int_0^{\pi/3} 2\sin 2x\cos x\,dx$

10. $\int_0^{\pi/4} \tan^3 x\,dx$ 11. $\int_0^{\pi/4} \sin^3 x\cos^3 x\,dx$

12. $\int_{\ln 2}^{\ln 5} \frac{2}{e^x - e^{-x}}\,dx$ 13. $\int_0^{\pi/4} \frac{\sin 2x}{1+\cos^2 x}\,dx$

14. $\int_0^1 x^2\sqrt{(1-x^2)}\,dx$ 15. $\int_0^1 2^x\,dx$

16. Find $\int \frac{2x-1}{(x+2)(x-3)^2}dx$, given that $x > 3$.

17. Show that $\int_0^3 \frac{x+3}{x^2+3}dx = \ln 2 + \frac{\pi\sqrt{3}}{3}$.

In questions 18 to 21 use the given substitutions to evaluate the definite integrals.

18. $\int_{\sqrt{2}}^2 \frac{1}{x\sqrt{(x^2-1)}}dx$, $x = \sec\theta$.

19. $\int_0^{1/2} \sqrt{\left(\frac{x}{1-x}\right)}dx$, $x = \sin^2\theta$.

20. $\int_0^1 \frac{1}{(1+x^2)}dx$, $x = \tan\theta$.

21. $\int_0^{\pi/6} \sec x\,dx$, $\sin x = u$.

22. Use integration by parts and the result of question 21 to show that

$$\int_0^{\pi/6} \sec^3 x\,dx = \tfrac{1}{3} + \tfrac{1}{4}\ln 3.$$

23. Show that the curve $y = \frac{3}{(2x+1)(1-x)}$ has only one turning point. Find the coordinates of this point and determine its nature. Sketch the curve. Find the area of the region enclosed by the curve and the line $y = 3$. (JMB)

24. Find the coordinates of the turning point of the curve $y = (x^2-4)/(x+1)^2$ and ascertain the nature of this turning point. Calculate the area of the region enclosed by the curve, the y-axis and the x-axis between $x = 0$ and $x = 2$. (AEB 1978)

25. If $f(x) = \frac{x^2+x+2}{(x^2+1)(x-1)}$ express $f(x)$ in partial fractions and hence evaluate $\int_2^3 f(x)\,dx$ and $\int_{-3}^{-2} f(x)\,dx$ each correct to 2 places of decimals. (SU)

26. The region enclosed by the loop of the curve $y^2(3-x) = x^2(3+x)$ is rotated about the x-axis through four right angles. Show that the volume of the solid of revolution is $18\pi(3\ln 2 - 2)$. (C)

27. (i) Given that $0 < a < b$, sketch the graph of $y = |x-a|$ for $-b \leqslant x \leqslant b$. Hence, or otherwise, find $\int_{-b}^{b} |x-a|\,dx$.

(ii) Find the value of $\int_1^2 \log_{10} x\,dx$ leaving your answer in terms of logarithms.

(iii) Using the substitution $x = \frac{1}{2}\sin\theta$, or otherwise, evaluate $\int_0^{1/4} \sqrt{(1-4x^2)}dx$.

(L)

28. Sketch the curve given in terms of a parameter t by $x = 2a + at^2$, $y = 2at$ where a is a positive constant. Find the equation of the tangent and of the normal to the curve at the point where $t = 1$. Determine the area of the finite region bounded by the curve and the straight line joining the points at which $t = \pm 1$. (L)

29. A curve is defined parametrically by the equations $x = \cos t$, $y = \sin^3 t$, $-\pi \leqslant t < \pi$. Show that the curve is symmetrical about each of the coordinate axes. Find dy/dx in terms of t and deduce the equation of the tangent to the curve at the point with parameter t_1. Sketch the curve. Show that the area, A, of the region enclosed by the curve is given by $A = 4\int_0^{\pi/2} \sin^4 t\, dt$ and use the relation $2\sin^2\theta = 1 - \cos 2\theta$ to deduce that $A = 3\pi/4$. (L)

30. The curve whose equation is $y = (1 - x)e^x$ meets the x-axis at A and the y-axis at B. The region bounded by the arc AB of the curve and the line segments OA and OB, where O is the origin, is rotated through a complete revolution about the x-axis. Show that the volume swept out is $\frac{1}{4}\pi(e^2 - 5)$. (JMB)

31. Obtain $\int x^2 \cos 2x\, dx$ and hence prove that $\int_0^{\pi} x^2 \cos 2x\, dx = \frac{1}{2}\pi$. Find the area of the region enclosed by the curve $y = x \sin x$, for $0 \leqslant x \leqslant \pi$, and the x-axis. The region is rotated through 2π radians about the x-axis. Find the volume of the solid of revolution thus generated. From your results, or otherwise, find the volume of the solid of revolution generated by rotation of the same region through 2π radians about the line $y = -\frac{1}{8}$. (O&C)

32. Express $y = \dfrac{2x^2 + 3x + 5}{(x + 1)(x^2 + 3)}$ in partial fractions and hence show that $\dfrac{dy}{dx} = -\dfrac{2}{3}$ when $x = 0$. Evaluate $\int_0^{\sqrt{3}} y\, dx$ and state the mean value of y in the interval $0 \leqslant x \leqslant \sqrt{3}$. (AEB 1979)

33. Determine the values of p and q for which $x^2 - 4x + 5 \equiv (x - p)^2 + q$ and hence evaluate $\int_2^3 \dfrac{1}{x^2 - 4x + 5} dx$. Calculate also $\int_2^3 \dfrac{2x - 4}{x^2 - 4x + 5} dx$ and deduce, or find otherwise, the value of $\int_2^3 \dfrac{2x}{x^2 - 4x + 5} dx$. (JMB)

34. (a) Using the formula for $\cos(A + B)$, express $\cos\theta$ in terms of $t = \tan\frac{1}{2}\theta$.

(b) Express $\dfrac{10(1 + t^2)}{(1 - t^2)(1 + 9t^2)}$ in partial fractions.

(c) With the help of (a) and (b), and using the substitution $t = \tan\frac{1}{2}\theta$, show that $\int_0^{\pi/3} \dfrac{5\, d\theta}{5\cos\theta - 4\cos^2\theta} = \log_e(2 + \sqrt{3}) + \frac{8}{9}\pi$. (W)

35. (a) Using integration by parts, or otherwise, evaluate

$$\int \log_e x \, dx \quad \text{and} \quad \int (\log_e x)^2 \, dx.$$

(b) Sketch the graph of the function $\log_e x$. Let A and C denote the points $(1, 0)$ and $(e, 1)$ on the graph, and let B denote the point $(e, 0)$. The region bounded by the graph and the lines, AB, BC is rotated about the x-axis; find the volume of the solid so formed. Show that the volume of the solid formed by rotating the same region about the y-axis is $\frac{1}{2}\pi(e^2 + 1)$. (W)

36. Evaluate (i) $\displaystyle\int_0^{\pi/2} \frac{\sin^3\theta}{2 - \sin^2\theta} d\theta$, (ii) $\displaystyle\int_0^{\sqrt{2}} x e^{-x^2} dx$.

37. Let $I = \displaystyle\int_0^{\pi} \frac{x \, dx}{1 + \sin x}$. Show, by means of the substitution $y = \pi - x$ that $I = \pi \displaystyle\int_0^{\pi/2} \frac{dx}{1 + \sin x}$. Hence, by means of the substitution $t = \tan\frac{1}{2}x$, or otherwise, evaluate I. (JMB)

38. Draw the circle $x^2 + y^2 = a^2$ and, on the same diagram, the line $x = a/2$. Find by integration the smaller area between the line and the circumference of the circle and the volume generated when this area is rotated through 4 right-angles about the x-axis. (SU)

39. The area A in the first quadrant bounded by the curve $(2 - x)y^2 = x$, the x-axis and the line $x = 1$ is rotated completely about the x-axis to form a solid of volume V. Find: (i) the volume V, (ii) the area A. (AEB 1976)

40. Find the area of the region enclosed between the curves $y = x^2/2a$ and $y = a - x^4/2a^3$ $(a > 0)$. Find also the volume of the solid obtained by rotating this region through two right angles about the y-axis. (O)

41. Sketch the curve $y^2 = x^2(4 - x^2)$. Find the area enclosed by one loop of the curve. Find also the volume of the solid generated when this loop is rotated through two right angles about the x-axis.

42. The domain of the function defined by $f(x) = e^{-x}\sin x$ is the set of real numbers in the interval $0 \leqslant x \leqslant 4\pi$. Show that stationary values of this function occur for values of x for which $\tan x = 1$. Determine the nature of these stationary values. Sketch the graph of the function. By integrating by parts twice, or otherwise, evaluate $\displaystyle\int_0^{\pi} f(x) \, dx$. Show that the ratio of $f(a + 2\pi)$ to $f(a)$ is independent of the value of a, and hence write down the value of $\displaystyle\int_{2\pi}^{3\pi} f(x) \, dx$. (JMB)

43. By interpreting $\int_a^b f(x)\,dx$ as an area, show that

(i) $\int_0^{\pi} \sin^5 x \cos^5 x\,dx = 0$, (ii) $\int_0^{\pi} e^{2x} \cos x\,dx < 0$,

(iii) for $n > 1$, $\frac{1}{n+1}\left(\frac{\pi}{4}\right)^{n+1} < \int_0^{\pi/4} \tan^n x\,dx < \frac{\pi}{8}$. (JMB)

44. (a) By considering $\int x^{1/2}\,dx$ between suitable limits, or otherwise, prove that $\frac{2}{3}n^{3/2} < \sqrt{1} + \sqrt{2} + \cdots + \sqrt{n} < \frac{2}{3}\{(n+1)^{3/2} - 1\}$.

(b) By considering an appropriate integral, prove that

$$\frac{1}{k}\left\{1 - \frac{1}{(n+1)^k}\right\} < \sum_{r=1}^{n} \frac{1}{r^{k+1}} < \frac{1}{k}\left\{k + 1 - \frac{1}{n^k}\right\}, \text{ where } k > 0. \quad \text{(O\&C)}$$

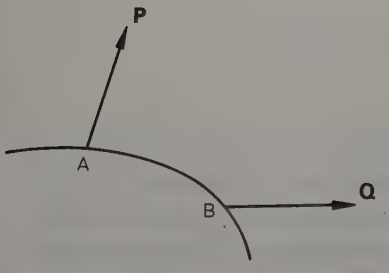
5 Coplanar forces; centre of gravity

5.1 Parallel forces, moments and couples

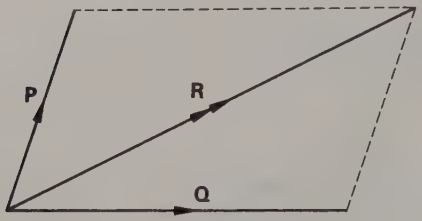
In Book 1 we considered the action of forces on bodies small enough to be treated as particles and represented by single points in space. Applying Newton's laws of motion, we found that if the forces acting on such a body have a non-zero resultant then the effect produced is an acceleration in the direction of the resultant. However, if the vector sum of the forces on the body is zero, the body either remains at rest or continues to move with unchanged velocity.

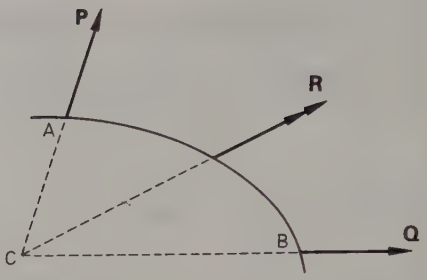
There are many situations in which this elementary approach is inadequate. From experience we know that the forces on a body may tend to produce rotation as well as the types of motion discussed in earlier work. A projectile, such as a cricket ball, may spin as it moves. A block placed on an inclined plane may topple rather than slide. Thus, when solving a problem in which the rotational effects of forces could be important, we must use a more realistic representation of the bodies involved. We will then find that the effect produced by a set of forces depends not only on the magnitude and direction of the resultant, but also on its line of action.

We now develop these ideas by considering the effect of coplanar forces on a variety of rigid bodies i.e. bodies whose shapes are not changed by the forces acting on them.

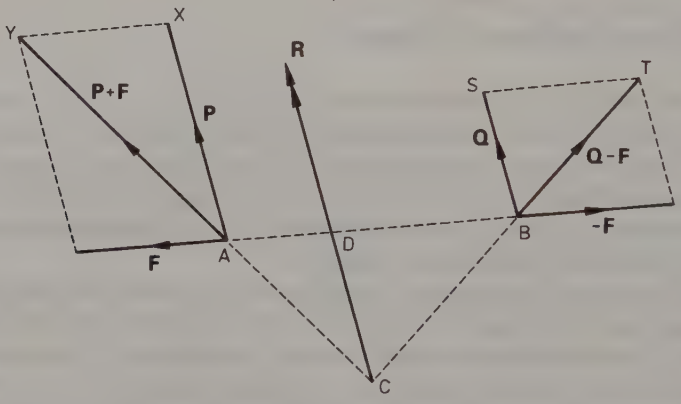


Let **P** and **Q** be *non-parallel* coplanar forces acting at points A and B of a rigid body. The magnitude and direction of their resultant **R** can be found using the parallelogram law. If the lines of action of **P** and **Q** intersect at the point C, then the line of action of **R** also passes through C.





To find the line of action of the resultant of *parallel* forces **P** and **Q** acting at points A and B of a rigid body, we introduce forces **F** and $-\mathbf{F}$ acting along AB. The application of these equal and opposite forces will have no effect on the body and leave the line of action of the resultant unaltered.



The diagram shows how the parallelogram law is used to construct a point C on the line of action of the resultant **R**, when **P** and **Q** are *like parallel forces* i.e. forces in the same sense.

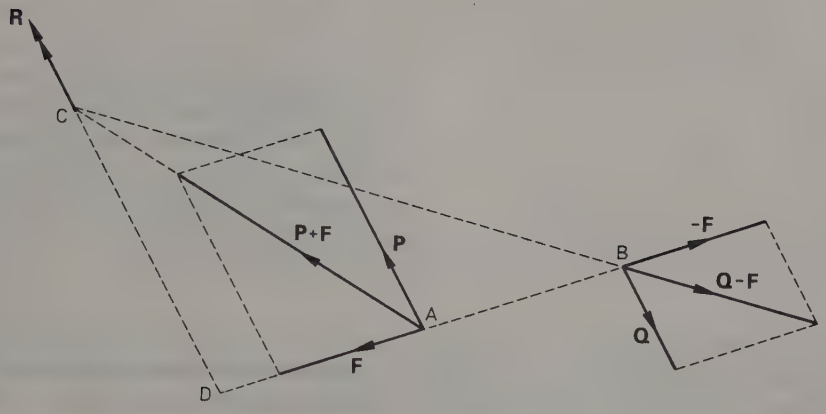
In similar triangles ADC, YXA: $\dfrac{AD}{DC} = \dfrac{YX}{XA} = \dfrac{F}{P}$

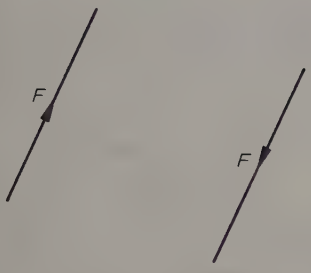
In similar triangles DBC, STB: $\dfrac{DC}{DB} = \dfrac{SB}{ST} = \dfrac{Q}{F}$

$$\therefore \quad \frac{AD}{DB} = \frac{AD}{DC} \times \frac{DC}{DB} = \frac{F}{P} \times \frac{Q}{F} = \frac{Q}{P}$$

By vector addition, the magnitude of **R** is $P + Q$.
Hence the resultant of the like parallel forces **P** and **Q** is of magnitude $P + Q$ and divides AB internally in the ratio $Q:P$.

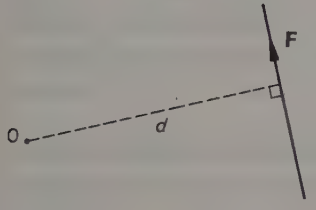
Using similar arguments it can be shown that the resultant of *unlike parallel forces* **P** and **Q**, where $P > Q$, is of magnitude $P - Q$ and divides AB externally in the ratio $Q:P$.





A pair of equal unlike parallel forces has zero resultant, but clearly does have a turning effect. Such a system is called a *couple*.

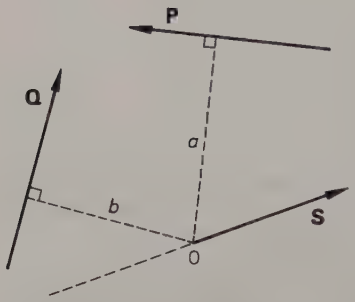
A quantity called the *moment of a force* or *torque* is used as a measure of "turning effect". The moment of a force about a given point is defined as the product of the magnitude of the force and its distance from the point. If the line of action of a force **F** is at a perpendicular distance d from a point O, then the moment of **F** about O is Fd.



[In three-dimensional work the quantity Fd is more accurately described as the moment or torque about an axis through O perpendicular to the plane containing O and the force **F**.]

The SI unit of torque is the newton metre (N m). Thus, if a force of magnitude 40 N acts along a line whose perpendicular distance from a point A is 2 m, then the moment of the force about A is 80 N m.

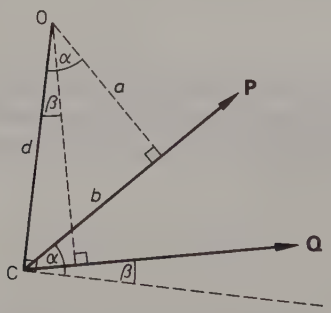
When finding the sum of the moments of a set of coplanar forces about a point, the moment of each force is taken to be positive or negative according to the sense of the rotation it would produce. The moment of a force about any point on its line of action is zero.

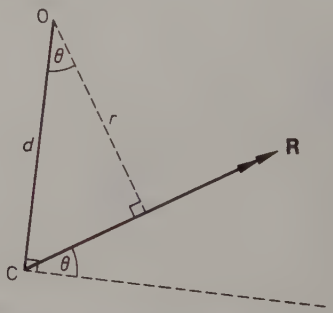


For instance, if forces **P**, **Q** and **S** act as shown in the diagram, then the sum of the anti-clockwise moments of **P**, **Q** and **S** about O is $Pa - Qb$, i.e. the combined turning effect of the system about O is $Pa - Qb$ anti-clockwise.

We now show that the sum of the moments of two coplanar forces about any point in their plane is equal to the moment of their resultant about the same point.

Suppose that **P** and **Q** are forces whose lines of action intersect at a point C. Let **R** be their resultant and let O be a point in the same plane such that $OC = d$.





Using the notation of the above diagrams, the sum of the moments of **P** and **Q** about O

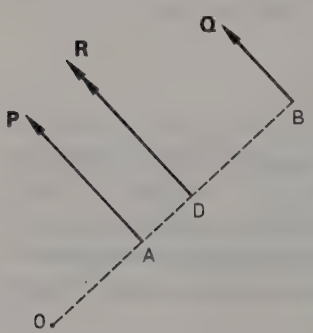
$$= Pa + Qb = Pd\cos\alpha + Qd\cos\beta = d(P\cos\alpha + Q\cos\beta)$$

The moment of **R** about O

$$= Rr = Rd\cos\theta = d(R\cos\theta).$$

Considering the components of **P**, **Q** and **R** in a direction perpendicular to OC, we have

$$P\cos\alpha + Q\cos\beta = R\cos\theta.$$



Hence the sum of the moments about O of **P** and **Q** is equal to the moment about O of **R**.

If **P** and **Q** are like parallel forces with resultant **R**, as shown in the diagram, then

$$\frac{AD}{DB} = \frac{Q}{P}$$

$$\therefore \quad P \times AD = Q \times DB$$

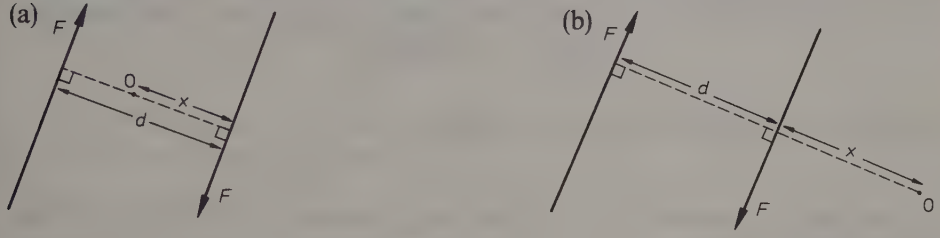
Thus the sum of the moments of **P** and **Q** about O

$= P \times OA + Q \times OB$
$= P(OD - AD) + Q(OD + DB)$
$= (P + Q)OD - P \times AD + Q \times DB$
$= R \times OD$
= the moment of **R** about O.

By extending these arguments to any set of coplanar forces with a non-zero resultant it is possible to establish the following general result called the *principle of moments.*

> The sum of the moments of a system of coplanar forces about any point in their plane is equal to the moment of their resultant about the same point

A couple has zero resultant but non-zero moment. Consider a couple consisting of a pair of equal and opposite forces of magnitude F with lines of action at a distance d apart. Suppose that O is any point in the plane of this couple.



In diagram (a) the sum of the moments about O

$= F(d - x) + Fx = Fd - Fx + Fx = Fd.$

In diagram (b) the sum of the moments about O

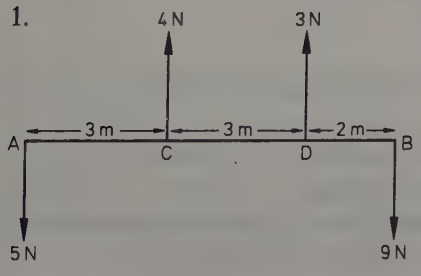
$= F(d + x) - Fx = Fd + Fx - Fx = Fd.$

Since both results are independent of x, the couple has the same moment, i.e. Fd clockwise, for all positions of O.

Hence the *moment of a couple* about any point in its plane is the product of the magnitude of one of the forces and the perpendicular distance between the forces.

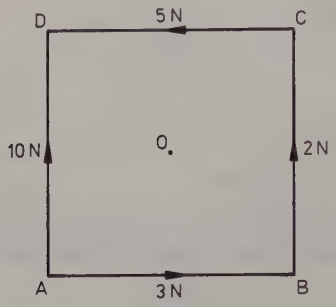
Exercise 5.1

1.



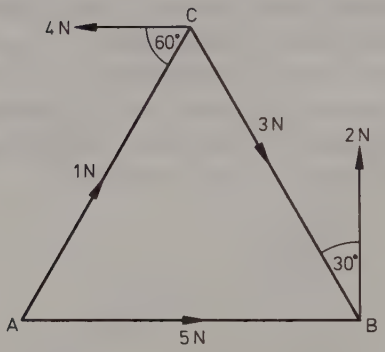
Forces act on a rod AB of length 8 m as shown in the diagram. Find the sum of the anti-clockwise moments of these forces about each of the points A, B, C and D.

2.



ABCD is a square of side 6 m with centre *O*. Forces act along the sides *AB*, *BC*, *CD*, *AD* as shown in the diagram. Find the sum of the anti-clockwise moments of these forces about each of the points *A*, *B*, *C*, *D* and *O*.

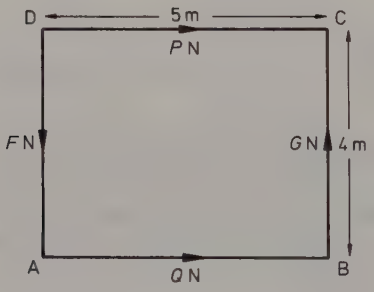
3.



ABC is an equilateral triangle of side 2 m. Find the sum of the anti-clockwise moments of the forces shown in the diagram about each of the points *A*, *B* and *C*. [You may leave your answers in surd form.]

In questions 4 to 9 forces **F**, **G**, **P** and **Q** act along the sides of the rectangle *ABCD* as shown in the diagram.

4.



The resultant of **P** and **Q** acts through a point *X* on *BC*. Find the magnitude of the resultant and the length of *BX* if (a) $P = Q = 3$, (b) $P = 6, Q = 2$.

5. The resultant of **F** and **G** acts through a point *Y* on *AB* produced. Find the magnitude of the resultant and the length of *AY* if
(a) $F = 1, G = 2$, (b) $F = 2, G = 3$.

6. Find the moment of the couple formed by forces **F** and **G** if $F = G = 10$.

7. If the forces **F** and **G** form a couple of moment 20 N m, find *F* and *G*.

8. The resultant of **F** and **P** acts through a point *S* on *AB*. Find the magnitude of the resultant and the length of *AS* if (a) $F = P = 5$, (b) $F = 12, P = 9$.

9. The resultant of **G** and **Q** acts through a point T on DA produced. Find the magnitude of the resultant and the length of AT if (a) $G = Q = 4$, (b) $G = 6, Q = 2.5$.

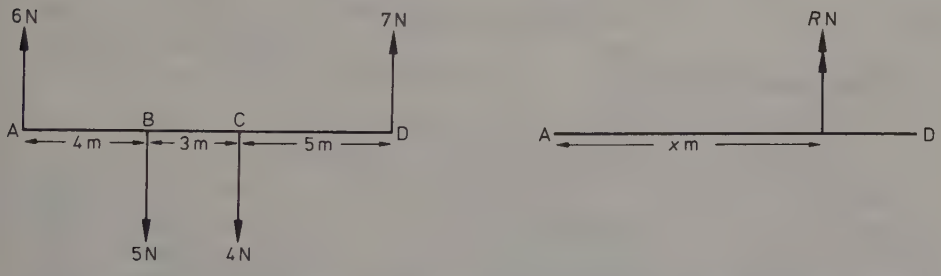
10. Four forces each of magnitude 3 N act along the sides BA, BC, CD, AD of a square $ABCD$, in the directions indicated by the order of the letters. Find the magnitude and the line of action of their resultant.

11. Three forces each of magnitude 10 N act along the sides AB, DE, AF of a regular hexagon $ABCDEF$ of side 2 m. Find the sum of the moments of these forces about each of the points A, B, C, D, E and F. Find also the magnitude and the line of action of the resultant of the three forces.

5.2 Systems of coplanar forces

We consider first a system of parallel coplanar forces with non-zero resultant. The magnitude and direction of the resultant is found by vector addition. Its line of action can be determined using the principle of moments.

Example 1 Find the magnitude and line of action of the resultant of the system of parallel forces shown in the diagram.



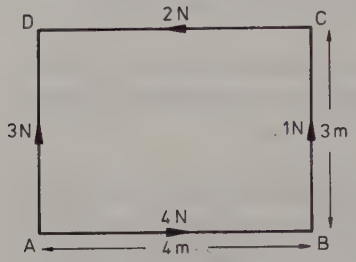
Let the resultant of the given system of forces be of magnitude R N acting at a distance x m from A.
Comparing magnitudes and moments about A↻

$$\uparrow \quad R = 6 - 5 - 4 + 7 = 4$$
$$A\circlearrowright \quad Rx = 0 - 5 \times 4 - 4 \times 7 + 7 \times 12 = 36$$
$$\therefore \quad R = 4 \quad \text{and} \quad x = 9.$$

Hence the resultant has magnitude 4 N and its line of action cuts AD at a point 9 m from A.

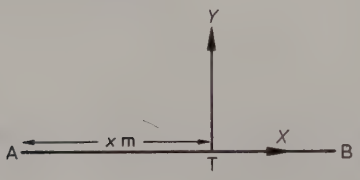
Similar methods can be applied to any set of coplanar forces with a non-zero resultant. The magnitude and direction of the resultant is found by means of a vector polygon or by resolving in two perpendicular directions. As before the line of action of the resultant is determined using the principle of moments.

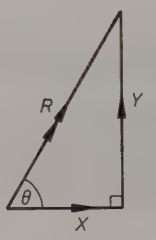
Example 2



The resultant of the system of forces shown in the diagram is a force **R** acting through a point T on AB. Find the magnitude and direction of **R** and the length of AT.

Let X and Y be the components of **R** parallel and perpendicular to AB.





Resolving parallel and perpendicular to AB

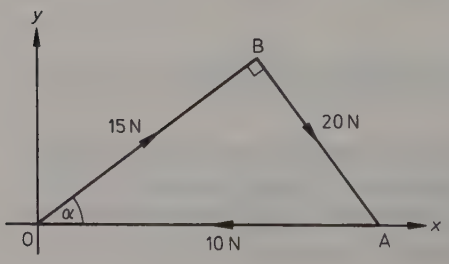
$$\rightarrow \qquad X = 4 - 2 \qquad \therefore \quad X = 2$$
$$\uparrow \qquad Y = 3 + 1 \qquad \therefore \quad Y = 4$$

Taking moments about A

$$Yx = 1 \times 4 + 2 \times 3 = 10 \qquad \therefore \quad x = 2{\cdot}5$$
$$\therefore \quad R = \sqrt{(X^2 + Y^2)} = \sqrt{(4 + 16)} = \sqrt{20} = 2\sqrt{5} \quad \text{and}$$
$$\tan\theta = \frac{Y}{X} = \frac{4}{2} = 2, \quad \theta \approx 63^\circ 26'.$$

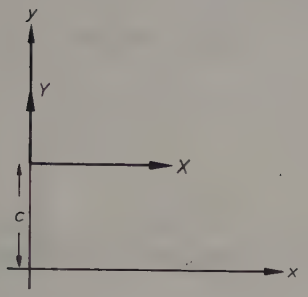
Hence the resultant has magnitude $2\sqrt{5}$ N and acts at $63^\circ 26'$ to AB. The length of AT is 2·5 m.

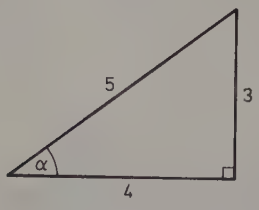
Example 3



Find the equation of the line of action of the resultant of the forces shown in the diagram, given that $\tan\alpha = \frac{3}{4}$ and that A is the point $(7, 0)$.

Let X and Y be the components of the resultant in the directions of the x- and y-axes respectively. Let the line of action cut the y-axis at the point $(0, c)$.





$\Rightarrow \quad \tan\alpha = \frac{3}{4}$
$\sin\alpha = \frac{3}{5},$
$\cos\alpha = \frac{4}{5}.$

Resolving in the directions of the x- and y- axes,

$$\rightarrow \qquad X = 15\cos\alpha + 20\sin\alpha - 10$$
$$\uparrow \qquad Y = 15\sin\alpha - 20\cos\alpha$$

$$\therefore \quad X = 12 + 12 - 10 = 14, \quad Y = 9 - 16 = -7.$$

Taking moments about O: $Xc = 20 \times 7\cos\alpha$

$$\therefore \quad c = \frac{20 \times 7\cos\alpha}{X} = \frac{20 \times 7 \times 4}{14 \times 5} = 8$$

Hence the line of action of the resultant has gradient $Y/X = -7/14 = -1/2$ and intercept on the y-axis $c = 8$,
$\therefore$ the equation of the line of action is

$$y = -\tfrac{1}{2}x + 8 \quad \text{i.e.} \quad x + 2y = 16.$$

Two systems of forces are said to be *equivalent* if their effects are the same. In particular, two sets of coplanar forces are equivalent if
(i) the sums of the components of the forces in any given direction are equal and
(ii) the sums of the moments of the forces about any given point are equal.
Thus, the *resultant* of a system of forces is the single force equivalent to the whole system.

It also follows from conditions (i) and (ii) that any two coplanar *couples* with the same moment are equivalent. Similarly a set of coplanar couples with moments $G_1, G_2, G_3, \ldots, G_n$ is equivalent to a single couple with moment

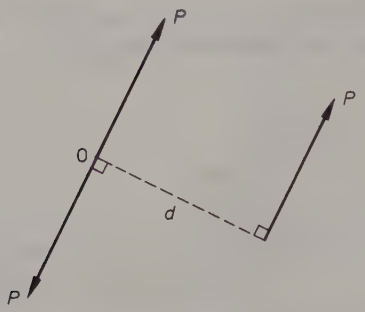
$$G_1 + G_2 + G_3 + \ldots + G_n.$$

We next consider the resultant of a force **P** acting at a point O and a couple of moment G in the same plane. The couple is equivalent to a pair of equal and opposite forces of magnitude P and perpendicular distance d apart, provided that

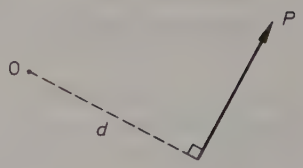
$$Pd = G.$$

Therefore the force and the couple are equivalent to the three forces shown in diagram (a) and hence to the single force shown in diagram (b).

(a)



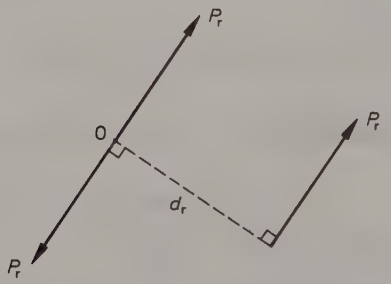
(b)



Thus the resultant of a force **P** acting at O and a couple of moment G is a force **P** acting at a distance d from O, where $d = G/P$.

Reversing the argument we may also deduce that a force **P** acting at a distance d from a point O is equivalent to a force **P** acting at O together with a couple of moment Pd.

This result can be used to find a simple equivalent system for any set of coplanar forces. Let $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$ be a system of coplanar forces acting at distances $d_1, d_2, \ldots, d_n$ respectively from a fixed point O in their plane. A typical force $\mathbf{P}_r$ acting at a distance d_r from O is equivalent to a force $\mathbf{P}_r$ acting at O together with a couple of moment $P_r d_r$. Applying this reasoning to all the forces in the system we obtain an equivalent system of forces $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$ acting at 0 together with couples of moments $P_1 d_1, P_2 d_2, \ldots, P_n d_n$.



If **R** is the vector sum of the forces and G is the sum of the moments of the original forces about O, then this new system can be replaced by a single force **R** acting at O and a couple of moment G.

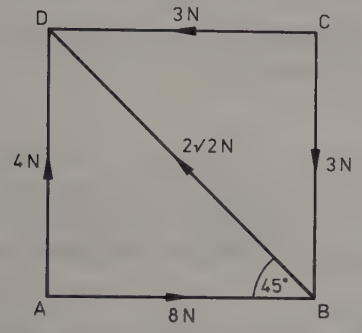
Thus, in general, any system of coplanar forces is equivalent to a single force acting through a given point together with a couple.

We consider further four possible types of system.

(1) $R = 0$, $G = 0$. The system is in equilibrium.
(2) $R = 0$, $G \neq 0$. The system is equivalent to a couple of moment G.
(3) $R \neq 0$, $G = 0$. The system is equivalent to a single force **R** acting through O.
(4) $R \neq 0$, $G \neq 0$. The system is equivalent to a single force **R** acting at a distance G/R from O.

Hence any system of coplanar forces which is not in equilibrium may be reduced either to a single force or to a couple.

Example 4 A square $ABCD$ of side 3 m has forces of magnitude 8, 3, 3, 4 and $2\sqrt{2}$ newtons acting along AB, CB, CD, AD and BD respectively. If the system is reduced to a force acting through A together with a couple, find the magnitude and direction of the force and the moment of the couple. If the system is reduced to a single force acting through a point E on AB at a distance d metres from A, find d.



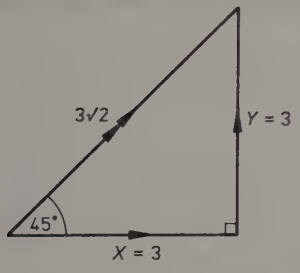
Suppose that the given system is equivalent to forces X and Y acting along AB and AD respectively, together with a couple of moment G in the sense $ABCD$.

Resolving in the directions of AB and AD:

$$\rightarrow \qquad X = 8 - 3 - 2\sqrt{2}\cos 45° = 3$$
$$\uparrow \qquad Y = 4 - 3 + 2\sqrt{2}\sin 45° = 3$$

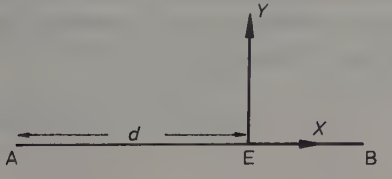
Taking moments about A ↻:

$$G = 3 \times 3 - 3 \times 3 + 2\sqrt{2} \times 3\sin 45° = 6$$



Hence the system is equivalent to a force of $3\sqrt{2}$ N acting along AC together with a couple of moment 6 N m in the sense $ABCD$.

The single equivalent force through E has moment G about A and components X and Y in the directions of AB and AD respectively.

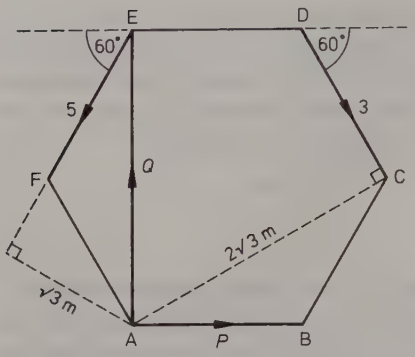


Taking moments about A ↻:

$$G = Yd$$
$$\text{i.e.} \quad 6 = 3d$$
$$\therefore \quad d = 2.$$

Example 5 $ABCDEF$ is a regular hexagon of side 2 m. Forces of magnitude P, 3, 5 and Q newtons act along AB, DC, EF and AE respectively. Show that this system of forces is not in equilibrium.

(a) If the system is equivalent to a couple, find its moment and the values of P and Q.
(b) If the system is equivalent to a single force through E, find P.



The sum of the moments of the forces about A (in N m)
$= 3 \times 2\sqrt{3} - 5 \times \sqrt{3} = \sqrt{3}$.
Since this sum is not zero, the forces are not in equilibrium.

(a) If the system is equivalent to a couple, it will have the same moment about any point in the plane,
$\therefore$ the moment of the couple is $\sqrt{3}$ N m.
The sum of the components of the forces in a couple is zero in any direction,
$\therefore$ resolving in the directions of AB and AE:

$$\rightarrow \qquad P + 3\cos 60° - 5\cos 60° = 0$$
$$\uparrow \qquad Q - 3\sin 60° - 5\sin 60° = 0$$

$\therefore \quad P = 2\cos 60° = 1 \quad$ and $\quad Q = 8\sin 60° = 4\sqrt{3}$.

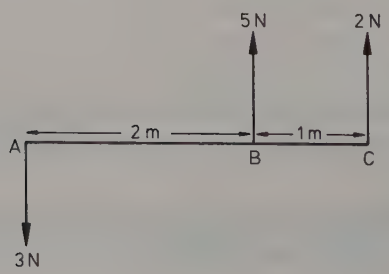
(b) If the system is equivalent to a single force through E, then the sum of the moments about E will be zero.
Taking moments about E: $\quad P \times 2\sqrt{3} - 3 \times \sqrt{3} = 0$

$$\therefore \quad P = \frac{3\sqrt{3}}{2\sqrt{3}} = 1{\cdot}5.$$

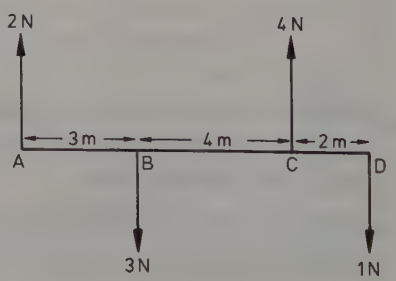
Exercise 5.2

In questions 1 to 4 the resultant of the given forces acts through a point T on AB or AB produced. In each case find the magnitude and direction of the resultant and the length of AT. [In question 4 $ABCDEF$ is a regular hexagon.]

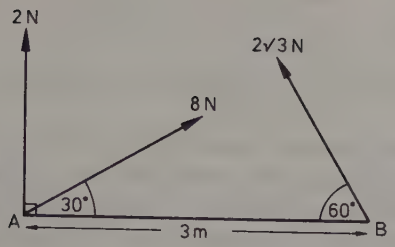
1.



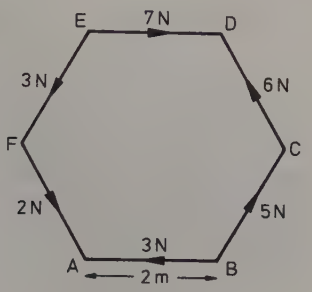
2.



3.



4.



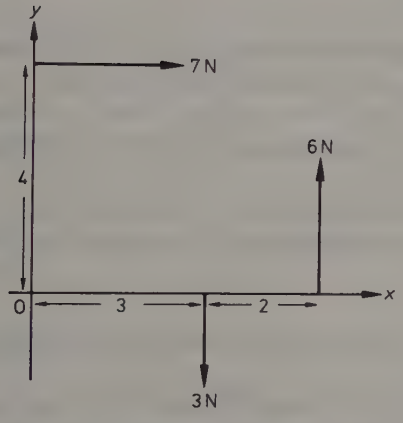
5. Forces of magnitude $8P, 7P, 5P$ and $3P$ act along the sides AB, BC, CD and DA respectively of a square $ABCD$ of side $2a$. The resultant of this system of forces cuts AB produced at E. Find

(a) the magnitude and direction of the resultant,

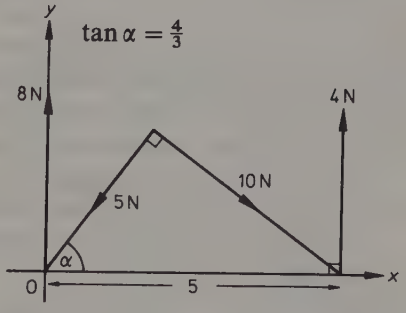
(b) the length of AE.

In questions 6 and 7 find the equation of the line of action of the resultant of the given forces.

6.



7.



In questions 8 to 19 a system of forces acts in the plane of perpendicular axes Ox and Oy. The sums of the components of the forces in the directions of Ox and Oy are X and Y respectively. The sums of the anti-clockwise moments of the forces about three non-collinear points A, B and C in the plane are G_A, G_B and G_C respectively. In each question use the given facts to decide which of the following statements could be true.

a: the system is in equilibrium,
b: the system is equivalent to a couple,
c: the system is equivalent to a single force.

8. $X = 0, Y = 0$.

9. $X = 0, Y \neq 0$.

10. $Y = 0, G_A = 0$.

11. $X = 0, G_B \neq 0$.

12. $Y \neq 0, G_C = 0$.

13. $G_A \neq 0, G_B = 0$.

14. $G_A = G_B = 0$. 15. $G_A = G_B \neq 0$. 16. $G_A = G_B \neq G_C$.

17. $G_A = G_B = G_C = 0$. 18. $G_A = G_B = G_C \neq 0$ 19. $X = 0$, $G_A = G_B = 0$.

20. In the square $OABC$ of side 2 m, O, A and B are the points $(0, 0)$, $(2, 0)$ and $(2, 2)$ respectively. Forces of magnitude 5, 8, 3 and $5\sqrt{2}$ newtons act along OA, BA, OC and AC respectively. Show that this system of forces is equivalent to a couple and find its moment. If a fifth force of magnitude $6\sqrt{2}$ newtons acts along OB, find the equation of the line of action of the resultant of the enlarged system.

21. In a rectangle $ABCD$, $AB = 12\,\text{m}$, $BC = 5\,\text{m}$. Forces of magnitude 6, P, 18, 10 and Q newtons act along AB, CB, CD, AD and DB respectively. Show that this system of forces is not in equilibrium. Find P and Q given that the system reduces to (a) a couple, (b) a single force acting along DC.

22. In a rectangle $ABCD$, $AB = 3\,\text{m}$, $BC = 2\,\text{m}$. Forces of magnitude 5, 8, 12, 6 and P newtons act along BA, BC, DC, AD and EA respectively, where E is a point on CD. Given that the forces are in equilibrium, find P and the distance DE. If the force along EA is replaced by an equal force acting through C, show that the system now reduces to a couple and find its moment.

23. $ABCDEF$ is a regular hexagon of side 1 m. Forces of magnitude 4, 6, 10, 8, P and 2 newtons act along AB, BC, CD, ED, FE and AF respectively, directions being indicated by the order of the letters.

(a) If the resultant passes through the centre of the hexagon find the value of P and the magnitude and direction of the resultant.

(b) The force in FE is replaced by another so that the new system reduces to a couple. Find the magnitude of the force and the moment of the couple indicating its sense. (SU)

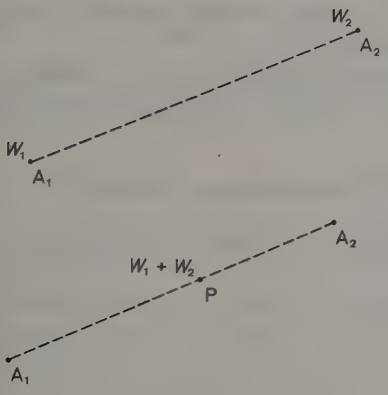
24. The points A, B and C have coordinates $(9a, 0)$, $(0, -4a)$ and $(6a, 4a)$ respectively referred to the coordinate axes Ox and Oy.

(a) Forces $12P$, $15P$, $5P$ and P act along $\overrightarrow{OA}$, $\overrightarrow{BC}$, $\overrightarrow{CA}$ and $\overrightarrow{OB}$ respectively. Calculate the magnitude of the resultant of these forces and the equation of the line of action of this resultant.

(b) Forces $15P$, S and T act along $\overrightarrow{BC}$, $\overrightarrow{OA}$ and $\overrightarrow{CA}$ respectively.
Given that these forces reduce to a couple, calculate (i) the values of S and T, in terms of P, (ii) the magnitude of the couple, in terms of a and P, (iii) the sense of the couple. (AEB 1979)

5.3 Centre of gravity; centre of mass

A solid body may be regarded as a tightly packed collection of small particles. The *weight* of the body is the resultant of the weights of the particles. For relatively small bodies on or near the earth's surface the weights of the constituent particles are assumed to form a system of parallel forces acting vertically downwards.

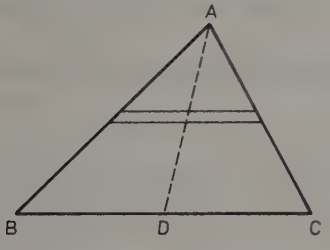


Suppose that at points A_1 and A_2 of a body there are particles of weights W_1 and W_2 respectively. As shown in §5.1, the resultant of the weights of these particles is a force of magnitude $W_1 + W_2$ acting through the point P which divides A_1A_2 in the ratio $W_2 : W_1$.

By extending this argument to include all the particles forming the body, we deduce that the resultant weight force is equal to the sum of the weights of the particles and acts through a fixed point in the body.

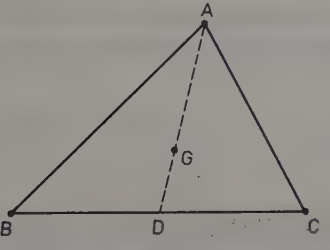
The *centre of gravity* of a body is the fixed point through which its weight acts.

The *symmetries* of a body can be used to determine its centre of gravity. If the particles in a body are evenly distributed about an axis or a plane of symmetry, then the centre of gravity of the body must lie on this axis or plane. Thus the centre of gravity of a thin uniform rod is at its mid-point and the centre of gravity of a uniform sphere is at its centre.



To find the centre of gravity of a *uniform triangular lamina ABC*, suppose that the triangle is divided into n strips parallel to the side BC. If the value of n is increased the strips become thin uniform rods with centres of gravity at their mid-points. Hence the centre of gravity of the whole lamina must lie on the line joining these mid-points i.e. on the median AD. Similarly the centre of gravity must also lie on the medians through B and C.

Thus the centre of gravity of a uniform triangular lamina is at the point of intersection of the medians.



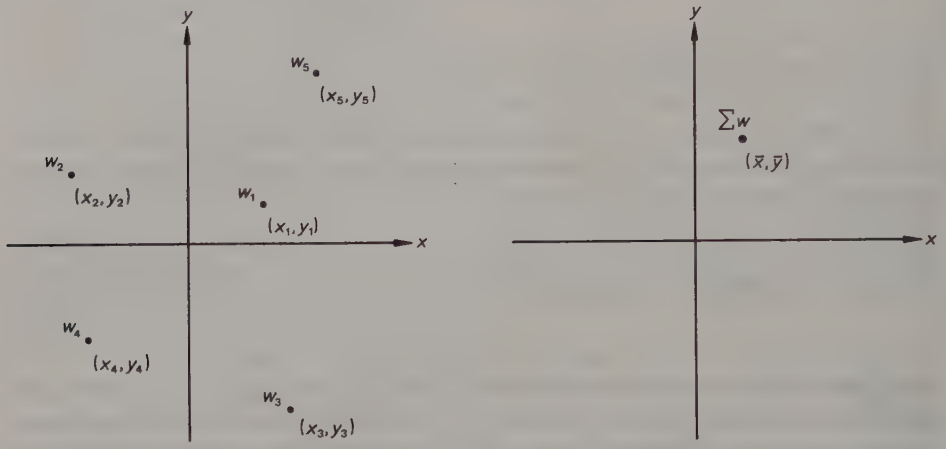
Consider now three particles each of weight W placed at the vertices of a triangle ABC. The resultant of the weights of the particles at B and C is a force of $2W$ acting at D, the mid-point of BC. Hence the resultant of the weights of all three particles is a force of $3W$ acting at the point G on AD which divides

AD in the ratio 2:1. Using similar arguments we find that G is also the point which divides the medians through B and C in the ratio 2:1.

Thus the centre of gravity of the three equal particles at A, B and C is at the point of intersection of the medians of triangle ABC, i.e. at the same point as the centre of gravity of the uniform lamina ABC.

[Note that we have also verified that this point divides each median in the ratio 2:1.]

In general the centre of gravity of a set of particles in a plane is found using the principle of moments.



Let particles with weights $w_1, w_2, \ldots, w_n$ have positions $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ relative to x- and y-axes in their plane. Suppose that the resultant weight of the set of particles acts through the point $(\bar{x}, \bar{y})$.

Assuming the x, y plane to be horizontal and applying the principle of moments we have,
for moments about the y-axis:

$$w_1x_1 + w_2x_2 + \cdots + w_nx_n = (w_1 + w_2 + \cdots + w_n)\bar{x}$$

and for moments about the x-axis:

$$w_1y_1 + w_2y_2 + \cdots + w_ny_n = (w_1 + w_2 + \cdots + w_n)\bar{y}.$$

Hence the centre of gravity of the particles is at the point $(\bar{x}, \bar{y})$,

where
$$\bar{x} = \frac{\sum wx}{\sum w}, \quad \bar{y} = \frac{\sum wy}{\sum w}.$$

This result can be expressed in vector notation by letting the position vectors of the particles be $\mathbf{r}_1 = x_1\mathbf{i} + y_1\mathbf{j}, \mathbf{r}_2 = x_2\mathbf{i} + y_2\mathbf{j}, \ldots$ and the position vector of the centre of gravity be $\bar{\mathbf{r}} = \bar{x}\mathbf{i} + \bar{y}\mathbf{j}$, so that

$$\bar{\mathbf{r}} = \frac{(w_1x_1 + w_2x_2 + \cdots)\mathbf{i}}{\sum w} + \frac{(w_1y_1 + w_2y_2 + \cdots)\mathbf{j}}{\sum w}$$

$$= \frac{w_1(x_1\mathbf{i} + y_1\mathbf{j}) + w_2(x_2\mathbf{i} + y_2\mathbf{j}) + \cdots}{\sum w}$$

Thus the position vector of the centre of gravity is $\bar{\mathbf{r}} = \dfrac{\sum w\mathbf{r}}{\sum w}$.

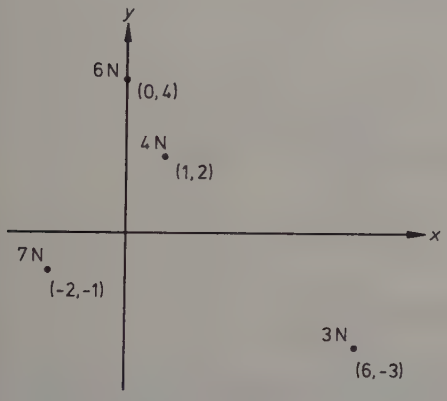
Note that in the case of three particles of weight W at the points $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ we have

$$\bar{x} = \frac{Wx_1 + Wx_2 + Wx_3}{3W} = \tfrac{1}{3}(x_1 + x_2 + x_3)$$

and similarly $\bar{y} = \tfrac{1}{3}(y_1 + y_2 + y_3)$.

Hence the centre of gravity of three equal particles at A, B, C and also of a uniform triangular lamina ABC has coordinates $(\tfrac{1}{3}\{x_1 + x_2 + x_3\}, \tfrac{1}{3}\{y_1 + y_2 + y_3\})$ and position vector $\tfrac{1}{3}\{\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3\}$.

Example 1 Find the coordinates of the centre of gravity of particles of weights 4 N, 7 N, 3 N and 6 N at the points (1, 2), (−2, −1), (6, −3) and (0, 4) respectively.



Let the centre of gravity of the particles be at the point $(\bar{x}, \bar{y})$, then the resultant weight of the particles has magnitude 20 N and acts through $(\bar{x}, \bar{y})$.

Taking moments about the y-axis,

$$20\bar{x} = 4 \times 1 + 7 \times (-2) + 3 \times 6 + 6 \times 0$$

$$\therefore \quad \bar{x} = \frac{4 - 14 + 18 + 0}{20} = \frac{8}{20} = \frac{2}{5}$$

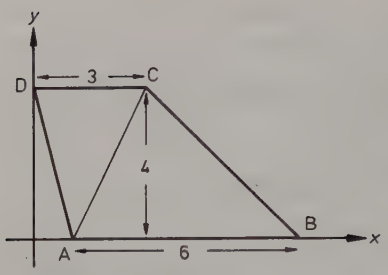
Taking moments about the x-axis,

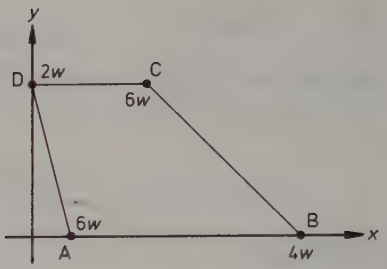
$$20\bar{y} = 4 \times 2 + 7 \times (-1) + 3 \times (-3) + 6 \times 4$$

$$\therefore \quad \bar{y} = \frac{8 - 7 - 9 + 24}{20} = \frac{16}{20} = \frac{4}{5}$$

Hence the centre of gravity has coordinates $\left(\dfrac{2}{5}, \dfrac{4}{5}\right)$.

Example 2 Find the coordinates of the centre of gravity of a uniform lamina in the form of a trapezium with vertices $A(1, 0)$, $B(7, 0)$, $C(3, 4)$ and $D(0, 4)$.





The weight of the lamina $ABCD$ is the resultant of the weights of lamina ABC and lamina ACD. Let w be the weight per unit area of the lamina, then the weight of ABC is $12w$ and is equivalent to three weights of $4w$ at points A, B and C.
Similarly the weight of ACD is $6w$ and is equivalent to three weights of $2w$ at points A, C and D.
Hence the centre of gravity $(\bar{x}, \bar{y})$ of the lamina $ABCD$ is the same as the centre of gravity of particles of weights $6w, 4w, 6w$ and $2w$ at the points A, B, C and D respectively.
Taking moments about the y-axis,

$$18w\bar{x} = 6w \times 1 + 4w \times 7 + 6w \times 3$$

Taking moments about the x-axis,

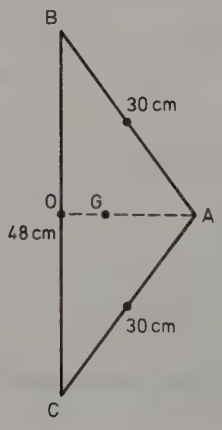
$$18w\bar{y} = 6w \times 4 + 2w \times 4$$

$$\therefore \bar{x} = \frac{52w}{18w} = \frac{26}{9}, \quad y = \frac{32w}{18w} = \frac{16}{9}.$$

Hence the centre of gravity has coordinates $\left(\frac{26}{9}, \frac{16}{9}\right)$.

The principle of moments is also used to find the centre of gravity of any body which can be divided into smaller bodies, each with known weight and centre of gravity.

Example 3 A thin uniform wire is bent to form a triangle ABC in which $AB = AC = 30\,\text{cm}$ and $BC = 48\,\text{cm}$. Find its centre of gravity.



From symmetry the centre of gravity G lies on OA, where O is the mid-point of BC.
Each of the wires AB, AC and BC has its centre of gravity at its mid-point.
By Pythagoras' theorem, $OA = 18\,\text{cm}$.

Letting w be the weight per cm of the wire, we have:

Body	Weight	Distance of centre of gravity from BC
Wire AB	$30w$	9
Wire AC	$30w$	9
Wire BC	$48w$	0
Whole body	$108w$	$\bar{x}$

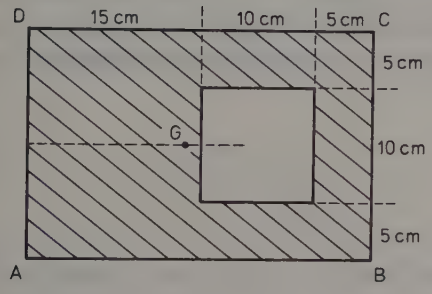
Taking moments about BC,

$$108w\bar{x} = 30w \times 9 + 30w \times 9 + 48w \times 0$$

$$\therefore \quad \bar{x} = \frac{2 \times 30w \times 9}{108w} = 5$$

$\therefore$ the centre of gravity is 5 cm from BC on the line OA.

Example 4



A square hole is cut in a uniform rectangular lamina as shown in the diagram. Find the centre of gravity of the resulting body.

From symmetry the centre of gravity G lies on the line joining the mid-points of AD and BC.

We use the positions of the centres of gravity of the complete rectangle and of the square removed to find the distance $\bar{x}$ of G from AD.
Letting w be the weight per unit area, we have:

Body	Weight	Distance of centre of gravity from AD
Rectangle	$600w$	15
Square	$100w$	20
Remainder	$500w$	$\bar{x}$

Taking moments about AD,

$$500w\bar{x} + 100w \times 20 = 600w \times 15$$

$$\therefore \quad 500w\bar{x} = 600w \times 15 - 100w \times 20$$

$$\therefore \quad \bar{x} = \frac{9000w - 2000w}{500w} = 14$$

$\therefore$ the centre of gravity is 14 cm from AD on the line joining the mid-points of AD and BC.

As shown earlier the centre of gravity of a set of particles in a plane is the point $\left(\frac{\sum wx}{\sum w}, \frac{\sum wy}{\sum w}\right)$. If $m_1, m_2, \ldots$ are the masses of the particles, their *centre of mass* is defined to be the point

$$\left(\frac{\sum mx}{\sum m}, \frac{\sum my}{\sum m}\right).$$

The weight w and the mass m of a typical particle are connected by the relationship $w = mg$, where g is the acceleration due to gravity. Thus, provided that g takes the same value for all the particles, $\frac{\sum wx}{\sum w} = \frac{\sum mgx}{\sum mg} = \frac{\sum mx}{\sum m}$ and similarly $\frac{\sum wy}{\sum w} = \frac{\sum my}{\sum m}$. It follows that, in general, the centre of gravity and the centre of mass of a body are at the same point.

[We note, however, that if a body is so large that the weights of its constituent particles are not parallel forces, then its centre of gravity cannot be found by elementary methods and does not necessarily coincide with the centre of mass.]

A *centroid* is a centre of area or volume. The centroid of a plane area coincides with the centre of mass of a uniform lamina of the same shape. The centroid of a three-dimensional figure coincides with the centre of mass of the corresponding uniform solid.

Exercise 5.3

1. Find the centre of gravity of two particles of weight 2 N and 3 N at points A and B respectively, given that $AB = 30$ cm.

2. Find the centre of gravity of two particles of weight 30 N and 70 N at points P and Q respectively, given that $PQ = 1{\cdot}5$ m.

3. A uniform triangular lamina ABC has $AB = AC = 17$ cm and $BC = 16$ cm. Find the distance of the centre of gravity from BC.

4. A uniform triangular lamina PQR has $PQ = 9$ cm, $QR = 15$ cm and $PR = 12$ cm. Find the distances of the centre of gravity from PR and PQ.

5. Find the coordinates of the centre of gravity of particles of weight 5 N, 7 N, 1 N and 3 N at the points $(1, 0)$, $(3, 1)$, $(6, 3)$ and $(0, 2)$ respectively.

6. Find the coordinates of the centre of mass of particles of mass 9 kg, 4 kg, 6 kg and 5 kg at the points $(4, 3)$, $(6, -6)$, $(-3, 0)$ and $(6, -3)$ respectively.

7. Find the position vector of the centre of mass of a uniform triangular lamina, given that the position vectors of its vertices are $3\mathbf{i} - 5\mathbf{j}$, $7\mathbf{i} + 2\mathbf{j}$ and $-\mathbf{i} + 6\mathbf{j}$.

8. Find the position vector of the centre of gravity of three particles of weight 7 N, 9 N and 4 N with position vectors $4\mathbf{i} + \mathbf{j}$, $3\mathbf{j}$ and $3\mathbf{i} + 4\mathbf{j}$.

9. A uniform lamina in the form of a quadrilateral has vertices $O(0,0)$, $A(4,-6)$, $B(7,0)$ and $C(0,6)$. If the weight per unit area of the lamina is w, show that it has the same centre of gravity as particles of weight $14w$, $7w$, $14w$ and $7w$ at O, A, B and C respectively. Hence find the centre of gravity of the lamina.

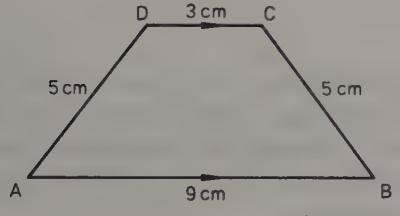
10. Use the method of the previous question to find the centre of gravity of a uniform lamina in the form of a pentagon with vertices $O(0,0)$, $A(3,0)$, $B(3,2)$, $C(0,4)$ and $D(-3,1)$.

11. A thin uniform wire is bent to form a triangle ABC in which $AB = 24\,\text{cm}$, $AC = 10\,\text{cm}$ and $\angle A = 90°$. Find the distances of the centre of gravity from AC and AB.

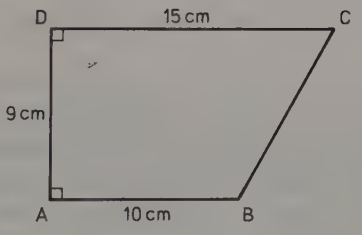
12. A thin uniform wire is bent into the shape of an isosceles trapezium $ABCD$ in which $AB = 24\,\text{cm}$, $AD = BC = 25\,\text{cm}$ and $CD = 10\,\text{cm}$. Find the distance of the centre of gravity from AB.

In questions 13 to 16 find the position of the centre of gravity of the uniform lamina shown in the given diagram.

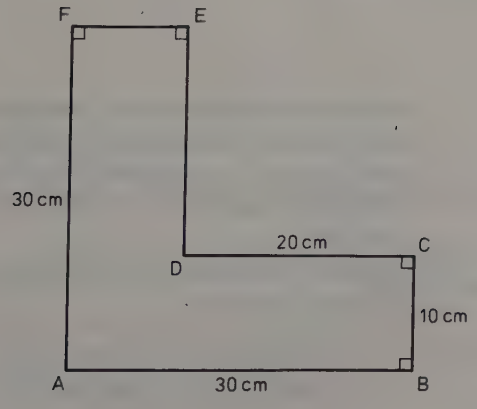
13.



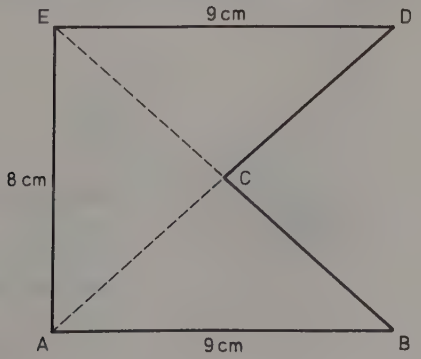
14.



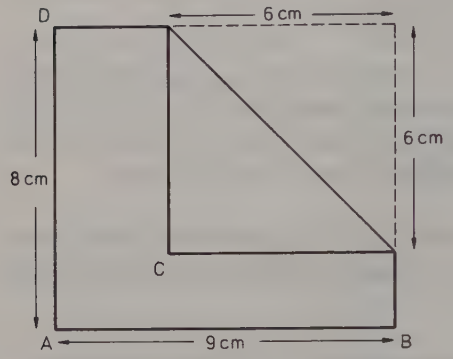
15.



16.

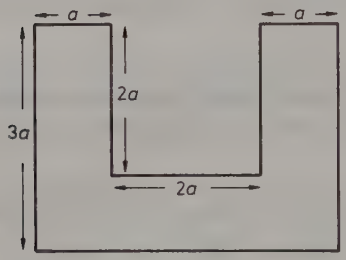


17.



Find the centre of gravity of a uniform rectangular lamina $ABCD$ folded as shown in the diagram.

18.

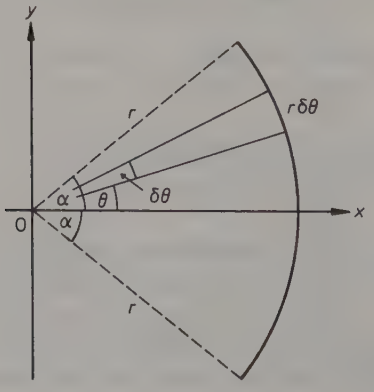


The diagram shows the central cross-section of a uniform cylindrical container weighing $2w$ per unit volume. Find its centre of gravity. If the container is now filled with a liquid weighing w per unit volume, find the new position of the centre of gravity.

5.4 Standard results by integration

As we saw in the previous section the principle of moments can be used to find the centre of gravity of any body which can be divided into smaller bodies with known weights and centres of gravity. In certain cases it is necessary to divide a body into a large number of small parts or elements. The sums of the moments of these elements about appropriate axes are then found by integration.

Example 1 Find the centre of gravity of a uniform wire in the form of an arc of a circle, radius r, subtending an angle 2α at the centre.



From symmetry the centre of gravity lies on the bisector of the angle 2α. As shown in the diagram, a typical element of arc subtends an angle $\delta\theta$ at O and is of length $r\,\delta\theta$.

Letting w be the weight per unit length of the wire, we have:

Body	Weight	Distance of centre of gravity from the y-axis
Element	$wr\,\delta\theta$	$r\cos\theta$
Whole arc	$wr\,2\alpha$	$\bar{x}$

Taking moments about the y-axis,

$$wr\,2\alpha\bar{x} \approx \sum(wr\,\delta\theta \times r\cos\theta)$$

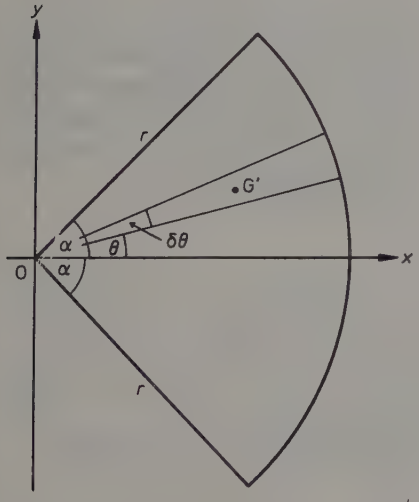
$$\therefore \quad 2wr\,\alpha\bar{x} = \int_{-\alpha}^{\alpha} wr^2\cos\theta\,d\theta$$

$$= wr^2\Big[\sin\theta\Big]_{-\alpha}^{\alpha} = 2wr^2\sin\alpha$$

$$\therefore \quad \bar{x} = \frac{r\sin\alpha}{\alpha}$$

Hence the centre of gravity of the wire is at a distance $(r\sin\alpha)/\alpha$ from the centre.

Example 2 Find the centre of gravity of a uniform lamina in the form of a sector of a circle, radius r, subtending an angle 2α at the centre.



From symmetry the centre of gravity lies on the bisector of the angle 2α. A typical element is a sector subtending an angle $\delta\theta$ at O and having area $\frac{1}{2}r^2\,\delta\theta$. This sector is approximately triangular with its centre of gravity G' at a distance $\frac{2}{3}r$ from O.

Letting w be the weight per unit area of the lamina, we have:

Body	Weight	Distance of centre of gravity from the y-axis
Element	$w\frac{1}{2}r^2\,\delta\theta$	$\frac{2}{3}r\cos\theta$
Whole sector	$w\frac{1}{2}r^2\,2\alpha$	$\bar{x}$

Taking moments about the y-axis,

$$w\tfrac{1}{2}r^2\,2\alpha\bar{x} \approx \sum(w\tfrac{1}{2}r^2\,\delta\theta \times \tfrac{2}{3}r\cos\theta)$$

$$\therefore \quad wr^2\,\alpha\bar{x} = \int_{-\alpha}^{\alpha} \tfrac{1}{3}wr^3\cos\theta\,d\theta$$

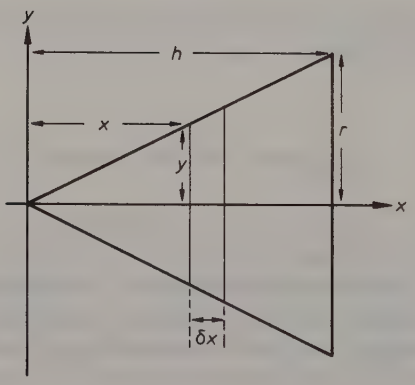
$$= \tfrac{1}{3}wr^3\Big[\sin\theta\Big]_{-\alpha}^{\alpha} = \tfrac{2}{3}wr^3\sin\alpha$$

$$\therefore \quad \bar{x} = \frac{2r\sin\alpha}{3\alpha}$$

Hence the centre of gravity of the lamina is at a distance $(\frac{2}{3}r\sin\alpha)/\alpha$ from the centre.

[This result can also be produced by showing that a uniform sector of radius r has the same centre of gravity as a uniform arc of radius $\frac{2}{3}r$.]

Example 3 Find the centre of gravity of a uniform solid right circular cone of base radius r, height h.



Let the axis of the cone lie along the x-axis, then from symmetry the centre of gravity is on the x-axis. If the cone is divided into elements by planes parallel to its base, then each element is approximately a disc, thickness δx, radius y, which has volume $\pi y^2\,\delta x$.

Letting w be the weight per unit volume of the cone, we have:

Body	Weight	Distance of centre of gravity from the y-axis
Element	$w\pi y^2\,\delta x$	x
Whole cone	$w\frac{1}{3}\pi r^2 h$	$\bar{x}$

Taking moments about the y-axis

$$w\tfrac{1}{3}\pi r^2 h\bar{x} \approx \sum(w\pi y^2\,\delta x \times x)$$

$$\therefore \quad \tfrac{1}{3}\pi r^2 hw\bar{x} = \int_0^h \pi w x y^2\,dx$$

From similar triangles, $\frac{y}{x} = \frac{r}{h}$

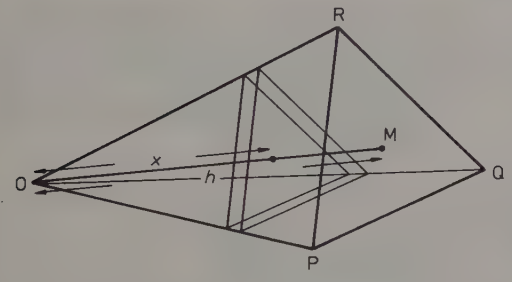
$$\therefore \quad \tfrac{1}{3}\pi r^2 hw\bar{x} = \int_0^h \frac{\pi r^2 w}{h^2} x^3 \, dx$$

$$= \frac{\pi r^2 w}{h^2}\left[\tfrac{1}{4}x^4\right]_0^h = \tfrac{1}{4}\pi r^2 h^2 w$$

$$\therefore \quad \bar{x} = \tfrac{3}{4}h$$

Hence the centre of gravity of the cone is on its axis at a distance $\frac{1}{4}h$ from its base.

Example 4 Find the centre of gravity of a uniform solid tetrahedron.



In tetrahedron $OPQR$, let M be the point of intersection of the medians of $\triangle PQR$. Let the length of OM be h and let the area of $\triangle PQR$ be A.

If the tetrahedron is divided into elements by planes parallel to the face PQR, then a typical element is a triangular lamina of thickness δx, which cuts OM at a distance x from O. Since this element is similar to $\triangle PQR$, its medians intersect on OM and its area is Ax^2/h^2.

Letting w be the weight per unit volume, we have:

Body	Weight	Distance of centre of gravity from 0
Element	$wAx^2\,\delta x/h^2$	x
Whole body	$w\frac{1}{3}Ah$	$\bar{x}$

Taking moments about an axis perpendicular to OM

$$w\tfrac{1}{3}Ah\bar{x} \approx \sum\left(\frac{wAx^2\,\delta x}{h^2} \times x\right)$$

$$\therefore \quad \tfrac{1}{3}Ahw\bar{x} = \int_0^h \frac{Aw}{h^2} x^3 \, \delta x$$

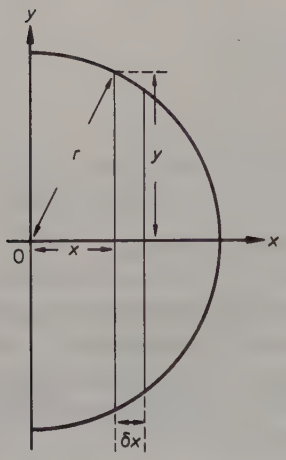
$$= \frac{Aw}{h^2}\left[\tfrac{1}{4}x^4\right]_0^h = \tfrac{1}{4}Awh^2$$

$$\therefore \quad \bar{x} = \tfrac{3}{4}h$$

Hence the centre of gravity of the tetrahedron lies one quarter of the way up a line joining the centroid of one face to the opposite vertex.

Example 5 Find the centre of gravity of a uniform solid hemisphere of radius r.

If we take x- and y-axes along and perpendicular to the axis of symmetry of the hemisphere, then its centre of gravity will be on the x-axis.



A typical element is approximately a disc of thickness δx, radius y at a distance x from the y-axis.
Letting w be the weight per unit volume of the hemisphere:

Body	Weight	Distance of centre of gravity from the y-axis
Element	$w\pi y^2\,\delta x$	x
Whole hemisphere	$w\frac{2}{3}\pi r^3$	$\bar{x}$

Taking moments about the y-axis,

$$w\tfrac{2}{3}\pi r^3\bar{x} \approx \sum(w\pi y^2\,\delta x \times x)$$

$$\therefore\quad \tfrac{2}{3}\pi r^3 w\bar{x} = \int_0^r \pi w x y^2\,dx$$

By Pythagoras' theorem, $x^2 + y^2 = r^2$,

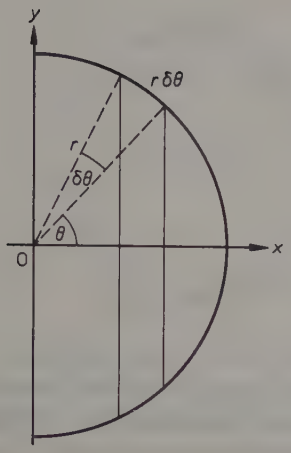
$$\therefore\quad \tfrac{2}{3}\pi r^3 w\bar{x} = \int_0^r \pi w(r^2x - x^3)\,dx$$

$$= \pi w\left[\tfrac{1}{2}r^2x^2 - \tfrac{1}{4}x^4\right]_0^r = \tfrac{1}{4}\pi r^4 w$$

$$\therefore\quad \bar{x} = 3r/8$$

Hence the centre of gravity of the hemisphere is on its axis of symmetry at a distance $3r/8$ from its plane face.

Example 6 Find the centre of gravity of a uniform hemispherical shell of radius r.



If we take x- and y-axes along and perpendicular to the axis of symmetry of the hemispherical shell, then its centre of gravity will be on the x-axis.

If the shell is divided into narrow bands by planes perpendicular to the x-axis, then a typical elemental band is approximately cylindrical with radius $r \sin\theta$ and width $r\,\delta\theta$. Letting w be the weight per unit area of the shell, we have:

Body	Weight	Distance of centre of gravity from the y-axis
Element	$w2\pi r^2 \sin\theta\,\delta\theta$	$r\cos\theta$
Whole body	$w2\pi r^2$	$\bar{x}$

Taking moments about the y-axis

$$w2\pi r^2\bar{x} \approx \sum(w2\pi r^2 \sin\theta\,\delta\theta \times r\cos\theta)$$

$$\therefore \quad 2\pi w r^2\bar{x} = \int_0^{\pi/2} 2\pi w r^3 \sin\theta\cos\theta\,d\theta$$

$$= \int_0^{\pi/2} \pi w r^3 \sin 2\theta\,d\theta$$

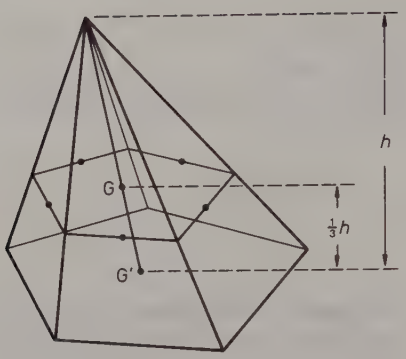
$$= \pi w r^3[-\tfrac{1}{2}\cos 2\theta]_0^{\pi/2} = \pi w r^3$$

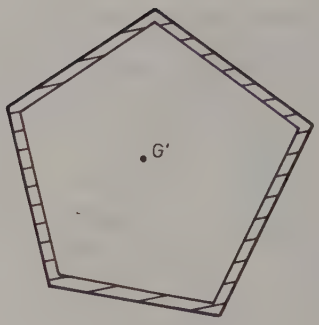
$$\therefore \quad \bar{x} = \frac{r}{2}$$

Hence the centre of gravity of the hemispherical shell is at a distance $\frac{1}{2}r$ from its base.

[For an alternative method see Exercise 5.6, question 11.]

It is not necessary to use integration to find the centres of gravity of hollow pyramids and cones.





Let us consider a thin hollow pyramid of height h, without base. Each of the faces is a triangular lamina with centre of gravity at a height $\frac{1}{3}h$ above the base. If the pyramid is divided into sections parallel to the base, we see that the centre of gravity must lie on the line joining the centres of gravity of these sections.

Hence the centre of gravity of the *hollow pyramid* lies one third of the way up a line joining the centroid of the base to the vertex.

As the number of sides of the base of the pyramid is increased indefinitely, it tends to become a cone. Hence the centre of gravity of a *hollow right circular cone*, without base, is on the axis of the cone at a distance of one third of the height of the cone from the base.

[Note that the results obtained in this section may be quoted without proof when finding the centres of gravity of composite bodies. A list of standard results will be found in the "Formulae for reference" section near the end of the book.]

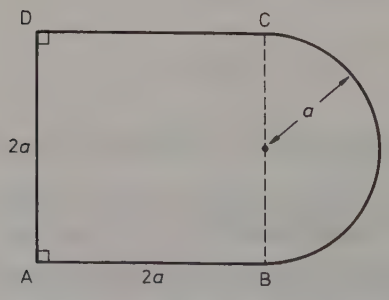
Exercise 5.4

1. Prove that the centre of gravity of a uniform wire in the form of a semi-circle of radius r is at a distance $2r/\pi$ from the centre.

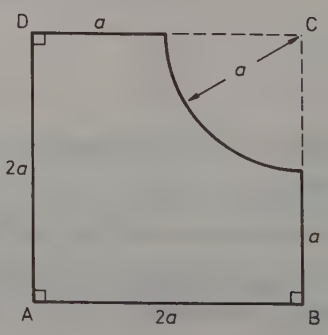
2. Prove that the centre of gravity of a uniform semi-circular lamina of radius r is at a distance $4r/3\pi$ from the centre.

In questions 3 to 6 find the distance from AD of the centre of gravity of the given uniform lamina.

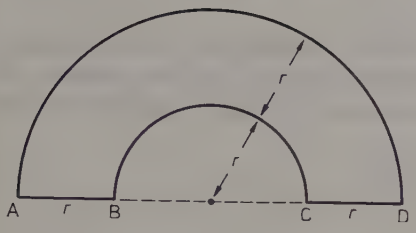
3.



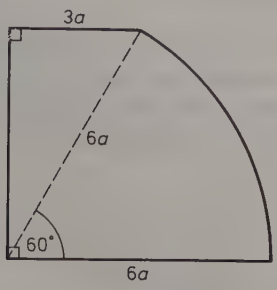
4.



5.



6.



7. A frustum is cut from a solid right circular cone of base radius r and height $2h$ by a plane parallel to the base and at a distance h from it. Find the distance of the centre of gravity of the frustum from the base.

8. A circular lamina of radius $2r$ is divided into two segments by a line at a distance r from the centre. Find the distance from the centre of the centre of gravity of (a) the larger segment, (b) the smaller segment.

9. A solid sphere of radius $3a$ is divided into two parts by a plane at a distance a from the centre. Find, by integration, the volume of the smaller part and the distance of its centre of gravity from the centre of the sphere.

10. Prove that the centre of gravity of a uniform solid pyramid with a square base lies one quarter of the height of the pyramid above the base and on the line joining the vertex to the point of intersection of the diagonals of the base.

11. Prove that the centre of gravity of a uniform solid hemisphere of radius r is a distance $3r/8$ from the centre.

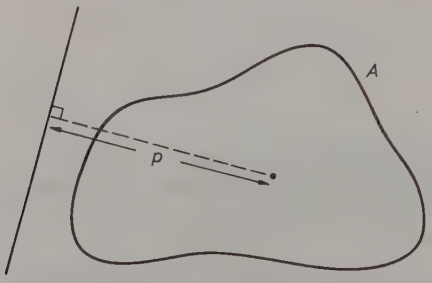
A child's toy is made up from a uniform and solid right circular cone and hemisphere. The radius of the cone is r, and its height $3r$. The radius of the hemisphere is r. The base of the cone and hemisphere are sealed together. The material from which the hemisphere is made is 3 times as heavy per unit volume as the cone material. Find the distance of the centre of gravity of the toy from the vertex of the cone. (SU)

12. Prove that the centre of gravity of a uniform thin hemispherical cup of radius r is at a distance $r/2$ from the centre.

A goblet consists of a uniform thin hemispherical cup of radius r, a circular base of the same material, thickness and radius as the cup, and an intervening stem of length r and whose weight is one-quarter of that of the cup. Show that the height of the centre of gravity above the base is $13r/14$. If the weight of the goblet is W and that of the amount of liquid that fills it is W', show that filling it raises the centre of gravity through a distance $\frac{39}{56}\left(\frac{W'}{W+W'}\right)r$. (W)

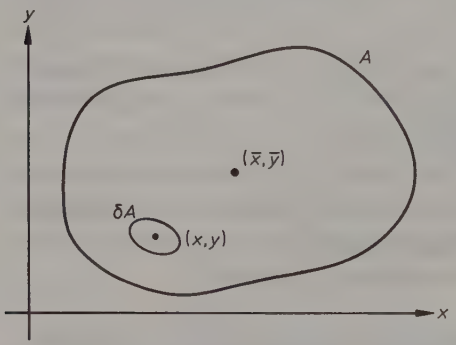
5.5 Centroids of areas and volumes

The *centroid* of a plane figure is its centre of area i.e. the point about which the area is evenly distributed. In simple cases we can use symmetry to find the centroid of a figure. However, to arrive at a more general method, we define a quantity called the *first moment of area.*



If the centroid of an area A is at a perpendicular distance p from an axis, then the first moment of A about this axis is pA.

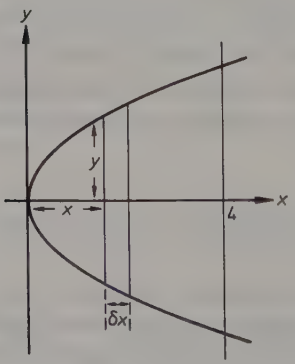
If an area is divided into a number of parts, then the sum of the first moments of the parts about any axis is equal to the first moment of the whole area about that axis.



Suppose that a region of the x, y plane has area A and centroid $(\bar{x}, \bar{y})$. If a typical element of area δA has centroid (x, y) then, comparing first moments about the coordinates axes, we have:

$$\bar{x}A = \sum x\,\delta A, \quad \bar{y}A = \sum y\,\delta A.$$

Example 1 Find the centroid of the region bounded by the curve $y^2 = x$ and the line $x = 4$.



A typical element of area is a strip of width δx, length $2y$ at a distance x from the y-axis. From symmetry the centroids of such strips and of the whole region lie on the x-axis.

Thus, we may write:

	Area	Coordinates of centroid
Element	$2y\,\delta x$	$(x, 0)$
Whole region	A	$(\bar{x}, 0)$

First moment of area about the y-axis

$$= \bar{x}A \approx \sum(x \times 2y\,\delta x)$$

$$\bar{x}A = \int_0^4 2xy\,dx$$

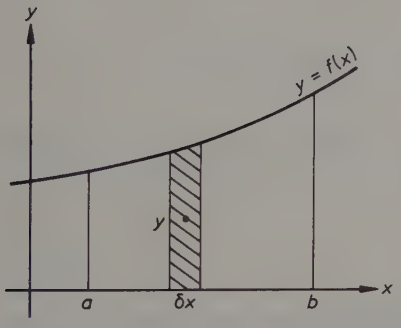
$$= \int_0^4 2x^{3/2}\,dx = 2\left[\frac{2}{5}x^{5/2}\right]_0^4 = \frac{128}{5}$$

But $$A = \int_0^4 2y\,dx$$

$$= \int_0^4 2x^{1/2}\,dx = 2\left[\frac{2}{3}x^{3/2}\right]_0^4 = \frac{32}{3}$$

$$\therefore \quad \bar{x} = \frac{128}{5} \Big/ \frac{32}{3} = \frac{128}{5} \times \frac{3}{32} = \frac{12}{5}$$

Hence the centroid of the region is the point $(12/5, 0)$.



Let us consider now the area bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$, $x = b$. A typical element of area is an approximately rectangular strip of width δx and height y.

	Area	Coordinates of centroid
Element	$y\,\delta x$	$(x, \frac{1}{2}y)$
Whole region	A	$(\bar{x}, \bar{y})$

First moment of area about the y-axis

$$= \bar{x}A \approx \sum(x \times y\,\delta x)$$

$$\therefore \quad \bar{x}A = \int_a^b xy\,dx$$

First moment of area about the x-axis

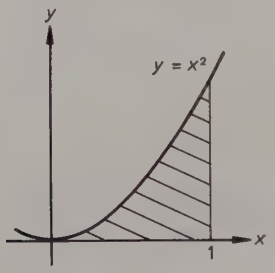
$$= \bar{y}A \approx \sum(\tfrac{1}{2}y \times y\,\delta x)$$

$$\therefore \quad \bar{y}A = \int_a^b \tfrac{1}{2}y^2\,dx$$

But $A = \int_a^b y\,dx$, so the centroid has coordinates

$$\bar{x} = \int_a^b xy\,dx \Big/ \int_a^b y\,dx \quad \bar{y} = \int_a^b \tfrac{1}{2}y^2\,dx \Big/ \int_a^b y\,dx$$

Example 2 Find the centroid of the area bounded by the curve $y = x^2$, the x-axis and the line $x = 1$.



Area of the given region

$$= \int_0^1 y\,dx = \int_0^1 x^2\,dx = \left[\frac{1}{3}x^3\right]_0^1 = \tfrac{1}{3}$$

First moment about the y-axis

$$= \int_0^1 xy\,dx = \int_0^1 x^3\,dx = \left[\frac{1}{4}x^4\right]_0^1 = \tfrac{1}{4}$$

First moment about the x-axis

$$= \int_0^1 \tfrac{1}{2}y^2\,dx = \int_0^1 \tfrac{1}{2}x^4\,dx = \left[\frac{1}{10}x^5\right]_0^1 = \frac{1}{10}$$

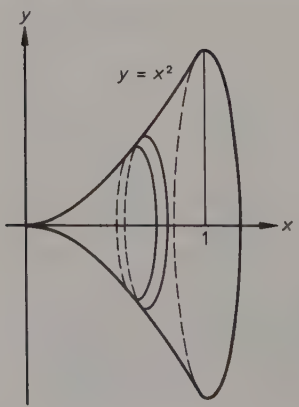
$$\therefore \quad \bar{x} = \int_0^1 xy\,dx \Big/ \int_0^1 y\,dx = \frac{1}{4} \div \frac{1}{3} = \frac{3}{4}$$

$$\therefore \quad \bar{y} = \int_0^1 \tfrac{1}{2}y^2\,dx \Big/ \int_0^1 y\,dx = \frac{1}{10} \div \frac{1}{3} = \frac{3}{10}$$

Hence the centroid of the area is the point $(\frac{3}{4}, \frac{3}{10})$.

The centroid of a solid three-dimensional figure is its centre of volume i.e. the point about which the volume is evenly distributed. The methods used to find centroids of volume are similar to those used to find centroids of area.

Example 3 Find the centroid of the volume obtained by rotating completely about the x-axis the area bounded by the curve $y = x^2$, the x-axis and the line $x = 1$.



A typical element of volume is approximately a disc of thickness δx, radius y at a distance x from the y-axis. From symmetry the centroids of such discs and of the whole solid lie on the x-axis.

Thus we may write:

	Volume	Coordinates of centroid
Element	$\pi y^2\,\delta x$	$(x, 0)$
Whole solid	V	$(\bar{x}, 0)$

First moment of volume about the y-axis

$$= \bar{x}V \approx \sum(x \times \pi y^2\,\delta x)$$

$$\therefore \quad \bar{x}V = \int_0^1 \pi x y^2\,dx$$

$$= \pi \int_0^1 x^5\,dx = \pi\left[\frac{1}{6}x^6\right]_0^1 = \frac{\pi}{6}$$

But $$V = \int_0^1 \pi y^2\,dx$$

$$= \pi \int_0^1 x^4\,dx = \pi\left[\frac{1}{5}x^5\right]_0^1 = \frac{\pi}{5}$$

$$\therefore \quad \bar{x} = \frac{\pi}{6}\Big/\frac{\pi}{5} = \frac{\pi}{6} \times \frac{5}{\pi} = \frac{5}{6}$$

Hence the centroid of the solid is at the point $(5/6, 0)$.

Exercise 5.5

Find the first moments about the y-axis and the x-axis of the areas bounded by the following curves and straight lines. In each case use your results to find the coordinates of the centroid of the area.

1. $y = x^2 + 1$, $y = 0$, $x = 0$ and $x = 2$.

2. $y = x(x + 2)$ for $x \geqslant 0$, $y = 0$ and $x = 3$.

3. $y = \dfrac{1}{x^2}$, $y = 0$, $x = 1$ and $x = 2$.

4. $y = x^2$ for $x \geqslant 0$, $x = 0$ and $y = 1$.

Find the centroids of the areas bounded by the following curves and straight lines.

5. $y^2 = x^3$ and $x = 4$.

6. $y^2 = 4x$ and $x = 1$.

7. $y = \cos x$ for $-\frac{1}{2}\pi \leqslant x \leqslant \frac{1}{2}\pi$ and $y = 0$.

8. $x^2y = 4$, $y = 1$ and $y = 4$.

9. $y = e^x$, $y = 0$, $x = 0$ and $x = 1$.

10. $y = \sin x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, $y = 0$ and $x = \frac{1}{2}\pi$.

Find the centroids of the solids formed by rotating completely about the x-axis the regions defined by the following inequalities.

11. $y \geqslant x(x - 4)$, $y \leqslant 0$.

12. $y^2 \leqslant 9x$, $y \geqslant 0$, $x \leqslant 1$.

13. $xy \leqslant 4$, $y \geqslant 0$, $1 \leqslant x \leqslant 2$.

14. $0 \leqslant y \leqslant e^{-x}$, $0 \leqslant x \leqslant 1$.

15. Sketch the curve $y^2 = x(3 - x)^2$. Find the centroid of the area enclosed by the loop.

16. Find the centroid of the solid formed by rotating through π radians about the y-axis the region bounded by the curve $y = 4x^2$ and the straight lines $y = 1$ and $y = 4$.

17. Use integration by parts to evaluate

$$\int_1^2 \ln x \, dx, \quad \int_1^2 x \ln x \, dx \quad \text{and} \quad \int_1^2 (\ln x)^2 \, dx.$$

Hence find, correct to 2 decimal places, the coordinates of the centroid of the area bounded by the curve $y = \ln x$, the x-axis and the line $x = 2$.

18. Find the area of the region R bounded by the curve $y = x^2$ and the straight line $y = x$. Show that the first moments of R about the x- and y-axes are given by

$$\int_0^1 \tfrac{1}{2}(x^2 - x^4)\, dx \quad \text{and} \quad \int_0^1 (x^2 - x^3)\, dx \text{ respectively.}$$

Hence find the centroid of the region R.

Exercise 5.6 (*miscellaneous*)

1. All forces in this question act in the plane of a triangle ABC in which $AB = 4a$, $AC = 3a$ and the angle $A = 90°$. Forces of magnitude $17P$, $15P$, $3P$ act along AB, BC, AC respectively in the directions indicated by the order of the letters. Calculate
(a) the magnitude of the resultant of these three forces and the tangent of the angle made by its line of action with AB,
(b) the distance from A of the point where the line of action of the resultant cuts AB. A couple G is now added to the system and the resultant of this enlarged system acts through the point B. Calculate the magnitude and sense of G. (L)

2. A rigid square lamina $ABCD$ of side a is subject to forces of magnitude $1, 2, 3, 1, 3\sqrt{2}$ and $\lambda\sqrt{2}$ units acting along AB, BC, CD, AD, AC and DB respectively

in the directions indicated by the order of the letters. Given that the direction of the resultant force is parallel to AC, find λ. With this value of λ, find the total moment about A of the forces acting on the lamina. Hence, or otherwise, find AE in terms of a, where E is the intersection of AB with the line of action of the resultant force.

(JMB)

3. (a) State conditions which will ensure that a system of forces in a plane will reduce to a couple.
   (b) State conditions which will ensure that a system of forces in a plane is in equilibrium.
   (c) $ABCD$ is a square of side a. Forces of size 1, 3, 3 and 7 act respectively along AB, BC, CD and DA in the directions indicated by the letters. (i) Find the sizes of the forces which must act along AC and BD so that the six forces will be equivalent to a couple, and find the moment of the couple. (ii) If the forces in DA, AC and BD are to be replaced so that the system is in equilibrium, how will this have to be done? (SU)

4. An equilateral triangle ABC has side $2a$ and D, E are the mid-points of BC, CA respectively. Fixed forces of magnitude $P, 2P, 4P$ act along AB, BC, CA respectively and variable forces of magnitude x, y act along AD, BE respectively, the direction of each force being indicated by the order of the letters. Find (in terms of P) the values of x and y if
(a) the system is equivalent to a force through B parallel to AD, stating the magnitude of this force;
(b) the system is equivalent to a couple, stating the moment of the couple. (O&C)

5. Forces $2, 4, 6, 2p, 2q$ and 18 newtons act along the sides AB, CD, ED, EF and AF respectively of a regular hexagon $ABCDEF$, the directions of the forces being indicated by the order of the letters. If the system is in equilibrium, find, by resolving parallel and perpendicular to AB, the values of p and q. Check your results by finding the moment of the forces about O, the centre of the hexagon.

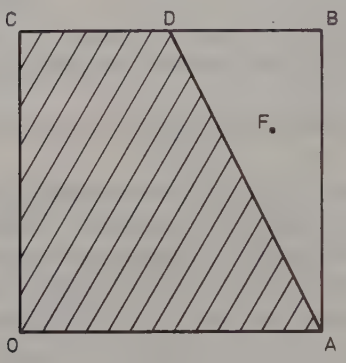
The forces along ED, EF and AF are now replaced by a coplanar force through O and a coplanar couple. If the resulting system is in equilibrium and if the length of each side of the hexagon is 2 metres, calculate

(a) the magnitude of this force through O,
(b) the magnitude of the couple. (L)

6. The centre of a regular hexagon $ABCDEF$ of side a is O. Forces of magnitude $P, 2P, 3P, 4P, mP$ and nP act along $\vec{AB}, \vec{BC}, \vec{CD}, \vec{DE}, \vec{EF}$, and $\vec{FA}$ respectively. Given that the resultant of these six forces is of magnitude $3P$ acting in a direction parallel to $\vec{EF}$,
(i) determine the values of m and n,
(ii) show that the sum of the moments of the forces about O is $9Pa\sqrt{3}$.
The mid-point of EF is M.
(iii) Find the equation of the line of action of the resultant referred to OM as x-axis and OA as y-axis.

The forces mP and nP acting along $\vec{EF}$ and $\vec{FA}$ are removed from the system. The remaining four forces and an additonal force Q, which acts through O, reduce to a couple. Calculate the magnitude of Q and the moment of the couple. (AEB 1979)

7.



The diagram shows a square $OABC$ of side a. The mid-point of BC is D. Show that, with respect to OA and OC as axes, the coordinates of the centroid F of the triangular region ABD are $(5a/6, 2a/3)$. Find the coordinates of the centre of mass of a uniform lamina in the form of the figure $OADC$. (JMB)

8. A uniform thin sheet of paper has the shape of a rectangle $OABC$, where $OA = 20\,\text{cm}$ and $OC = 12\,\text{cm}$. This side AB is folded down so as to lie entirely along OA. Find the distances of the centre of gravity of the folded sheet from OA and OC. (O&C)

9. A uniform solid hemisphere of radius a is cut into two parts by a plane parallel to the plane face of the hemisphere and at a distance $\frac{1}{2}a$ from it. Find the volume of each of the two parts and the position of the centre of mass of each of the two parts. (AEB 1980)

10. Prove that the centre of gravity of a uniform circular arc subtending an angle 2θ at the centre of a circle of radius a is $a \sin\theta/\theta$ from the centre. Deduce that the centre of gravity of a uniform sector bounded by that arc and the radii to its extremities is $\frac{2}{3}a \sin\theta/\theta$ from the centre.

Show also that the centre of gravity of a segment of a circular lamina cut off by a chord subtending a right angle at the centre of the circle is $\frac{2}{3}\sqrt{2}a/(\pi - 2)$ from the centre. (W)

11. Find by integration the distance of the centroid of a uniform solid hemisphere of radius a from its plane face.

A hollow sphere of external radius R and internal radius r is cut in half by a plane through its centre. Show that the distance of the centroid of each half from the centre of the sphere is

$$\frac{3(R + r)(R^2 + r^2)}{8(R^2 + Rr + r^2)}.$$

Deduce that the centroid of a hemispherical shell of radius R is at a distance $R/2$ from its centre. (L)

12. Find the area A of the loop of the curve $y^2 = x^2(2 - x)$. Find also the first moment about the y-axis of the volume generated when the area A is rotated through π radians about the x-axis. (AEB 1975)

13. Find the area of the region in the first quadrant enclosed between the curve $y = x^2 + 4$, the line $y = 8$ and the y-axis. The region is rotated through four right-angles about Oy to form a uniform solid. Find the volume and the coordinates of the centre of mass of the solid. (C)

14. Sketch on one diagram the graphs of $x^3y = 1$ and $y = \sqrt{x}$, for positive values of x. Show that the area of the finite region bounded by the straight lines $y = 0$ and $x = 2$ and an arc of each of the above curves is 25/24. Find also the coordinates of the centroid of the region. (C)

15. Find the area of the finite region between the curve $y = x^2 + 1$ and the line $y = 2x + 1$. The region is rotated through four right angles about Ox, to produce a uniform solid of revolution. Find the volume and the coordinates of the centre of gravity of this solid. (C)

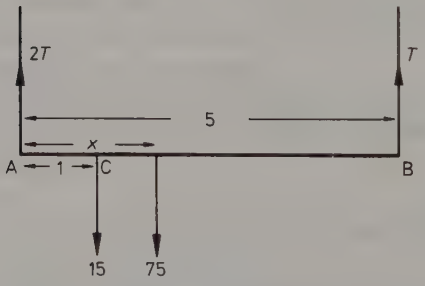
16. Sketch the arc of the curve given parametrically by $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, for which $0 \leqslant \theta \leqslant 2\pi$. Find the area of the finite region enclosed by this arc and the x-axis. Find also the coordinates of the centroid of the region. (L)

6 Equilibrium of rigid bodies

6.1 Parallel forces in equilibrium

If a set of coplanar forces is in equilibrium then
(i) the sum of the components of the forces in any given direction is zero and
(ii) the sum of the moments of the forces about any given point is zero.
These facts are used to find unknown quantities in many statics problems.

Example 1 A non-uniform rod AB of length 5 m and weight 75 N is supported in a horizontal position by two vertical strings attached to its ends. A weight of 15 N is attached to the rod at the point C, where $AC = 1$ m. If the tensions in the strings at A and B are $2T$ N and T N respectively, find the value of T. Find also the distance, x m, of the centre of gravity of the rod from A.



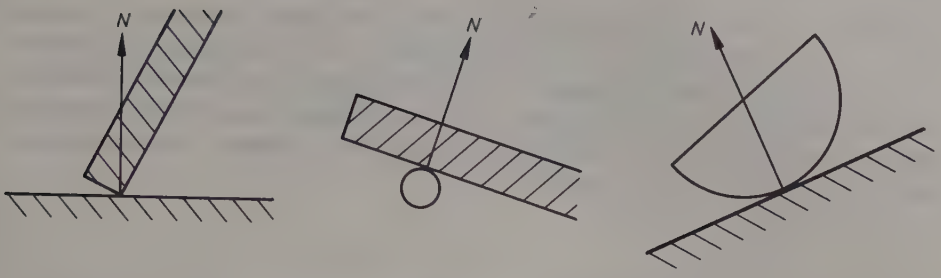
Considering the vertical forces on the rod,

$$\uparrow \qquad 2T - 15 - 75 + T = 0$$
$$\therefore \qquad 3T = 90$$
Hence $$T = 30.$$

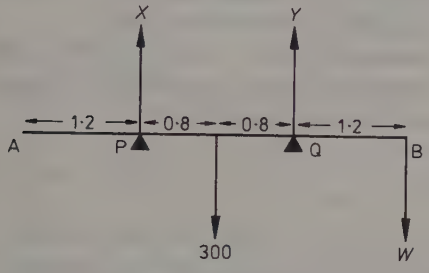
Taking moments clockwise about A,

$$\curvearrowright_A \qquad 15 \times 1 + 75 \times x - T \times 5 = 0$$
$$\therefore \qquad 75x = 5T - 15 = 135$$
Hence $$x = 1{\cdot}8.$$

When a rigid body rests on an object described as *smooth*, (e.g. a smooth support or a smooth table,) then the reaction on the body at any point of contact is assumed to be perpendicular to the surfaces in contact.



Example 2 A uniform plank AB of length 4 m and weight 300 N rests horizontally on smooth supports at P and Q, where $AP = QB = 1{\cdot}2$ m. Find the maximum weight that can be placed at B without tilting the plank.



Let us assume that the plank is in equilibrium with reactions of X N and Y N at P and Q respectively and a weight of W N at B.

Taking moments about Q,

$$X \times 1{\cdot}6 + W \times 1{\cdot}2 - 300 \times 0{\cdot}8 = 0$$

$\therefore \quad 1{\cdot}2W = 240 - 1{\cdot}6X$

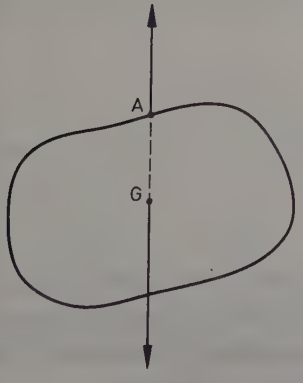
Since the reaction at P acts vertically upwards, $X \geqslant 0$,

$\therefore \quad 1{\cdot}2W \leqslant 240$ i.e. $W \leqslant 200$.

Hence the maximum weight that can be placed at B without tilting the plank is 200 N.

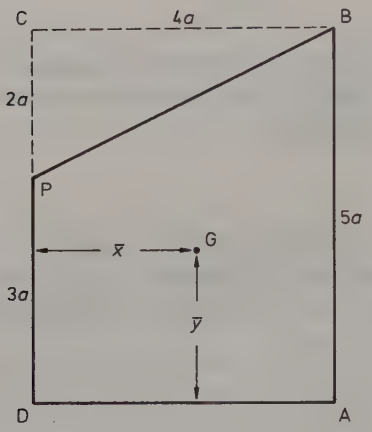
[Note that when a weight of 200 N is placed at B, the reaction at P is zero and the plank just fails to rotate about Q.]

When a body is in equilibrium under the action of *two forces*, these forces must be equal in magnitude and act in opposite directions along the same line.



In particular, when a body freely suspended from a point A is in equilibrium, the two forces acting on it, namely its weight and the force at A, must act along the same vertical line. We deduce that the body hangs in equilibrium with its centre of gravity G vertically below A.

Example 3 A uniform rectangular lamina $ABCD$ of weight $5W$ has $AB = 5a$ and $BC = 4a$. The point P lies on the edge DC and $DP = 3a$. The triangle BCP is cut away and the remaining trapezium is suspended from P. Find the angle that DP makes with the vertical when the trapezium hangs freely in equilibrium. Find also the moment of the couple which would be required to maintain the trapezium in equilibrium with D vertically below P.



We first find the position of G, the centre of gravity of the lamina $ABPD$. Rectangle $ABCD$ has area $20a^2$ and weight $5W$. Since triangle BCP has area $4a^2$, its weight must be W.

Body	Weight	Distance of centre of gravity from CD	Distance of centre of gravity from DA
Rectangle $ABCD$	$5W$	$2a$	$5a/2$
Triangle BCP	W	$4a/3$	$13a/3$
Remainder	$4W$	$\bar{x}$	$\bar{y}$

Taking moments about CD,

$$4W \times \bar{x} + W \times \frac{4a}{3} = 5W \times 2a$$

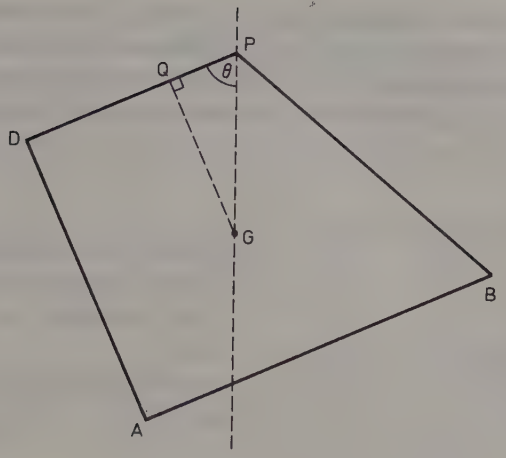
$$\therefore \quad \bar{x} = \frac{1}{4}\left(10a - \frac{4a}{3}\right) = \frac{13a}{6}$$

Taking moments about DA,

$$4W \times \bar{y} + W \times \frac{13a}{3} = 5W \times \frac{5a}{2}$$

$$\therefore \quad \bar{y} = \frac{1}{4}\left(\frac{25a}{2} - \frac{13a}{3}\right) = \frac{49a}{24}$$

When the trapezium hangs in equilibrium from P, the line PG must be vertical. Let θ be the angle DP then makes with the vertical and let Q be the foot of the perpendicular from G to DP.



$$GQ = \bar{x} = \frac{13a}{6}, \quad DQ = \bar{y} = \frac{49a}{24}, \qquad \therefore \quad QP = \frac{23a}{24}.$$

Thus $\tan\theta = \dfrac{GQ}{QP} = \dfrac{13}{6} \times \dfrac{24}{23} = \dfrac{52}{23}$ and $\theta \approx 66° 8'$.

Hence the angle made by DP with the vertical is $66° 8'$.

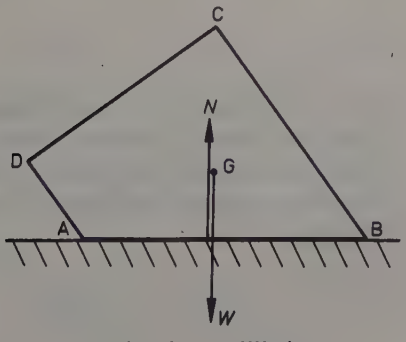
Suppose now that the trapezium is in equilibrium with D vertically below P and that a couple of moment M in the sense DAB is required to maintain this position. Taking moments anti-clockwise about P,

$$M - 4W \times \frac{13a}{6} = 0 \qquad \therefore \quad M = \frac{26Wa}{3}.$$

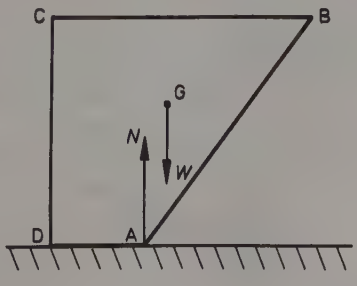
Hence the moment of the required couple is $26Wa/3$.

When a body rests in equilibrium on a horizontal plane, the weight of the body and the force exerted by the plane on the body must act along the same line, namely the vertical line through the centre of gravity.

In the case of a lamina with an edge AB resting on a smooth horizontal plane, the force N exerted by the plane on the lamina is the resultant of normal reactions at the points of contact along AB. Thus N must act through some point on AB. It follows that, if the vertical line through the centre of gravity G passes through a point on the edge AB, then the force N can act to maintain equilibrium. Otherwise equilibrium is not possible and the lamina will topple.

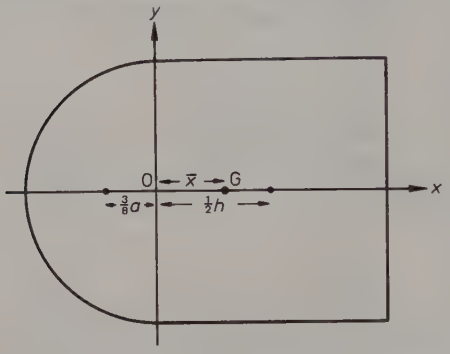


Lamina in equilibrium



Lamina will topple

Example 4 A uniform solid is formed by joining a hemisphere of radius a to a circular cylinder of radius a and height h, so that the plane face of the hemisphere coincides with a plane face of the cylinder. Show that this solid can rest in equilibrium on a horizontal plane with the curved surface of the cylinder touching the plane if $h^2 \geqslant \frac{1}{2}a^2$.



If we take x- and y-axes as shown in the diagram, then the centre of gravity G of the combined solid must lie on the x-axis.
Let w be the weight per unit volume of the solid.

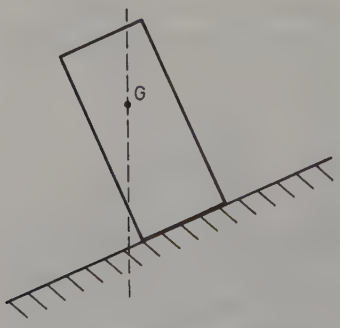
Body	Weight	x-coordinate of centre of gravity
Cylinder	$\pi a^2 hw$	$\frac{1}{2}h$
Hemisphere	$\frac{2}{3}\pi a^3 w$	$-\frac{3}{8}a$
Whole body	$(h + \frac{2}{3}a)\pi a^2 w$	$\bar{x}$

Taking moments about the y-axis,

$$(h + \tfrac{2}{3}a)\pi a^2 w\bar{x} = \pi a^2 hw \times \tfrac{1}{2}h - \tfrac{2}{3}\pi a^3 w \times \tfrac{3}{8}a$$
$$\therefore \quad (h + \tfrac{2}{3}a)\bar{x} = \tfrac{1}{2}h^2 - \tfrac{1}{4}a^2$$

When this solid is placed with the curved surface of the cylinder touching a horizontal plane, it will rest in equilibrium if the vertical line through G passes through a point of contact between the plane and the solid.
This condition will be satisfied if $\bar{x} \geqslant 0$ i.e. if $h^2 \geqslant \frac{1}{2}a^2$.

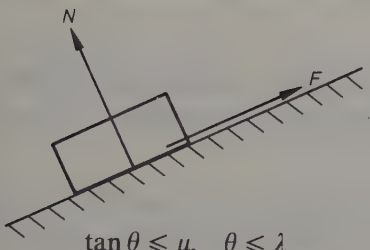
For a body to rest in equilibrium on a rough inclined plane the force exerted by the plane on the body must act along a vertical line through the centre of gravity of the body.



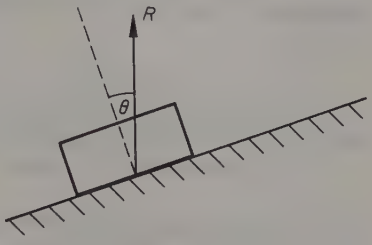
For the block shown in this diagram equilibrium is not possible. Since the vertical line through the centre of gravity G falls outside the base of the block, it will topple.

The reaction R between a body and a rough surface is the resultant of two components, the normal reaction N and a friction force F. According to the laws of friction, if μ is the coefficient of friction between the body and the surface, then $F \leqslant \mu N$. We can also express this condition in terms of λ, the angle of friction, and θ, the angle made by R with the normal to the surface.
Since $\tan \lambda = \mu$ and $\tan \theta = F/N$.

$$F \leqslant \mu N \Leftrightarrow \tan \theta \leqslant \mu \Leftrightarrow \theta \leqslant \lambda.$$



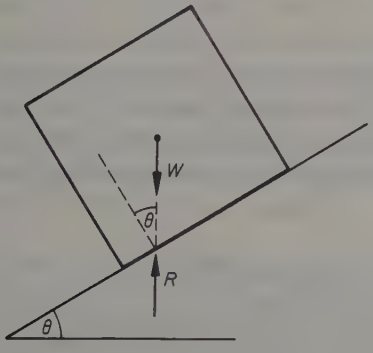
$\tan \theta \leqslant \mu, \quad \theta \leqslant \lambda$



$F \leqslant \mu N$

Example 5 A uniform solid cube is in equilibrium with one face resting on a rough inclined plane and one pair of opposite faces vertical. The coefficient of friction between the cube and the plane is μ. The angle of inclination of the plane is gradually increased. Show that equilibrium will be broken by sliding rather than toppling if $\mu < 1$.

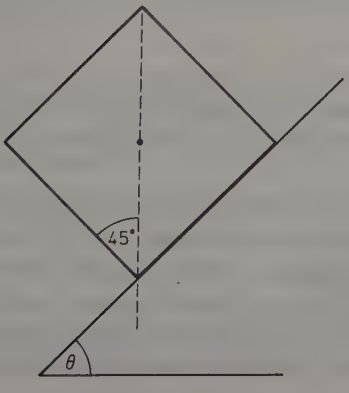
(1)



Let θ be the angle of inclination of the plane to the horizontal. When the cube is in equilibrium, as shown in diagram (1), the total reaction R acts vertically. The angle between its line of action and the normal to the plane is θ. When the cube is on the point of sliding, friction is limiting,

$$\therefore \qquad \tan \theta = \mu.$$

(2)



In diagram (2) the cube is on the point of toppling, since the vertical line through its centre of gravity passes through the lower edge of the base. Thus, when the cube is about to topple,

$$\tan \theta = \tan 45° = 1.$$

Hence equilibrium will be broken by sliding if $\tan\theta$ reaches the value μ before it reaches the value 1, i.e. if $\mu < 1$.

Exercise 6.1

1. A beam AB of length 4·5 m is supported in a horizontal position by two vertical cables attached to its ends. The tensions in the cables at A and B are 160 N and 200 N respectively. Find the weight of the beam and the distance of its centre of gravity from A.

2. A uniform rod AB of length 3 m and weight 50 N rests horizontally on smooth supports at A and B. A load of 24 N is attached to the rod at a point C, where $AC = 1$ m. Find the forces exerted on the rod by the supports.

3. A uniform bar of length 2·8 m and weight 80 N has loads of 20 N and 40 N attached to its ends A and B respectively. If the bar balances in a horizontal position when smoothly supported at C, find the distance of C from A.

4. A uniform rod $APQB$ of length 2 m rests horizontally on smooth supports at P and Q, where $PQ = 1{\cdot}2$ m. If the reaction at P is twice the reaction at Q, find the distance AP. Given that when a weight of 5 N is attached to the rod at B the reactions at P and Q are equal, find the weight of the rod.

5. A uniform plank ABC of weight 400 N is held in a horizontal position by a smooth support at B and a vertical rope attached at C, where $AB = 1$ m and $AC = 5$ m. If the rope is likely to break when the tension in it exceeds 750 N, find
(a) the maximum weight that can be placed on the plank at the mid-point of BC,
(b) the maximum distance that a man of weight 800 N can safely walk from B towards C.

6. A non-uniform beam $AXYB$ of length 4 m and weight 400 N is maintained in a horizontal position by smooth supports at X and Y, where $AX = YB = 1$ m. A boy of weight 500 N can just walk to the end A of the beam without overturning it. If the boy walks along the beam towards B, find his distance from B when the beam is just about to tip.

7. A uniform right circular solid cone of weight W is suspended by two vertical strings attached to the ends A and B of a diameter of its base. If the cone hangs in equilibrium with its vertex vertically below A, find the tensions in the strings.

8. $ABCD$ is a uniform rectangular lamina of weight 10 N. A particle of weight W N is attached to the lamina at B. When the system is freely suspended from A it rests in equilibrium with the mid-point of BC vertically below A. Find the value of W.

9. A uniform wire is bent into the form of a trapezium $ABCD$ in which $\angle ABC = \angle BCD = 90°$, $AB = 6a$, $BC = 4a$ and $CD = 3a$. Find the distances of

the centre of gravity from AB and BC. If the trapezium is freely suspended from A, find the angle which AB makes with the vertical.

10. A uniform triangular lamina ABC has $AB = 4a, BC = 5a, CA = 3a$. The lamina is folded through 180° along LN, where L and N are the mid-points of BC and AB respectively, so that the vertex B is at A. Find the coordinates of the centre of mass of the folded lamina relative to AN and AC as x-axis and y-axis respectively. The folded lamina is suspended from C and hangs in equilibrium under gravity. Find the tangent of the angle that CA makes with the vertical. (C)

11. A uniform wire is bent to form a semi-circular arc together with the diameter AB joining the ends of the arc. If the wire hangs in equilibrium from a smooth pivot at A, find the tangent of the angle that AB makes with the vertical.

12. A uniform wire AB of weight $2W$ is bent into a semi-circular arc. A particle of weight W is attached to the end B and the system is freely suspended from A. Find the tangent of the angle that AB makes with the vertical.

13. Three uniform rods AB, BC, CD, each of length $2a$ and weight W are rigidly joined together to form three sides of a square $ABCD$. If the system hangs freely from a smooth pivot at A, find the angle that AB makes with the vertical. Find also, in terms of W and a, the moment of the couple required to maintain the system in equilibrium with BC horizontal.

14. $ABCD$ is a uniform square lamina of side $6a$ and weight W. P is a point on the diagonal AC, such that $AP = 2PC$. The square having diagonal PC is cut away and the remaining body is freely suspended from B. Find the angle which BA makes with the vertical. Find also the moment of the couple required to maintain the body in equilibrium with BA horizontal.

15. $ABCD$ is a uniform rectangular lamina in which $BC = 2a$. P is the mid-point of BC and R is a point on AB such that $AR = a$ and $RB = ka$. The rectangle $BPQR$ is cut away and the remaining body is placed in a vertical plane with AR resting on a horizontal plane. Find the maximum value of k given that the lamina does not topple about R.

16. A uniform lamina of weight $3W$ in the form of a trapezium $ABCD$ has $\angle ABC = \angle BCD = 90°, AB = BC = a$ and $CD = 2a$. The lamina is placed in a vertical plane with AB resting on a horizontal plane. Find, in terms of W, the greatest weight that can be attached to the lamina at D without causing it to topple about A.

17. A solid hemisphere of radius a and a solid right circular cone of height h and base radius a are made from the same uniform material and are joined together with their plane faces completely in contact. O is the centre of the common base. Find the position of the centre of gravity of the whole body.

(a) The body is suspended freely from a point A on the edge of the common base and hangs in equilibrium under gravity. If $h = 3a$, find, correct to the nearest degree, the acute angle made by AO with the vertical.
(b) The body is found to rest in equilibrium when it is placed on a horizontal table with *any* point of its hemispherical surface in contact with the table. Find h in terms of a. (O&C)

18. A uniform thin hemispherical bowl rests in equilibrium with its axis of symmetry horizontal and its curved surface on a rough inclined plane. Find the angle between the plane and the horizontal. Find also the least possible value of the coefficient of friction between the bowl and the plane.

19. A uniform solid hemisphere rests in equilibrium with its curved surface on a rough plane inclined at an angle α to the horizontal, where $\sin\alpha = 0{\cdot}3$. Find the angle between the plane face of the hemisphere and the vertical.

20. A uniform right circular solid cone of radius a and height $3a$ rests in equilibrium with its base in contact with a rough inclined plane. The angle of inclination of the plane is then increased steadily.
(a) Assuming that the cone does not slide down the plane, find the angle of inclination of the plane to the horizontal when the cone is about to topple.
(b) If the coefficient of friction between the cone and the plane is $\frac{1}{2}$, find the angle of inclination of the plane to the horizontal when the cone is about to slide.

21. A uniform solid cylinder of radius a and height h is placed with one plane face resting on a rough inclined plane. The angle of inclination of the plane is then gradually increased. Show that the cylinder will slide before it topples if the coefficient of friction between the cylinder and the plane is less than $\frac{2a}{h}$.

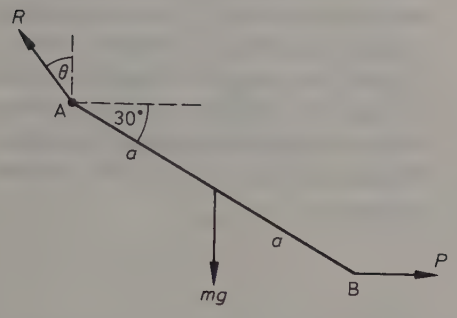
6.2 Three force problems

When a body is in equilibrium under the action of *three forces*, the vector sum of these forces must be zero. Hence three non-parallel forces in equilibrium can be represented in magnitude and direction by the sides of a triangle. We also know that the sum of the moments of coplanar forces in equilibrium about any point in their plane is zero.

Thus one approach to three force problems is to
(i) take moments about any suitable point, and
(ii) use a triangle of forces.

Example 1 A uniform rod AB of mass m, which is smoothly hinged at A, is maintained in equilibrium by a horizontal force P acting at B. Given that the rod is

inclined at 30° to the horizontal with B below A, find (a) an expression for P, (b) the magnitude and direction of the reaction at the hinge.



[Note that the reaction at a *smooth hinge* is a single force which may act in any direction. The system of reactions in a hinge which is *not smooth* may reduce to a single force together with a couple.]

Let the length of the rod be $2a$ and let the reaction at the hinge be of magnitude R acting at an angle θ to the vertical.

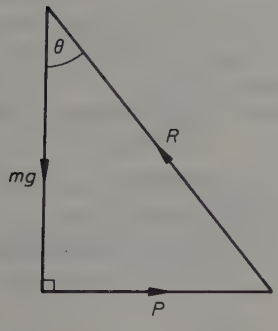
(a) The weight of the rod is mg and acts through the mid-point of the rod. Thus taking moments anti-clockwise about A, we have

$$P \times 2a \sin 30° - mg \times a \cos 30° = 0$$

$$\therefore \qquad Pa - \frac{\sqrt{3}}{2} mga = 0$$

Hence $P = \frac{\sqrt{3}}{2} mg.$

(b)



Using the triangle of forces,

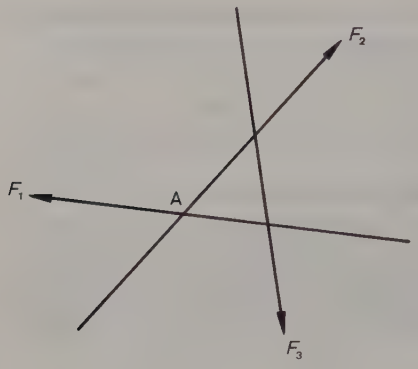
$$R = \sqrt{\{(mg)^2 + P^2\}} = mg\sqrt{(1 + \tfrac{3}{4})}$$

$$= \frac{\sqrt{7}}{2} mg$$

$$\tan\theta = \frac{P}{mg} = \frac{\sqrt{3}}{2} \qquad \therefore \quad \theta \approx 40° 54'$$

Hence the reaction at the hinge has magnitude $\frac{\sqrt{7}}{2} mg$ and makes an angle $\tan^{-1}\frac{\sqrt{3}}{2} \approx 40° 54'$ with the vertical.

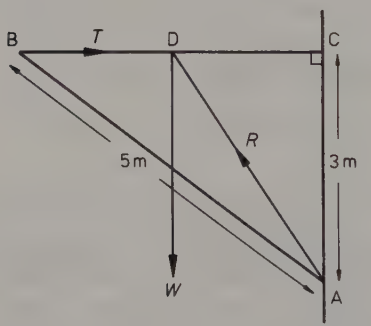
We now establish a property of three force systems which is a useful aid to problem solving. Suppose that a rigid body is in equilibrium under the action of three coplanar forces F_1, F_2 and F_3. Then either all three forces are parallel or the lines of action of two of them meet.



If F_1 and F_2 act along lines which intersect at a point A, then the moments of F_1 and F_2 about A are zero. Since F_1, F_2 and F_3 are in equilibrium, the moment of F_3 about A must also be zero. Hence the line of action of F_3 passes through A. We may deduce the following general principle.

> If a rigid body is in equilibrium under the action of three coplanar forces, then the lines of action of the forces are either parallel or concurrent.

Example 2 A uniform pole AB of length 5 m is smoothly hinged at A and supported at B by a light elastic string BC of natural length 3 m and modulus 60 N. The end C of the string is fixed 3 m vertically above A. Given that the system is in equilibrium with BC horizontal, find in newtons the tension T in the string, the weight W of the pole and magnitude R of the reaction at A.



The lines of action of T and W meet at D, the mid-point of BC,

$\therefore$ the third force R must also act through D.

By Pythagoras' theorem in $\triangle ABC$,

$$BC = 4\,\text{m},$$

$\therefore$ the extension in the elastic string

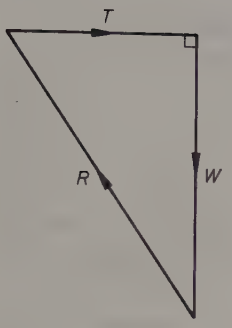
$$= (4 - 3)\,\text{m} = 1\,\text{m}.$$

By Hooke's law, $T = \dfrac{\lambda x}{l}$

where $\lambda = 60$, $l = 3$ and $x = 1$,

$$\therefore \quad T = \frac{60 \times 1}{3} = 20.$$

Hence the tension in the elastic string is 20 N.



Triangle of forces

Comparing the diagrams we see that the triangle of forces is similar to $\triangle ADC$. By Pythagoras' theorem in $\triangle ADC$,

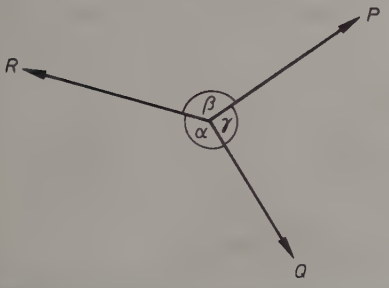
$$AD = \sqrt{(AC^2 + DC^2)} = \sqrt{(3^2 + 2^2)}\,\text{m} = \sqrt{13}\,\text{m},$$

$$\therefore \quad \frac{T}{2} = \frac{W}{3} = \frac{R}{\sqrt{13}}$$

$$\therefore \quad W = \frac{3}{2}T = 30$$

and $$R = \frac{\sqrt{13}}{2}T = 10\sqrt{13} \approx 36{\cdot}1.$$

Hence the weight of the pole is 30 N and the magnitude of the reaction at A is 36·1 N.



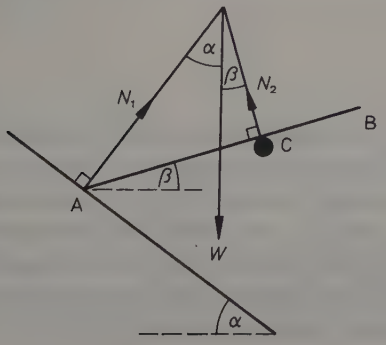
When solving three force problems in which angles rather than lengths are known, *Lami's theorem* may be useful. It states that for forces P, Q and R in equilibrium, as shown in the diagram.

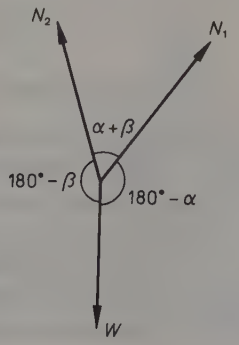
$$\frac{P}{\sin\alpha} = \frac{Q}{\sin\beta} = \frac{R}{\sin\gamma}$$

Example 3 A uniform rod AB of length $5a$ and weight W rests with the end A on a smooth plane inclined at an angle α to the horizontal, where $\sin\alpha = 3/5$. It is maintained in equilibrium by a smooth support at C. If the rod makes an angle β with the horizontal, where $\sin\beta = 7/25$, find

(a) the magnitudes of the reactions at A and C,
(b) the distance AC.

The reaction at A is perpendicular to the inclined plane and the reaction at C is perpendicular to the rod. Let their magnitudes be N_1 and N_2 respectively. Since the forces N_1, N_2 and W are in equilibrium, their lines of action are concurrent.





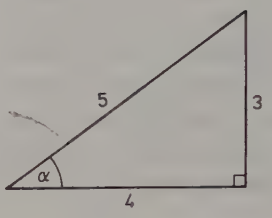
(a) By Lami's theorem, $\dfrac{N_1}{\sin(180° - \beta)} = \dfrac{N_2}{\sin(180° - \alpha)} = \dfrac{W}{\sin(\alpha + \beta)}$

$$\therefore \quad \frac{N_1}{\sin\beta} = \frac{N_2}{\sin\alpha} = \frac{W}{\sin(\alpha + \beta)}$$

$$\sin\alpha = \frac{3}{5} \Rightarrow \cos\alpha = \frac{4}{5}$$

$$\sin\beta = \frac{7}{25} \Rightarrow \cos\beta = \frac{24}{25}$$

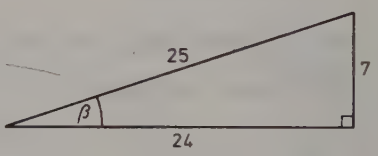
(1)



$$\therefore \quad \sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$= \frac{3}{5} \times \frac{24}{25} + \frac{4}{5} \times \frac{7}{25}$$

$$= \frac{72 + 28}{125} = \frac{100}{125} = \frac{4}{5}$$



Substituting these values in equation (1),

$$\frac{25}{7}N_1 = \frac{5}{3}N_2 = \frac{5}{4}W$$

$$\therefore \quad N_1 = \frac{7}{25} \times \frac{5}{4}W = \frac{7}{20}W \quad \text{and} \quad N_2 = \frac{3}{5} \times \frac{5}{4}W = \frac{3}{4}W$$

Hence the magnitudes of the reactions at A and C are $7W/20$ and $3W/4$ respectively.

(b) The distance of the centre of gravity of the rod from A is $5a/2$. Let the distance AC be x.

Taking moments anti-clockwise about A,

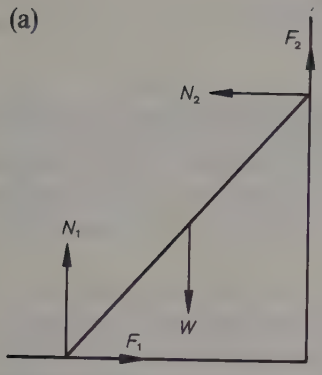
$$N_1 \times 0 + N_2 \times x - W \times \frac{5}{2}a\cos\beta = 0$$

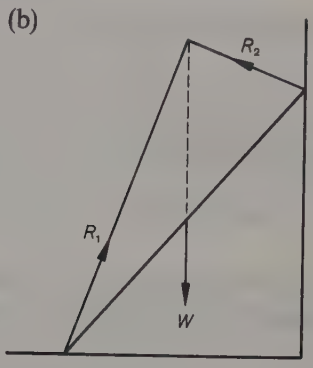
$$\therefore \quad \frac{3}{4}Wx - \frac{5}{2} \times \frac{24}{25}Wa = 0$$

$$\therefore \quad x = \frac{4}{3} \times \frac{5}{2} \times \frac{24}{25}a = \frac{16}{5}a$$

Hence the distance AC is $16a/5$.

In problems involving contact between rough surfaces the reaction at any point of contact can be treated as the resultant of the normal reaction and a frictional force or as a single force. For instance, a problem concerning a ladder resting on a rough horizontal floor and against a rough vertical wall may be solved using diagram (a) and the general methods to be described in §6.3. Alternatively, as indicated in diagram (b), the set of forces on the ladder can be reduced to a three force system by using the total reactions at the points of contact with the floor and the wall.

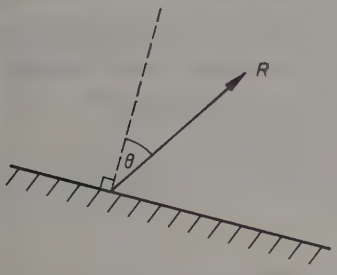




When using this latter approach, it is helpful to remember that if θ is the angle between the total reaction and the normal to the surfaces in contact, then

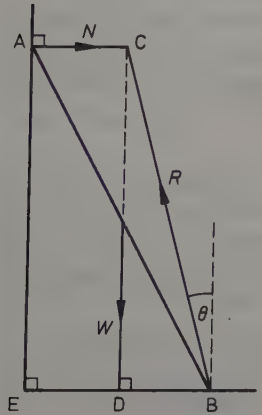
$$\tan\theta \leqslant \mu \quad \text{and} \quad \theta \leqslant \lambda,$$

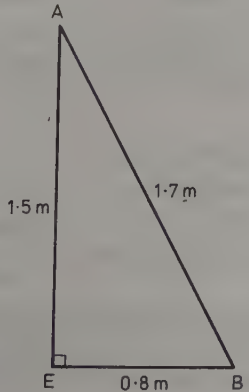
where μ is the coefficient of friction and λ is the angle of friction.

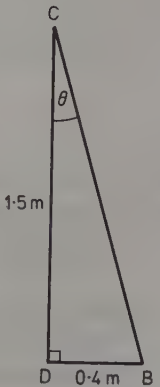


Example 4 A uniform rod AB of length 1·7 m is placed with the end A against a smooth vertical wall and the end B on rough horizontal ground at a distance of 0·8 m from the wall. If the coefficient of friction between the rod and the ground is μ, find the least value of μ for which equilibrium is possible.

If the rod is in equilibrium under the action of its weight W, the normal reaction N at A and the reaction R at B, then the lines of action of these three forces must be concurrent. Let C be their point of intersection and let θ be the angle the reaction R makes with the vertical.







By Pythagoras' theorem in $\triangle ABE$, $AE^2 + EB^2 = AB^2$

$\therefore \quad AE = \sqrt{\{(1{\cdot}7)^2 - (0.8)^2\}}\,\text{m} = 1{\cdot}5\,\text{m}.$

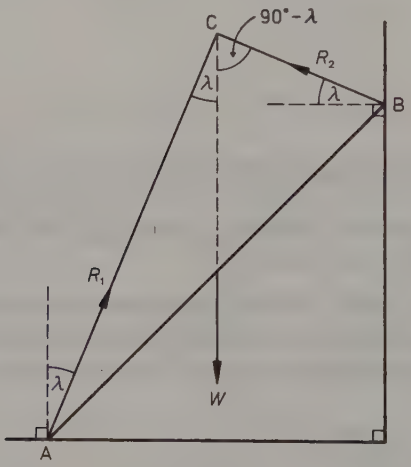
Since the weight of the rod acts vertically through its mid-point, D must be the mid-point of EB,

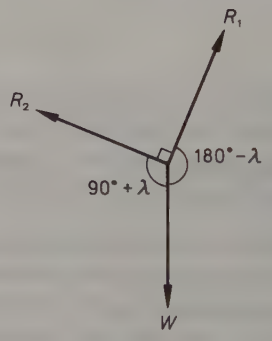
$$\therefore \quad \text{using } \triangle BCD, \quad \tan\theta = \frac{0{\cdot}4}{1{\cdot}5} = \frac{4}{15}.$$

Equilibrium is possible provided that $\tan\theta \leqslant \mu$.
Hence the least possible value of μ is 4/15.

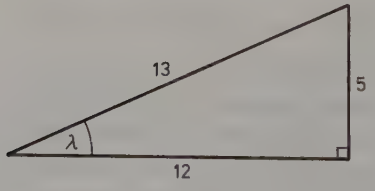
Example 5 A uniform ladder AB of weight W rests in limiting equilibrium with the end A on rough horizontal ground and the end B against a rough vertical wall. If the coefficient of friction at each end of the ladder is 5/12, find the magnitudes of the reactions at A and B.

Since the ladder is in limiting equilibrium, the reaction R_1 at A acts at an angle λ to the vertical and the reaction R_2 at B acts at an angle λ to the horizontal, where $\tan\lambda = 5/12$. If the lines of action of R_1 and R_2 meets at C, then the weight of the ladder also acts through C and $\angle ACB = \lambda + (90° - \lambda) = 90°$.





By Lami's theorem, $\dfrac{R_1}{\sin(90° + \lambda)} = \dfrac{R_2}{\sin(180° - \lambda)} = \dfrac{W}{\sin 90°}$



$$\therefore \quad \frac{R_1}{\cos\lambda} = \frac{R_2}{\sin\lambda} = W$$

Since $\tan\lambda = \frac{5}{12}$,

$\cos\lambda = \frac{12}{13}$ and $\sin\lambda = \frac{5}{13}$,

$\therefore \quad R_1 = W\cos\lambda = \frac{12}{13}W$ and $R_2 = W\sin\lambda = \frac{5}{13}W$.

Hence the reactions at A and B are $12W/13$ and $5W/13$ respectively.

Exercise 6.2

[Take g as $10\,\mathrm{m\,s^{-2}}$.]

1. A uniform rod AB of weight 20 N is smoothly hinged at B and maintained in equilibrium by a horizontal force of P N acting at A. If the rod is inclined at 45° to the horizontal with A above B, find (a) the value of P, (b) the magnitude and direction of the reaction at the hinge.

2. A non-uniform bar AB of length $2a$ and mass m is maintained in a horizontal position by two strings AC and BC attached to a fixed point C above the bar. If $\angle CAB = 60°$ and $\angle ABC = 30°$, find expressions for the tensions in the strings. Find also the distance of the centre of gravity of the bar from A.

3. A uniform rod AB of length 40 cm and weight 8 N rests on a rough peg at C, where $AC = 15$ cm, and against a smooth vertical wall at A. If the rod is in equilibrium at an angle α to the horizontal, where $\sin\alpha = 4/5$, find the magnitudes of the reactions at A and at C.

4. A uniform bar AB of length 2 m and mass 3 kg rests on a smooth peg at X, where $AX = 0{\cdot}5$ m and against a rough vertical wall at A. If the bar is in equilibrium at 60° to the horizontal, find the magnitude and direction of the force exerted by the wall on the bar.

5. A sphere of radius 40 cm and weight 30 N rests in equilibrium on a smooth plane inclined at an angle α to the horizontal, where $\tan\alpha = 8/15$. A horizontal string joins a point on the sphere to a fixed point on the plane. Find the length of the string and the tension in the string.

6. A rod AB of length 40 cm, mass 6 kg and centre of mass G, is suspended by a light inextensible string AXB of length 80 cm, which passes over a smooth peg at X. Find the tension in the string and the angle the rod makes with the vertical when it hangs in equilibrium if (a) $AG = GB$, (b) $AG:GB = 3:5$.

7. A uniform rod of weight W is in equilibrium with one end resting against a smooth vertical plane and the other end against a smooth plane inclined at an angle α to the horizontal, where $\tan\alpha = 5/12$. The line of intersection of the planes is horizontal. If the rod makes an angle θ with the vertical, find $\tan\theta$. Find also the magnitudes of the reactions at the vertical and inclined planes.

8. A non-uniform rod AB of weight $3W$ rests with the end A on a smooth plane inclined at 30° to the horizontal. It is maintained in equilibrium at 30° to the horizontal by a smooth peg at C, where $AC = 3a$. Find (a) the magnitudes of the reactions at A and C, (b) the distance of the centre of gravity of the rod from A.

9. A uniform rod AB of length 1 m is smoothly hinged at A and the end B is attached by means of a light elastic string of natural length 1 m, modulus 20 N to a point C which is 1 m vertically above A. Given that the tension in the string is 12 N,

find (a) the weight of the rod, (b) the magnitude and direction of the reaction at the hinge.

10. A uniform rod AB of weight $4W$ rests in equilibrium with the end A against a smooth vertical wall. A light elastic string XY of natural length 24 cm, modulus $15W$ is attached to a point X vertically above A and a point Y on the rod. Given that $XY = 32$ cm and $\angle XYB = 90°$, find the tension in the string and the length of the rod.

11. A smooth hemispherical bowl of radius $5\sqrt{5}$ cm is fixed with its rim horizontal. A uniform rod AB is in equilibrium with the end A resting inside the bowl and a point C of the rod against the rim, where $AC = 20$ cm. If the rod is inclined at an angle α to the horizontal, find $\tan \alpha$. Find also the length of the rod.

12. A uniform ladder of length 6·5 m rests with one end against a smooth vertical wall and the other end on rough horizontal ground at a distance of 2·5 m from the wall. If the foot of the ladder is about to slip, find the coefficient of friction between the ladder and the ground.

13. A uniform pole of length $2a$ and mass m rests in equilibrium at an angle α to the horizontal, with one end against a smooth vertical wall and the other end on rough horizontal ground. Find expressions for (a) the force exerted on the pole by the wall, (b) the least possible value of the coefficient of friction between the pole and the ground.

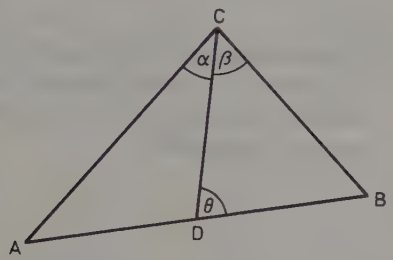
14. A uniform ladder of weight W and length $2l$ rests with its upper end against a smooth vertical wall. Its lower end stands on rough horizontal ground, and the coefficient of friction between the ladder and the ground is $\frac{1}{2}$. The ladder is in limiting equilibrium. Find the angle the ladder makes with the horizontal. Find also the reaction at the wall. A man of weight W climbs the ladder. Find how far up it he can go before the ladder will slip. Find also how far up the ladder he can go when a load of weight W is placed on the foot of the ladder. (L)

15. A uniform rod AB of length a and weight W has a light ring attached to the end A. The ring is free to slide along a straight wire inclined at 30° to the horizontal. The rod is maintained in equilibrium by a light inextensible string of length a attached to the end B. The other end of the string is attached to the wire at a fixed point C above A. (a) If the wire is smooth, find the angle θ which the string makes with the vertical. (b) If the string is vertical, find the least value of the coefficient of friction between the ring and the wire.

16. A heavy uniform rod of length $2a$ rests in equilibrium against a small smooth horizontal peg with one end of the rod on a rough horizontal floor. If the height of the peg above the floor is b and friction is limiting when the rod makes an angle $\frac{\pi}{6}$ with the floor, show that the coefficient of friction between the floor and the rod is $\frac{\sqrt{3}a}{(8b - 3a)}$. (O)

17. A uniform rod AB of length $10a$ rests in limiting equilibrium with the end A on horizontal ground and the end B against a vertical wall. The coefficient of friction at each end of the rod is μ. Given that A is at a distance $6a$ from the wall, find the value of μ.

18. (a)



In the given diagram D is the mid-point of AB. Using the sine rule or otherwise prove that

$$2\cot\theta = \cot\alpha - \cot\beta.$$

(b) A uniform ladder rests at an angle θ to the vertical, with one end on horizontal ground and the other end against a vertical wall. The coefficients of friction between the ladder and the ground and between the ladder and the wall are μ_1 and μ_2 respectively. Given that the ladder is on the point of slipping, use part (a) to prove that $\tan\theta = 2\mu_1/(1-\mu_1\mu_2)$.

6.3 General conditions for equilibrium

We have already established that if a body is in equilibrium under the action of a set of coplanar forces, then the sum of their components in any direction is zero and the sum of their moments about any point is zero. In this section we consider sets of conditions that are sufficient to ensure equilibrium.

Suppose that a system of coplanar forces is such that

(i) the sum of the components in a direction d_1 is zero,
(ii) the sum of the components in a direction d_2, not parallel to d_1, is zero, and
(iii) the sum of the moments about a point A in the plane is zero.

This system of forces is not equivalent to a single force, because no force can have zero components in two directions. The system does not reduce to a couple, because a couple has the same non-zero moment about every point in its plane. Thus, if a set of forces acting on a rigid body satisfies conditions (i), (ii) and (iii), it follows that the body is in equilibrium.

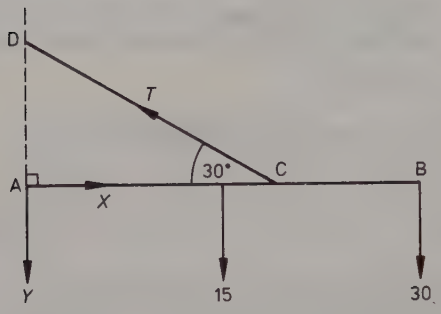
We can use this set of equilibrium conditions to solve problems involving forces in a *vertical plane* by

(i) resolving horizontally,
(ii) resolving vertically, and
(iii) taking moments about a suitable point.

In this way we obtain up to three equations which we then solve to find any unknown quantities.

Example 1 A uniform rod AB of weight 15 N and length 4 m is smoothly hinged at A and has a particle of weight 30 N attached to it at B. A light inextensible string,

attached to the rod at a point C, where $AC = 2{\cdot}5$ m, and to a point D vertically above A, keeps the rod in a horizontal position. If the angle between the rod and the string is 30°, find (a) the tension in the string, (b) the magnitude and direction of the reaction at the hinge.



Let the tension in the string be T N and let the horizontal and vertical components of the reaction at the hinge be X N and Y N respectively.

(a) Taking moments anti-clockwise about A,

$$T \times 2{\cdot}5 \sin 30° - 15 \times 2 - 30 \times 40 = 0$$

i.e. $$T \times \tfrac{5}{2} \times \tfrac{1}{2} - 30 - 120 = 0$$

$$\therefore \quad T = 150 \times \tfrac{4}{5} = 120.$$

Hence the tension in the string is 120 N

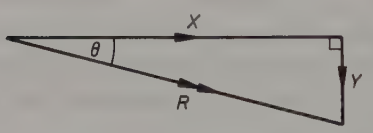
(b) Resolving horizontally and vertically,

$$\rightarrow \qquad X - T\cos 30° = 0 \qquad (1)$$

$$\downarrow \qquad Y + 15 + 30 - T\sin 30° = 0 \qquad (2)$$

From (1), $X = 120 \times \dfrac{\sqrt{3}}{2} = 60\sqrt{3}$

From (2), $Y = 120 \times \tfrac{1}{2} - 45 = 15$



If the resultant reaction R acts at an angle θ to the horizontal, then

$$R^2 = X^2 + Y^2, \tan\theta = \frac{Y}{X}.$$

$$\therefore \quad R = \sqrt{\{(60\sqrt{3})^2 + 15^2\}} = 15\sqrt{\{(4\sqrt{3})^2 + 1^2\}} = 15\sqrt{(48 + 1)} = 105,$$

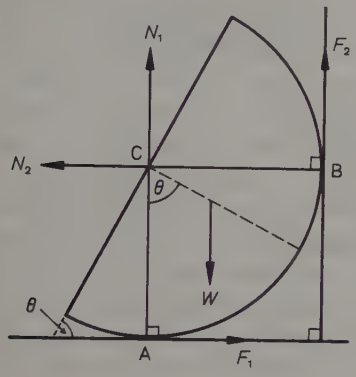
$$\tan\theta = \frac{15}{60\sqrt{3}} = \frac{1}{4\sqrt{3}} \quad \text{i.e.} \quad \theta \approx 8° 13'.$$

Hence the reaction at the hinge has magnitude 105 N and acts at 8° 13′ to the horizontal.

When a rigid body is kept in equilibrium by frictional forces, then at each point of contact with a rough surface the frictional force F acts along the surface, opposing any tendency to slide. It also satisfies the condition $F \leqslant \mu N$, where N is the normal

reaction and μ is the coefficient of friction. If the body is just about to slip, then friction is limiting at *every* point of contact and in each case $F = \mu N$.

Example 2 A uniform solid hemisphere rests in limiting equilibrium with its curved surface in contact with a horizontal floor and a vertical wall. The coefficient of friction at the floor is $\frac{1}{4}$ and at the wall is $\frac{1}{2}$. If θ is the angle of inclination of the plane face of the hemisphere to the horizontal, find $\sin\theta$.



Let the hemisphere have centre C and radius a. Let the forces on it be as shown in the diagram. Since the hemisphere is in limiting equilibrium, friction must be limiting at both A and B,

$\therefore \quad F_1 = \frac{1}{4}N_1, \quad F_2 = \frac{1}{2}N_2.$

Resolving horizontally and vertically,

$$\rightarrow \qquad F_1 - N_2 = 0 \qquad (1)$$
$$\uparrow \qquad N_1 + F_2 - W = 0 \qquad (2)$$

Substituting for F_1 in (1):

$$\tfrac{1}{4}N_1 - N_2 = 0 \qquad \therefore \quad N_1 = 4N_2$$

Substituting for N_1 and F_2 in (2):

$$4N_2 + \tfrac{1}{2}N_2 - W = 0 \qquad \therefore \quad N_2 = \tfrac{2}{9}W$$

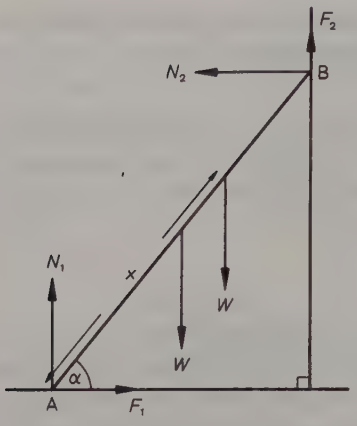
Hence $F_1 = N_2 = \frac{2}{9}W$ and $F_2 = \frac{1}{2}N_2 = \frac{1}{9}W$.

The centre of gravity of the hemisphere is on its axis of symmetry at a distance $\frac{3}{8}a$ from C.
Thus, taking moments clockwise about C,

$$W \times \tfrac{3}{8}a\sin\theta - F_1 \times a - F_2 \times a = 0$$
$$\therefore \qquad \tfrac{3}{8}aW\sin\theta - \tfrac{2}{9}aW - \tfrac{1}{9}aW = 0$$

Hence $\frac{3}{8}\sin\theta - \frac{3}{9} = 0$ i.e. $\sin\theta = \frac{8}{9}$.

Example 3 A uniform ladder AB of length $2a$ and weight W rests with the end A on horizontal ground and the end B against a vertical wall. The coefficients of friction at A and B are $\frac{2}{5}$ and $\frac{1}{2}$ respectively. The ladder makes an angle α with the horizontal, where $\tan\alpha = \frac{5}{4}$. If a man of weight W starts to climb the ladder, find his distance from A when the ladder begins to slip.



Let the reactions at A and B when the ladder is about to slip be as shown in the diagram. Since friction must be limiting at both A and B,

$$F_1 = \tfrac{2}{5}N_1, \quad F_2 = \tfrac{1}{2}N_2.$$

Resolving horizontally and vertically,

$$\rightarrow \qquad F_1 - N_2 = 0 \qquad (1)$$
$$\uparrow \qquad N_1 + F_2 - 2W = 0 \qquad (2)$$

Substituting for F_1 in (1),

$$\tfrac{2}{5}N_1 - N_2 = 0 \qquad \therefore \quad N_1 = \tfrac{5}{2}N_2$$

Substituting for N_1 and F_2 in (2),

$$\tfrac{5}{2}N_2 + \tfrac{1}{2}N_2 - 2W = 0 \qquad \therefore \quad N_2 = \tfrac{2}{3}W$$

Hence $F_2 = \tfrac{1}{2}N_2 = \tfrac{1}{3}W$

Let the man be a distance x from A when the ladder begins to slip.
Taking moments clockwise about A,

$$W \times x\cos\alpha + W \times a\cos\alpha - F_2 \times 2a\cos\alpha - N_2 \times 2a\sin\alpha = 0$$

Dividing by $\cos\alpha$,

$$Wx + Wa - 2F_2a - 2N_2a\tan\alpha = 0$$
$$\therefore \quad Wx + Wa - \tfrac{2}{3}Wa - \tfrac{4}{3}Wa \times \tfrac{5}{4} = 0$$
$$\therefore \quad x = \tfrac{5}{3}a + \tfrac{2}{3}a - a = \tfrac{4}{3}a.$$

Hence the man is at a distance of $4a/3$ from A when the ladder begins to slip.

The set of conditions discussed earlier, for the equilibrium of a rigid body under the action of coplanar forces, involved the sums of components in two directions and the sum of moments about one point. We now consider whether any other combinations of similar conditions will also ensure equilibrium.

To be certain that a set of forces is not equivalent to a couple, we need to know that the system has zero moment about at least one point. A system with zero moments about two points A and B may have a resultant whose line of action passes through A and B. We can rule out this possibility if we find that the components of the forces in some direction, not perpendicular to AB, have zero sum or that the system has zero moment about a third point not on AB.

We have then three sets of equilibrium conditions. A system of coplanar forces must be in equilibrium if

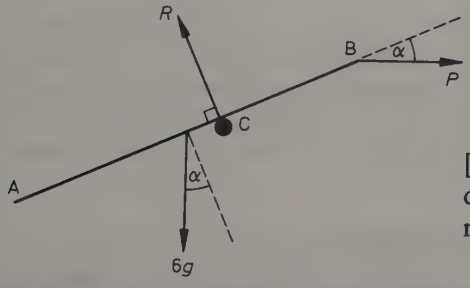
(a) the sums of the components in two non-parallel directions are each zero, and the sum of the moments about one point is zero;
or (b) the sums of the moments about two points A and B are each zero, and the sum of the components in a direction, not perpendicular to AB, is zero;
or (c) the sum of the moments about three points, not in a straight line, are each zero.

Thus when tackling problems we may
(a) resolve in two directions and take moments about one point;
or (b) take moments about two points A and B then resolve in a direction not perpendicular to AB;
or (c) take moments about three points which are not collinear.

Since each method involves putting three different restrictions on the system of forces concerned, we will obtain three *independent* equations connecting any unknown quantities i.e. three equations each giving information which could not be obtained from the other two. [Note that since these three equations ensure the equilibrium of the system, no new information can be gained from further equations formed by resolving or taking moments.]

The following examples show that, by careful choice of the points about which we take moments and of the directions in which we resolve, we may
(1) avoid equations involving quantities that we do not need to find,
(2) form equations containing as few unknowns as possible.

Example 4 A uniform rod AB of mass 6 kg rests on a smooth peg at a point C. The rod is kept in equilibrium at an angle α to the horizontal, where $\tan\alpha = 5/12$, by a horizontal force of P newtons acting at the upper end B of the rod. Find the value of P.
[Take g as $10\,\mathrm{m\,s^{-2}}$.]



[We try to avoid equations involving distances or the reaction at C, as we do not need to find these quantities.]

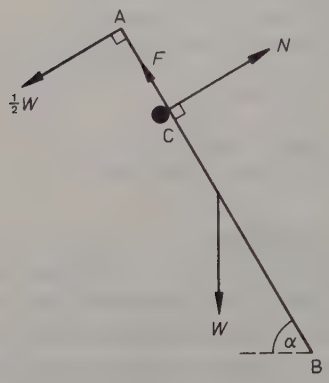
Resolving along the rod in the direction AB,

$$P\cos\alpha - 6g\sin\alpha = 0$$

$$\therefore \quad P = 6g\tan\alpha = 6 \times 10 \times \tfrac{5}{12} = 25.$$

Example 5 A uniform bar AB of weight W and length $4a$ rests on a rough peg at a point C, where $AC = a$. The bar is maintained in equilibrium, with A above B, by a force of magnitude $\frac{1}{2}W$ acting at A in a direction perpendicular to the bar. Find the

angle α, which the bar makes with the horizontal. Given that the bar is about to slip, find the coefficient of friction μ between the bar and the peg.



[In this example we try to form equations which simplify the working as much as possible.]

Taking moments clockwise about C,

$$W \times a\cos\alpha - \tfrac{1}{2}W \times a = 0 \qquad \therefore \cos\alpha = \tfrac{1}{2}$$

Hence $\alpha = 60°$.

Let F be the frictional force at C and let N be the normal reaction.
Taking moments clockwise about the mid-point of AB,

$$N \times a - \tfrac{1}{2}W \times 2a = 0 \qquad \therefore \quad N = W$$

Resolving along the bar in the direction BA,

$$F - W\sin\alpha = 0 \qquad \therefore \quad F = \frac{\sqrt{3}}{2}W$$

Since the bar is about to slip, $F = \mu N$.

Hence $\mu = \dfrac{F}{N} = \dfrac{\sqrt{3}}{2}$.

Exercise 6.3

[Take g as $10\,\mathrm{m\,s^{-2}}$.]

1. A uniform rod AB of weight 20 N and length 2 m is freely hinged at A. The rod is held in equilibrium at 30° to the horizontal, with B higher than A, by a light inextensible string of length 1·25 m. The string joins a point C on the rod to a fixed point D at a distance of 1·25 m vertically above A. If a particle of weight 5 N is suspended from B, find (a) the tension in the string, (b) the horizontal and vertical components of the reaction at A.

2. A uniform rod AB of mass 6 kg and length 3 m is freely hinged at A. A load of mass 2 kg is attached to the rod at B. The rod is kept in equilibrium by a light inextensible string CD of length 1·5 m, which is attached to a point C on the rod and

to a fixed point D on the same horizontal level as A. Given that $AC = 2\,\text{m}$ and $AD = 2{\cdot}5\,\text{m}$, find (a) the tension in the string, (b) the magnitude and direction of the reaction at the hinge.

3. A uniform square lamina $ABCD$ of mass 6 kg is hinged at A so that it is free to move in a vertical plane. It is maintained in equilibrium, with B vertically below A, by a horizontal force acting at C and a vertical force acting at D each of magnitude P newtons. Find (a) the value of P, (b) the magnitude and direction of the force exerted by the hinge on the lamina.

4. A uniform rod AB of weight 16 N is freely hinged at A. It is kept in equilibrium at an angle of 30° to the horizontal, with B lower than A, by a force acting at B. Find the magnitude and direction of the least force required at B to maintain equilibrium. Find also the corresponding horizontal and vertical components of the reaction at A.

5. A uniform rod AB of length 3 m and weight 40 N rests in equilibrium with the lower end A on a smooth plane inclined at an angle α to the horizontal, where $\tan\alpha = 3/4$, and a point C of the rod against a smooth peg. If the rod makes an angle θ with the horizontal, where $\tan\theta = 7/24$, find (a) the magnitudes of the reactions at A and C, (b) the distance AC.

6. A uniform rod AB of mass 13 kg is in limiting equilibrium with its lower end A on a rough horizontal plane and its upper end B attached to a light inextensible string inclined at an angle θ to the upward vertical, where $\tan\theta = 12/5$. The string passes over a smooth fixed pulley at a point vertically above A and supports at its other end a mass of 6·5 kg. Find (a) the coefficient of friction between the rod and the plane, (b) the angle the rod makes with the horizontal.

7. A smooth hemisphere of radius 60 cm is fixed with its plane face on a rough horizontal floor. A uniform rod AB of length 1 m and weight 10 N rests in equilibrium in a vertical plane with the end A on the floor and a point C of the rod in contact with the hemisphere, where $AC = 80\,\text{cm}$. Given that the rod is on the point of slipping, find the reaction at C and the coefficient of friction between the rod and the floor.

8. A uniform ladder AB of length 5 m and mass 12 kg rests with the end A on rough horizontal ground and the end B in contact with a smooth vertical wall. The foot of the ladder is at a distance of 1·4 m from the wall. Find the magnitudes of the reactions at A and B. If the ladder is about to slip, find the coefficient of friction between the ladder and the ground.

9. A uniform beam AB of length $2a$ and weight W rests with its end A against a smooth vertical wall and B on a rough horizontal plane. The vertical plane through AB is perpendicular to the wall. When it makes an angle of α with the wall the beam is on the point of slipping. Find the coefficient of friction between the beam and the horizontal plane. Find, also, what horizontal force applied to the mid-point of the

beam (in the vertical plane through AB) will just cause the beam to slip at B towards the wall. (SU)

10. A thin uniform hemispherical bowl, of weight W and radius a, rests in limiting equilibrium with its plane face at an angle θ to the horizontal. Its curved surface is in contact with a rough horizontal floor and a rough vertical wall. If the coefficient of friction at each point of contact is $\frac{1}{3}$, find the magnitudes of the frictional forces at the floor and the wall. Find also the value of $\sin\theta$.

11. A uniform solid hemisphere rests in limiting equilibrium with its plane face at an angle α to the horizontal, where $\sin\alpha = 8/13$. Its curved surface is in contact with rough horizontal ground and a rough vertical wall. If the coefficient of friction at both points of contact is μ, find the value of μ.

12. A uniform rod AB of weight W and length $4a$ rests with its end A on a rough horizontal plane and a point C of the rod, where $AC = 3a$, in contact with a smooth peg. Given that the rod makes an angle θ with the horizontal, where $\sin\theta = 3/5$, find the magnitude of the reaction at C and the horizontal and vertical components of the reaction at A. If μ is the coefficient of friction between the rod and the horizontal plane, show that $\mu \geqslant 24/43$.

13. A uniform ladder of weight W rests at $60°$ to the horizontal with its foot on rough horizontal ground and the other end against a smooth vertical wall. If the ladder is just about to slip when a man of weight $2W$ stands at the top, find the coefficient of friction between the ladder and the ground. Find the frictional force between the ladder and the ground when the man is standing at the mid-point of the ladder.

14. A uniform ladder of length $2a$ and weight W rests against a smooth vertical wall and has its foot on rough horizontal ground. The coefficient of friction between the ladder and the ground is $\frac{1}{2}$. If the ladder is placed at an angle α to the horizontal, where $\tan\alpha = \frac{3}{2}$, find the distance that a man of weight $3W$ can climb up the ladder before it slips.

15. A uniform pole AB of length 8 m and mass 60 kg rests in limiting equilibrium with the end A on horizontal ground and the end B against a vertical wall. If the coefficient of friction at each end of the pole is $\frac{1}{3}$, find the angle the pole makes with the horizontal. The pole is now placed at an angle α to the horizontal, where $\tan\alpha = 2$, and a body of mass m kg is attached to the pole at B without causing the pole to slip. Find the maximum value of m and the magnitude of the corresponding normal reaction at A.

16. A uniform circular hoop of weight 15 N hangs over a rough horizontal peg A. A horizontal force P N, acting in the plane of the hoop, is applied at the other end B of the diameter through A. If the hoop is on the point of slipping when AB is inclined at

an angle α to the downward vertical, where $\tan\alpha = 4/3$, find the value of P and the coefficient of friction between the hoop and the peg.

17. A uniform rod AB, of weight W and length $6a$, rests with its lower end A on a smooth horizontal plane and a point C of the rod against a smooth peg, where $AC = 4a$. The rod is kept in equilibrium at an angle α to the horizontal by a force of magnitude $\frac{1}{4}W$ acting along the rod. Find the magnitude of the reaction at A and the value of $\sin\alpha$.

18. A uniform ladder, of weight W and length l, rests in a vertical plane with one end against a smooth vertical wall, the wall being perpendicular to the vertical plane through the ladder. The other end of the ladder rests on horizontal ground, the coefficient of friction between the ladder and the ground being 1/4. The ladder is inclined at an angle θ to the horizontal, where $\tan\theta = 24/7$. A man of weight $10W$ climbs up the ladder. Show that the man can reach a height of $6l/7$ above the ground before the ladder begins to slip and calculate the force exerted by the ladder on the wall at this instant. (L)

19. A uniform ladder of length 7 m rests against a vertical wall with which it makes an angle of 45°, the coefficients of friction between the ladder and the wall and the ladder and the (horizontal) ground being $\frac{1}{3}$ and $\frac{1}{2}$ respectively. A boy whose weight is one half that of the ladder slowly ascends the ladder. How far along the ladder will he be when the ladder slips? [Assume the ladder is a uniform rod which is perpendicular to the line of intersection of the wall and ground.] (O&C)

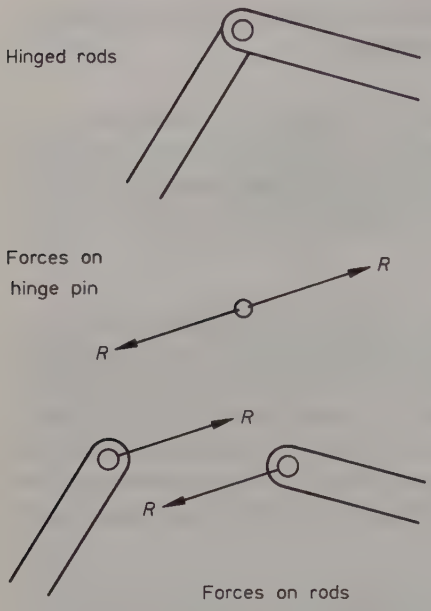
20. A uniform circular disc, of radius a and weight w, stands in a vertical plane upon a rough horizontal floor with the point P of its circumference in contact with an equally rough vertical wall and its plane at right angles to that of the wall. A particle of equal weight w is attached to a point Q of the disc between its centre O and P so that the disc is on the point of slipping. Draw a good diagram indicating clearly the normal reactions and the friction forces acting on the disc. Show that

$$OQ = \frac{2a\mu(1+\mu)}{1+\mu^2},$$

where μ is the coefficient of friction. (W)

6.4 Rigid body systems

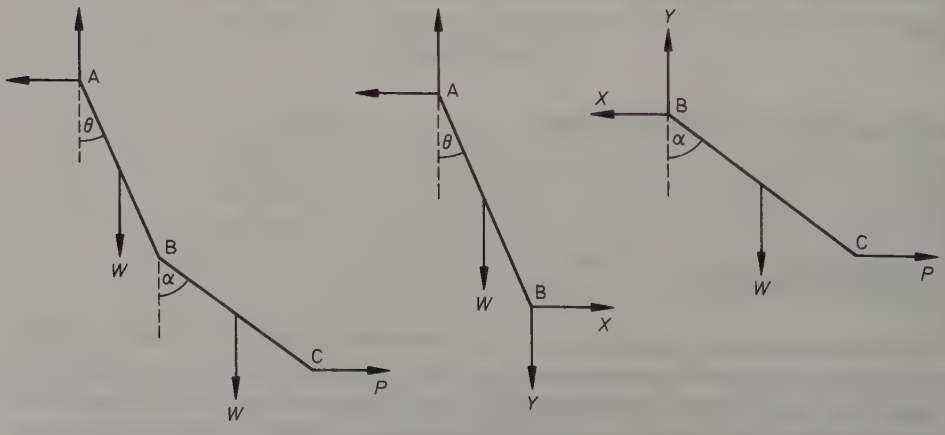
When two or more rigid bodies are in equilibrium under the action of a set of coplanar forces, then the whole system can be treated as a single body in equilibrium under the action of the external forces. Each separate body in the system is in equilibrium under the action of a set of forces which includes both the external forces on the body and the forces exerted on it by the other bodies in the system.



We first consider two rods connected by a smooth light hinge. Each rod exerts a force on the hinge and, according to Newton's third law, the hinge exerts an equal and opposite force on the rod. Thus, provided no external force acts on the hinge, the effect of the hinge is to set up equal and opposite forces on the two rods.

Example 1 Two equal uniform rods AB and BC, each of length $2a$ and weight W, are freely hinged together at B and hang freely from the point A. Equilibrium is maintained by a horizontal force P acting at C. Given that BC makes an angle α with the downward vertical, where $\tan\alpha = 4/3$, find the value of P and the horizontal and vertical components of the force exerted by the rod AB on the rod BC. Find also the angle made by AB with the vertical.

Let the horizontal and vertical components of the reaction at B be X and Y respectively and let θ be the angle AB makes with the vertical.



External forces Forces on AB Forces on BC

Consider first the forces on the rod BC.
Taking moments about B,

$$\circlearrowright \quad P \times 2a\cos\alpha - W \times a\sin\alpha = 0$$
$$\therefore \quad P = \tfrac{1}{2}W\tan\alpha = \tfrac{2}{3}W.$$

Resolving horizontally and vertically,

$$\leftarrow \quad X - P = 0 \qquad \therefore \quad X = P = \tfrac{2}{3}W$$
$$\uparrow \quad Y - W = 0 \qquad \therefore \quad Y = W$$

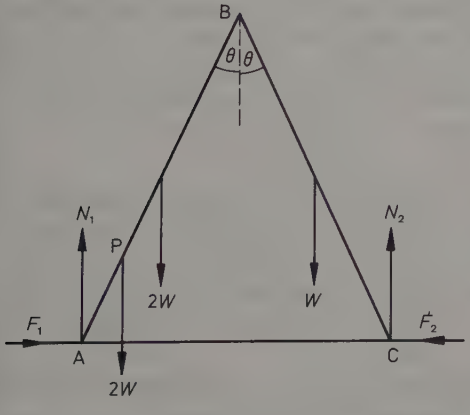
Hence the horizontal and vertical components of the force exerted by the rod AB on the rod BC are $\frac{2}{3}W$ and W respectively.
Consider now the forces on the rod AB.
Taking moments about A,

$$\circlearrowleft \quad W \times a\sin\theta + Y \times 2a\sin\theta - X \times 2a\cos\theta = 0$$
$$\therefore \quad Wa\sin\theta + 2Wa\sin\theta - \tfrac{4}{3}Wa\cos\theta = 0$$

$$\therefore \quad 3Wa\sin\theta = \tfrac{4}{3}Wa\cos\theta$$
$$\therefore \quad \tan\theta = \tfrac{4}{9} \quad \text{and} \quad \theta \approx 23°\,58'.$$

Hence AB makes an angle of 23° 58′ with the vertical.

Example 2 Two uniform rods AB and BC, each of length $2a$, have weights $2W$ and W respectively. The rods are freely jointed at B and rest in a vertical plane with A and C on a rough horizontal plane, the coefficient of friction at both points of contact being $\frac{1}{2}$. A particle of weight $2W$ is attached to a point P of the rod AB, where $AP = \frac{1}{2}a$. Given that the rods are inclined at an angle θ to the vertical, find the maximum value of θ for which equilibrium is possible.



Let the reactions at A and C be as shown in the diagram.

Consider first the forces on the whole system.
Resolving horizontally and vertically,

$$\rightarrow \quad F_1 - F_2 = 0 \qquad (1)$$
$$\uparrow \quad N_1 + N_2 - 5W = 0 \qquad (2)$$

Taking moments about A,

$$\circlearrowright \quad N_2 \times 4a\sin\theta - W \times 3a\sin\theta - 2W \times a\sin\theta - 2W \times \tfrac{1}{2}a\sin\theta = 0$$
$$\therefore \quad 4N_2 - 3W - 2W - W = 0 \quad \text{giving} \quad N_2 = \tfrac{3}{2}W.$$
From (2), $\qquad N_1 = 5W - N_2 = \tfrac{7}{2}W.$

Using the laws of friction, $F_1 \leqslant \mu N_1$ and $F_2 \leqslant \mu N_2$
Since $\mu = \frac{1}{2}$, $F_1 \leqslant \frac{1}{2}W$ and $F_2 \leqslant \frac{3}{4}W$
However, using (1), we may write $F_1 = F_2 = F$
$\therefore$ in any equilibrium position, $F \leqslant \frac{3}{4}W$ (3)
Taking moments about B for the forces on rod BC,

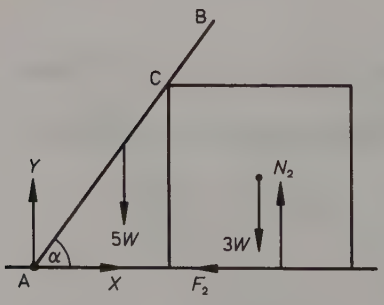
$$\circlearrowright \quad N_2 \times 2a\sin\theta - W \times a\sin\theta - F_2 \times 2a\cos\theta = 0$$
$$\therefore \quad 3Wa\sin\theta - Wa\sin\theta - 2Fa\cos\theta = 0$$
$$\therefore \quad 2Wa\sin\theta = 2Fa\cos\theta$$
Thus $\tan\theta = F/W$
$\therefore$ using (3), $\tan\theta \leqslant 3/4$

Hence the maximum value of θ for which equilibrium is possible is $\tan^{-1}(3/4) \approx 36° 52'$.

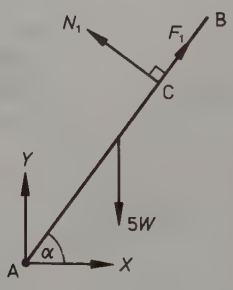
[Note that when θ takes this maximum value, slipping is about to occur at C.]

We now consider problems concerning two bodies in contact. By Newton's third law, the forces exerted by each body on the other are equal and opposite. Thus, as in the case of jointed bodies, unknown quantities can be found by resolving and taking moments, considering either the external forces acting on the system as a whole, or both internal and external forces acting on each body separately. In any given problem we try to simplify the working by avoiding equations involving quantities that we do not need to find.

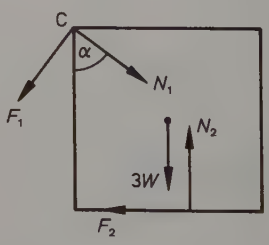
Example 3 A uniform cube of side $12a$ and weight $3W$ rests on a rough horizontal table. A uniform rod AB of length $20a$ and weight $5W$ is freely hinged at A to a fixed point on the table. A point C of the rod rests against the mid-point of one edge of the cube, this edge being perpendicular to the vertical plane containing the rod. When the rod is at an angle α to the horizontal, where $\sin\alpha = 4/5$, the cube is on the point of sliding across the table without toppling. If the coefficient of friction at C is $\frac{1}{2}$, find the coefficient of friction between the cube and the table.



Forces on whole system



Forces on rod



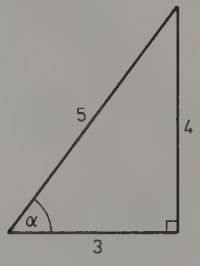
Forces on cube

Let F_1, N_1 and F_2, N_2 be the frictional forces and normal reactions at C and the base of the cube respectively. Consider first the forces on the rod.

Taking moments about A,

$$\circlearrowleft \qquad N_1 \times 12a \operatorname{cosec}\alpha - 5W \times 10a\cos\alpha = 0$$

Using a right-angled triangle,



if $\sin\alpha = \frac{4}{5}$, then $\operatorname{cosec}\alpha = \frac{5}{4}$, $\cos\alpha = \frac{3}{5}$

$\therefore \quad N_1 \times 12a \times \frac{5}{4} - 5W \times 10a \times \frac{3}{5} = 0$

Hence $N_1 = 2W$.
Since the cube is about to slide, $F_1 = \frac{1}{2}N_1 = W$.
Consider now the forces on the cube.
Resolving horizontally and vertically,

$$\leftarrow \qquad F_2 + F_1\cos\alpha - N_1\sin\alpha = 0 \qquad (1)$$

$$\uparrow \qquad N_2 - 3W - F_1\sin\alpha - N_1\cos\alpha = 0 \qquad (2)$$

From (1), $F_2 + \frac{3}{5}W - \frac{8}{5}W = 0 \quad \therefore \quad F_2 = W$

From (2), $N_2 - 3W - \frac{4}{5}W - \frac{6}{5}W = 0 \quad \therefore \quad N_2 = 5W$

Since the cube is on the point of sliding, the coefficient of friction between the cube and the table is given by F_2/N_2 and is equal to 1/5.

Exercise 6.4

1. Two uniform rods AB and BC, of weights 5 N and 10 N respectively, are freely hinged at B and hang freely from a fixed hinge at A. The system is kept in equilibrium by a horizontal force of magnitude 5 N acting at C. Find (a) the magnitudes of the reactions at the hinges A and B, (b) the angles the rods AB and BC make with the vertical.

2. Two equal uniform rods, each of weight W, are freely jointed at B and hang freely from a hinge at A. The end C is attached by means of a light inextensible string to a point D on the same horizontal level as A. When the system is in equilibrium the rod AB is inclined at 45° to the vertical and the rod BC is horizontal. Find the tension in the string and the angle it makes with the vertical.

3. Two uniform rods AB and BC, each of length $2a$, have weights W and $3W$ respectively. The rods are smoothly jointed at B. The ends A and C of the rods are smoothly hinged to fixed points at the same horizontal level and a distance $2a$ apart. If the system is in equilibrium with B below AC, find the horizontal and vertical components of the reaction at B.

4. Two equal uniform rods, AB and AC, each of length $2a$ and weight W, are freely jointed at A. The end C is freely hinged to a point on a rough horizontal plane, the end B rests on the plane and is about to slip, and both rods are in a vertical plane. If the coefficient of friction between B and the plane is μ and the angle ABC is θ, (i) prove that $\mu = \frac{1}{2}\cot\theta$, (ii) find, in terms of μ, the magnitude and direction of the reaction at the hinge at C, (iii) find the magnitude of the reaction at the joint A.
(SU)

5. Two uniform rods AB and BC, freely jointed together at B, are of equal length, but have weights $4W$ and W respectively. The end A is freely hinged to a fixed point on a rough vertical wall and the end C rests against the wall at a point vertically below A. The system rests in limiting equilibrium with each rod inclined at an angle α to the vertical, where $\tan\alpha = 2$. Find (a) the coefficient of friction between the rod BC and the wall, (b) the horizontal and vertical components of the reaction of AB on BC.

6. Two uniform rods AB and AC, of length $2a$, but of weights $2W$ and W respectively, are smoothly jointed at A. The rods rest in a vertical plane with B and C on a rough horizontal plane, the coefficient of friction between the rods and the plane being 0·4. Given that the system is in limiting equilibrium, determine whether slipping is about to occur at B or at C and find angle BAC.

7. Two equal uniform rods, AB and BC, each of mass $4m$, are smoothly hinged together at B and stand in a vertical plane with A and C on a rough horizontal floor. The angle ABC is 2α where $\tan\alpha = \frac{1}{2}$, and the coefficient of friction at both A and C is μ. A mass m is fixed to the mid-point of AB. Show that equilibrium is possible provided that $\mu \geqslant 9/34$. What happens if $\mu = \frac{1}{4}$?

8. Two smooth uniform spheres of weights W and $2W$ are each suspended from a fixed point O by a light inextensible string and rest against each other. The angle between the strings is 90° and the string supporting the heavier sphere makes an angle α with the vertical, where $\tan\alpha = \frac{3}{4}$. Find (a) the tensions in the strings, (b) the angle of inclination to the horizontal of the line joining the centres of the spheres.

9. A uniform rod AB of length $4a$ and weight W is freely hinged at A. A second uniform rod CD of length $5a$ and weight $2W$ is freely hinged at C, which lies at a distance $4a\sqrt{3}$ from A in the same horizontal plane. The rods rest in equilibrium in a vertical plane with the rod CD in contact with the end B of the rod AB. If the rods are on the point of slipping when $BC = 4a$, find the coefficient of friction between the rods.

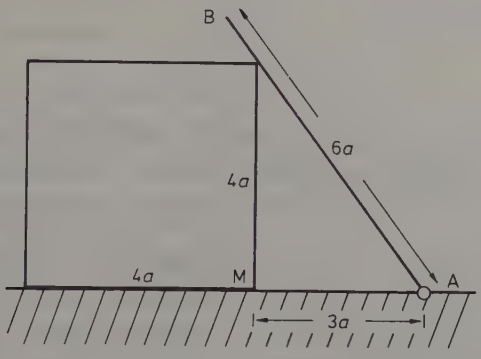
10. Two uniform circular cylinders rest in contact on a horizontal table. A third cylinder is supported by the first two. The cylinders are identical and their axes are horizontal and parallel. The coefficient of friction between any pair of cylinders and between the lower cylinders and the table is μ. Find the least value of μ for which equilibrium is possible.

11. A uniform rod OA of weight W and length $2a$ is pivoted smoothly to a fixed point at the end O. A light inextensible string, also of length $2a$, has one end attached to the end A of the rod, its other end being attached to a small heavy ring of weight $\frac{1}{2}W$ which is threaded on to a fixed rough straight horizontal wire passing through O. The system is in equilibrium with the string taut and the rod making an angle α with the downward vertical at O. By taking moments about O for the rod and ring

together, or otherwise, find the normal component R of the reaction exerted by the wire on the ring. Find also the frictional force F on the ring. The equilibrium is limiting when $\alpha = 60°$. Determine the coefficient of friction between the wire and the ring. (O&C)

12. Two uniform rods AB, AC, each of weight W and length $2a$, are smoothly jointed at A. The rods rest on a smooth circular cylinder of radius a with their plane perpendicular to the axis of the cylinder which is horizontal. When the rods are in equilibrium in a vertical plane with A vertically above the axis of the cylinder, angle $BAC = 2\theta$. Show that the force exerted by the cylinder on each rod is of magnitude $W \operatorname{cosec} \theta$. Show also that $\sin^3 \theta = \cos \theta$. Hence show that there is only one such position of equilibrium. (O&C)

13.



A uniform cube, of edge $4a$ and weight W, rests on a smooth horizontal surface (see diagram). A uniform smooth rod AB, of length $6a$ and weight w, has one end freely hinged to a point A of the surface. The rod rests against the mid-point of a horizontal edge of the top face of the cube. The rod is in a vertical plane through the centre of the cube. A light inelastic string of length $3a$ joins A to the mid-point M of the horizontal edge of the cube nearest to A. The system is in equilibrium. (i) Find the magnitude of the reaction between the rod and the cube. (ii) Find the magnitude of the reaction between the cube and the horizontal surface. (iii) Find the tension in the string AM. (iv) Show that $18w \leqslant 125W$. (C)

6.5 Light frameworks

The expression *light framework* is used to describe a system of rods whose weights are negligible compared with the other forces involved. It is also assumed that the rods are smoothly jointed together at their ends.

Let us consider the two resultant forces acting at the ends of a typical light rod AB in such a framework. If the rod is in equilibrium these forces must be equal and opposite. By taking moments about A and B we also find that both forces act along AB.

Forces on a rod in tension

Forces exerted by a rod in tension

Forces on a rod in thrust

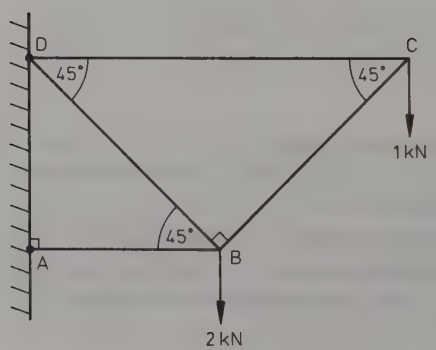
Forces exerted by a rod in thrust

As shown in the diagrams above, a rod may either be in *tension* or in *thrust* i.e. under *compression*. It follows from Newton's third law that at each end of the rod, the force exerted by the rod on the joint is equal and opposite to the force exerted by the joint on the rod.

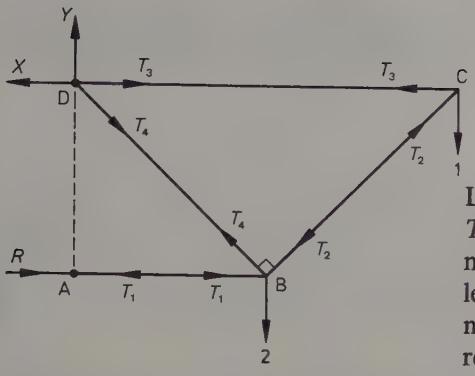
Elementary problems about light frameworks can be solved by considering the equilibrium of the external forces on the framework and the equilibrium of the forces at each joint. In general, unknown forces at joints are found by resolving in suitable directions. However, when dealing with three forces at a joint it is sometimes quicker to use a triangle of forces.

Whatever method is chosen, it is important to draw a large clear diagram showing the forces acting at each joint. Sometimes it is difficult to decide whether a particular rod is in tension or in thrust. In this case we may choose to indicate one type of stress in the diagram, but if calculation leads to a negative value for this stress, we deduce that it is of the other type.

Example 1



The diagram shows a framework of four smoothly jointed light rods. The framework is freely hinged at A and D to a vertical wall. Loads of 2 kN and 1 kN are hung from B and C respectively. Find the magnitudes of the reactions at A and B. Find also the stresses in the rods, stating whether they are tensions or thrusts.



Let the stresses in the rods be thrusts T_1, T_2 and tensions T_3, T_4 as shown. Let the magnitude of the reaction at A be R and let the horizontal and vertical components of the reaction at D be X and Y respectively.

Since there are only two forces acting at A, the reaction R must be a horizontal force such that

$$R - T_1 = 0 \qquad (1)$$

Consider now the forces on the whole framework.
Let $AB = AD = a$, then taking moments about D,

↻ $\quad R \times a - 2 \times a - 1 \times 2a = 0 \quad \therefore \quad R = 4$

Hence the magnitude of the reaction at A is 4 kN.
Resolving horizontally and vertically,

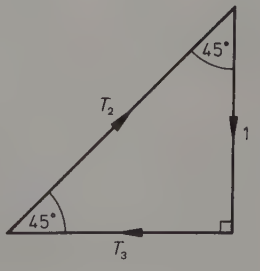
$$\leftarrow \quad X - R = 0 \quad \therefore \quad X = R = 4$$
$$\uparrow \quad Y - 2 - 1 = 0 \quad \therefore \quad Y = 3$$

Hence the magnitude of the reaction at D

$$= \sqrt{(X^2 + Y^2)}\,\text{kN} = \sqrt{(4^2 + 3^2)}\,\text{kN} = 5\,\text{kN}$$

From (1), $T_1 = R = 4$.
Using the triangle of forces for the joint C,

$$T_2 = \sqrt{2}, \quad T_3 = 1.$$



Resolving the forces at B in the direction BD,

↖ $\quad T_4 - 2\cos 45° - T_1 \cos 45° = 0$

$$\therefore \quad T_4 = 2 \times \frac{1}{\sqrt{2}} + 4 \times \frac{1}{\sqrt{2}} = 3\sqrt{2}$$

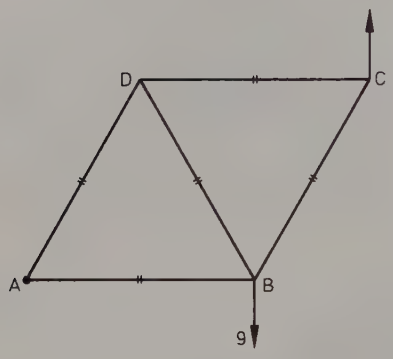
Hence the stresses in AB and BC are thrusts of magnitude 4 kN and $\sqrt{2}$ kN respectively. The stresses in BD and CD are tensions of magnitude $3\sqrt{2}$ kN and 1 kN respectively.

Exercise 6.5

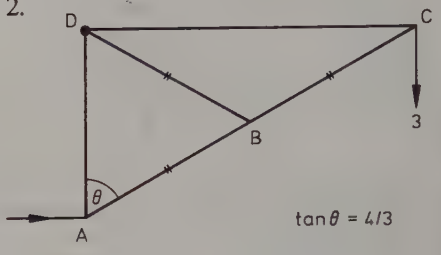
In questions 1 to 8 each diagram shows a light framework consisting of smooth jointed rods. The large dots represent points at which the framework is smoothly

hinged to a fixed point. External forces acting at other points are indicated by arrows and given in kilonewtons. Find the stresses in the rods, stating whether they are tensions or thrusts.

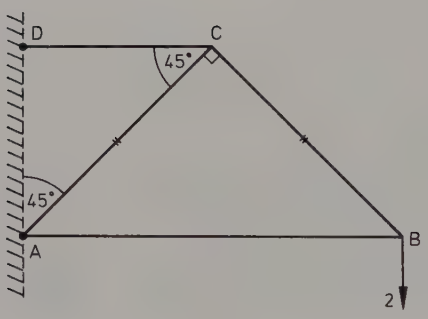
1.



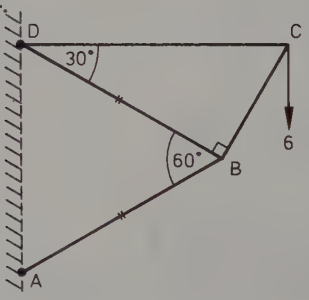
2.



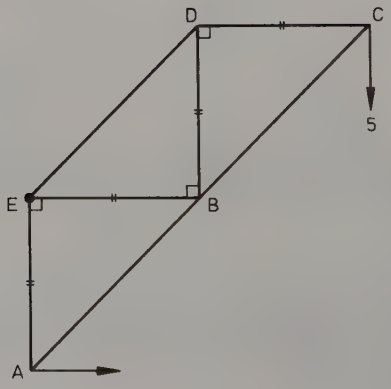
3.



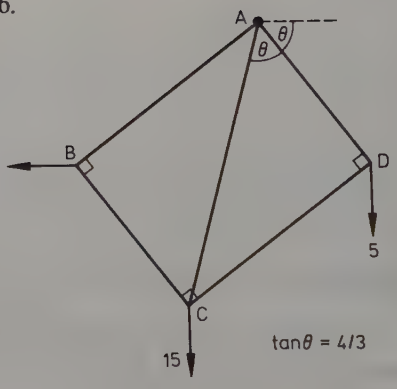
4.



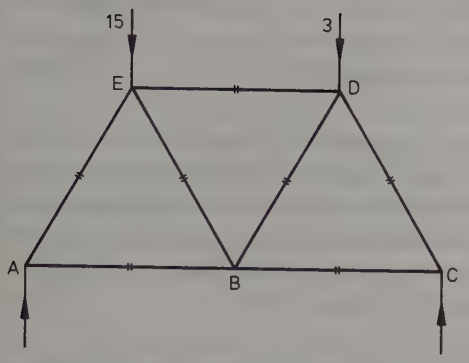
5.



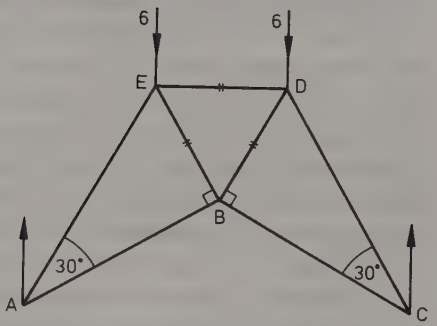
6.



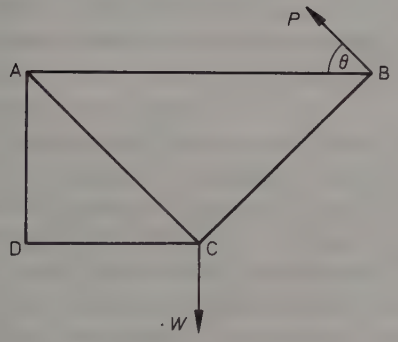
7.



8.



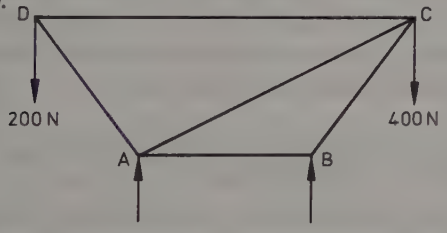
9.



The diagram represents a framework of five smoothly jointed light rods with $AB = \sqrt{2}BC = \sqrt{2}CA = 2CD = 2DA$. The framework is freely hinged to a support at D. A weight W is hung from C and the framework is supported with CD horizontally by a force P at B making an angle θ with the horizontal.

If the magnitude of the stress in CA is twice the magnitude of the stress in BC, find P and the stresses in the rods, specifying whether they are tensions or thrusts. (O)

10.



The diagram shows a smoothly jointed framework consisting of five light rigid rods with $AB = BC = AD = 5$ metres and $DC = 11$ metres. The framework is simply supported at A and B and is in equilibrium in a vertical plane with AB and DC horizontal and with loads of 400 N and 200 N at C and D respectively.
(i) Show that the magnitude of the force supporting the framework at B is 520 N.
(ii) Find the magnitude of the force acting in each member of the framework.
(iii) State which members of the framework are in tension. (AEB 1978)

Exercise 6.6 (*miscellaneous*)

[Take g as $10\,\mathrm{m\,s^{-2}}$.]

1. A uniform plank $ABCD$ of length $6a$ and weight W rests on two supports B and C; $AB = BC = CD = 2a$. A man of weight $4W$ wishes to stand on the plank at A, for

which purpose he places a counter-balancing weight Y at a point P between C and D at a distance x from D. Show that the least value of Y required is $7Wa/(4a - x)$. Show that, if the board is also to be in equilibrium when he is not standing on it, then the greatest value of Y that may be used is $Wa/(2a - x)$. (JMB)

2. A uniform lamina $ABCD$ is in the form of a trapezium in which $DC = AD = a$, $AB = 2a$ and $\angle BAD = \angle CDA = 90°$. When the lamina is freely suspended from a point E in BC the edge AB is vertical. Show that $BE:EC = 4:5$. Find the tangent of the angle which AB makes with the vertical when the lamina is freely suspended from B. The lamina is suspended by two vertical strings attached at C and D. Find the ratio of the tensions in the strings if CD is horizontal. (L)

3. Three equal uniform rods AB, BC, CA, each of weight w, are joined together to form a triangle ABC. A particle of weight W is attached to B and the system is suspended freely from the point A. In equilibrium the rod AB makes an angle θ with the vertical. Prove that $\tan\theta = \sqrt{3}w/(2W + 3w)$. (O)

4. A uniform thin sheet of cardboard of weight W is in the form of a square $ABCD$ of side $6a$. A cut is made along DO, where O is the point of intersection of the diagonals, and the triangular portion AOD is folded over along AO and stuck to the portion AOB with which it now coincides. Find the distances of the centre of gravity of the resulting object from AB and from BC and show that the centre of gravity lies on BD. If this object hangs freely from A, find the angle which BO makes with the horizontal. A particle of weight w is now attached to the object at D so that, in equilibrium, BO is horizontal. Find w. (L)

5. A uniform right circular solid cone of radius r and height h rests with its base on a rough plane which is initially horizontal. The inclination of the plane to the horizontal is gradually increased. Show that the cone will slide before it topples if the coefficient of friction between the plane and the cone is less than $4r/h$. (O)

6. If three non-parallel coplanar forces are in equilibrium, show that they must be concurrent.

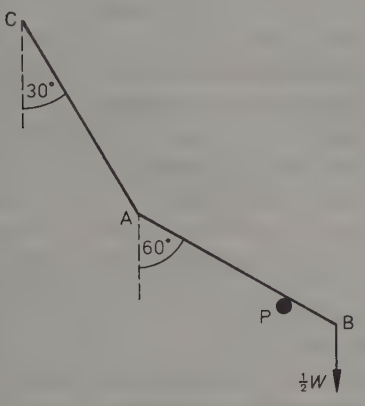
A uniform rod AB of length $2a$ and weight W has one end A in contact with a vertical wall and to the other end B is attached one end of a light string. The other end of the string is attached to the wall at C, the point at a height $2a$ vertically above A. The rod makes an angle α ($0° < \alpha < 90°$) with the downward vertical. Show that equilibrium is impossible if the wall is smooth. If $\alpha = 60°$ and the rod is in equilibrium, show that the least possible value of the coefficient of friction between the rod and the wall is $1/\sqrt{3}$. (L)

7. A uniform rectangular lamina $ABCD$ of mass 1 kg has $AB = 0{\cdot}8$ m and $AD = 0{\cdot}6$ m. The lamina is smoothly pivoted at A. It is kept in equilibrium in a vertical plane with the diagonal AC horizontal and with B as the lowest point by a force P acting at right angles to CD. Find the magnitude of P and the magnitude and direction of the force exerted on the lamina by the pivot if (a) the force P acts through

the mid-point of CD, (b) the force P acts at C. Find also the magnitude and the line of action of P when the force exerted by the pivot acts in the direction BA. (L)

8. A uniform rod AB, of length $2l$ and weight W, is smoothly hinged to a vertical wall at A. It is held at rest in a horizontal position by a light inextensible string attached at B and to a point C which is vertically above and at a distance l from A. Calculate the tension in the string and show that the reaction at A is equal in magnitude to the tension. When a load of weight $4W$ is attached to the rod at a point D the horizontal and vertical components of the reaction at A are equal in magnitude. Find AD. (L)

9. A thin uniform rod AB of length 3 m and mass 5 kg is freely pivoted to a fixed point A. A light elastic string BC of modulus 30 N has one end attached to B and its other end C fixed to a point at the same level as A, where $AC = 5$ m. When the system is in equilibrium $BC = 4$ m. Calculate (a) the tension in the string. (b) the natural length of the string. (c) the resultant force, in magnitude and direction, exerted on the rod at A. (L)

10. 

The diagram shows a uniform rod AB, of length $4c$ and weight W, resting on a rough fixed peg P, where $AP = 3c$. A particle of weight $\frac{1}{2}W$ hangs from B. One end of a light string is tied at A and the other end is tied at a fixed point C. The rod and the string are in the same vertical plane and are inclined at angles of 60° and 30° respectively to the vertical. (i) Show that the tension in the string is $\frac{1}{6}W\sqrt{3}$. (ii) Given that equilibrium is limiting, calculate the coefficient of friction between the peg and the rod. (iii) Given that $AC = 4c$ and that the string is elastic with modulus of elasticity $W\sqrt{3}$, calculate the natural length of the string. (AEB 1979)

11. A uniform square lamina $ABCD$ of weight W and side $2a$ rests in a vertical plane with the vertex A in contact with a rough horizontal plane. The lamina is kept in equilibrium by a force P acting at the vertex C in the direction of DC produced. Given that the height of the vertex D above the horizontal plane is $6a/5$, show that $P = W/10$. Find the coefficient of friction between the lamina and the plane if the lamina is in limiting equilibrium. (L)

12. A uniform rod AB, of weight W and length $2l$, rests in equilibrium with the end A on rough horizontal ground and with the end B in contact with a smooth vertical wall, which is perpendicular to the vertical plane containing the rod. If AB makes an angle α with the horizontal, where $\tan\alpha = 4/3$, find the least possible value of μ, the coefficient of friction between the rod and the ground, for equilibrium to be

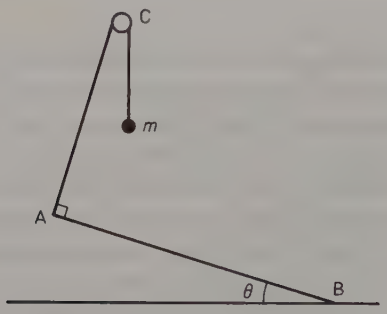
preserved. If $\mu = \frac{1}{2}$, find the distance from A to the highest point of the rod at which a load of weight W can be attached without equilibrium being disturbed. (L)

13. Prove that the position of the centre of gravity of a uniform solid hemisphere of radius a is distant $\frac{3}{8}a$ from the centre of its plane face.

A uniform solid hemisphere rests with its curved surface on a rough horizontal surface and also against a rough vertical surface. The coefficient of friction at both points of contact is $\frac{1}{4}$. If the hemisphere is just about to slip, show that its top surface is inclined at an angle of just under 52° to the horizontal. (SU)

14. A uniform bar, of length $3a$ and mass m, rests in limiting equilibrium at an angle α to the horizontal with one end A on rough horizontal ground and the other end B against a smooth vertical wall. A horizontal force P is now applied to the bar at a point C, where $AC = a$. Find in terms of m, α and g, the value of P which will just cause the bar to slip towards the wall and the corresponding reaction between the bar and the wall.

15.



AB is a uniform rod of mass M. The end B rests on a rough horizontal table (coefficient of friction μ), and a light inextensible string is attached to the end A, passes over a smooth pulley at C and carries a mass m at its other end. The system rests in equilibrium with the string at right angles to AB. Find an expression for θ in terms of M and m and show that equilibrium is not possible unless $2m < M$. Show also that $\mu \geqslant \dfrac{m\sqrt{(M^2 - 4m^2)}}{M^2 - 2m^2}$. (SU)

16. A hemisphere of radius a, whose curved surface is smooth, is fixed with its flat base on a rough horizontal table. A uniform rod AB of length $4a$ and weight W rests in equilibrium against the hemisphere, its point of contact with the hemisphere lying between A and B. The end A of the rod is in contact with the table and AB makes an acute angle θ with the horizontal. Prove that the frictional force F at A has magnitude $2W\sin^2\theta$ and find the normal reaction R at A. If the coefficient of friction between the end A and the table is $\frac{1}{2}$, prove that $3\tan^2\theta + 2\tan\theta - 1 \leqslant 0$ and deduce the maximum value of θ consistent with equilibrium. (O&C)

17. $ABCD$ is a rectangular lamina, freely pivoted at a point X on AB. The length of AB is $2a$ and of BC is a. Forces $3W$, $5W$, $2W$ and $7W$ act along the sides AB, BC, DC and AD respectively in the directions indicated by the order of the letters. Find the position of X so that the lamina is in equilibrium and, in this case, determine the reaction of the pivot on the lamina in magnitude and direction. If, instead, the lamina is pivoted at a point Y on CD and is in equilibrium, deduce from your results the position of Y. (O&C)

18. Explain, with the aid of a diagram, the meaning of the expression coefficient of friction.

A straight uniform rod AB of weight W rests in limiting equilibrium with the end A on horizontal ground and the end B against a vertical wall. The vertical plane containing AB is perpendicular to the wall. The coefficient of friction between the rod and the ground is 4/5. The coefficient of friction between the rod and the wall is 3/5. Given that the inclination of AB to the horizontal is α, calculate (i) the normal reactions at A and B in terms of W, (ii) the numerical value of $\tan\alpha$. (AEB 1980)

19. A uniform right circular heavy cylinder, of height h and radius a, rests with one plane face upon a rough horizontal plane. A gradually increasing horizontal force is applied to a point on the edge of the upper face, and the force passes through the axis of the cylinder. Show that the cylinder will slide before it topples if $\mu h < a$, where μ is the coefficient of friction between the cylinder and the horizontal plane. (O)

20. A uniform solid hemisphere of radius a has mass M. A concentric hemisphere of mass $M/8$ is removed. Show that the centre of mass of the remainder is at a distance $45a/112$ from the centre O of the plane base. A particle of mass M is attached to a point X on the outer edge of the plane base of the remainder. Find, relative to OX and the axis of symmetry of the hemisphere as axes, the coordinates of the centre of mass G of the combined solid (i.e. remainder together with attached mass). The combined solid is placed with its curved surface on a horizontal plane and rests in equilibrium. Find the tangent of the angle between the plane base of the solid and the horizontal. (C)

21. Prove by integration that the centre of mass of a uniform solid right circular cone of height h and base radius r is at a distance $\frac{3}{4}h$ from the vertex. Such a cone is joined to a uniform solid right circular cylinder, of the same material, with base radius r and height l, so that the plane base of the cone coincides with a plane face of the cylinder. Find the centre of mass of the solid thus formed. Show that, if $6l^2 \geqslant h^2$, this solid can rest in equilibrium on a horizontal plane, with the curved surface of the cylinder touching the plane. Given that $l = h$, show that the solid can rest in equilibrium with its conical surface touching the plane provided that $r \geqslant \frac{1}{4}h\sqrt{5}$. (JMB)

22. A toy consists of a solid hemisphere of radius a to which is glued a solid circular cylinder of radius a and height $2a$ so that the plane end of the hemisphere is in complete contact with a plane end of the cylinder. The cylinder is made of uniform material of density ρ, and the hemisphere is made of uniform material of density $k\rho$. The toy is designed so that if placed on a horizontal table with the hemisphere downwards and then tilted to one side, it will return to the vertical position. Show that $k > 8$. The toy is placed on a desk of slope α where $\sin\alpha = 1/8$, sufficiently rough to prevent slipping. It rests in equilibrium with the hemisphere in contact with the desk. Find an expression giving the (acute) angle β made by its axis of symmetry with the vertical. Hence deduce that $k \geqslant 13\frac{1}{2}$. (O&C)

23. Using a diagram, explain the meaning of the term angle of friction. A uniform straight pole AB of length 2·5 metres stands in limiting equilibrium with the end A on horizontal ground and the end B against a vertical wall. The vertical plane containing AB is perpendicular to the wall and A is at a distance 0·7 metres from the wall. The angles of friction between the pole and the ground and between the pole and the wall are equal. Show that the line of action of the resultant force exerted on the pole by the ground at A is perpendicular to the line of action of the resultant force exerted on the pole by the wall at B. Hence, or otherwise, show that the coefficient of friction at A and B is 1/7. Given that the pole weighs 50 N, find the magnitude of the resultant reactions (i) between the pole and the ground, (ii) between the pole and the wall. (AEB 1978)

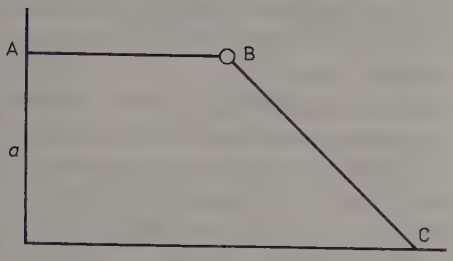
24. A uniform rod of weight W rests in contact with two parallel, horizontal, smooth pegs, passing over the higher peg and under the lower one. The lower end of the rod rests on a smooth horizontal plane. A force of magnitude P acts on the upper end of the rod in the direction of the rod and tending to move the rod upwards. The length of the rod is l, and its inclination to the horizontal is θ. If the upper and lower pegs are at distances a and b respectively from the lower end of the rod, prove that, so long as the rod remains in equilibrium, the reaction of the upper peg on the rod is $\dfrac{Wl\cos\theta - 2Pb\cot\theta}{2(a-b)}$. Given that $l > 2a$, determine which of the three forces exerted on the rod by the two pegs and the ground will first become zero as P is gradually increased from zero. (C)

25. A uniform rod AB, of length l and weight W, is smoothly hinged at A. The point C is vertically above A and is such that $AC = l$. An elastic string, of unstretched length $\frac{1}{2}l$, obeys Hooke's law and has modulus W. The ends of the string are attached to B and C so that the rod rests in equilibrium in an inclined position. Show that the tension in the string is $W/3$. Find the magnitude of the couple which when applied to the rod will maintain the system in equilibrium with $AB = BC$. (L)

26. A uniform rod AB of weight W and length $2a$ has the end A freely hinged to a fixed point. The other end B is freely hinged to a uniform rod BC of weight W and length $6a$. The ends A and C are joined by a light inelastic string whose length is such that AB is perpendicular to BC, and the system hangs in equilibrium. Show that AB is inclined at 45° to the vertical. Find the tension in AC and the horizontal and vertical components of the reaction on BC at B. (C)

27. Two uniform rods AB and BC of the same thickness and material, and of length 4 metres and 3 metres respectively, are freely hinged together and rest in a vertical plane with the ends A and C on a rough horizontal plane. The system is in limiting equilibrium when the angle ABC is 90°. Determine how equilibrium will be broken when the angle ABC is slightly increased beyond 90° and show that the coefficient of friction between the rods and the ground is 84/163. (W)

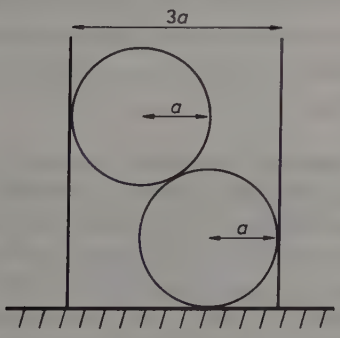
28.



A uniform rod AB of length a and weight W is smoothly jointed at B to a uniform rod BC of length $a\sqrt{2}$ and also of weight W. The system is in equilibrium with C resting on a rough horizontal floor at a distance $2a$ from a rough vertical wall, A resting against the wall, and AB horizontal at a height a above the floor. The plane of the rods is perpendicular to the floor and the wall. (i) Find the frictional force exerted by the wall at A. (ii) Find the vertical and horizontal components of the force exerted on BC at C. (iii) Show that equilibrium is possible only if the coefficient of friction at C is at least $\frac{2}{3}$. (iv) Find the least value of the coefficient of friction at A for equilibrium to be possible. (JMB)

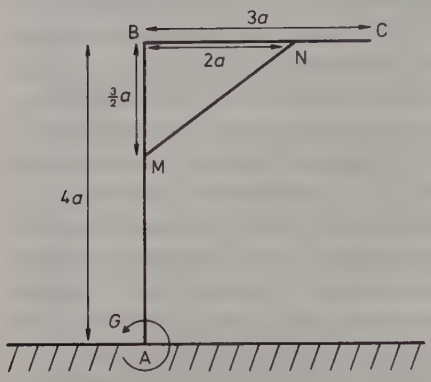
29. Three uniform straight rods, each of weight W, are smoothly jointed together to form an equilateral triangle ABC. The joint B is hinged to a fixed support and the system is free to rotate about B in a vertical plane. The system is held in equilibrium with BC horizontal and A uppermost by a horizontal light string which connects A with a fixed point. Show that the tension in the string is $W\sqrt{3}$ and find the magnitude and direction of the force exerted on the triangle by the support at B. Find the horizontal and vertical components of all the forces which are exerted on the rod AC, and indicate them clearly on your diagram. (JMB)

30. The diagram shows two parallel vertical walls at a distance $3a$ apart, with two uniform smooth cylinders, each of radius a and weight W. The axes of the cylinders are horizontal and parallel to the walls. Each cylinder is in contact with the other and with one wall; the lower cylinder rests on horizontal ground. Find the magnitudes of the reactions between the two cylinders and between each cylinder and the corresponding wall. (C)



31. A uniform sphere of radius a and weight $W/\sqrt{3}$ rests on a rough horizontal table. A uniform rod AB of weight $2W$ and length $2a$ is freely hinged at A to a fixed point on the table and leans against the sphere so that the centre of the sphere and the rod lie in a vertical plane. The rod makes an angle of $60°$ with the horizontal. Show that the frictional force between the rod and the sphere is $\frac{1}{3}W$. The coefficient of friction at each point of contact is μ. What is the smallest value of μ which makes equilibrium possible? (O&C)

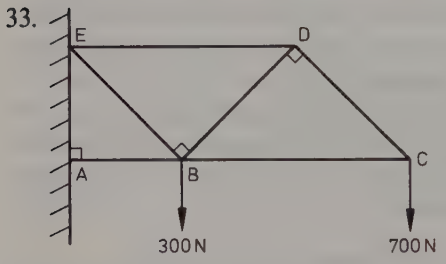
32.



The diagram shows a uniform post AB, of length $4a$ and weight W, smoothly hinged to the ground at A. A horizontal uniform beam BC, of length $3a$ and weight W, is smoothly hinged to AB at B. The beam is supported by a light inelastic rod MN smoothly jointed to AB at M and to BC at N, where $BM = \frac{3}{2}a$ and $BN = 2a$. A weight $3W$ hangs from C, and the system is maintained in equilibrium, with AB vertical, by an external couple of magnitude G applied to AB, as shown.

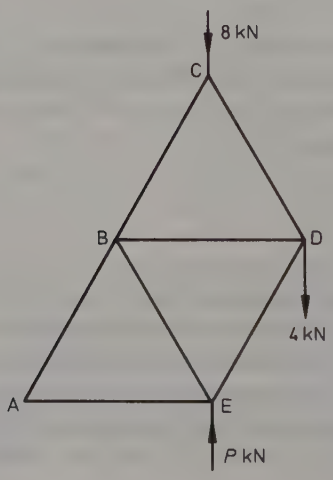
Find (i) the thrust in the rod MN, (ii) the horizontal and vertical components of the forces on the post AB at A and at B, (iii) the value of G. (C)

33.



The diagram shows a framework $ABCDE$, consisting of 6 light rigid rods which are smoothly jointed. The framework is smoothly hinged to a vertical wall at A and E and the triangles ABE, BED and BCD are all right-angled and isosceles. When loads of 300 N and 700 N are hung from B and C respectively, the framework is in equilibrium in a vertical plane. (i) Explain why the reation on the hinge at A is horizontal and find its magnitude. (ii) Find the magnitude and the direction of the reaction on the hinge at E. (iii) Find the magnitude of the force acting in each member of the framework and state which rods are in compression. (AEB 1978)

34.



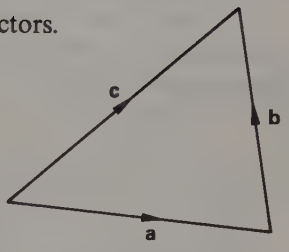
The diagram shows a framework of 7 light equal rods, smoothly jointed at their ends. It is freely hinged to a vertical wall at A. Forces of 8 kN and 4 kN are applied at C and D as shown. A force of size P kN is applied vertically upwards at E in order to keep the rod AE in a horizontal position. Calculate (i) the magnitude of P; (ii) the stresses in the rods, stating if they are in tension or under compression; (iii) the magnitude and direction of the reaction of the support at A on the frame. (SU)

7 Vectors and scalar product

7.1 Properties of vectors

We begin by reviewing some of the basic properties of vectors.

(1) The sum or resultant of two vectors may be found using the triangle law for vector addition



e.g. $\mathbf{a} + \mathbf{b} = \mathbf{c}$.

Vector addition is both commutative and associative

i.e. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}, \quad (\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$.

(2) For any scalar λ, the vector $\lambda\mathbf{a}$ is parallel to $\mathbf{a}$ and has magnitude $|\lambda|$ times the magnitude of $\mathbf{a}$.

It can be shown that $\lambda(\mu\mathbf{a}) = (\lambda\mu)\mathbf{a}$,

$(\lambda + \mu)\mathbf{a} = \lambda\mathbf{a} + \mu\mathbf{a}, \quad \lambda(\mathbf{a} + \mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b}$.

(3) Given two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$, which are not parallel, any vector $\mathbf{r}$ in the plane of $\mathbf{a}$ and $\mathbf{b}$ can be expressed in the form $\lambda\mathbf{a} + \mu\mathbf{b}$. The vectors $\lambda\mathbf{a}$ and $\mu\mathbf{b}$ are the *component vectors* of $\mathbf{r}$ in the directions of *base vectors* $\mathbf{a}$ and $\mathbf{b}$.

(i) If vectors $\lambda\mathbf{a} + \mu\mathbf{b}$ and $l\mathbf{a} + m\mathbf{b}$ are equal, then $\lambda = l$ and $\mu = m$.

(ii) If vectors $\lambda\mathbf{a} + \mu\mathbf{b}$ and $l\mathbf{a} + m\mathbf{b}$ are parallel, then for some scalar t, $\lambda\mathbf{a} + \mu\mathbf{b} = t(l\mathbf{a} + m\mathbf{b})$ and thus $\dfrac{\lambda}{l} = \dfrac{\mu}{m} = t$.

(4) Given three non-zero vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, which are not coplanar, any vector $\mathbf{r}$ can be expressed in the form $\lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c}$.

(i) If vectors $\lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c}$ and $l\mathbf{a} + m\mathbf{b} + n\mathbf{c}$ are equal, then $\lambda = l$, $\mu = m$ and $\nu = n$.

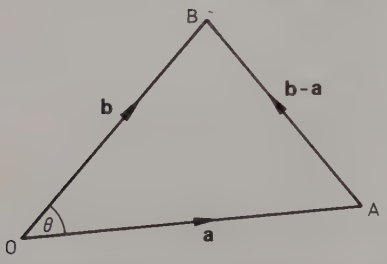
(ii) If vectors $\lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c}$ and $l\mathbf{a} + m\mathbf{b} + n\mathbf{c}$ are parallel, then for some scalar t, $\lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c} = t(l\mathbf{a} + m\mathbf{b} + n\mathbf{c})$ and thus $\dfrac{\lambda}{l} = \dfrac{\mu}{m} = \dfrac{\nu}{n} = t$.

(5) If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, where $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ are unit vectors in the directions of the x-, y- and z-axes respectively then,

(i) the magnitude of $\mathbf{r}$ is given by $r = \sqrt{(x^2 + y^2 + z^2)}$,

(ii) the unit vector in the direction of $\mathbf{r}$ is $\hat{\mathbf{r}} = \frac{x}{r}\mathbf{i} + \frac{y}{r}\mathbf{j} + \frac{z}{r}\mathbf{k}$.

(6)



If the position vectors of points A and B with respect to O are $\mathbf{a}$ and $\mathbf{b}$ respectively, then $\mathbf{OA} = \mathbf{a}$, $\mathbf{OB} = \mathbf{b}$ and $\mathbf{AB} = \mathbf{b} - \mathbf{a}$.

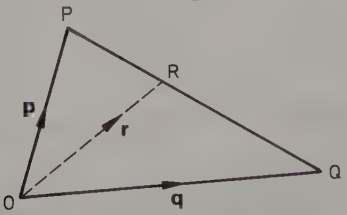
(i) The distance between A and B is given by $|\mathbf{b} - \mathbf{a}|$ (or $|\mathbf{a} - \mathbf{b}|$).

(ii) If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, then the cosine rule can be used to express $\cos\theta$ in terms of $|\mathbf{a}|, |\mathbf{b}|$ and $|\mathbf{a} - \mathbf{b}|$.

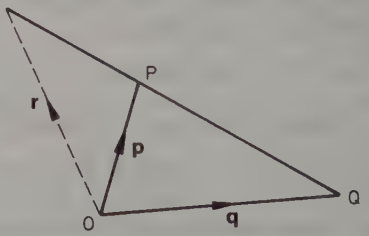
(iii) A point C, with position vector $\mathbf{c}$, lies on the straight line passing through A and B if and only if there is a scalar λ such that $\mathbf{AC} = \lambda\mathbf{AB}$ i.e. $\mathbf{c} - \mathbf{a} = \lambda(\mathbf{b} - \mathbf{a})$.

The fundamental ideas summarised in paragraphs (1) to (6) form the foundation for the work of this chapter.

(i)



(ii)



Let us consider three points P, Q and R such that R divides PQ in the ratio $\mu:\lambda$. We will observe the convention that (i) if λ and μ have the same sign then R lies between P and Q, and (ii) if λ and μ have opposite signs then R does not lie between P and Q.

In both cases $\quad \lambda\mathbf{PR} = \mu\mathbf{RQ}$

$\therefore \quad \lambda(\mathbf{OR} - \mathbf{OP}) = \mu(\mathbf{OQ} - \mathbf{OR})$

$\therefore \quad \lambda\mathbf{OR} - \lambda\mathbf{OP} = \mu\mathbf{OQ} - \mu\mathbf{OR}$

Hence $\quad (\lambda + \mu)\mathbf{OR} = \lambda\mathbf{OP} + \mu\mathbf{OQ}$

This result is called the *ratio theorem* and may also be stated as follows:

> The resultant of the vectors $\lambda\mathbf{OP}$ and $\mu\mathbf{OQ}$ is $(\lambda + \mu)\mathbf{OR}$, where R is the point which divides PQ in the ratio $\mu:\lambda$

If the position vectors of P, Q and R with respect to O are $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ respectively then

$$(\lambda + \mu)\mathbf{r} = \lambda\mathbf{p} + \mu\mathbf{q}.$$

Thus the point R which divides PQ in the ratio $\mu:\lambda$ has position vector

$$\mathbf{r} = \frac{\lambda\mathbf{p} + \mu\mathbf{q}}{\lambda + \mu}$$

In particular, if R is the *mid-point* of PQ, then $\mathbf{r} = \frac{1}{2}(\mathbf{p} + \mathbf{q})$.

Example 1 The points A and B have position vectors $\mathbf{a} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ respectively. Find the position vectors of the point P which divides AB internally in the ratio 1:4 and the point Q which divides AB externally in the ratio 3:2.

Observing the usual sign convention, P divides AB in the ratio 1:4 and Q divides AB in the ratio $3:-2$.
Thus the position vector of P

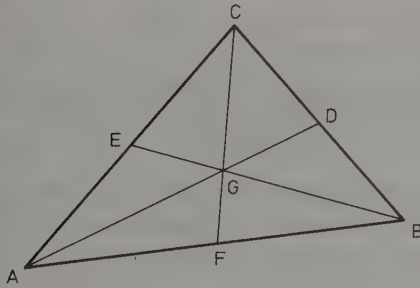
$$= \frac{4\mathbf{a} + \mathbf{b}}{4 + 1} = \frac{4(\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) + (6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})}{5} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}.$$

Similarly, the position vector of Q

$$= \frac{-2\mathbf{a} + 3\mathbf{b}}{-2 + 3} = -2(\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) + 3(6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) = 16\mathbf{i} - 12\mathbf{j} + 13\mathbf{k}.$$

[Note that the same result will be obtained for the position vector of Q if the ratio $-3:2$ is used instead of the ratio $3:-2$.]

Example 2 Prove that the medians of a triangle meet at a point which divides each median in the ratio 2:1.



In $\triangle ABC$ let D, E and F be the mid-points of BC, CA and AB respectively. If the position vectors of A, B and C are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, then the position vectors of D, E and F are $\frac{1}{2}(\mathbf{b} + \mathbf{c}), \frac{1}{2}(\mathbf{c} + \mathbf{a})$ and $\frac{1}{2}(\mathbf{a} + \mathbf{b})$ respectively.

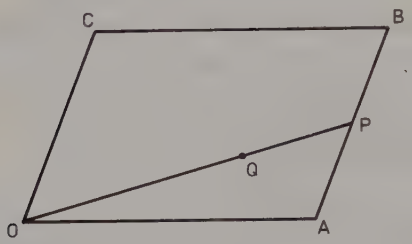
Let G be the point which divides AD in the ratio 2:1, then its position vector is $\dfrac{1 \times \mathbf{a} + 2 \times \frac{1}{2}(\mathbf{b} + \mathbf{c})}{1 + 2}$ i.e. $\frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$.
Since this expression is symmetrical in $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, G must also be the point which divides BE and CF in the ratio 2:1. Hence the medians meet at a point which divides each median in the ratio 2:1.

[As we saw in §5.3 the point G with position vector $\frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$ is the centroid of $\triangle ABC$.]

Let us consider again the points P, Q and R with position vectors $\mathbf{p}, \mathbf{q}$ and $\mathbf{r}$ respectively. It follows from the ratio theorem that if $(\lambda + \mu)\mathbf{r} = \lambda\mathbf{p} + \mu\mathbf{q}$, then R is the point which divides PQ in the ratio $\mu:\lambda$. If $\lambda + \mu = 1$, we obtain the following useful result.

> If $\mathbf{r} = \lambda\mathbf{p} + \mu\mathbf{q}$ for scalars λ and μ such that $\lambda + \mu = 1$, then R is the point which divides PQ in the ratio $\mu:\lambda$.

Example 3 In a parallelogram $OABC$, P is the mid-point of AB and Q divides OP in the ratio 2:1. Show that Q lies on AC and find the ratio $AQ:QC$.



Let the position vectors of A, B, C, P and Q with respect to O be $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{p}$ and $\mathbf{q}$ respectively.
P is the mid-point of AB

$$\therefore \quad \mathbf{p} = \tfrac{1}{2}(\mathbf{a} + \mathbf{b})$$

Q divides OP in the ratio 2:1

$$\therefore \quad \mathbf{q} = \tfrac{2}{3}\mathbf{p} = \tfrac{2}{3} \times \tfrac{1}{2}(\mathbf{a} + \mathbf{b}) = \tfrac{1}{3}\mathbf{a} + \tfrac{1}{3}\mathbf{b}$$

However, since $OABC$ is a parallelogram, $\mathbf{b} = \mathbf{a} + \mathbf{c}$,

$$\therefore \quad \mathbf{q} = \tfrac{1}{3}\mathbf{a} + \tfrac{1}{3}(\mathbf{a} + \mathbf{c}) = \tfrac{2}{3}\mathbf{a} + \tfrac{1}{3}\mathbf{c}$$

Hence, by the ratio theorem, Q is the point on AC such that $AQ:QC = 1:2$.

Exercise 7.1

1. The position vectors of the vertices A, B and C of parallelogram $ABCD$ are $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ respectively. Find, in terms of $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$, the position vector of D.

2. The position vectors of the vertices A, B and C of trapezium $ABCD$ are $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ respectively. Given that the sides BC and AD are parallel and that $AD = 2BC$, find the position vector of D in terms of $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$.

3. Find the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$ given that $|\mathbf{a}| = 3$, $|\mathbf{b}| = 8$ and $|\mathbf{a} - \mathbf{b}| = 7$.

4. The angle between the vectors $\mathbf{a}$ and $\mathbf{b}$ is 120°. If $|\mathbf{a}| = 5$ and $|\mathbf{b}| = 3$, find $|\mathbf{a} - \mathbf{b}|$ and $|\mathbf{a} + \mathbf{b}|$.

5. The points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively relative to an origin O, where $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = -4\mathbf{i} + s\mathbf{j} + t\mathbf{k}$. Find the possible values of s and t if (a) the points O, A and B are collinear, (b) $|\mathbf{AB}| = 7$ and $s = 2t$.

6. Given that $\mathbf{a} = 2\mathbf{i} + 6\mathbf{j}$ and $\mathbf{b} = 3\mathbf{i} + t\mathbf{j}$, find the value of t such that (a) $\mathbf{a}$ and $\mathbf{b}$ are parallel, (b) $\mathbf{a}$ and $\mathbf{b}$ are at right angles.

7. If the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, find in terms of $\mathbf{a}$ and $\mathbf{b}$ the position vectors of (a) the point P which divides AB internally in the ratio 5:3, (b) the point Q which divides AB externally in the ratio 3:1, (c) the point R such that A is the mid-point of BR.

8. The points A and B have position vectors $\mathbf{a} = 5\mathbf{i} + 4\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = -\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ respectively. Find the position vectors of (a) the point C on AB such that $AC = 2CB$, (b) the point D on AB produced such that $AD = 2BD$.

9. Write down, in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, the position vectors of the points $A(2, -5, 3)$ and $B(7, 0, -2)$. Hence find the coordinates of the point C which divides AB internally in the ratio 2:3 and the point D which divides AB externally in the ratio 3:8.

10. The vertices A, B and C of a triangle have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. Points P, Q and R have position vectors given respectively by $\mathbf{p} = \mathbf{a} - \mathbf{b} + \mathbf{c}$, $\mathbf{q} = \frac{3}{7}\mathbf{a} + \frac{4}{7}\mathbf{c}$, $\mathbf{r} = 3\mathbf{b} - 2\mathbf{c}$. Describe geometrically the positions of P, Q and R relative to A, B and C.

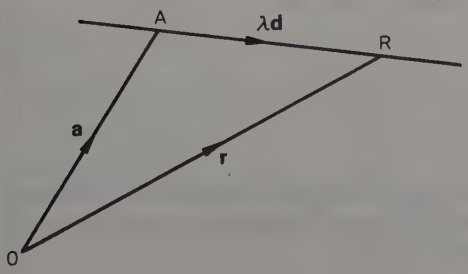
11. The diagonals AC and BD of a quadrilateral $ABCD$ intersect at a point P such that $AP:PC = 2:7$ and $BP:PD = 5:4$. Show that

$$\frac{7\mathbf{a} - 4\mathbf{b}}{7 - 4} = \frac{5\mathbf{d} - 2\mathbf{c}}{5 - 2}$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are the position vectors of A, B, C and D respectively. Explain the geometrical significance of this result.

12. The vertices A, B and C of a triangle have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. The point P divides BC internally in the ratio 3:1, the point Q is the mid-point of CA and the point R divides AB externally in the ratio 1:3. If the position vectors of P, Q and R are $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ respectively, express $\mathbf{r}$ in terms of $\mathbf{p}$ and $\mathbf{q}$. Hence show that R lies on PQ produced and find the ratio $PR:QR$.

7.2 Equations of lines and planes



The diagram shows a straight line which passes through a fixed point A with position vector $\mathbf{a}$ and which is parallel to a given vector $\mathbf{d}$. Let $\mathbf{r}$ be the position vector of a point R on the line.

Since **AR** is parallel to **d**, then, for some scalar λ,

$$\mathbf{AR} = \lambda\mathbf{d}$$
$$\therefore \quad \mathbf{r} - \mathbf{a} = \lambda\mathbf{d}$$
$$\text{i.e.} \quad \mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$$

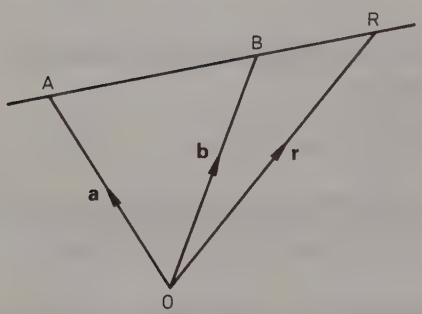
Thus every point on the line has a position vector of the form $\mathbf{a} + \lambda\mathbf{d}$. Moreover, each real value of the parameter λ corresponds to some point on the line.

The equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$ is called the *vector equation* of the line through A parallel to **d**. The vector **d** is a *direction vector* of the line.

Example 1 Find the vector equation of the line which passes through the point with position vector $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and which is parallel to the vector $2\mathbf{j} - \mathbf{k}$.

The vector equation of the given line is

$$\mathbf{r} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k} + \lambda(2\mathbf{j} - \mathbf{k})$$
$$\text{i.e.} \quad \mathbf{r} = 2\mathbf{i} + (2\lambda - 1)\mathbf{j} + (3 - \lambda)\mathbf{k}.$$



Consider now the line through two points A and B with position vectors **a** and **b** respectively. If a point R, with position vector **r**, lies on this line, then for some scalar λ,

$$\mathbf{AR} = \lambda\mathbf{AB}$$
$$\therefore \quad \mathbf{r} - \mathbf{a} = \lambda(\mathbf{b} - \mathbf{a})$$
$$\therefore \quad \mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}).$$

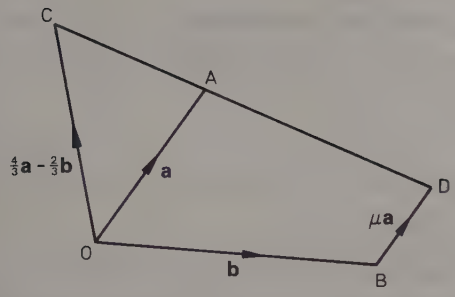
Hence the vector equation of the line through A and B is $\mathbf{r} = (1 - \lambda)\mathbf{a} + \lambda\mathbf{b}$.

Example 2 Find the vector equation of the straight line which passes through the points $A(1, 0, -2)$ and $B(2, 3, -1)$.

The position vectors of A and B are $\mathbf{i} - 2\mathbf{k}$ and $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ respectively.
Hence the vector equation of the line through A and B is

$$\mathbf{r} = (1 - \lambda)(\mathbf{i} - 2\mathbf{k}) + \lambda(2\mathbf{i} + 3\mathbf{j} - \mathbf{k})$$
$$\text{i.e.} \quad \mathbf{r} = (\lambda + 1)\mathbf{i} + 3\lambda\mathbf{j} + (\lambda - 2)\mathbf{k}.$$

Example 3 Three non-collinear points A, B and C have position vectors **a**, **b** and $\frac{4}{3}\mathbf{a} - \frac{2}{3}\mathbf{b}$ respectively, relative to an origin O. Find the position vector of the point D on CA produced, such that BD is parallel to OA.



The vector equation of the line through A and C is

$$\mathbf{r} = (1 - \lambda)\mathbf{a} + \lambda(\tfrac{4}{3}\mathbf{a} - \tfrac{2}{3}\mathbf{b})$$

i.e. $\mathbf{r} = (1 + \frac{1}{3}\lambda)\mathbf{a} - \frac{2}{3}\lambda\mathbf{b}$.

The vector equation of the line through B parallel to OA is

$$\mathbf{r} = \mathbf{b} + \mu\mathbf{a}$$

At the point D where these lines intersect

$$(1 + \tfrac{1}{3}\lambda)\mathbf{a} - \tfrac{2}{3}\lambda\mathbf{b} = \mu\mathbf{a} + \mathbf{b}$$

Since the vectors $\mathbf{a}$ and $\mathbf{b}$ are not parallel, it follows that

$$1 + \tfrac{1}{3}\lambda = \mu \quad \text{and} \quad -\tfrac{2}{3}\lambda = 1$$
$$\therefore \quad \lambda = -\tfrac{3}{2}, \quad \mu = \tfrac{1}{2}$$

Hence the position vector of D is $\frac{1}{2}\mathbf{a} + \mathbf{b}$.

In three dimensional space a pair of vector equations such as $\mathbf{r} = \mathbf{a} + \lambda\mathbf{p}$ and $\mathbf{r} = \mathbf{b} + \mu\mathbf{q}$ may represent

(i) the same line, (ii) parallel lines,
(iii) intersecting lines, (iv) skew lines.

Example 4 Show that the lines l_1 and l_2, with vector equations $\mathbf{r} = \mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} - 3\mathbf{k})$ and $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + \mu(3\mathbf{j} + 5\mathbf{k})$ respectively, intersect, and find the position vector of their point of intersection.

The equations of l_1 and l_2 may be written:

$$\mathbf{r} = \lambda\mathbf{i} - \lambda\mathbf{j} + (1 - 3\lambda)\mathbf{k}$$
$$\mathbf{r} = 2\mathbf{i} + (1 + 3\mu)\mathbf{j} + 5\mu\mathbf{k}$$

At any point of intersection

$$\lambda\mathbf{i} - \lambda\mathbf{j} + (1 - 3\lambda)\mathbf{k} = 2\mathbf{i} + (1 + 3\mu)\mathbf{j} + 5\mu\mathbf{k}$$

Thus

$$\lambda = 2 \tag{1}$$
$$-\lambda = 1 + 3\mu \tag{2}$$
$$1 - 3\lambda = 5\mu \tag{3}$$

From (1) and (2), $\lambda = 2, \mu = -1$.

Since these values also satisfy equation (3), the lines do have a point of intersection. Its position vector, given by substituting $\lambda = 2$ in the equation of l_1, is $2\mathbf{i} - 2\mathbf{j} - 5\mathbf{k}$.

[Note that we can check this result by substituting $\mu = -1$ in the equation of l_2.].

Example 5 Show that the lines l_1 and l_2, with vector equations

$$\mathbf{r} = \lambda\mathbf{i} + (3 - 2\lambda)\mathbf{j} + (2 + \lambda)\mathbf{k} \quad \text{and} \quad \mathbf{r} = (1 - 2\mu)\mathbf{i} + 4\mu\mathbf{j} + (1 - 2\mu)\mathbf{k}$$

respectively, are distinct parallel lines.

The equations of l_1 and l_2 may be written:

$$\mathbf{r} = 3\mathbf{j} + 2\mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$$
$$\mathbf{r} = \mathbf{i} + \mathbf{k} + \mu(-2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$$

Thus l_1 passes through a point A with position vector $\mathbf{a} = 3\mathbf{j} + 2\mathbf{k}$ and is parallel to the vector $\mathbf{p} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$.
Similarly l_2 passes through a point B with position vector $\mathbf{b} = \mathbf{i} + \mathbf{k}$ and is parallel to the vector $\mathbf{q} = -2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.
Since $\mathbf{q} = -2\mathbf{p}$, the vectors $\mathbf{p}$ and $\mathbf{q}$ are parallel.
Hence the lines l_1 and l_2 are parallel or coincident.
Since $\mathbf{AB} = \mathbf{b} - \mathbf{a} = \mathbf{i} - 3\mathbf{j} - \mathbf{k}$, which is not a scalar multiple of $\mathbf{p}$, the line through A and B is not parallel to $\mathbf{p}$.
Hence the lines l_1 and l_2 must be distinct parallel lines.

Example 6 Show that the lines with vector equations $\mathbf{r} = 2\lambda\mathbf{i} - 3\mathbf{j} + (\lambda - 2)\mathbf{k}$ and $\mathbf{r} = (\mu + 1)\mathbf{i} + (2 - \mu)\mathbf{j} + (2\mu - 5)\mathbf{k}$ do not intersect.

At any point of intersection

$$2\lambda\mathbf{i} - 3\mathbf{j} + (\lambda - 2)\mathbf{k} = (\mu + 1)\mathbf{i} + (2 - \mu)\mathbf{j} + (2\mu - 5)\mathbf{k}$$

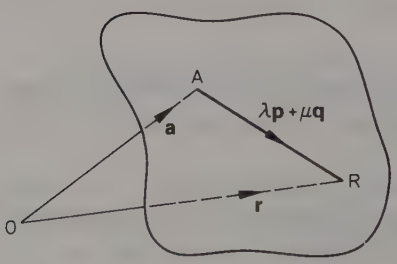
Thus
$$2\lambda = \mu + 1 \quad (1)$$
$$-3 = 2 - \mu \quad (2)$$
$$\lambda - 2 = 2\mu - 5 \quad (3)$$

From (2), $\mu = 5$
Using (1), $2\lambda = 5 + 1 \quad \therefore \quad \lambda = 3$

Testing these values in (3):

$$\lambda - 2 = 1, \quad 2\mu - 5 = 5$$

$\therefore$ no values of λ and μ can be found which satisfy all three equations.
Hence the given lines do not intersect.



Consider now a plane which passes through a point A with position vector $\mathbf{a}$ and which contains non-parallel vectors $\mathbf{p}$ and $\mathbf{q}$. Let $\mathbf{r}$ be the position vector of a point R in the plane.

Since **AR** lies in the plane, there are scalars λ and μ such that

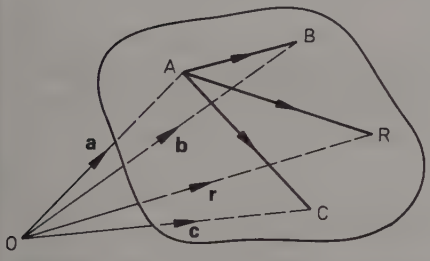
$$\mathbf{AR} = \lambda\mathbf{p} + \mu\mathbf{q}$$
$$\therefore \quad \mathbf{r} - \mathbf{a} = \lambda\mathbf{p} + \mu\mathbf{q}$$
$$\text{i.e.} \quad \mathbf{r} = \mathbf{a} + \lambda\mathbf{p} + \mu\mathbf{q}.$$

This equation is called the *vector equation* of the plane in *parametric form.*

Example 7 Write down the vector equation of the plane through the point with position vector $\mathbf{i} - 2\mathbf{k}$ which is parallel to the vectors $\mathbf{i} + \mathbf{j}$ and $\mathbf{k}$.

The vector equation of the given plane is

$$\mathbf{r} = \mathbf{i} - 2\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j}) + \mu\mathbf{k}$$
$$\text{i.e.} \quad \mathbf{r} = (\lambda + 1)\mathbf{i} + \lambda\mathbf{j} + (\mu - 2)\mathbf{k}.$$



Suppose that **r** is the position vector of a point R which lies in the plane containing three non-collinear points A, B and C with position vectors **a**, **b** and **c** respectively.

Since **AR** lies in the plane containing **AB** and **AC**, there are scalars λ and μ such that

$$\mathbf{AR} = \lambda\mathbf{AB} + \mu\mathbf{AC}$$
$$\therefore \quad \mathbf{r} - \mathbf{a} = \lambda(\mathbf{b} - \mathbf{a}) + \mu(\mathbf{c} - \mathbf{a})$$
$$\text{i.e.} \quad \mathbf{r} = (1 - \lambda - \mu)\mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$$

This is the vector equation of the plane through given points A, B and C.

Example 8 Find a vector equation of the plane through the points $A(1, 0, 0)$, $B(2, -6, 1)$ and $C(-3, 0, 4)$. Hence find the coordinates of the point of intersection of the plane ABC and the line with vector equation $\mathbf{r} = 2t\mathbf{i} + (1 - 5t)\mathbf{j} + (t - 2)\mathbf{k}$.

The points A, B and C have position vectors $\mathbf{i}, 2\mathbf{i} - 6\mathbf{j} + \mathbf{k}$ and $-3\mathbf{i} + 4\mathbf{k}$ respectively.

$\therefore$ the vector equation of the plane ABC is

$$\mathbf{r} = (1 - \lambda - \mu)\mathbf{i} + \lambda(2\mathbf{i} - 6\mathbf{j} + \mathbf{k}) + \mu(-3\mathbf{i} + 4\mathbf{k})$$
$$\text{i.e.} \quad \mathbf{r} = (1 + \lambda - 4\mu)\mathbf{i} - 6\lambda\mathbf{j} + (\lambda + 4\mu)\mathbf{k}.$$

The point of intersection of this plane with the line $\mathbf{r} = 2t\mathbf{i} + (1 - 5t)\mathbf{j} + (t - 2)\mathbf{k}$ is given by

$$1 + \lambda - 4\mu = 2t \qquad (1)$$
$$-6\lambda \qquad = -5t + 1 \qquad (2)$$
$$\lambda + 4\mu = t - 2 \qquad (3)$$

Adding (1) to (3), $1 + 2\lambda = 3t - 2$
$\therefore \quad 2\lambda = 3t - 3$ (4)
Substituting in (2), $-3(3t - 3) = -5t + 1$
i.e. $-9t + 9 = -5t + 1 \quad \therefore \quad t = 2$

Substituting in the equation of the line, $\mathbf{r} = 4\mathbf{i} - 9\mathbf{j}$.
Hence the coordinates of the point of intersection are $(4, -9, 0)$.

Exercise 7.2

1. Find the vector equation of the line with direction vector **d** which passes through the point with position vector **a** given that
(a) $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}, \mathbf{d} = 3\mathbf{i} - \mathbf{k}$,
(b) $\mathbf{a} = 4\mathbf{i} - 3\mathbf{k}, \mathbf{d} = \mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$.

2. Find the vector equation of the line which passes through the points with
(a) position vectors $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $-2\mathbf{i} + \mathbf{j} + \mathbf{k}$,
(b) position vectors $\mathbf{i} + 5\mathbf{j}$ and $3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$,
(c) coordinates $(0, 6, -8)$ and $(5, -7, 2)$,
(d) coordinates $(0, 0, 0)$ and $(5, -2, 3)$.

3. The vertices A, B and C of a triangle have position vectors **a**, **b** and **c** respectively. The point P divides BC internally in the ratio 2:1 and the point Q divides AC externally in the ratio 2:1. Find vector equations for the lines AP and BQ. Hence find the position vector of the point of intersection of the lines AP and BQ.

4. The position vectors of points A and B with respect to an origin O are **a** and **b** respectively. P is the mid-point of OA and $APBQ$ is a parallelogram. The lines AB and OQ intersect at T. Find the position vectors of P, Q and T.

5. The position vectors relative to an origin O of the points A and B are **a** and **b** respectively. The point P on AB is such that $3AP = PB$. The point Q on OP produced is such that AQ is parallel to OB. Find the position vector of the point of intersection of the lines OA and BQ.

6. The position vectors of four points A, B, C, D are $\mathbf{a} = 5\mathbf{j}, \mathbf{b} = -6\mathbf{i} + \mathbf{j}, \mathbf{c} = 6\mathbf{i} - \mathbf{j}$, $\mathbf{d} = -3\mathbf{i} + 5\mathbf{j}$ respectively. The lines AB and CD intersect at a point X. Find the ratio $AX:XB$.

7. Decide whether the following pairs of straight lines are intersecting, parallel or skew. If the lines intersect give the position vector of their point of intersection.
(a) $\mathbf{r} = (1 - \lambda)\mathbf{i} + 3\lambda\mathbf{j} + (2\lambda + 5)\mathbf{k}$,
$\mathbf{r} = 2\mu\mathbf{i} + (\mu - 4)\mathbf{j} + 3\mathbf{k}$.
(b) $\mathbf{r} = -7\mathbf{i} + 4\mathbf{j} + 3\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{k})$,
$\mathbf{r} = 5\mathbf{i} + 8\mathbf{j} + \mathbf{k} + \mu(-4\mathbf{i} + 3\mathbf{k})$.
(c) $\mathbf{r} = 2\lambda\mathbf{i} + 3\mathbf{j} - (2\lambda + 5)\mathbf{k}$,
$\mathbf{r} = (2 - 3\mu)\mathbf{i} - 4\mathbf{j} + 3\mu\mathbf{k}$.
(d) $\mathbf{r} = 2\lambda\mathbf{i} + (4\lambda + 1)\mathbf{j} + (5 - 3\lambda)\mathbf{k}$,
$\mathbf{r} = (\mu + 4)\mathbf{i} + (5\mu + 3)\mathbf{j} + (\mu - 6)\mathbf{k}$.

8. Decide which of the points A, B and C, with position vectors $\mathbf{a} = \mathbf{i} - 2\mathbf{j}$, $\mathbf{b} = 3\mathbf{i} - \mathbf{j} - \mathbf{k}$ and $\mathbf{c} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$ respectively, lie on the following lines and planes.
(a) $\mathbf{r} = (2\lambda + 5)\mathbf{i} + \lambda\mathbf{j} - (\lambda + 2)\mathbf{k}$,
(b) $\mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$,
(c) $\mathbf{r} = (2\lambda - \mu - 2)\mathbf{i} + (1 - \lambda + 2\mu)\mathbf{j} + \lambda\mathbf{k}$,
(d) $\mathbf{r} = -3\mathbf{i} - \mathbf{j} + 4\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} - \mathbf{k}) + \mu(3\mathbf{j} + 2\mathbf{k})$.

9. Write down in parametric form the vector equations of the planes through the given points parallel to the given pairs of vectors.
(a) $(1, -2, 0)$; $\mathbf{i} + 3\mathbf{j}$ and $-\mathbf{j} + 2\mathbf{k}$,
(b) the origin; $2\mathbf{i} - \mathbf{j}$ and $-\mathbf{i} + 2\mathbf{j} - 7\mathbf{k}$,
(c) $(3, 1, -1)$; $\mathbf{j}$ and $\mathbf{i} + \mathbf{j} + \mathbf{k}$.

10. Find a vector equation for the plane passing through the points with position vectors $2\mathbf{k}$, $\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $5\mathbf{i} + 2\mathbf{j}$.

11. Write down direction vectors for the lines l_1 and l_2 with equations $\mathbf{r} = (2 - \lambda)\mathbf{i} + 5\mathbf{j} + 3\lambda\mathbf{k}$ and $\mathbf{r} = 4\mathbf{i} + 5\mu\mathbf{j} - 2\mathbf{k}$ respectively. By writing down the position vector of a particular point on l_1, find a vector equation for the plane which contains l_1 and is parallel to l_2.

12. Find a vector equation for the plane through the points $A(1, 0, -2)$ and $B(3, -1, 1)$ which is parallel to the line with vector equation

$$\mathbf{r} = 3\mathbf{i} + (2\lambda - 1)\mathbf{j} + (5 - \lambda)\mathbf{k}.$$

Hence find the coordinates of the point of intersection of this plane and the line $\mathbf{r} = \mu\mathbf{i} + (5 - \mu)\mathbf{j} + (2\mu - 7)\mathbf{k}$.

13. The position vectors of three non-collinear points A, B and C, with respect to an origin O, are $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ respectively. Given that O does not lie in the plane ABC, show that
(a) if the point P with position vector $\mathbf{p} = \lambda\mathbf{a} + \mu\mathbf{b}$ lies on the line AB, then $\lambda + \mu = 1$;
(b) if the point Q with position vector $\mathbf{q} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$ lies in the plane ABC, then $\alpha + \beta + \gamma = 1$.

7.3 Forces and their lines of action

In this section we consider the application of vector methods to systems of forces given in vector form, often in terms of unit vectors $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$.

[Note that unless otherwise stated we will assume that vector quantities are measured in the appropriate SI units.]

Example 1 A force $\mathbf{P}$ of magnitude 28 N acts at the point A with position vector $2\mathbf{j} + \mathbf{k}$ in the direction of the vector $3\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$. A second force $\mathbf{Q}$ of

magnitude 27 N also acts at A, but in the direction of the vector $\mathbf{i} - 4\mathbf{j} - 8\mathbf{k}$. Find the magnitude of the resultant of the forces $\mathbf{P}$ and $\mathbf{Q}$. Obtain a vector equation for its line of action.

$|3\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}| = \sqrt{\{3^2 + (-2)^2 + 6^2\}} = \sqrt{\{9 + 4 + 36\}} = 7$
$|\mathbf{i} - 4\mathbf{j} - 8\mathbf{k}| = \sqrt{\{1^2 + (-4)^2 + (-8)^2\}} = \sqrt{\{1 + 16 + 64\}} = 9$

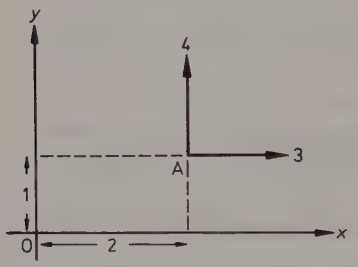
$\therefore$ the unit vectors in the directions of $\mathbf{P}$ and $\mathbf{Q}$ are $\frac{1}{7}(3\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})$ and $\frac{1}{9}(\mathbf{i} - 4\mathbf{j} - 8\mathbf{k})$ respectively.

Thus $\mathbf{P} = 28 \times \frac{1}{7}(3\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}) = 12\mathbf{i} - 8\mathbf{j} + 24\mathbf{k}$
and $\mathbf{Q} = 27 \times \frac{1}{9}(\mathbf{i} - 4\mathbf{j} - 8\mathbf{k}) = 3\mathbf{i} - 12\mathbf{j} - 24\mathbf{k}$
$\therefore \quad \mathbf{P} + \mathbf{Q} = 15\mathbf{i} - 20\mathbf{j} = 5(3\mathbf{i} - 4\mathbf{j})$
$\therefore \quad |\mathbf{P} + \mathbf{Q}| = 5\sqrt{\{3^2 + (-4)^2\}} = 5\sqrt{\{9 + 16\}} = 25$

Hence the resultant of $\mathbf{P}$ and $\mathbf{Q}$ acts in the direction of the vector $3\mathbf{i} - 4\mathbf{j}$ and has magnitude 25 N.
Since $\mathbf{P}$ and $\mathbf{Q}$ act at A, their resultant must also act at A. Thus a vector equation for its line of action is

$$\mathbf{r} = 2\mathbf{j} + \mathbf{k} + \lambda(3\mathbf{i} - 4\mathbf{j})$$
i.e. $\mathbf{r} = 3\lambda\mathbf{i} + (2 - 4\lambda)\mathbf{j} + \mathbf{k}.$

The approach of Example 1 can also be used to find the magnitude and direction of the resultant of a set of coplanar forces with different points of application. However, to find the line of action of the resultant we must use the principle of moments introduced in §5.1.



To illustrate how the moments of individual forces are calculated, we suppose that a force $\mathbf{F} = 3\mathbf{i} + 4\mathbf{j}$ acts at the point A with position vector $2\mathbf{i} + \mathbf{j}$.

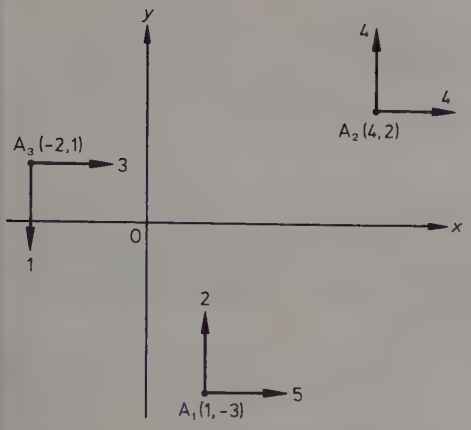
Taking anti-clockwise moments as positive, the moment of $\mathbf{F}$ about O

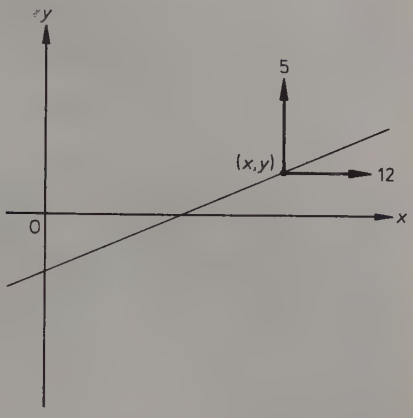
= (moment about O of the force $3\mathbf{i}$) + (moment about O of the force $4\mathbf{j}$)
$= -(3 \times 1) + (4 \times 2)\,\text{N m} = 5\,\text{N m}.$

Example 2 Coplanar forces $\mathbf{F}_1 = 5\mathbf{i} + 2\mathbf{j}$, $\mathbf{F}_2 = 4\mathbf{i} + 4\mathbf{j}$ and $\mathbf{F}_3 = 3\mathbf{i} - \mathbf{j}$ act at points A_1, A_2 and A_3 with position vectors $\mathbf{i} - 3\mathbf{j}, 4\mathbf{i} + 2\mathbf{j}$ and $-2\mathbf{i} + \mathbf{j}$ respectively. Find the magnitude of the resultant of these forces and the Cartesian equation of its line of action.

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = (5 + 4 + 3)\mathbf{i} + (2 + 4 - 1)\mathbf{j} = 12\mathbf{i} + 5\mathbf{j}$$
$\therefore |\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3| = \sqrt{(12^2 + 5^2)} = \sqrt{(144 + 25)} = 13.$

Hence the magnitude of the resultant of the three forces is 13 N.





Let the point with position vector $x\mathbf{i} + y\mathbf{j}$ be some point on the line of action of the resultant, then the anti-clockwise moment of the resultant about O is $5x - 12y$.

Taking moments anti-clockwise about O for the original forces,

$$\text{moment of } \mathbf{F}_1 = 5 \times 3 + 2 \times 1 = 17$$
$$\text{moment of } \mathbf{F}_2 = -4 \times 2 + 4 \times 4 = 8$$
$$\text{moment of } \mathbf{F}_3 = -3 \times 1 + 1 \times 2 = -1$$

$\therefore$ using the principle of moments,

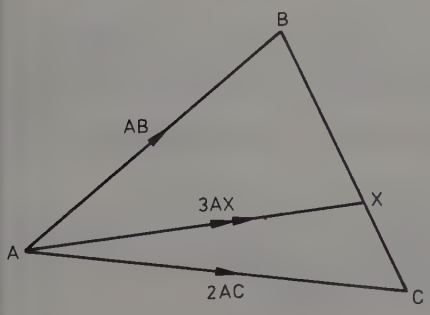
$$5x - 12y = 17 + 8 - 1$$

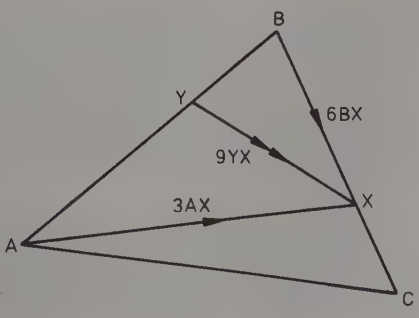
Hence the Cartesian equation of the line of action of the resultant is $5x - 12y = 24$.

The *ratio theorem* derived in §7.1 can be used to find resultant forces.

If two forces acting at a point O are represented in magnitude and direction by $\lambda\mathbf{OP}$ and $\mu\mathbf{OQ}$, then their resultant is represented by $(\lambda + \mu)\mathbf{OR}$, where R is the point which divides PQ in the ratio $\mu:\lambda$.

Example 3 Forces represented by **AB**, 2**AC** and 4**BC** act along the sides of a triangle ABC. Find their resultant.





By the ratio theorem, $\mathbf{AB} + 2\mathbf{AC} = 3\mathbf{AX}$,
where X is the point which divides BC in the ratio 2:1.

Since $BX:XC = 2:1$, we have $BC = \frac{3}{2}BX$,

$\therefore \quad \mathbf{AB} + 2\mathbf{AC} + 4\mathbf{BC} = 3\mathbf{AX} + 6\mathbf{BX}$.

By the ratio theorem, $3\mathbf{AX} + 6\mathbf{BX} = 9\mathbf{YX}$,

where Y is the point which divides AB in the ratio 6:3 i.e. 2:1. Hence the resultant of the three forces acts through X and is represented by $9\mathbf{YX}$, where X divides BC in the ratio 2:1 and Y divides AB in the ratio 2:1.

Exercise 7.3

1. Forces of magnitude 10 N and 5 N act in the directions of the vectors $3\mathbf{i} - 4\mathbf{j}$ and $4\mathbf{i} + 3\mathbf{j}$ respectively. Express each force as a vector in terms of $\mathbf{i}$ and $\mathbf{j}$. Hence find the resultant of the forces as a vector.

2. A force $\mathbf{R}$ of magnitude $4\sqrt{2}$ N acting in the direction of the vector $\mathbf{i} - \mathbf{j}$ is the resultant of two forces $\mathbf{P}$ and $\mathbf{Q}$ which act in the directions of the vectors $\mathbf{i} + \mathbf{j}$ and $\mathbf{i} - 7\mathbf{j}$ respectively. Find the magnitudes of $\mathbf{P}$ and $\mathbf{Q}$.

3. In a parallelogram $OABC$ the position vectors of A and B relative to O are $\mathbf{a} = 4\mathbf{i} + 3\mathbf{j}$ and $\mathbf{b} = 3\mathbf{j} + 3\mathbf{k}$ respectively. Forces of magnitude 20 N, $9\sqrt{2}$ N and 5 N act along OA, OB and OC respectively. Find the magnitude of their resultant.

4. In each of the following cases obtain a vector equation for the line of action of the resultant of forces $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$ acting at a point with position vector $\mathbf{a}$.
(a) $\mathbf{F}_1 = 2\mathbf{i} - 3\mathbf{j}$, $\mathbf{F}_2 = \mathbf{i} + 5\mathbf{j}$, $\mathbf{F}_3 = -\mathbf{i} + 2\mathbf{j}$; $\mathbf{a} = \mathbf{i} + \mathbf{j}$.
(b) $\mathbf{F}_1 = -3\mathbf{i} + 2\mathbf{k}$, $\mathbf{F}_2 = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{F}_3 = 4\mathbf{j}$; $\mathbf{a} = 2\mathbf{j} - \mathbf{k}$.
(c) $\mathbf{F}_1 = \mathbf{i} - 5\mathbf{j} + 8\mathbf{k}$, $\mathbf{F}_2 = 2\mathbf{i} - \mathbf{j}$, $\mathbf{F}_3 = -3\mathbf{i} + \mathbf{k}$; $\mathbf{a} = 3\mathbf{i}$.

5. Forces $\mathbf{P} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{Q} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$ act through points with position vectors $2\mathbf{i} - \mathbf{j}$ and $9\mathbf{i} - \mathbf{k}$ respectively. Write down vector equations for the lines of action of $\mathbf{P}$ and $\mathbf{Q}$. Hence find the position vector of the point through which both forces act. Find also a vector equation for the line of action of the resultant of $\mathbf{P}$ and $\mathbf{Q}$.

6. Forces $\mathbf{F}_1 = \mathbf{i} + 7\mathbf{j} + 4\mathbf{k}$ and $\mathbf{F}_2 = 5\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$ act through the origin O. A third force $\mathbf{F}_3 = \mathbf{i} - 4\mathbf{j} + 6\mathbf{k}$ acts through the point A with position vector $\mathbf{a} = 3\mathbf{i} + 5\mathbf{j}$. Show that the resultant of $\mathbf{F}_1$ and $\mathbf{F}_2$ also acts through A. Find the magnitude of the resultant of all three forces and obtain a vector equation for its line of action.

7. Coplanar forces $\mathbf{F}_1 = \mathbf{i} - 2\mathbf{j}$, $\mathbf{F}_2 = 3\mathbf{i} + 4\mathbf{j}$, $\mathbf{F}_3 = -5\mathbf{i} + \mathbf{j}$ act at points A_1, A_2, A_3 with position vectors $4\mathbf{i} + 5\mathbf{j}$, $3\mathbf{i} - 2\mathbf{j}$, $\mathbf{i} + 4\mathbf{j}$ respectively. Find the Cartesian equation of the line of action of the resultant of these forces.

8. Coplanar forces $\mathbf{F}_1 = 6\mathbf{i} + 2\mathbf{j}, \mathbf{F}_2 = -\mathbf{i} + 7\mathbf{j}, \mathbf{F}_3 = 4\mathbf{i} - 3\mathbf{j}$ act at points A_1, A_2, A_3 with position vectors $\mathbf{i} - \mathbf{j}, 4\mathbf{j}, 2\mathbf{i} + 3\mathbf{j}$ respectively. Find the magnitude of the resultant and the tangent of the angle it makes with the vector $\mathbf{i}$. Find also the Cartesian equation of its line of action.

9. The following sets of forces act along the edges of a triangle ABC. In each case find the resultant force, using a clear diagram to illustrate your answer.
(a) $2\mathbf{AB}, 3\mathbf{AC}, 6\mathbf{BC}$,
(b) $\mathbf{CB}, 3\mathbf{AB}, 5\mathbf{CA}$,
(c) $2\mathbf{BC}, \mathbf{BA}, 2\mathbf{AC}$,
(d) $6\mathbf{AB}, 4\mathbf{CA}, 3\mathbf{CB}$.

10. Three forces acting at a point P are represented by $\mathbf{PA}, \mathbf{PB}$ and $\mathbf{PC}$. Show that their resultant is given by $3\mathbf{PG}$, where G is the centroid of triangle ABC.

11. Forces represented by $3\mathbf{AB}, 2\mathbf{CB}, 4\mathbf{CD}$ and $6\mathbf{AD}$ act along the sides of a square $ABCD$. Show that their resultant is $15\mathbf{XY}$, where X divides AC in the ratio 2:3 and Y divides BD in the ratio 2:1.

12. Three forces represented by $\mathbf{PQ}, \mathbf{QR}$ and $\mathbf{RP}$ act along the sides of a triangle PQR. Show that these forces are equivalent to a couple whose moment is represented by twice the area of the triangle.

7.4 Use of vectors in kinematics and dynamics

Suppose that a particle P moving with constant velocity $\mathbf{v}$ has position vector $\mathbf{r}_0$ initially and position vector $\mathbf{r}$ at time t.

It follows that $$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}.$$

This equation gives the position vector of P at any instant and can be taken as the vector equation of the straight line path of the particle.

The next example shows the use of such equations in a problem concerning the *relative motion* of two ships.

Example 1 At noon ships A and B have position vectors $3\mathbf{i} + 2\mathbf{j}$ and $\mathbf{i} - 4\mathbf{j}$ respectively and constant velocities $4\mathbf{i} + 3\mathbf{j}$ and $3\mathbf{i} + 5\mathbf{j}$ respectively, where distances are measured in kilometres and times in hours. Find vectors to represent (a) the velocity of A relative to B, (b) the displacement of A relative to B, t hours after noon. Find also the least distance between the ships.

(a) The velocity of A relative to B

$$= (\text{velocity of } A) - (\text{velocity of } B)$$
$$= (4\mathbf{i} + 3\mathbf{j}) - (3\mathbf{i} + 5\mathbf{j}) = \mathbf{i} - 2\mathbf{j}.$$

(b) The position vector of A after t hours is given by

$$\mathbf{r}_A = 3\mathbf{i} + 2\mathbf{j} + t(4\mathbf{i} + 3\mathbf{j})$$

The position vector of B after t hours is given by

$$\mathbf{r}_B = \mathbf{i} - 4\mathbf{j} + t(3\mathbf{i} + 5\mathbf{j})$$

$\therefore$ the displacement of A relative to B after t hours is

$$\mathbf{r}_A - \mathbf{r}_B = 2\mathbf{i} + 6\mathbf{j} + t(\mathbf{i} - 2\mathbf{j})$$

i.e. $\mathbf{r}_A - \mathbf{r}_B = (2 + t)\mathbf{i} + (6 - 2t)\mathbf{j}$.

The distance between the ships

$$\begin{aligned} = |\mathbf{r}_A - \mathbf{r}_B| &= \sqrt{\{(2 + t)^2 + (6 - 2t)^2\}} \\ &= \sqrt{\{4 + 4t + t^2 + 36 - 24t + 4t^2\}} \\ &= \sqrt{\{5t^2 - 20t + 40\}} \\ &= \sqrt{\{5(t - 2)^2 + 20\}} \end{aligned}$$

Since $(t - 2)^2$ is never negative, this expression takes its least value when $t = 2$. Hence the least distance between the ships is $\sqrt{20}$ km i.e. $5\sqrt{2}$ km.

[Note that the least distance can also be found by writing

$$f(t) = 5t^2 - 20t + 40$$

Differentiating, $f'(t) = 10t - 20, \quad f''(t) = 10$

$\therefore$ $f'(t) = 0 \Leftrightarrow 10t - 20 = 0 \Leftrightarrow t = 2.$

Since $f(2) = 20$ and $f''(2) > 0$, the function $f(t)$ has a minimum value of 20.]

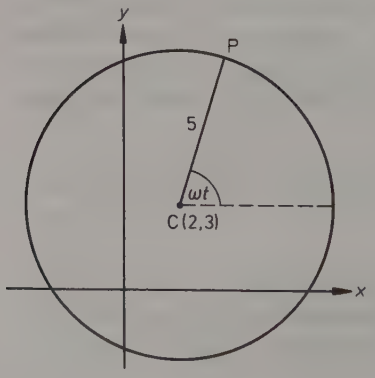
In general, if a moving particle P has position vector $\mathbf{r}$ at time t, then its velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ are given by

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} \quad \text{and} \quad \mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}$$

or, using a dot to indicate differentiation with respect to time,

$$\mathbf{v} = \dot{\mathbf{r}} \quad \text{and} \quad \mathbf{a} = \ddot{\mathbf{r}}.$$

Example 2 A particle P has position vector $\mathbf{r}$ at time t and moves in a circle with constant angular velocity ω. The centre of the circle is the point C with position vector $\mathbf{c} = 2\mathbf{i} + 3\mathbf{j}$, and when $t = 0$, $\mathbf{CP} = 5\mathbf{i}$. Find an expression for $\mathbf{r}$ and hence for the velocity $\mathbf{v}$ of the particle at time t. Find also the time at which P first moves parallel to the vector $\sqrt{3}\mathbf{i} - \mathbf{j}$.



Since P moves on a circle with centre C, the vector $\mathbf{CP}$ has constant magnitude and rotates with constant angular velocity ω. Initially $\mathbf{CP} = 5\mathbf{i}$ and $|\mathbf{CP}| = |5\mathbf{i}| = 5$. Thus, at time t, the

vector **CP** has magnitude 5 units and makes an angle ωt with the direction of **i**. It follows that

$$\mathbf{CP} = 5\cos\omega t\mathbf{i} + 5\sin\omega t\mathbf{j}$$
$$\therefore \quad \mathbf{r} - \mathbf{c} = 5\cos\omega t\mathbf{i} + 5\sin\omega t\mathbf{j}$$
$$\therefore \quad \mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + 5\cos\omega t\mathbf{i} + 5\sin\omega t\mathbf{j}$$

Differentiating with respect to t,

$$\mathbf{v} = \dot{\mathbf{r}} = -5\omega\sin\omega t\mathbf{i} + 5\omega\cos\omega t\mathbf{j}.$$

Thus when P is moving parallel to the vector $\sqrt{3}\mathbf{i} - \mathbf{j}$ there is a scalar λ such that

$$-5\omega\sin\omega t\mathbf{i} + 5\omega\cos\omega t\mathbf{j} = \lambda(\sqrt{3}\mathbf{i} - \mathbf{j})$$

i.e. $-5\omega\sin\omega t = \lambda\sqrt{3}$ and $5\omega\cos\omega t = -\lambda$

$$\therefore \quad \tan\omega t = \frac{\sin\omega t}{\cos\omega t} = \sqrt{3}$$

Hence P first moves parallel to the vector $\sqrt{3}\mathbf{i} - \mathbf{j}$ when $\omega t = \pi/3$ i.e. when $t = \pi/3\omega$.

[Note that at this instant P is not moving in the direction of the vector $\sqrt{3}\mathbf{i} - \mathbf{j}$, but in the opposite direction.]

In problems concerning the forces on moving objects *Newton's second law* is useful. It states that if a particle of mass m moves under the action of a force **F** with acceleration **a**, then

$$\mathbf{F} = m\mathbf{a}.$$

Example 3 A particle of mass 3 kg moves so that its position vector after t seconds is $\mathbf{r} = 8t\mathbf{i} - 5\mathbf{j} + 2t^3\mathbf{k}$. Find (a) the speed of the particle when $t = 1$, (b) the force acting on the particle when $t = 3$.

Let **v** and **a** be the velocity and acceleration of the particle after t seconds, then

$$\mathbf{v} = \dot{\mathbf{r}} = 8\mathbf{i} + 6t^2\mathbf{k}$$
and $\mathbf{a} = \ddot{\mathbf{r}} = 12t\mathbf{k}$.

(a) When $t = 1$, $\mathbf{v} = 8\mathbf{i} + 6\mathbf{k}$

$$\therefore \quad v = |8\mathbf{i} + 6\mathbf{k}| = \sqrt{(8^2 + 6^2)} = 10.$$

Hence, when $t = 1$, the speed of the particle is $10\,\mathrm{m\,s^{-1}}$.

(b) Let **F** be the force on the particle after t seconds, then by Newton's second law,

$$\mathbf{F} = 3\mathbf{a}.$$

When $t = 3$, $\mathbf{a} = 36\mathbf{k}$ $\quad\therefore\quad$ $\mathbf{F} = 108\mathbf{k}$.

Hence, when $t = 3$, the force acting on the particle is $108\mathbf{k}$ (in newtons).

Example 4 A particle of unit mass is acted upon at time t by a force **F**, where $\mathbf{F} = 4\mathbf{i} - 6t\mathbf{j}$. At time $t = 0$ the particle is at the point with position vector $\mathbf{i} + \mathbf{j}$ and its velocity is $5\mathbf{i}$. Find the position vector of the particle at time $t = 2$.

Let $\mathbf{r}$, $\mathbf{v}$ and $\mathbf{a}$ be the position vector, velocity and acceleration of the particle at time t.
By Newton's second law, $\mathbf{F} = m\mathbf{a}$.
Since the particle has unit mass, $\mathbf{F} = \mathbf{a}$.

$$\therefore \qquad \mathbf{a} = \frac{d\mathbf{v}}{dt} = 4\mathbf{i} - 6t\mathbf{j}$$

Integrating, $\qquad \mathbf{v} = 4t\mathbf{i} - 3t^2\mathbf{j} + \mathbf{c},$

where $\mathbf{c}$ is a constant vector.
When $t = 0$, $\mathbf{v} = 5\mathbf{i}$, $\qquad \therefore \quad \mathbf{c} = 5\mathbf{i}$

$$\therefore \qquad \mathbf{v} = \frac{d\mathbf{r}}{dt} = (4t + 5)\mathbf{i} - 3t^2\mathbf{j}$$

Integrating, $\qquad \mathbf{r} = (2t^2 + 5t)\mathbf{i} - t^3\mathbf{j} + \mathbf{d},$

where $\mathbf{d}$ is a constant vector.
When $t = 0$, $\mathbf{r} = \mathbf{i} + \mathbf{j}$, $\qquad \therefore \quad \mathbf{d} = \mathbf{i} + \mathbf{j}$
$\therefore \quad \mathbf{r} = (2t^2 + 5t + 1)\mathbf{i} + (1 - t^3)\mathbf{j}$

Hence, when $t = 2$, the position vector of the particle is

$$\mathbf{r} = (8 + 10 + 1)\mathbf{i} + (1 - 8)\mathbf{j} = 19\mathbf{i} - 7\mathbf{j}.$$

Exercise 7.4

1. Particles A and B are moving with constant velocities $5\mathbf{i} + 2\mathbf{j}$ and $2\mathbf{i} + \mathbf{j}$ respectively. Initially the particle A has position vector $3\mathbf{i} + 4\mathbf{j}$ and the particle B has position vector $5\mathbf{i} + 8\mathbf{j}$. Write down expressions for $\mathbf{r}_A$ and $\mathbf{r}_B$, the position vectors of the particles at time t. Find the vector $\mathbf{AB}$ at time t. Hence find the minimum distance between the particles.

2. Particles P and Q are moving with constant velocities $3\mathbf{i} - 2\mathbf{j}$ and $-\mathbf{i} + 4\mathbf{j}$ respectively. Initially the particle P has position vector $\mathbf{i} + 5\mathbf{j}$ and the particle Q has position vector $3\mathbf{i} + 2\mathbf{j}$. Find the velocity of P relative to Q and the displacement of P relative to Q at time t. Show that the particles collide and state the value of t when the collision occurs.

3. At noon ships A and B have position vectors $10\mathbf{i} + 3\mathbf{j}$ and $12\mathbf{i} - 15\mathbf{j}$ respectively. Both ships move with constant velocity, the velocity of A being $3\mathbf{i} + 8\mathbf{j}$ and the velocity of B relative to A being $-5\mathbf{i} + 4\mathbf{j}$. The unit of distance is the kilometre and the unit of time is the hour. Find an expression for the vector $\mathbf{AB}$, t hours after noon. Hence find the shortest distance between the ships. Find also the position vectors of the ships A and B when they are closest together.

4. A particle P is moving with constant angular velocity ω on a circle of radius 8 units, with centre at the origin. Given that initially P has position vector $8\mathbf{i}$, find its position vector $\mathbf{r}$ at time t. Hence show that $\ddot{\mathbf{r}} = -\omega^2\mathbf{r}$.

5. A particle P moves so that its position vector at time t is given by $\mathbf{r} = \mathbf{i} - 2\mathbf{j} + 3(\mathbf{i}\sin 2t + \mathbf{j}\cos 2t)$. Show that P is moving on a circle, stating the position vector of the centre of the circle and its radius. Find also the speed at which P is moving.

6. Two particles A and B are moving such that, at time t, their position vectors are $\mathbf{r}_A = 3(1 + \cos t)\mathbf{i} + (2\sin t)\mathbf{j}$ and $\mathbf{r}_B = (5 - 6t)\mathbf{i} + (3 + 4t)\mathbf{j}$. Find the value of t when the particles are first moving (a) in the same direction, (b) in opposite directions.

7. The position vector of a particle P at time t is $\mathbf{r} = (1 - t^2)\mathbf{i} + (3t - 5t^2)\mathbf{j}$. Find the time at which P is moving (a) directly towards the origin, (b) directly away from the origin.

8. A particle of mass m moves so that its position vector at time t is $\mathbf{r}$. Find its velocity $\mathbf{v}$ and the force $\mathbf{F}$ acting on it, given that

(a) $\mathbf{r} = 4t^2\mathbf{i} + (3t^3 - 2t)\mathbf{j}$,
(b) $\mathbf{r} = 5\mathbf{i} - 2t^3\mathbf{j} + (t^2 - 1)\mathbf{k}$,
(c) $\mathbf{r} = (2t - t^2)\mathbf{i} + (3\sin 2t)\mathbf{j}$,
(d) $\mathbf{r} = (te^{-t})\mathbf{i} - 2t\mathbf{j}$.

9. A particle of mass 5 kg moves so that its position vector after t seconds is $\mathbf{r} = (\cos 2t)\mathbf{i} + (4\sin 2t + 3)\mathbf{j}$. Find (a) the speed of the particle when $t = \pi/3$, (b) the force acting on the particle when $t = \pi/2$.

10. A particle of mass 2 kg starts from rest at the origin and moves under the action of the force $\mathbf{F}$, where $\mathbf{F} = 8\mathbf{i} + 20\mathbf{j}$. Find the velocity and the position vector of the particle after 3 seconds.

11. A particle of unit mass starts from rest at the point with position vector $2\mathbf{i} - 3\mathbf{j}$. At time t the force acting on the particle is $12t\mathbf{i} - 16\mathbf{j}$. Find the speed of the particle and its position vector at time $t = 2$.

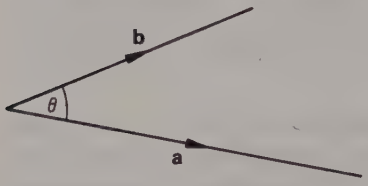
12. A particle of mass 3 kg is acted upon at time t by a force $\mathbf{F}$, where $\mathbf{F} = 6\mathbf{i} - 36t^2\mathbf{j} + 54t\mathbf{k}$. At time $t = 0$ the particle is at the point with position vector $\mathbf{i} - 5\mathbf{j} - \mathbf{k}$ and its velocity is $3(\mathbf{i} + \mathbf{j})$. Find the position vector of the particle at time $t = 1$.

13. A particle of mass 2 kg moves from rest at the origin under the action of two forces. One force $\mathbf{P}$ has magnitude 22 N and acts in the direction of the vector $2\mathbf{i} - 6\mathbf{j} + 9\mathbf{k}$. The other force $\mathbf{Q}$ has magnitude 30 N and acts in the direction of the vector $4\mathbf{j} - 3\mathbf{k}$. Find the acceleration of the particle and the distance it travels in the first 3 seconds of its motion.

14. A particle P is projected vertically upwards from a point O with a speed of $16\,\mathrm{m\,s^{-1}}$. At the same instant a second particle Q is projected horizontally from a point A, 25 m vertically above O, with speed $12\,\mathrm{m\,s^{-1}}$. Using $\mathbf{i}$ and $\mathbf{j}$ as unit vectors in the horizontal and upward vertical directions respectively, find expressions for the position vectors of P and Q with respect to O at time t after projection. Hence find the least distance between the particles.

[Take g as $10\,\mathrm{m\,s^{-2}}$.]

7.5 Scalar product



The *scalar product* (or dot product) of two vectors **a** and **b** is defined as the scalar $ab\cos\theta$, where θ is the angle between **a** and **b**.

The scalar product is denoted by **a . b**, which is read "**a** dot **b**".

Thus $$\mathbf{a}.\mathbf{b} = ab\cos\theta$$

The scalar product **a . a** is sometimes written as $\mathbf{a}^2$.

[Note that although a product of three real number a, b, c can be evaluated by writing $abc = (ab)c$, it is not possible to define a scalar product of three vectors in a similar way. Since **a . b** is a scalar, the expressions **(a . b) . c** and **a . b . c** are meaningless. Similarly, since **a . a** is a scalar, the expression $\mathbf{a}^3$ cannot be defined as **(a . a) . a**.]

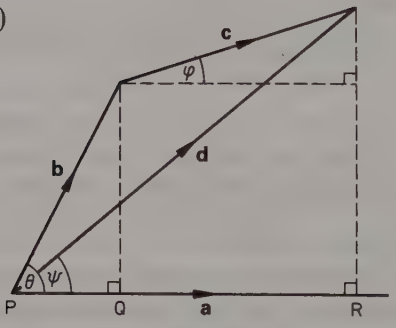
The commutative, associative and distributive properties of multiplication of real numbers can be used to establish various properties of the scalar product.

(1) $\mathbf{a}.\mathbf{b} = ab\cos\theta = ba\cos\theta = \mathbf{b}.\mathbf{a}$
Hence the scalar product is *commutative*.

(2) Since we cannot form products such as **(a . b) . c** there is no true associative law for the scalar product. However, for any scalar λ,

$$\lambda(\mathbf{a}.\mathbf{b}) = (\lambda\mathbf{a}).\mathbf{b} = \mathbf{a}.(\lambda\mathbf{b}) = \lambda ab\cos\theta.$$

(3)



For vectors **a**, **b**, **c** and **d** shown in the diagram,

$$\mathbf{a}.(\mathbf{b}+\mathbf{c}) = \mathbf{a}.\mathbf{d}$$

Since $PR = PQ + QR$

$$d\cos\psi = b\cos\theta + c\cos\phi$$

$$\therefore \quad ad\cos\psi = ab\cos\theta + ac\cos\phi$$

$$\therefore \quad \mathbf{a}.\mathbf{d} = \mathbf{a}.\mathbf{b} + \mathbf{a}.\mathbf{c}$$

$$\therefore \quad \mathbf{a}.(\mathbf{b}+\mathbf{c}) = \mathbf{a}.\mathbf{b} + \mathbf{a}.\mathbf{c}$$

Hence the scalar product is *distributive* over vector addition.

The definition of the scalar product gives rise to further useful properties.

(4) For any vector **a**, $\mathbf{a}.\mathbf{a} = a^2\cos 0° = a^2$.

(5) If non-zero vectors **a** and **b** are perpendicular, then

$$\mathbf{a}.\mathbf{b} = ab\cos 90° = 0.$$

Conversely, if non-zero vectors **a** and **b** are such that $\mathbf{a}.\mathbf{b} = 0$, then **a** is perpendicular to **b**.

These results can be summarised as follows:

(1) $\mathbf{a}.\mathbf{b} = \mathbf{b}.\mathbf{a}$, (2) $\lambda(\mathbf{a}.\mathbf{b}) = (\lambda\mathbf{a}).\mathbf{b} = \mathbf{a}.(\lambda\mathbf{b})$,

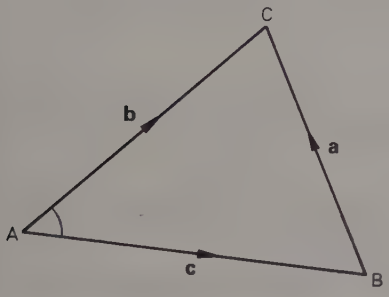
(3) $\mathbf{a}.(\mathbf{b}+\mathbf{c}) = \mathbf{a}.\mathbf{b} + \mathbf{a}.\mathbf{c}$, (4) $\mathbf{a}.\mathbf{a} = a^2$.

(5) $\mathbf{a}.\mathbf{b} = 0 \Leftrightarrow \mathbf{a} = 0$ or $\mathbf{b} = 0$ or $\mathbf{a}$ is perpendicular to $\mathbf{b}$.

The scalar product has important applications in geometrical proofs.

Example 1 Prove the cosine rule in the form

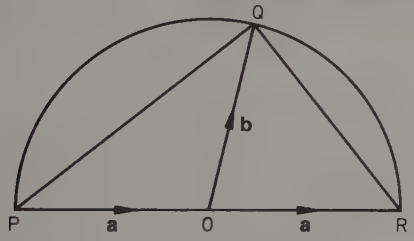
$$a^2 = b^2 + c^2 - 2bc\cos A.$$



In $\triangle ABC$, let $\mathbf{a} = \vec{BC}$, $\mathbf{b} = \vec{AC}$ and $\mathbf{c} = \vec{AB}$, then

$$\begin{aligned} a^2 &= \mathbf{a}.\mathbf{a} = (\mathbf{b}-\mathbf{c}).(\mathbf{b}-\mathbf{c}) \\ &= \mathbf{b}.(\mathbf{b}-\mathbf{c}) - \mathbf{c}.(\mathbf{b}-\mathbf{c}) \\ &= \mathbf{b}.\mathbf{b} - \mathbf{b}.\mathbf{c} - \mathbf{c}.\mathbf{b} + \mathbf{c}.\mathbf{c} \\ &= b^2 + c^2 - 2\mathbf{b}.\mathbf{c} \\ \therefore \quad a^2 &= b^2 + c^2 - 2bc\cos A. \end{aligned}$$

Example 2 Show that any angle inscribed in a semi-circle is a right angle.



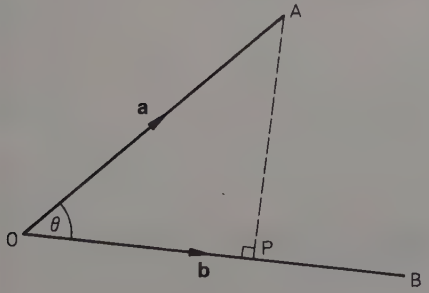
In the diagram, PR is a diameter of a circle with centre O. Let $\vec{PO} = \vec{OR} = \mathbf{a}$ and let Q be a point on the circle such that $\vec{OQ} = \mathbf{b}$, then $\vec{PQ} = \mathbf{a} + \mathbf{b}$, $\vec{QR} = \mathbf{a} - \mathbf{b}$.

$$\begin{aligned} \therefore \quad \vec{PQ}.\vec{QR} &= (\mathbf{a}+\mathbf{b}).(\mathbf{a}-\mathbf{b}) \\ &= \mathbf{a}.\mathbf{a} - \mathbf{a}.\mathbf{b} + \mathbf{b}.\mathbf{a} - \mathbf{b}.\mathbf{b} = a^2 - b^2 \end{aligned}$$

Since PO and OQ are radii of the circle, $a = b$,

$$\therefore \quad \vec{PQ}.\vec{QR} = 0$$

Hence $\vec{PQ}$ is perpendicular to $\vec{QR}$ i.e. $\angle PQR = 90°$.

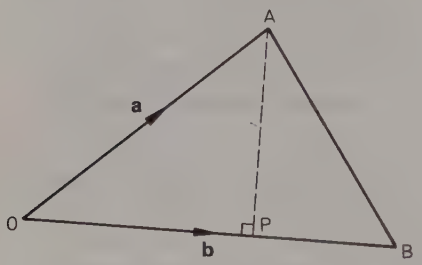


The diagram shows two vectors $\mathbf{a}$ and $\mathbf{b}$ represented by line segments $\vec{OA}$ and $\vec{OB}$ respectively. If the vector $\mathbf{a}$ is resolved along and perpendicular to $\mathbf{b}$, then its component in the direction of $\mathbf{b}$ is $a\cos\theta$.

Thus the component of **a** in the direction of **b** is $\dfrac{\mathbf{a}.\mathbf{b}}{b}$ or $\mathbf{a}.\hat{\mathbf{b}}$, where $\hat{\mathbf{b}}$ is the unit vector in the direction of **b**.

In geometrical work the quantity $\mathbf{a}.\hat{\mathbf{b}}$ is called the *projection* of **a** on **b**. In the diagram the length of this projection is represented by the distance OP.

Example 3 The points A and B have position vectors **a** and **b** with respect to an origin O. Show that the area of triangle OAB is given by $\frac{1}{2}\sqrt{\{a^2b^2 - (\mathbf{a}.\mathbf{b})^2\}}$.



The projection of **a** on **b** is

$$(\mathbf{a}.\mathbf{b})/b,$$

$\therefore$ if P is the foot of the perpendicular from A to OB,

$$OP^2 = (\mathbf{a}.\mathbf{b})^2/b^2$$

Using Pythagoras' theorem in $\triangle OAP$, $OP^2 + AP^2 = OA^2$

$$\therefore \quad AP^2 = OA^2 - OP^2 = a^2 - \frac{(\mathbf{a}.\mathbf{b})^2}{b^2}$$

Hence the area of $\triangle OAB = \frac{1}{2} \times OB \times AP$

$$= \tfrac{1}{2}\sqrt{\left\{b^2 \times \left(a^2 - \frac{(\mathbf{a}.\mathbf{b})^2}{b^2}\right)\right\}}$$
$$= \tfrac{1}{2}\sqrt{\{a^2b^2 - (\mathbf{a}.\mathbf{b})^2\}}.$$

When considering the unit vectors **i**, **j** and **k** the definition of scalar product leads to the results

$$\mathbf{i}.\mathbf{i} = \mathbf{j}.\mathbf{j} = \mathbf{k}.\mathbf{k} = 1, \quad \mathbf{i}.\mathbf{j} = \mathbf{j}.\mathbf{k} = \mathbf{k}.\mathbf{i} = 0$$

Thus for vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$,

$$\begin{aligned}\mathbf{a}.\mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}).(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\ &= a_1b_1\mathbf{i}.\mathbf{i} + a_1b_2\mathbf{i}.\mathbf{j} + a_1b_3\mathbf{i}.\mathbf{k} \\ &\quad + a_2b_1\mathbf{j}.\mathbf{i} + a_2b_2\mathbf{j}.\mathbf{j} + a_2b_3\mathbf{j}.\mathbf{k} \\ &\quad + a_3b_1\mathbf{k}.\mathbf{i} + a_3b_2\mathbf{k}.\mathbf{j} + a_3b_3\mathbf{k}.\mathbf{k}\end{aligned}$$

$\therefore$

$$\boxed{\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3}$$

If the angle between $\mathbf{a}$ and $\mathbf{b}$ is θ, then $\mathbf{a}.\mathbf{b} = ab\cos\theta$.

Thus
$$\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{ab} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{ab}$$

It follows that non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular if and only if
$$a_1b_1 + a_2b_2 + a_3b_3 = 0.$$

Example 4 Find the angle θ between the vectors $\mathbf{a} = 4\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} - 5\mathbf{j} - 4\mathbf{k}$.

$$\mathbf{a}.\mathbf{b} = 4 \times 3 + 5 \times (-5) + 3 \times (-4) = 12 - 25 - 12 = -25$$
$$a = \sqrt{\{4^2 + 5^2 + 3^2\}} = \sqrt{\{16 + 25 + 9\}} = \sqrt{50}$$
$$b = \sqrt{\{3^2 + (-5)^2 + (-4)^2\}} = \sqrt{\{9 + 25 + 16\}} = \sqrt{50}$$
$$\therefore \quad \cos\theta = \frac{\mathbf{a}.\mathbf{b}}{ab} = \frac{-25}{\sqrt{50} \times \sqrt{50}} = -\frac{25}{50} = -\frac{1}{2}$$

Hence the angle between the vectors is 120°.

Example 5 The points A, B and C have position vectors $\mathbf{a} = 5\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{c} = 7\mathbf{i} - 3\mathbf{j} + 10\mathbf{k}$ respectively. Show that $\angle ABC$ is a right angle.

$$\mathbf{BA} = \mathbf{a} - \mathbf{b} = 3\mathbf{i} + 4\mathbf{j} - \mathbf{k}, \quad \mathbf{BC} = \mathbf{c} - \mathbf{b} = 5\mathbf{i} - 2\mathbf{j} + 7\mathbf{k},$$
$$\therefore \quad \mathbf{BA}.\mathbf{BC} = 3 \times 5 + 4 \times (-2) + (-1) \times 7 = 15 - 8 - 7 = 0$$

Hence BA is perpendicular to BC i.e. $\angle ABC$ is a right angle.

Exercise 7.5

1. Use the properties of the scalar product to simplify
(a) $(\mathbf{p} + \mathbf{q}).(\mathbf{p} + \mathbf{q})$, (b) $(\mathbf{p} + \mathbf{q}).(\mathbf{p} - \mathbf{q})$,
(c) $\mathbf{p}.(\mathbf{q} + \mathbf{r}) - \mathbf{p}.(\mathbf{q} - \mathbf{r})$, (d) $|\mathbf{p} - \mathbf{q}|^2$.

2. Given that $\mathbf{x}$ is perpendicular to $\mathbf{y}$, simplify
(a) $\mathbf{x}.(\mathbf{x} + \mathbf{y})$, (b) $(2\mathbf{x} + \mathbf{y}).(\mathbf{x} - 3\mathbf{y})$.

3. Given that $|\mathbf{a}| = 7, |\mathbf{b}| = 4$ and $\mathbf{a}.\mathbf{b} = 8$, calculate $(\mathbf{a} + \mathbf{b}).(\mathbf{a} + \mathbf{b})$ and $(\mathbf{a} - \mathbf{b}).(\mathbf{a} - \mathbf{b})$. Hence find $|\mathbf{a} + \mathbf{b}|$ and $|\mathbf{a} - \mathbf{b}|$.

4. Given that $|\mathbf{x}| = 5, |\mathbf{y}| = 10$ and $\mathbf{x}.\mathbf{y} = 22$, calculate $|\mathbf{x} + \mathbf{y}|$ and $|\mathbf{x} - \mathbf{y}|$.

5. Show that if $\mathbf{a}$ and $\mathbf{b}$ are non-zero parallel vectors, then either $\mathbf{a}.\mathbf{b} = ab$ or $\mathbf{a}.\mathbf{b} = -ab$.

6. Show that if non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are of the same magnitude, then $(\mathbf{a} + \mathbf{b})$ is perpendicular to $(\mathbf{a} - \mathbf{b})$.

7. Show that if $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are non-zero vectors such that $\mathbf{a}.\mathbf{b} = \mathbf{a}.\mathbf{c}$, then either $\mathbf{b} = \mathbf{c}$ or $\mathbf{a}$ is perpendicular to $(\mathbf{b} - \mathbf{c})$.

8. Vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are such that $\mathbf{a}.\mathbf{c} = 3$ and $\mathbf{b}.\mathbf{c} = 4$. Given that the vector $\mathbf{d} = \mathbf{a} + \lambda\mathbf{b}$ is perpendicular to $\mathbf{c}$, find the value of λ.

9. Coplanar vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are such $|\mathbf{a}| = |\mathbf{b}| = 1$, $\mathbf{a}.\mathbf{b} = \frac{1}{2}$, $\mathbf{b}.\mathbf{c} = -\frac{1}{2}$ and $\mathbf{a}.\mathbf{c} = 2$. Find an expression for $\mathbf{c}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

10. Use vector methods to prove Pythagoras' theorem.

11. Use vector methods to prove that the diagonals of a rhombus bisect each other at right angles.

12. In triangle ABC the point D is the mid-point of BC. By letting $\vec{AB} = \mathbf{x}$ and $\vec{BD} = \mathbf{y}$, prove Apollonius' theorem, which states that

$$AB^2 + AC^2 = 2(AD^2 + BD^2).$$

13. In triangle ABC the altitudes through A and B intersect at a point H. Show that if $\vec{HA} = \mathbf{a}$, $\vec{HB} = \mathbf{b}$ and $\vec{HC} = \mathbf{c}$ then $\mathbf{a}.\mathbf{b} = \mathbf{a}.\mathbf{c}$ and $\mathbf{b}.\mathbf{a} = \mathbf{b}.\mathbf{c}$. Deduce that the altitudes of a triangle are concurrent.

14. Find the angles between the following pairs of vectors

(a) $8\mathbf{i} - \mathbf{j} + 4\mathbf{k}, 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$,
(b) $\mathbf{j} + 7\mathbf{k}, -5\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$,
(c) $2\mathbf{i} - 3\mathbf{j} + 8\mathbf{k}, 6\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}$,
(d) $\mathbf{i} - \mathbf{j} + \mathbf{k}, \mathbf{i} + \mathbf{k}$.

15. Find the cosine of the angle BAC, given that

(a) the points A, B and C have position vectors $\mathbf{i} - \mathbf{j}, 5\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and $\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ respectively;
(b) the points A, B and C have coordinates $(0, -2, 1), (1, -1, -2)$ and $(-1, 1, 0)$ respectively.

16. Find the component of the vector $\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ in the direction of

(a) $\mathbf{j}$, (b) $-\mathbf{k}$, (c) $-4\mathbf{i} + 3\mathbf{j}$ (d) $6\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$.

17. Given the points $A(1, -5, 3)$, $B(0, 2, -4)$, $C(-1, 1, 0)$ and $D(7, -2, -8)$, find the lengths of the projections of AB on AC and BD.

18. If $\mathbf{a} = \lambda\mathbf{i} + \mu\mathbf{j}$ and $\mathbf{b} = \mu\mathbf{i} - \lambda\mathbf{j}$, show that $\mathbf{a}$ is perpendicular to $\mathbf{b}$. Hence write down vectors perpendicular to

(a) $3\mathbf{i} + 4\mathbf{j}$, (b) $5\mathbf{i} - 2\mathbf{j}$, (c) $-2\mathbf{i} - 3\mathbf{j}$.

19. Points A and B have position vectors $4\mathbf{i} - 5\mathbf{j} - 2\mathbf{k}$ and $8\mathbf{i} - 5\mathbf{j} + 6\mathbf{k}$ respectively relative to an origin O. Find the angles of triangle AOB and also its area.

20. Given the points $A(7, 3, -1)$, $B(1, 5, 2)$ and $C(3, -1, 1)$, find the cosine of $\angle BAC$ and the area of triangle ABC.

21. Show that the lines with vector equations $\mathbf{r} = (1 + 4\lambda)\mathbf{i} + (1 - \lambda)\mathbf{j} + 2\lambda\mathbf{k}$ and $\mathbf{r} = (5 + 3\mu)\mathbf{i} + 2\mu\mathbf{j} + (2 - 5\mu)\mathbf{k}$ cut at right angles and give the position vector of their point of intersection.

22. The points A, B, C and D have coordinates $(3, -2, 0)$, $(-1, 2, 4)$, $(0, 1, -1)$ and $(5, -4, 4)$ respectively. Find (a) the point of intersection of the lines AB and CD, (b) the acute angle between these lines.

23. $ABCDA'B'C'D'$ is a cube with horizontal faces $ABCD$ and $A'B'C'D'$ and vertical edges AA', BB', CC', DD'. Taking $\mathbf{AB} = \mathbf{i}$, $\mathbf{AD} = \mathbf{j}$, $\mathbf{AA'} = \mathbf{k}$, obtain vectors (in terms of $\mathbf{i}, \mathbf{j}, \mathbf{k}$) perpendicular to the planes $ACC'A'$ and $CDA'B'$, and hence find the angle between these two planes. (O&C)

24. A plane p contains the vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. If $\mathbf{c} = \mathbf{i} + \lambda\mathbf{j} + \mu\mathbf{k}$, find the values of λ and μ such that $\mathbf{c}$ is perpendicular to the plane p. If $\mathbf{d} = -3\mathbf{i} + \mathbf{j} - 3\mathbf{k}$, find $\mathbf{c}.\mathbf{d}$. Hence find the sine of the angle between the plane p and the vector $\mathbf{d}$.

25. Prove that if the vectors $\mathbf{a}$ and $\mathbf{b}$ are differentiable functions of a scalar variable t, then $\dfrac{d}{dt}(\mathbf{a}.\mathbf{b}) = \mathbf{a}.\dfrac{d\mathbf{b}}{dt} + \dfrac{d\mathbf{a}}{dt}.\mathbf{b}$.

7.6 Further lines and planes

As shown in §7.2 the vector equation of the straight line which passes through the point with position vector $\mathbf{a}$ and has direction vector $\mathbf{d}$ is

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}.$$

Similarly the vector equation of the straight line passing through the points with position vectors $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{r} = (1 - \lambda)\mathbf{a} + \lambda\mathbf{b}.$$

The scalar product can be used to find vectors perpendicular to such lines.

Example 1 The points A, B and C have position vectors $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 4\mathbf{k}$, $\mathbf{b} = \mathbf{j} - 4\mathbf{k}$ and $\mathbf{c} = 6\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ respectively. Find the position vector of the point R on BC such that AR is perpendicular to BC. Hence find the perpendicular distance of A from the line BC.

Since R lies on BC its position vector is of the form

$$\mathbf{r} = (1 - \lambda)(\mathbf{j} - 4\mathbf{k}) + \lambda(6\mathbf{i} + 4\mathbf{j} + 5\mathbf{k})$$

i.e. $\mathbf{r} = 6\lambda\mathbf{i} + (3\lambda + 1)\mathbf{j} + (9\lambda - 4)\mathbf{k}$ (1)

Since AR is perpendicular to BC, $\mathbf{AR}.\mathbf{BC} = 0$.

But $\mathbf{BC} = \mathbf{c} - \mathbf{b} = 6\mathbf{i} + 3\mathbf{j} + 9\mathbf{k}$

and $\mathbf{AR} = \mathbf{r} - \mathbf{a} = (6\lambda - 3)\mathbf{i} + (3\lambda + 2)\mathbf{j} + (9\lambda - 8)\mathbf{k}$ (2)

$\therefore \quad 6(6\lambda - 3) + 3(3\lambda + 2) + 9(9\lambda - 8) = 0$

$\therefore \quad 36\lambda - 18 + 9\lambda + 6 + 81\lambda - 72 = 0$

$\therefore \quad 126\lambda - 84 = 0 \qquad \therefore \quad \lambda = \frac{2}{3}$

From (1), the position vector of R is $4\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$

From (2), $\mathbf{AR} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$

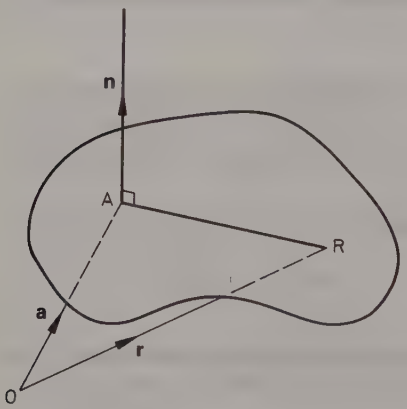
Hence the perpendicular distance of A from BC

$= |\mathbf{AR}| = \sqrt{\{1^2 + 4^2 + (-2)^2\}} = \sqrt{\{1 + 16 + 4\}} = \sqrt{21}.$

The parametric form of the vector equation of the plane which passes through a point A with position vector $\mathbf{a}$ and which contains non-parallel vectors $\mathbf{p}$ and $\mathbf{q}$ is

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{p} + \mu\mathbf{q}.$$

Another form of the equation of a plane is obtained by using the scalar product.



Consider a plane which passes through a point A with position vector $\mathbf{a}$ and which is perpendicular to a vector $\mathbf{n}$. If $\mathbf{r}$ is the position vector of a point R in the plane, then since $\mathbf{AR}$ is perpendicular to $\mathbf{n}$,

$$\mathbf{AR}.\mathbf{n} = 0$$

$$\therefore \quad (\mathbf{r} - \mathbf{a}).\mathbf{n} = 0$$

This is the vector equation of the plane through A perpendicular to $\mathbf{n}$, in scalar product form.

$$(\mathbf{r} - \mathbf{a}).\mathbf{n} = 0 \quad \Leftrightarrow \quad \mathbf{r}.\mathbf{n} - \mathbf{a}.\mathbf{n} = 0$$

$$\Leftrightarrow \quad \mathbf{r}.\mathbf{n} = \mathbf{a}.\mathbf{n}$$

$$\Leftrightarrow \quad \mathbf{r}.\mathbf{n} = p, \quad \text{where} \quad p = \mathbf{a}.\mathbf{n}.$$

Hence the equation of any plane perpendicular to a vector $\mathbf{n}$ may be written in the form $\mathbf{r}.\mathbf{n} = p$.

If the point D with position vector $\mathbf{d}$ is the foot of the perpendicular from the origin to the plane $\mathbf{r}.\mathbf{n} = p$, then the perpendicular distance of the origin from the plane is d.

Since D lies in the plane, $\mathbf{d}.\mathbf{n} = p$.

However, $\mathbf{d}$ is parallel to $\mathbf{n}$,

$\therefore$ either $\mathbf{d}.\mathbf{n} = dn$ or $\mathbf{d}.\mathbf{n} = -dn$.

Hence $dn = |p|$ giving $d = |p|/n$.

Thus the plane $\mathbf{r}.\mathbf{n} = p$ is perpendicular to the vector $\mathbf{n}$ and lies at a distance $\frac{|p|}{n}$ from the origin.

Example 2 Find the vector equation of the plane which passes through the point with position vector $2\mathbf{i} - \mathbf{j} + \mathbf{k}$ and is perpendicular to the vector $3\mathbf{i} + \mathbf{j} - 4\mathbf{k}$.

The vector equation of the given plane is

$$\begin{aligned} (\mathbf{r} - \{2\mathbf{i} - \mathbf{j} + \mathbf{k}\}).(3\mathbf{i} + \mathbf{j} - 4\mathbf{k}) &= 0 \\ \mathbf{r}.(3\mathbf{i} + \mathbf{j} - 4\mathbf{k}) - (2\mathbf{i} - \mathbf{j} + \mathbf{k}).(3\mathbf{i} + \mathbf{j} - 4\mathbf{k}) &= 0 \\ \mathbf{r}.(3\mathbf{i} + \mathbf{j} - 4\mathbf{k}) - (6 - 1 - 4) &= 0 \\ \text{i.e} \qquad \mathbf{r}.(3\mathbf{i} + \mathbf{j} - 4\mathbf{k}) &= 1. \end{aligned}$$

Example 3 Find in scalar product form the equation of the plane

$$\mathbf{r} = (1 + 3\lambda + 2\mu)\mathbf{i} + (1 + \lambda + 4\mu)\mathbf{j} + (\mu - \lambda)\mathbf{k}.$$

Rearranging the given equation, we have

$$\mathbf{r} = \mathbf{i} + \mathbf{j} + \lambda(3\mathbf{i} + \mathbf{j} - \mathbf{k}) + \mu(2\mathbf{i} + 4\mathbf{j} + \mathbf{k})$$

$\therefore$ the given plane passes through the point with position vectors $\mathbf{i} + \mathbf{j}$ and is parallel to the vectors $3\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $2\mathbf{i} + 4\mathbf{j} + \mathbf{k}$.
Let $\mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}$ be a vector perpendicular to $3\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $2\mathbf{i} + 4\mathbf{j} + \mathbf{k}$, then

$$\mathbf{n}.(3\mathbf{i} + \mathbf{j} - \mathbf{k}) = 0 \quad \text{i.e.} \quad 3n_1 + n_2 - n_3 = 0 \qquad (1)$$
$$\mathbf{n}.(2\mathbf{i} + 4\mathbf{j} + \mathbf{k}) = 0 \quad \text{i.e.} \quad 2n_1 + 4n_2 + n_3 = 0 \qquad (2)$$

Adding (1) to (2): $5n_1 + 5n_2 = 0 \quad \therefore \quad n_2 = -n_1$
Substituting in (1): $3n_1 - n_1 - n_3 = 0 \quad \therefore \quad n_3 = 2n_1$

$\therefore \quad n_1 : n_2 : n_3 = n_1 : -n_1 : 2n_1 = 1 : -1 : 2$

Thus one vector perpendicular to the given plane is $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. Hence the equation of the plane in scalar product form is

$$\begin{aligned} (\mathbf{r} - \{\mathbf{i} + \mathbf{j}\}).(\mathbf{i} - \mathbf{j} + 2\mathbf{k}) &= 0 \\ \mathbf{r}.(\mathbf{i} - \mathbf{j} + 2\mathbf{k}) - (\mathbf{i} + \mathbf{j}).(\mathbf{i} - \mathbf{j} + 2\mathbf{k}) &= 0 \\ \text{i.e.} \qquad \mathbf{r}.(\mathbf{i} - \mathbf{j} + 2\mathbf{k}) &= 0 \end{aligned}$$

Example 4 Find the angle between the planes $\mathbf{r}.(2\mathbf{i} - \mathbf{j} - 3\mathbf{k}) = 10$ and $\mathbf{r}.(\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) = 16$.

The given planes are perpendicular to the vectors $\mathbf{n}_1 = 2\mathbf{i} - \mathbf{j} - 3\mathbf{k}$ and $\mathbf{n}_2 = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ respectively.
$\therefore$ the angle θ between the planes is equal to the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$.

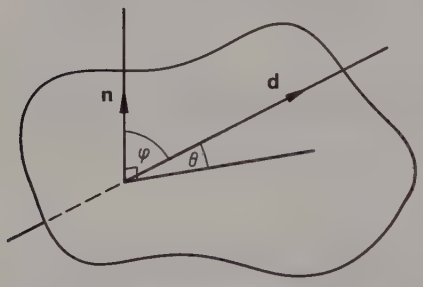
$$\mathbf{n}_1.\mathbf{n}_2 = 2 \times 1 + (-1) \times 3 + (-3) \times (-2) = 2 - 3 + 6 = 5$$
$$n_1 = \sqrt{\{2^2 + (-1)^2 + (-3)^2\}} = \sqrt{\{4 + 1 + 9\}} = \sqrt{14}$$
$$n_2 = \sqrt{\{1^2 + 3^2 + (-2)^2\}} = \sqrt{\{1 + 9 + 4\}} = \sqrt{14}$$

$$\therefore \quad \cos\theta = \frac{\mathbf{n}_1.\mathbf{n}_2}{n_1 n_2} = \frac{5}{\sqrt{14} \times \sqrt{14}} = \frac{5}{14}$$

Hence the angle between the planes is 69° 5′.

Example 5 Find the angle between the line $\mathbf{r} = 3\mathbf{k} + \lambda(7\mathbf{i} - \mathbf{j} + 4\mathbf{k})$ and the plane $\mathbf{r}.(2\mathbf{i} - 5\mathbf{j} - 2\mathbf{k}) = 8$.

The given line is parallel to the vector $\mathbf{d} = 7\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ and the given plane is perpendicular to the vector $\mathbf{n} = 2\mathbf{i} - 5\mathbf{j} - 2\mathbf{k}$.



If the angle between the line and the plane is θ and the angle between $\mathbf{d}$ and $\mathbf{n}$ is ϕ then

$$\theta + \phi = 90°,$$

$$\therefore \quad \sin\theta = \cos\phi = \frac{\mathbf{d}.\mathbf{n}}{dn}$$

$$\mathbf{d}.\mathbf{n} = 7 \times 2 + (-1) \times (-5) + 4 \times (-2) = 14 + 5 - 8 = 11$$
$$d = \sqrt{\{7^2 + (-1)^2 + 4^2\}} = \sqrt{\{49 + 1 + 16\}} = \sqrt{66}$$
$$n = \sqrt{\{2^2 + (-5)^2 + (-2)^2\}} = \sqrt{\{4 + 25 + 4\}} = \sqrt{33}$$

$$\therefore \quad \sin\theta = \frac{11}{\sqrt{66} \times \sqrt{33}} = \frac{11}{33\sqrt{2}} = \frac{1}{3\sqrt{2}}$$

Hence the angle between the line and the plane is 13° 38′.

Example 6 Find the position vector of the point of intersection of the line $\mathbf{r} = (2\mathbf{i} - \mathbf{k}) + \lambda(\mathbf{i} + 3\mathbf{j})$ and the plane $\mathbf{r}.(5\mathbf{i} - \mathbf{j} + 7\mathbf{k}) = 9$.

At the point of intersection

$$\{(2\mathbf{i} - \mathbf{k}) + \lambda(\mathbf{i} + 3\mathbf{j})\}.(5\mathbf{i} - \mathbf{j} + 7\mathbf{k}) = 9$$
$$\therefore \quad (2\mathbf{i} - \mathbf{k}).(5\mathbf{i} - \mathbf{j} + 7\mathbf{k}) + \lambda(\mathbf{i} + 3).(5\mathbf{i} - \mathbf{j} + 7\mathbf{k}) = 9$$
$$\therefore \quad (10 + 0 - 7) + \lambda(5 - 3 + 0) = 9$$
$$\therefore \quad 3 + 2\lambda = 9$$
$$\therefore \quad \lambda = 3$$

Hence the position vector of the point of intersection is

$$2\mathbf{i} - \mathbf{k} + 3(\mathbf{i} + 3\mathbf{j}) \quad \text{i.e.} \quad 5\mathbf{i} + 9\mathbf{j} - \mathbf{k}.$$

By writing $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ we can express the equations of lines and planes in Cartesian form.

Consider first the straight line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$, which passes through the point with position vector $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and is parallel to the vector $\mathbf{d} = d_1\mathbf{i} + d_2\mathbf{j} + d_3\mathbf{k}$.

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{d} \Leftrightarrow \left.\begin{aligned} x &= a_1 + \lambda d_1 \\ y &= a_2 + \lambda d_2 \\ z &= a_3 + \lambda d_3 \end{aligned}\right\}$$ These are the *parametric equations* of the line.

Assuming that d_1, d_2 and d_3 are non-zero, we have

$$\frac{x - a_1}{d_1} = \lambda, \quad \frac{y - a_2}{d_2} = \lambda, \quad \frac{z - a_3}{d_3} = \lambda$$

$$\therefore \quad \frac{x - a_1}{d_1} = \frac{y - a_2}{d_2} = \frac{z - a_3}{d_3}$$

These are the *Cartesian equations* of the line.
Any line parallel to the vector $d_1\mathbf{i} + d_2\mathbf{j} + d_3\mathbf{k}$ is said to have *direction ratios* $d_1 : d_2 : d_3$.

Thus the equations $\dfrac{x - a_1}{d_1} = \dfrac{y - a_2}{d_2} = \dfrac{z - a_3}{d_3}$ represent the straight line through the point (a_1, a_2, a_3) with direction ratios $d_1 : d_2 : d_3$.

By convention the Cartesian equations of a straight line may be given in the above form for all values of d_1, d_2, d_3 including zero. For instance, the equations $\dfrac{x - 2}{3} = \dfrac{y + 1}{0} = \dfrac{z}{-2}$ represent the straight line through the point $(2, -1, 0)$, with direction ratios $3 : 0 : -2$ i.e. in a direction parallel to the vector $3\mathbf{i} - 2\mathbf{k}$.

Consider now the plane which passes through the point with position vector $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and is perpendicular to the vector $\mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}$. Writing $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, its vector equation $(\mathbf{r} - \mathbf{a}).\mathbf{n} = 0$ becomes

$$\{(x - a_1)\mathbf{i} + (y - a_2)\mathbf{j} + (z - a_3)\mathbf{k}\}.(n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}) = 0.$$

Hence the Cartesian equation of this plane is

$$n_1(x - a_1) + n_2(x - a_2) + n_3(x - a_3) = 0.$$

Similarly the vector equation $\mathbf{r}.\mathbf{n} = p$ gives rise to the Cartesian form

$$n_1x + n_2y + n_3z - p = 0.$$

For instance, the Cartesian equation of the plane $\mathbf{r}.(3\mathbf{i} - \mathbf{j} + 4\mathbf{k}) = 5$ is $3x - y + 4z - 5 = 0$.
It follows that the equation $Ax + By + Cz + D = 0$ is the Cartesian equation of a plane perpendicular to the vector $A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$.

Thus the equation $Ax + By + Cz + D = 0$ represents a plane, the direction ratios of the normal to the plane being $A : B : C$.

Exercise 7.6

1. The equation of a line l is $\mathbf{r} = \lambda\mathbf{i} - (\lambda + 4)\mathbf{j} + \mathbf{k}$. Find the position vector with respect to the origin O of the point A on l, given that OA is perpendicular to l. Hence find the perpendicular distance of O from the line l.

2. The position vectors of points A and B are $3\mathbf{i} - 8\mathbf{j} + \mathbf{k}$ and $4\mathbf{j} - 2\mathbf{k}$ respectively. Find the position vector of the foot of the perpendicular from the origin O to AB. Hence find the area of the triangle OAB.

3. The points A and B have position vectors $2\mathbf{j} + 7\mathbf{k}$ and $5\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ respectively relative to an origin O. Find the position vector of the point R on AB such that OR is perpendicular to AB. Hence find the position vector of the reflection of O in AB.

4. A point A has position vector $2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$ and a line l has equation $\mathbf{r} = \mathbf{i} - \mathbf{j} + 7\mathbf{k} + \lambda(4\mathbf{i} - \mathbf{k})$. Find the position vector of a point R on l such that AR is perpendicular to l. Hence find the perpendicular distance of A from the line l.

5. Find, in scalar product form, the equation of the plane through the point with position vector $\mathbf{a}$ and perpendicular to the vector $\mathbf{n}$, given that
(a) $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}, \mathbf{n} = 4\mathbf{i} - \mathbf{k}$, (b) $\mathbf{a} = 3\mathbf{j} - 2\mathbf{k}, \mathbf{n} = \mathbf{i} - \mathbf{j} - 2\mathbf{k}$.

6. Find, in the form $\mathbf{r}.\mathbf{n} = p$, the equations of the planes through the given points and perpendicular to the given vectors
(a) $(0, 0, 1), \mathbf{k}$, (b) $(2, -1, 1), 3\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.

7. Find the perpendicular distances from the origin of the following planes
(a) $\mathbf{r}.(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) = 15$, (b) $\mathbf{r}.\mathbf{i} = 3$,
(c) $\mathbf{r}.(3\mathbf{i} - 4\mathbf{k}) = 20$, (b) $\mathbf{r}.(\mathbf{i} + \mathbf{j} + \mathbf{k}) = 0$.

8. Find a vector which is perpendicular to both $\mathbf{i} + 2\mathbf{k}$ and $3\mathbf{i} + \mathbf{j} + \mathbf{k}$. Hence find the vector equation in scalar product form of the plane parallel to these vectors which passes through the point with position vector $\mathbf{i} - \mathbf{j}$.

9. Find in scalar product form the equation of the plane

$$\mathbf{r} = (3 + 3\lambda + 4\mu)\mathbf{i} + (\lambda + 2\mu)\mathbf{j} + (1 + 3\lambda + 5\mu)\mathbf{k}.$$

10. Find in scalar product form the equation of the plane containing the lines

$$\mathbf{r} = (\lambda + 1)\mathbf{i} - 2\mathbf{j} + 3\mathbf{k} \text{ and } \mathbf{r} = 3\mu\mathbf{i} + (2\mu - 4)\mathbf{j} + (\mu + 2)\mathbf{k}.$$

11. Find the acute angles between the following pairs of planes.
(a) $\mathbf{r}.(\mathbf{i} - \mathbf{j}) = 4, \mathbf{r}.(\mathbf{j} + \mathbf{k}) = 1$,
(b) $\mathbf{r}.(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) = 5, \mathbf{r}.\mathbf{j} = 0$,
(c) $\mathbf{r}.(\mathbf{i} + \mathbf{j} + \mathbf{k}) = 1, \mathbf{r}.(\mathbf{i} - \mathbf{j} + \mathbf{k}) = 0$.

12. In each of the following cases find the sine of the angle between the given line and plane.
(a) $\mathbf{r} = \mathbf{i} - 3\mathbf{j} + \lambda(2\mathbf{i} - \mathbf{j} - \mathbf{k})$, $\mathbf{r}.(\mathbf{i} - 2\mathbf{j} - 7\mathbf{k}) = 10$,
(b) $\mathbf{r} = (2 + \lambda)\mathbf{i} - 3\mathbf{j} + (1 - \lambda)\mathbf{k}$, $\mathbf{r}.(4\mathbf{i} + \mathbf{j} - \mathbf{k}) = 6$,
(c) $\mathbf{r} = 3\lambda\mathbf{i} + 2\lambda\mathbf{j} - 6\lambda\mathbf{k}$, $\mathbf{r}.(4\mathbf{i} - 3\mathbf{k}) = 20$.

13. Find the position vectors of points of intersection of the line $\mathbf{r} = \lambda\mathbf{i} - 2\mathbf{j} + (2\lambda - 1)\mathbf{k}$ with the planes $\mathbf{r}.(2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) = 5$ and $\mathbf{r}.(\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}) = -2$.

14. Find the vector equation of the line through the point A with position vector $5\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ which is perpendicular to the plane $\mathbf{r}.(3\mathbf{i} - 4\mathbf{j} + \mathbf{k}) = 5$. Hence find the position vector of the foot of the perpendicular from A to this plane.

15. Find the Cartesian equations of the straight line which
(a) has vector equation $\mathbf{r} = \mathbf{i} - 2\mathbf{j} + \mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$,
(b) has vector equation $\mathbf{r} = \lambda\mathbf{i} + (2 - \lambda)\mathbf{j} + (3 + 2\lambda)\mathbf{k}$,
(c) passes through the point $(2, -3, 1)$ and has direction ratios $5 : -6 : 3$,
(d) passes through the point $(5, 0, -4)$ and has direction ratios $1 : 3 : 0$.

16. Write down the Cartesian equations of the planes
(a) $\mathbf{r}.(2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) = 5$, (b) $\mathbf{r}.(3\mathbf{i} + \mathbf{j} - 5\mathbf{k}) = 9$.

17. Find the Cartesian equation of the plane which passes through the point with position vector $\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and is perpendicular to the vector $5\mathbf{i} + \mathbf{j} + \mathbf{k}$.

18. Find the Cartesian equation of a plane which passes through the point $(4, 0, -1)$, given that the normal to the plane has direction ratios $2 : -1 : 5$.

19. The points A, B and C have position vectors $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{i} + 2\mathbf{j}$ and $4\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ respectively. Find a vector perpendicular to both $\mathbf{AB}$ and $\mathbf{AC}$. Hence find the equation of the plane containing A, B and C in (a) scalar product form, (b) Cartesian form.

20. Find the Cartesian equation of the plane containing the points $A(2, -1, 1)$, $B(1, -2, 0)$ and $C(-3, 6, -1)$.

21. Find the acute angle between

(a) the lines $\dfrac{x-1}{1} = \dfrac{y+2}{-2} = \dfrac{z}{2}$, $\dfrac{x+3}{3} = \dfrac{y+1}{0} = \dfrac{z-2}{-4}$;

(b) the planes $x - 2y - 3z = 1$, $2x - 4y + z = 3$;

(c) the line $\dfrac{x}{4} = \dfrac{y-1}{-1} = \dfrac{z+3}{-5}$, the plane $x - 2y + 4z + 3 = 0$.

22. The points A and B are fixed points with position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively relative to an origin O. If R is a variable point with position vector $\mathbf{r}$, describe

the locus of R in each of the following cases

(a) $\mathbf{r}.\mathbf{a} = 0$, (b) $(\mathbf{r} - \mathbf{a}).\mathbf{a} = 0$, (c) $(\mathbf{r} - \mathbf{a}).(\mathbf{r} - \mathbf{a}) = 1$,

(d) $\mathbf{r}.\mathbf{a} = \mathbf{r}.\mathbf{b}$, (e) $(\mathbf{r} - \mathbf{a}).\mathbf{r} = 0$, (f) $(\mathbf{r} - \mathbf{a}).(\mathbf{r} - \mathbf{b}) = 0$.

7.7 The scalar product in mechanics

In §7.5 the scalar product was used to find the component of a vector in a given direction and also to calculate the cosine of the angle between two vectors. These methods can be used in mechanics when dealing with vector quantities such as forces and velocities.

Example 1 Find the component of the force $\mathbf{F} = 2\mathbf{i} + 13\mathbf{j} - 8\mathbf{k}$ in the direction of the vector $\mathbf{d} = 2\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}$. Hence resolve $\mathbf{F}$ into component forces parallel and perpendicular to $\mathbf{d}$.

Let $\hat{\mathbf{d}}$ be the unit vector in the direction of $\mathbf{d}$.

$$d = |2\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}| = \sqrt{\{2^2 + (-6)^2 + 3^2\}} = \sqrt{\{4 + 36 + 9\}} = 7$$

$$\therefore \quad \hat{\mathbf{d}} = \frac{\mathbf{d}}{d} = \tfrac{2}{7}\mathbf{i} - \tfrac{6}{7}\mathbf{j} + \tfrac{3}{7}\mathbf{k}$$

Hence the component of $\mathbf{F}$ in the direction of $\mathbf{d}$

$$= \mathbf{F}.\hat{\mathbf{d}} = (2\mathbf{i} + 13\mathbf{j} - 8\mathbf{k}).(\tfrac{2}{7}\mathbf{i} - \tfrac{6}{7}\mathbf{j} + \tfrac{3}{7}\mathbf{k})$$

$$= \tfrac{4}{7} - \tfrac{78}{7} - \tfrac{24}{7} = -\tfrac{98}{7} = -14.$$

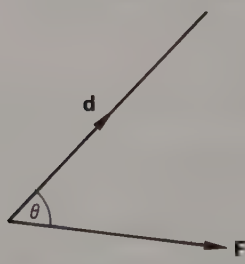
Resolving $\mathbf{F}$ parallel and perpendicular to $\mathbf{d}$, the component force parallel to $\mathbf{d}$

$$= (\mathbf{F}.\hat{\mathbf{d}})\hat{\mathbf{d}} = -14(\tfrac{2}{7}\mathbf{i} - \tfrac{6}{7}\mathbf{j} + \tfrac{3}{7}\mathbf{k}) = -4\mathbf{i} + 12\mathbf{j} - 6\mathbf{k}.$$

Thus the component force perpendicular to $\mathbf{d}$

$$= (2\mathbf{i} + 13\mathbf{j} - 8\mathbf{k}) - (-4\mathbf{i} + 12\mathbf{j} - 6\mathbf{k}) = 6\mathbf{i} + \mathbf{j} - 2\mathbf{k}.$$

We next show that the *work done by a force* can be expressed as a scalar product. We recall that if the point of application of a constant force $\mathbf{F}$ moves a distance s in a direction which makes an angle θ with the direction of $\mathbf{F}$, then the work done by the force is $Fs\cos\theta$.



If we now suppose that the point of application of a constant force $\mathbf{F}$ is given a displacement $\mathbf{d}$, where the angle between the vectors $\mathbf{F}$ and $\mathbf{d}$ is θ, then,

work done by $\mathbf{F} = Fd\cos\theta = \mathbf{F}.\mathbf{d}$.

[We will assume that forces, displacements and work are measured in newtons, metres and joules respectively.]

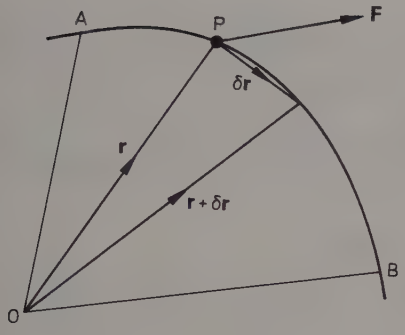
Example 2 A constant force $\mathbf{F} = 3\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}$ acts on a particle. Find the work done by $\mathbf{F}$ when the particle moves from the point with position vector $\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ to the point with position vector $4\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

The displacement of the particle

$$= (4\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) - (\mathbf{i} + 3\mathbf{j} - \mathbf{k}) = 3\mathbf{i} - \mathbf{j} + 4\mathbf{k}$$

$\therefore$ the work done by $\mathbf{F}$

$$= (3\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}).(3\mathbf{i} - \mathbf{j} + 4\mathbf{k}) = 9 + 4 - 8 = 5.$$



Consider now a variable force $\mathbf{F}$ acting on a particle P as it moves along a curve from a point A to a point B. If the path from A to B is regarded as a series of small displacements, then the work done by $\mathbf{F}$ in a typical displacement $\delta\mathbf{r}$ is approximately equal to $\mathbf{F}.\delta\mathbf{r}$.

$\therefore$ total work done by $\mathbf{F}$

$$\approx \sum_{A}^{B} \mathbf{F}.\delta\mathbf{r}$$

Letting $\delta\mathbf{r} \to 0$, we deduce that as the particle moves from A to B,

$$\boxed{\text{the work done by } \mathbf{F} \text{ is } \int_A^B \mathbf{F}.d\mathbf{r}} \qquad (1)$$

If $\mathbf{F}$ and $\mathbf{r}$ are given as functions of the time t, then we write

$$\int_A^B \mathbf{F}.d\mathbf{r} = \int_{t_1}^{t_2} \mathbf{F}.\frac{d\mathbf{r}}{dt}dt = \int_{t_1}^{t_2} \mathbf{F}.\mathbf{v}\,dt \qquad (2)$$

where t_1 and t_2 are the times at which the particle passes through A and B respectively.

Example 3 A particle moves along a curve so that its position vector at time t is given by $\mathbf{r} = t^3\mathbf{i} + 2t\mathbf{j}$. If one of the forces acting on the particle is a force $\mathbf{F}$, where $\mathbf{F} = 2\mathbf{i} - t\mathbf{j}$ at time t, find the work done by $\mathbf{F}$ in the interval from $t = 0$ to $t = 3$.

$$\mathbf{r} = t^3\mathbf{i} + 2t\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = 3t^2\mathbf{i} + 2\mathbf{j}$$

$\therefore$ the work done by $\mathbf{F}$ from $t = 0$ to $t = 3$

$$= \int_0^3 \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt$$

$$= \int_0^3 (2\mathbf{i} - t\mathbf{j}) \cdot (3t^2\mathbf{i} + 2\mathbf{j})\, dt$$

$$= \int_0^3 (6t^2 - 2t)\, dt$$

$$= \left[2t^3 - t^2\right]_0^3 = 54 - 9 = 45.$$

If a particle of mass m is moving with velocity $\mathbf{v}$ at time t, then its *kinetic energy* at that instant is $\frac{1}{2}mv^2$. We can establish the relationship between work and kinetic energy by considering the motion of this particle along a curved path AB. We will assume that the particle passes through A at time t_1 with velocity $\mathbf{v}_1$ and through B at time t_2 with velocity $\mathbf{v}_2$. If $\mathbf{F}$ is the resultant force acting on the particle at time t and W is the work done by $\mathbf{F}$ in the interval $t_1 \leqslant t \leqslant t_2$, then using our earlier results (1) and (2),

$$W = \int_A^B \mathbf{F} \cdot d\mathbf{r} = \int_{t_1}^{t_2} \mathbf{F} \cdot \mathbf{v}\, dt.$$

Using Newton's second law, $\mathbf{F} = m\mathbf{a} = m\dfrac{d\mathbf{v}}{dt}$

$\therefore$
$$W = \int_{t_1}^{t_2} m\frac{d\mathbf{v}}{dt} \cdot \mathbf{v}\, dt$$

But, using the product rule proved in Exercise 7.5 question 25,

$$\frac{d}{dt}(v^2) = \frac{d}{dt}(\mathbf{v} \cdot \mathbf{v}) = \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} + \frac{d\mathbf{v}}{dt} \cdot \mathbf{v} = 2\frac{d\mathbf{v}}{dt} \cdot \mathbf{v}$$

$\therefore$
$$\int \frac{d\mathbf{v}}{dt} \cdot \mathbf{v}\, dt = \tfrac{1}{2}v^2 + \text{constant}.$$

Hence
$$W = \left[\tfrac{1}{2}mv^2\right]_{t_1}^{t_2} = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2$$

Thus

work done by $\mathbf{F}$ = K.E. at B − K.E. at A

Example 4 A particle of mass 10 kg moves under the action of a force $\mathbf{F}$, so that its position vector after t seconds is given by $\mathbf{r} = 3t\mathbf{i} - \mathbf{j} + t^2\mathbf{k}$. Find the work done by $\mathbf{F}$ in the interval $1 \leqslant t \leqslant 5$.

$$\mathbf{r} = 3t\mathbf{i} - \mathbf{j} + t^2\mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = 3\mathbf{i} + 2t\mathbf{k}$$

$$\Rightarrow v^2 = 3^2 + (2t)^2 = 9 + 4t^2$$

$\therefore$ when $t = 1$, $v^2 = 13$ and when $t = 5$, $v^2 = 109$.

Work done by $\mathbf{F}$ = (K.E. when $t = 5$) − (K.E. when $t = 1$)
$= \frac{1}{2} \times 10 \times 109 - \frac{1}{2} \times 10 \times 13\,\text{J}$
$= 480\,\text{J}.$

The *power* of a force is the rate at which it does work. Let us suppose that at time t a force $\mathbf{F}$ acts at a point with position vector $\mathbf{r}$. If the point of application of $\mathbf{F}$ is given a displacement $\delta\mathbf{r}$ in a small interval of time δt, then the work done is approximately $\mathbf{F}.\delta\mathbf{r}$ and the average rate of working is $(\mathbf{F}.\delta\mathbf{r})/\delta t$.
$\therefore$ the power of the force $\mathbf{F}$ at time t

$$= \lim_{\delta t \to 0} \frac{\mathbf{F}.\delta\mathbf{r}}{\delta t} = \lim_{\delta t \to 0} \mathbf{F}.\frac{\delta\mathbf{r}}{\delta t} = \mathbf{F}.\frac{d\mathbf{r}}{dt}.$$

Hence if the point of application of a force $\mathbf{F}$ is moving with velocity $\mathbf{v}$, then

the power of $\mathbf{F}$ is $\mathbf{F}.\mathbf{v}$.

Example 5 At time t a particle of unit mass is moving with velocity $\mathbf{v}$ under the action of a force $\mathbf{F} = 4t\mathbf{i} + 3\mathbf{k}$. Given that when $t = 0$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$, find an expression for the velocity $\mathbf{v}$ at time t. Hence find an expression for the power of $\mathbf{F}$ at time t.

By Newton's second law, $\mathbf{F} = m\mathbf{a}$.
Since $\mathbf{F} = 4t\mathbf{i} + 3\mathbf{k}$, $m = 1$ and $\mathbf{a} = \dfrac{d\mathbf{v}}{dt}$,

we have $\dfrac{d\mathbf{v}}{dt} = 4t\mathbf{i} + 3\mathbf{k}$,

Integrating, $\mathbf{v} = 2t^2\mathbf{i} + 3t\mathbf{k} + \mathbf{c}$,

where $\mathbf{c}$ is a constant vector.

But when $t = 0$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$ $\quad\therefore\quad$ $\mathbf{c} = \mathbf{i} + 2\mathbf{j}$.
Hence $\mathbf{v} = (2t^2 + 1)\mathbf{i} + 2\mathbf{j} + 3t\mathbf{k}$.
The power of $\mathbf{F}$ at time $t = \mathbf{F}.\mathbf{v}$
$= (4t\mathbf{i} + 3\mathbf{k}).(\{2t^2 + 1\}\mathbf{i} + 2\mathbf{j} + 3t\mathbf{k})$
$= 4t(2t^2 + 1) + 3 \times 3t$
$= 8t^3 + 4t + 9t = 8t^3 + 13t.$

Exercise 7.7

1. Two particles A and B are moving with constant velocities $8\mathbf{i} + \mathbf{j} - 4\mathbf{k}$ and $4\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$ respectively. Find the angle between the paths of the particles. Find also the speed of A relative to B.

2. Two forces $\mathbf{F}_1 = 4\mathbf{j} - 5\mathbf{k}$ and $\mathbf{F}_2 = 2\mathbf{i} - 5\mathbf{j} - \mathbf{k}$ act on a particle. Find the magnitude of the resultant force acting on the particle and the angle the resultant makes with the force $\mathbf{F}_1$.

3. Find the component of the force $\mathbf{F}$ in the direction of the vector $\mathbf{d}$ if
(a) $\mathbf{F} = 3\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$, $\mathbf{d} = 6\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$,
(b) $\mathbf{F} = \mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$, $\mathbf{d} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$,
(c) $\mathbf{F} = 7\mathbf{i} - \mathbf{j} - 4\mathbf{k}$, $\mathbf{d} = 3\mathbf{i} - 5\mathbf{j} + 4\mathbf{k}$,
(d) $\mathbf{F} = 3\mathbf{i} + 5\mathbf{j} - \mathbf{k}$, $\mathbf{d} = 4\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}$.

4. Resolve the force $\mathbf{F}$ into component forces parallel and perpendicular to the vector $\mathbf{d}$ if
(a) $\mathbf{F} = 11\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$, $\mathbf{d} = 4\mathbf{i} - 3\mathbf{k}$,
(b) $\mathbf{F} = 5\mathbf{i} - 22\mathbf{j} + 19\mathbf{k}$, $\mathbf{d} = -5\mathbf{j} + 12\mathbf{k}$,
(c) $\mathbf{F} = 18\mathbf{i} + 35\mathbf{j} - 21\mathbf{k}$, $\mathbf{d} = 2\mathbf{i} - 6\mathbf{j} + 9\mathbf{k}$,
(d) $\mathbf{F} = 3\mathbf{i} - 4\mathbf{j} + 10\mathbf{k}$, $\mathbf{d} = 4\mathbf{i} - 8\mathbf{j} + \mathbf{k}$.

5. Find the work done when the point of application of a force $\mathbf{F}$ moves through a displacement $\mathbf{d}$, given that
(a) $\mathbf{F} = 3\mathbf{i} - 7\mathbf{j} + 6\mathbf{k}$, $\mathbf{d} = 5\mathbf{i} + \mathbf{j} - \mathbf{k}$,
(b) $\mathbf{F} = -2\mathbf{i} + 3\mathbf{j} - 8\mathbf{k}$, $\mathbf{d} = 6\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}$,
(c) $\mathbf{F} = 15\mathbf{i} + 7\mathbf{k}$, $\mathbf{d} = 10\mathbf{j} + 3\mathbf{k}$,
(d) $\mathbf{F} = 3\mathbf{i} - 9\mathbf{j} - 4\mathbf{k}$, $\mathbf{d} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

6. At time t a particle of mass m is moving with position vector $\mathbf{r}$ and velocity $\mathbf{v}$ under the action of a force $\mathbf{F}$. In each of the following cases find an expression for $\mathbf{v}$ and hence determine the work done by $\mathbf{F}$ in the given interval of time.
(a) $\mathbf{r} = 3t\mathbf{i} + t^2\mathbf{j}$, $0 \leqslant t \leqslant 2$,
(b) $\mathbf{r} = 5\mathbf{i} - t^2\mathbf{j} + 3t\mathbf{k}$, $2 \leqslant t \leqslant 4$,
(c) $\mathbf{r} = t^3\mathbf{i} - 4t\mathbf{j} + 3\mathbf{k}$, $1 \leqslant t \leqslant 3$,
(d) $\mathbf{r} = t\mathbf{i} - \mathbf{j} + (t^2 - 7t)\mathbf{k}$, $2 \leqslant t \leqslant 5$.

7. At time t a particle of mass m is moving with position vector $\mathbf{r}$ and velocity $\mathbf{v}$ under the action of a force $\mathbf{F}$. Find expressions for $\mathbf{F}$, $\mathbf{v}$ and the power of $\mathbf{F}$ at time t, given that
(a) $\mathbf{r} = 2t^2\mathbf{i} - 5t\mathbf{k}$,
(b) $\mathbf{r} = \mathbf{i} + t^3\mathbf{j} - 3t^2\mathbf{k}$,
(c) $\mathbf{r} = t\mathbf{i} - \mathrm{e}^{-t}\mathbf{j}$,
(d) $\mathbf{r} = \sin t\mathbf{j} + 3\cos t\mathbf{k}$.

8. A particle of unit mass is moving under the action of a constant force $\mathbf{F} = \mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$. Find the work done by $\mathbf{F}$ when the particle moves from the point A with position vector $5\mathbf{j} + 2\mathbf{k}$ to the point $\mathbf{B}$ with position vector $3\mathbf{i} + \mathbf{k}$. If the speed of the particle at A is $6\,\mathrm{m\,s^{-1}}$, find the speed of the particle at B.

9. A particle of mass 3 kg moves under the action of a constant force $\mathbf{F}$ with acceleration $\mathbf{a} = 3\mathbf{i} + 2\mathbf{k}$. Find the work done by $\mathbf{F}$ as the particle moves from the point A with position vector $4\mathbf{i} - 9\mathbf{j} + 21\mathbf{k}$ to the point B with position vector $10\mathbf{i} + 15\mathbf{j} - 7\mathbf{k}$. If the particle passes through A with velocity $12\mathbf{j} - 16\mathbf{k}$ and through

B with velocity $\lambda(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$, find the value of λ and the power of $\mathbf{F}$ at the points A and B.

10. A particle of unit mass is acted upon at time t by a force $\mathbf{F} = 2t\mathbf{i} - 3\mathbf{j}$. Given that the initial velocity of the particle is $\mathbf{j} + \mathbf{k}$, find an expression for its velocity $\mathbf{v}$ at time t. Hence find the work done by $\mathbf{F}$ in the interval $0 \leqslant t \leqslant 2$ and the power of $\mathbf{F}$ when $t = 2$.

11. A particle of mass 2 kg is acted upon at time t by a force $\mathbf{F} = 8\mathbf{i} - 4\cos t\mathbf{j} + 2t\mathbf{k}$. When $t = 0$ the velocity of the particle is $6\mathbf{i}$. Find expressions for the velocity of the particle and the power of $\mathbf{F}$ at time t.

12. A particle moves along a curve so that its position vector at time t is $\mathbf{r} = (3t - 2)\mathbf{i} + t^4\mathbf{j}$. If one of the forces acting on the particle is $\mathbf{F} = 5t^2\mathbf{i} - \mathbf{j}$, find by integration the work done by $\mathbf{F}$ in the interval $0 \leqslant t \leqslant 2$.

13. A particle moves along a curve with velocity $\mathbf{v} = \mathbf{i} + 10t^2\mathbf{j} - 3t\mathbf{k}$ at time t. If one of the forces acting on the particle is $\mathbf{F} = 2t\mathbf{i} + t^2\mathbf{j} - 2t\mathbf{k}$, find the power of $\mathbf{F}$ in terms of t. Hence find the work done by $\mathbf{F}$ in the interval from $t = 1$ to $t = 3$.

14. A particle moves on a curve with vector equation $\mathbf{r} = (\lambda - 12)\mathbf{i} - \lambda^2\mathbf{j} + \mathbf{k}$, where λ is a parameter. One of the forces acting on the particle is a variable force $\mathbf{F}$. Write down, in the form of a definite integral, an expression for the work done by $\mathbf{F}$ as the particle moves from the point where $\lambda = 1$ to the point where $\lambda = 2$. Evaluate this integral given that $\mathbf{F} = -4\mathbf{r}$.

Exercise 7.8 (*miscellaneous*)

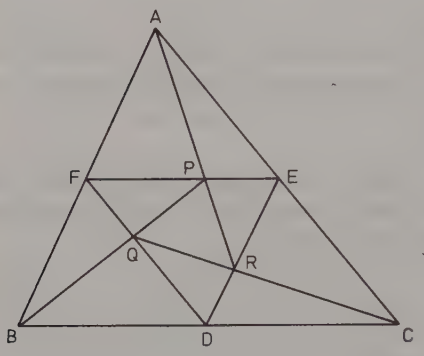
1. A cube $OPQRABCD$ has OA, PB, QC and RD perpendicular to the base $OPQR$; M is the centre of the face $ABCD$. The position vectors of A, B, C relative to O are $\mathbf{a}, \mathbf{b}, \mathbf{c}$ respectively. Find in terms of $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ the vectors $\mathbf{AB}, \mathbf{OM}, \mathbf{OD}$. If $|\mathbf{OP}| = 2$, express $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{OM}, \mathbf{OD}$ in terms of unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ along $\mathbf{OP}, \mathbf{OR}$, $\mathbf{OA}$ respectively. Find the cosine of the angle MOC. (L)

2. Points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively with respect to a given origin O. Prove that the position vector of the point P dividing the line segment AB in the ratio λ to μ is $(\mu\mathbf{a} + \lambda\mathbf{b})/(\lambda + \mu)$. The position vectors of A, B, C and D, the vertices of a plane quadrilateral, are $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ respectively. Find the position vectors of P, Q, R, S, T and U, the mid-points of AB, BC, CD, DA, AC and BD respectively. Hence show that the point with position vector $\frac{1}{4}(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d})$ lies on each of the lines PR, QS and TU and that these lines bisect each other. If $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}, \mathbf{b} = 6\mathbf{i} + 5\mathbf{j}, \mathbf{c} = 8\mathbf{i} - \mathbf{j}$ and $\mathbf{d} = \mathbf{j}$, where $\mathbf{i}$ and $\mathbf{j}$ are orthogonal unit vectors, show that the acute angle between AC and BD is $\cos^{-1}(5/13)$. Find a unit vector orthogonal to the line TU. (W)

3. The points A, B and C have position vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ with respect to an origin O. The point R in the plane ABC has position vector $\mathbf{r}$ where

$\mathbf{r} = \frac{1}{2}(\frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}) + \frac{1}{2}\mathbf{c}$. Use the ratio theorem to obtain a geometrical description of the position of R with reference to the points A, B and C. Illustrate your answer with a diagram. By writing $\mathbf{r}$ in the form $\lambda\mathbf{a} + \mu(m\mathbf{b} + n\mathbf{c})$ where $\lambda + \mu = 1$, obtain an alternative description of the position of R. Given that the point S has position vector $\frac{2}{3}\mathbf{a} + \frac{1}{12}\mathbf{b} + \frac{1}{4}\mathbf{c}$, show that S bisects the line segment AR. (JMB)

4.



The diagram shows a triangle ABC, with D, E, F as the mid-points of the sides. The points P, Q, R are chosen on EF, FD, DE so that RP, PQ, QR pass through A, B, C. If the position vectors of A, B, C are $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and if

$$\frac{PF}{EP} = \frac{QD}{FQ} = \frac{RE}{DR} = \lambda,$$

find the position vectors of P, Q, R and hence the value of λ. (O)

5. The position vectors of two points A and B relative to an origin O are $\mathbf{a}$ and $\mathbf{b}$ respectively. Show that the vector equation of the line bisecting angle AOB is $\mathbf{r} = \lambda(\hat{\mathbf{a}} + \hat{\mathbf{b}})$, where $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ are unit vectors in the directions of $\mathbf{a}$ and $\mathbf{b}$. Hence prove that the bisector of an interior angle of a triangle divides the opposite side into segments proportional to the adjacent sides.

6. OAD, OBE, OCF are three distinct straight lines. The position vectors with respect to O of the points A, B, C, D, E, F are $\mathbf{a}, \mathbf{b}, \mathbf{c}, p\mathbf{a}, q\mathbf{b}, r\mathbf{c}$ respectively, where p, q, r are scalars, no two of which are equal. Show that any point on BC has position vector given parametrically by $(1 - t)\mathbf{b} + t\mathbf{c}$. The intersections of the pairs of lines BC, EF; CA, FD; AB, DE (produced if necessary) are the points L, M, N respectively. Show that the position vector of L is $[q(1 - r)\mathbf{b} + r(q - 1)\mathbf{c}]/(q - r)$ and find the position vectors of M and N. Taking the values $p = 4, q = 3, r = 2$, show that L, M, N are collinear points. (C)

7. Three lines l_1, l_2 and l_3 have vector equations $\mathbf{r} = 2\mathbf{i} - \mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 3\mathbf{j} + 3\mathbf{k})$, $\mathbf{r} = \mathbf{i} - 3\mathbf{j} + \mu(2\mathbf{i} + 5\mathbf{j} + 4\mathbf{k})$ and $\mathbf{r} = 3\mathbf{i} + \nu(\mathbf{j} + 2\mathbf{k})$ respectively. Find the position vector of the point of intersection of l_1 and l_2. Hence show that the three lines are concurrent. Write down, in parametric form, a vector equation of the plane containing l_2 and l_3. Hence show that the three lines are coplanar.

8. Two planes p_1 and p_2 have vector equations $\mathbf{r} = (1 - \lambda)\mathbf{i} + (2\lambda + \mu)\mathbf{j} + (\mu - 1)\mathbf{k}$ and $\mathbf{r} = (s - t)\mathbf{i} + (2s - 3)\mathbf{j} + t\mathbf{k}$ respectively. Show that if a point in p_1 also lies in p_2, then $\mu = 4\lambda + 3$. Hence find a vector equation of the line of intersection of the two planes.

9. Given non-zero vectors $\mathbf{a}$ and $\mathbf{b}$, show that $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}-\mathbf{b}|$ if and only if $\mathbf{a}$ is perpendicular to $\mathbf{b}$. Deduce that a parallelogram is a rectangle if and only if its diagonals are equal in length.

10. Use vector methods to find the acute angle between two diagonals of a cube.

11. A tetrahedron $OABC$ with vertex O at the origin is such that $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$. Show that the line segments joining the mid-points of opposite edges bisect one another. Given that two pairs of opposite edges are perpendicular prove that $\mathbf{a}.\mathbf{b} = \mathbf{b}.\mathbf{c} = \mathbf{c}.\mathbf{a}$ and show that the third pair of opposite edges is also perpendicular. Prove also that, in this case, $OA^2 + BC^2 = OB^2 + AC^2$. (L)

12. ABC is a given triangle, and the centre of the circle through A, B, C is O. Relative to O, the position vectors of A, B, C are $\mathbf{a}, \mathbf{b}, \mathbf{c}$. If H is the point with position vector $\mathbf{a}+\mathbf{b}+\mathbf{c}$, prove that AH, BH, CH are perpendicular to BC, CA, AB respectively. If A', B', C' are the mid-points of BC, CA, AB respectively, prove that AA', BB' and CC' are concurrent in G, and find the position vector of G. Prove that O, G, H lie on a straight line. (O)

13. Let $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}, \mathbf{b} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$. Given that $\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$ and that $\mathbf{c}$ is perpendicular to $\mathbf{a}$, find the ratio of λ to μ. Let A, B be the points with position vectors $\mathbf{a}, \mathbf{b}$ respectively with respect to an origin O. Write down, in terms of $\mathbf{a}$ and $\mathbf{b}$, a vector equation of the line l through A, in the plane of O, A and B, which is perpendicular to OA. Find the position vector of P, the point of intersection of l and OB. (JMB)

14. The position vectors of A, B, C are $\mathbf{i}, 2\mathbf{j}, 3\mathbf{k}$ respectively. The perpendicular from the origin O to the plane ABC meets the plane at the point D. Find (i) the vector equation of the plane ABC, (ii) the vector equation of the line OD, (iii) the position vector of the point D. (AEB 1980)

15. A straight line passes through the point $(2, 0, -1)$ and is parallel to the vector $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$. The feet of the perpendiculars from the points $A(2, -2, 1)$ and $B(1, 7, 0)$ to this line are P and Q. Find (i) the coordinates of P and Q, (ii) the angles APB and AQB, (iii) the Cartesian equations of the planes PAQ and PBQ, (iv) the angle between the planes PAQ and PBQ.

16. Interpret geometrically the equations $\mathbf{r}.\mathbf{n} = p$ and $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ where $\mathbf{n}, \mathbf{a}$ and $\mathbf{b}$ are constant vectors, p is a constant scalar and t is a parameter which can take all real values. Find the value of t in terms of $\mathbf{a}, \mathbf{b}, \mathbf{n}$ and p if the value of $\mathbf{r}$ given by the second equation also satisfies the first. Explain the geometrical significance of the point given by this value of t, giving special attention to the cases when (i) $\mathbf{b}.\mathbf{n} = 0$ and (ii) $\mathbf{b} = \mathbf{n}$.

The point A has coordinates $(1, 1, 2)$. Determine the coordinates of the point A' which is the reflection of A in the plane $2x + 2y + z = 9$. (AEB 1979)

17. The lines l_1 and l_2 have equations $\mathbf{r}_1 = (2\lambda + 3)\mathbf{i} + (2 - 3\lambda)\mathbf{j} + \lambda\mathbf{k}$ and $\mathbf{r}_2 = (\mu - 4)\mathbf{i} + (2\mu + 1)\mathbf{j} + (4\mu - 3)\mathbf{k}$ respectively. Show that l_1 is perpendicular to l_2. Find the values of λ and μ such that the vector $\mathbf{r}_1 - \mathbf{r}_2$ is perpendicular to both l_1 and l_2. Hence find the position vectors of the ends of the common perpendicular to l_1 and l_2. Find also the length of this common perpendicular.

18. A pyramid has a square base $OABC$ and vertex V. The position vectors of A, B, C, V referred to O as origin are given by $\mathbf{OA} = 2\mathbf{i}, \mathbf{OB} = 2\mathbf{i} + 2\mathbf{j}, \mathbf{OC} = 2\mathbf{j}$, $\mathbf{OV} = \mathbf{i} + \mathbf{j} + 3\mathbf{k}$. (i) Express $\mathbf{AV}$ in terms of $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$. (ii) Using scalar products, or otherwise, find a vector $\mathbf{x}$ which is perpendicular to both $\mathbf{OV}$ and $\mathbf{AV}$. (iii) Calculate the angle between the vector $\mathbf{x}$, found in (ii), and $\mathbf{VB}$, giving your answer to the nearest degree. (iv) Write down the acute angle between VB and the plane OVA. (C)

19. The forces $\mathbf{F}_1$ and $\mathbf{F}_2$, where $\mathbf{F}_1 = 2\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ and $\mathbf{F}_2 = -\mathbf{i} + 3\mathbf{j}$, both act through the point with position vector $\mathbf{j} - \mathbf{k}$. Find the angle between the two forces and the magnitude of their resultant. Obtain, in both vector and Cartesian forms, equations for the line of action of the resultant.

20. Show that the resultant of two forces acting at a point O in directions OA and OB and represented in magnitude by λOA and μOB is represented by $(\lambda + \mu)OC$, where C is the point on AB dividing it internally in the ratio $\mu:\lambda$.

(a) $ABCDEF$ is a regular hexagon of side a. Forces of size AB, AC, AD, AE and AF act along AB, AC, AD, AE and AF respectively. Prove that the resultant of these forces has magnitude $6a$, and give its line of action.

(b) A and B are two fixed points in a plane and P is a variable point in the same plane. Two forces act through P, one of magnitude $2PA$ along PA and the other of magnitude PB along PB. If the resultant of these two forces is always equal to $3PA$ *in magnitude*, find the locus of P. (SU)

21. (i) The force $\mathbf{F}_1$, of magnitude 26 N, acts at the origin in the direction of the vector $(3\mathbf{i} + 4\mathbf{j} + 12\mathbf{k})$. A second force $\mathbf{F}_2$, of magnitude 30 N, also acts at the origin, but in the direction of the vector $(6\mathbf{j} - 8\mathbf{k})$. Calculate the magnitude and the direction of the resultant of the forces $\mathbf{F}_1$ and $\mathbf{F}_2$. Calculate also the cosine of the angle between $\mathbf{F}_1$ and $\mathbf{F}_2$.

(ii) A particle of mass 2 kg starts from rest at the origin and the force $\mathbf{F}_3$, where $\mathbf{F}_3 = (6\mathbf{i} + 26\mathbf{j})$ N, acts on it as it moves. Find the position vector of the particle 2 seconds later. (L)

22. The force acting at time t $(0 \leqslant t < 2)$ on a particle of unit mass is $24t^2\mathbf{i} + 6\mathbf{j}$. At time $t = 0$ the particle is at rest at the point with position vector $-2\mathbf{i} + 3\mathbf{j}$. Find the position vector of the particle at time $t = T$ $(0 \leqslant T < 2)$. For time $t \geqslant 2$ the force acting on the particle is $6\mathbf{j}$. Find the position vector of the particle at time $t = 3$. (L)

23. State the centre and radius of the circle which has vector equation $\mathbf{r} = 6\mathbf{i} + 8\mathbf{j} + 6(\mathbf{i}\cos\theta + \mathbf{j}\sin\theta)$. A particle P of mass 2 kg moves on this circle with

constant angular velocity $\pi/12$ radians per second. Write down the position vector of P at time t given that, at $t = 0$, P is at the point corresponding to $\theta = 0$. Calculate the components parallel to $\mathbf{i}$ and $\mathbf{j}$ of the resultant vector force on the particle when $t = 4$. Find the position vector of the particle when $t = 8$ given that the forces on the particle cease to act when $t = 4$. (L)

24. The position vector, relative to the origin O, of an object of mass m moving in a horizontal plane in which $\mathbf{i}$ and $\mathbf{j}$ are perpendicular unit vectors is given by

$$\mathbf{r} = \frac{\cos\theta}{1+\cos\theta}\mathbf{i} + \frac{\sin\theta}{1+\cos\theta}\mathbf{j}, \quad -\pi < \theta < \pi.$$

Obtain an expression for r, the magnitude of $\mathbf{r}$. Given that $r^2\frac{d\theta}{dt} = h$ where h is constant, show that the velocity of the object is $-h\sin\theta\mathbf{i} + h(1+\cos\theta)\mathbf{j}$. Show that the force acting on the object is of magnitude mh^2/r^2, and find the direction of this force. (JMB)

25. A particle is projected from the origin O with initial velocity $u\mathbf{i} + v\mathbf{j}$, where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the horizontal and upward vertical directions respectively, and moves freely under gravity. Show that its velocity and position vector with respect to O, at time t after projection, are given by $\dot{\mathbf{r}} = u\mathbf{i} + (v - gt)\mathbf{j}$ and $\mathbf{r} = ut\mathbf{i} + (vt - \frac{1}{2}gt^2)\mathbf{j}$ respectively.

For the remainder of this question components of velocities are measured in $\mathrm{m\,s^{-1}}$ and components of displacements in m. Take g as $10\,\mathrm{m\,s^{-2}}$.

Two particles, A and B, are projected simultaneously from the origin with velocities $-30\mathbf{i} + 90\mathbf{j}$ and $30\mathbf{i} + 110\mathbf{j}$ respectively. Both particles impinge on a plane which contains the origin, meeting the plane on a line of greatest slope which has the equation $\mathbf{r} = \lambda(3\mathbf{i} + \mathbf{j})$, where λ is a scalar. Show that the times of flight of A and B are equal and that the distance between the points of impact is $400\sqrt{10}$ m. Find the cosine of the acute angle between the direction of motion of A at the instant of impact and the line of greatest slope of the plane. (JMB)

26. At noon two ships A and B have the following position and velocity vectors:

	Position vector	*Velocity vector*
Ship A	$10\mathbf{i} + 5\mathbf{j}$	$-2\mathbf{i} + 4\mathbf{j}$
Ship B	$2\mathbf{i} - \mathbf{j}$	$2\mathbf{i} + 7\mathbf{j}$

where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the directions East and North respectively, and where the speeds are measured in kilometres per hour and the distances in kilometres. If they continue on their respective courses, (i) find the position vector of A after a time t hours; (ii) show that they will collide, and give the time of the collision; (iii) determine how far ship A will have travelled between noon and the collision. (SU)

27. A ship A is travelling on a course of 060° at a speed of $30\sqrt{3}$ km/h and a ship B is travelling on a course of 030° at 20 km/h. At noon B is 260 km due east of A.

Using unit base vectors **i** and **j** pointing east and north respectively find $\overrightarrow{AB}$ in terms of **i**, **j** and t at time t hours in the afternoon. Hence, or otherwise, calculate the least distance between A and B, to the nearest kilometre, and the time at which they are nearest to one another. (L)

28. An aeroplane A moves in space with a variable velocity **v** and is attacked by a guided missile B which moves with constant speed u, where $u > |v|$. The guidance system of B is such that, if from any instant the velocity of A stayed constant, B would hit A without having to change direction. Show that the velocity of A relative to B is

$$\left[\frac{\mathbf{r}.\mathbf{v} - \sqrt{\{(\mathbf{r}.\mathbf{v})^2 + (u^2 - \mathbf{v}^2)\mathbf{r}^2\}}}{\mathbf{r}^2}\right]\mathbf{r},$$

where **r** is the position vector of A relative to B. (O)

29. At time t a particle is in motion with velocity **v** and is being acted upon by a variable force **F**. Write down expressions for (i) the power at time t, (ii) the work done by **F** during the time interval $0 \leqslant t \leqslant T$.

The particle, of mass m, moves in a plane where **i** and **j** are perpendicular unit vectors so that its position vector at time t is given by $\mathbf{r} = 2a\cos 2t\mathbf{i} + a\sin 2t\mathbf{j}$, where a is a positive constant. Derive expressions for the velocity **v** and the force **F** at time t. Obtain an expression in terms of t for the power at time t and show that the work done by **F** during the interval $0 \leqslant t \leqslant T$ is $3ma^2(1 - \cos 4T)$. If T varies, find the maximum value of the work done by **F** and determine also the smallest value of T for which this maximum value is reached. (JMB)

30. A smooth fixed wire is in the shape of the curve given by

$$\mathbf{r} = a(\theta - \sin\theta)\mathbf{i} + a(1 + \cos\theta)\mathbf{j}, \; 0 \leqslant \theta \leqslant 2\pi,$$

where **i**, **j** are unit vectors in the horizontal and upward vertical directions respectively. A particle threaded on the wire is released from rest at the point A ($\theta = 0$). Write down, in terms of θ and its derivative, an equation expressing the conservation of energy during the motion. Show that the time taken for the particle to reach B ($\theta = 2\pi$) is $2\pi\sqrt{(a/g)}$. (JMB)

8 Impulse and momentum

8.1 The impulse of a force

When a constant force $\mathbf{F}$ acts for a time t, then the *impulse* of the force is defined as the vector quantity $\mathbf{F}t$.

The SI unit of impulse is the newton second (N s).

Example 1 A force $\mathbf{F} = 4\mathbf{i} - 3\mathbf{j}$ acts on a particle for 3 seconds. Find the magnitude of the impulse of the force.

If the impulse of the force is $\mathbf{I}$, then

$$\mathbf{I} = (4\mathbf{i} - 3\mathbf{j}) \times 3$$
$$\therefore \quad |\mathbf{I}| = 3|4\mathbf{i} - 3\mathbf{j}| = 3\sqrt{(4^2 + 3^2)} = 15.$$

Hence the magnitude of the impulse is 15 N s.

When a variable force $\mathbf{F}$ acts from time t_1 to time t_2, then the impulse of the force is defined as $\int_{t_1}^{t_2} \mathbf{F}\, dt$.

Example 2 A particle is moving under the action of a force $\mathbf{F} = 2\mathbf{i} - 3t^2\mathbf{j} + 4t\mathbf{k}$ at the time t. Find the impulse given to the particle in the interval $1 \leqslant t \leqslant 2$.

$$\text{Impulse} = \int_1^2 (2\mathbf{i} - 3t^2\mathbf{j} + 4t\mathbf{k})\, dt$$
$$= \Big[2t\mathbf{i} - t^3\mathbf{j} + 2t^2\mathbf{k}\Big]_1^2$$
$$= (4\mathbf{i} - 8\mathbf{j} + 8\mathbf{k}) - (2\mathbf{i} - \mathbf{j} + 2\mathbf{k})$$
$$= 2\mathbf{i} - 7\mathbf{j} + 6\mathbf{k}.$$

In some circumstances it is difficult to determine the exact nature of the forces involved. For example, in collisions between moving objects comparatively large

unknown forces act for very short intervals of time. We will now show that the impulse of such a force can be found by considering the effect it produces.

Suppose that at time t a particle of mass m is moving with velocity $\mathbf{v}$ under the action of a force $\mathbf{F}$. Let $\mathbf{I}$ be the impulse given to the particle by $\mathbf{F}$ in the interval $t_1 \leqslant t \leqslant t_2$. Whether $\mathbf{F}$ is constant or variable,

$$\mathbf{I} = \int_{t_1}^{t_2} \mathbf{F}\, dt$$

Using Newton's second law in its original form,

$$\mathbf{F} = \frac{d(m\mathbf{v})}{dt},$$

where $m\mathbf{v}$ is the *momentum* of the particle.

$$\therefore \quad \mathbf{I} = \int_{t_1}^{t_2} \frac{d(m\mathbf{v})}{dt}\, dt$$

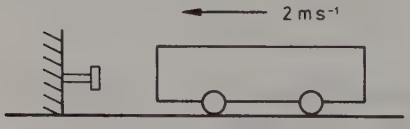
$$\mathbf{I} = \Big[m\mathbf{v} \Big]_{t_1}^{t_2}$$

$$\mathbf{I} = (m\mathbf{v})_{at\ t_2} - (m\mathbf{v})_{at\ t_1}$$

i.e. impulse = change in momentum

[Because of the relationship between impulse and momentum, both quantities are usually measured in newton seconds. An alternative unit of momentum is the kilogramme metre per second. However, since $1\,\mathrm{N} = 1\,\mathrm{kg\,m\,s^{-2}}$, it follows that $1\,\mathrm{N\,s} = 1\,\mathrm{kg\,m\,s^{-1}}$.]

Example 3 A truck of mass 1400 kg moving along a straight horizontal track at $2\,\mathrm{m\,s^{-1}}$ runs into fixed buffers and rebounds at a speed of $1\,\mathrm{m\,s^{-1}}$. If the truck and the buffers are in contact for 1·2 s, find the average force exerted by the buffers on the truck.



Before impact

After impact

Let the direction away from the buffers be positive and let F N be the average force exerted by the buffers.

Impulse = change in momentum

$$\therefore \quad F \times 1{\cdot}2 = 1400 \times 1 - 1400 \times (-2)$$

$$\therefore \quad F = \frac{1400 \times 3}{1{\cdot}2} = 3500$$

Hence the average force exerted by the buffers is 3500 N.

Example 4 A particle of mass 5 kg is moving with velocity $3\mathbf{i} - 7\mathbf{j}$ when it is given an impulse $\mathbf{I} = -7\mathbf{i} + 16\mathbf{j}$. If the velocity of the particle after the impulse is $\mathbf{v}$, find $\mathbf{v}$. Find also the change in kinetic energy.

Impulse = change in momentum

$\therefore \quad -7\mathbf{i} + 16\mathbf{j} = 5\mathbf{v} - 5(3\mathbf{i} - 7\mathbf{j})$

$\therefore \quad 5\mathbf{v} = -7\mathbf{i} + 16\mathbf{j} + 15\mathbf{i} - 35\mathbf{j}$

i.e. $\quad \mathbf{v} = \frac{1}{5}(8\mathbf{i} - 19\mathbf{j})$

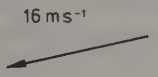
Initial speed $= \sqrt{(3^2 + 7^2)} = \sqrt{(9 + 49)} = \sqrt{58}$

New speed $= \frac{1}{5}\sqrt{(8^2 + 19^2)} = \frac{1}{5}\sqrt{(64 + 361)} = \frac{1}{5}\sqrt{425} = \sqrt{17}.$

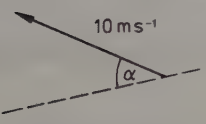
$\therefore$ there is a loss in kinetic energy of

$$(\tfrac{1}{2} \times 5 \times 58 - \tfrac{1}{2} \times 5 \times 17)\,\mathrm{J} \quad \text{i.e.} \quad 102{\cdot}5\,\mathrm{J}.$$

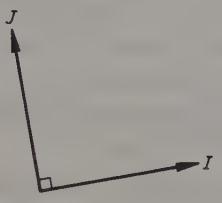
Example 5 A ball of mass 0·25 kg strikes a wall at a speed of $16\,\mathrm{m\,s^{-1}}$ and rebounds with a speed of $10\,\mathrm{m\,s^{-1}}$. If the angle between the velocities of the ball before and after impact is α, where $\tan\alpha = \frac{3}{4}$, find the magnitude of the impulse exerted by the wall on the ball.



Before impact



After impact



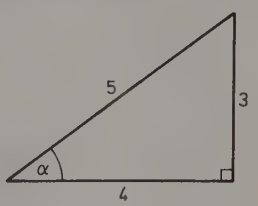
Components of the impulse

Resolving parallel and perpendicular to the initial velocity of the ball,

$\nearrow \qquad I = 0{\cdot}25(-10\cos\alpha) - 0{\cdot}25(-16)$

$\nwarrow \qquad J = 0{\cdot}25(10\sin\alpha)$

Using a right-angled triangle,



$$\sin\alpha = \tfrac{3}{5}, \quad \cos\alpha = \tfrac{4}{5}.$$

$\therefore \quad I = 0{\cdot}25(-8) - 0{\cdot}25(-16) = 2$

$J = 0{\cdot}25 \times 6 = 1{\cdot}5$

Hence the magnitude of the impulse exerted by the wall

$$= \sqrt{(2^2 + 1{\cdot}5^2)}\,\mathrm{N\,s} = 2{\cdot}5\,\mathrm{N\,s}.$$

When a jet of water is directed at a wall, the reaction of the wall on the water produces a *continuous change in momentum.* Provided that the water is being delivered at a steady rate and with constant velocity, the force exerted by the wall can be determined by considering the momentum lost by the water each second.

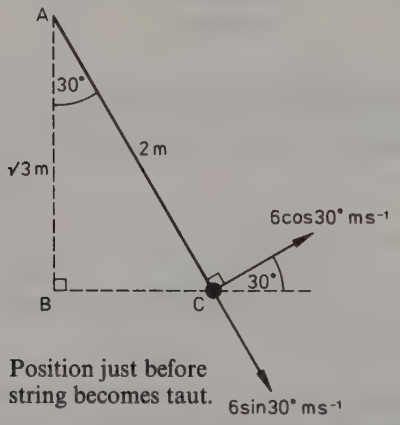
Example 6 A hose delivers water horizontally at a rate of 20 kg per second with a speed of 25 m s^{-1}. The water strikes a vertical wall and does not rebound. Find the force exerted on the wall.

Let F N be the force exerted on the wall, then the force exerted by the wall is also F N,
$\therefore$ impulse exerted in 1 second $= F \times 1$ N s $= F$ N s.
Mass of water delivered in 1 second $= 20$ kg
$\therefore$ loss of momentum in 1 second $= 20 \times 25$ N s $= 500$ N s
However, impulse = change in momentum,
$\therefore \quad F = 500$
Hence the force exerted on the wall is 500 N.

An instantaneous impulse acting along a string is called an *impulsive tension.* It may produce changes in momentum in the direction of the string, but has no effect perpendicular to the string. In particular, if a stationary particle is jerked into motion by the sudden tightening of a string attached to it, then the initial velocity of the particle will be in the direction of the string.

Example 7 One end of a light inextensible string of length 2 m is fixed at a point A on a smooth horizontal plane. The other end is attached to a particle of mass 2 kg. The particle is placed at a point B on the plane, where $AB = \sqrt{3}$ m, then projected horizontally with speed 6 m s^{-1} in a direction perpendicular to AB. Find the speed of the particle immediately after the string has become taut. Find also the impulsive tension in the string.

If C is the point on the path of the particle at which the string first becomes taut, $\angle CAB = \cos^{-1}(\sqrt{3}/2) = 30°$.



Position just before string becomes taut.

As shown in the diagram, just before the string tightens the velocity of the particle has components $6 \sin 30°$ m s^{-1} along AC and $6 \cos 30°$ m s^{-1} perpendicular to AC. When the string becomes taut the particle loses its momentum in the direction of AC, but its momentum perpendicular to AC is unchanged.

Hence just after the string has become taut, the speed of the particle is $6 \cos 30°$ m s^{-1} i.e. $3\sqrt{3}$ m s^{-1}.
The loss of momentum along AC is produced by the impulsive tension in the string,

$\therefore \quad$ impulsive tension $= 2 \times 6 \sin 30°$ N s $= 6$ N s.

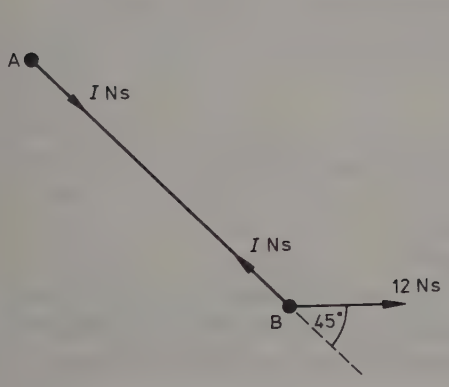
In problems which involve systems of two or more particles the impulse-momentum

relationship can be used both for the system as a whole and for particles taken separately. When considering the impulse given to the whole system, we will see that internal impulses occur in equal and opposite pairs and cancel each other out. Thus we may write:

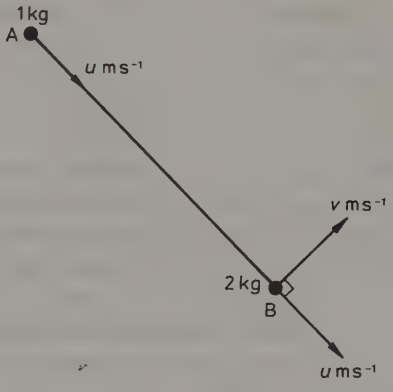
sum of external impulses = total change in momentum.

The next example concerns two particles connected by a light inextensible string. In this case the impulsive tensions acting on the particles are equal and opposite, so they can be disregarded when dealing with the system as a whole.

Example 8 Particles A and B, of masses 1 kg and 2 kg respectively, are connected by a light inextensible string. The particles are at rest on a smooth horizontal plane with the string taut. A horizontal impulse of magnitude 12 N s is applied to particle B in a direction inclined at 45° to the direction of AB. Find the velocities with which A and B start to move. Find also the impulsive tension in the string.



Impulse diagram



Velocity diagram

Let the initial velocity of B have components u and v along and perpendicular to AB respectively. Since the string is taut, A will start to move with speed u along AB.

For A and B together, impulse = change in momentum,

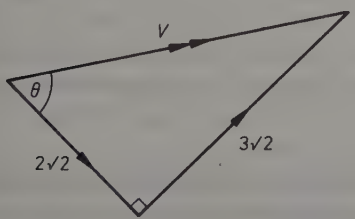
$\therefore$ resolving along and perpendicular to AB,

$$\searrow \qquad 12\cos 45° = 1 \times u + 2 \times u$$

$$\nearrow \qquad 12\sin 45° = 2 \times v$$

$$\therefore \qquad u = 2\sqrt{2} \quad \text{and} \quad v = 3\sqrt{2}$$

Thus the initial velocity of A is $2\sqrt{2}\,\text{m s}^{-1}$ along AB.



If the initial velocity of B is $V\,\text{m s}^{-1}$ at an angle θ to AB, then

$$V = \sqrt{\{(2\sqrt{2})^2 + (3\sqrt{2})^2\}} = \sqrt{26}$$
$$\theta = \tan^{-1}(3/2) \approx 56° 19'$$

Thus the initial velocity of B is $\sqrt{26}\,\text{m s}^{-1}$ at an angle of 56° 19′ to AB.

Let I N s be the impulsive tension in the string, then considering the motion of A,

$$I = 1 \times 2\sqrt{2}.$$

Hence the impulsive tension in the string is $2\sqrt{2}$ N s.

Exercise 8.1

1. When a constant force $\mathbf{F}$ acts for time t, the impulse of the force is $\mathbf{I}$.
(a) If $\mathbf{F} = 2\mathbf{i} - 7\mathbf{j}$ and $t = 4$, find $\mathbf{I}$.
(b) If $\mathbf{I} = 6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and $t = 3$, find $\mathbf{F}$.
(c) If $\mathbf{F} = -12\mathbf{i} + 5\mathbf{j}$ and $t = 5$, find $|\mathbf{I}|$.
(d) If $\mathbf{I} = 6\mathbf{i} + 12\mathbf{j} - 4\mathbf{k}$ and $t = 3{\cdot}5$, find $|\mathbf{F}|$.

2. Find the impulse of the variable force $\mathbf{F}$ in the stated interval of time.
(a) $\mathbf{F} = 5t\mathbf{i} + 3\mathbf{j}$, $0 \leqslant t \leqslant 2$.
(b) $\mathbf{F} = \mathbf{i} - 6t^2\mathbf{j} + 2t\mathbf{k}$, $1 \leqslant t \leqslant 3$.
(c) $\mathbf{F} = \cos 2t\mathbf{i} + 3\sin 2t\mathbf{j}$, $0 \leqslant t \leqslant \frac{1}{2}\pi$.

3. Find the momentum of
(a) a man of mass 80 kg running at $6\,\mathrm{m\,s^{-1}}$,
(b) a car of mass 1200 kg travelling at $25\,\mathrm{m\,s^{-1}}$,
(c) a bullet of mass 15 g moving at $800\,\mathrm{m\,s^{-1}}$.

4. A particle of mass m moves so that its position vector at time t is $\mathbf{r}$. Find, as a vector, the momentum of the particle at time T if
(a) $\mathbf{r} = 3\mathbf{i} + t^2\mathbf{j}$, $m = 2$, $T = 3$;
(b) $\mathbf{r} = -4t\mathbf{i} + t^3\mathbf{k}$, $m = 0{\cdot}5$, $T = 2$;
(c) $\mathbf{r} = 5\cos 3t\mathbf{i} + 2\mathbf{j} + 5\sin 3t\mathbf{k}$, $m = 4$, $T = \pi$.

5. A stone of mass 3 kg is falling at a speed of $18\,\mathrm{m\,s^{-1}}$ when it hits the ground. If the stone is brought to rest by the impact, find the impulse exerted by the ground.

6. A ball of mass 0·2 kg strikes a wall when moving horizontally at $12\,\mathrm{m\,s^{-1}}$. If the ball rebounds horizontally at $8\,\mathrm{m\,s^{-1}}$, find the impulse exerted by the wall.

7. A bullet of mass 0·03 kg is fired into a fixed block of wood at a speed of $400\,\mathrm{m\,s^{-1}}$. If the bullet is brought to rest in 0·02 s, find the average resistance exerted by the wood.

8. A particle of mass 5 kg is moving under the action of a constant force of magnitude 8 N. Initially the speed of the particle is $12\,\mathrm{m\,s^{-1}}$. Find its speed after 3 seconds if the force (a) acts in the direction of motion, (b) directly opposes motion.

9. A particle of mass 2 kg is projected across a rough horizontal plane with a speed of $21\,\mathrm{m\,s^{-1}}$. Find the time in which the particle comes to rest, given that the coefficient of friction is 2/7.

10. A particle of mass 0·5 kg is moving with velocity $\mathbf{u} = 6\mathbf{i} - 7\mathbf{j}$ when it is given an impulse $\mathbf{I} = 3\mathbf{i} + 11\mathbf{j}$. Find the new velocity of the particle and the average force required to bring it to rest in a further 3 seconds.

11. A particle of mass 0·8 kg is moving with velocity $4\mathbf{i} + 7\mathbf{j}$ when it strikes a fixed obstacle and rebounds with velocity $-6\mathbf{i} + 2\mathbf{j}$. Find the impulse exerted by the particle on the obstacle. Find also the kinetic energy lost by the particle in the collision.

12. A particle of mass 3 kg is moving with speed $6\,\mathrm{m\,s^{-1}}$ when it receives an impulse of magnitude 24 N s. Find the speed of the particle immediately after the impulse and the change in the kinetic energy of the particle if the direction of the impulse and the original direction of motion are (a) the same, (b) opposite, (c) perpendicular.

13. A ball of mass 0·2 kg strikes a wall at a speed of $12\,\mathrm{m\,s^{-1}}$ and rebounds with a speed of $3{\cdot}5\,\mathrm{m\,s^{-1}}$. If its path is diverted through 90° by the impact, find the magnitude of the impulse exerted on the wall.

14. An object of mass 4 kg strikes a barrier at a speed of $10\,\mathrm{m\,s^{-1}}$ and rebounds at a speed of $5\,\mathrm{m\,s^{-1}}$. If the angle between the velocities of the object before and after impact is 60°, find the magnitude of the impulse exerted on the barrier.

15. A hose is discharging water at the rate of 50 litres per second. The jet of water is travelling horizontally with a speed of $12\,\mathrm{m\,s^{-1}}$ when it strikes a vertical wall. Assuming that the water does not rebound, find the force exerted on the wall. [Take the mass of 1 litre of water to be 1 kg.]

16. A stream of water falling vertically from a pipe at the rate of 75 kg per second exerts a force of 900 N on the ground below. Assuming that the ground destroys the momentum of the water, find the speed at which the water hits the ground.

17. In a factory small screws are falling vertically onto a conveyor belt without bouncing at a rate of 48 kg per minute. Just before impact the screws are travelling at $2\,\mathrm{m\,s^{-1}}$. If the conveyor belt moves horizontally at $1{\cdot}5\,\mathrm{m\,s^{-1}}$, find the magnitude and direction of the average force it exerts on the screws.

18. One end of a light inextensible string of length $\sqrt{3}$ m is fixed at a point A on a smooth horizontal plane. The other end is attached to a particle of mass 3 kg. The particle is placed at a point B on the plane, where $AB = 1$ m, then projected horizontally with speed $4\,\mathrm{m\,s^{-1}}$. Find the speed of the particle immediately after the string has become taut and the impulsive tension in the string, given that the angle between the initial velocity of the particle and the direction BA is (a) 60°, (b) 90°.

19. Particles A and B, of masses 3 kg and 2 kg respectively, are connected by a light inextensible string. The particles are at rest on a smooth horizontal plane with the string taut. A horizontal impulse of magnitude 12·5 N s is applied to particle B in a direction inclined at an angle α to the direction of AB, where $\tan\alpha = 4/3$. Find the

velocities with which A and B start to move. Find also the impulsive tension in the string.

20. Particles A and B, both of mass m, are connected by a light inextensible string. A particle C of mass $2m$ is attached to the mid-point of the string. The particles are at rest on a smooth horizontal plane with the string taut and with AC at right angles to BC. A horizontal impulse of magnitude $12m$ is applied to C in a direction inclined at an angle α to AC, such that both parts of the string remain taut. Find the speeds with which the particles begin to move if (a) $\alpha = 45°$, (b) $\alpha = 60°$.

8.2 Conservation of momentum

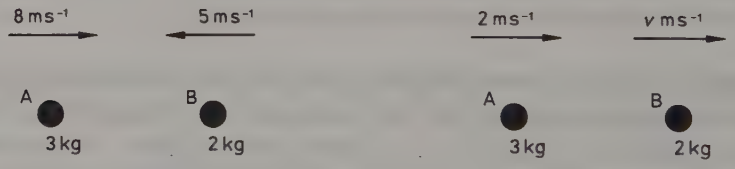
When two particles collide they exert equal and opposite impulses on each other. Therefore the changes in the momentum of the particles are also equal and opposite. It follows that the total momentum of the two particles together remains unchanged. A similar argument can be used in any situation in which equal and opposite impulses act on two bodies, e.g. when a bullet is fired from a gun or when a light inextensible string connecting two particles suddenly becomes taut.

These examples are special cases of a more general result called the *principle of conservation of linear momentum.*

> If no external force acts on a system of particles in a particular direction, then there can be no change in the total momentum of the system in that direction.

Example 1 Two particles A and B, of masses 3 kg and 2 kg respectively, are travelling in opposite directions along the same straight line when they collide. Just before the impact A is moving with speed $8\,\mathrm{m\,s^{-1}}$ and B with speed $5\,\mathrm{m\,s^{-1}}$. After the impact the speeds of A and B are $2\,\mathrm{m\,s^{-1}}$ and $v\,\mathrm{m\,s^{-1}}$ respectively. Find the value of v given that after the impact A and B are travelling in (a) the same direction, (b) opposite directions.

(a)



Momentum before impact = momentum after impact

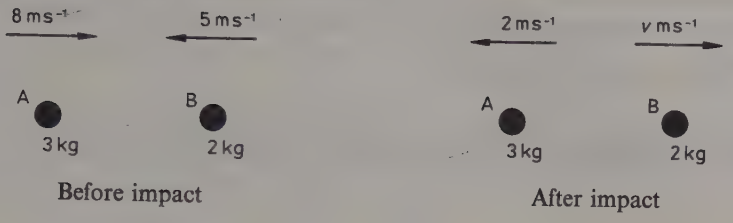
$\therefore$ taking the direction of motion of A as positive,

$$3 \times 8 + 2 \times (-5) = 3 \times 2 + 2 \times v$$

i.e. $$14 = 6 + 2v$$

Thus $$v = 4$$

(b)



Before impact

After impact

Momentum before impact = momentum after impact

$\therefore \quad 3 \times 8 + 2 \times (-5) = 3 \times (-2) + 2 \times v$

i.e. $14 = -6 + 2v$

Thus $v = 10.$

Example 2 A bullet of mass 0·02 kg is fired from a gun of mass 6 kg with a speed of 450 m s^{-1}. Find the speed of the gun's recoil. Find also the constant force required to bring the gun to rest in 0·5 seconds.

Let v m s^{-1} be the speed of the gun and let F N be the force required to bring the gun to rest in 0·5 seconds.

Momentum after firing = momentum before firing

$\therefore \quad 6 \times v + 0{\cdot}02 \times (-450) = 0$

i.e. $6v - 9 = 0$

$v = 1{\cdot}5$

Hence the speed of the gun's recoil is 1·5 m s^{-1}.

As the gun is brought to rest,

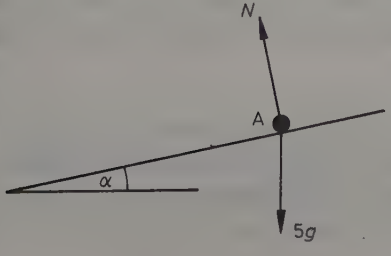
impulse = change in momentum

$\therefore \quad F \times 0{\cdot}5 = 6 \times 1{\cdot}5$

i.e. $F = 18$

Hence the constant force required is 18 N.

Example 3 A particle A of mass 5 kg slides from rest down a smooth plane inclined at an angle α to the horizontal, where $\sin \alpha = 1/6$. After 3 s it collides with a particle B of mass 1 kg which is sliding down the plane at 2·5 m s^{-1}. If the two particles coalesce, find their common speed immediately after impact.



The change in the momentum of A is equal to the impulse received.

$\therefore$ just before the collision the momentum of A down the plane

$= 5g \sin \alpha \times 3$ N s

$= 5 \times 9{\cdot}8 \times \frac{1}{6} \times 3$ N s

$= 24{\cdot}5$ N s

Momentum is conserved in the collision between A and B. Thus if $v\,\mathrm{m\,s^{-1}}$ is their common speed after impact,

$$24{\cdot}5 + 1 \times 2{\cdot}5 = (5 + 1)v$$
$$\therefore \qquad v = 4{\cdot}5$$

Hence the common speed of the particles after impact is $4{\cdot}5\,\mathrm{m\,s^{-1}}$.

Exercise 8.2

1. A railway truck of mass 12 000 kg is moving along a straight level track at $7\,\mathrm{m\,s^{-1}}$ when it collides with another truck of mass 8000 kg moving in the same direction at $2\,\mathrm{m\,s^{-1}}$. After the collision the trucks move on together. Find the common speed of the trucks after impact.

2. A bullet of mass 0·04 kg is fired horizontally into a block of wood of mass 1·4 kg which rests on a smooth horizontal surface. The bullet becomes embedded in the block and after impact they move on together at a speed of $20\,\mathrm{m\,s^{-1}}$. Find the speed at which the bullet is fired.

3. Two particles, A of mass 5 kg and B of mass 2 kg, are moving horizontally along the same straight line when they collide. Both come to rest instantaneously. If the velocity of A was $3\,\mathrm{m\,s^{-1}}$ just before the impact, find the velocity of B and the kinetic energy lost in the impact.

4. A gun of mass 4 kg fires a bullet of mass 20 g at a speed of $500\,\mathrm{m\,s^{-1}}$. Find the speed of the gun's recoil and the energy of the explosion in the gun.

5. Two particles A and B, of mass 3 kg and 5 kg respectively, collide. Just before impact A is moving with speed $8\,\mathrm{m\,s^{-1}}$ and B is moving with speed $4\,\mathrm{m\,s^{-1}}$ in the opposite direction. Immediately after the impact A rebounds with speed $7\,\mathrm{m\,s^{-1}}$. Find the speed of B immediately after impact.

6. Two particles X and Y, of mass 4 kg and 3 kg respectively, collide. Just before impact X is moving with speed $13\,\mathrm{m\,s^{-1}}$ and Y is moving with speed $5\,\mathrm{m\,s^{-1}}$ in the opposite direction. Immediately after the impact X moves on, with no change in direction, at $4\,\mathrm{m\,s^{-1}}$. Find the speed of Y immediately after impact.

7. A particle of mass m and velocity $4\mathbf{i} + \mathbf{j}$ collides with a particle of mass $2m$ and velocity $\mathbf{i} - 8\mathbf{j}$. If the particles coalesce on impact, find their common velocity just after the collision.

8. Two particles, A of mass $3m$ and B of mass $2m$, are moving with velocities $4\mathbf{i} - 6\mathbf{j}$ and $2\mathbf{i} + 7\mathbf{j}$ respectively when they collide. Immediately after the impact A moves on in the same direction as before, but with half its original speed. Find the velocity of B just after the collision.

9. A particle of mass 2 kg moving with speed $10\,\mathrm{m\,s^{-1}}$ collides with a stationary particle of mass 7 kg. Immediately after impact the particles move with the same speed but in opposite directions. Find the kinetic energy lost in the collision.

10. Two particles, of masses $3m$ and $2m$ moving in opposite directions with speeds u and $4u$ respectively, collide. Given that no kinetic energy is lost in the collision, find the speeds of the particles immediately after impact.

11. Three trucks A, B and C, each of mass 2 tonnes, are moving on a straight horizontal track in the same direction with speeds of 10, 5 and $3\,\mathrm{m\,s^{-1}}$ respectively. When any two trucks collide, they become coupled. Thus after two collisions the three trucks move on with a common speed. Find this speed and find also the total loss of kinetic energy in the collisions.

12. A truck A of mass 3000 kg moving at $4\,\mathrm{m\,s^{-1}}$ collides with a truck B of mass 2000 kg moving at $2\,\mathrm{m\,s^{-1}}$ in the same direction. If after impact truck A continues to move in the same direction but at a speed of $3\,\mathrm{m\,s^{-1}}$, find the new speed of truck B. If the trucks are in contact for 1·5 s, find the average force each exerts on the other.

13. A van of mass 1800 kg begins to tow a stationary car of mass 1200 kg. Just before the tow-rope tightens, the van is moving at a speed of $1\,\mathrm{m\,s^{-1}}$. Find the common speed of the vehicles just after the rope tightens. Find also the impulsive tension in the rope.

14. A bullet of mass 0·03 kg is fired from a gun with a horizontal velocity of $600\,\mathrm{m\,s^{-1}}$. If the gun is then brought to rest in 1·2 s by a horizontal force which rises uniformly from zero to P N and then falls uniformly to zero, find the value of P.

15. A particle of mass 2 kg slides from rest down a smooth plane inclined at an angle of 30° to the horizontal. After 5 seconds it collides with a particle of mass 3 kg which is sliding down the plane at a speed of $7\,\mathrm{m\,s^{-1}}$. If the two particles coalesce on impact find their common velocity (a) just after the collision, (b) 2 seconds after the collision.

16. Two small beads A and B, each of mass m, are threaded on a smooth straight wire inclined at an angle α to the horizontal. Bead A is released from rest and begins to slide down the wire. At same instant bead B is projected up the wire towards A with speed u. After a time t the two beads collide and are brought to rest by the impact. Find an expression for t in terms of u, g and α.

8.3 Work, energy and momentum

When faced with a problem in particle dynamics, we will now have to decide whether to consider work-energy relationships, impulse-momentum relationships or a

combination of both. The two fundamental equations are:

Work done = change in kinetic energy
Impulse = change in momentum

Example 1 A bullet of mass m is fired horizontally with velocity V into a fixed block of wood and penetrates a distance d into it. If the block exerts a constant resistance R, which stops the bullet in a time T, find expressions for R and T in terms of m, V and d.

Work done against resistance = loss of kinetic energy

$\therefore \quad Rd = \frac{1}{2}mV^2$

Hence $\quad R = mV^2/2d$

Impulse = change in momentum

$\therefore \quad RT = mV$

Hence $\quad T = \dfrac{mV}{R} = mV \times \dfrac{2d}{mV^2} = \dfrac{2d}{V}.$

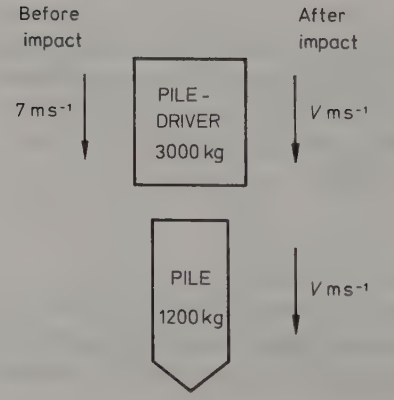
[Note that as the resistance is constant this problem could also be solved using Newton's second law and the constant acceleration formulae $v^2 = u^2 + 2as$, $v = u + at$.]

Example 2 A pile-driver of mass 3000 kg falls freely from a height of 2·5 m and strikes a pile of mass 1200 kg. After the blow the pile and the driver move on together. If the pile is driven a distance of 0·5 m into the ground, find the speed at which the pile starts to move into the ground and the average resistance of the ground to penetration.

Let v m s^{-1} be the downward velocity of the pile-driver just before impact.
Using the formula $v^2 = u^2 + 2as$, where $u = 0, a = 9{\cdot}8, s = 2{\cdot}5$, we have

$$v^2 = 0 + 2 \times 9{\cdot}8 \times 2{\cdot}5 = 49$$

$\therefore \quad v = 7$



Let V m s^{-1} be the common velocity of the pile and the driver after impact.
Using the principle of conservation of momentum,

$$7 \times 3000 + 0 \times 1200 = V(3000 + 1200)$$

$$\therefore \quad V = \frac{7 \times 3000}{4200} = 5$$

Hence the pile starts to move into the ground at 5 m s^{-1}.

Let R N be the average resistance of the ground, then the resultant force acting on pile and driver together is $(R - 4200g)$ N.

Work done = change in kinetic energy

$\therefore \quad (R - 4200g) \times 0{\cdot}5 = \frac{1}{2} \times 4200 \times 5^2$

$\therefore \quad R - 4200g = 4200 \times 25$

$\therefore \quad R = 4200(25 + 9{\cdot}8) = 146\,160$

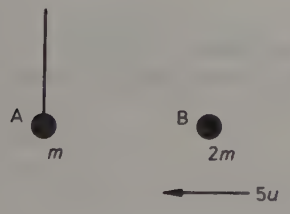
Hence the average resistance of the ground is 146 160 N.

When a particle moves in a vertical plane there are changes in its gravitational potential energy. Provided that no force other than gravity is doing work, we may then apply the principle of conservation of mechanical energy in the form:

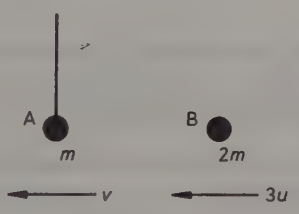
K.E. + P.E. = constant

Example 3 A particle A, of mass m, is attached to a fixed point by means of a light inextensible string and hangs in equilibrium. A particle B, of mass $2m$, is travelling horizontally with speed $5u$ when it collides with particle A. Immediately after the impact B continues to move in the same direction but with speed $3u$. If A swings to a height h above its original position before coming instantaneously to rest, find an expression for h in terms of u and g.

Let v be the speed of particle A after impact.



Just before impact



Just after impact

Using the principle of conservation of momentum,

$$m \times 0 + 2m \times 5u = m \times v + 2m \times 3u$$

$$10mu = mv + 6mu$$

$$\therefore \quad v = 4u$$

Hence the speed of A immediately after impact is $4u$.

When A is swinging freely its direction of motion is always at right angles to the string. Thus the tension in the string does no work and mechanical energy is conserved.

P.E. after impact = 0, K.E. after impact = $\frac{1}{2}m(4u)^2 = 8mu^2$.

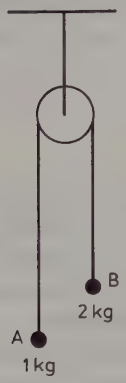
P.E. at height $h = mgh$, K.E. at height h = 0.

$\therefore \quad 0 + 8mu^2 = mgh + 0$

Hence $h = 8u^2/g$.

Example 4 Particles A and B, of masses 1 kg and 2 kg respectively, are attached to the ends of a light inextensible string passing over a smooth light pulley. The system is released from rest with both parts of the string taut and vertical. After A has travelled 1·2 m it hits and coalesces with a stationary particle C, of mass 2 kg. Find the new speed of the system and the impulsive tension in the string at the instant when C is picked up.

Taking the initial P.E. of each particle as zero, after the system has travelled 1·2 m the P.E. of A is $1{\cdot}2g$ J and the P.E. of B is $-1{\cdot}2 \times 2g$ J i.e. $-2{\cdot}4g$ J.



The initial K.E. of the system is zero. If v m s^{-1} is the speed of the particles after they have travelled 1·2 m, the K.E. acquired is $\frac{1}{2}(1+2)v^2$ J i.e. $1{\cdot}5v^2$ J.
Since no external force is doing work, the mechanical energy of the system remains constant.

$$\therefore \quad 1{\cdot}5v^2 + (1{\cdot}2g - 2{\cdot}4g) = 0$$
$$\text{i.e.} \quad 1{\cdot}5v^2 = 1{\cdot}2 \times 9{\cdot}8$$
$$\therefore \quad v = 2{\cdot}8$$

Thus, just before C is picked up the system is travelling at 2·8 m s^{-1}.
Let V m s^{-1} be the new speed of the system and J N s the impulsive tension in the string when C is picked up.

Impulse = change in momentum

Thus for B: $J = 2 \times 2{\cdot}8 - 2 \times V$

i.e. $J = 5{\cdot}6 - 2V$ (1)

For A and C: $J = (1 \times V + 2 \times V) - (1 \times 2{\cdot}8 + 2 \times 0)$

i.e. $J = 3V - 2{\cdot}8$ (2)

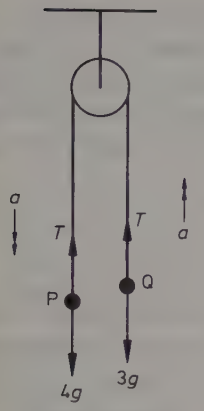
Subtracting (2) from (1): $0 = 8{\cdot}4 - 5V$

$\therefore \quad V = 1{\cdot}68$

Substituting in (1): $J = 5{\cdot}6 - 2 \times 1{\cdot}68 = 2{\cdot}24$.

Hence the new speed of the system is 1·68 m s^{-1} and the impulsive tension in the string is 2·24 N s.

Example 5 A particle P of mass 4 kg is connected to a particle Q of mass 3 kg by means of a light inextensible string which passes over a smooth light pulley. The system is released from rest with both parts of the string taut and vertical. After 3·5 s particle P hits the ground without rebounding. Assuming that particle Q does not hit the pulley, find the further time which elapses before the string is again taut. Find also the speed with which P leaves the ground.



Initial position

Let $a\,\mathrm{m\,s^{-2}}$ be the acceleration of the particles and let $T\,\mathrm{N}$ be the tension in the string.
Using Newton's second law,

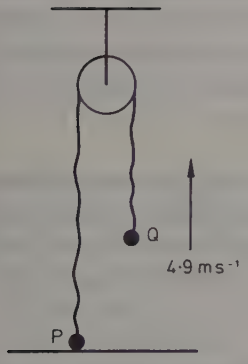
for P: $\quad 4g - T = 4a \qquad (1)$
for Q: $\quad T - 3g = 3a \qquad (2)$
(1) + (2): $\quad g = 7a$

$$\therefore \quad a = \frac{g}{7} = 1{\cdot}4$$

Using the formula $v = u + at$, where $u = 0$, $a = 1{\cdot}4$, $t = 3{\cdot}5$,

$$v = 0 + 1{\cdot}4 \times 3{\cdot}5 = 4{\cdot}9$$

Hence when P hits the ground, Q has an upward velocity of $4{\cdot}9\,\mathrm{m\,s^{-1}}$.

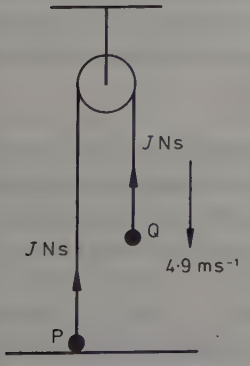


When P hits the ground

While P is on the ground Q moves freely under gravity. Hence the time which elapses before Q returns to the position where the string slackened is given by the formula $s = ut + \frac{1}{2}at^2$, where $s = 0$, $u = 4{\cdot}9$ and $a = -9{\cdot}8$.

$$\therefore \quad 0 = 4{\cdot}9t - 4{\cdot}9t^2$$
$$\therefore \quad t = 0 \text{ or } t = 1.$$

Hence the string is again taut after a further time of 1 s.
Since there is no loss of energy in this time, it follows that as the string tightens Q has a downward velocity of $4{\cdot}9\,\mathrm{m\,s^{-1}}$.



As the string becomes taut again

Let $V\,\mathrm{m\,s^{-1}}$ be the speed with which P is jerked into motion and let $J\,\mathrm{N\,s}$ be the impulsive tension in the string.

Impulse = change in momentum

Thus for P: $\quad J = 4V$
and for Q: $\quad J = 3 \times 4{\cdot}9 - 3V$
Subtracting: $\quad 0 = 7V - 3 \times 4{\cdot}9$
$\therefore \quad V = 2{\cdot}1$

Hence P leaves the ground at a speed of $2{\cdot}1\,\mathrm{m\,s^{-1}}$.

If we look again at the worked examples in this chapter we will see that a collision or the sudden tightening of a string often results in a loss of kinetic energy. This happens because some of the kinetic energy of the bodies involved is converted into other forms such as heat, sound or light. It is therefore important to stress that the principle of conservation of mechanical energy cannot usually be applied to a system when impulsive forces are acting.

Exercise 8.3

1. A car of mass 1200 kg travelling in a straight line at a speed of $20\,\mathrm{m\,s^{-1}}$ is brought to rest in a distance of 50 m by a constant braking force. Find the magnitude of this force and the time taken to stop the car.

2. A bullet of mass 25 g is fired horizontally with velocity $600\,\mathrm{m\,s^{-1}}$ into a fixed block of wood and penetrates a distance of 50 cm into it. Find the constant resistance exerted by the wood and the time taken to bring the bullet to rest.

3. A railway truck of mass 3000 kg moving at $4\,\mathrm{m\,s^{-1}}$ collides with a truck of mass 2000 kg moving in the opposite direction at $1\,\mathrm{m\,s^{-1}}$. If the trucks move on together after the collision, find their common speed just after impact. If a constant retarding force of 400 N acts on the trucks after impact, find the distance they travel before coming to rest.

4. A particle is projected at a speed of $7\,\mathrm{m\,s^{-1}}$ across a rough horizontal plane. After travelling a distance of 2 m it collides with a stationary particle of equal mass. The two particles coalesce and travel a further distance d m before coming to rest. If the coefficient of friction between the particles and the plane is 5/12, find the value of d.

5. A stone of mass 5 kg is dropped from a point 10 m above horizontal ground. Find the impulse exerted by the ground on the stone if (a) it comes to rest without rebounding, (b) it rises to a maximum height of 10 cm after impact.

6. A particle of mass 2 kg is projected vertically upwards from a point A at a speed of $21\,\mathrm{m\,s^{-1}}$. At the same instant a particle of mass 1 kg is dropped from a point 30 m vertically above A. Find the time which elapses before the two particles collide. If the particles coalesce on impact, find the speed at which they pass through A.

7. A pile-driver of mass 1500 kg falls freely from a height of 1·6 m and strikes a pile of mass 500 kg. After the blow the pile and the driver move on together. If the pile is driven a distance of 0·3 m into the ground, find the speed at which the pile starts to move into the ground and the average resistance of the ground to penetration.

8. A pile-driver of mass 1200 kg falls freely from a height of 3·6 m and strikes without rebounding a pile of mass 800 kg. The blow drives the pile a distance of 36 cm into the ground. Find the resistance of the ground, assumed to be constant. Find also the time for which the pile is in motion.

9. A particle P is released from rest at a point A on a smooth plane inclined at 30° to the horizontal. When it has reached a point B on the plane, P collides with a particle Q, of equal mass, moving up the plane at $7\ \text{m s}^{-1}$. After the collision P moves up the plane coming to rest instantaneously at A. Given that Q is brought to rest by the impact, find the distance AB.

10. A bullet of mass 30 g is fired into a stationary block of wood of mass 5 kg and becomes embedded in it. The block is freely suspended from a fixed point and after impact swings to a height of 40 cm above its initial position. Find, correct to 3 significant figures, the velocity of the bullet just before impact.

11. Two beads, A of mass m and B of mass $2m$, are threaded on a smooth circular wire fixed in a vertical plane. Initially A is at rest at the lowest point O of the wire and B is released from rest at a height h above O. Just before B collides with A it is moving with velocity v and immediately after impact A starts to move with velocity v. Find, in terms of h, the height above O at which each bead first comes to rest.

12. A particle P is attached to one end of a light inextensible string of length 1·25 m, the other end being attached to a fixed point A. The string is taut and inclined at an angle of 60° to the downward vertical when P is released from rest. As P reaches its lowest point B, it strikes a stationary particle Q, of equal mass, which is free to move horizontally. If the speed of Q just after impact is $2{\cdot}1\ \text{m s}^{-1}$, find the maximum height above B to which P rises after the collision.

13. Particles A and B, of masses 1 kg and 3 kg respectively are attached to the ends of a light inextensible string passing over a smooth light pulley. The system is released from rest with both parts of the string taut and vertical. After A has travelled 1·8 m it hits and coalesces with a stationary particle C, of mass 4 kg. Find the new velocity of the system and the impulsive tension in the string at the instant when C is picked up.

14. Particles A and B, of masses 1 kg and 2 kg respectively, are connected by a light inextensible string which passes over a smooth light pulley. The system is released from rest with the string taut and the hanging parts vertical. After 4·5 s A picks up a stationary particle of mass 3 kg. Find the velocity of the system immediately after impact and the further time which elapses before the system first comes instantaneously to rest.

15. A particle A of mass 5 kg is connected to a particle B of mass 2 kg by means of a light inextensible string which passes over a smooth light pulley. The system is released from rest with both parts of the string taut and vertical. After 3·5 s particle A hits the ground without rebounding. Assuming that particle B does not hit the pulley, find the further time which elapses before the string is again taut. Find also the speed with which A leaves the ground.

16. Particles P and Q, of masses m and $2m$ respectively, are connected by a light inextensible string which passes over a smooth light pulley. The system is released from rest with the string taut and both particles at a height h above the ground.

Assuming that Q hits the ground without rebounding and that P does not hit the pulley, find expressions for (a) the greatest height above the ground reached by P, (b) the impulsive tension in the string when Q leaves the ground again.

17. A particle A, of mass 4 kg, rests on a smooth plane inclined at an angle of 30° to the horizontal. It is connected by a light inextensible string passing over a smooth pulley fixed at the top of the plane to a particle B, of mass 3 kg, which hangs freely. The system is released from rest with both parts of the string taut. When B has fallen a distance of 6·3 m. A picks up a particle C, of mass 3·5 kg, which was previously at rest. Find the velocity of the system immediately after C has been picked up. Find also the loss in kinetic energy as C is picked up, giving your answer correct to 3 significant figures.

18. A pile of mass M is being driven into the ground by a pile-driver of mass m. The pile-driver is released from rest at a height h above the top of the pile and falls freely under gravity on to the top of the pile. The pile-driver rebounds vertically from the top of the pile and, after the first impact, it comes first to rest at a point $\frac{3}{4}h$ below the original point of release. (i) Find the kinetic energy lost by the pile-driver in the first impact. (ii) Find the kinetic energy gained by the pile in the first impact, assuming that the momentum of the system is conserved in the impact. (iii) Given that the pile is driven a distance $\frac{1}{16}h$ into the ground by the first impact, show that the frictional resistance, assumed constant, between the pile and the ground is $(36m^2 + M^2)/g/M$. (C)

19. Two particles, A and B, of masses $2m$ and $3m$ respectively, are attached to the ends of a light inextensible string of length c and are placed close together on a horizontal table. The particle A is projected vertically upwards with speed $\sqrt{(6gc)}$. (i) Show that, at the instant immediately after the string tightens, B is moving with velocity $\frac{4}{5}\sqrt{(gc)}$. (ii) State the impulse of the tension in the string. (iii) Find the height to which A rises above the table before it comes to instantaneous rest. (iv) Calculate the loss in kinetic energy due to the tightening of the string. (AEB 1979)

20. A cube of mass 100 g and side 14 cm is at rest on a smooth horizontal table. A bullet of mass 20 g is fired with velocity 400 m s^{-1} into the centre of a vertical face of the cube. The bullet travels through the cube in a direction perpendicular to this face and emerges undeflected with velocity 200 m s^{-1}. Find the velocity of the cube as the bullet emerges. Find the kinetic energy gained by the cube and the kinetic energy lost by the bullet. If the cube moves a distance of d cm before the bullet emerges, write down the distance moved by the bullet as it passes through the cube. By considering the work done on the cube and on the bullet, determine the value of d and the magnitude of the constant force exerted by the cube on the bullet.

8.4 Newton's law of restitution

In a collision between two elastic bodies the impulses exerted by each body on the other produce changes in momentum and the bodies bounce away from each other.

If both bodies are moving along the line of action of these impulses at the instant the collision occurs, then the impact is said to be *direct*. Otherwise it is described as *indirect* or *oblique*. Both types of elastic impact were investigated by Sir Isaac Newton and as a result of his experimental work he was able to formulate a law of impact which is known as *Newton's law of restitution* or as *Newton's experimental law*. In its general form the law concerns the indirect impact of two moving bodies. However, we first consider its application to the direct impact of one body with a fixed plane.

Our everyday experience tells us that if several objects made of different materials are dropped from a given height on to horizontal ground, then the various objects will probably rebound to different heights after impact. A lump of plasticine will become flattened and show little tendency to rebound, whereas a "superball" will regain most of its original height. The behaviour of most objects will come somewhere between these two extremes.

Further observation shows that when a body strikes a fixed plane there is a period of time during which the body is compressed and its speed is reduced to zero. This is followed by a period of restitution in which the body regains its original shape and starts to rebound. While the body is being compressed some of its original kinetic energy is stored as elastic potential energy and this is then released as the body rebounds. The remaining kinetic energy is converted into other forms such as heat energy. Thus, the behaviour of a particular body during impact depends on its elasticity.

Experimental evidence suggests that in a series of direct collisions between a given elastic body a fixed plane the ratio of the speed of the body after impact to its speed before impact is approximately constant. Thus, in its simplest form, Newton's law of restitution states that if the speed of the body before impact is u and its speed after impact is v, then

$$v = eu,$$

where e is constant for that particular body. The constant e is known as the *coefficient of restitution* between the body and the plane and its value always lies between 0 and 1. If $e = 0$, collisions between the body and the plane are said to be *inelastic*, but if $e = 1$ they are described as *perfectly elastic*.

Example 1 A ball dropped from a height of 9 m above horizontal ground rebounds to a height of 4 m. Find the coefficient of restitution between the ball and the ground.

Substituting $u = 0, a = g, s = 9$ in the formula $v^2 = u^2 + 2as$,
we have: $v^2 = 0 + 2g \times 9$
$\therefore \quad v = 3\sqrt{(2g)}$

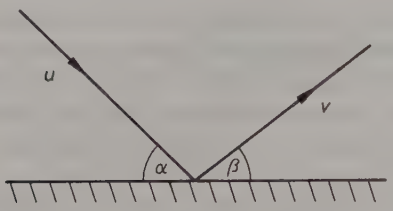
Hence the speed of the ball just before impact is $3\sqrt{(2g)}\,\mathrm{m}^{-1}$.
Substituting $v = 0, a = -g, s = 4$ in the formula $v^2 = u^2 + 2as$,
we have: $0 = u^2 - 2g \times 4$
$\therefore \quad u = 2\sqrt{(2g)}$

Hence the speed of the ball just after impact is $2\sqrt{(2g)}\,\mathrm{m\,s}^{-1}$.

Thus, if e is the coefficient of restitution,

$$2\sqrt{(2g)} = e \times 3\sqrt{(2g)}$$
$$\therefore \quad e = \tfrac{2}{3}.$$

When considering indirect impacts between an elastic body and a smooth fixed plane, it is found that Newton's law of restitution can be applied to motion perpendicular to the plane.



Let us suppose that a body strikes a smooth fixed plane while travelling with speed u in a direction inclined at an angle α to the plane, then rebounds with speed v at an angle β to the plane.

Since the plane is smooth no force acts along the plane and therefore there is no change in the component of velocity along the plane

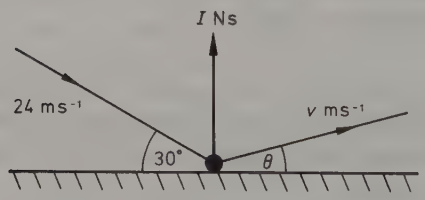
i.e. $$v\cos\beta = u\cos\alpha.$$

However, the impulse of the normal reaction of the plane changes the component of velocity perpendicular to the plane in accordance with Newton's law,

i.e. $$v\sin\beta = eu\sin\alpha,$$

where e is the coefficient of restitution.

Example 2 A particle of mass 2 kg strikes a fixed smooth plane while moving at a speed of $24\,\mathrm{m\,s^{-1}}$ in a direction making an angle of 30° with the plane. If the coefficient of restitution between the particle and the plane is $\frac{1}{4}$, find the speed at which the particle rebounds. Find also the impulse exerted by the plane on the particle.



Let the velocity of the particle after impact be $v\,\mathrm{m\,s^{-1}}$ at an angle θ to the plane and let the impulse exerted by the plane be I N s.

There is no change in velocity parallel to the plane

$$\therefore \quad v\cos\theta = 24\cos 30° = 12\sqrt{3}.$$

By Newton's law of restitution perpendicular to the plane,

$$v\sin\theta = \tfrac{1}{4} \times 24\sin 30° = 3$$

Squaring and adding: $v^2\cos^2\theta + v^2\sin^2\theta = (12\sqrt{3})^2 + 3^2$

$$\therefore \quad v = \sqrt{(432 + 9)} = 21.$$

Hence the speed of the particle after impact is $21\,\mathrm{m\,s^{-1}}$.

Considering the motion of the particle perpendicular to the plane,

impulse = change in momentum

$$\therefore \qquad I = 2 \times v \sin\theta + 2 \times 24 \sin 30° = 30.$$

Hence the impulse exerted by the plane is 30 N s.

When applied to the direct impact of two moving bodies, Newton's law of restitution states that the ratio of the relative speed of separation after impact to the relative speed of approach before impact is constant.

Before impact ($u_1 > u_2$) After impact ($v_1 < v_2$)

Let us suppose that two spheres of masses m_1 and m_2 collide when moving along the same straight line with velocities u_1 and u_2 respectively. Let the velocities of the spheres after impact be v_1 and v_2 respectively. Provided that all velocities are measured in the same direction, as shown in the diagram, the relative speed of approach is $(u_1 - u_2)$ and the relative speed of separation is $(v_2 - v_1)$.

By Newton's law of restitution

$$v_1 - v_2 = -e(u_1 - u_2)$$

where e is the coefficient of restitution.

Assuming that no external impulse acts during the impact, we can also apply the principle of conservation of momentum to give:

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

Example 3 A sphere A, of mass $2m$ and moving with speed $6u$, collides directly with a sphere B of mass $3m$ moving in the same direction with speed $3u$. After the impact the speed of B is $5u$. Find the coefficient of restitution between the spheres.

Let the speed of A after impact be v, as shown in the diagram.

Using the principle of conservation of momentum,

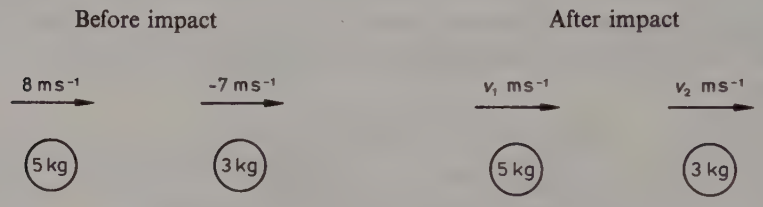
$$2m \times 6u + 3m \times 3u = 2m \times v + 3m \times 5u$$
$$\therefore \quad 12mu + 9mu = 2mv + 15mu$$
$$\therefore \quad v = 3u.$$

Using Newton's law of restitution,

$$v - 5u = -e(6u - 3u)$$
$$\therefore \quad 3u - 5u = -3eu$$
$$\therefore \quad e = \tfrac{2}{3}$$

Hence the coefficient of restitution between the spheres is $\frac{2}{3}$.

Example 4 A sphere of mass 5 kg, moving at $8\,\mathrm{m\,s^{-1}}$, collides directly with a sphere of mass 3 kg moving in the opposite direction at $7\,\mathrm{m\,s^{-1}}$. If the coefficient of restitution between the spheres is 0·6, find their speeds after impact. Find also the magnitude of the impulse exerted on each sphere during the collision.



Let the velocities of the spheres after impact be $v_1\,\mathrm{m\,s^{-1}}$ and $v_2\,\mathrm{m\,s^{-1}}$, as shown in the diagram.
Using conservation of momentum,

$$5 \times 8 - 3 \times 7 = 5v_1 + 3v_2$$
$$\therefore \quad 5v_1 + 3v_2 = 19 \qquad (1)$$

Using Newton's law of restitution,

$$v_1 - v_2 = -0{\cdot}6\{8 - (-7)\}$$
$$\therefore \quad v_1 - v_2 = -9 \qquad (2)$$

Adding 3 × (2) to (1): $8v_1 = -8$
Hence $v_1 = -1$ and $v_2 = 8$.

Thus after impact the speed of the 5 kg sphere is $1\,\mathrm{m\,s^{-1}}$ and the speed of the 3 kg sphere is $8\,\mathrm{m\,s^{-1}}$. The directions of both spheres are reversed in the collision.

Considering the motion of the 3 kg sphere,

impulse = change in momentum
$= 3 \times 8 - 3 \times (-7)\,\mathrm{N\,s} = 45\,\mathrm{N\,s}$

Hence the magnitude of the impulse exerted on each sphere is 45 N s.

Exercise 8.4

1. A ball of mass 0·2 kg is dropped from a point 10 m above horizontal ground. If the coefficient of restitution between the ball and the ground is $\frac{1}{2}$, find (a) the height above the ground to which the ball rebounds, (b) the loss of kinetic energy in the impact, (c) the impulse exerted by the ground.

2. A ball dropped from a height of 8 m above horizontal ground rebounds to a height of 4·5 m. Find the coefficient of restitution between the ball and the ground. If the ball is then dropped from a height of 12·8 m and allowed to bounce twice, find the maximum height it reaches after the second bounce.

3. A particle is projected vertically upwards with speed $7\,\mathrm{m\,s^{-1}}$ from a point A, which is 2 m above a horizontal plane. After striking this plane the particle first comes instantaneously to rest at A. Find the coefficient of restitution between the particle and the plane.

4. A particle of mass 3 kg strikes a fixed smooth plane while moving at a speed of $20\,\mathrm{m\,s^{-1}}$ in a direction making an angle α with the plane, where $\sin\alpha = 3/5$. If the coefficient of restitution between the particle and the plane is 2/3, find the speed at which the particle rebounds. Find also the impulse exerted by the plane on the particle.

5. A particle strikes a fixed smooth plane while travelling in a direction inclined at 45° to the plane. After impact its direction makes an angle of 30° with the plane. Find the coefficient of restitution between the particle and the plane. If on another occasion the particle is moving in a direction inclined at 60° to the plane, find its direction after impact with the plane.

6. A particle strikes a smooth fixed plane and rebounds in a direction at right angles to its direction just before impact. If the coefficient of restitution is $\frac{1}{3}$, find the acute angle between the initial direction of motion of the particle and the plane.

7. A particle is projected with speed $6{\cdot}5\,\mathrm{m\,s^{-1}}$ at an angle θ to the horizontal, where $\tan\theta = 12/5$, from a point A on a smooth horizontal plane. The particle strikes the plane again at B, then rebounds and strikes the plane for a second time at C. The coefficient of restitution between the particle and the plane is 5/9. Find the speed and direction with which the particle rebounds from the plane at B. Show that the distance AC is 100/21 m.

8. A sphere of mass $4m$ and moving with speed $5u$, collides directly with a sphere of mass $3m$, which is at rest. If the coefficient of restitution between the spheres is 0·4, find their speeds after impact. Find also the loss in kinetic energy due to the impact.

9. A sphere A, of mass 1 kg and moving with speed $12\,\mathrm{m\,s^{-1}}$, collides directly with a sphere B, of mass 4 kg, moving in the same direction with speed $2\,\mathrm{m\,s^{-1}}$. Given that sphere A is brought to rest by the collision, find the coefficient of restitution between the spheres.

10. A sphere moving with speed $7\,\mathrm{m\,s^{-1}}$ collides directly with an identical sphere moving in the opposite direction with speed $2\,\mathrm{m\,s^{-1}}$. Find the speeds of the spheres after impact if the coefficient of restitution is (a) $\frac{1}{3}$, (b) $\frac{2}{3}$. In each case state whether the spheres are travelling in the same direction or in opposite directions.

11. A sphere of mass m moving with speed $9u$ collides directly with a sphere of mass M moving in the same direction with speed $5u$. After impact the spheres move in the same direction with speeds $6u$ and $7u$. Find the coefficient of restitution between the spheres and the ratio $m:M$.

12. Three perfectly elastic particles A, B, C, with masses 4 kg, 2 kg, 3 kg respectively, lie at rest in a straight line on a smooth horizontal table. Particle A is projected towards B with speed $15\,\mathrm{m\,s^{-1}}$ and after A has collided with B, B collides with C. Find the velocities of the particles after the second collision and state whether there will be a third collision.

13. Three identical spheres A, B, C, lie at rest in a straight line on a smooth horizontal table. Particle A is projected towards B with speed u. After the collision between A and B, B moves on to hit C. The coefficient of restitution between any two of the spheres is $\frac{1}{2}$. Find the speeds of the spheres after the second collision and state whether there will be a third collision.

14. A sphere A of mass $2m$, moving on a smooth horizontal floor with speed $5u$, collides directly with a sphere B of mass m, which is at rest. If the coefficient of restitution between the spheres is 4/5, find the speed of B after impact. Sphere B goes on to strike a vertical wall, which is at right angles to the direction of motion of the spheres. A is then brought to rest by a second collision with B. Find the coefficient of restitution between B and the wall.

15. A small smooth sphere moves on a horizontal table and strikes an identical sphere lying at rest on the table at a distance d from a vertical wall, the impact being along the line of centres and perpendicular to the wall. Prove that the next impact between the spheres will take place at a distance $2de^2/(1 + e^2)$ from the wall, where e is the coefficient of restitution for all impacts involved. (L)

16. The coefficient of restitution between two small, smooth spheres A and B is e. The mass of A is m and of B is em. Show that, if A is at rest and B is projected with velocity u to collide directly with A, B is reduced to rest by the collision. If, instead, B is at rest and A is projected with the same speed to hit B directly, find the velocities of A and B after the collision. Show that the loss of kinetic energy is the same in each collision and is greatest when $e = \frac{1}{2}$. (SU)

17. Two small spheres of masses m and $2m$ are connected by a light inextensible string of length $2a$. When the string is taut and horizontal, its mid-point is fixed and the spheres are released from rest. The coefficient of restitution between the spheres is $\frac{1}{2}$. Show that the first impact brings the heavier sphere to rest, and the second impact brings the lighter sphere to rest. Find the velocity of each sphere immediately after the third impact. (L)

18. Three small smooth spheres A, B, C, of equal radius but of mass $m, 2m$ and m respectively, lie at rest and separated from one another on a smooth horizontal table in the order ABC with their centres in a straight line. Sphere A is projected with speed V directly towards sphere B. The coefficient of restitution at each collision is e, where $0 < e < 1$. Find, in terms of V and e, the speeds of the three spheres (a) after the first collision (between A and B), (b) after the second collision (between B and C). Hence show that there will be at least three collisions if $e^2 - 7e + 1 > 0$. (O&C)

Exercise 8.5 (*miscellaneous*)

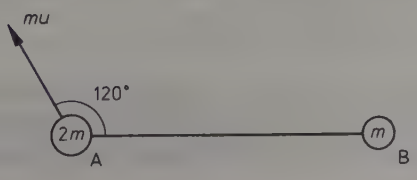
1. Two particles of masses $2m$ and $3m$ are moving towards each other with speeds $4u$ and $2u$ respectively. The direction of motion of the heavier particle is reversed by the impact and its speed after impact is u. Find (i) the magnitude of the impulse, (ii) the loss of energy. (C)

2. A particle of mass 4 kg moves under the action of a force $\mathbf{F}$ in the interval of time from $t = 0$ to $t = 2$. When $t = 0$ the velocity of the particle is $2\mathbf{i} - 3\mathbf{j}$. Given that $\mathbf{F} = 6t\mathbf{i} + 2\mathbf{j}$, find (a) the impulse of $\mathbf{F}$, (b) the velocity of the particle when $t = 2$, (c) the gain in kinetic energy.

3. A particle of mass m moves in a horizontal plane Oxy with speed v along the x-axis in the positive direction. It is subjected to a horizontal impulse $\mathbf{I}$ which turns its direction of motion through 30° in an anticlockwise sense and reduces its speed to $v/\sqrt{3}$. Find the vector $\mathbf{I}$. At the same instant an impulse $-\mathbf{I}$ is applied to a particle of mass $3m$ which is at rest. Find the magnitude and direction of the resultant velocity of this particle. (JMB)

4. A pump raises water from an underground reservoir through a height of 10 m and delivers it at a speed of $9\,\mathrm{m\,s^{-1}}$ through a circular pipe of internal diameter 20 cm. Taking 1 litre of water to have a mass of 1 kg and g to have the value $9{\cdot}8\,\mathrm{m\,s^{-2}}$ find (i) the mass of water raised per second, correct to 3 sig. fig.; (ii) the kinetic energy imparted to the water each second, correct to 3 sig. fig.; (iii) the effective power of the pump correct to 2 sig. fig.; (iv) the actual power of the pump if it is 70% efficient. If the water hits a vertical wall horizontally and does not rebound calculate the magnitude of the force exerted on the wall, correct to 2 sig. fig. (SU)

5. A light inextensible string AB has a particle of mass $2m$ attached at A and a particle of mass m attached at B. The particles are placed on a smooth horizontal table with the string taut. A horizontal impulse is applied to the particle at A of magnitude mu in a direction which makes an angle of 120° with AB. (i) Show that B



starts to move along BA with speed $u/6$. (ii) Find the components of the initial velocity of the particle at A along BA and perpendicular to BA and state the magnitude of the impulse of the tension in the string. (AEB 1977)

6. A composite particle of mass $m_1 + m_2$ is formed as a result of a head-on collision between two particles of masses m_1, m_2 travelling with speeds u_1, u_2 respectively. Find the speed of the composite particle in terms of m_1, m_2, u_1, u_2. Show that the collision leads to a loss of kinetic energy of amount

$$\frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right)(u_1 + u_2)^2. \qquad \text{(W)}$$

7. A gun of mass M fires a shell of mass m horizontally. The recoil of the gun is opposed by a constant force F which brings the gun to rest in t seconds. Show that the energy of the explosion is $\dfrac{F^2t^2(M+m)}{2Mm}$. (O)

8. A particle of mass m is dropped from rest from a point A which is at a height h above a horizontal surface. After hitting the surface the particle rebounds vertically and reaches a maximum height of $\frac{1}{2}h$ above the surface at the point B. Find expressions for the loss of energy and the impulse on the particle at the impact. At the instant of the particle's impact with the surface, a second particle, of mass $4m$, is dropped from rest at A. Show that the two particles collide at B. The two particles coalesce when they collide at B. When the combined particle hits the surface, one-half of its kinetic energy is lost in the impact. Show that, after the impact, the combined particle reaches a greatest height of $41h/100$ above the surface. (C)

9. Two beads, one of mass 3 g and the other of mass 2 g, are threaded on a smooth circular wire which is fixed with its plane vertical. The heavier bead is initially at rest at the lowest point L of the wire. The lighter bead is released from rest and is moving with speed $2\ \text{m s}^{-1}$ when it strikes the other bead. There is no loss of energy in the impact. Find the speed of each bead immediately after impact. In the subsequent motion each bead comes to rest instantaneously before reaching the highest point of the wire. Find, to the nearest millimetre, the height above L at which each bead comes first to rest. (C)

10. A pile-driver of mass 2000 kg is raised 1·6 m above a pile of mass 1500 kg and then dropped on to it. After impact the driver and the pile move together and the pile is driven 0·2 m into the ground. Find (i) the velocity of the pile-driver just before it hits the pile, (ii) the common velocity of pile and driver immediately after impact, (iii) the loss of energy on impact, stating the units in which it is measured, (iv) the mean resistance of the ground to penetration. (AEB 1980)

11. A particle A of mass $3m$ lies on a smooth horizontal rectangular table at a distance $5a$ from a point C at the edge of the table. The particle is attached to one end of a light inelastic string of length $6a$ which passes over a small smooth pulley at C.

To the other end of the string is attached a particle B of mass $2m$. The horizontal floor is at a depth $4a$ below C. The particle B is released from rest at C and motion takes place in a vertical plane perpendicular to the edge of the table. Show that the speed of the particles immediately after the string becomes taut is $\frac{2}{5}\sqrt{(2ga)}$. Find (i) the magnitude of the impulse which acts on the particle A, (ii) the loss of kinetic energy of the system due to the string becoming taut, (iii) the speed of the particle B just before it hits the floor. (C)

12. Two masses, one 3 kg and the other 2 kg, are connected by a light inextensible string which passes over a fixed smooth pulley and the system is released from rest. Find the acceleration of the system and show that the tension in the string is 23·52 N. After falling 2 metres the heavier mass strikes a fixed horizontal plane and is brought to rest. If the lighter mass does not reach the pulley, show that after 4/7 seconds the 3 kg mass is again in motion. Find the speed with which it begins to ascend and the impulsive tension in the string. (W)

13. Two particles A and B, each of mass m and connected by a light inextensible string of length $4a$, are resting on a smooth horizontal table with $AB = 2a$. A particle C of mass $3m$ is attached to the midpoint of the string and rests on the table so that $AC = BC$. The particles A and B are then simultaneously projected along the table with equal velocities perpendicular to AB and in the sense away from C. Show that, when the string tightens, the initial velocities of the particles A and B are each inclined at angle $\frac{1}{3}\pi$ with the line AB. (O)

14. A particle P of mass $2m$ is connected to a particle Q of mass $3m$ by means of a light inextensible string which passes over a small smooth pulley. The particles are released from rest with the string taut and its hanging parts vertical. (i) Calculate the acceleration of P and the tension in the string. At the instant when P is moving with velocity v it collides and coalesces with a stationary particle of mass $3m$ to form a particle P_1. (ii) Prove that the velocity of P_1 immediately after the collision is $5v/8$. (iii) Calculate the magnitude of the impulse of the tension in the string. (iv) Calculate the loss in kinetic energy of the whole system due to the collision.

The string breaks and particle Q hits a horizontal floor when moving with speed u and bounces to a height $u^2/(6g)$ above the floor. (v) Calculate the coefficient of restitution between Q and the floor. (AEB 1979)

15. A bullet of mass m travelling horizontally with speed u strikes perpendicularly a wooden block of mass M which is free to move horizontally in the line of motion of the bullet. The resistance to penetration is assumed uniform and the bullet comes to rest after penetrating a distance a into the block. Show that the ultimate common velocity of the system is $mu/(m + M)$ and that the total loss of kinetic energy during the penetration is $mMu^2/2(m + M)$. Deduce the value of the resistance and show that (i) the block moves a distance $am/(m + M)$ before the bullet comes to rest relative to the block, (ii) the penetration by the bullet ceases after a time $2a/u$. (W)

16. A particle of mass m is attached to one end of a light inextensible string of length a, the other end being attached to a fixed point O. Initially the string is taut and

makes an angle α ($\leqslant 90°$) with the downward vertical at O. The particle is released and, when it is at its lowest point A, coalesces with a stationary particle of mass m. The combined body next comes to rest with the string making an angle β with the downward vertical at O. Show that $1 - \cos\beta = \frac{1}{4}(1 - \cos\alpha)$.

The motion continues so that each time the body passes through A it coalesces with a stationary particle of mass m. If $\alpha = 90°$, (a) prove that the speed of the body just before passing through A for the fourth time is $\frac{1}{4}\sqrt{(2ag)}$, (b) find the angle (correct to the nearest degree) made by the string with the downward vertical at O when the body next comes to instantaneous rest after passing through A for the fourth time, (c) find the ratio of the tensions in the string just before and just after the body passes through A for the fourth time. (O&C)

17. Two equal particles are projected at the same instant from points A and B on horizontal ground, the first from A with speed u at an angle of elevation α and the second from B with speed v at an angle of elevation β. They collide directly when they are moving horizontally in opposite directions. Find v in terms of u, α and β, and show that $AB = \dfrac{u^2 \sin\alpha \sin(\alpha + \beta)}{g \sin\beta}$. The particles coalesce on impact. Find, in terms of u, α and β, the speed of the combined particle immediately after the collision. In the case when $\alpha = 30°$ and $\beta = 60°$ find AC, where C is the point where the combined particle returns to the ground. (JMB)

18. A ball is thrown vertically upwards from the floor with velocity V. It rebounds from the ceiling, which is at height h above the floor, and then rebounds from the floor. After this second rebound, it just reaches the ceiling again. If the coefficient of restitution between the ball and ceiling is e, and between the ball and floor is f, prove that

$$V^2 = \frac{2gh(1 - f^2 + e^2 f^2)}{e^2 f^2}.$$ (SU)

19. A particle of mass m strikes a fixed smooth plane at speed u while moving in a direction inclined at an angle α to the plane, then rebounds at an angle β to the plane. Find expressions for (a) the coefficient of restitution between the particle and the plane, (b) the speed of the particle after impact, (c) the impulse exerted on the plane.

20. A particle of mass m, projected from a point A with speed u at an angle of elevation α, strikes a smooth vertical wall at a point B and after rebounding returns to A. Show that the total time taken to move from A to B and then back to A is $(2u \sin\alpha)/g$. Given that e is the coefficient of restitution between the particle and the wall, find expressions for (a) the loss of kinetic energy on impact, (b) the distance of A from the wall.

21. Two elastic spheres of masses m and M collide directly. Their velocities before impact are 12 m/s and 6 m/s respectively in the same direction, and after impact they are 5 m/s and 8 m/s respectively also in the same direction. Find the coefficient of restitution and also the ratio $m:M$. Find also the velocities of the spheres after

impact if they collide directly when travelling in opposite directions, the velocity of the heavier sphere being 4 m/s and that of the lighter sphere 14 m/s. (L)

22. Three small spheres A, B and C, of equal radii and masses m, $2m$ and $3m$ respectively, are placed with their centres in a straight line on a smooth horizontal table with B between A and C. The sphere A is given a velocity $5u$ in the direction $\overrightarrow{AB}$ and, as a result of the ensuing collision between A and B, B moves towards C with speed $3u$. Calculate (i) the magnitude and the sense of the velocity of A after the collision, (ii) the coefficient of restitution between A and B, (iii) the loss in kinetic energy due to the collision between A and B.

The sphere B strikes C and as a result C receives an impulse of $4mu$. Calculate (iv) the velocities of B and C after their collision, (v) the coefficient of restitution between B and C. (AEB 1978)

23. A small sphere A, of mass m moving with velocity $2u$ on a smooth horizontal plane, impinges directly on a small sphere B, of equal radius and mass $2m$ moving with velocity u in the same direction on the plane. Given that after impact B moves with velocity $3u/2$, calculate (i) the coefficient of restitution between A and B, (ii) the loss in kinetic energy due to the impact. The sphere B continues to move with velocity $3u/2$ until it hits a vertical wall from which it rebounds and is then brought to rest by a second impact with the approaching sphere A. Calculate (iii) the coefficient of restitution between B and the wall, (iv) the final speed of the sphere. (AEB 1980)

24. Particles A and B of mass m and $2m$ respectively lie at rest on a smooth horizontal plane. They are projected towards each other both with speed u. Show that, if the coefficient of restitution between the particles is $\frac{3}{4}$, their directions of motion are reversed on impact and find their speeds after impact. Find, also, the loss of kinetic energy during impact. If A subsequently hits a smooth vertical plane normally, show that A and B will collide again only if the coefficient of restitution between A and the vertical plane is greater than $1/8$. (SU)

25. Two particles A, B of masses M, m respectively are attached to a fixed point O by light inextensible strings of equal length. The particle A is held so that its string is taut and horizontal. It is then released so that it strikes the particle B which was hanging in equilibrium. Prove that if OB just reaches the horizontal then $Me = m$, where e is the coefficient of restitution between the particles. (O)

26. Two particles, A of mass $2m$ and B of mass m, moving on a smooth horizontal table in opposite directions with speeds $5u$ and $3u$ respectively, collide directly. Find their velocities after the collision in terms of u and the coefficient of restitution e. Show that the magnitude of the impulse exerted by B on A is $16mu(1 + e)/3$. Find the value of e for which the speed of B after the collision is $3u$. Moving at this speed B subsequently collides with a stationary particle C of mass km, and thereafter remains attached to C. Find the velocity of the combined particle and find the range of values of k for which a third collision will occur. (JMB)

27. Identical beads P and Q are threaded on a smooth circular hoop fixed in a horizontal plane. Initially Q is at rest at a point A of the hoop when P collides with it. Denoting the coefficient of restitution between P and Q by e, find the values of e for which the next collision takes place at A. (JMB)

9 Series and expansions

9.1 Summation of finite series

There are various standard results that can be used to find the sums of finite series. Two of these are the formula for the sums of arithmetic and geometric progressions obtained in Book 1.

> The sum to n terms of the A.P. with first term a and common difference d is $\frac{1}{2}n\{2a + (n-1)d\}$.
> The sum to n terms of the G.P. with first term a and common ratio r is $a(1-r^n)/(1-r)$.

The series $\sum_{r=1}^{n} r = 1 + 2 + 3 + \ldots + n$ is an arithmetic progression with first term 1 and common difference 1.

$$\therefore \qquad \sum_{r=1}^{n} r = \tfrac{1}{2}n\{2 + (n-1)\} = \tfrac{1}{2}n(n+1).$$

The series $\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2$ can be summed using the identity

$$(r+1)^3 - r^3 \equiv 3r^2 + 3r + 1.$$

Substituting $r = 1, 2, 3, \ldots, n$, we have

$$\begin{aligned} 2^3 - 1^3 &= 3.1^2 + 3.1 + 1 \\ 3^3 - 2^3 &= 3.2^2 + 3.2 + 1 \\ 4^3 - 3^3 &= 3.3^2 + 3.3 + 1 \\ &\ldots\ldots\ldots\ldots \\ (n+1)^3 - n^3 &= 3n^2 + 3n + 1 \end{aligned}$$

Adding together all n equations, we find that most of the terms on the left may be

cancelled out to give:

$$(n+1)^3 - 1^3 = 3\sum_{r=1}^{n} r^2 + 3\sum_{r=1}^{n} r + n$$

$$\therefore \quad 3\sum_{r=1}^{n} r^2 = (n+1)^3 - 1^3 - 3\sum_{r=1}^{n} r - n$$

$$= (n+1)^3 - 1 - \tfrac{3}{2}n(n+1) - n$$

$$= n^3 + \tfrac{3}{2}n^2 + \tfrac{1}{2}n = \tfrac{1}{2}n(2n^2 + 3n + 1)$$

Hence $$\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1).$$

By a similar method it can be shown that

$$\sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2 = \left(\sum_{r=1}^{n} r\right)^2$$

To summarise, we now have three further important results:

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1), \quad \sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1), \quad \sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2$$

Example 1 Find the sum of the first n terms of the series

$$1.3.5 + 2.4.6 + 3.5.7 + \ldots.$$

The rth term of the series $= r(r+2)(r+4) = r^3 + 6r^2 + 8r$

$\therefore$ the sum of the first n terms

$$= \sum_{r=1}^{n} (r^3 + 6r^2 + 8r)$$

$$= \sum_{r=1}^{n} r^3 + 6\sum_{r=1}^{n} r^2 + 8\sum_{r=1}^{n} r$$

$$= \tfrac{1}{4}n^2(n+1)^2 + 6.\tfrac{1}{6}n(n+1)(2n+1) + 8.\tfrac{1}{2}n(n+1)$$

$$= \tfrac{1}{4}n(n+1)\{n(n+1) + 4(2n+1) + 16\}$$

$$= \tfrac{1}{4}n(n+1)(n^2 + n + 8n + 4 + 16)$$

$$= \tfrac{1}{4}n(n+1)(n^2 + 9n + 20)$$

$$= \tfrac{1}{4}n(n+1)(n+4)(n+5)$$

Example 2 Find the sum of the series $(n+1)^2 + (n+2)^2 + (n+3)^2 + \ldots + (2n)^2$.

The sum of the series $= \sum_{r=1}^{2n} r^2 - \sum_{r=1}^{n} r^2$

$$= \tfrac{1}{6}.2n(2n+1)(4n+1) - \tfrac{1}{6}n(n+1)(2n+1)$$

$$= \tfrac{1}{6}n(2n+1)\{2(4n+1) - (n+1)\}$$

$$= \tfrac{1}{6}n(2n+1)(7n+1).$$

Example 3 Find the sum of the series $1^3 + 3^3 + 5^3 + \ldots + (2n-1)^3$.

$$1^3 + 2^3 + 3^3 + \ldots + (2n)^3 = \sum_{r=1}^{2n} r^3$$

$$= \tfrac{1}{4}(2n)^2(2n+1)^2$$

$$= n^2(2n+1)^2$$

$$2^3 + 4^3 + 6^3 + \ldots + (2n)^3 = 2^3(1^3 + 2^3 + 3^3 + \ldots + n^3)$$

$$= 8\sum_{r=1}^{n} r^3$$

$$= 2n^2(n+1)^2$$

$$\therefore \quad 1^3 + 3^3 + 5^3 + \ldots + (2n-1)^3$$
$$= n^2(2n+1)^2 - 2n^2(n+1)^2$$
$$= n^2\{(2n+1)^2 - 2(n+1)^2\} = n^2(2n^2 - 1)$$

In the next example we use the *method of differences* to find the sum of the given series. We express each term of the series as a difference, in such a way that most terms of the series cancel each other out. A similar process was used earlier in the section to show that $\sum_{r=1}^{n} \{(r+1)^3 - r^3\} = (n+1)^3 - 1^3$.

Example 4 Show that $\dfrac{1}{r} - \dfrac{1}{r+1} \equiv \dfrac{1}{r(r+1)}$. Hence evaluate $\sum_{r=1}^{n} \dfrac{1}{r(r+1)}$.

$$\frac{1}{r} - \frac{1}{r+1} \equiv \frac{(r+1)-1}{r(r+1)} \equiv \frac{1}{r(r+1)}$$

$$\therefore \quad \sum_{r=1}^{n} \frac{1}{r(r+1)} = \sum_{r=1}^{n} \left(\frac{1}{r} - \frac{1}{r+1}\right)$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots$$

$$\ldots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

[Note that as $n \to \infty, \dfrac{1}{n+1} \to 0, \quad \therefore \quad 1 - \dfrac{1}{n+1} \to 1$. Thus the sum to infinity of this series is 1.]

We can extend the method of Example 4 to other series by using partial fractions.

Example 5 Find

$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{1}{1.2.3} + \frac{1}{2.3.4.} + \ldots + \frac{1}{n(n+1)(n+2)}.$$

Let $\dfrac{1}{r(r+1)(r+2)} \equiv \dfrac{A}{r} + \dfrac{B}{r+1} + \dfrac{C}{r+2}$

then $1 \equiv A(r+1)(r+2) + Br(r+2) + Cr(r+1)$

Putting $r = 0$: $1 = 2A \quad \therefore \quad A = \frac{1}{2}$

Putting $r = -1$: $1 = -B \quad \therefore \quad B = -1$

Putting $r = -2$: $1 = 2C \quad \therefore \quad C = \frac{1}{2}$

Hence $\dfrac{1}{r(r+1)(r+2)} \equiv \dfrac{1}{2r} - \dfrac{1}{r+1} + \dfrac{1}{2(r+2)}$

$$\therefore \sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{1}{2.1} - \frac{1}{2} + \frac{1}{2.3}$$
$$+ \frac{1}{2.2} - \frac{1}{3} + \frac{1}{2.4}$$
$$+ \frac{1}{2.3} - \frac{1}{4} + \frac{1}{2.5}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+ \frac{1}{2(n-1)} - \frac{1}{n} + \frac{1}{2(n+1)}$$
$$+ \frac{1}{2n} - \frac{1}{n+1} + \frac{1}{2(n+2)}$$

$$\therefore \sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{1}{2.1} - \frac{1}{2} + \frac{1}{2.2} + \frac{1}{2(n+1)} - \frac{1}{n+1} + \frac{1}{2(n+2)}$$
$$= \frac{1}{4} - \frac{1}{2(n+1)} + \frac{1}{2(n+2)}$$

[Note that the results obtained in this section can also be proved by the method of mathematical induction, as shown in §1.4.]

Exercise 9.1

1. Use standard results to find the sums of the following series.

(a) $\sum_{r=1}^{n} (2r+1)$ (b) $\sum_{r=1}^{n} (r-1)(r+1)$ (c) $\sum_{r=1}^{n} 2^r$

(d) $\sum_{r=1}^{n} r^2(r+1)$ (e) $\sum_{r=1}^{n} 3^{1-r}$ (f) $\sum_{r=1}^{n} (r+4)^3$

2. Use partial fractions to find the sums of the following series.

(a) $\sum_{r=1}^{n} \dfrac{1}{(r+1)(r+2)}$ (b) $\sum_{r=1}^{n} \dfrac{1}{r(r+2)}$ (c) $\sum_{r=1}^{n} \dfrac{1}{(r+1)(r+2)(r+3)}$

3. Find the sums of the following series (i) to $2n$ terms, (ii) to $(2n + 1)$ terms.
(a) $1 - 1 + 1 - 1 + \ldots$ (b) $1 - 2 + 3 - 4 + \ldots$

4. Show that $r(r + 1)(r + 2) - (r - 1)r(r + 1) \equiv 3r(r + 1)$.

Hence find the sum of the series $\sum_{r=1}^{n} r(r + 1)$.

5. Use the method of question 4 to show that

$$\sum_{r=1}^{n} r(r + 1)(r + 2) = \tfrac{1}{4}n(n + 1)(n + 2)(n + 3).$$

6. Show that $(r + 1)^4 - r^4 \equiv 4r^3 + 6r^2 + 4r + 1$. Use this identity and the standard formulae for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to prove that $\sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n + 1)^2$.

7. Use the method of question 6 to find $\sum_{r=1}^{n} r^4$.

8. Use the method of differences to find the sum to n terms of the series $1(1!) + 2(2!) + 3(3!) + \ldots$.

[Hint: express each term in the form $p! - q!$.]

9. Find the sum of (a) the integers from $2n$ to $3n$ inclusive, (b) the squares of the integers from $2n$ to $3n$ inclusive.

10. Find the sum of (a) the first n odd positive integers, (b) the squares of the first n odd positive integers.

11. Find the sums of the following series
(a) $21^3 + 22^3 + 23^3 + \ldots + 35^3$,
(b) $1^3 + 3^3 + 5^3 + \ldots + 25^3$,
(c) $(n + 1)^3 + (n + 2)^3 + (n + 3)^3 + \ldots + (2n)^3$,
(d) $1^3 - 2^3 + 3^3 - 4^3 + \ldots + (2n - 1)^3 - (2n)^3$.

In questions 12 to 19 write down an expression for the rth term of the given series and hence find the sum of the first n terms of the series.

12. $1.4 + 2.5 + 3.6 + \ldots$

13. $2^2 + 5^2 + 8^2 + 11^2 + \ldots$

14. $1.2 + 3.4 + 5.6 + \ldots$

15. $1.2.5 + 2.3.6 + 3.4.7 + \ldots$

16. $\dfrac{1}{1.4} + \dfrac{1}{2.5} + \dfrac{1}{3.6} + \ldots$

17. $\dfrac{1}{1.3} + \dfrac{1}{3.5} + \dfrac{1}{5.7} + \ldots$

18. $\dfrac{1}{1.3.5} + \dfrac{1}{3.5.7} + \dfrac{1}{5.7.9} + \ldots$

19. $\dfrac{1}{2.3.4} + \dfrac{2}{3.4.5} + \dfrac{3}{4.5.6} + \ldots$

20. Find the sum of the series $1 + x + x^2 + \ldots + x^n$, where $x \neq 1$. By differentiating your result, find the sum of the series $1 + 2x + 3x^2 + \ldots + nx^{n-1}$ for $x \neq 1$.

21. Find the sum $S = x + 2x^2 + 3x^3 + \ldots + nx^n$, where $x \neq 1$, by considering the expression $S - xS$.

22. Find the sum $S = 1 - 3a + 5a^2 - \ldots + (2n + 1)(-a)^n$, where $a \neq -1$, by considering the expression $(1 + a)S$.

9.2 Maclaurin's theorem

Under certain conditions a function $f(x)$ can be expressed as an infinite series of ascending powers of x. Assuming that such a series exists, we may write

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots + a_rx^r + \ldots$$

Differentiating with respect to x, we have

$$\begin{aligned}
f'(x) &= a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots + ra_rx^{r-1} + \ldots \\
f''(x) &= 2a_2 + 3.2a_3x + 4.3a_4x^2 + \ldots + r(r-1)a_rx^{r-2} + \ldots \\
f'''(x) &= 3.2a_3 + 4.3.2a_4x + \ldots + r(r-1)(r-2)a_rx^{r-3} + \ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
f^{(r)}(x) &= r(r-1)(r-2)\ldots3.2a_r + \ldots
\end{aligned}$$

Substituting $x = 0$, we find that

$$\begin{aligned}
f(0) &= a_0 & \therefore\ a_0 &= f(0) \\
f'(0) &= a_1 & \therefore\ a_1 &= f'(0) \\
f''(0) &= 2a_2 & \therefore\ a_2 &= \frac{f''(0)}{2!} \\
f'''(0) &= 3.2a_3 & \therefore\ a_3 &= \frac{f'''(0)}{3!}
\end{aligned}$$

and $f^{(r)}(0) = r(r-1)(r-2)\ldots3.2a_r \quad \therefore\ a_r = \dfrac{f^{(r)}(0)}{r!}$

Thus $$\boxed{f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^{(r)}(0)}{r!}x^r + \ldots}$$

This result is called *Maclaurin's theorem*[†] and the series obtained is known as the *Maclaurin series* for $f(x)$.

It is possible to find a Maclaurin series for any function $f(x)$ whose derivatives $f'(0), f''(0), f'''(0), \ldots$ can be determined. The series obtained may converge to the

[†]*Maclaurin, Colin* (1698–1746) Scottish mathematician. He became professor of Mathematics at Aberdeen at the age of 19. His most important work was his *Treatise on fluxions* (1742).

sum $f(x)$ for all values of x. However, for many functions, Maclaurin's theorem holds only within a restricted range of values of x.

We now use Maclaurin's theorem to obtain various important expansions.

Let $f(x) = (1 + x)^n$, where n is a real number,

then
$$f'(x) = n(1 + x)^{n-1},$$
$$f''(x) = n(n - 1)(1 + x)^{n-2},$$
$$f'''(x) = n(n - 1)(n - 2)(1 + x)^{n-3},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$f^{(r)}(x) = n(n - 1)(n - 2)\ldots(n - r + 1)(1 + x)^{n-r},$$

$\therefore$ $f(0) = 1$, $f'(0) = n$, $f''(0) = n(n - 1)$, $f'''(0) = n(n - 1)(n - 2)$

and
$$f^{(r)}(0) = n(n - 1)(n - 2)\ldots(n - r + 1).$$

Thus
$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r + \ldots$$

This result was introduced in Book 1 as an extension of the binomial theorem. When n is a positive integer the expansion is valid for all values of x. It can be shown that, for other values of n, the result holds only when $|x| < 1$, i.e. when $-1 < x < 1$.

If
$$f(x) = e^x, \quad f'(x) = e^x, \quad f''(x) = e^x, \quad \ldots\ldots, \quad f^{(r)}(x) = e^x,$$
then
$$f(0) = 1, \quad f'(0) = 1, \quad f''(0) = 1, \quad \ldots\ldots, \quad f^{(r)}(0) = 1$$

Thus
$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^r}{r!} + \ldots$$

This expansion is valid for all values of x.

The function $\ln x$ is not defined for $x = 0$ and therefore has no Maclaurin series. However, an expansion of $\ln(1 + x)$ can be obtained.

If
$$f(x) = \ln(1 + x),$$
$$f'(x) = (1 + x)^{-1},$$
$$f''(x) = -(1 + x)^{-2},$$
$$f'''(x) = 2(1 + x)^{-3},$$
$$f^{(4)}(x) = -3.2(1 + x)^{-4},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$f^{(r)}(x) = (-1)^{r+1}(r - 1)!(1 + x)^{-r},$$

then
$$f(0) = \ln 1 = 0, \quad f'(0) = 1, \quad f''(0) = -1, \quad f'''(0) = 2, \quad f^{(4)}(0) = -3!, \quad \ldots\ldots, \quad f^{(r)}(0) = (-1)^{r+1}(r - 1)!$$

Thus
$$\ln(1 + x) = 0 + x - \frac{x^2}{2!} + \frac{2x^3}{3!} - \frac{3!x^4}{4!} + \ldots + \frac{(-1)^{r+1}(r-1)!x^r}{r!} + \ldots$$

i.e. $$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^{r+1}\frac{x^r}{r} + \ldots$$

This result is valid for $-1 < x \leqslant 1$.

If

$$\begin{aligned} f(x) &= \sin x, \\ f'(x) &= \cos x, \\ f''(x) &= -\sin x, \\ f'''(x) &= -\cos x, \\ f^{(4)}(x) &= \sin x, \\ f^{(5)}(x) &= \cos x, \\ &\ldots\ldots\ldots\ldots \end{aligned} \quad \text{then} \quad \begin{aligned} f(0) &= 0 \\ f'(0) &= 1 \\ f''(0) &= 0 \\ f'''(0) &= -1 \\ f^{(4)}(0) &= 0 \\ f^{(5)}(0) &= 1 \\ &\ldots\ldots\ldots\ldots \end{aligned}$$

Since we can now see a pattern emerging, we may write

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

A similar result can be obtained for $\cos x$:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

The expansions of $\sin x$ and $\cos x$ are valid for all values of x (in radians).

Exercise 9.2

In questions 1 to 9 use Maclaurin's theorem to expand the given function in ascending powers of x as far as the term in x^4.

1. $\cos x$
2. $\tan x$
3. $\sec x$
4. $\ln(\cos x)$
5. $\sin^2 x$
6. $\ln(1 + \sin x)$
7. $\tan^{-1} x$
8. $e^{\sin x}$
9. $\sin^{-1} x$

10. If $y = e^x \sin x$, show that $\dfrac{d^2y}{dx^2} = 2\left(\dfrac{dy}{dx} - y\right)$. By further differentiation of this result, find the Maclaurin expansion of the function $e^x \sin x$ as far as the term in x^6.

9.3 The binomial series

The *binomial series* is the expansion of $(1 + x)^n$ in ascending powers of x, as follows:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$
$$\ldots + \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r + \ldots$$

where n is a real number and $|x| < 1$.

Simple applications of this expansion were discussed in Book 1, §16.7. We now consider further uses of the series.

First we recall that the expression $(a+x)^n$ cannot be expanded directly using the binomial series. Provided that $|x| < |a|$, we write

$$(a+x)^n = a^n\left(1+\frac{x}{a}\right)^n = a^n\left\{1 + n\left(\frac{x}{a}\right) + \frac{n(n-1)}{2!}\left(\frac{x}{a}\right)^2 + \ldots\right\}$$

Example 1 Expand $(8+3x)^{1/3}$ in ascending powers of x as far as the term in x^3, stating the values of x for which the expansion is valid. Hence obtain an approximate value for $\sqrt[3]{8{\cdot}72}$.

$$(8+3x)^{1/3} = 8^{1/3}\left(1+\frac{3}{8}x\right)^{1/3}$$
$$= 2\left\{1 + \frac{1}{3}\left(\frac{3}{8}x\right) + \frac{1}{2!}\times\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{3}{8}x\right)^2 + \frac{1}{3!}\times\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)\left(\frac{3}{8}x\right)^3 - \ldots\right\}$$
$$= 2\left\{1 + \frac{1}{8}x - \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{2}{3}\cdot\frac{3^2}{8^2}x^2 + \frac{1}{6}\cdot\frac{1}{3}\cdot\frac{2}{3}\cdot\frac{5}{3}\cdot\frac{3^3}{8^3}x^3 - \ldots\right\}$$
$$= 2\left\{1 + \frac{1}{8}x - \frac{1}{8^2}x^2 + \frac{5}{3.8^3}x^3 - \ldots\right\}$$
$$= 2 + \frac{1}{4}x - \frac{1}{32}x^2 + \frac{5}{768}x^3 - \ldots$$

The expansion is valid for $|\frac{3}{8}x| < 1$ i.e. for $|x| < \frac{8}{3}$.
Substituting $x = 0{\cdot}24$, we have

$$(8{\cdot}72)^{1/3} = 2 + \tfrac{1}{4}(0{\cdot}24) - \tfrac{1}{32}(0{\cdot}24)^2 + \tfrac{5}{768}(0{\cdot}24)^3 - \ldots$$
$$= 2 + 0{\cdot}06 - 0{\cdot}0018 + 0{\cdot}00009 - \ldots$$
$$\approx 2{\cdot}05829.$$

Hence the approximate value of $\sqrt[3]{8{\cdot}72}$ is 2·05829.

[To decide whether it is reasonable to give this result to 5 decimal places, we may consider the sum, S_n, of the first n terms in the expansion, for $n = 1, 2, 3, \ldots$.

$$S_1 = 2$$
$$S_2 = 2{\cdot}06$$
$$S_3 = 2{\cdot}0582$$
$$S_4 = 2{\cdot}05829$$
$$S_5 = 2{\cdot}0582846$$

Each term in the expansion of $(8{\cdot}72)^{1/3}$ is smaller than the previous term but opposite in sign. It follows that $\sqrt[3]{8{\cdot}72}$ lies between S_4 and S_5. Thus the difference between 2·05829 and the true value of $\sqrt[3]{8{\cdot}72}$ must be less than 1×10^{-5}.]

Example 2 Given that the first three terms in the expansion in ascending powers of x of $(1 - 8x)^{1/4}$ are the same as the first three terms in the expansion of $(1 + ax)/(1 + bx)$, find the values of a and b. Hence find an approximation to $(0{\cdot}6)^{1/4}$ in the form p/q, where p and q are integers.

$$(1 - 8x)^{1/4} = 1 + \frac{1}{4}(-8x) + \frac{1}{2!} \times \frac{1}{4}\left(\frac{1}{4} - 1\right)(-8x)^2 + \ldots$$

$$= 1 - 2x - \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{4}\cdot 8^2x^2 - \ldots$$

$$= 1 - 2x - 6x^2 - \ldots$$

$$\frac{1 + ax}{1 + bx} = (1 + ax)(1 + bx)^{-1}$$

$$= (1 + ax)\left\{1 + (-1)bx + \frac{1}{2!}(-1)(-2)b^2x^2 + \ldots\right\}$$

$$= (1 + ax)(1 - bx + b^2x^2 + \ldots)$$
$$= 1 + ax - bx - abx^2 + b^2x^2 + \ldots$$
$$= 1 + (a - b)x + (b^2 - ab)x^2 + \ldots$$

Since the first three terms of the expansions are the same.

$$a - b = -2, \quad b^2 - ab = -6$$
i.e. $$b - a = 2, \quad b(b - a) = -6$$

Hence $b = -3$ and $a = -5$.

Thus
$$(1 - 8x)^{1/4} \approx \frac{1 - 5x}{1 - 3x}.$$

Substituting $x = 0{\cdot}05$, we have

$$(1 - 0{\cdot}4)^{1/4} \approx \frac{1 - 0{\cdot}25}{1 - 0{\cdot}15} = \frac{0{\cdot}75}{0{\cdot}85}.$$

Hence $(0{\cdot}6)^{1/4}$ is approximately equal to 15/17.

Example 3 Assuming that x is so small that terms in x^3 and higher powers may be neglected, find a quadratic approximation to $\sqrt{\left(\frac{1 - x}{1 + 2x}\right)}$.

$$\sqrt{\left(\frac{1-x}{1+2x}\right)} = (1-x)^{1/2}(1+2x)^{-1/2}$$

$$(1-x)^{1/2} = 1 + \frac{1}{2}(-x) + \frac{1}{2!} \times \frac{1}{2}\left(-\frac{1}{2}\right)(-x)^2 + \ldots$$

$$= 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \ldots$$

$$(1+2x)^{-1/2} = 1 + \left(-\frac{1}{2}\right)(2x) + \frac{1}{2!} \times \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(2x)^2 + \ldots$$

$$= 1 - x + \frac{3}{2}x^2 - \ldots$$

$\therefore$ neglecting terms in x^3 and higher powers,

$$\sqrt{\left(\frac{1-x}{1+2x}\right)} \approx \left(1 - \frac{1}{2}x - \frac{1}{8}x^2\right)\left(1 - x + \frac{3}{2}x^2\right)$$

$$\approx 1 - x + \frac{3}{2}x^2 - \frac{1}{2}x + \frac{1}{2}x^2 - \frac{1}{8}x^2$$

Hence $\sqrt{\left(\frac{1-x}{1+2x}\right)} \approx 1 - \frac{3}{2}x + \frac{15}{8}x^2.$

Example 4 Expand $\frac{1}{1+x+2x^2}$ in ascending powers of x up to and including the term in x^3.

$$\frac{1}{1+y} = (1+y)^{-1}$$

$$= 1 + (-1)y + \frac{(-1)(-2)}{2!}y^2 + \frac{(-1)(-2)(-3)}{3!}y^3 + \ldots$$

$$= 1 - y + y^2 - y^3 + \ldots$$

Substituting $y = x + 2x^2$, we obtain

$$\frac{1}{1+x+2x^2} = 1 - (x+2x^2) + (x+2x^2)^2 - (x+2x^2)^3 + \ldots$$

$$= 1 - x - 2x^2 + x^2 + 4x^3 - x^3 + \ldots$$
$$= 1 - x - x^2 + 3x^3 + \ldots$$

Suppose that we now wish to expand $\frac{1}{(1-x)(1+2x)}$ in ascending powers of x. We could use the method of Example 4:

$$\frac{1}{(1-x)(1+2x)} = (1+x-2x^2)^{-1} = 1 - (x-2x^2) + (x-2x^2)^2 - \ldots$$

or we could write:

$$\frac{1}{(1-x)(1+2x)} = (1-x)^{-1}(1+2x)^{-1}$$
$$= (1+x+x^2+\ldots)(1-2x+4x^2-\ldots)$$

We may quickly obtain the first few terms of the expansion using either of these methods. However, if a general formula for the term in x^r is required, we must adopt a different approach. This further method involves the use of partial fractions. (See §2.6)

Example 5 Expand $\dfrac{1}{(1-x)(1+2x)}$ in ascending powers of x, giving the first three terms and the term in x^r. State the values of x for which the expansion is valid.

$$(1-x)^{-1} = 1 + (-1)(-x) + \frac{(-1)(-2)}{2!}(-x)^2 + \ldots + \frac{(-1)(-2)\ldots(-r)}{r!}(-x)^r + \ldots$$

Thus $(1-x)^{-1} = 1 + x + x^2 + \ldots + x^r + \ldots$

Similarly $(1+2x)^{-1} = 1 - 2x + 4x^2 + \ldots + (-2)^r x^r + \ldots$

Let $\dfrac{1}{(1-x)(1+2x)} \equiv \dfrac{A}{1-x} + \dfrac{B}{1+2x}$

then $1 \equiv A(1+2x) + B(1-x)$

Putting $x = 1$: $1 = 3A \quad \therefore A = \frac{1}{3}$

Putting $x = -\frac{1}{2}$: $1 = \frac{3}{2}B \quad \therefore B = \frac{2}{3}$

$$\therefore \frac{1}{(1-x)(1+2x)} = \tfrac{1}{3}(1-x)^{-1} + \tfrac{2}{3}(1+2x)^{-1}$$

$$= \tfrac{1}{3}(1 + x + x^2 + \ldots + x^r + \ldots) + \tfrac{2}{3}(1 - 2x + 4x^2 + \ldots + (-2)^r x^r + \ldots)$$
$$= 1 - x + 3x^2 + \ldots + \{\tfrac{1}{3} + \tfrac{2}{3}(-2)^r\}x^r + \ldots$$
$$= 1 - x + 3x^2 + \ldots + \tfrac{1}{3}\{1 - (-2)^{r+1}\}x^r + \ldots$$

The expansion of $(1-x)^{-1}$ is valid for $|x| < 1$ and the expansion of $(1+2x)^{-1}$ is valid for $|2x| < 1$, i.e. $|x| < \frac{1}{2}$.

Thus the expansion of $\dfrac{1}{(1-x)(1+2x)}$ is valid when $|x| < \frac{1}{2}$.

Exercise 9.3

In questions 1 to 9 expand the given function in ascending powers of x up to and including the term in x^3.

1. $(1+x)^{-3}$ 2. $(1+2x)^{1/2}$ 3. $(4+x)^{-3/2}$

4. $\dfrac{1+2x}{1-2x}$ 5. $\dfrac{1-x}{(2+x)^2}$ 6. $\dfrac{1-2x}{\sqrt{(1-4x)}}$

7. $\dfrac{1}{1+x+x^2}$ 8. $\dfrac{1}{\sqrt{(1+x-2x^2)}}$ 9. $\sqrt[3]{(1+x+3x^2)}$

In questions 10 to 15 assume that x is so small that terms in x^3 and higher powers of x may be neglected. Hence find a quadratic approximation to the given function, stating the values of x for which your answer is valid.

10. $\sqrt{\left(\dfrac{1-x}{1+x}\right)}$ 11. $\sqrt[3]{\left(\dfrac{8+x}{1-3x}\right)}$ 12. $\dfrac{(1+4x)^{1/4}}{(1+5x)^{1/5}}$

13. $\dfrac{1}{(1+x)(3-x)}$ 14. $\dfrac{1}{(1-x)(1+2x)^2}$ 15. $\dfrac{1}{(1-2x)\sqrt{(1-x)}}$

In questions 16 to 21 use partial fractions to find the first five non-zero terms in the expansion of the given function in ascending powers of x. State the values of x for which the expansion is valid.

16. $\dfrac{3-5x}{(1-3x)(1+x)}$ 17. $\dfrac{4x}{(1-x)(3+x)}$ 18. $\dfrac{1+x}{(1+x^2)(1-x)}$

19. $\dfrac{4-x}{(1-x)^2(1+2x)}$ 20. $\dfrac{2}{(1+x)(1+x^2)}$ 21. $\dfrac{8(2x-1)}{(x-2)^2(x^2+2)}$

In questions 22 to 24 expand the given function in ascending powers of x, giving the first three non-zero terms and the term in x^r.

22. $\dfrac{3}{(1+x)(1-2x)}$ 23. $\dfrac{4x}{(1-x)(1+3x)}$ 24. $\dfrac{4+3x}{(2-x)(1+2x)}$

25. Expand $(1+2x)^{1/4}$ in ascending powers of x as far as the term in x^3, stating the values of x for which the expansion is valid. Hence obtain approximate values for (a) $\sqrt[4]{1{\cdot}4}$, (b) $\sqrt[4]{1{\cdot}08}$. Obtain the values of $\sqrt[4]{1{\cdot}4}$ and $\sqrt[4]{1{\cdot}08}$ by calculator, (using the function $x^{1/y}$ if available). Explain why the error in answer (a) is greater than the error in answer (b).

26. Expand $(a+2x)\sqrt{(1-4x)} + \dfrac{b}{(1-x)^2}$ in ascending powers of x as far as the term in x^3. Given that the coefficients of x and x^2 are zero, find the first term in the expansion and the coefficient of x^3.

27. Given that the first three terms in the expansion in ascending powers of x of $(1+x+x^2)^n$ are the same as the first three terms in the expansion of $\left(\dfrac{1+ax}{1-3ax}\right)^3$,

find the non-zero values of n and a. Show that the coefficients of x^3 in the two expansions differ by 7·5.

28. If x is so small that x^3 and higher powers of x may be neglected, find the values of a and b such that

$$\sqrt{(1+4x)} \approx \frac{1+ax}{1+bx}.$$

By letting $x = 0{\cdot}04$ find an approximation to $\sqrt{29}$ in the form p/q, where p and q are integers.

9.4 Exponential and logarithmic series

As shown in §9.2 the expansion in ascending powers of x of the exponential function e^x is as follows:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^r}{r!} + \ldots \text{ for all real values of } x.$$

Example 1 Expand e^{2x} in ascending powers of x giving the first three terms and the term in x^r.

$$e^{2x} = 1 + (2x) + \frac{(2x)^2}{2!} + \ldots + \frac{(2x)^r}{r!} + \ldots$$

$$= 1 + 2x + 2x^2 + \ldots + \frac{2^r}{r!}x^r + \ldots$$

Example 2 Find the first three terms in the expansion of e^{x+x^2} in ascending powers of x.

$$e^{x+x^2} = 1 + (x + x^2) + \frac{(x+x^2)^2}{2!} + \ldots$$

$$= 1 + x + x^2 + \frac{1}{2}x^2 + \text{higher powers of } x$$

$$= 1 + x + \frac{3}{2}x^2 + \ldots$$

Example 3 Expand e^{2-x} in ascending powers of x as far as the term in x^3.

$$e^{2-x} = e^2 \times e^{-x} = e^2\left\{1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \ldots\right\}$$

$$= e^2\left\{1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \ldots\right\}$$

Example 4 Find the coefficient of x^r in the expansion of $(x^2 + 2x + 3)e^x$ in ascending powers of x.

Considering the terms $\ldots + \dfrac{x^{r-2}}{(r-2)!} + \dfrac{x^{r-1}}{(r-1)!} + \dfrac{x^r}{r!} + \ldots$ in the expansion of e^x, we find that the terms in x^r in the expansion of $(x^2 + 2x + 3)e^x$ must be

$$\ldots + x^2 \times \frac{x^{r-2}}{(r-2)!} + 2x \times \frac{x^{r-1}}{(r-1)!} + 3 \times \frac{x^r}{r!} + \ldots$$

Hence the coefficient of x^r in the expansion

$$= \frac{1}{(r-2)!} + \frac{2}{(r-1)!} + \frac{3}{r!}$$

$$= \frac{r(r-1)}{r!} + \frac{2r}{r!} + \frac{3}{r!}$$

$$= \frac{r^2 - r + 2r + 3}{r!}$$

$$= \frac{r^2 + r + 3}{r!}$$

In §9.2 we also obtained the logarithmic series:

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{r+1}\frac{x^r}{r} + \ldots \text{ where } -1 < x \leqslant 1.$$

Replacing x by $-x$ in this expansion gives:

$$\ln(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots - \frac{x^r}{r} - \ldots \text{where } -1 \leqslant x < 1.$$

When these series are used to find numerical values of logarithms, large numbers of terms are needed to produce accurate results. A much better expansion for this purpose is given by:

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1 + x) - \ln(1 - x)$$

$$= \left(x - \frac{x^2}{2} + \frac{x^2}{3} - \ldots\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots\right)$$

i.e.

$$\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right)$$

We demonstrate the advantages of using this series rather than the expansion of $\ln(1 + x)$ by calculating a sequence of approximate values for $\ln 1{\cdot}5$.

	$\ln(1+x),\ x=\frac{1}{2}$	$\ln\left(\frac{1+x}{1-x}\right),\ x=\frac{1}{5}$
1 term	0·5	0·4
2 terms	0·375	0·40533333
3 terms	0·41666667	0·40546133
4 terms	0·40104167	0·40546499
5 terms	0·40729167	0·40546510
6 terms	0·40468750	0·40546511

Using the figures on the left we can give $\ln 1{\cdot}5$ to 1 decimal place, but not to 2. However, the figures on the right give a result correct to 8 decimal places.

We now use the laws of logarithms to obtain some further expansions.

Example 5 Expand $\ln(2 + 3x)$ in ascending powers of x as far as the term in x^3. State the values of x for which the expansion is valid.

$$\ln(2+3x) = \ln 2\left(1+\frac{3}{2}x\right)$$

$$= \ln 2 + \ln\left(1+\frac{3}{2}x\right)$$

$$= \ln 2 + \left(\frac{3}{2}x\right) - \frac{1}{2}\left(\frac{3}{2}x\right)^2 + \frac{1}{3}\left(\frac{3}{2}x\right)^3 - \ldots$$

$$= \ln 2 + \frac{3}{2}x - \frac{9}{8}x^2 + \frac{9}{8}x^3 - \ldots$$

The expansion is valid for $-1 < \frac{3}{2}x \leqslant 1$ i.e. $-\frac{2}{3} < x \leqslant \frac{2}{3}$.

Example 6 Expand $\ln\dfrac{1+x}{\sqrt{(1-2x)}}$ in ascending powers of x as far as the term in x^3, stating the values of x for which the expansion is valid.

$$\ln\frac{1+x}{\sqrt{(1-2x)}} = \ln(1+x) - \frac{1}{2}\ln(1-2x)$$

$$= \left\{x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \ldots\right\}$$

$$-\frac{1}{2}\left\{(-2x) - \frac{1}{2}(-2x)^2 + \frac{1}{3}(-2x)^3 - \ldots\right\}$$

$$= \left\{x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \ldots\right\} + \left\{x + x^2 + \frac{4}{3}x^3 + \ldots\right\}$$

$$= 2x + \frac{1}{2}x^2 + \frac{5}{3}x^3 + \ldots$$

The expansion of $\ln(1+x)$ is valid for $-1 < x \leqslant 1$ and the expansion of $\ln(1-2x)$ is valid for $-1 < -2x \leqslant 1$, i.e. $-\frac{1}{2} \leqslant x < \frac{1}{2}$.

Hence the expansion of $\ln\dfrac{1+x}{\sqrt{(1-2x)}}$ is valid for $-\frac{1}{2} \leqslant x < \frac{1}{2}$.

Example 7 Assuming that x is so small that x^3 and higher powers may be neglected, obtain a quadratic approximation to the function

$$\frac{1+e^x}{1+\ln(1+x)}.$$

$$1+e^x = 1+\left(1+x+\frac{x^2}{2!}+\ldots\right) \approx 2+x+\frac{1}{2}x^2$$

$$1+\ln(1+x) = 1+\left(x-\frac{x^2}{2}+\ldots\right) \approx 1+x-\frac{1}{2}x^2$$

Using the binomial expansion,

$$(1+y)^{-1} = 1+(-1)y+\frac{(-1)(-2)}{2!}y^2+\ldots = 1-y+y^2-\ldots$$

Substituting $y = x-\frac{1}{2}x^2$,

$$\begin{aligned}\frac{1}{1+\ln(1+x)} &\approx \left\{1+\left(x-\frac{1}{2}x^2\right)\right\}^{-1}\\ &= 1-\left(x-\frac{1}{2}x^2\right)+\left(x-\frac{1}{2}x^2\right)^2-\ldots\\ &= 1-x+\frac{1}{2}x^2+x^2+\ldots\\ &\approx 1-x+\frac{3}{2}x^2\end{aligned}$$

$$\therefore\quad \frac{1+e^x}{1+\ln(1+x)} \approx \left(2+x+\frac{1}{2}x^2\right)\left(1-x+\frac{3}{2}x^2\right)$$

$$\approx 2-2x+3x^2+x-x^2+\frac{1}{2}x^2$$

$$\text{i.e.}\quad \frac{1+e^x}{1+\ln(1+x)} \approx 2-x+\frac{5}{2}x^2.$$

Exercise 9.4

1. Find the first three non-zero terms and the general terms in the expansions in ascending powers of x of:

(a) e^{-x} (b) e^{3x} (c) e^{x^2} (d) e^{1+x}

2. Find the first four terms in the expansions of:
(a) $(1 + 2x)e^{2x}$ (b) $(1 - x)^2e^{-x}$ (c) $e^{(x^2-x)}$

3. Use the exponential series to calculate the values of e and $1/e$ correct to 4 decimal places.

4. Find the first three non-zero terms and the general terms in the expansions of the following functions, stating the values of x for which the expansions are valid:
(a) $\ln(1 + 3x)$ (b) $\ln(1 - \frac{1}{2}x)$ (c) $\ln(3 + x)$ (d) $\ln(e - x)$

5. Find the first four terms in the expansions of:

(a) $\ln\left(\dfrac{1+x}{1+x^2}\right)$ (b) $\ln(1 + x - 2x^2)$ (c) $\ln\dfrac{(1-x)^2}{\sqrt{(1+4x)}}$

6. Use the expansion of $\ln\left(\dfrac{1+x}{1-x}\right)$ to calculate the value of $\ln 3$ correct to 4 decimal places.

7. Find the coefficient of x^r in the expansions of each of the following functions in ascending powers of x:
(a) $(1 - x)e^x$ (b) $(1 + x - 3x^2)e^{2x}$ (c) $(1 + x)^2e^{x/2}$

8. Given that x is so small that terms in x^3 and higher powers of x may be neglected, find quadratic approximations to the following functions:
(a) $e^x \ln(1 + x)$ (b) $(1 + e^{-x})^2$ (c) $(1 + x)^8e^{-x/2}$

(d) $\dfrac{e^{3x} - e^x}{e^{2x}}$ (e) $\dfrac{1 - \ln(1 - x)}{1 - x}$ (f) $\dfrac{\ln(1 - 2x)}{1 - e^x}$

9. Show that, if $n > 1$, $\ln\left(\dfrac{n+1}{n-1}\right) = 2\left(\dfrac{1}{n} + \dfrac{1}{3n^3} + \dfrac{1}{5n^5} + \ldots\right)$. Hence calculate the value of $\ln 2$ correct to 4 decimal places.

10. The expansion of $(1 - x)^n$ in ascending powers of x is the same as the expansion of e^{ax+x^2} as far as the term in x^2. Find the values of a and n.

11. The expansion of $(3x + ax^2)/(3 + bx)$ in ascending powers of x is the same as the expansion of $\ln(1 + x)$ as far as the term in x^3. Find the values of a and b. Hence find a rational approximation to $\ln 1{\cdot}2$.

12. (a) Use the expansion of e^x to show that if n is a positive integer, then $x^ne^{-x} \to 0$ as $x \to \infty$.
(b) By letting $x = e^y$, show that $\dfrac{1}{x}\ln x \to 0$ as $x \to \infty$.

13. Show that, if $n > 1$, $\ln\left(1+\frac{1}{n}\right)^n = 1 - \frac{1}{2n} + \frac{1}{3n^2} - \dots$

Use the expansion of e^x to deduce that

$$\left(1+\frac{1}{n}\right)^n = e\left(1 - \frac{1}{2n} + \frac{11}{24n^2} - \dots\right).$$

Deduce that, as $n \to \infty$, $\left(1+\frac{1}{n}\right)^n \to e$.

9.5 The expansions of sin x and cos x

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots, \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

for all real values of x.
The expansions of $\sin x$ and $\cos x$ are used in the same ways as the various series discussed in previous sections.

Example 1 Expand $e^{\sin x}$ in ascending powers of x, neglecting terms in x^4 and higher powers of x.

$$\sin x = x - \frac{x^3}{3!} + \dots = x - \frac{1}{6}x^3 + \dots$$

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \dots = 1 + y + \frac{1}{2}y^2 + \frac{1}{6}x^3 + \dots$$

Substituting $y = x - \frac{1}{6}x^3$,

$$e^{\sin x} = 1 + \left(x - \frac{1}{6}x^3\right) + \frac{1}{2}\left(x - \frac{1}{6}x^3\right)^2 + \frac{1}{6}\left(x - \frac{1}{6}x^3\right)^3 + \dots$$

$$= 1 + x - \frac{1}{6}x^3 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$$

$$= 1 + x + \frac{1}{2}x^2 + \dots$$

Example 2 Expand $\sec x$ in ascending powers of x, as far as the term in x^4.

$$\sec x = (\cos x)^{-1} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right)^{-1} = \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots\right)^{-1}$$

Using the binomial expansion, $(1+y)^{-1} = 1 - y + y^2 - \dots$

Thus, substituting $y = -\frac{x^2}{2} + \frac{x^4}{24}$,

$$\sec x = 1 - \left(-\frac{x^2}{2} + \frac{x^4}{24}\right) + \left(-\frac{x^2}{2} + \frac{x^4}{24}\right)^2 - \dots$$

$$= 1 + \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^4}{4} + \dots$$

$$= 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \dots$$

Exercise 9.5

1. Evaluate, correct to 4 decimal places,
(a) $\sin 0{\cdot}3$ (b) $\cos 0{\cdot}4$ (c) $\cos 2$ (d) $\sin 1$

2. Find the first three non-zero terms in the expansions of the following functions in ascending powers of x:
(a) $\sin 3x$ (b) $\cos 2x$ (c) $\ln(1 + \sin x)$
(d) $e^{\cos x}$ (e) $x \operatorname{cosec} x$ (f) $\tan x$

3. Find approximations to the following functions, given that powers of x higher than the fourth are negligible:
(a) $\sin x(x + \cos x)$ (b) $\sin\{\ln(1 + 2x)\}$
(c) $\cos(\sin 2x) - e^{-2x^2}$ (d) $e^{2x}\cos^2 x$

Exercise 9.6 (*miscellaneous*)

1. Find the sums of the following series
(a) $\sum_{r=2}^{n} r(r-1)$ (b) $\sum_{r=0}^{n} (2r+1)(2r+3)$
(c) $\sum_{r=n}^{2n} \frac{1}{r(r+1)}$ (d) $\sum_{r=2}^{n} \frac{1}{r^2 - 1}$

2. Find the sum to n terms of each of the following series
(a) $1.2^2 + 2.3^2 + 3.4^2 + \dots$ (b) $1.2.3 + 3.4.5 + 5.6.7 + \dots$

3. Find the sum of all possible products of the form rs^2, where r and s are distinct positive integers such that $1 \leqslant r, s \leqslant n$.

4. The sum of the first n terms of the series $1 + 2x^2 + 3x^4 + \dots$, where $x^2 \neq 1$, is S. Find an expression for S by considering $S - x^2 S$.

5. Using the remainder theorem, or otherwise, factorise $x^3 + 6x^2 + 11x + 6$. Express $\dfrac{4x+6}{x^3 + 6x^2 + 11x + 6}$ in partial fractions. Hence show that

$$\sum_{n=0}^{18} \frac{4n+6}{n^3 + 6n^2 + 11n + 6} = 2\frac{43}{140}. \qquad \text{(L)}$$

6. (a) Find the sum of the first n terms of the series whose rth term is $2^r + 2r - 1$.
(b) If x is so small that terms in x^n, $n \geqslant 3$, can be neglected and $\dfrac{3 + ax}{3 + bx} = (1 - x)^{1/3}$, find the values of a and b. Hence, without the use of tables, find an approximation in the form p/q, where p and q are integers, for $\sqrt[3]{0{\cdot}96}$. (AEB 1979)

7. Find the expansions of $\left(1 + \dfrac{1}{x^2}\right)^{1/2}$ and $\left(1 - \dfrac{1}{x^2}\right)^{1/2}$ in ascending powers of $1/x^2$ up to and including the term in $1/x^4$. Hence, or otherwise, show that, if the positive number x is very large compared with 1, then $(x^2 + 1)^{1/2} - (x^2 + 1)^{1/2} \approx \dfrac{1}{x}$. (C)

8. Expand $\sqrt{(4 - x)}$ as a series in ascending powers of x up to and including the terms in x^2. If terms in x^n, $n \geqslant 3$, can be neglected, find the quadratic approximation to $\sqrt{\left(\dfrac{4 - x}{1 - 2x}\right)}$. State the range of values of x for which this approximation is valid. (AEB 1980)

9. Expand the function $(2 - x)\sqrt{(1 + 2x + 2x^2)}$ in ascending powers of x as far as the term in x^3.

10. Use partial fractions to expand the following functions in ascending powers of x, giving the first three non-zero terms and the term in x^r. State the values of x for which the expansions are valid.

(a) $\dfrac{3x}{(1 + 2x)(1 - x)}$ (b) $\dfrac{x}{(x - 3)(x - 2)}$ (c) $\dfrac{1 - x - x^2}{(1 - 3x)(1 - 2x)(1 - x)}$

11. Expand the following functions in ascending powers of x, as far as the term in x^4. State the values of x for which the expansions are valid.

(a) $\dfrac{1 - 2x}{\sqrt{(1 + 2x^2)}}$ (b) $\dfrac{1 + 2x^2}{(1 - 2x)^2}$ (c) $\dfrac{3}{(1 + 2x^2)(1 - 2x)}$

12. (a) Expand $e^{x/2} \log_e(1 + x)$ in ascending powers of x as far as the term in x^4 and hence show that, for certain values of x to be stated, $e^{x/2} \log_e(1 + x) + e^{-x/2} \log_e(1 - x) = ax^4 + \ldots$ and give the value of a.
(b) Expand $(1 + x)^{-1/2}$ in ascending powers of x, giving the first four terms and the general term. Hence, without using tables or calculator, obtain the value of $1/\sqrt{101}$, correct to 6 decimal places, showing all working. (SU)

13. (a) Draw rough sketches of the following graphs, showing quite clearly their general shape, and where they cross any axis:
(i) $y = e^x$ (ii) $y = e^{-x}$ (iii) $y = \log_e x$ (iv) $y = \log_e(1 + x)$

(b) Expand $\log_e \dfrac{(1+2x)^2}{1-3x}$ in ascending powers of x, as far as the term in x^4, giving the range of values of x for which the expansion is valid. What is the nth term?

(c) Prove that the coefficient of x^n in the expansion of $e^{2x}(x^2+3x-7)$ is $\dfrac{2^{n-2}(n^2+5n-28)}{n!}$ (SU)

14. Obtain the series expansion of $\dfrac{e^{2x}}{1+x}$ in ascending powers of x up to and including the term in x^3. Using these terms find an approximation to the value of $\displaystyle\int_0^{1/4} \frac{e^{2x}dx}{1+x}$, giving your answer correct to three places of decimals. (O)

15. State the series expansion in ascending powers of x, including the general term, of $\log_e(1+x)$ for $|x|<1$. Show that

(i) $\log_e\left(\dfrac{1+x}{1-x}\right) = 2\left[x+\frac{1}{3}x^3+\frac{1}{5}x^5+\ldots\right]$,

(ii) $\log_e m = 2\left[\left(\dfrac{m-1}{m+1}\right)+\frac{1}{3}\left(\dfrac{m-1}{m+1}\right)^3+\frac{1}{5}\left(\dfrac{m-1}{m+1}\right)^5+\ldots\right]$.

Hence evaluate $\log_e 2$ to four decimal places, showing all your working.

If $\log_e\left(\dfrac{1+x}{1-x}\right)-\left(\dfrac{e^{2x}-1}{e^x}\right) = ax^3+bx^5$, when terms in x^n, $n \geqslant 6$, are neglected, find the values of a and b. (AEB 1978)

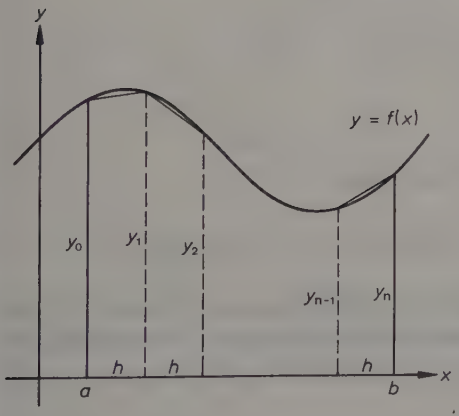
16. Given that $1-px+x^2 \equiv (1-\alpha x)(1-\beta x)$, obtain expressions for $\alpha^2+\beta^2$ and $\alpha^3+\beta^3$ in terms of p. Hence, or otherwise, find the first three terms in the expansion of $\log_e(1-px+x^2)$ in ascending powers of x. (JMB)

17. Find the values of positive constants a, b and c given that the expansions of $e^{-ax}\sin 2x$ and $\cos bx \ln(1+cx)$ in ascending powers of x are the same as far as the term in x^3.

10 Some numerical methods

10.1 Numerical integration

When a function $f(x)$ is difficult to integrate, it may be possible to estimate the value of the definite integral $\int_a^b f(x)\,dx$ by numerical methods. Such methods are also useful when, instead of a formula for $f(x)$, we have a set of values obtained experimentally.



The integral $\int_a^b f(x)\,dx$ is represented by the area under the curve $y = f(x)$ between $x = a$ and $x = b$. Suppose that the area is divided into n strips of width h, as shown in the diagram. The ordinates $y_0, y_1, y_2, \ldots, y_n$ represent the values of $f(x)$ at $x_0 = a, x_1 = a + h, x_2 = a + 2h, \ldots, x_n = b$.

An approximate value for the area is obtained by regarding each strip as a trapezium, so that

$$\int_a^b f(x)\,dx \approx \tfrac{1}{2}h(y_0 + y_1) + \tfrac{1}{2}h(y_1 + y_2) + \ldots + \tfrac{1}{2}h(y_{n-1} + y_n)$$

i.e.

$$\int_a^b f(x)\,dx \approx \tfrac{1}{2}h(y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n)$$

Thus we have the *trapezium rule* for n strips ($n + 1$ ordinates)

$$\int_a^b f(x)\,dx \approx \tfrac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\},$$

where $h = (b - a)/n$.

Example 1 Use the trapezium rule with 5 ordinates to find an approximate value for $\int_0^2 \frac{1}{1 + x^2}\,dx$.

Using 5 ordinates in the interval from $x = 0$ to $x = 2$ gives 4 strips of width $h = 0{\cdot}5$.

n	x_n	y_n (rounded to 5 d.p.)	
0	0	1	
1	0·5		0·8
2	1		0·5
3	1·5		0·307 69
4	2	0·2	
		1·2	1·607 69
		3·215 38	× 2
		4·415 38	3·215 38

By the trapezium rule,

$$\int_0^2 \frac{1}{1 + x^2}\,dx \approx \tfrac{1}{2}h\{(y_0 + y_4) + 2(y_1 + y_2 + y_3)\}$$
$$\approx \tfrac{1}{2} \times 0{\cdot}5 \times 4{\cdot}415\,38$$
$$\approx 1{\cdot}104$$

This integral can also be evaluated directly as follows:

$$\int_0^2 \frac{1}{1 + x^2}\,dx = \Big[\tan^{-1} x\Big]_0^2 = \tan^{-1} 2 = 1{\cdot}107\,15 \text{ (to 5 d.p.)}$$

Thus the error in the value obtained using the trapezium rule is less than 0·5%. The error may be reduced by increasing the number of ordinates used. For instance, we can obtain a better approximation by using 11 ordinates i.e. 10 intervals of width $h = 0{\cdot}2$.

n	x_n	y_n (rounded to 5 d.p.)	
0	0	1	
1	0·2		0·961 54
2	0·4		0·862 07
3	0·6		0·735 29
4	0·8		0·609 76
5	1		0·5
6	1·2		0·409 84
7	1·4		0·337 84

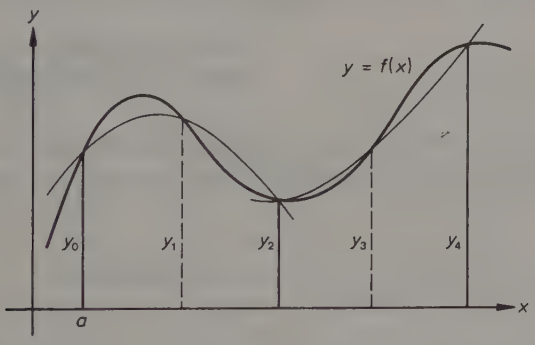
8	1·6		0·280 90
9	1·8		0·235 85
10	2	0·2	
		1·2	4·933 09
		9·866 18	× 2
		11·066 18	9·866 18

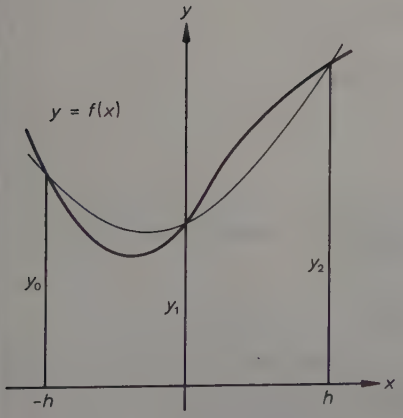
By the trapezium rule, $\int_0^2 \frac{1}{1+x^2}\,dx \approx \frac{1}{2}h\{(y_0 + y_{10}) + 2(y_1 + y_2 + \ldots + y_9)\}$

$$\approx \tfrac{1}{2} \times 0{\cdot}2 \times 11{\cdot}066\,18$$
$$\approx 1{\cdot}1066$$

Using 21 ordinates the value 1·1070 is obtained. However, rounding errors in the values of y_n will limit further improvements in accuracy.

Evaluating the integral $\int_a^b f(x)\,dx$ by the trapezium rule is equivalent to replacing the curve $y = f(x)$ by a series of straight lines joining the tops of the ordinates $y_0, y_1, y_2, \ldots$ A better estimate of the area under the curve is usually obtained using *Simpson's rule.*† In this case the curve $y = f(x)$ is regarded as a series of parabolic arcs joining the tops of the ordinates y_0, y_1 and y_2; y_2, y_3 and y_4;....





We consider first the integral $\int_{-h}^{h} f(x)\,dx$, letting y_0, y_1, y_2 be the ordinates at $x_0 = -h, x_1 = 0$ and $x_2 = h$. If $y = px^2 + qx + r$ is the parabola which joins the tops of the ordinates y_0, y_1, y_2, then

$$y_0 = ph^2 - qh + r,$$
$$y_1 = r,$$
$$y_2 = ph^2 + qh + r.$$

†*Simpson, Thomas* (1710–1761) English mathematician known as 'the oracle of Nuneaton'. The rule which bears his name was used by earlier mathematicians, but Simpson was credited with its discovery when he published it in 1743.

The area under this parabola between $x = -h$ and $x = h$

$$= \int_{-h}^{h} (px^2 + qx + r)\,dx = \left[\tfrac{1}{3}px^3 + \tfrac{1}{2}qx^2 + rx\right]_{-h}^{h}$$

$$= (\tfrac{1}{3}ph^3 + \tfrac{1}{2}qh^2 + rh) - (-\tfrac{1}{3}ph^3 + \tfrac{1}{2}qh^2 - rh)$$

$$= \tfrac{2}{3}ph^3 + 2rh$$

$$= \tfrac{1}{3}h(2ph^2 + 6r)$$

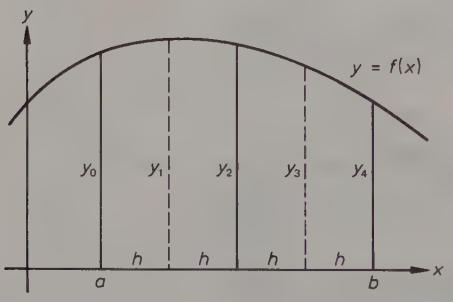
$$= \tfrac{1}{3}h(y_0 + 4y_1 + y_2)$$

Thus $$\int_{-h}^{h} f(x)\,dx \approx \tfrac{1}{3}h(y_0 + 4y_1 + y_2).$$

Since this result depends only on the values of the ordinates y_0, y_1, y_2 and the interval h between them, we may also write:

$$\int_a^b f(x)\,dx \approx \tfrac{1}{3}h(y_0 + 4y_1 + y_2)$$

where $h = \frac{1}{2}(b - a)$ and $y_0 = f(a)$, $y_1 = f(a + h)$, $y_2 = f(b)$.
This is Simpson's rule in its simplest form. The rule for 2 strips (3 ordinates) can be extended to any even number of strips (or any odd number of ordinates.)



In the case of 4 strips, as shown in the diagram, we apply the basic rule twice:

$$\int_a^b f(x)\,dx \approx \tfrac{1}{3}h(y_0 + 4y_1 + y_2) + \tfrac{1}{3}h(y_2 + 4y_3 + y_4)$$

Thus Simpson's rule for 4 strips (5 ordinates) is

$$\int_a^b f(x)\,dx \approx \tfrac{1}{3}h(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4)$$

More generally, Simpson's rule states that:

$$\int_a^b f(x)\,dx \approx \tfrac{1}{3}h(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \ldots + 4y_{n-1} + y_n)$$

i.e.
$$\int_a^b f(x)\,dx \approx \tfrac{1}{3}h\{(y_0 + y_n) + 4(y_1 + y_3 + \ldots) + 2(y_2 + y_4 + \ldots)\}$$
where n is even and $h = (b - a)/n$.

Example 2 Use Simpson's rule with 5 ordinates to find an approximate value for $\int_0^2 \frac{1}{1+x^2}\,dx$.

Using 5 ordinates in the interval from $x = 0$ to $x = 2$ gives 4 strips of width $h = 0{\cdot}5$.

n	x_n	y_n (rounded to 5 d.p.)		
0	0	1		
1	0·5		0·8	
2	1			0·5
3	1·5		0·307 69	
4	2	0·2		
		1·2	1·107 69	0·5
		4·430 76	× 4	× 2
		1·0	4·430 76	1·0
		6·630 76		

By Simpson's rule,

$$\int_0^2 \frac{1}{1+x^2}\,dx \approx \tfrac{1}{3}h\{(y_0 + y_4) + 4(y_1 + y_3) + 2y_2\}$$

$$\approx \tfrac{1}{3} \times 0{\cdot}5 \times 6{\cdot}630\,76$$
$$\approx 1{\cdot}105$$

This is closer to the true value of the integral than the result obtained in Example 1 using the trapezium rule. Accuracy can again be improved by increasing the number of ordinates to 11.

n	x_n	y_n (rounded to 5 d.p.)		
0	0	1		
1	0·2		0·961 54	
2	0·4			0·862 07
3	0·6		0.735 29	
4	0·8			0·609 76
5	1		0·5	
6	1·2			0·409 84
7	1·4		0·337 84	
8	1·6			0·280 90
9	1·8		0·235 85	
10	2	0·2		
		1·2	2·770 52	2·162 57
		11·082 08	× 4	× 2
		4·325 14	11·082 08	4·325 14
		16·607 22		

By Simpson's rule,

$$\int_0^2 \frac{1}{1+x^2}\,dx \approx \tfrac{1}{3}h\{(y_0 + y_{10}) + 4(y_1 + y_3 + \ldots) + 2(y_2 + y_4 + \ldots)\}$$
$$\approx \tfrac{1}{3} \times 0{\cdot}2 \times 16{\cdot}607\,22$$
$$\approx 1{\cdot}1071$$

[Unless more accurate values of y_n can be obtained, there is little point in making further increases in the number of ordinates.]

Under certain conditions an approximate value for $\int_a^b f(x)\,dx$ can be obtained by expanding $f(x)$ in ascending powers of x, then integrating term by term. Clearly the method cannot be used unless the expansion of $f(x)$ is valid throughout the interval $a \leqslant x \leqslant b$.

Since the binomial series for $(1 + x^2)^{-1}$ is valid only when $|x| < 1$, this expansion will not produce a good approximation to the value of $\int_0^2 \frac{1}{1+x^2}\,dx$. We consider instead $\int_0^{0.4} \frac{1}{1+x^2}\,dx$. Using the binomial expansion,

$$(1 + x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \cdots$$

$$\therefore \quad \int_0^{0{\cdot}4} \frac{1}{1+x^2}\,dx \approx \int_0^{0{\cdot}4} (1 - x^2 + x^4 - x^6 + \ldots)\,dx$$
$$\approx \left[x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\right]_0^{0{\cdot}4}$$
$$\approx 0{\cdot}4 - \frac{(0{\cdot}4)^3}{3} + \frac{(0{\cdot}4)^5}{5} - \frac{(0{\cdot}4)^7}{7}$$
$$\approx 0{\cdot}3805.$$

Exercise 10.1

Use the trapezium rule to obtain approximate values of the following integrals, giving your answers to 3 decimal places.

1. $\int_1^3 \frac{1}{1+x}\,dx$, 5 ordinates.
2. $\int_0^3 \sqrt{(1 + x^2)}\,dx$, 4 ordinates.
3. $\int_0^2 10^x\,dx$, 5 strips.
4. $\int_1^8 \lg x\,dx$, 7 strips.
5. $\int_0^{\pi/6} \tan x\,dx$, 7 ordinates.
6. $\int_0^4 xe^{-x}\,dx$, 5 ordinates.

Questions 7 to 12. Use sketches to decide whether the results obtained in questions 1 to 6 are greater than or less than the true values of the given integrals.

Use Simpson's rule to obtain approximate values of the following integrals, giving your answers to 4 decimal places.

13. $\int_0^1 \frac{1}{1+x^3}$, 5 ordinates.

14. $\int_0^{\frac{1}{2}} \sqrt{(1-x^2)}\,dx$, 4 strips.

15. $\int_0^{\pi/3} \sqrt{(\sin x)}\,dx$, 6 strips.

16. $\int_0^4 xe^{-x}\,dx$, 5 ordinates.

17. $\int_1^2 x \lg x\,dx$, 11 ordinates.

18. $\int_0^2 \sqrt{(1+x^3)}\,dx$, 11 ordinates.

19. Expand $(1+x^2)^{12}$ in ascending powers of x as far as the term in x^8. Hence find an approximate value for the integral $\int_0^{0.5} (1+x^2)^{12}\,dx$. Using the trapezium rule with 6 ordinates, obtain a second estimate for the value of this integral. Give your answers to 4 decimal places.

20. Use Simpson's rule with 5 ordinates to estimate the value of $\int_0^{0.4} \ln(1+x)dx$. By expanding $\ln(1+x)$ in ascending powers of x as far as the term in x^3 and integrating term by term, obtain a second estimate for the value of this integral. Give your answers to 4 decimal places.

21. Values of a continuous function $f(x)$ were obtained experimentally as follows:

x	1	1·5	2	2·5	3
$f(x)$	8·01	6·02	4·69	3·80	3·27

Estimate the value of $\int_1^3 f(x)\,dx$ using (a) the trapezium rule, (b) Simpson's rule.

22. A car of mass 1000 kg starts from rest and accelerates continuously along a straight road for 20 seconds. After t seconds the resultant force acting on the car is P newtons. The table below gives values of P at intervals of 5 seconds.

t	0	5	10	15	20
P	800	740	590	460	360

Use the trapezium rule to estimate (a) the speed of the car after 10 seconds, (b) the speed of the car after 20 seconds. Use the trapezium rule and your answers to (a) and (b) to estimate the distance travelled in the first 20 seconds.

23. A particle moves in a straight line under the action of a variable force of magnitude F newtons. The value of F after the particle has travelled a distance of s metres is given in the table below.

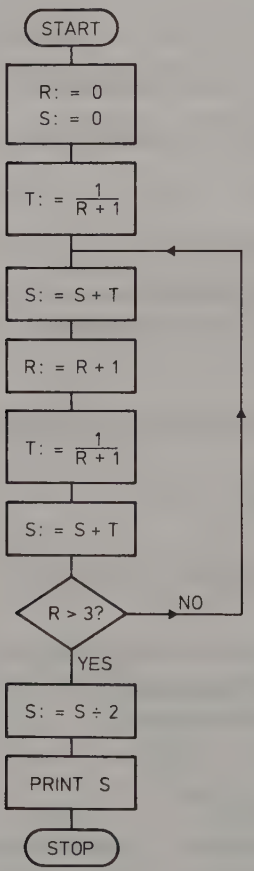
s	0	1	2	3	4	5	6
F	54	61	67	72	76	79	81

Estimate the work done when the particle has travelled a distance of 6 metres using (a) the trapezium rule, (b) Simpson's rule.

24. Construct a flow diagram to evaluate $\int_0^1 e^{-x^2}\,dx$ using the trapezium rule with 6 ordinates. Test your procedure using a pocket calculator.

25. Construct a flow diagram to evaluate $\int_2^3 \sqrt{(x^2-4)}\,dx$ using Simpson's rule with 11 ordinates. Test your procedure using a pocket calculator.

26.



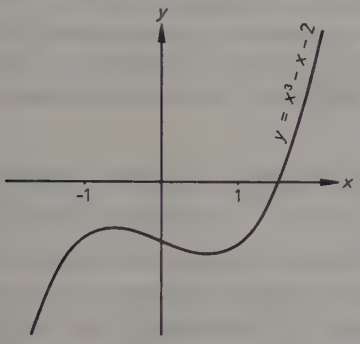
The following flow diagram is designed for the calculation of an approximate value of $\int_0^4 \frac{1}{1+x}\,dx$ using the trapezium rule. Carry out the procedure indicated, tabulating the successive value of R, S and T in simplified fractional form as you proceed. State the printed value of S giving two decimal places in your answers.

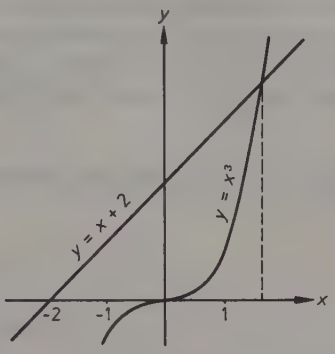
By referring to a sketch graph of $y = \frac{1}{x+1}$ for $0 \leqslant x \leqslant 4$, give a reason why the printed value of S will be an over-estimate of the value of the integral. (JMB)

27. Show that the formula $\int_0^{2h} f(x)\,dx \approx \frac{1}{3}h\{f(0) + 4f(h) + f(2h)\}$ is exact when $f(x)$ is a cubic polynomial in x. Use the formula given above to obtain an approximate value of I where $I = \int_0^{\frac{1}{2}} \frac{1}{\sqrt{(1-x^2)}}\,dx$, giving your result to three decimal places. Show, by direct integration, that $I = \pi/6$. (L)

10.2 Approximate solutions to equations

The roots of the equation $f(x) = 0$ are the values of x at which the curve $y = f(x)$ cuts the x-axis. Thus approximate values for the roots can be obtained by drawing the graph of $y = f(x)$. However, it is sometimes better to estimate the roots by rearranging the equation and finding the points of intersection of two graphs.

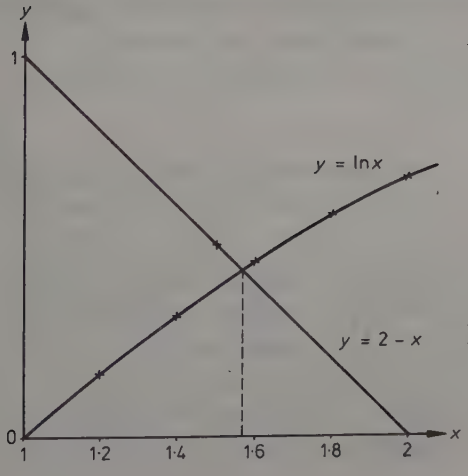




Consider, for instance, the equation $x^3 - x - 2 = 0$. An accurate graph of $y = x^3 - x - 2$ shows that the equation has one real root approximately equal to 1·5. Rearranging the equation as $x^3 = x + 2$, we see that we can also find this root by drawing the graphs of $y = x^3$ and $y = x + 2$.

Example 1 By drawing the graphs of $y = \ln x$ and $y = 2 - x$ in the interval $1 \leqslant x \leqslant 2$, find the solution of the equation $\ln x + x - 2 = 0$, correct to 1 decimal place.

x	1·0	1·2	1·4	1·6	1·8	2·0
$\ln x$	0	0·182	0·336	0·470	0·588	0·693



x	1	1·5	2
$2 - x$	1	0·5	0

At the points of intersection of the graphs of $y = \ln x$ and $y = 2 - x$ the approximate value of x is 1·6.

Hence if $\ln x + x - 2 = 0$, then $x \approx 1{\cdot}6$.

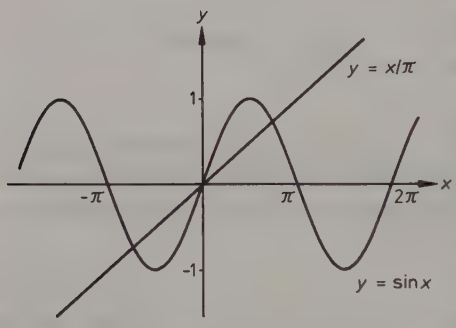
We can check that this result is accurate to one decimal place as follows:

If $x = 1{\cdot}55, \ln x \approx 0{\cdot}438$ and $2 - x = 0{\cdot}45$ $\therefore$ $\ln x < 2 - x$.
If $x = 1{\cdot}65, \ln x \approx 0{\cdot}501$ and $2 - x = 0{\cdot}35$ $\therefore$ $\ln x > 2 - x$.

Hence $\ln x = 2 - x$ for some value of x between 1·55 and 1·65 i.e. when $x = 1{\cdot}6$ correct to 1 decimal place.

Since drawing an accurate graph is rather time-consuming, it is often quicker to use a sketch to find the approximate locations of the roots of an equation, then to improve the accuracy of these rough estimates by numerical methods. First we consider the use of a sketch to determine the number of real roots of an equation.

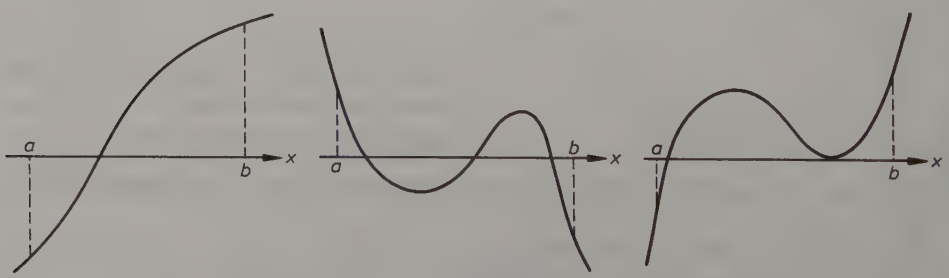
Example 2 Show graphically that the equation $\pi \sin x = x$ has three real roots.



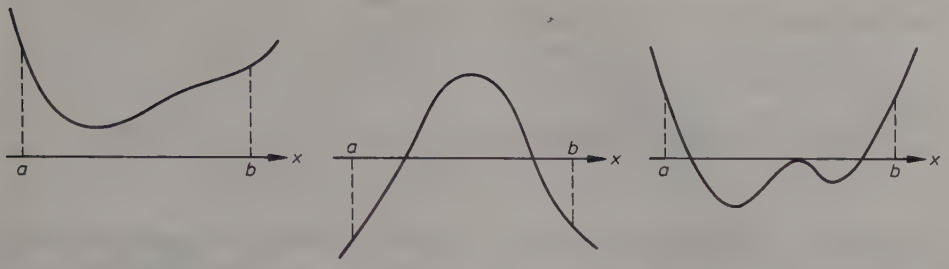
The roots of the equation $\pi \sin x = x$ are the x-coordinates of the points of intersection of the graphs of $y = \sin x$ and $y = x/\pi$. The curve $y = \sin x$ always lies between the lines $y = -1$ and $y = 1$. The straight line $y = x/\pi$ passes through the origin and the points $(-\pi, -1)$, $(\pi, 1)$. Thus, as shown in the sketch, the graphs intersect at three points. Hence the equation $\pi \sin x = x$ has three real roots.

Roots of the equation $f(x) = 0$ can be located by considering the sign changes of the function $f(x)$. Each of the following diagrams shows a function $f(x)$ which is continuous between $x = a$ and $x = b$. We see that if $f(a)$ and $f(b)$ are *opposite* in sign then, in the interval $a < x < b$, the equation $f(x) = 0$ must have at least one root, but may have any *odd* number of roots. If $f(a)$ and $f(b)$ have the *same* sign then, in the interval $a < x < b$, the equation $f(x) = 0$ may have no roots, but could have any *even* number of roots.

[Some of the roots may be coincident.]

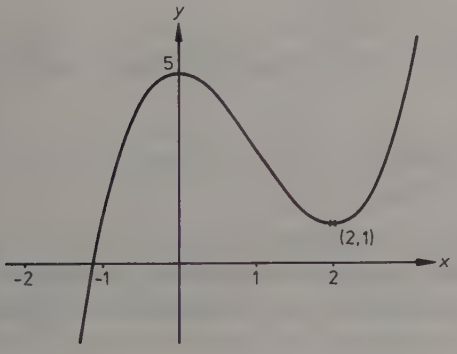


$f(a)$ and $f(b)$ opposite in sign



$f(a)$ and $f(b)$ have the same sign

Example 3 Show that the equation $x^3 - 3x^2 + 5 = 0$ has only one real root and that this root lies between -2 and -1.



Let $y = x^3 - 3x^2 + 5$,

then $\dfrac{dy}{dx} = 3x^2 - 6x = 3x(x - 2)$

and $\dfrac{d^2y}{dx^2} = 6x - 6 = 6(x - 1)$

$\therefore \dfrac{dy}{dx} = 0$ when $x = 0$ or $x = 2$.

When $x = 0$, $y = 5$ and $\dfrac{d^2y}{dx^2} < 0$.

When $x = 2, y = 1$ and $\dfrac{d^2y}{dx^2} > 0$.

Hence on the curve $y = x^3 - 3x^2 + 5$ there is a maximum point at $(0, 5)$ and a minimum point at $(2, 1)$.
Since both turning points are above the x-axis, if follows that the equation $x^3 - 3x^2 + 5 = 0$ has only one real root.
When $x = -2$, $y = -15$ and when $x = -1$, $y = 1$
$\therefore$ y changes sign in the interval $-2 < x < -1$.
Hence the root of the given equation lies between -2 and -1.

Example 4 Show that the equation $x^3 + x - 16 = 0$ has only one real root α and find the integer n such that $n \leqslant \alpha < n + 1$.

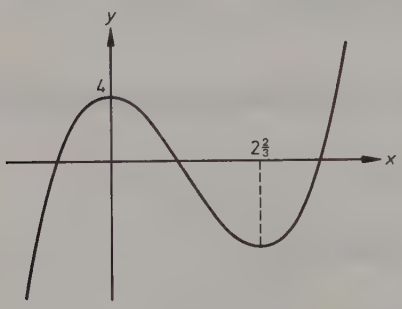
Let $f(x) = x^3 + x - 16$, then $f'(x) = 3x^2 + 1$.
Since $f'(x)$ is always positive, the curve $y = f(x)$ has no turning points and cannot cut the x-axis more than once. For large negative values of x, $f(x)$ is negative and for large positive values of x, $f(x)$ is positive,
$\therefore$ the curve $y = f(x)$ cuts the x-axis exactly once.
Hence the equation $x^3 + x - 16 = 0$ has only one real root α.
When $x \leqslant 0$, $f(x) < 0$ $\therefore$ $\alpha > 0$.

x	1	2	3
$f(x)$	-14	-6	14

The table shows that $f(x)$ changes sign between $x = 2$ and $x = 3$. Thus

$$n \leqslant \alpha < n + 1 \quad \text{for} \quad n = 2.$$

Example 5 Show that the equation $x^3 - 4x^2 + 4 = 0$ has one negative root and two positive roots. Find pairs of successive integers between which these roots lie.



Let $f(x) = x^3 - 4x^2 + 4$ and consider the curve $y = f(x)$.

$$\frac{dy}{dx} = 3x^2 - 8x = x(3x - 8)$$

$$\frac{d^2y}{dx^2} = 6x - 8$$

$$\therefore \quad \frac{dy}{dx} = 0 \text{ when } x = 0 \text{ or } x = 2\tfrac{2}{3}$$

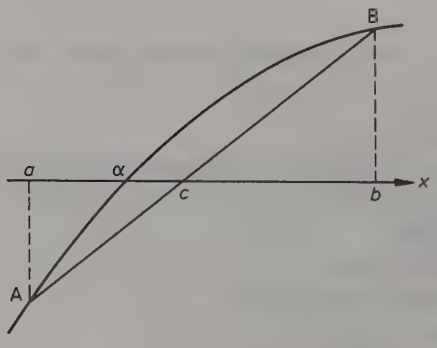
When $x = 0$, $y = 4$ and $\dfrac{d^2y}{dx^2} < 0$. When $x = 2\frac{2}{3}$, $y \approx -5\frac{1}{2}$ and $\dfrac{d^2y}{dx^2} > 0$.

Hence at $x = 0$ there is a maximum point above the x-axis and at $x = 2\frac{2}{3}$ there is a minimum point below the x-axis. It follows that the equation $x^3 - 4x^2 + 4 = 0$ must have one negative root and two positive roots. Testing various integral values of x, we have:

x	-1	0	1	2	3	4
$f(x)$	-1	4	1	-4	-5	4

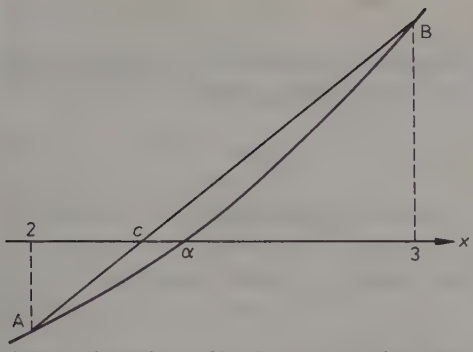
Using the sign changes shown in the table, we deduce that the roots of the given equation lie between -1 and 0, 1 and 2, 3 and 4.

When it has been shown that an equation $f(x) = 0$ has a root α in the interval $a < x < b$, then *linear interpolation* can be used to estimate the value of α.



Let us suppose that A and B are the points on the curve $y = f(x)$ at which $x = a$ and $x = b$ respectively. If the straight line joining A to B cuts the x-axis at $(c, 0)$, then c is taken as an approximate value of the root α.

We illustrate the method by letting $f(x) = x^3 + x - 16$. As shown in Example 4, the equation $x^3 + x - 16 = 0$ has a root α in the interval $2 < x < 3$ and the



curve $y = x^3 + x - 16$ passes through the points $A(2, -6)$ and $B(3, 14)$. If the straight line joining A to B cuts the x-axis at $(c, 0)$, then using similar triangles, we find that c divides the interval from $x = 2$ to $x = 3$ in the ratio 6:14

Hence, by linear interpolation,

$$\alpha \approx 2 + \frac{6}{6 + 14} = 2{\cdot}3.$$

[Note that since $f(2{\cdot}3) \approx -1{\cdot}5$, the true value of α lies between 2·3 and 3.]

If the equation $f(x) = 0$ is known to have a root so small that, for values of x close to the root, x^3 and higher powers of x are negligible, then the root can be estimated using a *quadratic approximation* to $f(x)$.

Example 6 Given that the equation $(1 - x)e^x = 0{\cdot}7(2x + 1)$ has a small root, use a quadratic approximation to estimate its value.

$$(1 - x)e^x = (1 - x)\left(1 + x + \frac{x^2}{2!} + \ldots\right)$$

$$= 1 + x + \tfrac{1}{2}x^2 - x - x^2 + \ldots$$

$$\therefore \quad (1 - x)e^x \approx 1 - \tfrac{1}{2}x^2.$$

Hence the small root of the equation $(1 - x)e^x = 0{\cdot}7(2x + 1)$ is approximately equal to the small root of the equation $1 - \frac{1}{2}x^2 = 0{\cdot}7(2x + 1)$.
Multiplying both sides of this equation by 10, we have

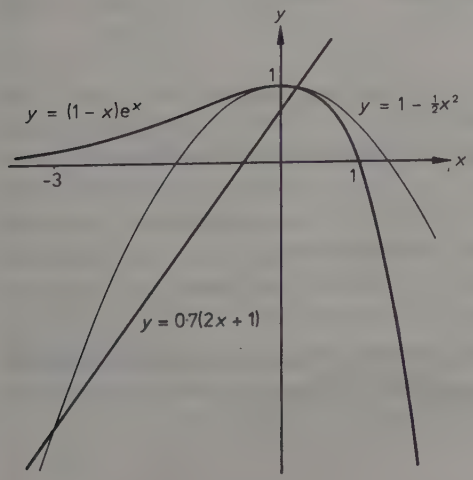
$$10 - 5x^2 = 7(2x + 1)$$

i.e. $5x^2 + 14x - 3 = 0$

$$(5x - 1)(x + 3) = 0$$

$$\therefore \quad x = 0{\cdot}2 \quad \text{or} \quad x = -3.$$

Hence the approximate value of the small root of the given equation is 0·2.



[We see from the sketch that the approximation $(1 - x)e^x \approx 1 - \frac{1}{2}x^2$ is valid for only a small range of values of x. Thus the second root, $x = -3$, obtained using this approximation has no real significance.]

Exercise 10.2

In questions 1 to 10 obtain the roots of the given equations by drawing suitable accurate graphs. Give your answers correct to one decimal place.

1. $x^2 - 2x - 2 = 0$
2. $x^3 + x - 1 = 0$
3. $x^3 - 3x + 3 = 0$
4. $x^3 - 2x^2 + 1 = 0$
5. $x^3 + x^2 - 2x - 1 = 0$
6. $e^x = 3 - x$
7. $2\sin x = x$
8. $4\cos x = x$
9. $\dfrac{4}{x^2} = 2x + 1$
10. $(x - 2)\ln x = x - 1$

In questions 11 to 14 use a sketch to find the number of real roots of the given equation.

11. $e^{-x} = x - 1$
12. $2e^x = 5x + 1$
13. $\cos 2x = 1 - x$
14. $\sin \pi x = \frac{1}{2}(x - 1)$

In questions 15 to 22 determine the number of real roots of the given equation and find pairs of successive integers between which the roots lie.

15. $x^3 - 6x^2 + 36 = 0$
16. $x^3 + 5x - 10 = 0$
17. $x^3 - 12x + 12 = 0$
18. $x^3 + 2x^2 - 4x - 4 = 0$
19. $2x^3 + 4x - 15 = 0$
20. $x^3 - x - 3 = 0$
21. $e^x = x + 3$
22. $\sin x = \ln x$

23. Show that the equation $x^3 - 7x - 12 = 0$ has one real root and show that this root lies between 3 and 4. Use linear interpolation to estimate the root, giving your answer to one decimal place.

24. Show that the equation $x^4 - 4x - 4 = 0$ has one negative root α and one positive root β. Show that $-1 < \alpha < 0$ and that $1 < \beta < 2$. Use linear interpolation to estimate the values of α and β, giving your answers correct to one decimal place.

25. Show that the equation $x^4 + 3x^3 + x^2 - 9 = 0$ has one negative root α and one positive root β. Find α and the integer k such that $k < \beta < k + 1$. Use linear interpolation to find an approximate value for β, giving your answer correct to one decimal place.

In questions 26 to 31 use linear interpolation between suitable successive integers to obtain estimates for the roots of the given equations, giving your answers correct to one decimal place.

26. $2x^3 - 25 = 0$

27. $x^3 + 3x^2 - 14x - 10 = 0$

28. $4x^3 - 9x^2 + 3 = 0$

29. $x^3 + 3x^2 - 4x + 3 = 0$

30. $\ln x = x^2 - 6x + 8$

31. $6\cos x + x = 0$

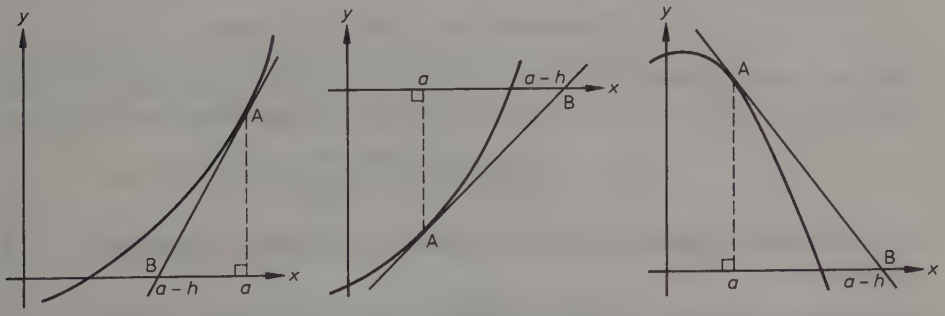
32. Draw an accurate graph of $y = 2x^3 - 5x + 2$ for values of x between -2 and $+2$. Estimate the roots of the equation $2x^3 - 5x + 2 = 0$ (a) from your graph, (b) by linear interpolation between suitable successive integers. Why does linear interpolation give poor approximations to the roots in this case?

33. Each of the following equations has a root so small that, for values of x close to the root, x^3 and higher powers of x are negligible. Use quadratic approximations to estimate these roots.

(a) $2(1 - x)^5 = 18x + 5$
(b) $8\ln(1 + x) = 3x + 1$
(c) $10(1 - \cos x) = 2 - 3x$
(d) $5\sqrt[3]{(1 + 3x)} = 14x + 3$

10.3 The Newton-Raphson method

If $x = a$ is an approximate value of a root of an equation $f(x) = 0$, then the tangent to the curve $y = f(x)$ at $x = a$ can be used to find a second approximation to the root. As shown in the diagrams, the x-coordinate of the point of intersection of this tangent with the x-axis is usually closer to the true root than the first approximation $x = a$.



Let A be the point $(a, f(a))$ on the curve $y = f(x)$ and let the point of intersection of the tangent at A with the x-axis be the point B with coordinates $(a - h, 0)$, then:

$$\text{gradient of } AB = \frac{f(a) - 0}{a - (a - h)} = \frac{f(a)}{h}.$$

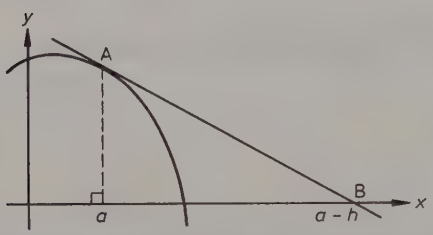
However, since AB is the tangent to the curve $y = f(x)$ at $x = a$, its gradient is also equal to $f'(a)$,

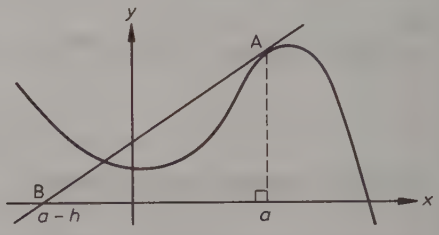
$$\therefore \qquad f'(a) = \frac{f(a)}{h} \quad \text{which gives} \quad h = \frac{f(a)}{f'(a)}.$$

Thus the tangent at A cuts the x-axis where $x = a - \dfrac{f(a)}{f'(a)}$. This result forms the basis of the procedure known as the *Newton-Raphson*† *method* (or simply Newton's method).

> If $x = a$ is a first approximation to a root of the equation $f(x) = 0$, then a second approximation to the root is given by $x = a - \dfrac{f(a)}{f'(a)}$.

The following diagrams show that the method may fail to produce a better approximation to the root when the value of $f'(a)$ is small or when the first approximation to the root is poor.





Example 1 Taking $x = 2{\cdot}3$ as a first approximation to the positive root of the equation $3 \sin x = x$, use the Newton-Raphson method to find a second approximation to this root, giving your answer to 2 decimal places.

Let $f(x) = 3 \sin x - x$, then $f'(x) = 3 \cos x - 1$.
Working to 4 decimal places,

$$f(2{\cdot}3) \approx -0{\cdot}0629 \quad \text{and} \quad f'(2{\cdot}3) \approx -2{\cdot}9988.$$

By the Newton-Raphson method, if $x = 2{\cdot}3$ is a first approximation to the positive root of the equation $f(x) = 0$, then a second approximation is given by

$$x = 2{\cdot}3 - \frac{f(2{\cdot}3)}{f'(2{\cdot}3)} \approx 2{\cdot}3 - \frac{0{\cdot}0629}{2{\cdot}9988} \approx 2{\cdot}28.$$

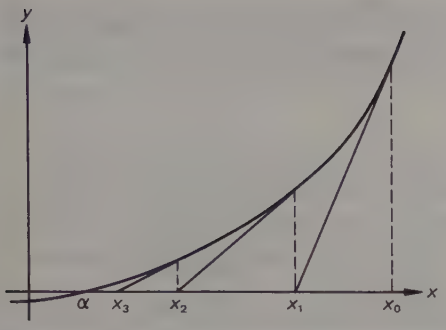
Hence the second approximation to the positive root of the equation $3 \sin x = x$ is $x = 2{\cdot}28$.

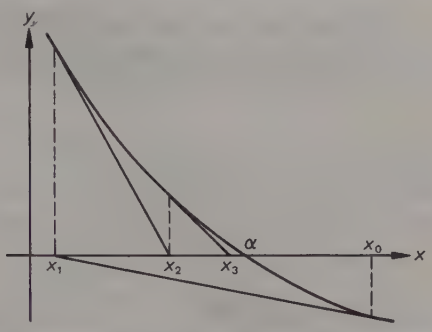
If x_0 is a first approximation to a root α of the equation $f(x) = 0$, then the Newton-Raphson method can be used to produce a sequence of approximations $x_1, x_2, x_3, \ldots$ by writing:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

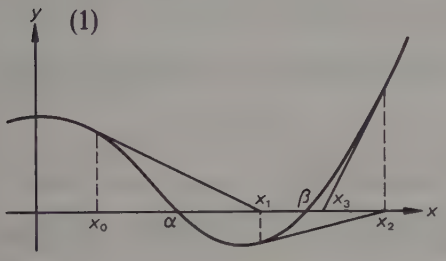
The following diagrams show that, in general, the sequence $x_0, x_1, x_2, x_3, \ldots$ converges quickly to the required root.

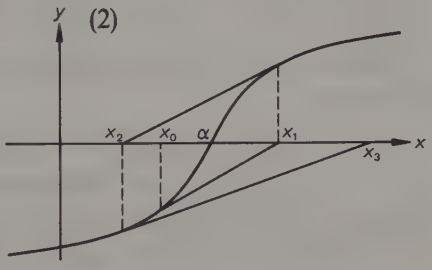
†*Raphson, Joseph* (1648–1715) English mathematician. His account of the Newton-Raphson method was published in 1690 and was based on earlier work by Newton. Raphson produced many other mathematical works including a *History of Fluxions* (1715).





The next two diagrams show that in certain circumstances the method fails. In (1) the sequence $x_0, x_1, x_2, x_3, \ldots$ converges to a different root β and in (2) the sequence diverges.





Example 2 The equation $x^3 - 3x^2 - 1 = 0$ has one real root. Starting with $x = 3$, use two iterations of Newton's method to find an approximate value for this root, correct to 3 decimal places.

Let $f(x) = x^3 - 3x^2 - 1$, then $f'(x) = 3x^2 - 6x$.
Using Newton's method and working to 5 decimal places,

let $x_0 = 3$, then $f(x_0) = -1$, $f'(x_0) = 9$

$$\therefore \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3 - \frac{(-1)}{9} \approx 3{\cdot}11111$$

Let $x_1 = 3{\cdot}11$, then $f(x_1) \approx 0{\cdot}063\,93$, $f'(x_1) = 10{\cdot}3563$

$$\therefore \quad x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 3{\cdot}11 - \frac{0{\cdot}063\,93}{10{\cdot}3563} \approx 3{\cdot}103\,83$$

Hence an approximate value for the root is 3·104.

Checking the accuracy of this result we find that

$$f(3{\cdot}1035) \approx -0{\cdot}003 \quad \text{and} \quad f(3{\cdot}1045) \approx 0{\cdot}007,$$

$\therefore$ the root is 3·104 correct to 3 decimal places.

[Note that when evaluating a function $f(x)$ by calculator it is usually best to store the value of x in the memory. If $f(x)$ is a polynomial more accurate results will be obtained by using nested multiplication as described in Book 1, §4.4.]

To demonstrate how rapidly the Newton-Raphson process converges we now obtain further approximations to the root of the equation $x^3 - 3x^2 - 1 = 0$, giving the results in the form of a table.

n	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1}
0	3	-1	9	3·111 1111
1	3·11	0·063 931	10·3563	3·103 8269
2	3·1038	$-0·000\,0350$	10·277 9233	3·103 8034
3	3·103 8034	$-0·000\,000\,03$	10·277 9662	3·103 8034

When using the Newton-Raphson process it is usually reasonable to assume that if $|x_{n+1} - x_n| < \varepsilon$, then the error in the approximation x_{n+1} is less than ε.

In the given table the values of x_3 and x_4 agree to 7 decimal places. It follows that $|x_4 - x_3| < 1 \times 10^{-7}$ and thus the error in the approximation x_4 is less than 1×10^{-7}.

Closer examination of the results shows that, allowing for rounding errors in the calculations,

$$|x_4 - x_3| = |f(x_3)/f'(x_3)| < 1 \times 10^{-8}$$

$$\therefore \quad 3·103\,8034 < x_4 < 3·103\,803\,41$$

Thus the true value of the root is unlikely to exceed 3·103 803 42. Therefore $x = 3·103\,8034$ correct to 7 decimal places.

Exercise 10.3

In questions 1 to 8 use the Newton-Raphson method to find a second approximation to a root of the given equation taking the stated value of a as a first approximation. Give your answers to 2 decimal places.

1. $x^3 - 4x^2 + 4 = 0, a = 1.$
2. $x^3 - 3x^2 + 5 = 0, a = -1.$
3. $x^3 - x + 2 = 0, a = -1·5.$
4. $x^3 + x - 16 = 0, a = 2·3.$
5. $\ln x + x - 2 = 0, a = 1·6.$
6. $e^x + x - 3 = 0, a = 0·8.$
7. $2 \sin x = x, a = -2.$
8. $3 \cos x = x, a = 1·2.$

In questions 9 to 14 use two iterations of the Newton-Raphson method to find an approximate value for a root of the given equation, taking the stated value of x_0 as a first approximation. Give your answers to 3 decimal places.

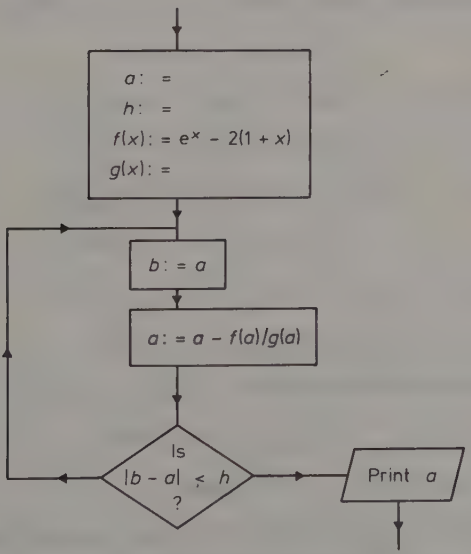
9. $x^3 - 5x - 8 = 0, x_0 = 3.$
10. $x^3 - 9x^2 + 20 = 0, x_0 = 2.$
11. $4x^3 + 6x^2 - 5x - 6 = 0, x_0 = -1.$
12. $x^3 + x + 8 = 0, x_0 = -2.$
13. $e^{-x} = x - 1, x_0 = 1·2.$
14. $\sin x = \ln x, x_0 = 2·3.$

15. Show that the equation $x^3 + 2x - 1 = 0$ has only one real root and that this root lies between 0 and 1. Find the approximate value of this root correct to 3 decimal places.

16. Show that the equation $x^3 - 12x - 6 = 0$ has one positive and two negative roots. Use linear interpolation between suitable successive integers to find a first approximation to the positive root. Hence find the approximate value of this root correct to 4 decimal places.

17. Give, on the same diagram, a sketch of the graph of $y = 3e^{x/2}$ and of the graph of $y = 4x + 6$. State the number of roots of the equation $3e^{x/2} = 4x + 6$. Taking 4 as a first approximation to one root α of this equation, find a second approximation to α, giving three significant figures in your answer and showing clearly how this answer has been obtained. Using a suitable integer as first approximation to another root β of the equation, find a second approximation to β, again giving three significant figures in your answer and showing clearly how this answer has been determined. (C)

18. Show that the equation $e^x = 2(1 + x)$ has only one positive root. Find the nearest integer to this root. Rewrite the flow chart shown below and complete it to specify a process which will find the root of the above equation, with an error of less than 0·01, starting from the integer nearest to the root. Evaluate the root to this degree of accuracy.



(AEB 1978)

10.4 Other iterative processes

An *iterative process* is a repetitive procedure designed to produce a sequence of approximations to some numerical quantity. The Newton-Raphson method described in the previous section is one example of such a process.

Interval bisection is another iterative process used to obtain approximations to a root α of an equation $f(x) = 0$. To use this method it is necessary to locate an interval containing the root, i.e. to find values x_0 and x_1 such that $x_0 < \alpha < x_1$. The length of this interval is halved by taking $x_2 = \frac{1}{2}(x_0 + x_1)$ and then determining whether α lies between x_0 and x_2 or between x_2 and x_1. The new interval is then bisected to produce a further approximation x_3. The bisection process is repeated until the interval containing α is small enough to ensure a result to the required degree of accuracy. We will see in the following example that this method converges much more slowly than the Newton-Raphson method.

Example 1 Use interval bisection to obtain a sequence of approximations to the positive root of the equation $x^2 - 3 = 0$. Hence evaluate $\sqrt{3}$ correct to 2 decimal places.

Let $f(x) = x^2 - 3$, then the positive root of the equation $f(x) = 0$ is $\sqrt{3}$.
Since $f(1) < 0$ and $f(2) > 0$, we have $1 < \sqrt{3} < 2$.
Taking $x_0 = 1$ and $x_1 = 2$ as first approximations to $\sqrt{3}$,

let $x_2 = \frac{1}{2}(x_0 + x_1) = \frac{1}{2}(1 + 2) = 1{\cdot}5$, then $f(x_2) < 0$

$\therefore \quad 1{\cdot}5 < \sqrt{3} < 2$

Let $x_3 = \frac{1}{2}(1{\cdot}5 + 2) = 1{\cdot}75$, then $f(x_3) > 0$

$\therefore \quad 1{\cdot}5 < \sqrt{3} < 1{\cdot}75$

Let $x_4 = \frac{1}{2}(1{\cdot}5 + 1{\cdot}75) = 1{\cdot}625$, then $f(x_4) < 0$

$\therefore \quad 1{\cdot}625 < \sqrt{3} < 1{\cdot}75$

Let $x_5 = \frac{1}{2}(1{\cdot}625 + 1{\cdot}75) = 1{\cdot}6875$, then $f(x_5) < 0$

$\therefore \quad 1{\cdot}6875 < \sqrt{3} < 1{\cdot}75$

Let $x_6 = \frac{1}{2}(1{\cdot}6875 + 1{\cdot}75) = 1{\cdot}71875$, then $f(x_6) < 0$

$\therefore \quad 1{\cdot}7187 < \sqrt{3} < 1{\cdot}75$

Let $x_7 = \frac{1}{2}(1{\cdot}7187 + 1{\cdot}75) = 1{\cdot}73435$, then $f(x_7) < 0$

$\therefore \quad 1{\cdot}7187 < \sqrt{3} < 1{\cdot}7344$

Let $x_8 = \frac{1}{2}(1{\cdot}7187 + 1{\cdot}7344) = 1{\cdot}72655$, then $f(x_8) < 0$

$\therefore \quad 1{\cdot}7265 < \sqrt{3} < 1{\cdot}7344$

Hence $\sqrt{3} = 1{\cdot}73$ correct to 2 decimal places.

In general, an iterative process is based on a rule or *algorithm* for obtaining an approximation x_{n+1} from earlier approximations $x_0, x_1, x_2, \ldots, x_n$. In many cases this rule can be expressed as an iterative formula giving x_{n+1} in terms of x_n. In section §10.3 we showed that the Newton-Raphson method can be used to produce such a formula. We now discuss an alternative method.

Let us consider the general iterative formula

$$x_{n+1} = f(x_n)$$

If the sequence $x_0, x_1, x_2, x_3, \ldots$ converges to a finite limit α, then as $n \to \infty$, $x_{n+1} \to \alpha$ and $f(x_n) \to f(\alpha)$

giving $\alpha = f(\alpha)$.

Thus, if the iteration $x_{n+1} = f(x_n)$ is convergent, it will produce a sequence of approximations to a root of the equation $x = f(x)$.

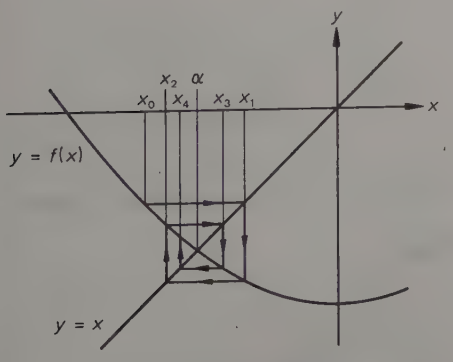
When solving an equation we may be able to obtain a suitable iterative formula by writing the equation in the form $x = f(x)$. For any particular equation several different rearrangements will be possible, but not all the corresponding iterative processes will converge to the required root. It is difficult to predict which rearrangements will be most successful. However, it can be shown that, in general, the iteration $x_{n+1} = f(x_n)$ converges to a root α, provided that (i) x_0 is a fairly good approximation to α, and that (ii) $-1 < f'(\alpha) < 1$.

To illustrate the various possibilities we consider the equation $x^2 - 4x - 8 = 0$. This has one root α between -2 and -1 and another root β between 5 and 6. We will attempt to calculate these roots using iterative formulae based on the rearrangements:

(a) $x = \frac{1}{4}x^2 - 2$, (b) $x = \dfrac{8}{x-4}$, (c) $x = 2\sqrt{(x+2)}$.

(a) The following table shows the sequences obtained using the formula $x_{n+1} = \frac{1}{4}x_n^2 - 2$, taking x_0 as -2, -1, 5 and 6, then working to 3 decimal places.

x_0	-2	-1	5	6
x_1	-1	$-1{\cdot}75$	4·25	7
x_2	$-1{\cdot}75$	$-1{\cdot}234$	2·516	10·25
x_3	$-1{\cdot}234$	$-1{\cdot}619$	$-0{\cdot}417$	24·266
x_4	$-1{\cdot}619$	$-1{\cdot}345$	$-1{\cdot}957$	145·210
x_5	$-1{\cdot}345$	$-1{\cdot}548$	$-1{\cdot}043$	⋮
x_6	$-1{\cdot}548$	$-1{\cdot}401$	$-1{\cdot}728$	⋮



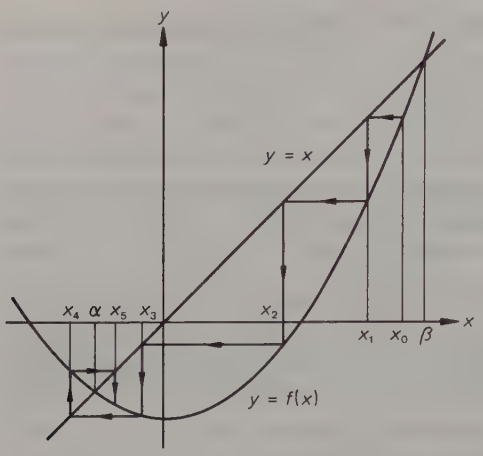
$x_0 = -2$, $f(x) = \frac{1}{4}x^2 - 2$

The sketch graphs show that three of the sequences converge slowly to the root α and that the other sequence diverges. Writing $f(x) = \frac{1}{4}x^2 - 2$, we find that

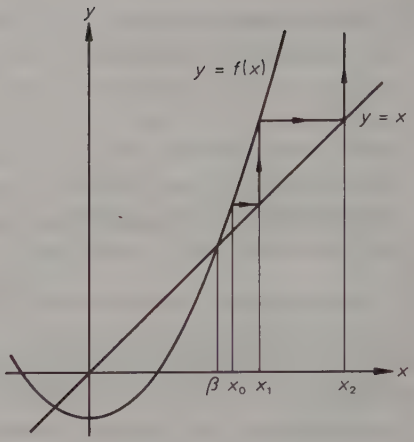
$$f'(\alpha) \approx f'(-1{\cdot}5) = -0{\cdot}75$$

and $$f'(\beta) \approx f'(5{\cdot}5) = 2{\cdot}75.$$

These results tend to confirm that the rate of convergence depends on the value of $f'(x)$.



$x_0 = 5, f(x) = \frac{1}{4}x^2 - 2$

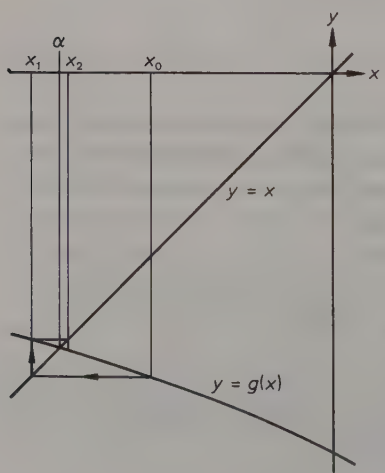


$x_0 = 6, f(x) = \frac{1}{4}x^2 - 2$

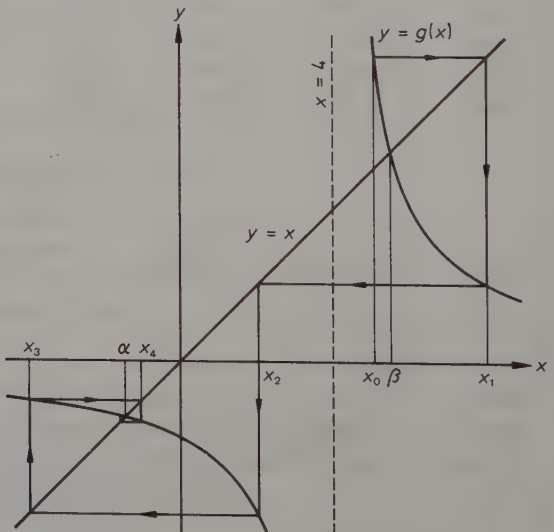
(b) Next we work with the formula $x_{n+1} = \dfrac{8}{x_n - 4}$

x_0	-2	-1	5	6
x_1	$-1{\cdot}333$	$-1{\cdot}6$	8	4
x_2	$-1{\cdot}500$	$-1{\cdot}429$	2	—
x_3	$-1{\cdot}455$	$-1{\cdot}474$	-4	
x_4	$-1{\cdot}467$	$-1{\cdot}461$	-1	
x_5	$-1{\cdot}463$	$-1{\cdot}465$	$-1{\cdot}6$	
x_6	$-1{\cdot}464$	$-1{\cdot}464$	$-1{\cdot}429$	

In this case three of the sequences converge fairly quickly to the root α. In the remaining sequence the value of x_2 is undefined. Writing $g(x) = 8/(x - 4)$, we have $g'(\alpha) \approx -0{\cdot}25$ and $g'(\beta) \approx -4$.



$x_0 = -1, g(x) = 8/(x - 4)$

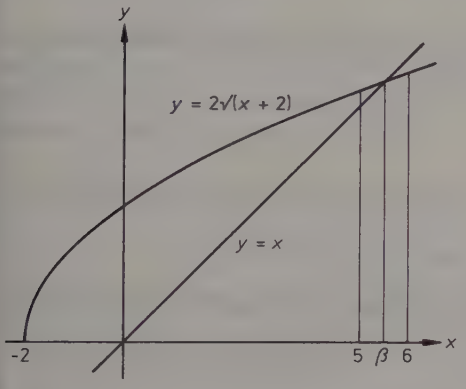


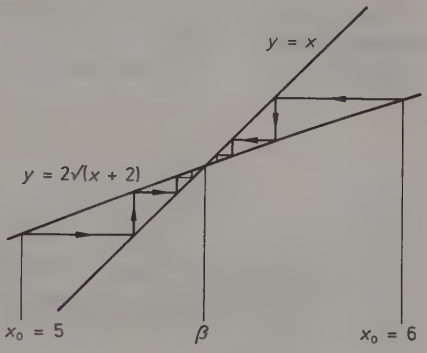
$x_0 = 5, g(x) = 8/(x - 4)$

(c) Since the rearrangement $x = 2\sqrt{(x+2)}$ is not valid for the negative root α of the original equation, we hope to use the formula $x_{n+1} = 2\sqrt{(x_n + 2)}$ to obtain the positive root β.

x_0	5	6
x_1	5·292	5·657
x_2	5·401	5·534
x_3	5·441	5·490
x_4	5·456	5·474
x_5	5·461	5·468
x_6	5·463	5·466

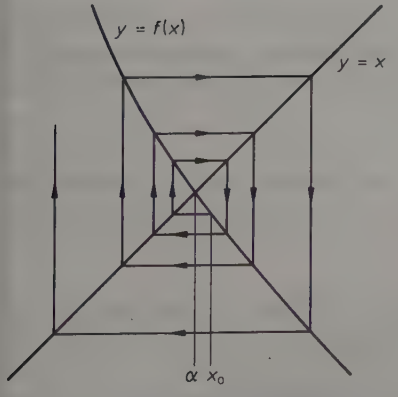
Both sequences converge to the root β, one from below, the other from above. Letting $h(x) = 2\sqrt{(x+2)}$, we find that $h'(\beta) \approx 0{\cdot}37$.





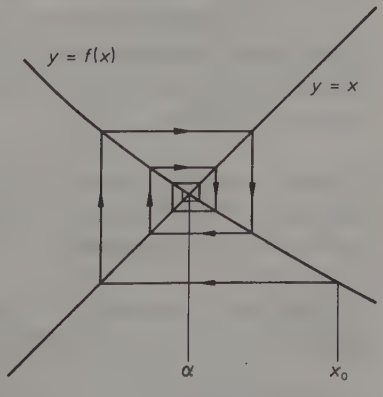
Returning to the general formula $x_{n+1} = f(x_n)$, we conclude that there are four ways in which the sequence of approximations $x_0, x_1, x_2, \ldots$ may behave close to a root α of the equation $x = f(x)$.

(1)



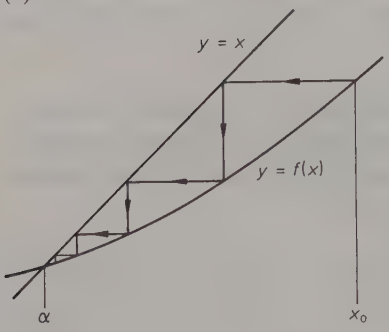
$f'(\alpha) < -1$

(2)



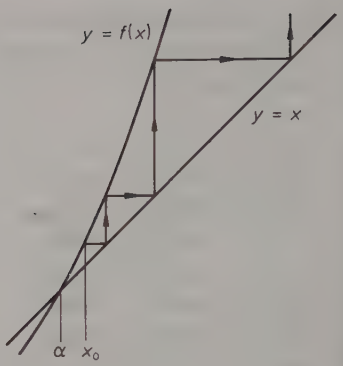
$-1 < f'(\alpha) < 0$

(3)



$0 < f'(\alpha) < 1$

(4)



$f'(\alpha) > 1$

[Note that in cases (1) and (4) the sequence may later converge to a different root of the equation $x = f(x)$.]

Exercise 10.4

1. Use the method of interval bisection to find the positive roots of the following equations correct to 2 decimal places.
(a) $x^2 - 5 = 0$, (b) $x^3 - 2 = 0$, (c) $x^3 - 4x^2 + 2 = 0$.

2. The iterative formula $x_{n+1} = \frac{1}{2}(x_n + a/x_n)$ gives successive approximations to the square root of a. Use this formula and the given values of x_0 to calculate correct to 3 decimal places.
(a) $\sqrt{2}, x_0 = 1$, (b) $\sqrt{10}, x_0 = 3$, (c) $\sqrt{13}, x_0 = 4$.

3. The iterative formula $x_{r+1} = \frac{1}{2}(x_r + N/x_r^2)$ can be used to find an approximate cube root of a number N. Letting $x_0 = 2$, find $\sqrt[3]{5}$ correct to 3 significant figures.

4. By considering the function $f(x) = N - 1/x$ and using the Newton-Raphson method, establish the formula $x_{r+1} = x_r(2 - Nx_r)$ for obtaining successive approximations to the reciprocal of N. Taking $x_0 = 1$, find the reciprocal of 1·3 correct to 3 significant figures.

5. Show that the equation $x = \ln(8 - x)$ has a root between 1 and 2. Calculate this root correct to 3 decimal places (a) using the formula $x_{n+1} = \ln(8 - x_n)$, (b) using the Newton-Raphson method.

6. Show that the equation $x = \cos x - 3$ has a root between -4 and -3. Use an iterative method to calculate this root correct to 3 decimal places.

7. Use the formula $x_{r+1} = (1 - x_r^2)/5$, where $x_0 = 0$, to find the positive root of the equation $x^2 + 5x - 1 = 0$, correct to 4 decimal places. Discuss with the aid of

sketch graphs the behaviour of the sequence $x_0, x_1, x_2, \ldots$ when $x_0 = -6$ and when $x_0 = -5$.

8. Show graphically, or otherwise, that the equation $x^3 - x - 1 = 0$ has only one real root and find the integer n such that the root α satisfies $n < \alpha < n + 1$. An iterative process for finding this root is defined by

$$x_1 = 1, \quad x_{m+1} = (x_m + 1)^{1/3} \quad \text{for all} \quad m \in \mathbb{N}.$$

Obtain, to 3 places of decimals, the values of x_2 and x_3. Show, on a sketch graph, the line $y = x$ and the curve $y = (x + 1)^{1/3}$, indicating on this graph the relation between x_1, x_2, x_3 and the root α. (L)

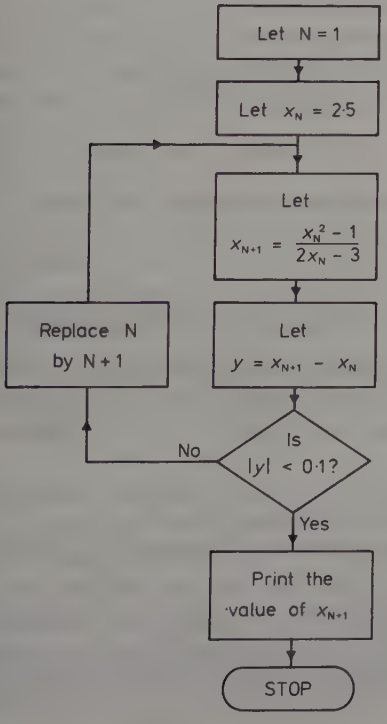
9. Verify by sketching graphs of $y = x$ and $y = 2(1 + 1/x)$ that, in general, the sequence defined by $x_{n+1} = 2(1 + 1/x_n)$ converges to the root $1 + \sqrt{3}$ of the equation $x^2 - 2x - 2 = 0$. Hence find $\sqrt{3}$ to 4 significant figures, taking a suitable integer as first approximation.

10. Demonstrate graphically that neither

$$x_{n+1} = \tfrac{2}{5}(4 - x_n^3) \quad \text{nor} \quad x_{n+1} = (8 - 5x_n)/2x_n^2$$

can be used to find the real root of the equation $2x^3 + 5x - 8 = 0$. Derive a suitable iteration formula and hence obtain the root to 3 decimal places.

11.



The flow diagram shown represents the steps involved in finding an estimate of a solution of the equation $x^2 - 3x + 1 = 0$, using an iterative method. Follow the steps to their conclusion and obtain the estimate of the root. Indicate clearly everything that is to be printed and work to three decimal places. Indicate the purpose of the decision box.

(L)

12. The equation $f(x) = 0$ is known to have precisely one root in the interval $a \leqslant x \leqslant b$. Construct a flow diagram giving an algorithm to estimate this root, using the method of interval bisection. The following are given: the numbers a and b (with $a < b$), the function f, and the positive number δ which is the maximum permissible magnitude of the error in the root. Demonstrate the working of your flow diagram by applying it to the equation $\cos x - x = 0$ with $a = 0$, $b = 1$ and $\delta = 0{\cdot}1$, giving your answer correct to one decimal place. (C)

Exercise 10.5 (miscellaneous)

1. Values of a continuous function f were found experimentally as given below.

t	0	0·3	0·6	0·9	1·2	1·5	1·8
$f(t)$	2·72	3·00	3·32		4·06	4·48	4·95

Use linear interpolation to estimate $f(0{\cdot}9)$. Then use Simpson's rule with seven ordinates to estimate $\int_0^{1\cdot8} f(t)\,dt$, tabulating your working and giving your answer to two places of decimals. (JMB)

2. The following pairs of values of x and y satisfy approximately a relation of the form $y = ax^n$, where a and n are integers. By plotting the graph of $\lg y$ against $\lg x$ find the values of the integers a and n.

x	0·7	0·9	1·1	1·3	1·5
y	1·37	2·92	5·32	8·80	13·50

Estimate the value of the integral $\int_{0\cdot7}^{1\cdot5} y\,dx$ (a) by Simpson's rule, using five ordinates and clearly indicating your method, (b) by using the relation $y = ax^n$ with the values found for a and n. (L)

3. Find an approximate value of $\int_0^{1/2} \sqrt{(1 - x^2)}\,dx$ by using the trapezoidal rule with intervals of 0·1. Show, by working out the integral by substitution, that the magnitude of the error in your approximation is less than 0·001. Draw a diagram to illustrate the reason for this error, having regard to its sign. (AEB 1979)

4. If $f(x) = a + bx + cx^2 + kx^3$, where a, b, c, k are constants, show that $\int_{-h}^{h} f(x)\,dx = \frac{1}{3}h\{f(-h) + 4f(0) + f(h)\}$. Using Simpson's rule with 5 ordinates, obtain an approximate value, to 3 decimal places, for $\int_0^{0\cdot4} \sqrt{(1 + x^2)}\,dx$. Check your result by expanding $\sqrt{(1 + x^2)}$ in ascending powers of x as far as the term in x^6 and integrating term by term. (O&C)

5. Plot the graph of $y = x^3$ for x from -3 to $+3$. Use the graph to obtain solutions to the following equations (i) $x^3 - 2x - 5 = 0$, (ii) $2x^3 - 4x + 1 = 0$. By writing the equation $5x^3 + 2x^2 - 1 = 0$ in terms of $1/x$, obtain its solution, correct to one decimal place. (AEB 1979)

6. Sketch the graph of $y = \tan x$ for values of x between 0 and 2π. Use your sketch graph, together with additional straight line graphs, to solve the following problems:
(i) If $a > 0$ and $b > 0$, find the number of roots, between 0 and 2π, of the equation $\tan x = ax + b$.
(ii) If m can take all real values, find the set of values of m such that the equation $\tan x = mx + \pi$ has exactly three roots lying in the interval $0 < x < 2\pi$. (C)

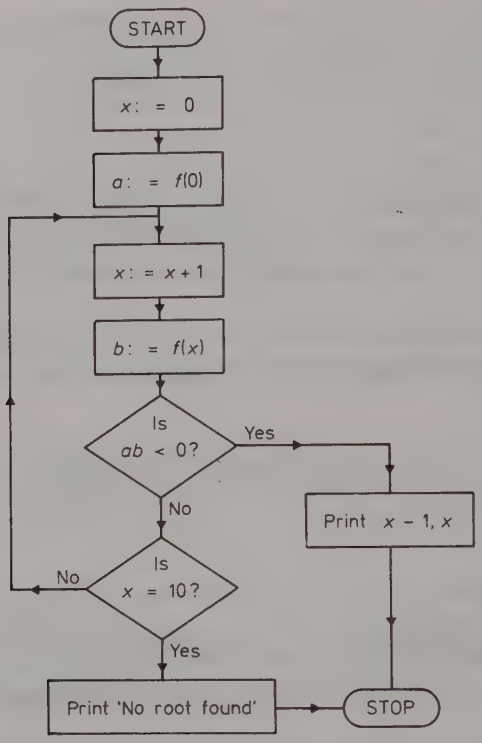
7. Sketch the graph of $y = x^2e^{-x}$ giving the coordinates of the turning points. Explain, from your graph, why, if $0 < k < 4/e^2$, the equation $ke^x = x^2$ has three real roots. (O&C)

8. By considering the graph of $y = x^3 - 3px - q$, where p and q are real constants, show that the condition for the equation $x^3 - 3px - q = 0$ to have three distinct real roots is $4p^3 > q^2$. Find the additional condition necessary for two of the roots to be positive. (JMB)

9. The least positive root of the equation $x^3 - 10x^2 + 12x + 16 = 0$ is denoted by α. Use linear interpolation between two suitable successive integers to obtain an estimate for α, giving your answer to one decimal place. Will your answer be an underestimate or overestimate of the exact root? Explain your reasoning. (O&C)

10. Sketch the graph of $y = \cos x$ for $0 \leqslant x \leqslant 4\pi$. Use your sketch to find the number of positive roots of the equation $\cos x = x/10$. Prove that the equation of the tangent to the curve $y = \cos x$ at $(\frac{1}{2}\pi, 0)$ is $y = \frac{1}{2}\pi - x$. By using $\frac{1}{2}\pi - x$ as an approximation for $\cos x$ (valid for x near $\frac{1}{2}\pi$), calculate an estimate of the smallest positive root of the equation $\cos x = x/10$, giving your answer to 2 decimal places. Show on a new sketch the part of the curve $y = \cos x$ near $x = \frac{1}{2}\pi$ and the tangent at $x = \frac{1}{2}\pi$. Hence determine whether the estimate of the root calculated above is larger or smaller than the correct value. (C)

11. The given flow diagram is designed to search for a root of the equation $f(x) = 0$ in the range $0 < x < 10$. (i) If x reaches 10 explain why the message printed has to be "No root found" and not "There is no root between 0 and 10". (ii) Work through the flow diagram in the case where $f(x) = e^x - 3x - 4$. (iii) Extend the flow diagram so that, when a root is located between two consecutive integers, a further search is carried out which will narrow the range in which the root lies to 0·1.



(C)

12. Using the same axes draw accurate graphs of $y = \ln x$ and $y = 3 - x$ in the interval $1 \leqslant x \leqslant 4$. Deduce that the equation $x + \ln x - 3 = 0$ has a root near $x = 2{\cdot}2$. Clearly showing your method, obtain alternative approximations to the root of the equation (a) by linear interpolation between $x = 2{\cdot}2$ and $x = 2{\cdot}3$, (b) by one application of the Newton-Raphson procedure using $x = 2{\cdot}2$ as the initial value. (L)

13. Given that $P(x) = x^3 - 6x^2 + 9x + a$ for all real x, find the values of x for which $P'(x) = 0$. Determine the values of the constant a such that the equation $P(x) = 0$ has a repeated root. By sketching graphs of $y = P(x)$ for these values of a, or otherwise, find the set of values of a for which $P(x) = 0$ has only one real root. Given now that $a = -5$, and that 4 is an approximation to the root of the equation $P(x) = 0$, use Newton's method once to obtain another approximation to the root, giving your answer correct to one decimal place. (JMB)

14. Use linear interpolation to find an approximation x_1 to the root of the equation $f(x) = 0$ lying between 3 and 4, where $f(x) \equiv x^4 - 4x^3 + x + 12$. Use one application of the Newton-Raphson process applied to $f(x)$ starting with $x = 4$ to find another approximation x_2 to the same root. Use one application of the same process starting with $x = 3$ to find another approximation x_3 to the same root. Correct to 2 decimal places the root is $x = 3{\cdot}69$. Indicate on a diagram how each of the approximations is related to the graph of $f(x)$ for $3 \leqslant x \leqslant 4$. (C)

15. Explain, with the use of diagrams, the use of Newton's method for obtaining the numerical solutions of an equation. Draw a diagram to show how the method can break down.

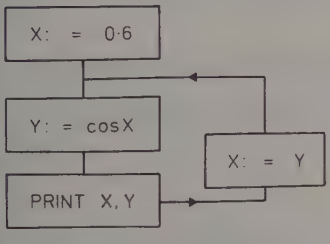
Obtain the real factors, with integer coefficients, of $x^4 - 4x^3 + 3x^2 - 4x + 12$. Hence or otherwise obtain the greater real root of the equation

$$x^4 - 4x^3 + 3x^2 - 4x + 11 = 0,$$

correct to 3 decimal places. (AEB 1979)

16. Show that the equation $x^3 + 3x - 3 = 0$ has only one real root and that it lies between $x = 0{\cdot}8$ and $x = 1$. Obtain approximations to the root (a) by performing one application of the Newton-Raphson procedure using $x = 0{\cdot}8$ as the first approximation, (b) by performing two iterations, using the procedure defined by $x_{n+1} = (3 - x_n^3)/3$, and starting with $x = 0{\cdot}8$. (L)

17. Using the same axes and scales, draw graphs of $y = \cos x$ and $y = x$ for the interval $0 \leqslant x \leqslant 1{\cdot}4$. (A scale of 10 cm to a unit on each axis is recommended.) On your graph mark as A, B, C and D the first four points whose coordinates are printed when the following section of a flow diagram is executed.



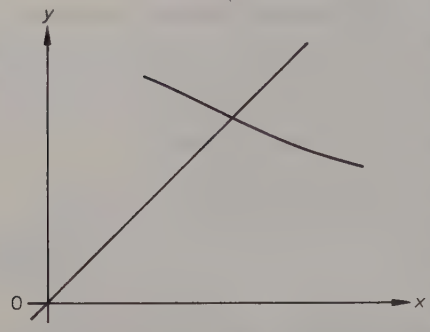
The procedure indicated in the flow diagram is an iterative method of solving a particular equation. Write down this equation and obtain from your graph an approximation solution to it. (JMB)

18. Given that $f(x) = x^3 + x^2 + 2x - 2$, show that the equation $f(x) = 0$ has only one root and that this root lies between 0 and 1. Use the iterative procedure

$$x_{n+1} = \frac{2}{x_n^2 + x_n + 2}, \quad x_1 = 1,$$

to obtain x_2 and x_3. By means of a sketch, illustrate how the successive values of x_n converge to the desired root. (L)

19. A solution of the equation $x = \phi(x)$ is to be attempted by using the iteration $x_{n+1} = \phi(x_n)$, starting with an initial estimate x_1. Draw sketch graphs showing $y = x$ and $y = \phi(x)$ to illustrate the following possibilities regarding convergence towards, or divergence from, the root $x = \alpha$. (i) $x_1 > \alpha$ and successive iterates steadily decrease, with the value α as a limit. (ii) $x_1 > \alpha$ and successive iterates are alternately less than α and greater than α, but approach α as a limit. (iii) $x_1 > \alpha$ and successive iterates get steadily larger. Use an iterative method to find a non-zero root of the equation $x = \tan^{-1}(2x)$ correct to 2 significant figures. (C)

20. 

The equation $\sin x = \cos^{-1} x$ has a root in the neighbourhood of $x = \frac{1}{4}\pi$, and the iteration $x_{n+1} = \cos(\sin x_n)$ is to be employed to calculate this root. The diagram shows the appropriate parts of the graphs of $y = x$ and $y = \cos(\sin x)$. Explain with reference to this diagram why successive iterates are alternately greater than and less than the root. Use the given iteration to determine the root correct to 2 decimal places, showing clearly that you have achieved this degree of accuracy. Construct a flow diagram to show how the iteration may be continued until successive iterates differ by less than a given quantity δ. (C)

21. Show that the equation $x^3 - 3x + 1 = 0$ has one negative root and two positive roots. Rearrange the equation in the form $x = F(x)$ in such a way that the iteration $x_{n+1} = F(x_n)$ will converge to the larger positive root, showing clearly that the condition for convergence is satisfied by your function F. Show by means of a sketch how the iterates converge to the root if the initial estimate x_1 is taken to be 1. Use your iteration to calculate the root correct to two decimal places, taking 1·5 as your initial approximation. (C)

11 Differential equations

11.1 Differential equations and their solution curves

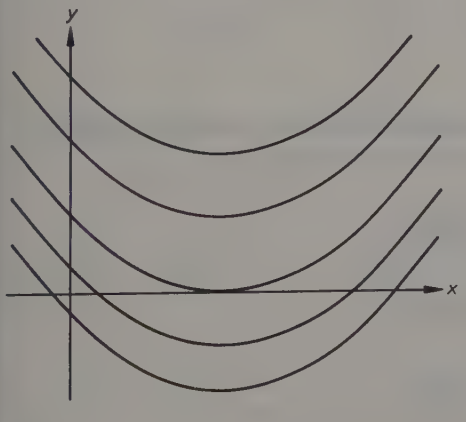
An equation connecting a function and one or more of its derivatives is called a *differential equation*. We shall see in §11.4 that many physical laws, such as the laws of natural growth or decay, can be expressed as differential equations.

The *order* of a differential equation is the order of the highest derivative which appears in it. For instance, the equations $\frac{dx}{dt} = -4x$ and $\left(\frac{dy}{dx}\right)^2 = x + y$ are first order equations and the equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} = e^{2x}$ is a second order equation.

Let us consider a simple differential equation of the first order:

$$\frac{dy}{dx} = x - 1.$$

Any equation connecting x and y which satisfies this differential equation is a solution. Thus $y = \frac{1}{2}x^2 - x$, $y = \frac{1}{2}x^2 - x + 5$ and $y = \frac{1}{2}x^2 - x - 28$ are all solutions. The *general solution* of the differential equation is $y = \frac{1}{2}x^2 - x + c$, where c is an arbitrary constant. Since c can take any real value, this general solution can be represented graphically by an infinite set of curves called the *solution curves* or *integral curves* of the equation. Each member of this "family" of curves represents a particular solution, or *particular integral*, of the differential equation.



The solution curves of $\frac{dy}{dx} = x - 1$.

Given the equation of a family of curves, we can obtain the corresponding differential equation by differentiating and then eliminating any arbitrary constants.

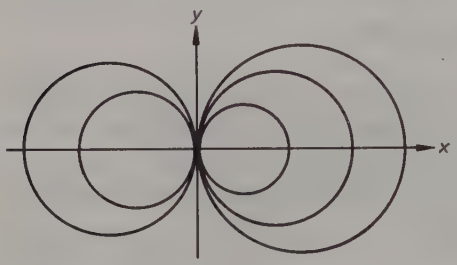
Example 1 Find the differential equation representing the family of curves, $y = Ax^3$, where A is an arbitrary constant.

If $y = Ax^3$, then $\dfrac{dy}{dx} = 3Ax^2$

$$\therefore \quad x\frac{dy}{dx} = 3Ax^3 = 3y$$

Hence the required differential equation is $x\dfrac{dy}{dx} = 3y$.

Example 2 Sketch some of the curves given by the equation $x^2 + y^2 - 2kx = 0$. Obtain the differential equation which represents this family of curves.



The equation $x^2 + y^2 - 2kx = 0$ may be written in the form

$$(x - k)^2 + y^2 = k^2.$$

Hence each curve in the family is a circle, centre $(k, 0)$ and radius $|k|$.

Differentiating the equation $\quad x^2 + y^2 - 2kx = 0$

we obtain $\quad 2x + 2y\dfrac{dy}{dx} - 2k = 0$

$$\therefore \quad 2x^2 + 2xy\frac{dy}{dx} = x^2 + y^2 \quad (= 2kx)$$

Hence this family of circles is represented by the differential equation

$$2xy\frac{dy}{dx} = y^2 - x^2.$$

Example 3 Given that $y = Ae^x + Be^{2x}$, find a differential equation connecting x and y which is independent of the arbitrary constants A and B.

If $y = Ae^x + Be^{2x}$, then $\quad \dfrac{dy}{dx} = Ae^x + 2Be^{2x}$

and $\quad \dfrac{d^2y}{dx^2} = Ae^x + 4Be^{2x}$

$$\therefore \quad \frac{dy}{dx} - y = Be^{2x} \quad \text{and} \quad \frac{d^2y}{dx^2} - y = 3Be^{2x}$$

$$\therefore \quad \frac{d^2y}{dx^2} - y = 3\left(\frac{dy}{dx} - y\right)$$

Hence the required differential equation is

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0.$$

In Examples 1 and 2 equations involving one arbitrary constant gave rise to differential equations of the first order. However, in Example 3 it was necessary to introduce two derivatives to eliminate the two arbitrary constants. This suggests that in general an equation containing n arbitrary constants should lead to an nth order differential equation. Conversely, it seems reasonable to assume that the general solution of a differential equation of order n should contain n arbitrary constants.

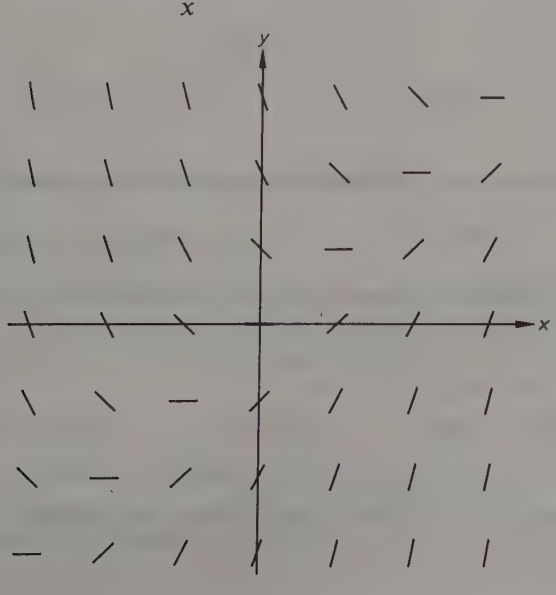
We now consider ways of sketching the solution curves of a differential equation when the general solution of the equation is not available. We take as an example the differential equation

$$\frac{dy}{dx} = x - y.$$

A general impression of the solution curves can be gained by calculating values of dy/dx at various points of the x, y plane, then plotting the directions on a sketch graph.

y \ x	−3	−2	−1	0	1	2	3
3	−6	−5	−4	−3	−2	−1	0
2	−5	−4	−3	−2	−1	0	1
1	−4	−3	−2	−1	0	1	2
0	−3	−2	−1	0	1	2	3
−1	−2	−1	0	1	2	3	4
−2	−1	0	1	2	3	4	5
−3	0	1	2	3	4	5	6

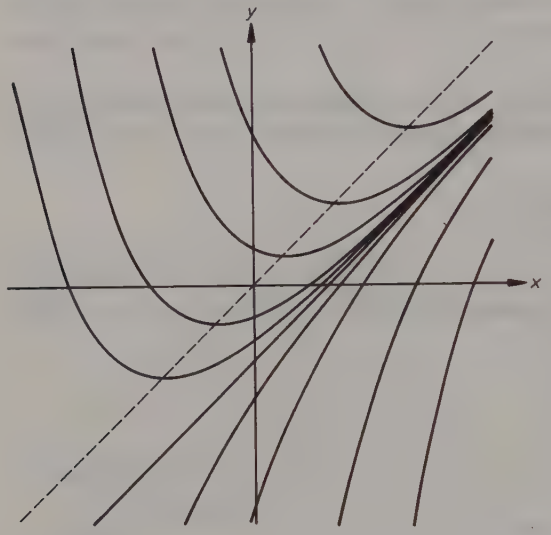
Table showing values of $\frac{dy}{dx}$



Sketch graph showing directions of the solution curves

We see that $\frac{dy}{dx} = 0$ on the line $y = x$. Above this line $\frac{dy}{dx} < 0$ and below the line $\frac{dy}{dx} > 0$. Hence the points on the line $y = x$ are minimum points on the solution curves.

The diagram also suggests that the line $y = x - 1$ could be one of the solution curves. Substituting $y = x - 1$ in the differential equation confirms that this is so.



It is now possible to make a rough sketch of the solution curves.

Exercise 11.1

1. Verify that the following are solutions of the differential equation

$$x\frac{dy}{dx} = x^2 + y,$$

(a) $y = x^2$, (b) $y = x^2 - x$, (c) $y = x^2 + 5x$.

2. Verify that the following are solutions of the differential equation $\frac{d^2y}{dx^2} = 1 - y$,

(a) $y = 1 + 2\cos x$, (b) $y = 1$, (c) $y = 1 - 3\sin x$.

3. Decide which of the following are solutions of the differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = 6y.$$

(a) $y = 3e^{2x}$, (b) $y = 6e^{x}$, (c) $y = 2e^{3x}$, (d) $y = e^{2x} - e^{-3x}$.

4. Given that $y = ax + b$ is a solution of the differential equation $\frac{dy}{dx} = 4x - 2y$, find a and b.

5. Given that $y = ax^2 + bx + c$ is a solution of the differential equation

$$\frac{d^2y}{dx^2} = y - x^2,$$

find a, b and c.

In questions 6 to 11 sketch some of the curves represented by the given equation and obtain a differential equation for y which does not contain the arbitrary constant A.

6. $y = x^3 + A$ 7. $y = x^2 - Ax$ 8. $y = A/x$

9. $y = Ae^{-x}$ 10. $x^2 + y^2 = A^2$ 11. $y^2 = 4Ax$

In questions 12 to 19 form differential equations by eliminating the arbitrary constant A, B and C.

12. $y = x + \frac{A}{x}$ 13. $y = x^2 \ln x + Ax^2$

14. $y^2 = A \cos x$ 15. $y = x^A$

16. $y = A \cos x + B \sin x$ 17. $y = Ae^{-x} + Be^{3x}$

18. $y^2 = Ax^2 + Bx + C$ 19. $y = (Ax + B)e^x + C$

In questions 20 to 25 sketch the solution curves of the given differential equation.

20. $\frac{dy}{dx} = \frac{x}{y}$ 21. $\frac{dy}{dx} = \frac{y}{x}$ 22. $\frac{dy}{dx} = -\frac{x}{2y}$

23. $\frac{dy}{dx} = x + y$ 24. $x\frac{dy}{dx} = x - y$ 25. $2y\frac{dy}{dx} = x + y$

11.2 Elementary methods of solution

Differential equations of the form $\frac{dy}{dx} = f(x)$ may be solved by direct integration.

Example 1 Solve the differential equation $\frac{dy}{dx} = 6x^2 - 5$.

By integration the general solution is $y = 2x^3 - 5x + c$, where c is an arbitrary constant.

Example 2 Solve the differential equation $\cos^2 x \dfrac{dy}{dx} = \sin x$.

Rearranging, we have: $\dfrac{dy}{dx} = \dfrac{\sin x}{\cos^2 x} = \sec x \tan x$

Integrating, we obtain the general solution $y = \sec x + c$, where c is an arbitrary constant.

Sometimes we require a solution which satisfies certain given conditions. We then use these conditions to determine the values of any arbitrary constants in the general solution.

Example 3 Solve the differential equation $e^x \dfrac{dy}{dx} + x = 0$, given that $y = 2$ when $x = 0$.

Rearranging we obtain: $\dfrac{dy}{dx} = -xe^{-x}$

Integrating by parts using the formula

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx,$$

where $u = x$ and $\dfrac{dv}{dx} = -e^{-x}$

$\dfrac{du}{dx} = 1$ $\qquad v = e^{-x}$

$$\int (-xe^{-x}) dx = xe^{-x} - \int e^{-x} dx$$

$$= xe^{-x} + e^{-x} + c$$

$\therefore$ the general solution of the differential equation is

$$y = xe^{-x} + e^{-x} + c.$$

Since $y = 2$ when $x = 0$, $\qquad 2 = 0 \times 1 + 1 + c \qquad \therefore \quad c = 1$

Hence the required solution is $y = xe^{-x} + e^{-x} + 1$.

When the integration of a differential equation leads to a general solution involving logarithmic functions, any restrictions on the values of x and y should be clearly stated.

Example 4 Solve the differential equation $\dfrac{dy}{dx} = \dfrac{x}{x-1}$, given that $y = 2$ when $x = 2$.

Rearranging: $\dfrac{dy}{dx} = \dfrac{x}{x-1} = 1 + \dfrac{1}{x-1}$

[Since we require a solution valid for $x = 2$, we will assume that x is greater than 1 when integrating.]

For $x > 1$, the general solution of this differential equation is

$$y = x + \ln(x - 1) + c$$

Since $y = 2$ when $x = 2$, $\quad 2 = 2 + \ln 1 + c \quad \therefore \quad c = 0$

Hence the required solution is $y = x + \ln(x - 1)$.

Second order differential equations of the form $\dfrac{d^2y}{dx^2} = f(x)$ can also be solved by direct integration.

Example 5 Solve the differential equation $\dfrac{d^2y}{dx^2} = 6x^3 + \dfrac{1}{x^3}$

Integrating we have: $\dfrac{dy}{dx} = \dfrac{3}{2}x^4 - \dfrac{1}{2x^2} + A$

Integrating again we obtain the general solution

$$y = \frac{3}{10}x^5 + \frac{1}{2x} + Ax + B,$$

where A and B are arbitrary constants.

Example 6 Solve the differential equation $\dfrac{d^2x}{dt^2} = -\sin t + \cos t$, subject to the conditions that $x = 0$, $\dfrac{dx}{dt} = 4$ when $t = 0$.

Integrating: $\dfrac{dx}{dt} = \cos t + \sin t + A$

$$x = \sin t - \cos t + At + B$$

Since $\dfrac{dx}{dt} = 4$ when $t = 0$, $4 = 1 + 0 + A \quad \therefore \quad A = 3$

Since $x = 0$ when $t = 0$, $\quad 0 = 0 - 1 + 0 + B \quad \therefore \quad B = 1$

Hence the required solution is $x = \sin t - \cos t + 3t + 1$.

To solve differential equations of the form $\dfrac{dy}{dx} = f(y)$, we use the fact that $\dfrac{dy}{dx} = 1 \Big/ \dfrac{dx}{dy}$.

Example 7 Solve the differential equation $\dfrac{dy}{dx} = y^2$.

The equation may be written as: $\dfrac{dx}{dy} = \dfrac{1}{y^2}$

Integrating with respect to y: $x = -\dfrac{1}{y} + c$

$\therefore$ $\dfrac{1}{y} = c - x$

Hence the general solution is $y = \dfrac{1}{c - x}$.

A substitution is sometimes used to reduce a differential equation to a simpler form.

Example 8 Use the substitution $y = vx$, where v is a function of x, to solve the differential equation $x\dfrac{dy}{dx} = x + y$, given that $y = -1$ when $x = 1$.

If $y = vx$, then using the product rule, $\dfrac{dy}{dx} = v + \dfrac{dv}{dx}$

Substituting in the given differential equation we have,

$$x\left(v + x\frac{dv}{dx}\right) = x + vx$$

i.e. $vx + x^2\dfrac{dv}{dx} = x + vx$

$\therefore$ $\dfrac{dv}{dx} = \dfrac{1}{x}$

For $x > 0$, the general solution of this differential equation is

$v = \ln x + c$

$\therefore$ $y = vx = x(\ln x + c)$

Since $y = -1$ when $x = 1$, $-1 = 1(0 + c)$ $\therefore$ $c = -1$

Hence the required solution is $y = x(\ln x - 1)$.

In Example 8 we again dealt with an integration involving logarithms by considering a restricted set of values of x. However, in some problems it is better to avoid placing restrictions on the values of x and y by using modulus signs, as described in §3.4.

For instance, in the case of the differential equation

$$\frac{dy}{dx} = \frac{1}{x - 5}$$

the general solution may be given as

$$y = \ln|x - 5| + c \qquad (1)$$

Suppose now that we require the particular solution which satisfies the condition $y = -2$ when $x = 6$.
Substituting these values in the general solution (1),

$$-2 = \ln 1 + c \qquad \therefore \quad c = -2$$

Hence $$y = \ln|x - 5| - 2.$$

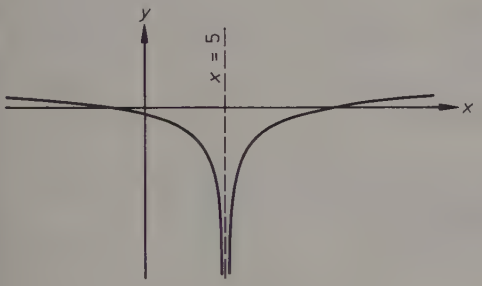
But, using the method of Examples 4 and 8,

for $x > 5$, $$y = \ln(x - 5) + c \tag{2}$$

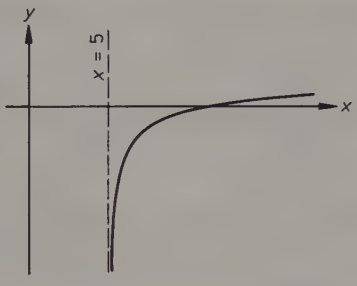
As before, since $y = -2$ when $x = 6$, we have $c = -2$

Hence $$y = \ln(x - 5) - 2.$$

Thus, although equation (1) provides a neat general solution, equation (2) leads to a simpler particular integral.



$y = \ln|x - 5| - 2$



$y = \ln(x - 5) - 2$

The sketch graphs show that solution (1) represents a family of curves with two branches, whereas solution (2) produces single curves lying to the right of the line $x = 5$. A question requiring a particular integral is probably best answered by a solution of type (2).

Exercise 11.2

Solve the following differential equations giving y in terms of x.

1. $\dfrac{dy}{dx} = 3x^2 + 1$

2. $\dfrac{dy}{dx} = \cos \frac{1}{2}x$

3. $x\dfrac{dy}{dx} = x^2 + 1$

4. $(x - 1)\dfrac{dy}{dx} = x + 1$

5. $\dfrac{d^2y}{dx^2} + \cos x = 2x$

6. $e^x\dfrac{d^2y}{dx^2} = 1 - e^{2x}$

7. $\dfrac{dy}{dx} + 2y^2 = 0$

8. $\dfrac{dy}{dx} = \cos^2 y$

9. $\dfrac{dy}{dx} = e^{-y}$

10. $3\dfrac{dy}{dx} = y^4$

Solve the following differential equations, subject to the given conditions.

11. $\dfrac{dy}{dx} = (x + 1)^2$; $y = 0$ when $x = 2$

12. $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{(2x + 3)}}$; $y = 4$ when $x = 3$

13. $\sec x \dfrac{dy}{dx} = x$; $y = 0$ when $x = \pi$

14. $(1 + x^2)\dfrac{dy}{dx} = 2x$; $y = 0$ when $x = -1$

15. $\dfrac{d^2y}{dx^2} = \sin 2x$; $y = 0$, $\dfrac{dy}{dx} = -\dfrac{1}{4}$ when $x = 0$

16. $x^2\dfrac{d^2y}{dx^2} = 2(x^2 - 1)$; $y = \dfrac{dy}{dx} = 0$ when $x = 1$

17. $\dfrac{dy}{dx} + 2y = 0$; $y = 3$ when $x = 0$

18. $\dfrac{dy}{dx} = y - 1$; $y = 0$ when $x = 0$

19. $\dfrac{dy}{dx} = 1 + y^2$; $y = 1$ when $x = 0$

20. $2\dfrac{dy}{dx} = y^2 - 1$; $y = 2$ when $x = 0$

21. Use the substitution $y = 1/z$ to solve the differential equation $x^2\dfrac{dy}{dx} = y^2$.

22. Use the substitution $y = vx$, where v is a function of x, to solve the differential equation $x\dfrac{dy}{dx} - y = x^2 \cos x$.

23. Use the substitution $y = vx^2$, where v is a function of x, to solve the differential equation $x\dfrac{dy}{dx} - 2y = x$.

24. Use the substitution $dy/dx = p$, where p is a function of x, to solve the differential equation $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 1$, given that $y = \dfrac{dy}{dx} = 2$ when $x = 1$.

25. If $z = xe^y$, where y is a function of x, find an expression for dz/dx. Hence find y in terms of x given that $xe^y \dfrac{dy}{dx} + e^y = 2x$ and that $y = 0$ when $x = 2$.

11.3 Equations with separable variables

If a first order differential equation can be expressed in the form

$$f(y)\frac{dy}{dx} = g(x)$$

then the variables x and y are said to be *separable.*
Integrating both sides of the equation with respect to x:

$$\int f(y)\frac{dy}{dx}\,dx = \int g(x)\,dx$$

As shown in §4.3, $\dfrac{dy}{dx}\,dx$ may be replaced by dy.

Thus $$\int f(y)\,dy = \int g(x)\,dx$$

Example 1 Find the general solution of the equation $2y\dfrac{dy}{dx} = \cos x$.

Integrating both sides of the equation: $\displaystyle\int 2y\,dy = \int \cos x\,dx$

Hence the general solution of the equation is

$$y^2 = \sin x + c.$$

Example 2 Find the general solution of the equation

$$\frac{dy}{dx} = 2xy^2 - y^2.$$

Rearranging: $\dfrac{dy}{dx} = (2x - 1)y^2$

$$\frac{1}{y^2}\frac{dy}{dx} = 2x - 1$$

$$\therefore \quad \int \frac{1}{y^2}\,dy = \int (2x - 1)\,dx \qquad \therefore \quad -\frac{1}{y} = x^2 - x + c$$

Hence the general solution of the equation is

$$y = -1/(x^2 - x + c).$$

Example 3 Find y in terms of x given that $2\dfrac{dy}{dx} = e^{x-2y}$ and that $y = 1$ when $x = 2$.

Rearranging: $$2\frac{dy}{dx} = e^x \times e^{-2y}$$

$$2e^{2y}\frac{dy}{dx} = e^x$$

$\therefore$ $$\int 2e^{2y}\,dy = \int e^x\,dx$$

$\therefore$ $$e^{2y} = e^x + c$$

Since $y = 1$ when $x = 2$, $\quad e^2 = e^2 + c \quad \therefore \quad c = 0$

Thus $$e^{2y} = e^x$$

$\therefore$ $$2y = x$$

Hence $$y = \tfrac{1}{2}x.$$

Example 4 Express y in terms of x given that $\dfrac{dy}{dx} = xy - x$ and that $y = 4$ when $x = 0$.

Rearranging: $$\frac{dy}{dx} = x(y-1)$$

$$\frac{1}{(y-1)}\frac{dy}{dx} = x$$

$\therefore$ $$\int \frac{1}{y-1}\,dy = \int x\,dx$$

$\therefore$ $$\ln|y-1| = \tfrac{1}{2}x^2 + c$$

$\therefore$ $$|y-1| = e^{x^2/2+c}$$

Thus $$y - 1 = Ae^{x^2/2}, \quad \text{where} \quad |A| = e^c$$

Since $y = 4$ when $x = 0$, $3 = Ae^0 = A$

Hence $y - 1 = 3e^{x^2/2}$

i.e. $$y = 1 + 3e^{x^2/2}.$$

Example 5 Find the general solution of the equation

$$(x^2 - 1)\frac{dy}{dx} = xy.$$

Rearranging: $$\frac{1}{y}\frac{dy}{dx} = \frac{x}{x^2-1}$$

$\therefore$ $$\int \frac{1}{y}\,dy = \int \frac{x}{x^2-1}\,dx$$

$$\therefore \quad \ln|y| = \tfrac{1}{2}\ln|x^2 - 1| + c$$
$$\therefore \quad 2\ln|y| - \ln|x^2 - 1| = 2c$$

Using the laws of logarithms:

$$\ln\left|\frac{y^2}{x^2 - 1}\right| = 2c$$

$$\therefore \quad \frac{y^2}{x^2 - 1} = A, \quad \text{where} \quad \ln|A| = 2c$$

Hence the general solution of the equation is $y^2 = A(x^2 - 1)$.

Example 6 Find the general solution of the equation

$$2x\frac{dy}{dx} - (2y + 1)(x + 1) = 0$$

Rearranging:
$$2x\frac{dy}{dx} = (2y + 1)(x + 1)$$

$$\frac{2}{(2y + 1)}\frac{dy}{dx} = 1 + \frac{1}{x}$$

$$\therefore \quad \int \frac{2}{2y + 1}\,dy = \int\left(1 + \frac{1}{x}\right)dx$$

$$\therefore \quad \ln|2y + 1| = x + \ln|x| + c$$
$$\therefore \quad \ln|2y + 1| - \ln|x| = x + c$$

i.e.
$$\ln\left|\frac{2y + 1}{x}\right| = x + c$$

$$\therefore \quad \left|\frac{2y + 1}{x}\right| = e^{x+c}$$

Thus
$$\frac{2y + 1}{x} = Ae^x, \quad \text{where} \quad |A| = e^c$$

$$\therefore \quad 2y + 1 = Axe^x$$

Hence the required general solution is $y = \frac{1}{2}(Axe^x - 1)$.

Example 7 Find y in terms of x given that $x(x - 1)\dfrac{dy}{dx} = y$ and that $y = 1$ when $x = 2$.

Rearranging:
$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x(x - 1)}$$

$$\therefore \qquad \int \frac{1}{y}\,dy = \int \frac{1}{x(x-1)}\,dx$$

$$\therefore \qquad \int \frac{1}{y}\,dy = \int \left\{\frac{1}{x-1} - \frac{1}{x}\right\} dx$$

$$\therefore \qquad \ln|y| = \ln|x-1| - \ln|x| + c$$

i.e. $\ln|y| - \ln|x-1| + \ln|x| = c$

Thus $$\ln\left|\frac{xy}{x-1}\right| = c$$

$$\therefore \qquad \frac{xy}{x-1} = A, \quad \text{where} \quad \ln|A| = c$$

Since $y = 1$ when $x = 2$, $\qquad 2 = A$

Hence $$\frac{xy}{x-1} = 2$$

i.e. $$y = 2\left(1 - \frac{1}{x}\right).$$

Exercise 11.3

In questions 1 to 10 find the general solutions of the given differential equations, expressing y in terms of x in each case.

1. $3y^2\dfrac{dy}{dx} = 2x - 1$

2. $\dfrac{dy}{dx} = 6xy^2$

3. $\dfrac{dy}{dx} = e^y \sin x$

4. $\dfrac{dy}{dx} = e^{x-y}$

5. $\dfrac{dy}{dx} = x \sec y$

6. $\dfrac{dy}{dx} = 3\cos^2 y$

7. $x\dfrac{dy}{dx} = y$

8. $(1-x)\dfrac{dy}{dx} = y$

9. $\dfrac{dy}{dx} = \dfrac{4xy}{x^2+1}$

10. $\dfrac{dy}{dx} = \dfrac{2y}{x^2-1}$

In questions 11 to 20 find expressions for y in terms of x.

11. $x^2\dfrac{dy}{dx} - y^2 = 0$; $y = -1$ when $x = 1$.

12. $\frac{dy}{dx} + 2xy = 0$; $y = 5$ when $x = 0$.

13. $\cot x \frac{dy}{dx} = y$; $y = 2$ when $x = 0$.

14. $\frac{dy}{dx} = xe^{-2y}$; $y = 0$ when $x = 0$.

15. $\frac{dy}{dx} - 2xy = 2x$; $y = 1$ when $x = 0$.

16. $x\frac{dy}{dx} = xy + y$; $y = 1$ when $x = 1$.

17. $(1 + x^3)\frac{dy}{dx} = 3x^2 \tan y$; $y = \frac{1}{2}\pi$ when $x = 0$.

18. $x \cos y \frac{dy}{dx} = 1 + \sin y$; $y = 0$ when $x = 1$.

19. $x\frac{dy}{dx} = 2y(y - 1)$; $y = 2$ when $x = \frac{1}{2}$.

20. $2x\frac{dy}{dx} = 1 - y^2$; $y = 0$ when $x = 1$.

In questions 21 to 24 find the general solutions of the given differential equations, expressing y in terms of x.

21. $(1 - x)\frac{dy}{dx} = xy$

22. $(x^2 - 1)\frac{dy}{dx} = (x^2 + 1)y$

23. $\frac{dy}{dx} = e^x(1 + y^2)$

24. $e^y\frac{dy}{dx} + 2x = 2xe^y$

25. Find y in terms of x given that $e^{2x}y\frac{dy}{dx} + 2x = 0$ and that $y = 1$ when $x = 0$.

26. Find y in terms of x given that $xy\frac{dy}{dx} = \sqrt{(y^2 - 9)}$ and that $y = 5$ when $x = e^4$.

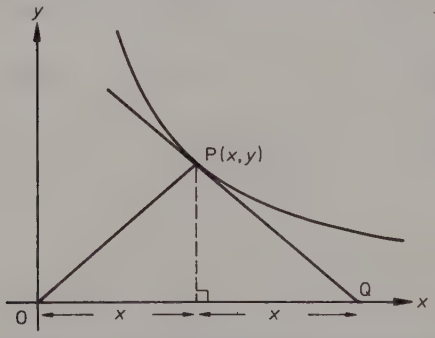
27. Use the substitution $x + y = z$ to obtain the general solution of the differential equation $(x + y - 1)\frac{dy}{dx} = x - y + 1$.

28. Use the substitution $y = vx$, where v is a function of x, to obtain the general solution of the differential equation $xy\dfrac{dy}{dx} = 2x^2 - y^2$.

11.4 Formulation of differential equations

In our first example information about the tangent to a curve is expressed in the form of a differential equation. This equation is then solved to find the equation of the curve.

Example 1 The tangent at any point P on a curve in the first quadrant cuts the x-axis at Q. Given that $OP = PQ$, where O is the origin, and that the point $(1, 4)$ lies on the curve, find the equation of the curve.



Since $OP = PQ$ the triangle OPQ is isosceles. Thus if P is the point (x, y), the coordinate of Q must be $(2x, 0)$.

$\therefore$ the gradient of PQ

$$= \frac{y - 0}{x - 2x} = -\frac{y}{x}$$

However, since PQ is the tangent at P its gradient is also given by dy/dx.

$$\therefore \quad \frac{dy}{dx} = -\frac{y}{x}$$

$$\text{i.e.} \quad \frac{1}{y}\frac{dy}{dx} = -\frac{1}{x}$$

$$\therefore \int \frac{1}{y}dy = -\int \frac{1}{x}dx$$

Since the curve lies in the first quadrant, $x > 0$ and $y > 0$

$$\therefore \quad \ln y = -\ln x + c$$

$$\text{i.e.} \quad \ln y + \ln x = c$$

$$\therefore \quad \ln xy = c$$

$$\therefore \quad xy = k, \quad \text{where} \quad \ln k = c$$

Since the curve passes through the point $(1, 4)$, $1 \times 4 = k$

Hence the equation of the curve is $xy = 4$.

Many scientific laws can be expressed as differential equations. For instance, as we saw in §3.1, Newton's law of cooling and certain laws of growth and decay lead to

differential equations of the form $dx/dt = \pm kx$, where k is a positive constant. The solutions of such equations are used to make predictions about the behaviour of the variables involved.

Example 2 A liquid is being heated in an oven maintained at a constant temperature of 180°C. It is assumed that the rate of increase in the temperature of the liquid is proportional to $(180 - \theta)$, where θ°C is the temperature of the liquid at time t minutes. If the temperature of the liquid rises from 0°C to 120°C in 5 minutes, find the temperature of the liquid after a further 5 minutes.

The rate of increase in the temperature of the liquid is $d\theta/dt$,

$$\therefore \quad \frac{d\theta}{dt} = k(180 - \theta), \text{ where } k \text{ is constant.}$$

Rearranging, $$\frac{1}{(180 - \theta)}\frac{d\theta}{dt} = k$$

$$\therefore \quad \int \frac{1}{180 - \theta} d\theta = \int k\, dt$$

$$\therefore \quad \int \frac{-1}{180 - \theta} d\theta = -\int k\, dt$$

Thus for $\theta < 180$, $\ln(180 - \theta) = -kt + c$

$\therefore$ $180 - \theta = e^{-kt+c}$

Writing $A = e^c$, $180 - \theta = Ae^{-kt}$

Hence $\theta = 180 - Ae^{-kt}$

Letting $t = 0$ when $\theta = 0$, $0 = 180 - A$

$\therefore$ $\theta = 180(1 - e^{-kt})$

Since $\theta = 120$ when $t = 5$, $120 = 180(1 - e^{-5k})$

$\therefore$ $\frac{2}{3} = 1 - e^{-5k}$

i.e. $e^{-5k} = \frac{1}{3}$

Hence when $t = 10$, $\theta = 180(1 - e^{-10k})$

$= 180(1 - \{e^{-5k}\}^2)$

$= 180(1 - \{\frac{1}{3}\}^2) = 160$

Thus the temperature of the liquid after a further 5 minutes is 160°C.

Exercise 11.4

1. The tangent at any point P on a curve cuts the x-axis at the point Q. Given that $\angle OPQ = 90°$, where O is the origin, and that the point $(1, 2)$ lies on the curve, find the equation of the curve.

2. The tangent at any point P on a curve cuts the x-axis at A and the y-axis at B. Given that $2\vec{AP} = \vec{PB}$ and that the curve passes through the point $(1, 1)$, find the equation of the curve.

3. At time t, the rate of increase in the concentration C of micro-organisms in a controlled environment is equal to k times the concentration, where k is a positive constant. When $t = 0, C = C_0$. Write down a differential equation involving C, t and k. Hence find C in terms of C_0, t and k. Find also, in terms of k, the time at which the concentration has increased by 50% from its value at $t = 0$. (L)

4. A body is placed in a room which is kept at a constant temperature. The temperature of the body falls at a rate $k\theta$ C per minute, where k is a constant and θ is the difference between the temperature of the body and that of the room at time t. Express this information in the form of a differential equation and hence show that $\theta = \theta_0 e^{-kt}$, where θ_0 is the temperature difference at time $t = 0$. The temperature of the body falls 5°C in the first minute and 4°C in the second minute. Show that the fall of temperature in the third minute is 3·2°C. (L)

5. A rectangular tank with its base horizontal is filled with water to a depth h at time $t = 0$. Water leaks out of the tank from a small hole in the base at a rate proportional to the square root of the depth of the water. If the depth of water is $\frac{1}{2}h$ at time t, find the further time it will take before the tank is empty. (O)

6. A radioactive substance decays so that the rate of decrease of mass at any time is proportional to the mass present at that time. Denoting by x the mass remaining at time t, write down a differential equation satisfied by x, and show that $x = x_0 e^{-kt}$, where x_0 is the initial mass and k is a constant. The mass is reduced to 4/5 of its initial value in 30 days. Calculate, correct to the nearest day, the time required for the mass to be reduced to half its initial value. A mass of 625 milligrammes of the substance is prepared. Determine the mass which is present 90 days after the preparation. (JMB)

7. According to Newton's law, the rate of cooling of a body in air is proportional to the difference between the temperature T of the body and the temperature T_0 of the air. If the air temperature is kept constant at 20°C and the body cools from 100°C to 60°C in 20 minutes, in what further time will the body cool to 30°C? (O)

8. Two liquids, X and Y, are flowing into a trough at the constant rate of 10 and 20 litres per minute respectively. The liquid in the trough is stirred continuously and pumped out at the rate of 30 litres per minute. Initially the trough contains 200 litres of X and 100 litres of Y. After t minutes the tank contains x litres of X. By considering the change in x in a small interval of time δt, show that

$$\frac{dx}{dt} = 10 - \frac{x}{10}.$$

Hence find an expression for x in terms of t. Find, correct to the nearest litre, the quantity of liquid X in the trough after 10 minutes. After how long, to the nearest second, will there be less of liquid X in the mixture than of liquid Y?

9. A tank being cleaned initially contains 3900 litres of water and 100 kg of dissolved dye. The mixture is kept uniform by stirring. Pure water is run in at the rate

of 130 litres per minute and the mixture is removed at the same rate. If the mass of dissolved dye remaining after t minutes is x kg, show that

$$\frac{dx}{dt} = -\frac{x}{30}.$$

How long will it be before half the dye has been removed? Determine how much dye remains after 30 minutes. (L)

10. In a certain chemical reaction, in which a compound X is formed from a compound Y, the masses of X, Y present at time t are x, y respectively. The sum of the masses of X and Y is a, where a is constant, and at any time the rate at which x is increasing is proportional to the product of the two masses at that time. Show that $\frac{dx}{dt} = kx(a - x)$, where k is constant. If $x = a/5$ at $t = 0$ and $x = a/2$ at $t = \ln 2$, show that $k = 2/a$. Hence find t, correct to three significant figures, when $x = 99a/100$. (O & C)

11. A plant grows in a pot which contains a volume V of soil. At time t the mass of the plant is m and the volume of soil utilised by the roots is αm, where α is constant. The rate of increase of the mass of the plant is proportional to the mass of the plant times the volume of soil not yet utilised by the roots. Obtain a differential equation for m, and verify that it can be written in the form $V\beta\frac{dt}{dm} = \frac{1}{m} + \frac{\alpha}{V - \alpha m}$, where β is a constant. The mass of the plant is initially $V/4\alpha$. Find, in terms of V and β, the time taken for the plant to double its mass. Find also the mass of the plant at time t. (JMB)

12. At any instant, a spherical meteorite is gaining mass because of two effects: (i) mass is condensing onto it at a rate which is proportional to the surface area of the meteorite at that instant, (ii) the gravitational field of the meteorite attracts mass onto itself, the rate being proportional to the meteorite mass at that instant. Assuming that the two effects can be added together and that the meteorite remains spherical and of constant density, show that its radius r at time t satisfies the differential equation $\frac{dr}{dt} = A + Br$, where A and B are constants. If $r = r_0$ at $t = 0$, show that $r = r_0e^{Bt} + \frac{A}{B}(e^{Bt} - 1)$. (L)

11.5 The equations $d^2y/dx^2 \pm n^2y = \text{constant}$

We now consider some second order differential equations which have important applications especially in mechanics.

Type 1: $\dfrac{d^2y}{dx^2} + n^2y = 0$, where n is constant.

Multiplying both sides by $2\dfrac{dy}{dx}$ we obtain:

$$2\frac{dy}{dx}\frac{d^2y}{dx^2} + n^2\left(2y\frac{dy}{dx}\right) = 0.$$

But $$2\frac{dy}{dx}\frac{d^2y}{dx^2} = \frac{d}{dx}\left\{\left(\frac{dy}{dx}\right)^2\right\} \quad \text{and} \quad 2y\frac{dy}{dx} = \frac{d}{dx}(y^2),$$

so integrating both sides with respect to x:

$$\left(\frac{dy}{dx}\right)^2 + n^2y^2 = \text{constant}$$

Letting $y = a$ when $\dfrac{dy}{dx} = 0$, this becomes

$$\left(\frac{dy}{dx}\right)^2 + n^2y^2 = n^2a^2$$

$$\therefore \qquad \frac{dy}{dx} = \pm n\sqrt{(a^2 - y^2)}$$

Hence $$\int \frac{1}{\sqrt{(a^2 - y^2)}}\,dy = \pm\int n\,dx$$

$$\therefore \qquad \sin^{-1}\frac{y}{a} = \pm(nx + b)$$

$$\therefore \qquad \frac{y}{a} = \sin\{\pm(nx + b)\}$$

$$\therefore \qquad y = \pm a\sin(nx + b)$$

Since a is an arbitrary constant which may take any real value, positive or negative, there is no loss of generality in writing:

$$y = a\sin(nx + b)$$

Letting $b = c + \frac{1}{2}\pi$, we obtain:

$$y = a\sin(nx + c + \tfrac{1}{2}\pi)$$

i.e. $$y = a\cos(nx + c)$$

But $$a\cos(nx + c) = a\cos nx\cos c - a\sin nx\sin c,$$

thus the solution can also be expressed in the form

$$y = A\cos nx + B\sin nx$$

Thus the general solution of the differential equation

$$\frac{d^2y}{dx^2} + n^2y = 0$$

may be expressed in three possible forms:

$$y = a\sin(nx + b), \quad y = a\cos(nx + c)$$

or $$y = A\cos nx + B\sin nx,$$

where a, b, c, A and B are arbitrary constants.

Example 1 Solve the differential equation $\frac{d^2y}{dx^2} + 9y = 0$ given that $y = 1$ when $x = 0$ and that $y = 2$ when $x = \frac{1}{2}\pi$.

The general solution of the given equation is

$$y = A\cos 3x + B\sin 3x$$

Since $y = 1$ when $x = 0$, $\quad 1 = A + 0$ i.e. $A = 1$

Since $y = 2$ when $x = \frac{1}{2}\pi$, $\quad 2 = 0 - B$ i.e. $B = -2$

Hence the required solution is $y = \cos 3x - 2\sin 3x$.

Example 2 Solve the differential equation $9\frac{d^2y}{dx^2} + 4y = 0$ given that $y = \frac{dy}{dx} = 2$ when $x = 0$.

Rearranging the equation, $\quad \frac{d^2y}{dx^2} + \frac{4}{9}y = 0$

Hence the general solution is

$$y = A\cos\tfrac{2}{3}x + B\sin\tfrac{2}{3}x$$

$$\therefore \quad \frac{dy}{dx} = -\tfrac{2}{3}A\sin\tfrac{2}{3}x + \tfrac{2}{3}B\cos\tfrac{2}{3}x$$

Since $y = 2$ when $x = 0$, $\quad 2 = A$

Since $\frac{dy}{dx} = 2$ when $x = 0$, $\quad 2 = \frac{2}{3}B$

$\therefore \quad A = 2$ and $B = 3$

Hence the required solution is $y = 2\cos\frac{2}{3}x + 3\sin\frac{2}{3}x$.

Type 2: $\frac{d^2y}{dx^2} + n^2y = k$, where n and k are constant.

Rearranging: $$\frac{d^2y}{dx^2} + n^2\left(y - \frac{k}{n^2}\right) = 0$$

Letting $z = y - \frac{k}{n^2}$, we find that $\frac{d^2z}{dx^2} = \frac{d^2y}{dx^2}$

so the equation becomes: $\dfrac{d^2z}{dx^2} + n^2z = 0$

$\therefore$ the general solution is $z = A\cos nx + B\sin nx$

i.e. $y - \dfrac{k}{n^2} = A\cos nx + B\sin nx$

Thus the general solution of the differential equation

$$\frac{d^2y}{dx^2} + n^2y = k$$

is $y = A\cos nx + B\sin nx + k/n^2$,

where A and B are arbitrary constants.

Type 3: $\dfrac{d^2y}{dx^2} - n^2y = 0$, where n is constant.

Since $y = e^{nx}$ is a solution of this equation, we try the substitution $y = e^{nx}v$, where v is a function of x.

Differentiating: $\dfrac{dy}{dx} = e^{nx}\dfrac{dv}{dx} + ne^{nx}v$

$\therefore$ $\dfrac{d^2y}{dx^2} = e^{nx}\dfrac{d^2v}{dx^2} + 2ne^{nx}\dfrac{dv}{dx} + n^2e^{nx}v$

Hence the differential equation becomes

$$e^{nx}\frac{d^2v}{dx^2} + 2ne^{nx}\frac{dv}{dx} + n^2e^{nx}v - n^2e^{nx}v = 0$$

i.e. $\dfrac{d^2v}{dx^2} + 2n\dfrac{dv}{dx} = 0$

Integrating with respect to x,

$$\frac{dv}{dx} + 2nv = a$$

Rearranging: $\dfrac{dv}{dx} = a - 2nv$

$\therefore$ $\dfrac{1}{(a-2nv)}\dfrac{dv}{dx} = 1$

$\therefore$ $\displaystyle\int \frac{-2n}{(a-2nv)}\,dv = \int(-2n)\,dx$

$\therefore$ $\ln|a - 2nv| = -2nx + c$

i.e. $|a - 2nv| = e^{-2nx+c}$

Thus $a - 2nv = ke^{-2nx}$, where $|k| = e^c$.

$\therefore$ $$v = \frac{a}{2n} - \frac{k}{2n}e^{-2nx}$$

$\therefore$ $$y = e^{nx}v = \frac{a}{2n}e^{nx} - \frac{k}{2n}e^{-nx}$$

Thus the general solution of the differential equation

$$\frac{d^2y}{dx^2} - n^2y = 0$$

is of the form $y = Ae^{nx} + Be^{-nx}$,

where A and B are arbitrary constants.

Example 3 Solve the differential equation $\frac{d^2y}{dx^2} - 100y = 0$, given that $y = 6$, $\frac{dy}{dx} = 40$ when $x = 0$.

The general solution of the given equation is

$$y = Ae^{10x} + Be^{-10x}$$

$\therefore$ $$\frac{dy}{dx} = 10Ae^{10x} - 10Be^{-10x}$$

Since $y = 6$ when $x = 0$, $6 = A + B$

Since $\frac{dy}{dx} = 40$ when $x = 0$, $40 = 10A - 10B$

$\therefore$ $A = 5$ and $B = 1$

Hence the required solution is $y = 5e^{10x} + e^{-10x}$.

Type 4: $\frac{d^2y}{dx^2} - n^2y = k$, where n and k are constant.

By the method applied to equations of type 2 it can be shown that

$$y + \frac{k}{n^2} = Ae^{nx} + Be^{-nx}$$

Thus the general solution of the differential equation

$$\frac{d^2y}{dx^2} - n^2y = k$$

is $$y = Ae^{nx} + Be^{-nx} - \frac{k}{n^2},$$

where A and B are arbitrary constants.

Exercise 11.5

[In this exercise the general solutions of the given differential equations may be quoted without proof.]

Write down the general solutions of the following differential equations:

1. $\frac{d^2y}{dx^2} = -16y$

2. $4\frac{d^2y}{dx^2} + 7y = 0$

3. $\frac{d^2y}{dx^2} + 9y = 18$

4. $\frac{d^2y}{dx^2} = 10 - 10y$

5. $\frac{d^2y}{dx^2} = 36y$

6. $\frac{d^2y}{dx^2} - 8y = 0$

7. $9\frac{d^2y}{dx^2} = 1 + 27y$

8. $\frac{d^2y}{dx^2} - 4y + 1 = 0$

Solve the following differential equations subject to the given conditions.

9. $\frac{d^2y}{dx^2} + 4y = 0$; $y = 4$ when $x = 0$ and $y = 3$ when $x = 3\pi/4$.

10. $\frac{d^2y}{dx^2} - 16y = 0$; $y = 10$ and $\frac{dy}{dx} = -16$ when $x = 0$.

11. $36\frac{d^2y}{dx^2} + y = 0$; $y = 6$ and $\frac{dy}{dx} = 0$ when $x = 0$.

12. $9\frac{d^2y}{dx^2} + 16y = 0$; $y = -5$ when $x = 0$ and $y = 4$ when $x = 3\pi/8$.

13. $4\frac{d^2y}{dx^2} - 25y = 0$; $y = -9$ and $\frac{dy}{dx} = 12\frac{1}{2}$ when $x = 0$.

14. $\frac{d^2y}{dx^2} - 12y = 0$; $y = 4/e^2$ when $x = 1/\sqrt{3}$ and $y = 4$ when $x = 0$.

15. $9\frac{d^2y}{dx^2} - 49y = 28$; $y = 8\frac{3}{7}$ and $\frac{dy}{dx} = -7$ when $x = 0$.

16. $4\frac{d^2y}{dx^2} + 25y = 40$; $y = 5{\cdot}6$ and $\frac{dy}{dx} = 22{\cdot}5$ when $x = 0$.

17. $\frac{d^2y}{dx^2} + 3y = 6$; $y = 2$ when $x = 0$ and $y = 3$ when $x = \pi/2\sqrt{3}$.

18. $16\dfrac{d^2y}{dx^2} - 81y = 243$; $y = -1$ and $\dfrac{dy}{dx} = 0$ when $x = 0$.

19. $\dfrac{d^2y}{dx^2} - 18y = 54$; $y = -3 + \sqrt{2}$ and $\dfrac{dy}{dx} = 3$ when $x = 0$.

20. $\dfrac{d^2y}{dx^2} + 12y = 6$; $\dfrac{dy}{dx} = 24$ when $x = \pi/4\sqrt{3}$ and $\dfrac{dy}{dx} = -18$ when $x = 0$.

Exercise 11.6 (miscellaneous)

1. Sketch some of the straight lines represented by the equation $y = k(2x - k)$, where k is any constant. Obtain the differential equation for this family of lines.

2. Find a second order differential equation of which the general solution is $y = Axe^x + Be^{-x}$, where A and B are arbitrary constants.

3. The differential equation of a family of curves is $\dfrac{dy}{dx} = y - x^2$. Find the equations of the loci of points where $\dfrac{dy}{dx} = 0$ and where $\dfrac{d^2y}{dx^2} = 0$. Hence sketch the integral curves of the differential equation.

4. (a) Find, in any form, the solution of the differential equation $\dfrac{du}{dt} - 7u + 3 = 0$, given that $u = 0$ when $t = 5$.

   (b) Solve the differential equation $\dfrac{dy}{dx} = x\sqrt{(x^2 + 9)}$ given that $y = -9$ when $x = 0$. (C)

5. (a) Solve the differential equation $\dfrac{dx}{dt} = \dfrac{1}{x + 2} - \dfrac{3}{3x + 5}$, given that $x = 1$ when $t = 2$.

   (b) Solve the differential equation $\dfrac{dy}{dx} = \dfrac{2y^2 + 3}{4y}$, given that $y = 1$ when $x = -1$. Given your answer in a form expressing y in terms of x. (C)

6. The temperature y degrees of a body, t minutes after being placed in a certain room, satisfies the differential equation $6\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} = 0$. By using the substitution $z = dy/dt$, or otherwise, find y in terms of t, given that $y = 63$ when $t = 0$ and $y = 36$ when $t = 6 \ln 4$. Find after how many minutes the rate of cooling of the body will have fallen below one degree per minute, giving your answer correct to the nearest minute. How cool does the body get? (O&C)

7. By substituting $z = (1 + x)y$, solve the differential equation

$$\frac{dy}{dx} + \frac{y}{1+x} = \cos x,$$

given that $y = 0$ when $x = 0$. In your answer give y in terms of x.

8. (a) Solve the differential equation $xy\dfrac{dy}{dx} = 1 + y^2$, given that, when $x = 2$, $y = 0$. Find the equation of the tangent to the curve at the point in the first quadrant where $x = 4$.
   (b) If $Ax^2 - By^2 = C$, where A, B and C are constants, prove that

$$xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0. \qquad \text{(SU)}$$

9. (a) Solve the differential equation $dy/dx = \sin x \cos^2 y$, given that, when $x = \pi/2$, $y = \pi/4$.
   (b) A curve passes through the origin and is such that $dy/dx = e^{2x-y}$. Obtain the equation of the curve in the form $y = f(x)$ and give a rough sketch of its graph. (SU)

10. (i) Find y in terms of x given that $(1 + x)\dfrac{dy}{dx} = (1 - x)y$ and that $y = 1$ when $x = 0$.
    (ii) A curve passes through $(2, 2)$ and has gradient at (x, y) given by the differential equation $ye^{y^2}\dfrac{dy}{dx} = e^{2x}$. Find the equation of the curve. Show that the curve also passes through the point $(1, \sqrt{2})$ and sketch the curve. (L)

11. If y is a solution of the differential equation $\dfrac{dy}{dx} = \dfrac{1+y^2}{1+x^2}$ deduce that

$$y = x + k(1 + xy)$$

where k is a constant.

(You may assume that $\tan^{-1} a - \tan^{-1} b = \tan^{-1}\left(\dfrac{a-b}{1+ab}\right) + n\pi$, n an integer.) (W)

12. Obtain the general solution of the differential equation $2x^2\dfrac{dy}{dx} + x^2y^2 = y^2$.
Find the three particular solutions such that (a) when $x = 1$, $y = 1$, (b) when $x = 1$, $y = \frac{2}{3}$, (c) when $x = 1$, $y = \frac{1}{2}$. Show that each of the integral curves given by these particular solutions has a maximum point on the line $x = 1$.
Sketch the three curves on the same axes. (L)

13. In a chemical reaction there are present, at time, t, x kg of substance X and y kg of substance Y, and initially there is 1 kg of X and 2 kg of Y. The variables x and y satisfy the equations $\frac{dx}{dt} = -x^2y, \frac{dy}{dt} = -xy^2$. Find $\frac{dy}{dx}$ in terms of x and y, and express y in terms of x. Hence obtain a differential equation in x and t only, and so find an expression for x in terms of t. (JMB)

14. (a) The equation of the family of all parabolas whose axes lie along the x-axis has the form $y^2 = 4a(x - h)$ where each parabola is specified by values of a and h. Form a differential equation (*not* involving a or h) satisfied by any point (x, y) lying on any one of these parabolas.

(b) The tangent to a curve at the point (x, y) cuts the x-axis at Q. If $OQ = kx$ (where O is the origin of the coordinate system) show that points on the curve satisfy the differential equation

$$\frac{dy}{dx} = \frac{y}{x - kx}.$$

By solving this differential equation show that the equation of the curve has the form $y = Ax^{1/(1-k)}$ where A is an arbitrary constant. (AEB 1975)

15. (i) Obtain the general solution of the differential equation $x \ln x \frac{dy}{dx} = 1$ (a) where $0 < x < 1$ and (b) where $x > 1$.

(ii) Sketch the family of integral curves for the differential equation

$$\frac{dy}{dx} = x(y + 1).$$

(iii) Find the equations of the two curves which pass through the point $(1, 1)$ and which satisfy the differential equation

$$\tan(y - 1)\frac{dy}{dx} = \tan(x - 1).$$ (L)

16. According to Newton's law of cooling, the rate at which a hot object cools is proportional to the difference between the temperature of the object and the temperature of the surrounding air (assumed to be constant). If an object cools from 100° to 80° in 10 minutes, and from 80° to 65° in another 10 minutes, find the temperatureof the surrounding air and the temperature of the object after a further 10 minutes. (O)

17. (a) Given that $\frac{dy}{dx} = 9y^2 - 4$ and that $y = 1$ when $x = 2$, find an equation expressing x in terms of y.

(b) In a certain country, the price p of a particular commodity increases with the time t at a rate equal to kp where k is a positive constant. Write down a differential equation expressing this information. Show that if $p = 1$ when $t = 0$ and $p = \alpha$ when $t = 1$, then, at time t, $p = \alpha^t$. (C)

18. (a) Solve the differential equation $\frac{dy}{dx} = 3y + 2$ given that $y = 1$ when $x = 2$, expressing your answer in a form giving y as a function of x.

(b) The rate of decomposition of a radioactive substance is proportional to the mass of the substance remaining. Write down a differential equation which expresses this law, stating the meaning of each symbol used. If one third of the mass is left after 12 years, how much was left after 3 years? (C)

19. Social scientists studying the growth of population in an urban region use a model in which the rate of growth of population is proportional to $P(N - P)$, where P is the current size of the population (regarded as a continuous variable) and N is a constant. Set up a differential equation for this model and find its general solution expressing P as a function of time t. In a particular region, the population at a certain time was 100 000. Fifty years later the population was 200 000 and fifty years after that it was 300 000. Using the above model determine the value of N in this case. What is the significance of N? (C)

20. A water reservoir is in the shape of the surface obtained by rotating a parabola $y = kx^2$ about its vertical axis, $x = 0$. Show that when the central depth of the water in the reservoir is h, the surface area A of the water is proportional to h and the volume V is proportional to h^2. In a dry period the rate of loss of water, due entirely to evaporation, is λA per day. Obtain a differential equation for h after t days and deduce that the depth decreases at a constant rate. If, in addition to the evaporation a constant volume C of water is now used per day, obtain the new differential equation and show that the time to empty the reservoir from a depth H will be $\int_0^H \frac{\pi h\, dh}{Ck + \pi\lambda h}$ days. Verify that when $H = 20$, $\lambda = \frac{1}{2}$, $C = 500$ and $k = 0{\cdot}0025$ this amounts to just under 35 days. (AEB 1979)

21. (i) An electrically charged body loses its charge, Q coulombs, at a rate kQ coulombs per second, where k is a constant. Write down a differential equation involving Q and t, where t seconds is the time since the discharge started. Solve this equation for Q, given that the initial charge was Q_0 coulombs. If $Q_0 = 0{\cdot}001$, and $Q = 0{\cdot}0005$ when $t = 10$, find the value of Q when $t = 20$.

(ii) Solve the differential equation $\frac{d^2y}{dx^2} - 25y = 0$, given that $y = 4, \frac{dy}{dx} = 10$ when $x = 0$. (L)

22. Find the solution of the differential equation $\frac{d^2y}{dx^2} + 9y = 18$, for which y has

a maximum at $(\frac{1}{2}\pi, 6)$. Find the minimum value of y, and the values of x for which $y = 0$. (O)

23. (i) Solve the differential equation $\dfrac{dy}{dx} = \left(\dfrac{x}{y}\right)e^{x+y}$ in the form $f(y) = g(x)$, given that $y = 0$ when $x = 0$.

(ii) Solve the differential equation $\dfrac{d^2y}{dx^2} + 4y = 16$, given that $y = 4$ and $\dfrac{dy}{dx} = -4$ when $x = 0$. For which values of x does the graph of y have points of inflexion? (O)

24. (i) Find the general solution of the differential equation

$$(1 + x)\frac{dy}{dx} = xy.$$

(ii) By using the substitution $z = dy/dx$, or otherwise, solve the differential equation $\dfrac{d^3y}{dx^3} + 4\dfrac{dy}{dx} = 0$, given that y takes the values $0, 0, 1$ for $x = 0$, $\frac{1}{4}\pi$, $\frac{1}{2}\pi$ respectively. (O&C)

25. Prove that, if $y = \cos^2(x^2)$, then

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} + 16x^3y = 8x^3.$$

Prove that changing the independent variable in this differential equation from x to t, where $t = x^2$, gives the equation

$$\frac{d^2y}{dt^2} + ky = C,$$

where k, C are constants and hence, or otherwise, find the general solution of the first equation in terms of x. What is the general solution of the differential equation

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} - 16x^3y = 8x^3?$$ (O&C)

12 Further particle dynamics

12.1 Acceleration as a function of displacement

Let us consider a particle moving along a straight line so that at time t its displacement from a fixed point on the line is s. The velocity v and the acceleration a of the particle at time t are given by:

$$v = \frac{ds}{dt} \quad \text{and} \quad a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

Suppose now that the acceleration a is regarded as a function of the displacement s, then, using the chain rule,

$$a = \frac{dv}{dt} = \frac{dv}{ds} \times \frac{ds}{dt} = \frac{dv}{ds} \times v = v\frac{dv}{ds}$$

$\therefore$ if $a = f(s)$, then $$v\frac{dv}{ds} = f(s),$$

giving $$\int v\,dv = \int f(s)\,ds$$

$\therefore$ $$\tfrac{1}{2}v^2 = \int f(s)\,ds$$

i.e. $$v^2 = 2\int f(s)\,ds$$

Thus if the acceleration a is a function of the displacement s, the relationship $a = v\dfrac{dv}{ds}$ may be used to obtain an expression for v^2 in terms of s.

Example 1 A particle of mass m moves in a straight line under the action of a force directed towards a fixed point O of the line. The magnitude of the force is $3ms^2$, where

s is the displacement of the particle from O at time t. Given that the particle starts from rest at a distance $2a$ from O, find the speed of the particle as it reaches O.

If v is the velocity of the particle at time t, then its acceleration is $v\frac{dv}{ds}$ in the direction of the displacement s. Since the force of magnitude $3ms^2$ acts in the opposite direction,

by Newton's second law, $mv\frac{dv}{ds} = -3ms^2$

$$\therefore \quad v\frac{dv}{ds} = -3s^2$$

$$\therefore \quad \int v\,dv = -\int 3s^2\,ds$$

$$\tfrac{1}{2}v^2 = -s^3 + c$$

But $v = 0$ when $s = 2a$,

$$\therefore \quad 0 = -8a^3 + c \quad \text{i.e.} \quad c = 8a^3$$

Thus $v^2 = 2(8a^3 - s^3)$,

and when $s = 0$, $v^2 = 16a^3$.

Hence when the particle reaches O its speed is $4a^{3/2}$.

Example 2 Assuming that the earth is a sphere of radius R, the acceleration due to gravity of an object at a distance x from the centre of the earth is inversely proportional to x^2 when above the earth's surface and equal to g on the surface. Given that an object projected vertically upwards from the earth's surface with speed u rises to a maximum height R above the surface, find u in terms of g and R.

The acceleration of the object is inversely proportional to the square of the displacement x and in the opposite direction,

$$\therefore \quad v\frac{dv}{dx} = -\frac{k}{x^2},$$

where v is the velocity and k is constant.

Since the acceleration is $-g$ when $x = R$,

$$-g = -\frac{k}{R^2} \quad \text{giving} \quad k = gR^2.$$

$$\therefore \quad v\frac{dv}{dx} = -\frac{gR^2}{x^2}$$

$$\therefore \quad \int v\,dv = -\int \frac{gR^2}{x^2}\,dx$$

$$\tfrac{1}{2}v^2 = \frac{gR^2}{x} + c$$

Given that $v = 0$ when $x = 2R$,

$$0 = \tfrac{1}{2}gR + c \quad \text{i.e.} \quad c = -\tfrac{1}{2}gR$$

$$\therefore \quad v^2 = \frac{2gR^2}{x} - gR$$

Since $v = u$ when $x = R$, $\quad u^2 = \dfrac{2gR^2}{R} - gR = gR$

Hence $\quad u = \sqrt{(gR)}$.

Exercise 12.1

In questions 1 to 4 a particle moves in a straight line so that after t seconds its displacement from a fixed point on the line is s metres and its velocity is $v\,\mathrm{m\,s^{-1}}$.

1. Given that the acceleration of the particle is $1/(1 + t)\,\mathrm{m\,s^{-2}}$ and that the particle starts from rest, find its speed after 2 seconds.

2. Given that the particle is subject to a retardation of $5e^{-t}\,\mathrm{m\,s^{-2}}$ and that its initial speed is $4\,\mathrm{m\,s^{-1}}$, find the time at which the particle first comes to rest.

3. Given that the acceleration of the particle is $2s\,\mathrm{m\,s^{-2}}$ and that $v = 2$ when $s = 0$, find the speed of the particle when $s = 4$.

4. Given that the particle is subject to a retardation of $2\sin^2 s\,\mathrm{m\,s^{-2}}$ and that $v = 1$ when $s = \pi/4$, find, correct to 3 significant figures, the speed of the particle when $s = \pi/3$.

5. A particle of mass m moving along a straight line is subject to a force $4mx$ acting towards a fixed point O on the line, where x is the displacement of the particle from O. Given that the particle starts from rest at a distance a from O, find the speed of the particle as it passes through O.

6. The acceleration due to gravity of an object at a distance x from the centre of the earth is inversely proportional to x^2 when above the earth's surface and equal to g on the surface. The earth is assumed to be a sphere of radius R. If an object falls from rest at a height $2R$ above the earth's surface, find its speed when it reaches a height R above the surface.

7. A particle of mass m is moving in a straight line under the action of a force $4m/x^3$ directed towards a fixed point O of the line, x being the displacement of the particle from O. Given that initially the particle is at a distance h from O, moving away from O at a speed of $2/h$, find expressions for (a) dx/dt in terms of x, and (b) x in terms of t and h.

8. A particle of mass m is moving in a straight line under the action of a force of magnitude $\frac{1}{4}e^{2x}$, where x is the displacement of the particle from a fixed

point O on the line. Initially the particle is at a point A travelling towards O at speed u. Given that the force on the particle acts in the direction of $\vec{OA}$ and that $OA = \ln(2u)$, find an expression for dx/dt in terms of x. Hence show that $x = \ln\left(\dfrac{2u}{1+ut}\right)$.

12.2 Acceleration as a function of velocity

We consider again a particle moving in a straight line so that at time t its displacement from a fixed point on the line is s and its velocity is v. If the acceleration of the particle is a function of the velocity, $f(v)$ say, then we may write:

(1) $\dfrac{dv}{dt} = f(v)$, or (2) $v\dfrac{dv}{ds} = f(v)$.

Equation (1) is used when a relationship between velocity and time is required. However, equation (2) may be used to obtain a relationship between velocity and displacement.

Example 1 A particle of mass m is projected vertically upwards from a point O with speed u and after time t its velocity is v. Given that the particle moves under gravity in a medium which exerts a resistance kv per unit mass, where k is a positive constant, find the time taken by the particle to reach its greatest height.

As the particle rises to its highest point the forces acting on it are its weight mg and the resistance mkv, both directed vertically downwards.

Thus, by Newton's second law, $-mg - mkv = m\dfrac{dv}{dt}$

$$\therefore \quad -(g+kv) = \frac{dv}{dt}$$

$$1 = -\frac{1}{(g+kv)}\frac{dv}{dt}$$

$$\therefore \quad \int dt = -\int \frac{1}{g+kv}\,dv$$

$$t = -\frac{1}{k}\ln(g+kv) + c$$

Since $v = u$ when $t = 0$, $c = \dfrac{1}{k}\ln(g+ku)$

$$\therefore \quad t = -\frac{1}{k}\ln(g+kv) + \frac{1}{k}\ln(g+ku) = \frac{1}{k}\ln\left(\frac{g+ku}{g+kv}\right)$$

$$\therefore \quad \text{when } v = 0,\ t = \frac{1}{k}\ln\left(\frac{g+ku}{g}\right)$$

Hence the time taken by the particle to reach its greatest height is $\dfrac{1}{k}\ln\left(1+\dfrac{ku}{g}\right)$.

Example 2 A particle of mass m moves in a straight line under the action of a retarding force $mv^4/2k^4$, where k is a positive constant and v is the velocity of the particle. Initially the particle is at a distance k^2 from a fixed point A, travelling away from A at speed k. Find the speed of the particle when its displacement from A is x. Find also the time which elapses before the particle reaches the point B, where $AB = 4k^2$.

Writing the acceleration of the particle in the form $v\,dv/dx$ and using Newton's second law we have:

$$-\frac{mv^4}{2k^4} = mv\frac{dv}{dx}$$

$$\therefore \quad \frac{1}{k^4} = -\frac{2}{v^3}\frac{dv}{dx}$$

$$\therefore \quad \int \frac{1}{k^4}\,dx = \int\left(-\frac{2}{v^3}\right)dv$$

Thus
$$\frac{x}{k^4} = \frac{1}{v^2} + c$$

Since $v = k$ when $x = k^2$, $\quad \frac{1}{k^2} = \frac{1}{k^2} + c \quad$ i.e. $\quad c = 0$

$$\therefore \quad \frac{x}{k^4} = \frac{1}{v^2}$$

$$v^2 = \frac{k^4}{x}$$

Hence when the displacement of the particle from A is x, its speed is $k^2/\sqrt{x}$.
$\therefore$ if x is the displacement from A at time t,

$$\frac{k^2}{\sqrt{x}} = \frac{dx}{dt}$$

$$\therefore \quad \int k^2\,dt = \int x^{1/2}\,dx$$

$$k^2t = \tfrac{2}{3}x^{3/2} + C$$

Since $x = k^2$ when $t = 0$, $\quad C = -\tfrac{2}{3}k^3$

$$\therefore \quad k^2t = \tfrac{2}{3}(x^{3/2} - k^3)$$

Thus when $x = 4k^2$, $\quad k^2t = \tfrac{2}{3}(8k^3 - k^3) = 14k^3/3$

Hence the time which elapses before the particle reaches B is $14k/3$.

Example 3 A car of mass m moving on a straight horizontal road is subject to a constant resistance R. The engine is working at a constant rate kR. Given that in time t the car accelerates to a speed v from rest, where $v < k$, find an expression for t in terms of m, R, k and v. Find also an expression for the distance x travelled by the car in this time.

If the tractive force exerted by the engine is F when the car is travelling at speed v, then its rate of working $= kR = Fv$,

$$\therefore \qquad F = kR/v$$

Using Newton's second law, the equation of motion is

$$\frac{kR}{v} - R = m\frac{dv}{dt}$$

$$\therefore \qquad R\left(\frac{k-v}{v}\right) = m\frac{dv}{dt} \tag{1}$$

$$\therefore \qquad \int R\,dt = \int m\left(\frac{v}{k-v}\right)dv$$

$$= \int m\left(\frac{k}{k-v} - 1\right)dv$$

$$\therefore \qquad Rt = m\{-k\ln(k-v) - v\} + c$$

Since $v = 0$ when $t = 0$, $\quad c = mk\ln k$

$$\therefore \qquad Rt = m\{-k\ln(k-v) - v + k\ln k\}$$

Hence
$$t = \frac{m}{R}\left\{k\ln\left(\frac{k}{k-v}\right) - v\right\}$$

Substituting $\dfrac{dv}{dt} = v\dfrac{dv}{dx}$ in equation (1),

$$R\left(\frac{k-v}{v}\right) = mv\frac{dv}{dx}$$

$$\therefore \qquad \int R\,dx = \int m\left(\frac{v^2}{k-v}\right)dv$$

$$\begin{array}{r} -v-k \\ -v+k\,\overline{)\,v^2} \\ \underline{v^2 - kv} \\ kv \\ \underline{kv - k^2} \\ k^2 \end{array}$$

By long division

$$\frac{v^2}{k-v} = -v - k + \frac{k^2}{k-v}$$

$$\therefore \qquad \int R\,dx = \int m\left\{\frac{k^2}{k-v} - v - k\right\}dv$$

$$\therefore \qquad Rx = m\{-k^2\ln(k-v) - \tfrac{1}{2}v^2 - kv\} + C$$

Since $v = 0$ when $x = 0$, $\quad C = mk^2\ln k$

$$\therefore \qquad Rx = m\{-k^2\ln(k-v) - \tfrac{1}{2}v^2 - kv + k^2\ln k\}$$

Hence
$$x = \frac{m}{R}\left\{k^2\ln\left(\frac{k}{k-v}\right) - \tfrac{1}{2}v^2 - kv\right\}.$$

In certain types of problem it is quicker to use definite integrals rather then indefinite integrals. For instance, if in Example 3 we had been asked to find the time which elapses as the car accelerates from speed $\frac{1}{3}k$ to speed $\frac{2}{3}k$, we could have deduced from equation (1) that:

$$\int_{t_1}^{t_2} R\,dt = \int_{k/3}^{2k/3} m\left(\frac{v}{k-v}\right)dv$$

where $t = t_1$ when $v = \frac{1}{3}k$ and $t = t_2$ when $v = \frac{2}{3}k$

$$\therefore \quad R\int_{t_1}^{t_2} dt = m\int_{k/3}^{2k/3}\left(\frac{k}{k-v} - 1\right)dv$$

$$\therefore \quad R\Big[t\Big]_{t_1}^{t_2} = m\Big[-k\ln(k-v) - v\Big]_{k/3}^{2k/3}$$

$$\therefore \quad R(t_2 - t_1) = m(-k\ln\tfrac{1}{3}k - \tfrac{2}{3}k) - m(-k\ln\tfrac{2}{3}k - \tfrac{1}{3}k)$$

$$= mk(\ln\tfrac{2}{3}k - \ln\tfrac{1}{3}k + \tfrac{1}{3} - \tfrac{2}{3})$$

$$= mk(\ln 2 - \tfrac{1}{3})$$

Hence the time taken is $\dfrac{mk}{R}(\ln 2 - \frac{1}{3})$.

Exercise 12.2

1. A particle of unit mass, moving in a straight line with speed v at time t, is subject to a retarding force of magnitude $(3 + 2v)$. Given that initially $v = 66$, find the value of t when $v = 6$.

2. A particle travelling in a straight line is subject to a retardation of $(1 + v^2)$, where v is its speed. If the initial speed of the particle is u, find an expression for the distance it travels before coming to rest.

3. A particle is moving in a straight line so that when the displacement of the particle from a fixed point on the line is x, its velocity is v. Given that the acceleration of the particle is $k(1 - v)$, where k is constant and that $x = 0$ when $v = 0$, find an expression for x in terms of v.

4. A particle is projected horizontally from a point O on a horizontal plane with an initial velocity $2a/b$ where a and b are positive constants. The particle is subject to a retardation $a + bv$ where v is the velocity at time t from the commencement of the motion. Show that $bv = a(3e^{-bt} - 1)$ and, using this result or otherwise, find the distance gone by the particle when it comes to instantaneous rest. (O&C)

5. The retardation of a particle moving in a straight line with speed v is proportional to v^3. The initial speed of the particle is u and s is the displacement of the particle from its initial position at time t. Given that $v = \frac{1}{2}u$ when $s = 2/u$, find an expression for v in terms of u and s. Hence find an expression for t in terms of u and s.

6. Show that the acceleration of an object moving along a straight line may be written as $v\,dv/ds$.

A vehicle of mass 2500 kg moving on a straight course is subject to a single resisting force in the line of motion of magnitude kv newtons, where v metres per second is the velocity and k is constant. At 100 km/h this force is 2000 N. The vehicle is slowed down from 100 km/h to 50 km/h. Find (i) the distance travelled, (ii) the time taken. (AEB 1980)

7. A particle falls from rest and, at any moment, it experiences a resistance of size kmv, where m is the mass of the particle, v is its velocity and k is a constant.

(i) Show that, from the equation of motion we can deduce

$$\int \frac{dv}{g - kv} = \int dt.$$

(ii) Using this equation, show that the velocity at any time is

$$\frac{g}{k}(1 - e^{-kt}).$$

(iii) Show that, no matter for how long the particle falls, its velocity never exceeds a certain value. If this value is V, show that k is g/V. (SU)

8. A particle of mass m moving under gravity is subject to a resistance of magnitude mkv^2, where v is its speed and k is constant. Initially the particle is travelling vertically downwards at speed u, where $u^2 < g/k$. Find the speed of the particle when it has fallen a distance s.

9. A train, when braking, is subject to a retardation of $\left(1 + \frac{v}{100}\right)\text{m s}^{-2}$, where v is its speed at any time. The brakes are applied when it is travelling at $20\,\text{m s}^{-1}$.

(i) Show that it takes just over 18 seconds to come to rest.

(ii) Show that it will travel approximately 177 m after the brakes are applied.

$$\left[\text{It may prove useful to know that } \frac{v}{v + 100} \text{ can be written as } 1 - \frac{100}{v + 100}\right].$$

(iii) If it then accelerates with acceleration $\left(1 + \frac{v^2}{100}\right)\text{m s}^{-2}$, find, correct to 3 significant figures, how long it takes to attain a speed of $20\,\text{m s}^{-1}$. (SU)

10. A particle of mass m moves, in the direction of x increasing, along the positive x-axis under the action of two forces. These are (i) a force toward the origin O of magnitude kv, and (ii) a force away from O of magnitude kU^2/v, where k and U are positive constants and v is the speed of the particle at time t. Write down the equation of motion of the particle in the form of a differential equation in v and t, and show that the time taken for the particle to accelerate from speed $\frac{1}{4}U$ to speed $\frac{1}{2}U$ is $\frac{m}{2k}\ln\frac{5}{4}$.

Write the equation of motion in the form of a differential equation in v and x, and hence show that the distance travelled by the particle while it accelerates from speed $\frac{1}{4}U$ to speed $\frac{1}{2}U$ is given by

$$\frac{m}{k}\int_{U/4}^{U/2}\left(\frac{U^2}{U^2-v^2}-1\right)dv.$$

Show that this distance is $\frac{mU}{4k}\left(2\ln\frac{9}{5}-1\right)$. (C)

11. A particle of mass m moves in a straight line under the action of a retarding force of magnitude $k(4+v^2)$, where v is the speed of the particle at time t and k is constant. If the initial speed of the particle is u, find an expression of the time taken by the particle to come to rest. Find also an expression, in terms of m, k and u, for the distance travelled in this time.

12. An engine of mass m kilograms, driven by a constant tractive force of P newtons, experiences a resistance which is k times its speed in metres/second. Show that, when moving horizontally from rest, its speed t seconds after the start is $\frac{P}{k}(1-e^{-kt/m})$ metres/second. Hence deduce that it will have travelled mP/ek^2 metres in m/k seconds. What power is being developed by the engine when the time is m/k seconds after the start? (W)

13. A train is pulled along a level track by an engine which exerts a constant pull P newtons at all speeds and the total resistance to motion varies as the square of the speed $v\,\mathrm{m\,s^{-1}}$. The mass of the engine and train combined is 161 700 kg, the maximum speed attained is $120\,\mathrm{km\,h^{-1}}$ and the power then developed is 1078 kW. Show that (a) $P = 32\,340$, (b) $5\times10^4 v\frac{dv}{ds} = 10^4 - 9v^2$, where s metres is the distance travelled. Hence show that the distance travelled from rest while attaining a speed of $20\,\mathrm{m\,s^{-1}}$ is about 1240 metres. (C)

14. A train of mass M moves on a straight horizontal track. At speeds less than V the resultant force on the train is constant and equal to P; at speeds not less than V the rate of working of the force is constant and equal to PV. Show that a speed $v(>V)$ is attained from rest in a time $M(V^2+v^2)/2PV$ and that the distance travelled in this time is $M(V^3+2v^3)/6PV$. (W)

15. A car of mass m moving on a level road is subject to a constant resistance R. The engine is working at a constant rate H. Find the time taken for the speed of the car to increase from u to $2u$, where $2Ru < H$. Find also, in terms of m, R, H and u, the distance travelled in this time.

16. A car of mass m moving on a level road is subject to a resisting force kv, where v is the speed of the car at time t and k is constant. When the engine of the car is working at full power S, the car can travel at a steady speed U. Find the

time taken by the car to accelerate under full power from rest to speed V, where $V < U$. Find also the distance travelled in this time. Give both answers in terms of m, S, U and V.

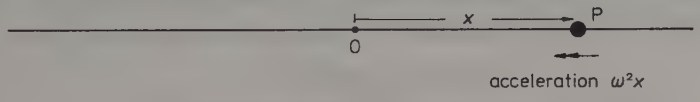
12.3 Simple harmonic motion

Any motion which satisfies a differential equation of the form

$$\ddot{x} = -\omega^2 x$$

where ω is constant, is described as *simple harmonic motion* or S.H.M.

For a particle moving in a straight line, the variable x is the displacement of the particle from a fixed point on the line. However, x may also be a displacement measured along a curved path. In the case of circular motion or the rotation of a rigid body x could represent an angle.



The diagram shows a particle P executing simple harmonic motion along a straight line. If x is the displacement of P from a fixed point O and $\ddot{x} = -\omega^2 x$, then the magnitude of the acceleration of P is a constant multiple of the distance OP. Since when $x > 0, \ddot{x} < 0$ and when $x < 0, \ddot{x} > 0$, the acceleration is always directed towards O. This leads to an alternative definition of simple harmonic motion.

When a particle moves in a straight line with acceleration always directed towards a fixed point of the line and proportional to its distance from that point, the particle is said to be executing simple harmonic motion.

The differential equation $\ddot{x} = -\omega^2 x$, i.e. $\dfrac{d^2x}{dt^2} = -\omega^2 x$, is of a type discussed in §11.5. In this case the method of solution can be simplified by expressing the acceleration of the particle in the form $v\,dv/dx$, where v is the velocity at time t, to give:

$$v\frac{dv}{dx} = -\omega^2 x$$

$$\therefore \quad \int 2v\,dv = -\int 2\omega^2 x\,dx$$

$$v^2 = -\omega^2 x^2 + c$$

If $v = 0$ when $x = a$, we find that

$$v^2 = \omega^2(a^2 - x^2) \tag{1}$$

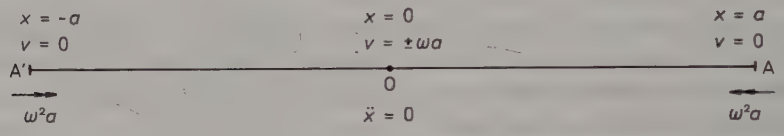
As shown in §11.5, this leads to the general solution:

$$x = a\sin(\omega t + \varepsilon) \tag{2}$$

or the equivalent form:

$$x = A\cos\omega t + B\sin\omega t \tag{3}$$

From equations (1) and (2) we deduce that the particle is oscillating between points A and A', which lie on opposite sides of O at a distance a from O. The point O is called the *centre* or *central point* of the oscillation and a is the *amplitude.*



From equation (1) we see that the particle is momentarily at rest when it reaches A or A', and achieves its maximum speed ωa when passing through O.

Since $\quad x = A\cos\omega t + B\sin\omega t$

and $\quad v = -A\omega\sin\omega t + B\omega\cos\omega t,$

when ωt is increased to $\omega t + 2\pi$ the same values of x and v are obtained. This means that at times t and $t + 2\pi/\omega$ the particle passes through the same point with the same velocity. Hence $2\pi/\omega$ is the time taken to perform a complete oscillation and is called the *period* of the motion. In particular, $2\pi/\omega$ is the time taken by the particle to move from A to A' and back again to A.

The *frequency*, i.e. the number of complete oscillations per unit time, is $\omega/2\pi$. Summarising the properties of S.H.M. we have:

$\ddot{x} = -\omega^2 x,$
$v^2 = \omega^2(a^2 - x^2)$, where a is the amplitude,
$x = a\sin(\omega t + \varepsilon)$, or
$x = A\cos\omega t + B\sin\omega t,$
$T = 2\pi/\omega$, where T is the period of oscillations.

When solving problems involving S.H.M. standard formulae may be quoted without proof. It is usually best to begin be determining a and ω.

Example 1 A particle is performing simple harmonic oscillations between points A and A' which are 6 m apart. When the particle is at a distance $\sqrt{5}$ m from the mid-point O of AA' its speed is 10 m s^{-1}. Find (a) the period of the motion, (b) the maximum speed of the particle, (c) its maximum acceleration.

Let the equation of motion of the particle be

$$\ddot{x} = -\omega^2 x,$$

then

$$v^2 = \omega^2(a^2 - x^2),$$

where x m is the displacement of the particle from O, v m s^{-1} is its speed and a m is the amplitude of the motion.
But $AA' = 6\,\text{m} = 2a\,\text{m}$, which gives $a = 3$

$$\therefore \quad v^2 = \omega^2(9 - x^2)$$

Since $v = 10$ when $x = \sqrt{5}$, $100 = \omega^2(9 - 5)$ $\quad \therefore \quad \omega = 5$

(a) The period of the motion $= \dfrac{2\pi}{\omega}$ seconds $= \dfrac{2\pi}{5}$ seconds.

(b) The speed of the particle is greatest when $x = 0$
$\therefore$ the maximum speed $= \omega a\,\text{m s}^{-1} = 15\,\text{m s}^{-1}$.

(c) The acceleration of the particle is greatest when the magnitude of $\omega^2 x$ is greatest,
$\therefore$ the maximum acceleration $= \omega^2 a\,\text{m s}^{-2} = 75\,\text{m s}^{-2}$.

In S.H.M. problems involving time we need to find suitable expressions for x and v in terms of t. The forms of these expressions depend on the starting point chosen for the motion. One way of approaching such problems is to write:

$$x = A\cos\omega t + B\sin\omega t, \tag{3}$$
$$v = -A\omega\sin\omega t + B\omega\cos\omega t, \tag{4}$$

then to use any given conditions to determine A and B. We first apply this method to two important special cases.

Case I: $x = a$ when $t = 0$.

Substituting in (3): $a = A$
Since $v = 0$ when $x = a$, from (4): $B = 0$

$\therefore$ $\boxed{x = a \quad \text{when} \quad t = 0 \;\Rightarrow\; x = a\cos\omega t.}$

Case II: $x = 0, v > 0$ when $t = 0$.

Substituting in (3): $0 = A$
$\therefore$ $x = B\sin\omega t$
Since the amplitude of the motion is a and $v > 0$ when $t = 0$, we must have $B = a$.

$\therefore$ $\boxed{x = 0, v > 0 \quad \text{when} \quad t = 0 \;\Rightarrow\; x = a\sin\omega t.}$

Example 2 A particle moving in a straight line performs simple harmonic oscillations about a point O with amplitude 2 m and maximum speed 1 m s^{-1}. If P

and Q are the two points on the line which lie at a distance $\sqrt{2}$ m from O, find the time taken by the particle to move directly from P to Q.

Let the equation of motion be $\ddot{x} = -\omega^2 x$, then, assuming that $x = a$ when $t = 0$, $x = a\cos\omega t$.
Since the amplitude of the motion is 2 m, $a = 2$.
The maximum speed of the particle $= \omega a \text{ m s}^{-1} = 1 \text{ m s}^{-1}$

$\therefore \quad \omega = \frac{1}{2}$
Thus $x = 2\cos\frac{1}{2}t$.

Let P and Q be the points at which $x = \sqrt{2}$ and $x = -\sqrt{2}$ respectively.

For $x = \sqrt{2}$: $\sqrt{2} = 2\cos\frac{1}{2}t$
i.e. $\cos\frac{1}{2}t = 1/\sqrt{2}$

$\therefore$ the particle first passes through P when $\frac{1}{2}t = \frac{1}{4}\pi$ i.e. when $t = \frac{1}{2}\pi$.

For $x = -\sqrt{2}$: $-\sqrt{2} = 2\cos\frac{1}{2}t$
i.e. $\cos\frac{1}{2}t = -1/\sqrt{2}$

$\therefore$ the particle first passes through Q when $\frac{1}{2}t = \frac{3}{4}\pi$ i.e. when $t = \frac{3}{2}\pi$.
Hence the particle moves directly from P to Q in a time of π seconds.

Example 3 A particle is performing simple harmonic motion about a point O with amplitude 5 m and period $\frac{1}{2}\pi$ seconds. If P is the point at which the speed of the particle is 10 m s^{-1}, find the time taken by the particle to move directly from O to P.

Let the equation of motion of the particle be $\ddot{x} = -\omega^2 x$, and let $x = a\sin\omega t$, so that the particle is at O when $t = 0$.
Since the amplitude of the motion is 5 m, $a = 5$.
The period of oscillation $= \dfrac{2\pi}{\omega}\text{s} = \dfrac{\pi}{2}\text{s}, \qquad \therefore \quad \omega = 4.$
Thus $x = 5\sin 4t$ and $v = 20\cos 4t$.

Substituting $v = 10$: $10 = 20\cos 4t$
$\therefore$ $\cos 4t = \frac{1}{2}$

$\therefore$ the particle first passes through P when $4t = \pi/3$. Hence the particle moves directly from O to P in a time of $\pi/12$ seconds.

Example 4 A particle moves on a straight line through a fixed point O so that at time t seconds its displacement from O is x metres and its equation of motion is $\ddot{x} = -9x$. Given that $x = 5$ and $\dot{x} = -6$ when $t = \pi/6$, find the position and speed of the particle when $t = 2\pi/3$.

The general solution of the equation of motion is

$$x = A\cos 3t + B\sin 3t$$
$$\therefore \quad \dot{x} = -3A\sin 3t + 3B\cos 3t$$

Since $x = 5$ and $\dot{x} = -6$ when $t = \pi/6$,

$$5 = A\cos\tfrac{1}{2}\pi + B\sin\tfrac{1}{2}\pi$$

i.e. $5 = B$

and $-6 = -3A\sin\tfrac{1}{2}\pi + 3B\cos\tfrac{1}{2}\pi$

i.e. $-6 = -3A$

$\therefore$ $A = 2$ and $B = 5$

Thus $x = 2\cos 3t + 5\sin 3t$,

$\dot{x} = -6\sin 3t + 15\cos 3t$.

$\therefore$ when $t = 2\pi/3$, $x = 2\cos 2\pi + 5\sin 2\pi = 2$

and $\dot{x} = -6\sin 2\pi + 15\cos 2\pi = 15$

i.e. the particle is 2 m from O moving at $15\,\mathrm{m\,s^{-1}}$.

Exercise 12.3

[Standard formulae may be quoted without proof.]

1. A particle moves in a straight line with simple harmonic motion of amplitude 2·5 m. If the period of oscillation is π seconds, find the maximum speed of the particle and its maximum acceleration.

2. A particle moves in a straight line with simple harmonic motion about the point O as centre. The maximum speed of the particle is $6\,\mathrm{m\,s^{-1}}$ and its maximum acceleration is $18\,\mathrm{m\,s^{-2}}$. Find (a) the amplitude of the motion, (b) the period of the motion, (c) the speed of the particle when it is 1 m from O.

3. A particle of mass 3 kg moves in a straight line with simple harmonic motion between two points A and A' which are 15 m apart. Given that when the particle is at a distance of 3 m from A its speed is $2\,\mathrm{m\,s^{-1}}$, find the period of the motion. Find also the greatest force exerted on the particle during the motion.

4. A mass of 10 kg moves with simple harmonic motion. When it is 2 m from the centre of the oscillation, the velocity and acceleration of the body are $12\,\mathrm{m\,s^{-1}}$ and $162\,\mathrm{m\,s^{-2}}$ respectively. Calculate (i) the number of oscillations per minute; (ii) the amplitude of the motion; (iii) the force being applied to the body when it is at the extremities of its motion. (SU)

5. A particle P moves in a straight line so that its acceleration is always directed towards a point O in its path and is of magnitude proportional to the distance OP. When P is at the point A, where $OA = 1$ m, its speed is $3\sqrt{3}$ m/s and when P is at the point B, where $OB = \sqrt{3}$ m, its speed is 3 m/s. Calculate the maximum speed attained by P and the maximum value of OP. Show that P takes $\pi/18$ seconds to move directly from A to B. Find, in m/s correct to 2 significant figures, the speed of P one second after it passes O. (L)

6. A particle, A, is performing simple harmonic oscillations about a point O with amplitude 2 m and period 12π s. Find the least time from the instant when A passes through O until the instant when (i) its displacement is 1 m, (ii) its velocity is half that at O, (iii) its kinetic energy is half that at O. (AEB 1978)

7. A particle P is describing simple harmonic motion in the horizontal line $ADCB$, where $AD = DC = \frac{1}{2}CB$. The speed of P as it passes through C is 5 m/s and P is instantaneously at rest at A and B. Given that P performs 3 complete oscillations per second, calculate (i) the distance AB, (ii) the speed of P as it passes through D, (iii) the distance of P from C at an instant when the acceleration of P is 18π m/s^2, (iv) the time taken by P to go directly from D to A. (AEB 1978)

8. A particle P moves in a straight line with simple harmonic motion of period 2 seconds and maximum speed 4 m/s. Find the speed of P when it is at the point A which is $2/\pi$ metres from O, the centre of the path. Find also the time taken by P to move directly from O to A. When P is passing through O it strikes and adheres to a stationary particle which is free to move. If each particle is of mass 2 kg, find the kinetic energy lost in the collision. (L)

9. A particle moves in a straight line so that at time t its displacement from a fixed point on the line is x and its equation of motion is $\ddot{x} = -4x$. Given that $x = 3$ and $\dot{x} = -6$ when $t = \pi/4$, find (a) x in terms of t, (b) the values of x and $\dot{x}$ when $t = 3\pi/4$, (c) the least positive value of t for which $x = 0$.

10. A particle moves in a straight line so that at time t its displacement from a fixed point on the line is x and its equation of motion is $\ddot{x} = -16x$. Given that $x = 3$ and $\dot{x} = 4\sqrt{3}$ when $t = 0$, find (a) x in terms of t, (b) the least positive value of t for which $\dot{x} = 0$, (c) the amplitude of the oscillation.

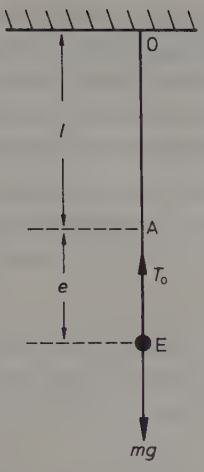
11. A particle P moves round a circle, diameter AB, with constant angular speed ω. A second particle Q moves along AB so that PQ is always perpendicular to AB, i.e. Q is the projection of P on AB. Show that the motion of Q is simple harmonic.

12.4 Forces producing simple harmonic motion

To show that a given set of forces acting on a particle produce simple harmonic motion we must prove that the equation of motion of the particle can be expressed in the form $\ddot{x} = -\omega^2 x$.

Since the tension in an elastic string is proportional to the extension in the string, it may be possible for a particle attached to an elastic string or a spring to perform simple harmonic motion.

Let us suppose that a light elastic string of natural length l and modulus of elasticity λ has one end fastened to a fixed point O. A particle of mass m is



attached to the other end. Let e be the extension in the string and T_0 the tension when the particle hangs in equilibrium at the point E.

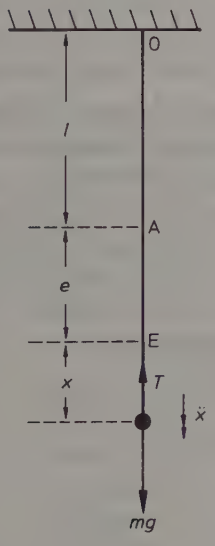
By Hooke's law, $T_0 = \dfrac{\lambda e}{l}$

Since the particle is in equilibrium,

$$T_0 - mg = 0$$

$$\therefore \quad \frac{\lambda e}{l} = mg$$

Consider now the forces on the particle when its displacement from E is x vertically downwards.



By Hooke's law, $T = \dfrac{\lambda(x + e)}{l}$

where T is the tension in the string. Applying Newton's second law vertically downwards:

$$mg - T = m\ddot{x}$$

$$\therefore \quad \frac{\lambda e}{l} - \frac{\lambda(x + e)}{l} = m\ddot{x}$$

$$\therefore \quad -\frac{\lambda x}{l} = m\ddot{x}$$

Hence the equation of motion of the particle is $\ddot{x} = -\dfrac{\lambda}{lm}x$. Since this equation is of the form $\ddot{x} = -\omega^2 x$, where $\omega = \sqrt{(\lambda/lm)}$, the particle can perform simple harmonic motion with centre E and period $2\pi\sqrt{(lm/\lambda)}$. The amplitude a of the motion is the maximum distance below E reached by the particle.

Provided that $a < e$ the string will remain taut throughout the motion and complete simple harmonic oscillations will be possible. However, if $a > e$ the particle will perform simple harmonic motion when it is below A, but move freely under gravity when the string becomes slack.

[Note that in the case of a particle suspended from a spring of natural length l, complete S.H.M. involving both extension and compression of the spring is theoretically possible when $a > e$.]

Example 1 A light elastic string of natural length l and modulus mg has one end fastened to a fixed point O. The other end of the string is attached to a particle of

mass m which hangs in equilibrium at a point E. Find the distance OE. The particle is now pulled down to a point A at a distance $5l/2$ vertically below O. If at time $t = 0$ the particle is released from rest at A, show that the subsequent motion is simple harmonic and find the speed of the particle as it passes through E. Find also the time at which the particle first passes through the point B which lies at a distance $7l/4$ vertically below O.

When the particle is at E let the tension in the string be T_0 and the extension e.

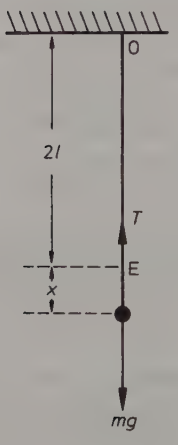
By Hooke's law: $T_0 = \dfrac{mg \times e}{l}$

But since the particle is in equilibrium, $T_0 = mg$

$$\therefore \quad \frac{mg \times e}{l} = mg$$

$$e = l$$

Hence $OE = l + e = 2l$.



Let T be the tension in the string when the displacement of the particle from E is x vertically downwards.

By Hooke's law: $T = \dfrac{mg(x + l)}{l}$

Applying Newton's second law vertically downwards:

$$mg - T = m\ddot{x}$$

$$\therefore \quad mg - \frac{mg(x + l)}{l} = m\ddot{x}$$

$$\therefore \quad \ddot{x} = -\frac{g}{l}x$$

Since this equation is of the form $\ddot{x} = -\omega^2 x$, where $\omega = \sqrt{(g/l)}$, the particle performs simple harmonic motion with centre E.
The speed v of the particle is given by the formula

$$v^2 = \omega^2(a^2 - x^2)$$

When the particle is at A, $x = \frac{1}{2}l$ and $v = 0$ $\quad \therefore \quad a = \frac{1}{2}l$

Hence $\quad v^2 = \dfrac{g}{l}(\frac{1}{4}l^2 - x^2).$

$$\therefore \quad \text{when } x = 0, v^2 = \frac{g}{l} \times \frac{l^2}{4} = \frac{gl}{4}$$

Hence as the particle passes through E its speed is $\frac{1}{2}\sqrt{(gl)}$.

Since $x = \frac{1}{2}l$ and $v = 0$ when $t = 0$,

$$x = \tfrac{1}{2}l\cos\omega t, \quad \text{where} \quad \omega = \sqrt{(g/l)}.$$

When $x = -\frac{1}{4}l$, $\quad -\frac{1}{4}l = \frac{1}{2}l\cos\omega t$

i.e. $\quad \cos\omega t = -\frac{1}{2}$

$\therefore$ the particle first passes through B when $\omega t = 2\pi/3$

i.e. when $t = \dfrac{2\pi}{3}\sqrt{\left(\dfrac{l}{g}\right)}$.

We include the following example as a reminder that many problems involving particles attached to elastic strings can be solved using energy considerations rather than the theory of simple harmonic motion.

Example 2 An elastic string of natural length l and modulus $4mg$ has one end fastened to a fixed point. The other end of the string is attached to a particle of mass m. If the particle is released from rest at O, find the distance it falls before coming instantaneously to rest at a point A.

Let x be the extension in the string when the particle reaches A. The kinetic energy of the particle is zero when it is released from O and when it comes to rest at A.
Since $OA = l + x$, the loss in gravitational potential energy is $mg(l + x)$.
The energy stored in the elastic string as it is stretched to the length $(l + x)$ is $4mgx^2/2l$ i.e. $2mgx^2/l$.
Thus, using the principle of conservation of mechanical energy,

$$\frac{2mgx^2}{l} = mg(l + x)$$

$\therefore \quad 2x^2 = l(l + x)$

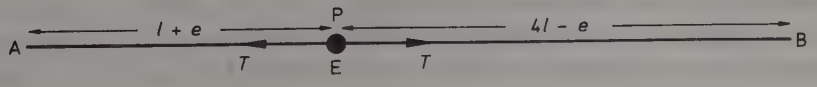
$$2x^2 - lx - l^2 = 0$$

$$(x - l)(2x + l) = 0$$

Since $x > 0$, we must have $x = l$.
Hence the particle falls a distance $2l$ before coming to rest.

In the next example we consider oscillations in a horizontal plane.

Example 3 A particle P of mass m lies on a smooth horizontal table and is attached to two fixed points A, B on the table by two elastic strings each of natural length l. The strings AP, PB have moduli $2mg$ and mg respectively and $AB = 5l$. If E is the equilibrium position of the particle, find AE. Given that the particle is released from rest at the point C on AB such that $AC = l$, show that it performs simple harmonic motion, stating the period and the amplitude of the oscillations.



Let T be the tension in both strings when the particle is in equilibrium at E.
If $AE = l + e$, then the extensions in the strings AP and PB are e and $3l - e$ respectively.

By Hooke's law for AP: $T = \dfrac{2mge}{l}$

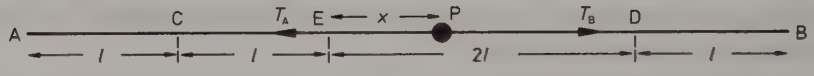
By Hooke's law for BP: $T = \dfrac{mg(3l - e)}{l}$

$$\therefore \quad \frac{2mge}{l} = \frac{mg(3l - e)}{l}$$

$$2e = 3l - e$$

$$\therefore \quad e = l$$

Hence $AE = l + e = 2l$.



Suppose now that the displacement of the particle P from E is x, as shown in the diagram.
Assuming that both strings are taut, i.e. that P lies between the points C and D, the extension in AP is $(l + x)$ and the extension in PB is $(2l - x)$.
Let T_A and T_B be the tensions in the strings AP and PB respectively, then applying Hooke's law,

$$T_A = \frac{2mg(l + x)}{l}, \quad T_B = \frac{mg(2l - x)}{l}$$

Using Newton's second law in the direction AB,

$$T_B - T_A = m\ddot{x}$$

$$\therefore \quad \frac{mg(2l - x)}{l} - \frac{2mg(l + x)}{l} = m\ddot{x}$$

$$-\frac{3mgx}{l} = m\ddot{x}$$

$$\therefore \quad \ddot{x} = -\frac{3g}{l}x$$

Thus the equation of motion of the particle is of the form $\ddot{x} = -\omega^2 x$, where $\omega = \sqrt{(3g/l)}$. Hence when both strings are taut the particle performs simple harmonic motion with centre E and period $2\pi\sqrt{(l/3g)}$.
Since the particle starts from rest at C, where $CE = l$, the amplitude of the simple harmonic motion is l.
The diagram shows that complete oscillations of this amplitude are possible with both strings taut. Thus when the particle is released at C it performs simple harmonic motion of period $2\pi\sqrt{(l/3g)}$ and amplitude l.

These examples illustrate the fact that in simple harmonic motion the centre of oscillation is an equilibrium position. If a displacement x from this equilibrium position is considered, then the equation of motion is obtained in the standard form

$$\ddot{x} = -\omega^2 x.$$

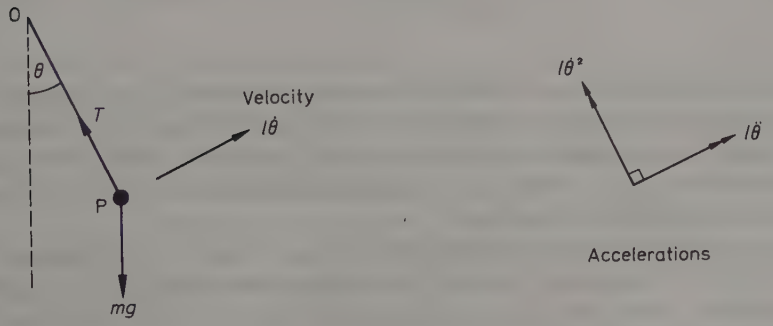
However, if a displacement from a point which is not an equilibrium position is used, then the simple harmonic motion equation takes the more general form

$$\ddot{x} + \omega^2 x = \text{constant}.$$

Solutions to this type of differential equation were discussed in §11.5.

Finally, we show that the motion of a *simple pendulum* is approximately simple harmonic. A simple pendulum consists of a particle, known as the "bob", suspended from a fixed point by a light inextensible string. When the particle swings in a vertical plane with the string taut, the bob moves along a circular arc.

It was shown in Book 1 that when a particle moves in a circle with variable speed v, then the acceleration of the particle has components v^2/r towards the centre of the circle and dv/dt in the direction of motion. These results can now be used to find the equation of motion of a simple pendulum.



Let the bob of the pendulum be a particle P of mass m and let the length of the string be l. At the instant when the string makes an angle θ with the downward vertical, P is moving perpendicular to the string with velocity $l\dot{\theta}$. Hence the acceleration of P has components $l\dot{\theta}^2$ and $l\ddot{\theta}$ along and perpendicular to the string.

Applying Newton's second law perpendicular to the string,

$$mg \sin\theta = -ml\ddot{\theta}$$

Hence
$$\ddot{\theta} = -\frac{g}{l}\sin\theta$$

However, if θ is small, $\sin\theta \approx \theta$

$\therefore$
$$\ddot{\theta} \approx -\frac{g}{l}\theta$$

Thus small oscillations of a simple pendulum are approximately simple harmonic. The period of complete oscillations is $2\pi\sqrt{\frac{l}{g}}$.

A pendulum which beats seconds is called a *seconds pendulum*. Such a pendulum swings through its equilibrium position once every second and therefore each complete oscillation takes 2 seconds. Hence the length of a seconds pendulum is found by writing:

$$2\pi\sqrt{\frac{l}{g}} = 2.$$

Exercise 12.4

1. A particle of mass m is attached to one end of a light elastic string of length a and modulus $\frac{3}{2}mg$. The other end of the string is attached to a fixed point O and the particle hangs in equilibrium under gravity at the point E. Find the distance OE. If the particle is a *further* distance x below E show that the resultant force acting on the particle is proportional to x. The particle is pulled down to the point at a distance a below E and released from rest. Show that, in the subsequent motion and while the string is taut, the particle executes Simple Harmonic Motion and that its distance below E at time t after being released is

$$a\cos\left\{\left(\frac{3g}{2a}\right)^{1/2}t\right\}. \qquad \text{(O\&C)}$$

2. A light elastic string of natural length l has one end fastened to a fixed point O. The other end of the string is attached to a particle of mass m. When the particle hangs in equilibrium the length of the string is $\frac{7}{4}l$. The particle is displaced from equilibrium so that it moves vertically with the string taut. Show that the motion is simple harmonic with period $\pi\sqrt{(3l/g)}$.

At time $t = 0$ the particle is released from rest at a point A at a distance $\frac{3}{2}l$ vertically below O. Find (i) the depth below O of the lowest point L of the motion, (ii) the time taken to move from A to L, (iii) the depth below O of the particle at time $t = \frac{1}{3}\pi\sqrt{\frac{3l}{g}}$. (C)

3. A particle of mass m is attached to one end of a light elastic string of natural length l and modulus $2mg$. The other end of the string is fixed at a point A. The

particle rests on a support B vertically below A, with $AB = 5l/4$. Find the tension in the string and the reaction exerted on the particle by the support B. The support B is suddenly removed. Show that the particle will execute simple harmonic motion and find (i) the depth below A of the centre of oscillation, (ii) the period of the motion. (JMB)

4. A light elastic spring AB of natural length l has the end A attached to a fixed point and hangs vertically under gravity with a particle of mass m attached at B. The system is set in motion and performs small vertical oscillations. Show that the motion is simple harmonic. The particle of mass m at B is replaced by a particle of mass αm, and it is found that the period of oscillation is doubled. Find the value of α. (C)

5. One end of an elastic string of modulus mg and natural length a is attached to a fixed point O. To the other end A are attached two particles P and Q, P having mass $2m$ and Q having mass m. The particles hang down in equilibrium under gravity. If Q falls off, show that P subsequently performs simple harmonic motion and state the period and amplitude of this motion. If on the other hand P falls off, find the distance from O of the highest point reached by Q. (O&C)

6. An elastic string of natural length $2a$ and modulus λ has its ends attached to two points A, B on a smooth horizontal table. The distance AB is $4a$ and C is the mid-point of AB. A particle of mass m is attached to the mid-point of the string. The particle is released from rest at D, the mid-point of CB. Denoting by x the displacement of the particle from C, show that the equation of motion of the particle is

$$\frac{d^2x}{dt^2} + \frac{2\lambda}{ma}x = 0.$$

Find the maximum speed of the particle and show that the time taken for the particle to move from D directly to the mid-point of CD is $\frac{\pi}{3}\left(\frac{ma}{2\lambda}\right)^{1/2}$. (JMB)

7. A particle P of mass m is attached by two light elastic strings to two fixed points A and B on a smooth horizontal table. Each string is of natural length l and $AB = 4l$. If the strings AP and PB have moduli of elasticity λ and 3λ respectively, find the length of AP when the particle is in equilibrium. If the particle is given a small displacement from the equilibrium position along the line AB, show that while both strings are taut the particle executes simple harmonic motion. Find the period of the oscillations. Given that the string PB just becomes slack in the ensuing motion, find the maximum speed of the particle.

8. Two fixed points A and B on a smooth horizontal table are at a distance $5l$ apart. A particle of mass m lies between A and B. It is attached to A by means of a light elastic string of modulus mg and natural length l and to B by means of a light elastic string of modulus $2mg$ and natural length $2l$. If O is the point at which the particle would rest in equilibrium, find the distance OA. The particle is projected towards O

from the mid-point of AB with speed u. Show that while both strings remain taut the particle performs simple harmonic motion. Find the period and the amplitude of this motion. Show that the particle will perform complete simple harmonic oscillations if $2u^2 \leqslant 3gl$.

9. A particle of mass m moving on a smooth horizontal plane is attached to a point O of the plane by an elastic string of natural length l and modulus λ. Initially the particle is at the point A which lies at a distance l from O and is moving away from O at speed u. If the particle first comes to rest at the point B, use energy considerations to find the distance AB. Show that when the particle is moving in the line AB its equation of motion is of the form $\ddot{x} = -\omega^2 x$, where x is its displacement from A. Find the time taken by the particle to move directly from A to B. Find also the further time which elapses before the particle next comes to rest.

10. A particle of mass m is attached to two elastic strings each of natural length a. The first string has modulus $3mg$ and its other end is attached to a fixed point A. The second string has modulus $6mg$ and its other end is attached to a fixed point B which is at a distance $3a$ vertically below A. Show that the particle can rest in equilibrium at a point O between A and B with both strings taut. Find the distance y of O from A. The particle is in motion in the line AB with both strings taut. At time t, the particle is at P where $OP = x$. Prove that $\dfrac{d^2x}{dt^2} = -\dfrac{9g}{a}x$.

The particle is projected vertically upwards from the point at a distance $2a$ below A with speed u. Find the maximum value of u for which the upper string does not become slack in the subsequent motion. For this maximum value of u find the time taken for the particle to travel from O to its highest point. (O&C)

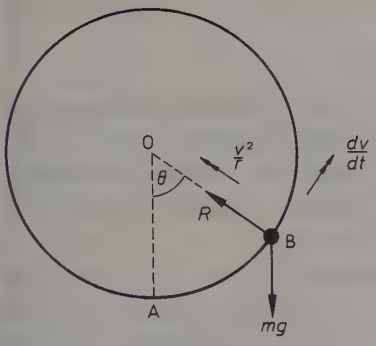
11. Find, to the nearest millimetre, the length of a seconds pendulum, (a) at Greenwich, where $g \approx 9{\cdot}81$, (b) at the equator, where $g \approx 9{\cdot}78$, (c) at the North Pole, where $g \approx 9{\cdot}83$, (d) on the surface of the moon, where $g \approx 1{\cdot}62$.

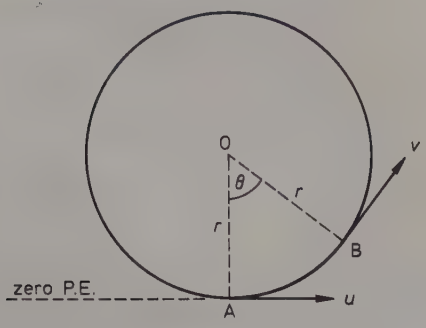
12. A musician wishes to use a simple pendulum as a makeshift metronome. Regarding a complete oscillation as two "beats", find, to the nearest centimetre, the length of pendulum required to produce (a) 50 beats per minute, (b) 80 beats per minute, (c) 100 beats per minute, (d) 160 beats per minute. [Take g as $9{\cdot}8\,\mathrm{m\,s^{-2}}$.]

12.5 Motion in a vertical circle

It was established in Book 1 that the acceleration of a particle moving in a circle, radius r, with variable speed v has components v^2/r towards the centre of the circle and dv/dt in the direction of motion.

When dealing with motion in a vertical circle we can obtain equations of motion by applying Newton's second law in these two directions. However, it may sometimes be more convenient to use an energy equation.





We consider first a small bead of mass m sliding on a smooth circular wire, of centre O and radius r, fixed in a vertical plane. Let u be the speed of the bead as it passes through the lowest point A of the wire and let v be the speed of the bead when it is at a point B such that $\angle AOB = \theta$. Since the wire is smooth the reaction R of the wire on the bead is perpendicular to the wire. Using Newton's second law radially and tangentially,

$$R - mg\cos\theta = m\frac{v^2}{r}$$

$$mg\sin\theta = -m\frac{dv}{dt}$$

The reaction R is perpendicular to the direction of motion and therefore does no work. This means that we can apply the principle of conservation of mechanical energy:

$$\text{(K.E. + P.E.) at } A = \text{(K.E. + P.E.) at } B$$

$$\therefore \qquad \tfrac{1}{2}mv^2 = \tfrac{1}{2}mv^2 + mgr(1 - \cos\theta) \qquad (1)$$

As θ increases the speed of the bead decreases. If the bead comes to rest before it reaches the highest point of the wire it will slide back down the wire and perform oscillations about A. However, if the speed of the bead is greater than zero for all values of θ, it will execute complete circles.

From (1): $\quad v^2 = u^2 - 2gr(1 - \cos\theta)$

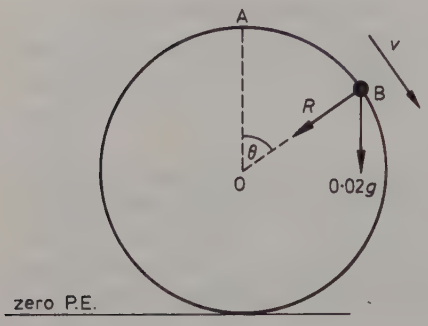
$\therefore$ if $\quad v^2 > 0$ for all θ

then $\quad u^2 > 2gr(1 - \cos\theta)$ for all θ

Since the greatest value taken by $(1 - \cos\theta)$ is 2, the condition for performing complete circles is

$$u^2 > 4gr.$$

Example 1 A small ring of mass 0·02 kg is threaded on a smooth circular wire of centre O fixed in a vertical plane. The ring is slightly displaced from rest at the highest point A of the wire. Find the magnitude and direction of the reaction of the wire on the ring when the radius to the ring makes an angle of 45° with (a) the upward vertical, (b) the downward vertical.



Let v m s^{-1} be the speed of the ring when it is at a point B such that $\angle AOB = \theta$. Let the reaction of the wire on the ring be R N towards O and let the radius of the wire be r m.

Using the principle of conservation of mechanical energy,

(K.E. + P.E.) at A = (K.E. + P.E.) at B

$\therefore \quad 0 + 0{\cdot}02g \times 2r = \frac{1}{2} \times 0{\cdot}02 \times v^2 + 0{\cdot}02g \times r(1 + \cos\theta)$

$\therefore \quad 4gr = v^2 + 2gr(1 + \cos\theta)$

Thus $\quad v^2 = 2gr(1 - \cos\theta).$

Applying Newton's second law towards O,

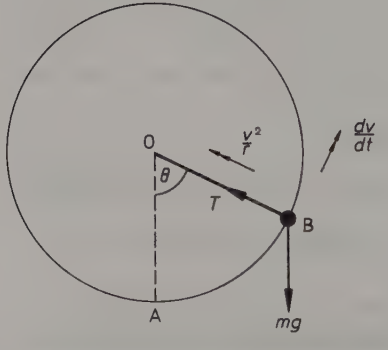
$$R + 0{\cdot}02g\cos\theta = 0{\cdot}02 \times \frac{v^2}{r}$$

$\therefore \quad R + 0{\cdot}02g\cos\theta = 0{\cdot}02 \times 2g(1 - \cos\theta)$

Thus $\quad R = 0{\cdot}02g(2 - 3\cos\theta).$

(a) When $\theta = 45°$, $R = 0{\cdot}02 \times 9{\cdot}8 \times (2 - 3\cos 45°) \approx -0{\cdot}024$.
Hence when the radius to the ring makes an angle of 45° with the upward vertical the reaction of the wire is 0·024 N *away from O*.

(b) When $\theta = 135°$, $R = 0{\cdot}02 \times 9{\cdot}8 \times (2 - 3\cos 135°) \approx 0{\cdot}808$.
Hence when the radius to the ring makes an angle of 45° with the downward vertical the reaction of the wire is 0·808 N *towards O*.



Suppose now that a particle of mass m, suspended from a fixed point O by a light inextensible string of length r, moves in a vertical circle. Let u be the speed of the particle as it passes through its lowest point A and let v be the speed of the particle when it is at a point B such that $\angle AOB = \theta$.

Applying Newton's second law towards O,

$$T - mg\cos\theta = m\frac{v^2}{r} \qquad (1)$$

As in the case of a bead sliding on a circular wire, it can also be shown that

$$v^2 = u^2 - 2gr(1 - \cos\theta) \tag{2}$$

$$\therefore \quad T = \frac{m}{r}\{u^2 - 2gr(1 - \cos\theta)\} + mg\cos\theta$$

i.e.
$$T = \frac{mu^2}{r} - 2mg + 3mg\cos\theta \tag{3}$$

Thus when the particle passes through A its speed v and the tension T in the string take their maximum values. As θ increases, both v and T decrease, but the exact nature of the subsequent motion depends on the magnitude of u. We now consider in more detail the following possibilities:

(i) v reaches zero before T,
(ii) T reaches zero before v,
(iii) both v and T are positive for all values of θ.

Case (i): The particle comes to rest with the string taut.

Substituting $v = 0$ in (1) and (2), then rearranging:

$$T = mg\cos\theta \tag{4}$$
$$2gr - u^2 = 2gr\cos\theta \tag{5}$$

Using (4): $T > 0$ when $v = 0$

$\Leftrightarrow$ $\cos\theta > 0$ when $v = 0$

Hence the particle comes to rest at a point in the lower half of the circle.

Using (5): $\cos\theta > 0$ when $v = 0$

$\Leftrightarrow$ $2gr - u^2 > 0$

$\Leftrightarrow$ $2gr > u^2$

Hence the particle performs oscillations below the level of O with the string taut if and only if $u^2 < 2gr$.

Case (ii): The string becomes slack while the particle is moving.

Substituting $T = 0$ in (1) and (3), then rearranging:

$$v^2 = -gr\cos\theta \tag{6}$$
$$2gr - u^2 = 3gr\cos\theta \tag{7}$$

Using (6): $v^2 > 0$ when $T = 0$

$\Leftrightarrow$ $-1 < \cos\theta < 0$ when $T = 0$

Hence the string can become slack only at a point in the upper half of the circle.

Using (7): $-1 < \cos\theta < 0$ when $T = 0$

$\Leftrightarrow$ $-3gr < 3gr\cos\theta < 0$ when $T = 0$

$\Leftrightarrow$ $-3gr < 2gr - u^2 < 0$

$\Leftrightarrow$ $0 < u^2 - 2gr < 3gr$

$\Leftrightarrow$ $2gr < u^2 < 5gr$

Hence the string becomes slack at a point above the level of O if and only if $2gr < u^2 < 5gr$. The particle will then leave its circular path and move freely under gravity until the string is again taut.

Case (iii): *The particle performs complete circles*

Using (2): $v > 0$ for all θ

$\Leftrightarrow$ $u^2 > gr(2 - 2\cos\theta)$ for all θ

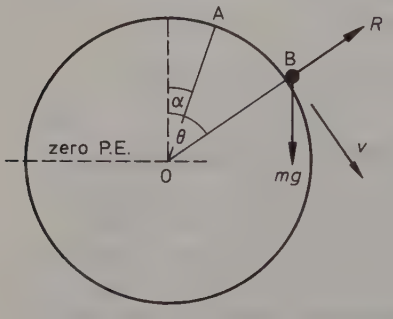
Using (3): $T > 0$ for all θ

$\Leftrightarrow$ $u^2 > gr(2 - 3\cos\theta)$ for all θ

Since $\cos\theta$ takes values from -1 to $+1$, both conditions are satisfied if and only if $u^2 > 5gr$. Hence the particle performs complete circles if and only if $u^2 > 5gr$.

Lastly we illustrate, by means of a worked example, a further way in which motion in a vertical circle may arise.

Example 2 A particle of mass m is released from rest at a point A on the outer surface of a smooth fixed sphere of centre O and radius r. Given that OA makes an angle α with the upward vertical, find an expression for the speed at which the particle is travelling when it leaves the surface of the sphere.



Let v be the speed of the particle when it reaches a point B such that OB makes an angle θ with the upward vertical. Since the sphere is smooth the reaction R of the sphere on the particle acts in the direction OB.

Since the reaction does no work mechanical energy is conserved during the motion,

$\therefore$ (K.E. + P.E.) at A = (K.E. + P.E.) at B

$\therefore$ $0 + mg \times r\cos\alpha = \frac{1}{2}mv^2 + mg \times r\cos\theta.$

i.e. $$2gr\cos\alpha = v^2 + 2gr\cos\theta \qquad (1)$$

Applying Newton's second law towards O,

$$mg\cos\theta - R = m\frac{v^2}{r} \qquad (2)$$

The particle will leave the surface of the sphere when R vanishes.
Putting $R = 0$ in equation (2) gives $gr\cos\theta = v^2$

$\therefore$ from (1): $2gr\cos\alpha = v^2 + 2v^2$

i.e. $$v^2 = \tfrac{2}{3}gr\cos\alpha$$

Hence the speed at which the particle leaves the surface is $\sqrt{(\frac{2}{3}gr\cos\alpha)}$.

Exercise 12.5

1. A particle P of mass 2 kg is suspended from a fixed point O by a light inextensible string of length 2·5 m and hangs in equilibrium at a point A. The particle is projected from A with horizontal speed 5 m s^{-1}. Given that when $\angle AOP = \theta$, the speed of the particle is v m s^{-1} and the tension in the string is T N, find expressions for v^2 and T in terms of θ. Hence find the values of v and T when $\theta = 40°$. Find also to the nearest degree, the greatest value attained by θ in the subsequent motion.

2. A small bead of mass 0·5 kg moves on a smooth circular wire of radius 2 m which is fixed in a vertical plane. When the radius to the bead makes an angle of 60° with the downward vertical the reaction between the bead and the wire is of magnitude 12 N. Find the speed of the bead at this instant and the magnitude of the reaction when the bead passes through the lowest point of the wire.

3. A particle of mass m is suspended from a fixed point O by a light inextensible string of length a. When the particle is in its lowest position it is given a horizontal speed of $\sqrt{(3ag)}$. If v is the speed of the particle when the string makes an angle θ with the upward vertical, find $\cos\theta$ and an expression for v^2 at the instant when the string becomes slack.

4. A particle of mass m is placed inside a smooth hollow sphere of radius a. The particle is set in motion at the lowest point of the sphere with speed $\sqrt{(\lambda ag)}$. Show that the particle will perform complete circles if $\lambda > 5$.

5. A small ring P of mass m is threaded on a smooth circular wire, of centre O and radius r, fixed in a vertical plane. Initially the ring is slightly displaced from the highest point of the wire. Find the magnitude of the force exerted by the ring on the wire when OP makes an angle θ with the upward vertical. Find the range of values of $\cos\theta$ for which this force acts towards O.

6. A particle of mass m slides down the smooth outside surface of a fixed sphere of radius a. At the top of the sphere its velocity is horizontal and of magnitude u. If θ is the angle that the radius to the particle makes with the upward vertical, show that the reaction between the particle and the sphere is

$$mg(3\cos\theta - 2) - mu^2/a.$$

Show that the particle, when just displaced from rest at the top, leaves the surface when $\theta = \cos^{-1}\frac{2}{3}$, and find its speed at that instant. (W)

7. A light rod OP of length a is free to rotate in a vertical plane about a fixed point O, and a mass m is attached to it at P. Denote by θ the angle between OP and the downward vertical. When $\theta = 0$, P is moving horizontally with speed u. Find the tension in the rod as a function of θ, and show that the value of the function is positive throughout the motion if $u^2 > 5ag$. Find the least value of the tension when $u^2 = 6ag$. Show that, if $u^2 < 2ag$, P remains below the level of O and deduce that, in this case also, the value obtained for the tension is positive throughout the motion. (JMB)

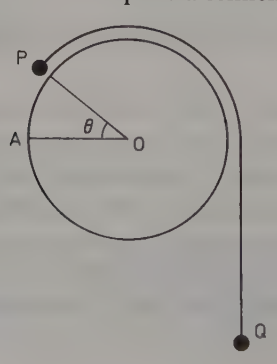
8. A particle P is projected horizontally with speed u from the lowest point A of the smooth inside surface of a fixed hollow sphere of internal radius a.
(i) In the case when $u^2 = ga$ show that P does not leave the surface of the sphere. Show also that, when P has moved halfway along its path from A towards the point at which it first comes to rest, its speed is $\sqrt{\{ga(\sqrt{3} - 1)\}}$.
(ii) Find u^2 in terms of ga in the case when P leaves the surface at a height $3a/2$ above A, and find, in terms of a and g, the speed of P as it leaves the surface. (JMB)

9. A particle of mass m, attached to a fixed point O by an inelastic string of length l, is allowed to fall from a point in the horizontal through O at a distance $l\cos\alpha$ from O. Find the speed of the particle when it first begins to move in a circle. In the subsequent motion, show that, when the string first makes an angle θ ($< \frac{1}{2}\pi - \alpha$) with the downward vertical, the speed v of the particle is $\sqrt{\{2gl(\cos\theta - \sin^3\alpha)\}}$ and the tension in the string is $3mv^2/2l + mg\sin^3\alpha$. Will the tension vanish during the subsequent motion? (W)

10. One end of a light inextensible string of length a is attached to a particle of mass m. The other end is attached to a fixed point O which is at a height $5a/2$ above the horizontal ground. Initially the string is taut and horizontal. The particle is then projected vertically downward with velocity $(2ag)^{\frac{1}{2}}$. When the string has turned through an angle θ ($< \pi$) find the velocity v of the particle and show that the tension in the string is $mg(2 + 3\sin\theta)$.
(a) If the string can withstand a tension of at least $5mg$, prove that the string will not break.
(b) If the string can withstand a tension of at most $7mg/2$, find the values of θ and v when the string breaks. In this case find the time to reach the vertical through O. Hence show that the particle strikes the ground at the point vertically below O. (O&C)

11. Two small smooth pegs O and C are fixed at the same level and $OC = a$. A light inextensible string of length $3a$ has one end attached to O and a particle P of mass m hanging at its other end vertically below O. The particle is projected with speed u parallel to OC. Find the speed of P when OP first makes an acute angle θ with the downward vertical and show that the tension in the string is then $mg(3\cos\theta - 2) + mu^2/3a$. Show that the string will reach a horizontal position if $u^2 \geqslant 6ga$. Given that $u^2 > 6ga$, find the tension when, after the string has struck the peg C, the moving portion CP makes an angle ϕ above the horizontal, and show that P will complete a semicircle with centre C if $u^2 \geqslant 12ga$. (JMB)

12.



A particle P of mass m is at rest at a point A on the smooth outer surface of a fixed sphere of centre O and radius a, OA being horizontal. The particle is attached to one end of a light inextensible string which is taut and passes over the sphere, its other end carrying a particle Q of mass $2m$ which hangs freely. The string lies in the vertical plane containing OA. The system is released from rest,

and after time t the angle POA is θ, as shown in the diagram. Show that, while P remains in contact with the sphere,

$$3a\left(\frac{d\theta}{dt}\right)^2 = 2g(2\theta - \sin\theta).$$

Find the tension in the string and the reaction of the sphere on P in terms of m, g and θ. (JMB)

Exercise 12.6 (*miscellaneous*)

1. A particle is in motion along a straight line. When at a distance x from a fixed point of the line its velocity is v. Show that its acceleration is $v\,dv/dx$.

The gravitational force per unit mass at a distance $x(> a)$ from the earth's centre is ga^2/x^2, where a is the radius of the earth, and g is constant. A particle is projected vertically upwards from the earth's surface at A with speed u. Show that if $u^2 < 2ag$, the particle rises to a maximum height $au^2/(2ag - u^2)$ above A. What happens if $u^2 > 2ag$? (O&C)

2. A man pushes a loaded cart of total mass m along a straight horizontal path against a constant resisting force λm. The cart starts from rest, and the man exerts a force, in the direction of the motion, of magnitude $Ke^{-\alpha x}$, where x is the distance moved, α is a positive constant and K is a constant greater than λm. Write down a differential equation, involving x and the speed v, for the motion of the cart (while it continues to move), and hence obtain v in terms of x and the given constants. (JMB)

3. A particle moves in a straight line in such a manner that when the particle is at the point P, at a distance x from some fixed point O on the line, the only force acting on the particle is in the direction **OP** and has magnitude $m\mu\left\{\frac{a}{x^3} - \frac{1}{x^2}\right\}$, where m is the mass of the particle and μ a positive constant.

If the particle is projected from the point $x = a$ with speed V in the direction **OP** find the set of values to which V must belong if the particle reaches the point $x = 2a$ but does not reach the point $x = 4a$. (C)

4. A particle moves along the x-axis so that its displacement x from the origin O at time t satisfies the differential equation $\frac{d^2x}{dt^2} + 9x - 18 = 0$. Its speed when $t = \pi/12$ is $21/\sqrt{2}$ in the direction of the positive x-axis and its speed when $t = \pi/6$ is 15 in the *opposite* direction. Find x at any time t. What is the greatest distance from O reached by the particle and what is its greatest speed? If the motion starts at $t = 0$, find the time at which the particle first passes through O. [The general solution of the differential equation may be quoted.] (O&C)

5. A particle is projected vertically upwards under (constant) gravity g from a point A on the earth's surface with an initial speed k. At time t after projection, the

particle is at height x above A and its speed is v; the air resistance is gv/k per unit mass. Show that, before the particle reaches its highest point, $\frac{dv}{dt} = -\frac{g}{k}(v + k)$, and $v = k(2e^{-gt/k} - 1)$. Hence find the corresponding value of x as a function of t. (O&C)

6. A bullet of mass 40 g is fired horizontally with velocity 800 m/s into a fixed block of material which exerts a constant resistance, R newtons, which stops it in 0·5 m. Show that $R = 2{\cdot}56 \times 10^4$ and find the time taken. An identical bullet is fired at the same velocity into a second fixed block which offers a resistance which varies as the square root of the speed of the bullet. The initial value of this resistance is $2{\cdot}56 \times 10^4$ N. Write down the equation of motion for the bullet and find the time taken to stop it. Explain why the bullet could never be stopped by a material which offered a resistance proportional to the velocity. (AEB 1979)

7. With the usual notation, prove that $\frac{dv}{dt} = v\frac{dv}{ds}$. A particle P of mass m moves in a straight line and starts from a point O with velocity u. When $OP = x$, where $x \geqslant 0$, the velocity v of P is given by $v = u + \frac{x}{T}$, where T is a positive constant. Show that, at any instance, the force acting on P is proportional to v. Given that the velocity of P at the point A is $3u$, calculate (i) the distance OA in terms of u and T, (ii) the time, in terms of T, taken by P to move from O to A, (iii) the work done, in terms of m and u, in moving P from O to A. (AEB 1980)

8. A point P is travelling in the positive direction on the x-axis with acceleration proportional to the square of its speed v. At time $t = 0$ it passes through the origin with speed gT and with acceleration g. Show that $dv/dx = v/(gT^2)$, and hence obtain an expression for v in terms of x, g and T. Prove that, at time t,

$$x = gT^2 \ln\left(\frac{T}{T-t}\right).$$

Sketch the graph of x against t for $0 \leqslant t < T$. (L)

9. A particle is projected vertically upwards with speed u from a point O. The air resistance is proportional to the fourth power of the velocity and is initially equal to the weight of the particle. If v is the speed of the particle when it is at a height z above O, show that $\frac{d}{dz}(\frac{1}{2}v^2) + g\left(1 + \frac{v^4}{u^4}\right) = 0$.

Solve this equation by writing w for v^2 and so prove that the particle is instantaneously at rest when its height above O is $\pi u^2/8g$. (W)

10. A body of mass m falls from rest under gravity through a resisting medium. The resistance is kv^2 per unit mass where k is constant and v is the velocity when the body has fallen a distance x.

(i) Given that u is the limiting value of the velocity, i.e. the velocity for which there would be zero acceleration, establish the differential equation $v\dfrac{dv}{dx} = k(u^2 - v^2)$.

(ii) By solving this differential equation obtain an expression for v in terms of x. (JMB)

11. A locomotive of mass M works at a constant rate H. Its motion is resisted by a force Mkv, where v is the speed at time t and k is constant. If $c^2 = H/(Mk)$, show that the equation of motion of the locomotive can be written $v\dfrac{dv}{dt} = k(c^2 - v^2)$.

Prove that the locomotive will increase its speed from $\frac{1}{3}c$ to $\frac{1}{2}c$ in time $\{1/(2k)\}\ln(32/27)$, and in this interval of time will travel a distance $\dfrac{c}{2k}\ln(1{\cdot}5) - \dfrac{c}{6k}$. (O&C)

12. A car of mass M is driven by an engine working at constant power and the resistance to motion is proportional to the square of the speed. If R is the resistance at full speed V on a horizontal road, prove that the distance covered as the speed increases from v_1 to v_2 is $\dfrac{MV^2}{3R}\log_e\left(\dfrac{V^3 - v_1^3}{V^3 - v_2^3}\right)$. (O)

13. (a) A particle travelling in a straight line is subject to a retardation of $(v^2 + 25)\,\mathrm{m\,s^{-2}}$, where $v\,\mathrm{m\,s^{-1}}$ is its speed at t seconds. If its initial speed is $12\,\mathrm{m\,s^{-1}}$, show that it travels a distance of $\log_e(13/5)$ metres before coming to rest.

(b) A particle moving with SHM oscillates between points A and B which are 10 m apart. When it is 1 m from A, its speed is $16\,\mathrm{m\,s^{-1}}$. Find (i) the periodic time; (ii) the maximum speed it attains; (iii) the maximum acceleration; (iv) the shortest time it takes to travel from A to C, where AC is $\frac{1}{4}AB$. (SU)

14. A particle moves along a straight line with simple harmonic motion of amplitude 0·5 m and of period 4 seconds. Find the maximum speed in m/s and the maximum acceleration in $\mathrm{m/s^2}$ of the particle. Find also in m/s the speed of the particle when it is 0·25 m from its central position. (Answers may be left in terms of π.) Find also the least time which elapses between two instants when the speed of the particle is half its maximum speed. (L)

15. A particle of mass m moves in simple harmonic motion about O in a straight line, under the action of a restoring force of magnitude proportional to the distance from O. At time $t = 0$ the speed of the particle is zero. After 1 second the speed of the particle is $2\,\mathrm{m\,s^{-1}}$, after a further second the speed is $2\sqrt{3}\,\mathrm{m\,s^{-1}}$ and subsequently the particle passes through O for the first time. Show that the speed of particle when it passes through O is $4\,\mathrm{m\,s^{-1}}$ and find (i) the period and amplitude of the motion, (ii) the time at which the speed of the particle is equal to $2\,\mathrm{m\,s^{-1}}$ after it first passes through O.

A stationary particle of mass m is placed at O and the moving particle collides and coalesces with this particle. The combined particle moves under the action of the

same restoring force as before. Find the period of the subsequent motion and the speed when the displacement is $12/\pi$ m from O. (C)

16. At time t the displacement from the origin of a particle moving along the x-axis under the action of a single force is given by $x = a\cos(\omega t + \alpha)$ where a, ω and α are constants. Find the acceleration of the particle in terms of its displacement from the origin and describe the force acting on the particle. What is the periodic time of this motion?

Two particles, each moving in simple harmonic motion, pass through their centres of oscillation at time $t = 0$. They are next at their greatest distances from their centres at times $t = 2$ s and $t = 3$ s respectively, having been at the same distance from their respective centres at time $t = 1$ s. Show that the amplitudes of their motions are in the ratio $1:\sqrt{2}$. When will the particles next be at their centres, simultaneously? When will the initial conditions next be repeated? (W)

17. A particle moves on a straight line through a fixed point O so that at time t its displacement from O is x and its equation of motion is $\ddot{x} = -\omega^2 x$. If when $t = 0$, $x = d$ and $\dot{x} = u$, show that $x = d\cos\omega t + (u/\omega)\sin\omega t$. Given that the first three times at which the particle passes through the point $x = d$ are $t = 0, t = 2$ and $t = 8$, find d in terms of u.

18. A rough horizontal platform moves horizontally in simple harmonic motion. A particle P of mass 2 kg rests on the platform without slipping and is at O, the centre of oscillations when the motion starts; the initial speed of P is $4\,\text{m s}^{-1}$. The period of the S.H.M. is 8 s. Find (i) the amplitude of the motion correct to 3 sig. fig., (ii) the distance OP, 1 s after the motion starts, correct to 3 sig. fig., (iii) an expression for the time taken for the particle to be first displaced a distance 1 m from O. Do not evaluate the expression.

Draw a diagram showing the forces acting on P with its acceleration and obtain the maximum frictional force acting on P. Hence give the range of values of the coefficient of friction in order that P should not slip. (SU)

19. The end A of a light elastic string AB, obeying Hooke's law and of natural length 0·5 m, is fixed. When a particle of mass 2 kg is attached to the string at B and hangs freely under gravity, the extension of the string in the equilibrium position is 0·075 m. Calculate, in newtons, the modulus of elasticity of the string. The particle is now pulled down vertically a further 0·1 m and released. Show that, until the string becomes slack, the motion of the particle is simple harmonic. Show that the time that elapses before the particle first passes through the equilibrium position is $(\pi\sqrt{3})/40$ s and find the speed of the particle when it is in this position. [Take g as $10\,\text{m/s}^2$]. (L)

20. One end of a light elastic string of natural length l and modulus λ is attached to a fixed point A, and a small mass m, attached to the other end, hangs in equilibrium at B. A small mass $2m$ is made to adhere to the under side of the first mass, and the combined mass is released from rest at B. Show that the combined mass oscillates

about a point C which is at a depth $l + \dfrac{3mgl}{\lambda}$ below A, and that the equation of motion is

$$\frac{d^2x}{dt^2} = -\frac{\lambda}{3ml}x,$$

where x is the depth below C of the combined mass at time t. Show that the speed of the combined mass when $x = 0$ is $2g\sqrt{(ml/3\lambda)}$. Find the force of adhesion between the masses in terms of m, g, λ and x. (JMB)

21. A particle P of mass m lies on a smooth horizontal table and is attached to two fixed points A, B on the table by two light elastic strings, each obeying Hooke's law and of natural lengths $2l, 3l$ and of moduli $2mg, mg$, respectively. If $AB = 7l$, show that, when P is in equilibrium, $AP = 5l/2$. The particle P is held at rest at the point C in the line AB, where $AC = 3l$ and C lies between A and B, and is then released. Show that the motion of P is simple harmonic of period $\pi\sqrt{(3l/g)}$. Find the maximum speed of P during the motion. (O&C)

22. One end of a light elastic string, of natural length $2a$ and modulus $4mg$, is attached to a fixed point A of a smooth horizontal table. A particle of mass m is attached to the other end of the string. The particle is released from rest from a point C of the table, where $AC = 3a$. Show that the particle reaches C again after time $(4 + \pi)\sqrt{(2a/g)}$. (L)

23. A scale pan of mass m is suspended from a fixed point by an elastic spring, of modulus $2mg$ and natural length l. When the scale pan is hanging at rest, a particle of mass $\frac{1}{2}m$ falls vertically on to the scale pan, striking it with speed $\sqrt{(lg)}$. Find the amplitude of the oscillations which the scale pan starts to make if the particle rebounds, the coefficient of restitution between the particle and the scale pan being $\frac{1}{4}$. (L)

24. A particle rests on the *outside* of a fixed smooth vertical hoop, of radius a, at its highest point A, and a second particle rests on the *inside* of the hoop at its lowest point B. The particle at A is just disturbed from rest and the particle at B is given a horizontal velocity v such that both particles leave the hoop at the same point P. Show that (i) the radius OP is inclined to the vertical at an angle $\cos^{-1}\frac{2}{3}$, (ii) $v = 2\sqrt{(ag)}$. (W)

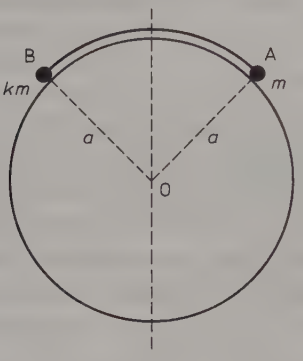
25. A smooth wire in the form of a circle of centre O and radius a is fixed in a vertical plane. Two beads, P of mass $2m$ and Q of mass m, are threaded on to the wire. Initially Q is at rest at A, the lowest point of the wire, and P is held at a point B on the wire, where the angle AOB is $\frac{1}{3}\pi$.

The particle P is then released. Find the speed of P when the angle AOP is θ and P has not yet reached A, and find the reaction then exerted on P by the wire. Show that, when P reaches A, its speed is $\sqrt{(ga)}$. Show that, when the particles collide, the impulse of the reaction between them is of magnitude $\frac{2}{3}m(1 + e)\sqrt{(ga)}$, where e is the

coefficient of restitution. In the case when $e = 1/5$ find the height above A which Q subsequently reaches before coming instantaneously to rest. (JMB)

26. A particle is at rest at the lowest point A inside a smooth hollow sphere of radius a. It is projected horizontally with velocity $\sqrt{(7ga/2)}$. Show that it loses contact with the sphere at a height $3a/2$ above A and will next strike the sphere at A. (C)

27.



Two particles A and B of masses m and km respectively, where $k < 1$, are joined by a light inextensible string of length $\frac{1}{2}\pi a$. The string is taut and the particles are held, at the same horizontal level, in contact with a fixed smooth cylinder of radius a. The axis of the cylinder is horizontal and the string is in a vertical plane perpendicular to the axis of the cylinder. The system is released from rest. If θ is the angle made with the vertical by the radius OA of the cylinder when the particle A is about to lose contact with the cylinder, show that $(3 + k)\cos\theta + 2k\sin\theta = (1 + k)\sqrt{2}$. (C)

28. Two small equal beads, threaded on a vertical smooth circular wire of radius a, are connected by a light inextensible string of length a. One bead is held at the highest point of the wire with the string taut and is then released. Find the speed of the beads at the instant the string first becomes slack. (O)

29. A particle of mass m is projected horizontally with velocity $\mathbf{u}$ from a point O in a medium producing a resisting force $-mk\dot{\mathbf{r}}$ where k is a positive constant and $\mathbf{r}$ is the position vector of the particle relative to O. The acceleration due to gravity is $\mathbf{g}$. Write down the equation of motion (involving $\ddot{\mathbf{r}}$) and show that at time t

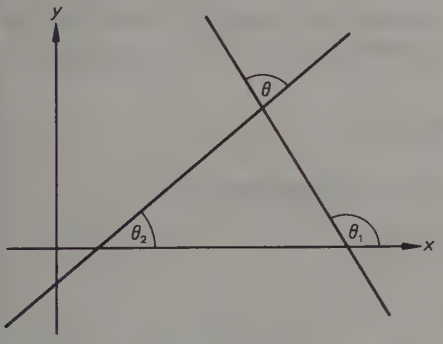
(i) $\dfrac{d}{dt}(\mathbf{r}e^{kt}) = (\mathbf{u} + \mathbf{g}t)e^{kt}$,

(ii) $\mathbf{r} = \dfrac{1}{k}(1 - e^{-kt})\mathbf{u} + \dfrac{1}{k^2}(e^{-kt} + kt - 1)\mathbf{g}$.

If k is so small that k^2 may be neglected compared with k, show that approximately $\mathbf{r} = \mathbf{u}t(1 - \frac{1}{2}kt) + \mathbf{g}t^2(\frac{1}{2} - \frac{1}{6}kt)$. (JMB)

13 Coordinate geometry

13.1 Angles, distances and areas



The diagram shows two lines which make angles θ_1 and θ_2 with the positive direction of the x-axis. If the gradients of these lines are m_1 and m_2 respectively, then

$$m_1 = \tan\theta_1, \qquad m_2 = \tan\theta_2.$$

One angle between the lines is θ, where $\theta = \theta_1 - \theta_2$,

$$\therefore \qquad \tan\theta = \tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}.$$

Hence the angle θ between two lines with gradients m_1 and m_2 is given by

$$\boxed{\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}}$$

Example 1 Find the acute angle between the straight lines $3x - y + 2 = 0$ and $x - 2y - 1 = 0$.

The equations of the lines may be re-written as:

$y = 3x + 2$ and $y = \frac{1}{2}x - \frac{1}{2}$.

Hence the gradients of the lines are 3 and $\frac{1}{2}$.

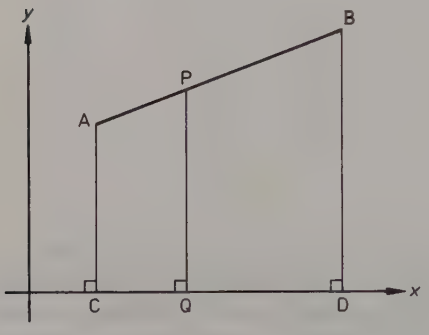
If θ is one angle between the lines, then

$$\tan\theta = \frac{3 - \frac{1}{2}}{1 + 3 \times \frac{1}{2}} = \frac{2\frac{1}{2}}{2\frac{1}{2}} = 1$$

$\therefore$ the acute angle between the lines is 45°.

[Note that the angle between two curves at any point of intersection is defined as the angle between the tangents to the curves at the point.]

Consider now the line joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$. Let P be the point which divides AB in the ratio $\mu:\lambda$. Then if C, D and Q are the feet of the perpendiculars from A, B and P to the x-axis, Q must divide CD in the ratio $\mu:\lambda$. It follows that the x-coordinate of P



$$= x_1 + \frac{\mu}{\lambda + \mu}(x_2 - x_1) = \frac{\lambda x_1 + \mu x_2}{\lambda + \mu}$$

A similar expression is obtained for the y-coordinate of P.

i.e. The point dividing the line AB in the ratio $\mu:\lambda$ has coordinates

$$\left(\frac{\lambda x_1 + \mu x_2}{\lambda + \mu}, \frac{\lambda y_1 + \mu y_2}{\lambda + \mu}\right).$$

This formula can be used when P divides AB externally in a given ratio if μ and λ are taken to have opposite signs.

[We recall that the corresponding formula for the position vector of a point which divides a line in the ratio $\mu:\lambda$ was obtained in §7.1.]

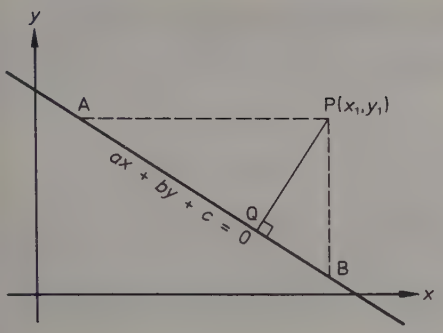
Example 2 If A and B are the points $(2, 1)$ and $(-4, 4)$ respectively, find the coordinates of the point C which divides AB internally in the ratio 2:1 and of the point D which divides AB externally in the ratio 3:2.

The coordinates of C are

$$\left(\frac{1 \times 2 + 2(-4)}{1 + 2}, \frac{1 \times 1 + 2 \times 4}{1 + 2}\right) \quad \text{i.e.} \quad (-2, 3)$$

Regarding D as the point which divides AB in the ratio $3:-2$, the coordinates of D are

$$\left(\frac{(-2)2 + 3(-4)}{-2 + 3}, \frac{(-2)1 + 3 \times 4}{-2 + 3}\right) \quad \text{i.e.} \quad (-16, 10).$$



Next we derive a formula for the perpendicular distance from a point $P(x_1, y_1)$ to a straight line $ax + by + c = 0$. In the diagram A and B are points on the line $ax + by + c = 0$, such that AP is parallel to the x-axis and BP is parallel to the y-axis. The point Q is the foot of the perpendicular from P to the line and so PQ is the required perpendicular distance.

Substituting $y = y_1$ in the equation $ax + by + c = 0$, we find that the x-coordinate of A is $-(by_1 + c)/a$.

$$\therefore \qquad AP = \left| x_1 - \left\{ -\frac{(by_1 + c)}{a} \right\} \right| = \left| \frac{ax_1 + by_1 + c}{a} \right|$$

Similarly $BP = \left| \dfrac{ax_1 + by_1 + c}{b} \right|$

$$\therefore \qquad AB = \sqrt{\{AP^2 + BP^2\}}$$

$$= \sqrt{\left\{ \frac{(ax_1 + by_1 + c)^2}{a^2} + \frac{(ax_1 + by_1 + c)^2}{b^2} \right\}}$$

$$= \sqrt{\left\{ \frac{(ax_1 + by_1 + c)^2(a^2 + b^2)}{a^2 b^2} \right\}}$$

$$= \left| \frac{(ax_1 + by_1 + c)\sqrt{(a^2 + b^2)}}{ab} \right|$$

However, the area of $\triangle APB = \frac{1}{2} \times PQ \times AB = \frac{1}{2} \times AP \times BP$

$$\therefore \qquad PQ = \frac{AP \times BP}{AB}$$

$$= \left| \frac{(ax_1 + by_1 + c)^2}{ab} \times \frac{ab}{(ax_1 + by_1 + c)\sqrt{(a^2 + b^2)}} \right|$$

$$= \left| \frac{ax_1 + by_1 + c}{\sqrt{(a^2 + b^2)}} \right|$$

i.e. The perpendicular distance from the point (x_1, y_1) to the line $ax + by + c = 0$ is $\left| \dfrac{ax_1 + by_1 + c}{\sqrt{(a^2 + b^2)}} \right|$

Example 3 Find the distance of the point $(3, -5)$ from the line $2x - y = 1$.

The distance of the point $(3, -5)$ from the line $2x - y - 1 = 0$ is

$$\left|\frac{2 \times 3 - (-5) - 1}{\sqrt{\{2^2 + (-1)^2\}}}\right| = \frac{10}{\sqrt{5}} = 2\sqrt{5}.$$

Example 4 Find the locus of points equidistant from the lines $y = 2x$ and $2x + 4y - 3 = 0$.

Let $P(x_1, y_1)$ be a point equidistant from the lines $2x - y = 0$ and $2x + 4y - 3 = 0$, then

$$\left|\frac{2x_1 - y_1}{\sqrt{\{2^2 + (-1)^2\}}}\right| = \left|\frac{2x_1 + 4y_1 - 3}{\sqrt{\{2^2 + 4^2\}}}\right|$$

$$\therefore \quad \frac{1}{\sqrt{5}}|2x_1 - y_1| = \frac{1}{\sqrt{20}}|2x_1 + 4y_1 - 3|$$

$$\therefore \quad 2|2x_1 - y_1| = |2x_1 + 4y_1 - 3|$$

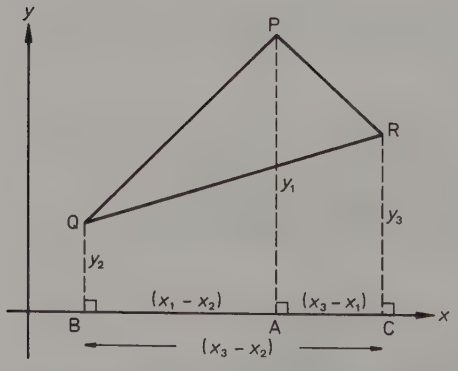
Thus either
$$2(2x_1 - y_1) = 2x_1 + 4y_1 - 3$$
$$4x_1 - 2y_1 = 2x_1 + 4y_1 - 3$$
$$2x_1 - 6y_1 + 3 = 0$$

or
$$2(2x_1 - y_1) = -(2x_1 + 4y_1 - 3)$$
$$4x_1 - 2y_1 = -2x_1 - 4y_1 + 3$$
$$6x_1 + 2y_1 - 3 = 0$$

Hence the locus of all points P equidistant from the given lines is the pair of straight lines

$$2x - 6y + 3 = 0, \qquad 6x + 2y - 3 = 0.$$

It follows that these lines are the bisectors of the angles between the given lines.



Finally we consider the area of the triangle whose vertices are $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$.

As shown in the diagram, the area of ΔPQR = area $PQBA$ + area $PACR$ − area $QBCR$.

Since each of these areas is a trapezium,

$$\text{area of } \Delta PQR = \tfrac{1}{2}(x_1 - x_2)(y_1 + y_2) + \tfrac{1}{2}(x_3 - x_1)(y_1 + y_3) - \tfrac{1}{2}(x_3 - x_2)(y_2 + y_3)$$
$$= \tfrac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}$$

It is found that, although this result holds for any points P, Q and R, it sometimes gives a negative value for the area. We must therefore include modulus signs in the general formula.

$$\text{Area of triangle} = \tfrac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$$

[Note that this formula is rarely needed in elementary work. Problems on area usually involve right-angled triangles, isosceles triangles or other triangles whose heights and bases are easily calculated.]

Exercise 13.1

1. Find the tangent of the acute angle between the following pairs of lines
(a) $y = 2x - 3$, $y = 3x + 1$ (b) $3x - 2y + 1 = 0$, $2x - y - 7 = 0$
(c) $x + 3y - 2 = 0$, $5x - 6y + 2 = 0$ (d) $7x + 4y + 5 = 0$, $y = 0$
(e) $y = 4x$, $2x - 8y - 5 = 0$ (f) $4x + 3y + 2 = 0$, $x = 0$.

2. Find the coordinates of the points which divide the lines joining the given pairs of points internally in the stated ratio.
(a) $(7, 3)$, $(1, -6)$; 1:2 (b) $(2, -4)$, $(-8, 1)$; 2:3
(c) $(0, 6)$, $(2\frac{1}{2}, -4)$; 4:1 (d) $(-3, 5)$, $(1, -1)$; 5:3

3. Find the coordinates of the points which divide the lines joining the given pairs of points externally in the stated ratio.
(a) $(5, -1)$, $(-2, 3)$; 1:2 (b) $(2, -3)$, $(0, 5)$; 2:3
(c) $(4, 3)$, $(-2, 0)$; 4:1 (d) $(-3, 4)$, $(1, -1)$; 5:3

4. Find the distances between the given points and the given lines.
(a) $(1, 2)$, $4x - 3y - 3 = 0$ (b) $(-3, -1)$, $5x + 12y + 1 = 0$
(c) $(2, 5)$, $x - y + 5 = 0$ (d) $(0, 0)$, $2x + 3y = 13$
(e) $(-6, 10)$, $x = 3$ (f) $(7, -4)$, $3y - x - 1 = 0$

5. Find the areas of the triangles with the following vertices:
(a) $(2, 3)$, $(5, -1)$, $(2, -1)$ (b) $(-3, 4)$, $(5, 1)$, $(-3, -2)$
(c) $(-1, 3)$, $(2, -3)$, $(4, 5)$ (d) $(3, -1)$, $(-2, 0)$, $(1, 4)$
(e) $(5, 7)$, $(-4, -11)$, $(0, -3)$ (f) $(-2, 6)$, $(3, -7)$, $(4, 5)$

6. Find the equations of the bisectors of the angles between the given pairs of lines
(a) $3x - y - 3 = 0$, $x + 3y + 1 = 0$ (b) $2x + 3y = 2$, $3x + 2y = 3$
(c) $3x + 6y - 1 = 0$, $x - 2y + 1 = 0$ (d) $y = 7x - 1$, $y = x + 5$

7. Find the angles of the triangle PQR with vertices $P(1, 4)$, $Q(3, -2)$ and $R(5, 2)$.

8. Find the angles of the triangle ABC with vertices $A(1, -3)$, $B(4, 6)$ and $C(-1, 1)$.

9. A triangle has vertices $A(-4, 1)$, $B(3, 0)$ and $C(1, 2)$. If the internal bisector of $\angle B$ meets AC at P, use the fact that $AP : PC = AB : BC$ to find the coordinates of P.

10. A triangle has vertices $P(1, -2)$, $Q(5, 1)$ and $R(6, 10)$. Find the coordinates of the point on QR which is equidistant from the sides PQ and PR.

11. Find the tangent of the acute angle between the parabola $y^2 = 4x$ and the circle $x^2 + y^2 = 5$ at their points of intersection.

12. The curves $y = x^2$ and $y^2 = 8x$ intersect at the origin and at a point A. Find the angle between the curves at A.

13. Find the incentre of the triangle with vertices $(-1, 4)$, $(2, -2)$ and $(7, 8)$.

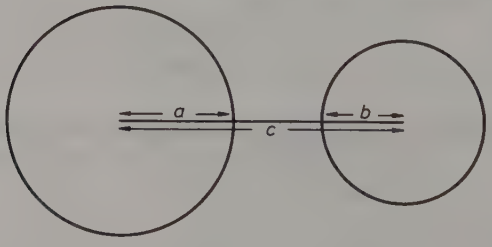
14. Find the incentre of the triangle whose sides have equations $x - y + 1 = 0$, $x + 7y + 9 = 0$ and $7x - y - 2 = 0$.

15. The points $A(-8, 9)$ and $C(1, 2)$ are opposite vertices of a parallelogram $ABCD$. The sides BC, CD of the parallelogram lie along the lines $x + 7y - 15 = 0$, $x - y + 1 = 0$, respectively. Calculate (i) the coordinates of D, (ii) the tangent of the acute angle between the diagonals of the parallelogram, (iii) the length of the perpendicular from A to the side CD, (iv) the area of the parallelogram. (JMB)

13.2 Further work on circles

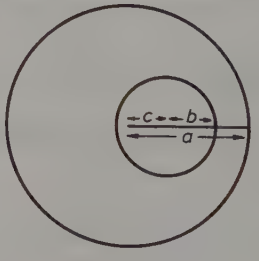
When a problem involves two circles, of radii a and b, the relative positions of the circles can be determined by finding the distance c between their centres. Assuming that $a > b$, there are various different possibilities.

(1)



$c > a + b$

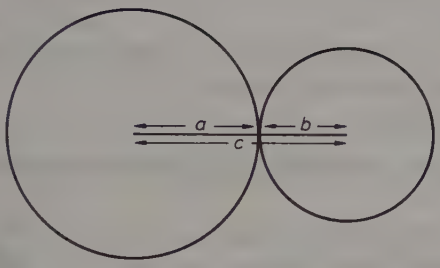
(2)



$c < a - b$

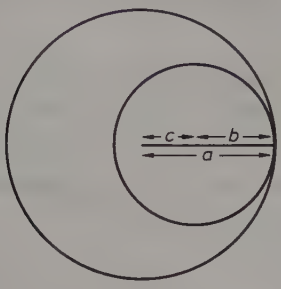
In diagrams (1) and (2) the circles do not intersect.

(3)



$c = a + b$

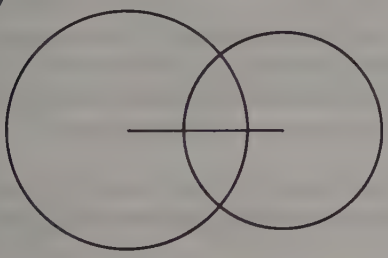
(4)



$c = a - b$

In diagrams (3) and (4) the circles touch either externally or internally.

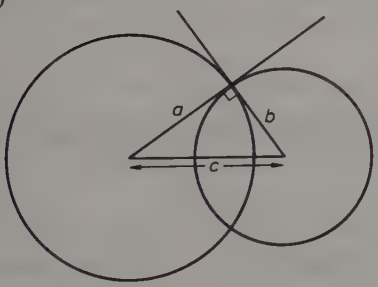
(5)



$a - b < c < a + b$

In diagram (5) the circles have two distinct points of intersection.

(6)



$c^2 = a^2 + b^2$

Two circles which cut at right-angles are called *orthogonal* circles. Diagram (6) shows two such circles.

The *common chord* of two circles is the line joining their points of intersection. The equation of this chord can be obtained without finding the coordinates of the points of intersection.

Example 1 The circles $x^2 + y^2 + 4x - 3y + 1 = 0$ and $x^2 + y^2 + x - y - 2 = 0$ intersect at the points A and B. Find the equation of the common chord AB.

$$x^2 + y^2 + 4x - 3y + 1 = 0 \quad (1)$$
$$x^2 + y^2 + x - y - 2 = 0 \quad (2)$$

Subtracting (2) from (1) we obtain:

$$3x - 2y + 3 = 0$$

Since the coordinates of A and B satisfy equations (1) and (2), they must also satisfy this new equation, which represents a straight line.
Hence $3x - 2y + 3 = 0$ is the equation of AB.

We now extend the method of Example 1 by considering two intersecting circles with equations:

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{(i)}$$
$$x^2 + y^2 + 2Gx + 2Fy + C = 0 \quad \text{(ii)}$$

From these equations we can form a new equation:

$$x^2 + y^2 + 2gx + 2fy + c + k(x^2 + y^2 + 2Gx + 2Fy + C) = 0 \quad \text{(iii)}$$
i.e. $$(1 + k)x^2 + (1 + k)y^2 + 2(g + kG)x + 2(f + kF)y + c + kC = 0$$

This is the equation of a circle for all values of k, except $k = -1$. Moreover, any pair of values of x and y which satisfy equations (i) and (ii) must also satisfy equation (iii). Hence, in general, equation (iii) represents a circle passing through the points of intersection of circles (i) and (ii). When $k = -1$ equation (iii) represents a straight line through the points of intersection i.e. the common chord of the circles.

Example 2 The circles $x^2 + y^2 + 3x - y - 5 = 0$ and $x^2 + y^2 - 2x + y - 1 = 0$ intersect at points A and B. Find the equation of the circle which passes through the origin and the points A and B.

Any circle which passes through A and B has an equation of the form:

$$x^2 + y^2 + 3x - y - 5 + k(x^2 + y^2 - 2x + y - 1) = 0$$

If this circle also passes through the origin,

$$-5 + k(-1) = 0 \quad \text{i.e.} \quad k = -5$$

Hence the equation of the circle which passes through the origin and the points A and B is

$$x^2 + y^2 + 3x - y - 5 - 5(x^2 + y^2 - 2x + y - 1) = 0$$

i.e. $$4x^2 + 4y^2 - 13x + 6y = 0.$$

Exercise 13.2

1. Two circles have equations $x^2 + y^2 - 2x - 6y - 54 = 0$ and
$x^2 + y^2 - 8x + 2y + 13 = 0$.
Show that one circle lies entirely inside the other.

2. Prove that the circles whose equations are $x^2 + y^2 - 2x - 2y - 2 = 0$, $x^2 + y^2 - 6x - 10y + 33 = 0$ lie entirely outside one another.

3. A circle C_1 has equation $x^2 + y^2 - 32x - 24y + 300 = 0$ and a second circle C_2 has as diameter the line joining the points $(8, 0)$ and $(0, 6)$. Show that the circles C_1 and C_2 touch externally.

4. A circle C has equation $x^2 + y^2 - 4x + 2y = 40$. Find the equations of the circles with centre (3, 1) which touch circle C. Find also the coordinates of the points of contact.

5. Show that the circles $x^2 + y^2 - 2x - 2y - 2 = 0$ and
$$x^2 - y^2 - 8x - 10y + 32 = 0$$
touch externally and find the coordinates of the point of contact.

6. Find the equations of the two circles with centres $A(1, 1)$ and $B(9, 7)$ which have equal radii and touch each other externally. Find also the equations of the common tangents to these circles.

7. Two circles with centres $(3, -5)$ and $(-6, 7)$ pass through the point $(4, 2)$. Find the equations of these circles. Find also the equation and the length of their common chord.

8. Prove that the circles whose equations are $x^2 + y^2 - 4y - 5 = 0$, $x^2 + y^2 - 8x + 2y + 1 = 0$ cut orthogonally and find the equation of the common chord.

9. Find the equation of the circle with centre $(3, -2)$ and radius 5 units. Find also the equation of the circle with centre $(-7, 3)$ which intersects the original circle at right angles.

10. Show that any circle which passes through the points $(1, 0)$ and $(-1, 0)$ has an equation of the form $x^2 + y^2 - 2\lambda y - 1 = 0$. Prove also that if the circles given by $\lambda = \lambda_1$ and $\lambda = \lambda_2$ cut orthogonally then $\lambda_1 \lambda_2 = -1$.

11. Show that if the circles $x^2 + y^2 + 2gx + 2fy + c = 0$ and
$$x^2 + y^2 + 2Gx + 2Fy + C = 0$$
cut orthogonally, then $2gG + 2fF = c + C$.

12. The circles $x^2 + y^2 - 2x = 0$ and $x^2 + y^2 + 4x - 6y - 3 = 0$ intersect at the points A and B. Find (a) the equation of AB, (b) the equation of the circle which passes through A, B and the point $(1, 2)$.

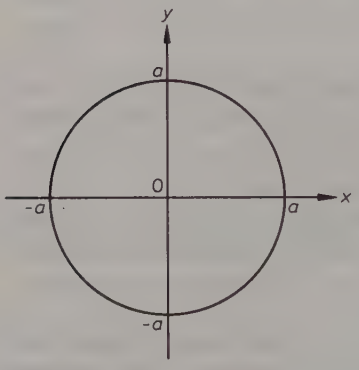
13. Given that the circles $x^2 + y^2 - 3y = 0$ and $x^2 + y^2 + 5x - 8y + 5 = 0$ intersect at P and Q, find the equation of the circle which passes through P, Q and (a) the point $(1, 1)$, (b) the origin, (c) touches the x-axis.

14. The circle $x^2 + y^2 + 3x - 5y - 4 = 0$ and the straight line $y = 2x + 5$ intersect at the points A and B. Find the equation of the circle which passes through A, B and (a) the point $(3, 1)$, (b) the origin, (c) has its centre on the y-axis.

15. Show that the circles $x^2 + y^2 + 8x + 2y + 8 = 0$, $x^2 + y^2 - 2x + 2y - 2 = 0$ have three common tangents, and find their equations. (C)

13.3 Loci using parametric forms

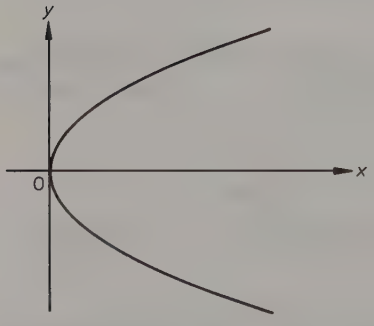
Many problems in coordinate geometry are best solved using the parametric equations of curves. The basic techniques involved were introduced in Book 1 and used to investigate the properties of various curves including the parabola $y^2 = 4ax$. We now consider further applications of these methods. As clear diagrams are often useful in this type of work, we begin by giving sketches of several important curves together with the most commonly used parametric forms of their equations.



The circle $x^2 + y^2 = a^2$.
Parametric equations:

$$x = a\cos\theta,$$
$$y = a\sin\theta.$$

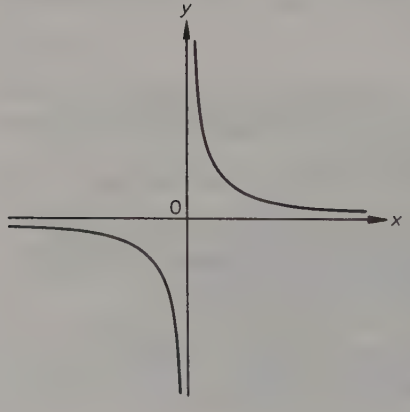
This is the circle with centre O and radius a.



The parabola $y^2 = 4ax$.
Parametric equations:

$$x = at^2,$$
$$y = 2at.$$

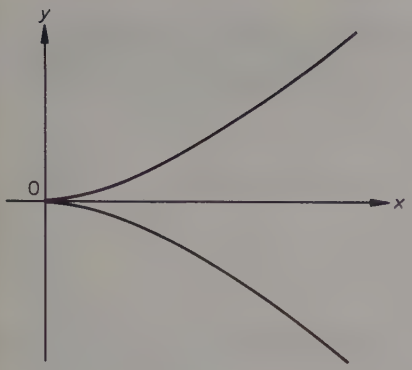
This parabola has vertex O and is symmetrical about the x-axis.



The rectangular hyperbola $xy = c^2$.
Parametric equations:

$$x = ct,$$
$$y = c/t.$$

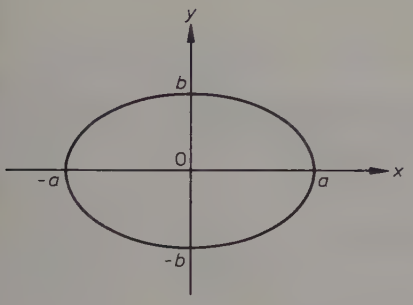
The x- and y-axes are asymptotes to the curve and the curve is symmetrical about the lines $y = \pm x$.



The semi-cubical parabola $ay^2 = x^3$.
Parametric equations:

$$x = at^2,$$
$$y = at^3.$$

The curve has a *cusp* at O and is symmetrical about the x-axis.



The ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.
Parametric equations:

$$x = a\cos\theta,$$
$$y = b\sin\theta.$$

The curve is symmetrical about the x- and y-axes.

Example 1 Find the equation of the chord of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ joining the points $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$.

$$\text{Gradient of } PQ = \frac{b\sin\theta - b\sin\phi}{a\cos\theta - a\cos\phi}$$

$$= \frac{2b\cos\tfrac{1}{2}(\theta+\phi)\sin\tfrac{1}{2}(\theta-\phi)}{-2a\sin\tfrac{1}{2}(\theta+\phi)\sin\tfrac{1}{2}(\theta-\phi)} = -\frac{b\cos\tfrac{1}{2}(\theta+\phi)}{a\sin\tfrac{1}{2}(\theta+\phi)}$$

Thus the equation of the chord PQ is

$$y - b\sin\theta = -\frac{b\cos\tfrac{1}{2}(\theta+\phi)}{a\sin\tfrac{1}{2}(\theta+\phi)}(x - a\cos\theta)$$

i.e. $ay\sin\tfrac{1}{2}(\theta+\phi) - ab\sin\theta\sin\tfrac{1}{2}(\theta+\phi)$

$$= -bx\cos\tfrac{1}{2}(\theta+\phi) + ab\cos\theta\cos\tfrac{1}{2}(\theta+\phi)$$

$\therefore\quad bx\cos\tfrac{1}{2}(\theta+\phi) + ay\sin\tfrac{1}{2}(\theta+\phi)$

$$= ab\{\cos\theta\cos\tfrac{1}{2}(\theta+\phi) + \sin\theta\sin\tfrac{1}{2}(\theta+\phi)\}$$

$$= ab\cos\{\theta - \tfrac{1}{2}(\theta+\phi)\} = ab\cos\tfrac{1}{2}(\theta-\phi).$$

Hence the equation of the chord PQ may be written:

$$\frac{x\cos\tfrac{1}{2}(\theta+\phi)}{a} + \frac{y\sin\tfrac{1}{2}(\theta+\phi)}{b} = \cos\tfrac{1}{2}(\theta-\phi).$$

Example 2 Find the equations of the tangent and the normal to the curve $xy = c^2$ at the point $P(ct, c/t)$. Given that the normal at P meets the curve again at Q, find the coordinates of Q. If the tangent at P meets the y-axis at R, find the equation of the locus of the mid-point M of PR.

The parametric equations of the curve are

$$x = ct, \qquad y = \frac{c}{t}$$

Differentiating with respect to t we have

$$\frac{dx}{dt} = c, \qquad \frac{dy}{dt} = -\frac{c}{t^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \left(-\frac{c}{t^2}\right)\Big/c = -\frac{1}{t^2}$$

Hence the gradients of the tangent and the normal to the curve at P are $-1/t^2$ and t^2 respectively.

$\therefore$ the equation of the tangent at P is

$$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$$

i.e. $$t^2y - ct = -x + ct$$

i.e. $$x + t^2y = 2ct \qquad (1)$$

The equation of the normal at P is

$$y - \frac{c}{t} = t^2(x - ct)$$

i.e. $$ty - c = t^3x - ct^4$$

i.e. $$t^3x - ty = c(t^4 - 1) \qquad (2)$$

Let $(cu, c/u)$ be a point which lies on the curve $xy = c^2$ and on the normal at P, then from equation (2)

$$t^3 \times cu - t \times \frac{c}{u} = c(t^4 - 1)$$

$\therefore$ $t^3u^2 + (1 - t^4)u - t = 0$

$\therefore$ $(u - t)(t^3u + 1) = 0$

$\therefore$ either $u = t$ or $u = -1/t^3$

Since the solution $u = t$ gives the point P, the remaining solution $u = -1/t^3$ must give the point Q.

Hence the coordinates of Q are $(-c/t^3, -ct^3)$.

Substituting $x = 0$ in equation (1) we have,

$$t^2y = 2ct \quad \text{i.e.} \quad y = 2c/t$$

$\therefore$ the coordinates of the point R are $(0, 2c/t)$.

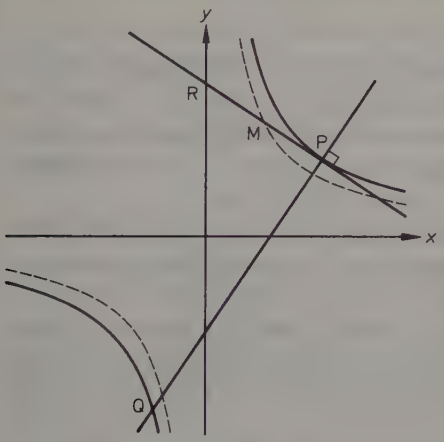
Thus the coordinates of the mid-point M of PR are

$$\left(\frac{1}{2}\{ct+0\}, \frac{1}{2}\left\{\frac{c}{t}+\frac{2c}{t}\right\}\right) \quad \text{i.e.} \quad \left(\frac{1}{2}ct, \frac{3c}{2t}\right)$$

Hence as t varies the parametric equations of the locus of M are

$$x = \frac{1}{2}ct, \qquad y = \frac{3c}{2t}$$

Eliminating t from these equations we obtain the Cartesian equation of the locus, namely $xy = \frac{3}{4}c^2$.



As shown in the diagram the locus of M is another rectangular hyperbola.

Exercise 13.3

1. Write down parametric coordinates for a point on each of the following curves and sketch the curves.

(a) $y^2 = 12x$, (b) $xy = 9$, (c) $y^2 = x^3$,
(d) $4xy = 25$, (e) $4y^2 = x^3$, (f) $y^2 = 10x$.

2. Write down parametric coordinates for a point on each of the following curves and sketch the curves.

(a) $\frac{x^2}{9} + \frac{y^2}{4} = 1$, (b) $x^2 + y^2 = 9$, (c) $\frac{x^2}{5} + y^2 = 1$,
(d) $4x^2 + 9y^2 = 4$, (e) $9x^2 + y^2 = 9$, (f) $x^2 + (y-1)^2 = 25$.

3. Find the equations of the tangent and normal to the curve $xy = 16$ at the point $(4t, 4/t)$.

4. Find the equations of the tangent and normal to the curve $x = t$, $y = 1/t$ at the point where $t = 2$.

5. Find the equations of the tangent and normal to the curve $x = 2\cos\theta$, $y = \sin\theta$ at the point where $\theta = \frac{1}{3}\pi$.

6. P is the point $(5\cos\theta, 4\sin\theta)$ on the curve $x^2/25 + y^2/16 = 1$. Given that S and S' are the points with coordinates $(-3, 0)$ and $(3, 0)$, show that the value of $PS + PS'$ is independent of θ.

7. Find the equations of the tangents to the curve $xy = c^2$ which have gradient -4 and write down the coordinates of their points of contact.

8. The points $P(cp, c/p)$, $Q(cq, c/q)$ and $R(cr, c/r)$ lie on the curve $xy = c^2$. Show that if $\angle RPQ$ is a right angle, then $p^2qr = -1$. Prove further that QR is perpendicular to the tangent at P to the curve.

9. The parametric equations of a curve are $x = 2t$, $y = 2/t$. The tangent to the curve at $P(2t, 2/t)$ meets the x-axis at A; the normal to the curve at P meets the y-axis at B. The point Q divides AB in the ratio 3:1. Find the parametric equations of the locus of Q as P moves on the given curve. (JMB)

10. Let P and Q be points on the rectangular hyperbola $xy = a^2$, with coordinates $(ap, a/p)$ and $(aq, a/q)$. Obtain the equation of the line PQ, and deduce, or obtain otherwise, the equation of the tangent at P to the hyperbola. Show that, if the line PQ meets the coordinate axes at A and B, then the mid-point M of AB is the mid-point of PQ also. Obtain the coordinates of the point T where the tangents at P and Q meet, and show that the line MT passes through the origin. (W)

11. Prove that the normal to the hyperbola $xy = c^2$ at the point $P(ct, c/t)$ has equation $y = t^2x + \dfrac{c}{t} - ct^3$. If the normal at P meets the line $y = x$ at N, and O is the origin, show that $OP = PN$ provided that $t^2 \neq 1$. The tangent to the hyperbola at P meets the line $y = x$ at T. Prove that $OT \,.\, ON = 4c^2$. (O&C)

12. The normal at the point $P(ct, c/t)$ on the rectangular hyperbola $xy = c^2$ meets the curve again at Q. If the normal meets the x-axis at A and the y-axis at B, show that the mid-point of AB is also the mid-point of PQ.

13. Show that the tangent at $P(cp, c/p)$ to the rectangular hyperbola $xy = c^2$ has the equation $p^2y + x = 2cp$. The perpendicular from the origin to this tangent meets it at N, and meets the hyperbola again at Q and R. Prove that (i) the angle QPR is a right angle, (ii) as p varies, the point N lies on the curve whose equation is $(x^2 + y^2)^2 = 4c^2xy$. (C)

14. The tangents at the points $P(cp, c/p)$ and $Q(cq, c/q)$ on the rectangular hyperbolar $xy = c^2$ intersect at the point R. Given that R lies on the rectangular hyperbola $xy = \frac{1}{2}c^2$, find the equation of the locus of the mid-point M of PQ as p and q vary.

15. Find the equation of the tangent to the curve $ay^2 = x^3$ at the point (at^2, at^3) and prove that, apart from one exceptional case, the tangent meets the curve again. Find the coordinates of the point of intersection. What is the exceptional case? (O&C)

16. The tangents to the curve $9y^2 = x^3$ at the points $P(9p^2, 9p^3)$ and $Q(9q^2, 9q^3)$ intersect at the point R. Find the coordinates of R. If p and q vary in such a way that $\angle PRQ$ is always a right angle, find the equation of the locus of R.

17. Show that the equation of the tangent to the curve $x = a\cos t$, $y = b\sin t$ at the point $P(a\cos p, b\sin p)$ is $\frac{x}{a}\cos p + \frac{y}{b}\sin p = 1$. This tangent meets the curve $x = 2a\cos\theta$, $y = 2b\sin\theta$ at the points Q and R, which are given by $\theta = q$ and $\theta = r$ respectively. Show that p differs from each of q and r by $\pi/3$. (JMB)

18. Show that if the tangents to the ellipse $x^2/a + y^2/b = 1$ at the points $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$ intersect at the point R, then the coordinates of R are $\left(\frac{a\cos\frac{1}{2}(\theta+\phi)}{\cos\frac{1}{2}(\theta-\phi)}, \frac{b\sin\frac{1}{2}(\theta+\phi)}{\cos\frac{1}{2}(\theta-\phi)}\right)$. If P and Q move on the ellipse in such a way that $\phi = \theta + \frac{1}{2}\pi$, find the equation of the locus of R.

19. Find the equation of the normal to the curve $x^2 - y^2 = 1$ at the point $P(\sec\theta, \tan\theta)$, where $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$. If this normal cuts the x-axis at A and the y-axis at B, show that P is the centre of the circle which passes through A, B and the origin O. If the line OP cuts this circle again at the point Q, find the equation of the locus of Q as θ varies.

20. (a) The points $A(d, 0)$, $B(-d, 0)$ and $P(d\cos\theta, d\sin\theta)$ lie on the circle $x^2 + y^2 = d^2$. Show that the equation of the tangent to the circle at P is $x\cos\theta + y\sin\theta = d$. Show that the sum of the perpendicular distances from A and B to this tangent is independent of θ, and calculate the product of these distances.
(b) The point $P(p, p^3)$ lies on the curve $y = x^3$. Show that the equation of the tangent to the curve at P is $y - 3p^2x + 2p^3 = 0$. Show that when $p = 3$ this tangent passes through the point $A(7/3, 9)$, and find the other two values of p for which the tangent passes through A. (W)

21. A curve has the parametric equations $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, where $0 \leqslant \theta \leqslant \pi$. A point P of the curve has parameter ϕ, where $\phi \neq 0$.

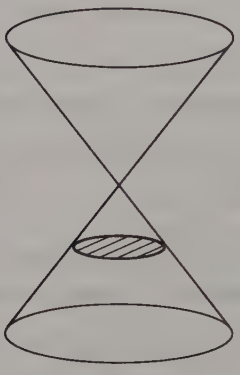
(i) Show that, at P, $\frac{dy}{dx} = \frac{\sin\phi}{1 - \cos\phi}$.

(ii) The normal to the curve at P meets the x-axis at G, and O is the origin. Show that $OG = a\phi$.

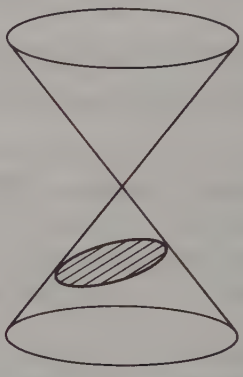
(iii) The tangent to the curve at P meets at K the line through G parallel to the y-axis. Show that $GK = 2a$. (C)

13.4 Conic sections

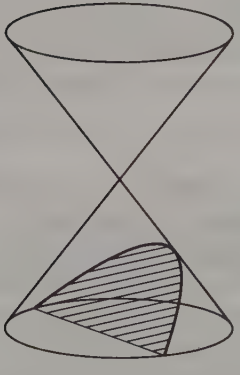
Some of the curves we have been studying arise naturally as cross-sections of a circular cone. The following diagrams show the sections produced when a double cone of semi-vertical angle θ is cut by a plane which does not pass through the vertex. The nature of the curve obtained in each case is determined by the value of the angle α between the plane and the axis of the cone, where $0 \leqslant \alpha \leqslant \frac{1}{2}\pi$.



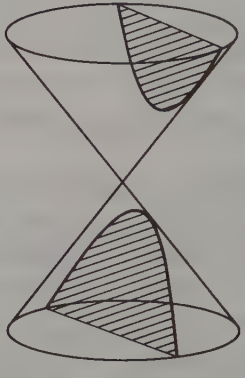
Circle, $\alpha = \frac{1}{2}\pi$



Ellipse, $\alpha > \theta$



Parabola, $\alpha = \theta$

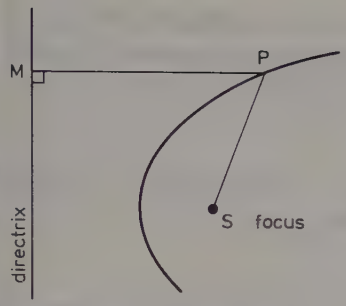


Hyperbola, $\alpha < \theta$

The complete set of conics includes the sections produced by planes through the vertex of the double cone, namely:

for $\alpha < \theta$, a pair of straight lines;
for $\alpha = \theta$, a single straight line;
for $\alpha > \theta$, a single point.

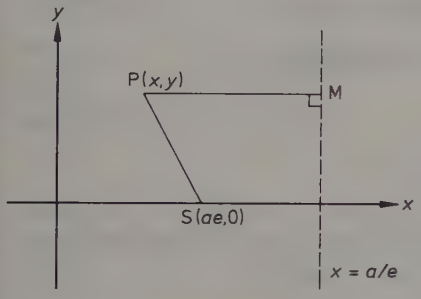
It can be shown that the definition of a parabola as a conic section is consistent with the locus definition used in Book 1. We now consider the corresponding locus definitions of the ellipse and hyperbola.



Suppose that P is a point which moves so that its distance from a fixed point S, the *focus*, is a constant multiple of its distance from a fixed line called the *directrix*. If M is the foot of the perpendicular from P to the directrix then

$$PS = ePM$$

where e is a positive constant called the *eccentricity*. The nature of the locus of P depends on the value taken by e, as follows:



for an ellipse, $0 < e < 1$;
for a parabola, $e = 1$;
for a hyperbola, $e > 1$.

The standard form of the equation of the ellipse is obtained by taking the point $(ae, 0)$ as focus and the line $x = a/e$ as directrix.

$$PS^2 = e^2PM^2$$

$$\therefore \quad (x - ae)^2 + y^2 = e^2\left(\frac{a}{e} - x\right)^2$$

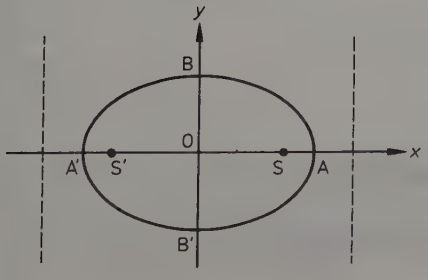
$$x^2 - 2aex + a^2e^2 + y^2 = a^2 - 2aex + e^2x^2$$

$$x^2(1 - e^2) + y^2 = a^2(1 - e^2)$$

i.e.
$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$$

Hence the equation of the ellipse is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \text{where} \quad b^2 = a^2(1 - e^2).$$



From this equation we see that the ellipse is symmetrical about both x- and y-axes. It cuts the x-axis at the points $A(a, 0)$ and $A'(-a, 0)$. It cuts the y-axis at the points $B(b, 0)$ and $B'(-b, 0)$. AA' is called the *major axis* and BB' the *minor axis*. The origin O is the *centre*

of the ellipse and any chord through O is a *diameter*. From symmetry of the curve it is clear that the same locus would have been produced using the point $S'(-ae, 0)$ as focus and the line $x = -a/e$ as directrix. Thus the ellipse is said to have foci $(\pm ae, 0)$ and directrices $x = \pm ae$.

[Note that in terms of a and b the coordinates of the foci are $(\pm\sqrt{\{a^2 - b^2\}}, 0)$ and the equations of the directrices are $x = \pm a^2/\sqrt{(a^2 - b^2)}$.]

Example 1 Find the eccentricity, the foci and the directrices of the ellipse $\frac{x^2}{9} + y^2 = 1$.

For the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we have $b^2 = a^2(1 - e^2)$.

Substituting $a = 3$, $b = 1$ we find that for the given ellipse,

$$1 = 9(1 - e^2)$$
$$\therefore \quad 1 = 9 - 9e^2$$
$$\therefore \quad e^2 = 8/9$$

Hence the eccentricity of the ellipse is $\frac{2}{3}\sqrt{2}$. Thus the foci are the points $(\pm 2\sqrt{2}, 0)$ and the directrices are the lines $x = \pm\frac{9}{4}\sqrt{2}$.

The standard form of the equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad \text{where} \quad b^2 = a^2(e^2 - 1).$$

Using the methods applied to the ellipse it can be shown that this hyperbola has foci $(\pm ae, 0)$ and directrices $x = \pm a/e$.

From the equation we see that the hyperbola is also symmetrical about both x- and y-axes. It cuts the x-axis at the points $A(a, 0)$ and $A'(-a, 0)$, but does not cut the y-axis.

Rearranging the equation we have:

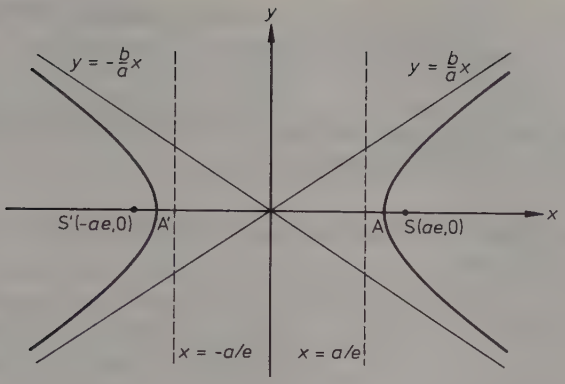
$$\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1$$

$$\therefore \quad \frac{y^2}{x^2} = \frac{b^2}{a^2} - \frac{b^2}{x^2}$$

$$\therefore \quad \text{as } |x| \to \infty, \quad \frac{y^2}{x^2} \to \frac{b^2}{a^2} \quad \text{and} \quad \frac{y}{x} \to \pm\frac{b}{a}$$

i.e. as $|x|$ increases the curve approaches the lines $y = \pm\frac{b}{a}x$.

Hence the lines $y = \pm\frac{b}{a}x$ are asymptotes to the hyperbola.



A hyperbola with perpendicular asymptotes is called a *rectangular hyperbola.* Since the lines $y = \pm\frac{b}{a}x$ are perpendicular when $a = b$, the equation of a rectangular hyperbola is of the form

$$x^2 - y^2 = a^2.$$

When the asymptotes are used as x- and y-axes the equation of the rectangular hyperbola takes the more familiar form $xy = c^2$.

Exercise 13.4

1. Find the eccentricities, the foci and the directrices of the following ellipses.

(a) $\frac{x^2}{25} + \frac{y^2}{9} = 1$, (b) $16x^2 + 25y^2 = 100$, (c) $\frac{x^2}{4} + y^2 = 1$,

(d) $\frac{x^2}{9} + \frac{y^2}{5} = 1$, (e) $\frac{x^2}{4} + \frac{y^2}{9} = 1$, (f) $4x^2 + y^2 = 1$.

2. Find the eccentricities, the foci, the directrices and the asymptotes of the following hyperbolas.

(a) $\frac{x^2}{16} - \frac{y^2}{9} = 1$, (b) $x^2 - y^2 = 4$, (c) $x^2 - 4y^2 = 4$

3. Find the eccentricities of the following ellipses

(a) $x = 2\cos\theta$, $y = \sin\theta$, (b) $x = 5\cos\theta$, $y = 3\sin\theta$,
(c) $x = 2\cos\theta$, $y = 3\sin\theta$, (d) $x = 5\sin\theta$, $y = 3\sin\theta$.

4. Find in the form $x^2/a^2 + y^2/b^2 = 1$ the equations of the ellipses with (a) eccentricity 1/2, foci $(\pm 2, 0)$, (b) eccentricity 3/5, foci $(\pm 9, 0)$.

5. Use the locus definition of the ellipse to find the equation of the ellipse with eccentricity $\frac{2}{3}$, focus $(2, 1)$ and directrix $x = -\frac{1}{2}$.

6. Find the equations of the tangent and the normal to each of the following ellipses at the given point:

(a) $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1; (3, 0)$, (b) $\dfrac{x^2}{8} + \dfrac{y^2}{2} = 1; (-2, 1)$,

(c) $9x^2 + 16y^2 = 40; (2, \frac{1}{2})$, (d) $4x^2 + 5y^2 = 120; (-5, -2)$.

7. Find the equations of the tangents at the point (x_1, y_1) to the following curves:

(a) $4x^2 + 9y^2 = 36$, (b) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, (c) $\dfrac{x^2}{3} - \dfrac{y^2}{2} = 1$.

8. Use the locus definition of the hyperbola to find the equation of the hyperbola with eccentricity $\sqrt{2}$, focus $(2k, 2k)$ and directrix $x + y = k$.

9. The line $y = mx + c$ is a tangent to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1 = 0$, $(a > b > 0)$. Show that $c^2 = a^2m^2 + b^2$. The perpendicular distances from the points $(\sqrt{(a^2 - b^2)}, 0)$ and $(-\sqrt{(a^2 - b^2)}, 0)$ to any tangent to the ellipse are p_1, p_2. Show that $p_1p_2 = b^2$. (C)

10. A line with gradient m is drawn through the fixed point $C(h, 0)$, where $0 < h < a$, to meet the ellipse $x^2/a^2 + y^2/b^2 = 1$ at the points P and Q. Prove that the mid-point R of PQ has the coordinates $\left(\dfrac{a^2hm^2}{a^2m^2 + b^2}, \dfrac{-b^2hm}{a^2m^2 + b^2}\right)$. Show that, as m varies, R always lies on the curve whose equation is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{hx}{a^2}$. (C)

11. The ellipse $x^2/a^2 + y^2/b^2 = 1$ intersects the positive x-axis at A and the positive y-axis at B. Determine the equation of the perpendicular bisector of AB. (i) Given that this line intersects the x-axis at P and that M is the mid-point of AB, prove that the area of triangle PMA is $b(a^2 + b^2)/8a$. (ii) If $a^2 = 3b^2$, find, in terms of b, the coordinates of the points where the perpendicular bisector of AB intersects the ellipse. (JMB)

12. Prove that the hyperbola H_1, with equation $x^2 - y^2 = a^2$, cuts the hyperbola H_2, with equation $xy = c^2$, at right angles at two points P and Q. If the distance between the tangents to H_1 at P and Q is equal to the distance between the tangents to H_2 at P and Q, find the relation between a and c. (O)

Exercise 13.5 (*miscellaneous*)

1. A straight line parallel to the line $2x + y = 0$ intersects the x-axis at A and the y-axis at B. The perpendicular bisector of AB cuts the y-axis at C. Prove that the gradient of the line AC is $-\frac{3}{4}$. Find also the tangent of the acute angle between the line AC and the bisector of the angle AOB, where O is the origin. (JMB)

2. The triangle ABC has vertices $A(0, 12)$, $B(-9, 0)$, $C(16, 0)$. Find the equations of the internal bisectors of the angles ABC and ACB. Hence, or otherwise, find the equation of the inscribed circle of the triangle ABC. Find also the equation of the circle passing through A, B and C. (C)

3. Write down the perpendicular distance from the point (a, a) to the line $4x - 3y + 4 = 0$. The circle, with centre (a, a) and radius a, touches the line $4x - 3y + 4 = 0$ at the point P. Find a, and the equation of the normal to the circle at P. Show that P is the point $(1/5, 8/5)$. Show that the equation of the circle which has centre P and which passes through the origin is $5(x^2 + y^2) - 2x - 16y = 0$. (L)

4. Find the centre and the radius of the circle C which passes through the points $(4, 2)$, $(2, 4)$ and $(2, 6)$. If the line $y = mx$ is a tangent to C, obtain the quadratic equation satisfied by m. Hence or otherwise find the equations of the tangents to C which pass through the origin O. Find also (i) the angle between the two tangents, (ii) the equation of the circle which is the reflection of C in the line $y = 3x$. (AEB 1977)

5. Show that the equation of the tangent to the curve $y^2 = 4ax$ at the point $(at^2, 2at)$ is $ty = x + at^2$ and the equation of the tangent to the curve $x^2 = 4by$ at the point $(2bp, bp^2)$ is $y = px - bp^2$. The curves $y^2 = 32x$ and $x^2 = 4y$ intersect at the origin and at A. Find the equation of the common tangent to these curves and the coordinates of the points of contact B and C between the tangent and the curves. Calculate the area of the triangle ABC. (AEB 1976)

6. On the same diagram sketch the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 10x = 0$. The line $ax + by + 1 = 0$ is a tangent to both these circles. State the distances of the centres of the circles from this tangent. Hence, or otherwise, find the possible values of a and b and show that, if 2ϕ is the angle between the common tangents, then $\tan\phi = \frac{3}{4}$. (L)

7. Find the centre and radius of each of the circles C_1 and C_2 whose equations are $x^2 + y^2 - 16y + 32 = 0$ and $x^2 + y^2 - 18x + 2y + 32 = 0$ respectively and show that the circles touch externally. Find the coordinates of their point of contact and show that the common tangent at that point passes through the origin. The other tangents from the origin, one to each circle, are drawn. Find, correct to the nearest degree, the angle between these tangents. (SU)

8. Two circles, C_1 and C_2, have equations $x^2 + y^2 - 4x - 8y - 5 = 0$ and $x^2 + y^2 - 6x - 10y + 9 = 0$, respectively. Find the x-coordinates of the points P and Q at which the line $y = 0$ cuts C_1, and show that this line touches C_2. Find the tangent of the acute angle made by the line $y = 0$ with the tangents to C_1 at P and Q. Show that, for all values of the constant λ, the circle C_3 whose equation is $\lambda(x^2 + y^2 - 4x - 8y - 5) + x^2 + y^2 - 6x - 10y + 9 = 0$ passes through the points of intersection of C_1 and C_2. Find the two possible values of λ for which the line $y = 0$ is a tangent to C_3. (JMB)

9. The circles whose equations are $x^2 + y^2 - x + 6y + 7 = 0$
and $x^2 + y^2 + 2x + 2y - 2 = 0$

intersect at the points A and B. Find (i) the equation of the line AB, (ii) the coordinates of A and B. Show that the two given circles intersect at right angles and obtain the equation of the circle which passes through A and B and which also passes through the centres of the two circles. (AEB 1978)

10. A curve C_1 has equation $2y^2 = x$; a curve C_2 is given by the parametric equations $x = 4t$, $y = 4/t$. Sketch C_1 and C_2 on the same diagram and calculate the coordinates of their point of intersection, P. The tangent to C_1 at P crosses the x-axis at T and meets C_2 again at Q. (a) Show that T is a point of trisection of PQ. (b) Find the area of the finite region bounded by the x-axis, the line PT and the curve C_1. (L)

11. Show that the tangent at the point P, with parameter t, on the curve $x = ct$, $y = c/t$ has equation $x + t^2y = 2ct$. This tangent meets the x-axis in a point Q and the line through P parallel to the x-axis cuts the y-axis in a point R. Show that, for any position of P on the curve, QR is a tangent to the curve with parametric equation $x = ct$, $y = c/(2t)$. (L)

12. Prove that the equation of the normal to the rectangular hyperbola $xy = c^2$ at the point $P(ct, c/t)$ is $ty - t^3x = c(1 - t^4)$. The normal at P and the normal at the point $Q(c/t, ct)$, where $t > 1$, intersect at the point N. Show that $OPNQ$ is a rhombus, where O is the origin. Hence, or otherwise, find the coordinates of N. If the tangents to the hyperbola at P and Q intersect at T, prove that the product of the lengths of OT and ON is independent of t. (JMB)

13. The line of gradient $m(\neq 0)$ through the point $A(a, O)$ is a tangent to the rectangular hyperbola $xy = c^2$ at the point P. Find m in terms of a and c, and show that the coordinates of P are $(\frac{1}{2}a, 2c^2/a)$. The line through A parallel to the y-axis meets the hyperbola at Q, and the line joining Q to the origin O intersects AP at R. Given that OQ and AP are perpendicular to each other, find the numerical value of c^2/a^2 and the numerical value of the ratio $AR{:}RP$. (JMB)

14. The point P in the first quadrant lies on the curve with parametric equations $x = t^2$, $y = t^3$. The tangent to the curve at P meets the curve again at Q and is normal to the curve at Q. Find the coordinates of P and of Q. (L)

15. Sketch the parabola whose parametric equations are $x = t^2$, $y = 2t$ and on the same diagram sketch the curve with parametric equations $x = 10(1 + \cos\theta)$, $y = 10\sin\theta$. These curves touch at the origin and meet again at two other points A and B. The normals at A and B to the parabola meet at P and the tangents to the other curve at A and B meet at Q. Calculate the length of PQ. (L)

16. Obtain an equation of the tangent, at the point with parameter t, to the curve $\mathscr{C}$ whose parametric equations are given by

$$x = 2\sin^3 t, \quad y = 2\cos^3 t, \quad 0 \leqslant t \leqslant \pi/2.$$

Show that, if the tangent meets the coordinate axes in points R and S, then RS is of constant length. Sketch the curve $\mathscr{C}$. Find the area of the finite region enclosed by the curve $\mathscr{C}$ and the coordinate axes. (L)

17. Sketch the curve whose parametric equations are $x = a\cos\phi$, $y = b\sin\phi$, $0 \leqslant \phi \leqslant 2\pi$, where a and b are positive constants. The point P is given by $\phi = \pi/4$. Find (a) the equation of the tangent to the curve at P, (b) the equation of the normal to the curve at P. By evaluating a suitable integral, calculate the area of the region in the first quadrant between the curve and the coordinate axes. Hence deduce the area of the region enclosed by the curve. (L)

18. Prove that the equation of the tangent at the point (x_1, y_1) on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is $\dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} = 1$. The tangent at the point $(2\cos\theta, \sqrt{3}\sin\theta)$ on the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$ passes through the point $P(2, 1)$. Show that $\sqrt{3}\cos\theta + \sin\theta = \sqrt{3}$. Without using tables, calculator or slide rule, find all the solutions of this equation which are in the range $0° \leqslant \theta < 360°$. Hence obtain the coordinates of the points of contact, Q and R, of the tangents to the ellipse from P. Verify that the line through the origin and the point P passes through the mid-point of the line QR. (JMB)

19. Prove that the equation of the normal at $(\alpha\cos\phi, \beta\sin\phi)$ to the ellipse $\dfrac{x^2}{\alpha^2} + \dfrac{y^2}{\beta^2} = 1$ is $\alpha x\sec\phi - \beta y\operatorname{cosec}\phi = \alpha^2 - \beta^2$. P is the point $(a\cos\theta, b\sin\theta)$ on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. M and N are the feet of the perpendiculars from P to the axes. Find the equation of MN. Prove that, for variable θ, MN is always normal to a fixed concentric ellipse and find the equation of this ellipse. (O&C)

20. Prove that the equation of the tangent to the curve $xy^2 = c^3$ $(c > 0)$ at the point $P(ct^2, c/t)$ is $2t^3y + x = 3ct^2$. Prove that the parameters of the points of contact of the tangents which pass through the point $Q(h, k)$ $(k \neq 0)$ satisfy a cubic equation and, by considering the turning values of this cubic, or otherwise, prove that there are three distinct tangents to the curve which pass through Q if $hk^2 < c^3$

and $h > 0$. State a necessary and sufficient condition on the parameters t_1, t_2, t_3 for the tangents at the corresponding points to be concurrent.

When Q lies on the curve, the tangent at Q cuts the curve again at R; if the parameter of Q is t, determine the parameter of R. Prove that, if the tangents Q_1, Q_2, Q_3 are concurrent and cut the curve again at R_1, R_2, R_3 then the tangents at R_1, R_2, R_3 are concurrent. (O&C)

14 Complex numbers

14.1 Algebraic form

We are already familiar with various types of real number, such as integers, rationals and irrationals. However, since the square of any non-zero real number is positive, there are no solutions within the real number system of the equations such as $x^2 + 2 = 0$ and $x^2 + 2x + 2 = 0$. To deal with these equations we must extend the number system to include the square roots of negative numbers, e.g. $\sqrt{(-1)}$, $-\sqrt{(-5)}$, $\sqrt{(-7/3)}$. Numbers of the form $\pm\sqrt{(-k^2)}$, where k is real, are called *imaginary numbers*. If the number $\sqrt{(-1)}$ is denoted by i, so that $i^2 = -1$, then any imaginary number can be expressed as a multiple of i. For instance:

$$\sqrt{(-4)} = \sqrt{\{4 \times (-1)\}} = \sqrt{4} \times \sqrt{(-1)} = 2i,$$
$$\sqrt{(-7)} = \sqrt{\{7 \times (-1)\}} = \sqrt{7} \times \sqrt{(-1)} = i\sqrt{7}.$$

The set of all numbers of the form $a + ib$, where a and b are real, is called the set of *complex numbers* and denoted by $\mathbb{C}$. The number a is called the *real part* of $a + ib$ and b is the *imaginary part* of $a + ib$. Since $a + ib$ is real when $b = 0$, the set of real numbers $\mathbb{R}$ is a subset of $\mathbb{C}$.

Let us now consider the roots of the quadratic equation $ax^2 + bx + c = 0$, given by $x = \dfrac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$. As we know, when $b^2 - 4ac \geqslant 0$, this equation has real roots, but when $b^2 - 4ac < 0$ the formula for x involves the square root of a negative number. We can write this square root as a multiple of i, thus obtaining the roots of the quadratic equations in the form of complex numbers.

Example 1 Find the roots of the equations (a) $x^2 + 9 = 0$, (b) $x^2 + 2x + 2 = 0$, (c) $2x^2 - 2x + 5 = 0$.

(a) $x^2 + 9 = 0 \Rightarrow x^2 = -9 \Rightarrow x = \pm\sqrt{(-9)} = \pm 3i.$

(b) $x^2 + 2x + 2 = 0 \Rightarrow x = \dfrac{-2 \pm \sqrt{(4-8)}}{2}$

$$= \frac{-2 \pm \sqrt{(-4)}}{2}$$

$$= \frac{-2 \pm 2i}{2} = -1 \pm i.$$

(c) $2x^2 - 2x + 5 = 0 \Rightarrow x = \dfrac{2 \pm \sqrt{(4-40)}}{4}$

$$= \frac{2 \pm \sqrt{(-36)}}{4}$$

$$= \frac{2 \pm 6i}{4} = \frac{1}{2} \pm \frac{3}{2}i.$$

Each of the equations in Example 1 has a pair of roots of the form $p \pm iq$. The numbers $p + iq, p - iq$ are called *conjugate complex numbers* and $p - iq$ is described as the *conjugate* of $p + iq$. Further consideration of the general formula for the roots of a quadratic equation with real coefficients shows that complex roots must always occur in conjugate pairs. Thus if $p + iq$ is one root of a given quadratic equation, then the other root is $p - iq$.

It is assumed that complex numbers can be combined according to the usual laws of algebra, together with the property $i^2 = -1$.

Example 2 Express, in the form $a + ib$, (a) $(2 - 3i) + (-1 + 4i)$, (b) $(2 - 3i) - (-1 + 4i)$, (c) $(2 - 3i)(-1 + 4i)$, (d) $(2 - 3i)(2 + 3i)$.

(a) $(2 - 3i) + (-1 + 4i) = (2 - 1) + (-3 + 4)i = 1 + i,$

(b) $(2 - 3i) - (-1 + 4i) = (2 + 1) + (-3 - 4)i = 3 - 7i,$

(c) $(2 - 3i)(-1 + 4i) = -2 + 8i + 3i - 12i^2$

$= -2 + 8i + 3i + 12 = 10 + 11i,$

(d) $(2 - 3i)(2 + 3i) = 4 + 6i - 6i - 9i^2 = 4 + 9 = 13.$

This last result illustrates the fact that the product of a pair of conjugate complex numbers is real. In general,

$$(a + ib)(a - ib) = a^2 + b^2.$$

This property is used to simplify a quotient of complex numbers.

Example 3 Express $\dfrac{-1 + 2i}{1 + 3i}$ in the form $a + ib$.

Multiplying numerator and denominator by the conjugate of $1 + 3i$:

$$\frac{(-1 + 2i)(1 - 3i)}{(1 + 3i)(1 - 3i)} = \frac{-1 + 3i + 2i - 6i^2}{1 - 3i + 3i - 9i^2} = \frac{-1 + 5i + 6}{1 + 9} = \frac{5 + 5i}{10}$$

$$\therefore \quad \frac{-1 + 2i}{1 + 3i} = \frac{1}{2} + \frac{1}{2}i$$

We next consider two *equal* complex numbers $a + ib$ and $c + id$, where a, b, c, d are real.

$$\begin{aligned} a + ib = c + id &\Rightarrow a - c = i(d - b) \\ &\Rightarrow (a - c)^2 = i^2(d - b)^2 \\ &\Rightarrow (a - c)^2 = -(d - b)^2 \\ &\Rightarrow a - c = d - b = 0 \\ &\Rightarrow a = c \quad \text{and} \quad b = d \end{aligned}$$

Hence $a + ib$ and $c + id$ are equal if and only if $a = c$ and $b = d$. This means that if two complex numbers are known to be equal, we may equate real and imaginary parts.

Example 4 Find the square roots of the complex number $5 - 12i$, giving your answers in the form $a + ib$.

Let $(a + ib)^2 = 5 - 12i$
then $a^2 - b^2 + 2abi = 5 - 12i$
Equating real and imaginary parts:

$$a^2 - b^2 = 5, \quad 2ab = -12$$
i.e. $$a^2 - b^2 = 5, \quad ab = -6$$

By inspection, either $a = 3, b = -2$ or $a = -3, b = 2$.
$\therefore$ the square roots of $5 - 12i$ are $3 - 2i$ and $-3 + 2i$.

[Note that the equations $a^2 - b^2 = 5, ab = -6$ could also be solved by eliminating b to obtain a quadratic equation in a^2.]

It is often convenient to use a single letter, such as z, to represent a complex number. The complex conjugate of z is then denoted by z^* (or sometimes $\bar{z}$). The real and imaginary parts of z are denoted by $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ respectively.

Thus, if $z = x + iy$, $z^* = x - iy$, $\mathrm{Re}(z) = x$ and $\mathrm{Im}(z) = y$.

Exercise 14.1

1. Express the following in terms of i:
(a) $\sqrt{(-25)}$, (b) $\sqrt{(-5)}$, (c) $\sqrt{9} - \sqrt{(-16)}$, (d) $2 - \sqrt{(-18)}$

2. Write down the roots of the following equations:
(a) $x^2 - 4 = 0$, (b) $x^2 + 4 = 0$, (c) $x^2 + 2 = 0$,
(d) $4x^2 + 9 = 0$, (e) $x^3 + 3x = 0$, (f) $x^4 - 1 = 0$.

3. Find the roots of the following equations:
(a) $x^2 - 2x + 2 = 0$, (b) $x^2 - 6x + 10 = 0$, (c) $x^2 - 4x + 13 = 0$,
(d) $4x^2 - 4x + 5 = 0$, (e) $2x^2 + 3x + 2 = 0$, (f) $x^2 - x + 1 = 0$.

4. Write down the conjugates of the following complex numbers:
(a) $3 + 5i$, (b) $4 - 7i$, (c) $3i$, (d) -4, (e) $-1 + i$.

5. Find $z_1 + z_2$ and $z_1 - z_2$ in each of the following cases:
(a) $z_1 = 5 + 8i, z_2 = 2 + i$, (b) $z_1 = 2 + 3i, z_2 = 1 - 4i$,
(c) $z_1 = 3 - i, z_2 = 5 - 3i$, (d) $z_1 = -1 + 2i, z_2 = 3 + 5i$.

6. Express in the form $a + ib$:
(a) $(3 + i)(2 + 3i)$, (b) $(1 - 2i)(5 + i)$, (c) $(2 - i)(3 - 2i)$,
(d) $(3 - 4i)(3 + 4i)$, (e) $(2 - 5i)^2$, (f) $(1 - i)^3$.

7. Simplify (a) i^3, (b) i^5, (c) i^8, (d) $\frac{1}{i^2}$, (e) $\frac{1}{i}$, (f) $\frac{1}{i^3}$.

8. Express in the form $a + ib$:

(a) $\frac{3 + 4i}{1 - 2i}$, (b) $\frac{2i}{3 + i}$, (c) $\frac{3 - 2i}{1 - i}$,

(d) $\frac{\sqrt{3} - i}{\sqrt{3} + i}$, (e) $\frac{1}{(1 + i)^2}$, (f) $\frac{1}{(2 - i)(1 + 2i)}$.

9. Express in the form $a + ib$:

(a) $\frac{1}{2 + 3i} + \frac{1}{2 - 3i}$, (b) $\frac{1}{3 + i} - \frac{1}{1 + 7i}$, (c) $3 + 4i + \frac{25}{3 + 4i}$.

10. Find the values of a and b given that
(a) $(a + ib)(2 - i) = a + 3i$, (b) $(a + i)(1 + ib) = 3b + ia$.

11. Find the square roots of the following complex numbers
(a) $3 - 4i$, (b) $2i$, (c) $8 + 6i$, (d) $21 - 20i$.

12. Find the values of real numbers p and q given that one root of the equation $x^2 + px + q = 0$ is
(a) $2 + i$, (b) $-1 + 3i$, (c) $4i$, (d) $3 - 5i$.

13. Find the complex factors of the following quadratic expressions.
(a) $x^2 + 1$, (b) $x^2 + 25$, (c) $9x^2 + 4$,
(d) $x^2 + 2x + 5$, (e) $x^2 + 6x + 13$, (f) $4x^2 - 12x + 25$.

14. Given that the complex number $a + ib$ is conjugate to its own square, find all possible pairs of values of a and b.

15. Given that $\frac{5}{x + iy} + \frac{2}{1 + 3i} = 1$, where x and y are real, find x and y.

16. If $(1 + i)z - iw + i = iz + (1 - i)w - 3i = 6$, find the complex numbers z, w, expressing each in the form $a + bi$ where a, b are real. (O&C)

17. Express $(6 + 5i)(7 + 2i)$ in the form $a + ib$. Write down $(6 - 5i)(7 - 2i)$ in a similar form. Hence find the prime factors of $32^2 + 47^2$. (JMB)

18. The roots of the quadratic equation $z^2 + pz + q = 0$ are $1 + i$ and $4 + 3i$. Find the complex numbers p and q. It is given that $1 + i$ is also a root of the equation $z^2 + (a + 2i)z + 5 + ib = 0$, where a and b are real. Determine the values of a and b. (JMB)

19. Given that $z = (p + i)^4$, where p is real, find the values of p for which (a) z is real, (b) z is a real multiple of i.

20. Find the two square roots of $5 + 12i$. Hence find the roots of the equation $z^2 - (1 + 6i)z - 10 = 0$.

21. Prove that for any complex number z,

$$\mathrm{Re}(z) = \tfrac{1}{2}(z + z^*) \quad \text{and} \quad \mathrm{Im}(z) = \frac{1}{2i}(z - z^*).$$

14.2 Roots of polynomial equations

In the previous section we saw that a quadratic equation with real coefficients has either real roots or a pair of conjugate complex roots. This result can be extended to polynomial equations of higher degree by considering further the properties of conjugate complex numbers.

For two complex numbers $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, we have

$$\begin{aligned}(z_1 + z_2)^* &= \{(x_1 + iy_1) + (x_2 + iy_2)\}^* \\ &= \{(x_1 + x_2) + i(y_1 + y_2)\}^* \\ &= (x_1 + x_2) - i(y_1 + y_2) \\ &= (x_1 - iy_1) + (x_2 - iy_2) = z_1^* + z_2^* \\ (z_1 z_2)^* &= \{(x_1 + iy_1)(x_2 + iy_2)\}^* \\ &= \{(x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)\}^* \\ &= (x_1 x_2 - y_1 y_2) - i(x_1 y_2 + x_2 y_1) \\ &= (x_1 - iy_1)(x_2 - iy_2) = z_1^* z_2^*\end{aligned}$$

i.e. $\boxed{(z_1 + z_2)^* = z_1^* + z_2^*, \qquad (z_1 z_2)^* = z_1^* z_2^*}$

It follows that $(z^2)^* = (z^*)^2$ and $(z^3)^* = (z^*)^3$.
Thus, if $f(z) = az^3 + bz^2 + cz + d$, where a, b, c, d are real, then

$$\begin{aligned}\{f(z)\}^* &= \{az^3 + bz^2 + cz + d)^* \\ &= (az^3)^* + (bz^2)^* + (cz)^* + d^* \\ &= a(z^3)^* + b(z^2)^* + cz^* + d \\ &= a(z^*)^3 + b(z^*)^2 + cz^* + d = f(z^*)\end{aligned}$$

More generally, it can be shown that $(z^n)^* = (z^*)^n$, where n is a positive integer.

Hence if $P(z)$ is a polynomial of degree n with real coefficients, then

$$\{P(z)\}^* = P(z^*)$$

Suppose now that α is a complex root of the polynomial equation $P(z) = 0$.

$$P(\alpha) = 0 \Leftrightarrow \{P(\alpha)\}^* = \{0\}^* \Leftrightarrow P(\alpha^*) = 0$$

$\therefore$ if α is a root of the equation $P(z) = 0$, its conjugate α^* is also a root.

Thus complex roots of polynomial equations with real coefficients occur in conjugate pairs.

Much of the general theory of polynomial equations is too difficult to discuss here. However, because of its importance, we will state one basic theorem.

Every polynomial equation of degree n has exactly n roots of the form $a + ib$, where a and b are real. Some of these roots may be repeated.

If the polynomial equation $P(z) = 0$ has a real root a, then $P(z)$ has a linear factor $z - a$. Thus, if the equation $P(z) = 0$ has r real roots, then $P(z)$ has r real linear factors.

Similarly, if the equation has a complex root $a + ib$, then since $a - ib$ is also a root, the polynomial $P(z)$ has factors $z - a - ib$ and $z - a + ib$. The product of these factors is $z^2 - 2az + a^2 + b^2$, which is a quadratic factor with real coefficients. Thus, if the equation $P(z) = 0$ has k pairs of conjugate complex roots, then $P(z)$ has k real quadratic factors which cannot be split into real linear factors.

Hence any polynomial with real coefficients is a product of real linear and quadratic factors.

When solving problems it is useful to remember that the formulae for sums and products of roots of polynomial equations remain valid for equations with complex roots.

Example 1 Find the real root of the equation $3z^3 - 10z^2 + 7z + 10 = 0$ given that one root is $2 - i$.

Since $2 - i$ is a root, its complex conjugate $2 + i$ is also a root.

The equation has three roots with sum $-\dfrac{(-10)}{3}$ i.e. $\dfrac{10}{3}$

$\therefore$ the real root of the equation $= \frac{10}{3} - (2 - i + 2 + i) = -\frac{2}{3}$

One cubic equation of special interest is the equation $z^3 = 1$, whose roots are the *cube roots of unity*.

$$z^3 - 1 = 0$$
$$\Rightarrow \quad (z-1)(z^2 + z + 1) = 0$$
$$\Rightarrow \quad \text{either } z - 1 = 0 \quad \text{or} \quad z^2 + z + 1 = 0$$

Hence the complex cube roots of unity are the roots of the equation $z^2 + z + 1 = 0$. Let ω be one of these complex roots, then

$$\omega^3 = 1 \quad \text{and} \quad \omega^2 + \omega + 1 = 0.$$

But $(\omega^2)^3 = \omega^6 = (\omega^3)^2 = 1^2 = 1$,
$\therefore$ ω^2 must also be a complex cube root of unity.

Thus the cube roots of 1 are 1, ω, ω^2, where $\omega^3 = 1$ and $\omega^2 + \omega + 1 = 0$.

By solving the equation $z^2 + z + 1 = 0$ we find that the complex cube roots of 1 are $\frac{1}{2}(-1 \pm i\sqrt{3})$. However, when solving problems involving these roots it is often easier to use ω and its properties, rather than the numerical values of the roots.

Example 2 If ω is a complex cube root of unity, find the value of $\omega^3 + \omega^4 + \omega^5$.

Since $\omega^3 = 1$ and $\omega^2 + \omega + 1 = 0$,
$\omega^3 + \omega^4 + \omega^5 = \omega^3(1 + \omega + \omega^2) = 0$.

Example 3 Given that ω is a complex cube root of unity find the equation whose roots are i, $i\omega$, $i\omega^2$.

The roots of the required equation are given by $z = i\alpha$, where α is a cube root of unity.
$\therefore$ $z^3 = (i\alpha)^3 = i^3\alpha^3 = -i$
Hence the equation with roots i, $i\omega$, $i\omega^2$ is $z^3 + i = 0$.

Exercise 14.2

1. Prove that a number z is real if and only if $z = z^*$.

2. Prove that if z is a complex number and k is real, $(kz)^* = kz^*$.

3. If z_1 and z_2 are two complex numbers, prove that

(a) $(z_1 - z_2)^* = z_1^* - z_2^*$, (b) if $z_2 \neq 0$, then $\left(\frac{z_1}{z_2}\right)^* = \frac{z_1^*}{z_2^*}$.

4. Prove by induction that if z is a complex number then $(z^n)^* = (z^*)^n$, where n is a positive integer.

5. Find the real root of the equation $z^3 + z + 10 = 0$ given that one root is $1 - 2i$.

6. Given that $3 + i$ is a root of the equation $z^3 - 3z^2 - 8z + 30 = 0$, find the remaining roots.

7. Given that $1 + i$ is a root of the equation $z^3 - 2z + k = 0$, find the other two roots and the value of the real constant k.

8. Given that $2 - 3i$ is a root of the equation $z^3 + pz^2 + qz + 13 = 0$, find the other two roots and the value of the real constants p and q.

9. Show that $z = i$ is a root of the equation $z^4 + z^3 + z - 1 = 0$. Find the three other roots.

10. Show that $z = -1 + i$ is a root of the equation $z^4 - 2z^3 - z^2 + 2z + 10 = 0$. Find the remaining roots.

11. Given that $1, \omega_1, \omega_2$ are the roots of the equation $z^3 = 1$, express ω_1 and ω_2 in the form $a + ib$. Hence show that
(a) $(\omega_1)^2 = \omega_2$, (b) $(\omega_2)^2 = \omega_1$, (c) $1/\omega_1 = \omega_2$.

12. Express the roots of the equation $z^3 - \alpha^3 = 0$ in terms of α and ω, where ω is a complex cube root of unity. Use your answer to find the roots of the following equations in the form $a + ib$:
(a) $z^3 - 27 = 0$, (b) $z^3 + 8 = 0$, (c) $z^3 - i = 0$.

13. Find the roots of the following equations in the form $a + ib$:
(a) $z^3 - 8 = 0$, (b) $z^3 + 1 = 0$, (c) $(z + 1)^3 = 1$.

14. Evaluate the following expressions given that ω is a complex cube root of unity.
(a) $\omega + \omega^3 + \omega^5$, (b) $\dfrac{1}{\omega^2 + \omega^4}$, (c) $\dfrac{\omega^2}{\omega + \omega^3}$.

15. Given that ω is a complex cube root of unity, show that $\omega^* = \omega^2$. Hence write down in terms of ω the conjugates of
(a) $1 + \omega$, (b) $1 - \omega^2$, (c) $3 + 4\omega + 5\omega^2$.

16. Given that ω is a complex cube root of unity, simplify the expression

$$(a + b + c)(a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c).$$

17. Given that $1, \omega, \omega^2$ are the cube roots of unity, find the equation whose roots are $\dfrac{1}{3}, \dfrac{1}{2 + \omega}, \dfrac{1}{2 + \omega^2}$.

18. Given that ω denotes either one of the non-real roots of the equation $z^3 = 1$, show that (i) $1 + \omega + \omega^2 = 0$, and (ii) the other non-real root is ω^2.

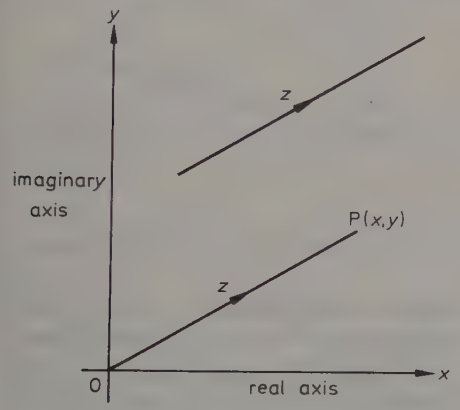
Show that the non-real roots of the equation $\left(\frac{1-u}{u}\right)^3 = 1$ can be expressed in the form $A\omega$ and $B\omega^2$, where A and B are real numbers, and find A and B.

(JMB)

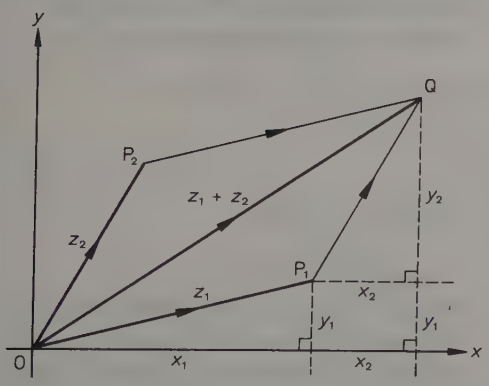
14.3 Geometrical representation

The complex number $z = x + iy$ is completely specified by the ordered pair of real numbers (x, y). Thus to any complex number z there corresponds a unique point $P(x, y)$ in the x, y plane. This suggests that z may be represented by

(1) the point P with coordinates (x, y),
or (2) the vector $\overrightarrow{OP}$, where O is the origin,
or (3) any vector equal to $\overrightarrow{OP}$ in magnitude and direction.



A diagram showing complex numbers represented by points or vectors is called an *Argand*† *diagram.*
Since real numbers are represented along the x-axis and imaginary numbers along the y-axis, in an Argand diagram these axes are often referred to as the *real axis* and the *imaginary axis* respectively.



If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$. Let z_1, z_2 and $z_1 + z_2$ be represented by the vectors $\overrightarrow{OP_1}$, $\overrightarrow{OP_2}$ and $\overrightarrow{OQ}$ respectively. The diagram shows that $\overrightarrow{P_1Q}$ is equal to $\overrightarrow{OP_2}$ in magnitude and direction

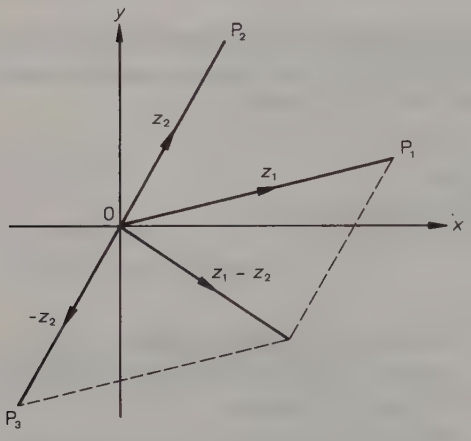
$$\therefore \quad \overrightarrow{OQ} = \overrightarrow{OP_1} + \overrightarrow{P_1Q} = \overrightarrow{OP_1} + \overrightarrow{OP_2}.$$

Thus the sum of two complex numbers z_1 and z_2 is represented in an Argand diagram by the sum of the corresponding vectors $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$.

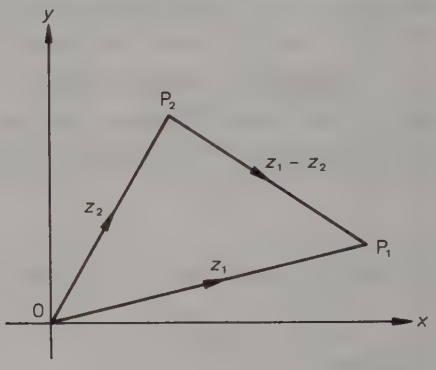
†*Argand, Jean Robert* (1768–1822) Swiss mathematician. He published the geometrical representation of complex numbers in an "Essai" in 1806.

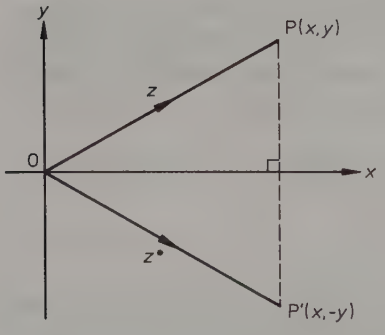
There are two ways of obtaining a geometrical representation of $z_1 - z_2$. If the complex number $(-z_2)$ is represented by the vector $\overrightarrow{OP_3}$, then $z_1 - z_2$ is represented by the sum of the vectors $\overrightarrow{OP_1}$ and $\overrightarrow{OP_3}$, as shown in diagram (i). However, since $\overrightarrow{OP_1} - \overrightarrow{OP_2} = \overrightarrow{P_2P_1}$, $z_1 - z_2$ can also be represented by $\overrightarrow{P_2P_1}$, as shown in diagram (ii).

(i)

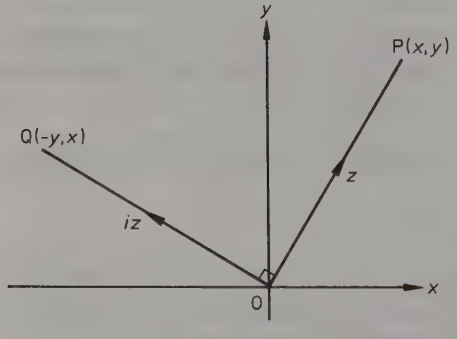


(ii)





Let the complex number $z = x + iy$ and its conjugate $z^* = x - iy$ be represented by $\overrightarrow{OP}$ and $\overrightarrow{OP'}$ respectively. Since the coordinates of P and P' are (x, y) and $(x, -y)$ respectively, P' must be the reflection of P in the real axis.

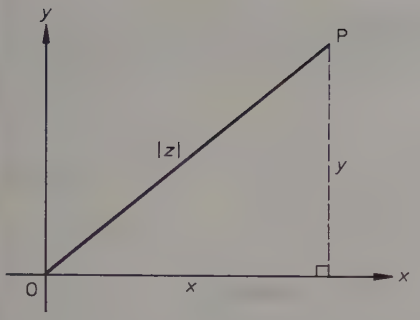


If $z = x = iy$, then

$$iz = i(x + iy) = -y + ix.$$

Thus if z and iz are represented by the vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$, then the coordinates of P and Q are (x, y) and $(-y, x)$ respectively.

Since the gradient of OP is y/x and the gradient of OQ is $-x/y$, OP must be perpendicular to OQ. Hence multiplication of a complex number z by i results in an anti-clockwise rotation of the vector $\overrightarrow{OP}$ through $\frac{1}{2}\pi$ radians.

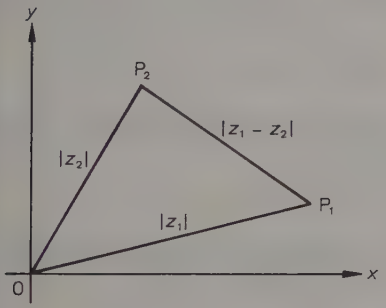


The *modulus* of a complex number z is defined as the length of the corresponding vector $\overrightarrow{OP}$ and is denoted by $|z|$.

Thus if $z = x + iy$, $|z| = \sqrt{(x^2 + y^2)}$.

For instance, $|3 + 4i| = \sqrt{\{3^2 + 4^2\}} = 5$

and $|1 - 2i| = \sqrt{\{1^2 + (-2)^2\}} = \sqrt{5}$.



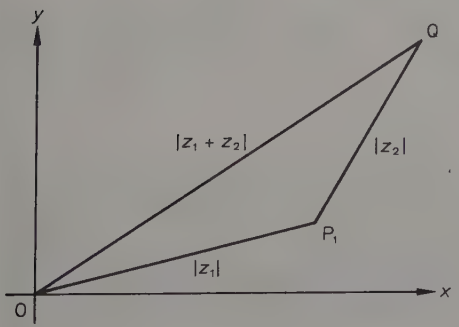
If complex numbers z_1 and z_2 are represented by the vectors $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ respectively, then $z_1 - z_2$ is represented by $\overrightarrow{P_2P_1}$.

$\therefore \qquad |z_1 - z_2| = |\overrightarrow{P_2P_1}|$

Thus $|z_1 - z_2|$ is the distance between the points P_1 and P_2.

Example 1 Given that the complex numbers $2 + 5i$ and $3 - 2i$ are represented in an Argand diagram by the points A and B, find the length of AB.

$$AB = |(2 + 5i) - (3 - 2i)|$$
$$= |-1 + 7i| = \sqrt{\{(-1)^2 + 7^2\}} = \sqrt{50} = 5\sqrt{2}.$$



If complex numbers z_1, z_2 and $z_1 + z_2$ are represented by the vectors $\overrightarrow{OP_1}$, $\overrightarrow{OP_2}$ and $\overrightarrow{OQ}$, then as shown earlier, z_2 is also represented by $\overrightarrow{P_1Q}$. We now use triangle OP_1Q to produce a set of results called the *triangle inequalities*.

Since $OQ \leqslant OP_1 + P_1Q$, $\qquad |z_1 + z_2| \leqslant |z_1| + |z_2|$

Since $OP_1 \leqslant P_1Q + OQ$, $\qquad |z_1| \leqslant |z_2| + |z_1 + z_2|$

$\therefore \qquad |z_1| - |z_2| \leqslant |z_1 + z_2|$

Since $P_1Q \leqslant OP_1 + OQ$, $\quad |z_2| \leqslant |z_1| + |z_1 + z_2|$

$\therefore \quad |z_2| - |z_1| \leqslant |z_1 + z_2|$

Combining these three results, we have

$$\boxed{||z_1| - |z_2|| \leqslant |z_1 + z_2| \leqslant |z_1| + |z_2|}$$

[For an alternative method of proof see Exercise 14.4 question 14.]

Exercise 14.3

1. Write down the complex numbers corresponding to the following points in an Argand diagram.
(a) $(3, 2)$, (b) $(-2, 4)$, (c) $(0, 5)$, (d) $(1, -2)$, (e) $(-2, 0)$.

2. Represent the following complex numbers by vectors in an Argand diagram and find the modulus of the complex number in each case.
(a) $4 + 3i$, (b) $2 - 5i$, (c) $3i$, (d) $-1 - i$, (e) -4.

3. Represent the complex number $z = 2 + i$ by a vector in an Argand diagram. Find and represent in the same diagram $-z, z^*, -z^*, iz$ and $-iz$.

4. Represent the complex number $z = 1 + i$ by a vector in an Argand diagram. Find and represent in the same diagram z^2, z^3 and $1/z$.

5. If $z_1 = 1 + 3i$ and $z_2 = 3 - i$, find z_1z_2 and z_1/z_2 in the form $a + ib$. Show on an Argand diagram the vectors representing z_1, z_2, z_1z_2 and z_1/z_2.

6. Find and represent in an Argand diagram the square roots of
(a) -1, (b) $-2i$, (c) $3 + 4i$, (d) $15 + 8i$.

7. In each of the following cases represent z_1, z_2 and $z_1 + z_2$ in an Argand diagram and verify that

$$||z_1| - |z_2|| \leqslant |z_1 + z_2| \leqslant |z_1| + |z_2|.$$

(a) $z_1 = 12 + 5i, z_2 = -3 + 4i$, (b) $z_1 = -1 - i, z_2 = 7 - i$.

8. If $z_1 = 1 + 2i$, find the set of values of z_2 for which
(a) $|z_1 + z_2| = |z_1| + |z_2|$, (b) $|z_1 + z_2| = |z_1| - |z_2|$,
(c) $|z_1 + z_2| = |z_2| - |z_1|$.

9. If z_1 and z_2 are complex numbers, show geometrically that

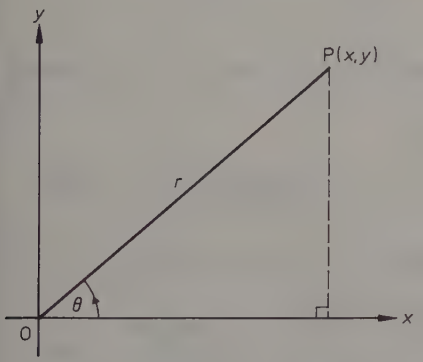
$$||z_1| - |z_2|| \leqslant |z_1 - z_2| \leqslant |z_1| + |z_2|.$$

10. In each of the following cases find the distance between points in the Argand diagram representing the given pair of complex numbers.
(a) $2 + i, 5 - 3i$, (b) $1 - 2i, 3 + 7i$, (c) $2 - 5i, -1 - 3i$.

11. The points A, B, C and D in the Argand diagram correspond to the complex numbers $8 - i, 3 + 11i, -9 + 6i, -4 - 6i$. Prove that $ABCD$ is a square.

12. Given that $z = 1 + i\sqrt{2}$, express in the form $a + ib$ each of the complex numbers $p = z + \frac{1}{z}, q = z - \frac{1}{z}$. In an Argand diagram, P and Q are the points which represent p and q respectively, O is the origin, M is the midpoint of PQ and G is the point on OM such that $OG = \frac{2}{3}OM$. Prove that the angle PGQ is a right angle. (JMB)

14.4 Modulus-argument form



Let us suppose that a complex number $z = x + iy$ is represented in an Argand diagram by the vector $\overrightarrow{OP}$, where r is the length of $\overrightarrow{OP}$ and θ is the angle $\overrightarrow{OP}$ makes with the positive direction of the real axis.

From the diagram we see that

$$x = r\cos\theta, \qquad y = r\sin\theta,$$
$$r^2 = x^2 + y^2, \quad \tan\theta = y/x.$$

As we already know, the length r is called the modulus of z and written $|z|$,

$$\therefore \qquad \boxed{|z| = r = \sqrt{(x^2 + y^2)}}$$

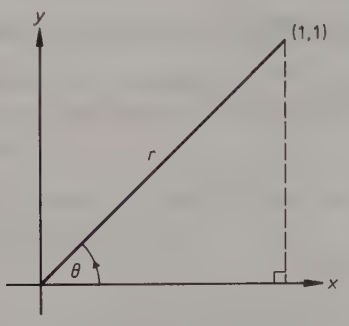
The angle θ is called the *argument* of z (or less commonly, the amplitude), positive values of θ being measured anti-clockwise from the real axis and negative values clockwise. The problem with this definition is that for any given z there is an infinite set of possible values for θ. To avoid confusion we usually work with the value of θ for which $-\pi < \theta \leqslant \pi$. This is called the *principal argument* of z and denoted by arg z.

$$\therefore \qquad \boxed{\arg z = \theta, \quad \text{where} \quad \sin\theta = \frac{x}{r}, \quad \cos\theta = \frac{y}{r} \quad \text{and} \quad -\pi < \theta \leqslant \pi.}$$

In practice the formula $\tan\theta = y/x$ is often used to find the principal argument of a complex number z, despite the fact that it leads to two possible values for θ in the permitted range. The correct value of $\arg z$ is then chosen with the aid of a sketch showing the approximate position of the corresponding vector $\overrightarrow{OP}$.

Example 1 Find the moduli and principal arguments of the complex numbers (a) $1 + i$, (b) $-1 - i\sqrt{3}$, (c) -5.

(a)

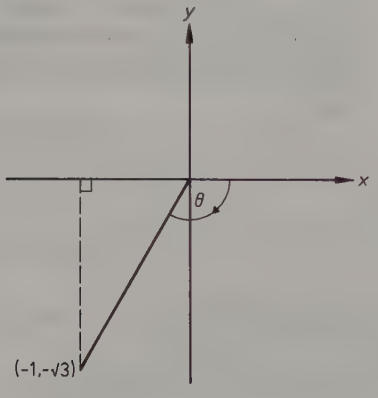


$|1 + i| = \sqrt{(1^2 + 1^2)} = \sqrt{2}$
If $\arg(1 + i) = \theta$,
then $\tan\theta = \frac{1}{1} = 1 = \tan\frac{1}{4}\pi$
Since the point representing $1 + i$ lies in the first quadrant, $0 < \theta < \frac{1}{2}\pi$.

$\therefore \quad \theta = \frac{1}{4}\pi$

Hence $|1 + i| = \sqrt{2}$ and $\arg(1 + i) = \frac{1}{4}\pi$.

(b)



$|-1 - i\sqrt{3}| = \sqrt{\{(-1)^2 + (-\sqrt{3})^2\}}$
$= \sqrt{\{1 + 3\}} = 2$

If $\arg(-1 - i\sqrt{3}) = \theta$, then

$$\tan\theta = \frac{-\sqrt{3}}{-1} = \sqrt{3} = \tan\tfrac{1}{3}\pi$$

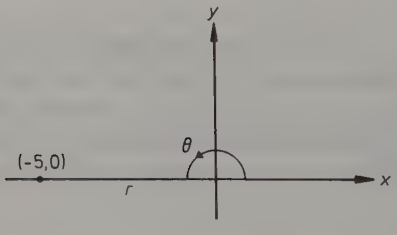
Since the point representing $-1 - i\sqrt{3}$ lies in the third quadrant,

$$-\pi < \theta < -\tfrac{1}{2}\pi.$$

$\therefore \quad \theta = \frac{1}{3}\pi - \pi = -\frac{2}{3}\pi.$

Hence $|-1 - i\sqrt{3}| = 2$ and $\arg(-1 - i\sqrt{3}) = -\frac{2}{3}\pi$.

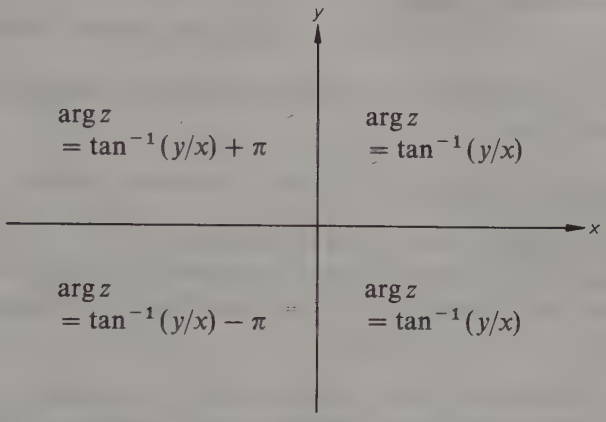
(c)



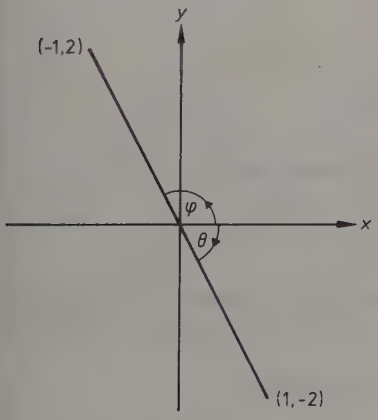
Since the point representing -5 lies on the negative part of the real axis, at a distance of 5 units from the origin, $|-5| = 5$ and $\arg(-5) = \pi$.

When obtaining the argument of a complex number by calculator, care must be taken to use the inverse tangent function properly. Since this function always takes

values between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$, it gives the correct value of arg z only when z is represented by a point in the first or fourth quadrants. When the point representing z lies in the second or third quadrants, it will be necessary to add $\pm\pi$ to the value obtained for $\tan^{-1}(y/x)$.



Example 2 Find in radians the principal arguments of the complex numbers $1 - 2i$ and $-1 + 2i$.



If $\arg(1-2i)=\theta$, then $\tan\theta=\dfrac{-2}{1}=-2$

Since the point $(1,-2)$ lies in the 4th quadrant,

$\arg(1-2i)=\tan^{-1}(-2)\approx -1{\cdot}107$

If $\arg(-1+2i)=\phi$, then

$$\tan\phi=\frac{2}{-1}=-2$$

Since the point $(-1,2)$ lies in the 2nd quadrant,

$\arg(-1+2i)=\tan^{-1}(-2)+\pi\approx 2{\cdot}034$.

If a complex number $z = x + iy$ has modulus r and argument θ, then $x = r\cos\theta$ and $y = r\sin\theta$

$$\therefore \qquad z = r(\cos\theta + i\sin\theta)$$

This is called the *modulus-argument* form of the number.

Using the results of Example 1 the numbers $1 + i$, $-1 - i\sqrt{3}$ and -5 have expressions in modulus-argument form as follows:

$$1 + i = \sqrt{2}(\cos\tfrac{1}{4}\pi + i\sin\tfrac{1}{4}\pi),$$
$$-1 - i\sqrt{3} = 2\{\cos(-\tfrac{2}{3}\pi) + i\sin(-\tfrac{2}{3}\pi)\},$$
$$-5 = 5(\cos\pi + i\sin\pi).$$

Example 3 Write down the moduli and arguments of (a) $3(\cos 2\alpha + i\sin 2\alpha)$, (b) $\cos\frac{2}{5}\pi + i\sin\frac{2}{5}\pi$, (c) $a(\cos\phi - i\sin\phi)$.

(a) Since $r(\cos\theta + i\sin\theta)$ has modulus r and argument θ, $3(\cos 2\alpha + i\sin 2\alpha)$ has modulus 3 and argument 2α.
(b) Since $\cos\frac{2}{5}\pi + i\sin\frac{2}{5}\pi = 1(\cos\frac{2}{5}\pi + i\sin\frac{2}{5}\pi)$, this number has modulus 1 and argument $\frac{2}{5}\pi$.
(c) Since $a(\cos\phi - i\sin\phi) = a\{\cos(-\phi) + i\sin(-\phi)\}$, its modulus is a and its argument is $(-\phi)$.

We now investigate the general properties of complex numbers in modulus-argument form.
If $z = r(\cos\theta + i\sin\theta)$, then $z^* = r(\cos\theta - i\sin\theta)$.
$\therefore\quad zz^* = r^2(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = r^2(\cos^2\theta + \sin^2\theta) = r^2$.
Since z^* may also be written as $r\{\cos(-\theta) + i\sin(-\theta)\}$, the modulus of z^* is r and its argument is $(-\theta)$. It follows that:

$$zz^* = |z|^2, |z^*| = |z| \text{ and, for } -\pi < \arg z < \pi, \arg z^* = -\arg z.$$

If $z = r(\cos\theta + i\sin\theta)$, then $\dfrac{1}{z} = \dfrac{1}{r(\cos\theta + i\sin\theta)}$

$$\therefore\quad \frac{1}{z} = \frac{\cos\theta - i\sin\theta}{r(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)}$$

$$= \frac{\cos\theta - i\sin\theta}{r(\cos^2\theta + \sin^2\theta)} = \frac{1}{r}\{\cos(-\theta) + i\sin(-\theta)\}$$

$$\therefore\quad \left|\frac{1}{z}\right| = \frac{1}{|z|} \text{ and, for } -\pi < \arg z < \pi, \arg\left(\frac{1}{z}\right) = -\arg z.$$

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then

$$\begin{aligned} z_1z_2 &= r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1r_2\{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\} \\ &= r_1r_2\{\cos\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\} \end{aligned}$$

Hence z_1z_2 has modulus r_1r_2 and argument $\theta_1 + \theta_2$.

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{r_2(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)}$$

$$= \frac{r_1\{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)\}}{r_2(\cos^2\theta_2 + \sin^2\theta_2)}$$

$$= \frac{r_1}{r_2}\{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\}$$

Hence z_1/z_2 has modulus r_1/r_2 and argument $\theta_1 - \theta_2$.

$\therefore$

$$\boxed{|z_1 z_2| = |z_1||z_2| \quad \text{and} \quad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}}$$

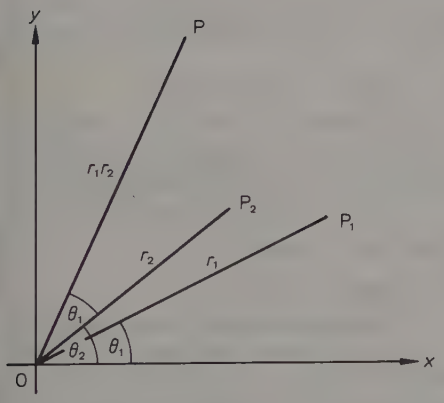
If $\theta_1 = \arg z_1$ and $\theta_2 = \arg z_2$, it follows that $z_1 z_2$ has argument

$$\theta_1 + \theta_2 = \arg z_1 + \arg z_2.$$

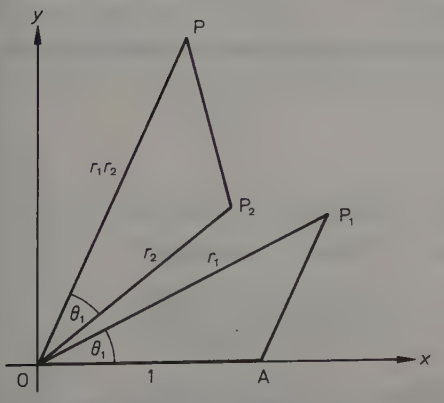
However, since $-\pi < \theta_1 \leqslant \pi$ and $-\pi < \theta_2 \leqslant \pi$, $\theta_1 + \theta_2$ may itself lie outside the interval from $-\pi$ to π. In such cases $\arg z_1 + \arg z_2$ does not give the principal argument of $z_1 z_2$ and it is necessary to add or subtract 2π to obtain $\arg(z_1 z_2)$. Similar adjustments may be needed when finding $\arg(z_1/z_2)$.

$\therefore$

$$\boxed{\begin{aligned} &\arg(z_2 z_2) = \arg z_1 + \arg z_2 \ (\pm 2\pi \text{ if necessary}), \\ &\arg(z_1/z_2) = \arg z_1 - \arg z_2 \ (\pm 2\pi \text{ if necessary}). \end{aligned}}$$



Let z_1 and z_2 be represented in an Argand diagram by the vectors $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ respectively. When z_2 is multiplied by z_1 the vector $\overrightarrow{OP_2}$ is rotated through an angle θ_1 and its length is multiplied by r_1 to produce the vector $\overrightarrow{OP}$ representing $z_1 z_2$.



In this diagram we see that if A is the point $(1, 0)$, then triangles OAP_1 and OP_2P are similar.

[There are similar geometrical representations of the quotient z_1/z_2.]

Example 4 If $z_1 = 6(\cos\frac{5}{12}\pi + i\sin\frac{5}{12}\pi)$ and $z_2 = 3(\cos\frac{1}{4}\pi + i\sin\frac{1}{4}\pi)$, find z_1z_2 and z_1/z_2 in the form $a + ib$.

$$z_1z_2 = 6 \times 3\{\cos(\tfrac{5}{12}\pi + \tfrac{1}{4}\pi) + i\sin(\tfrac{5}{12}\pi + \tfrac{1}{4}\pi)\}$$

$$= 18\{\cos\tfrac{2}{3}\pi + i\sin\tfrac{2}{3}\pi\} = 18\left\{-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right\} = -9 + 9\sqrt{3}i$$

$$\frac{z_1}{z_2} = \frac{6}{3}\{\cos(\tfrac{5}{12}\pi - \tfrac{1}{4}\pi) + i\sin(\tfrac{5}{12}\pi - \tfrac{1}{4}\pi)\}$$

$$= 2\{\cos\tfrac{1}{6}\pi + i\sin\tfrac{1}{6}\pi\} = 2\left\{\frac{\sqrt{3}}{2} + \frac{1}{2}i\right\} = \sqrt{3} + i.$$

Exercise 14.4

In questions 1 to 4 find the modulus and principal argument of each of the given complex numbers. Give the argument in radians, either as a multiple of π or correct to 3 decimal places.

1. (a) $\sqrt{3} + i$ (b) $1 - i$ (c) $4i$ (d) -3

2. (a) $3 - 4i$ (b) $-2 + i$ (c) $-1 - 3i$ (d) $5 - 3i$

3. (a) $\dfrac{10}{\sqrt{3} - i}$ (b) $\dfrac{1}{1 + i\sqrt{3}}$ (c) $\dfrac{2 + 3i}{5 + i}$ (d) $\dfrac{7 - i}{-4 - 3i}$

4. (a) $5(\cos\frac{1}{3}\pi + i\sin\frac{1}{3}\pi)$ (b) $\cos\frac{3}{2}\pi + i\sin\frac{3}{2}\pi$
   (c) $\cos\frac{2}{3}\pi - i\sin\frac{2}{3}\pi$ (d) $-2(\cos\frac{1}{4}\pi + i\sin\frac{1}{4}\pi)$

5. Express the following in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $-\pi < \theta \leqslant \pi$,
(a) $\sqrt{3} - 3i$ (b) 8 (c) $-2 - 2i$ (d) $-i$

6. If $z = r(\cos\theta + i\sin\theta)$, where $r > 0$ and $0 < \theta < \frac{1}{2}\pi$, find in terms of r and θ the modulus and principal argument of
(a) $-z$ (b) iz (c) z^2 (d) z/z^*

7. Simplify the following expressions:
   (a) $(\cos\frac{1}{4}\pi + i\sin\frac{1}{4}\pi)(\cos\frac{3}{4}\pi + i\sin\frac{3}{4}\pi)$ (b) $(\cos\frac{5}{12}\pi + i\sin\frac{5}{12}\pi)^2$
   (c) $(\cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi)(\cos\frac{7}{12}\pi + i\sin\frac{7}{12}\pi)$ (d) $(\cos\frac{2}{9}\pi + i\sin\frac{2}{9}\pi)^3$
   (e) $\dfrac{(\cos\frac{1}{3}\pi + i\sin\frac{1}{3}\pi)}{(\cos\frac{5}{6}\pi + i\sin\frac{5}{6}\pi)}$ (f) $\dfrac{(\cos\frac{1}{4}\pi + i\sin\frac{1}{4}\pi)^2}{(\cos\frac{1}{6}\pi + i\sin\frac{1}{6}\pi)}$

8. Find the moduli and principal arguments of w, z, wz and w/z, given that:
(a) $w = 10i$, $z = 1 + i\sqrt{3}$, (b) $w = -2\sqrt{3} + 2i$, $z = 1 - i$.

9. Given that $z = 2/(1 + \cos 2\theta - i \sin 2\theta)$, where $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$, find expressions for $|z|$ and $\arg z$ in terms of θ.

10. Given that $z = \cos\theta + i\sin\theta$, where $0 < \theta < \pi$, find the moduli and principal arguments of $1 - z$, $z - 1$, $1/(1 - z)$, $z/(z - 1)$.

11. If $z_1 = 1 + i\sqrt{3}$ and $z_2 = 2i$, find $|z_1|$ and $\arg z_1$, $|z_2|$ and $\arg z_2$. Using an Argand diagram, deduce that $\arg(z_1 + z_2) = 5\pi/12$. Hence show that $\tan(5\pi/12) = 2 + \sqrt{3}$.

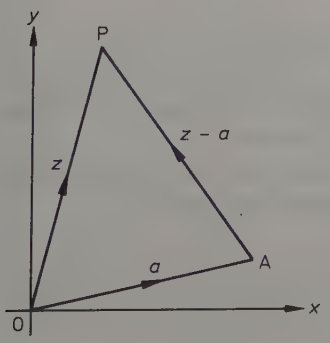
12. Two non-zero complex numbers z_1 and z_2 are such that $|z_1 + z_2| = |z_1 - z_2|$. Represent $z_1, z_2, z_1 + z_2$ and $z_1 - z_2$ by vectors on an Argand diagram. Hence, or otherwise, find the possible values of $\arg(z_1/z_2)$. (JMB)

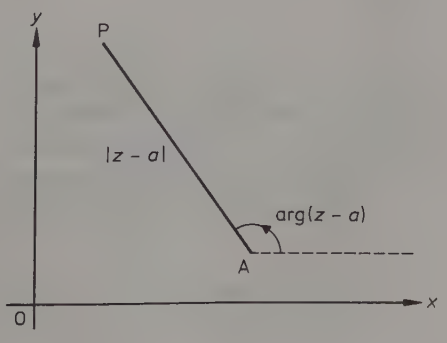
13. (a) Express the complex number $z = \dfrac{1 + i}{3 - 2i}$ in the form $a + bi$ $(a, b \in \mathbb{R})$, and find the argument of z, giving your answer in radians correct to 3 significiant figures. Write down the argument of z^2, and find the exact value of $|z^2|$.
(b) The conjugate of the non-zero complex number z is denoted by z^*. In an Argand diagram with origin O, the point P represents z and Q represents $1/z^*$. Prove that O, P and Q are collinear, and find the ratio $OP{:}OQ$ in terms of $|z|$. (C)

14. Show that, for any complex number z, (a) $zz^* = |z|^2$, (b) $z + z^* = 2\,\mathrm{Re}(z)$, (c) $\mathrm{Re}(z) \leqslant |z|$. Hence show that
(i) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1 z_2^*)$,
(ii) $|z_1 + z_2| \leqslant |z_1| + |z_2|$.

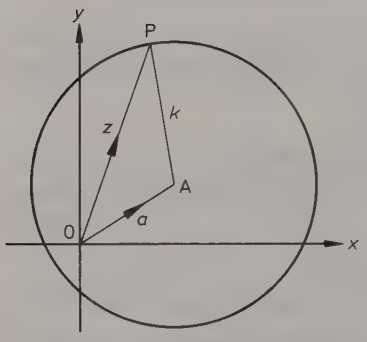
14.5 Loci in the Argand diagram

Let us consider again a complex number $z = x + iy$ represented by the vector $\overrightarrow{OP}$, where P is the point (x, y). If z varies subject to some given condition, then the corresponding set of points in the Argand diagram is called the *locus* of P. Since many loci can be defined using either distances or angles, equations of loci in the Argand diagram often involve moduli or arguments of complex variables.



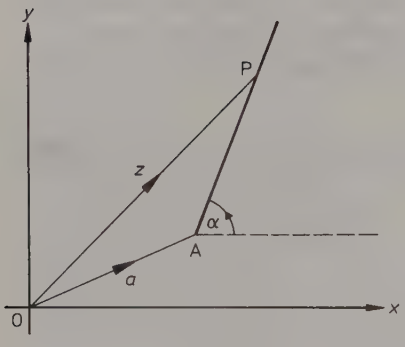


If a is constant complex number represented by a fixed vector $\overrightarrow{OA}$, then $z - a$ is represented by $\overrightarrow{AP}$. Hence $|z - a|$ is represented by the distance AP and $\arg(z - a)$ is the angle $\overrightarrow{AP}$ makes the positive direction of the real axis.



It follows that if $|z - a| = k$, where k is a positive constant, then the distance of P from A is constant. Thus

> $|z - a| = k$ is the equation of the circle with centre at the point A and radius k.



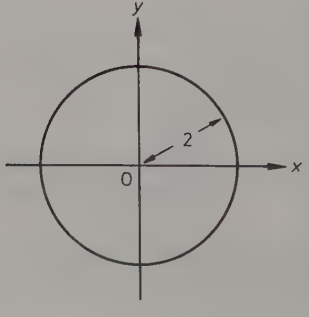
However, if $\arg(z - a) = \alpha$, where α is constant and $-\pi < \alpha \leqslant \pi$, then the direction of $\overrightarrow{AP}$ is constant. Thus

> $\arg(z - a) = \alpha$ is the equation of a *half-line* with end-point A, inclined at an angle α to the real axis.

Example 1 Sketch the loci defined by the equations (a) $|z| = 2$, (b) $|z - 3 + 2i| = 5$.

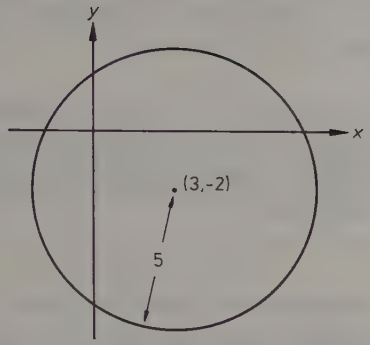
Let z be represented in the Argand diagram by the point P.

(a)



If $|z| = 2$, then the distance of P from the origin is 2 units.
Hence the locus of P is a circle with centre the origin and radius 2.

(b)

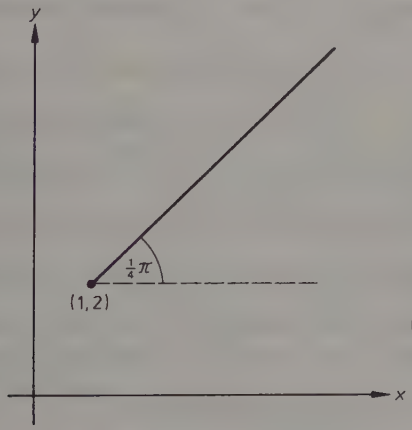


If $|z - 3 + 2i| = 5$,
then $|z - (3 - 2i)| = 5$,
i.e. the distance of P from the point representing $3 - 2i$ is 5 units.
Hence the locus of P is a circle with centre $(3, -2)$ and radius 5.

Example 2 Sketch the loci defined by the equations (a) $\arg(z - 1 - 2i) = \frac{1}{4}\pi$, (b) $\arg(z + 2) = -\frac{2}{3}\pi$.

Let z be represented in the Argand diagram by the point P.

(a)

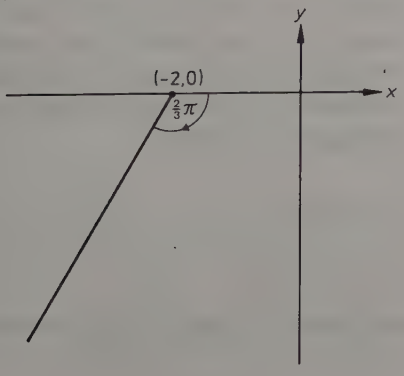


$z - 1 - 2i = z - (1 + 2i)$
Thus, if A is the point representing $1 + 2i$, $\arg(z - 1 - 2i)$ is the angle which $\overrightarrow{AP}$ makes with the real axis.
Hence the equation

$$\arg(z - 1 - 2i) = \tfrac{1}{4}\pi$$

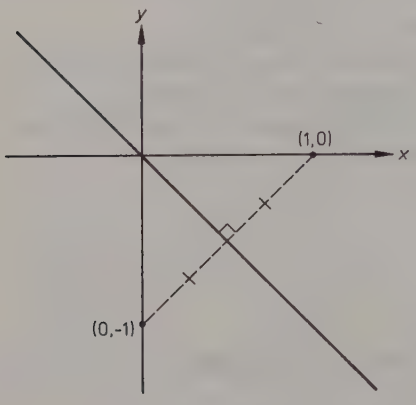
represents the half-line with end-point $(1, 2)$, inclined at an angle $\frac{1}{4}\pi$ to the real axis.

(b)



Using similar arguments the equation $\arg(z + 2) = -\frac{2}{3}\pi$ represents the half-line with end-point $(-2, 0)$, inclined to the real axis at an angle $\frac{2}{3}\pi$, measured clockwise.

Example 3 Sketch the locus of the point $P(x, y)$ representing the complex number $z = x + iy$, given that $|z - 1| = |z + i|$. Write down the Cartesian equation of the locus.



$|z - 1|$ and $|z + i|$ are the distances between the point P and the points representing 1 and $-i$ respectively. Hence the equation

$$|z - 1| = |z + i|$$

represents the locus of points equidistant from $(1, 0)$ and $(0, -1)$ i.e. the perpendicular bisector of the line joining the points $(1, 0)$ and $(0, -1)$. From the sketch it is clear that the Cartesian equation of the locus is $y = -x$.

Example 4 Find the Cartesian equation of the locus of the point $P(x, y)$ representing the complex number z, given that $2|z - 3i| = |z|$. Show that the locus is a circle, giving its centre and radius.

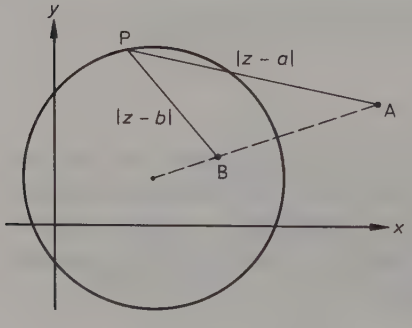
$$\begin{aligned} 2|z - 3i| = |z| &\Rightarrow 2|x + iy - 3i| &&= |x + iy| \\ &\Rightarrow 2|x + i(y - 3)| &&= |x + iy| \\ &\Rightarrow 4\{x^2 + (y - 3)^2\} &&= x^2 + y^2 \\ &\Rightarrow 4x^2 + 4y^2 - 24y + 36 &&= x^2 + y^2 \\ &\Rightarrow 3x^2 + 3y^2 - 24y + 36 = 0 \end{aligned}$$

Hence the Cartesian equation of the locus is

$$x^2 + y^2 - 8y + 12 = 0$$

Rearranging: $x^2 + (y - 4)^2 = 4$

This equation represents a circle with centre $(0, 4)$ and radius 2.



In general, if a point P moves so that the ratio of its distances from two fixed points A and B is constant, then the locus of P is a circle. This locus is referred to as Apollonius' circle and is represented in the Argand diagram by an equation of the form

$$|z - a| = k|z - b| \quad \text{or} \quad \left|\frac{z - a}{z - b}\right| = k, \quad (k \neq 1).$$

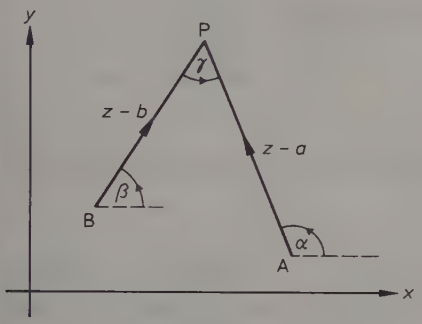
As shown in Example 3, when $k = 1$, the equation represents the perpendicular bisector of the line joining A to B.

Equations involving $\arg\left(\frac{z-a}{z-b}\right)$ are more difficult to interpret.

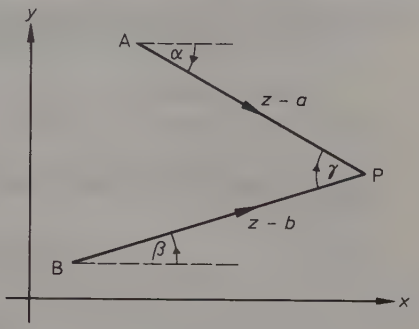
If $\arg(z-a)=\alpha$, $\arg(z-b)=\beta$ and $\arg\left(\frac{z-a}{z-b}\right)=\gamma$,

then $$\gamma=\alpha-\beta(\pm 2\pi \text{ if necessary}).$$

Thus γ is the angle which the vector $\overrightarrow{AP}$ makes with the vector $\overrightarrow{BP}$. If the turn from $\overrightarrow{BP}$ to $\overrightarrow{AP}$ is anti-clockwise then γ is positive; if the turn is clockwise then γ is negative.



$$\arg\left(\frac{z-a}{z-b}\right)>0$$

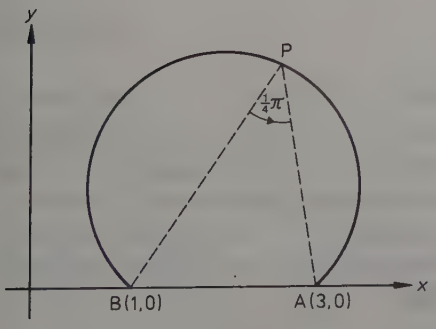


$$\arg\left(\frac{z-a}{z-b}\right)<0$$

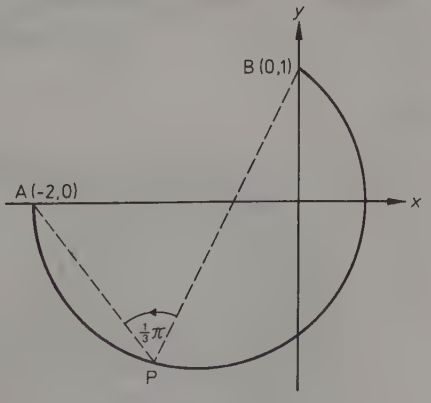
It follows that the equation $\arg\left(\frac{z-a}{z-b}\right)=\lambda$, where λ is constant, represents a circular arc with end-points A and B.

For instance, if $\arg\left(\frac{z-3}{z-1}\right)=\frac{1}{4}\pi$, then the locus of P is a circular arc with end-points $A(3,0)$ and $B(1,0)$, such that $\angle APB=\frac{1}{4}\pi$.

Similarly, if $\arg\left(\frac{z+2}{z-i}\right)=\frac{1}{3}\pi$, then the locus of P is a circular arc with end-points $A(-2,0)$ and $B(0,1)$, such that $\angle APB=\frac{1}{3}\pi$. Since in both cases the given arguments are positive, the arcs must be drawn so that the turn from $\overrightarrow{BP}$ to $\overrightarrow{AP}$ is anti-clockwise.



$$\arg\left(\frac{z-3}{z-1}\right)=\tfrac{1}{4}\pi$$



$$\arg\left(\frac{z+2}{z-i}\right)=\tfrac{1}{3}\pi$$

We use a worked example to illustrate the use of equations involving parameters to represent loci in the Argand diagram.

Example 5 If z is represented in an Argand diagram by the point P and $z - 6i = \lambda iz$, where λ is real, find the Cartesian equation of the locus of P as λ varies.

Let $z = x + iy$, then the equation of the locus becomes

$$x + iy - 6i = \lambda i(x + iy)$$

i.e. $x + i(y - 6) = -\lambda y + i\lambda x$

Equating real and imaginary parts,

$$x = -\lambda y, \qquad y - 6 = \lambda x$$

$\therefore$ provided that $x \neq 0$ and $y \neq 0$, $\dfrac{x}{-y} = \dfrac{y - 6}{x} = \lambda$

i.e. $$x^2 = -y^2 + 6y$$

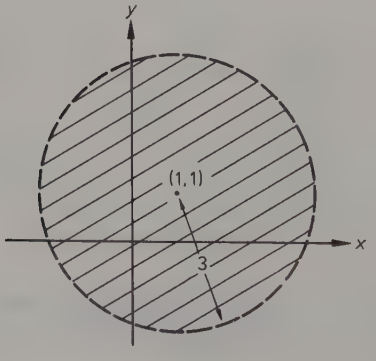
Hence the Cartesian equation of the locus of P is $x^2 + y^2 - 6y = 0$, which represents a circle with centre $(0, 3)$ and radius 3.

[Investigating the excluded values $x = 0, y = 0$, we find that there is no finite value of λ such that $z = 0$ satisfies the equation $z - 6i = \lambda iz$. This means that strictly speaking the origin $(0, 0)$ is not part of the locus.]

Inequalities involving the modulus or argument of a complex variable can be used to represent regions of an Argand diagram.

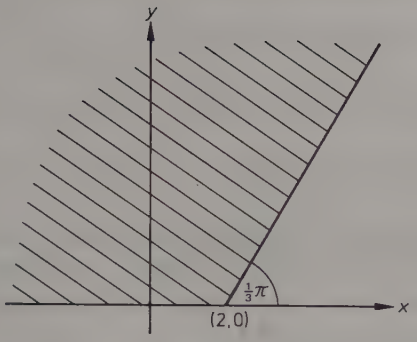
Example 6 Shade in separate Argand diagrams the regions in which
(a) $|z - 1 - i| < 3$, (b) $\frac{1}{3}\pi \leqslant \arg(z - 2) \leqslant \pi$.

(a)



Since $z - 1 - i = z - (1 + i)$, the equation $|z - 1 - i| = 3$ represents a circle with centre $(1, 1)$ and radius 3. Hence the inequality $|z - 1 - i| < 3$ represents the interior of this circle, but not the circle itself.

(b)



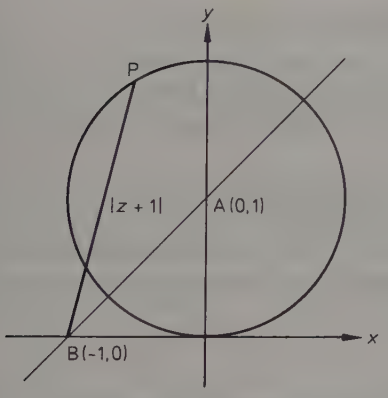
The equations $\arg(z-2)=\frac{1}{3}\pi$ and $\arg(z-2)=\pi$ represent half-lines with end-point $(2,0)$. Hence the inequality

$$\tfrac{1}{3}\pi \leqslant \arg(z-2) \leqslant \pi$$

represents these two lines and the region between them.

A geometrical approach is sometimes helpful when solving problems.

Example 7 Given that z is a complex number which varies such that $|z-i|=1$, find the greatest and least values of $|z+1|$.



Let z be represented in an Argand diagram by the point P. Given that $|z-i|=1$, the locus of P is a circle with centre $A(0,1)$ and radius 1. Since $|z+1|$ is the distance of P from the point $B(-1,0)$, its greatest and least values occur when P lies on the straight line through A and B. As $AB=\sqrt{2}$ and the radius of the circle is 1, these greatest and least values must be $\sqrt{2}+1$ and $\sqrt{2}-1$.

Exercise 14.5

In questions 1 to 11 sketch in separate Argand diagrams the loci defined by the given equations.

1. (a) $|z|=5$, (b) $|z-2|=3$.

2. (a) $|z-3i|=2$, (b) $|z+4|=4$.

3. (a) $|z-1+i|=1$, (b) $|z-2-3i|=4$.

4. (a) $|z+1+2i|=3$, (b) $|z+3-4i|=5$.

5. (a) $\arg z = \frac{1}{4}\pi$, (b) $\arg(z - i) = \frac{1}{3}\pi$.

6. (a) $\arg(z + 1 - 3i) = -\frac{1}{6}\pi$, (b) $\arg(z - 3 + 2i) = \pi$.

7. (a) $\arg(z + 2 + i) = \frac{1}{2}\pi$, (b) $\arg(z - 1 - i) = -\frac{3}{4}\pi$.

8. (a) $|z + 1| = |z - 3|$, (b) $|z| = |z - 6i|$.

9. (a) $\left|\frac{z - i}{z - 1}\right| = 1$, (b) $\left|\frac{z - 2 - 3i}{z + 2 + i}\right| = 1$.

10. (a) $\arg\left(\frac{z - 1}{z + 1}\right) = \frac{1}{3}\pi$, (b) $\arg\left(\frac{z + 1}{z - 1}\right) = \frac{1}{3}\pi$.

11. (a) $\arg\left(\frac{z - 3}{z - 2i}\right) = \frac{1}{4}\pi$, (b) $\arg\left(\frac{z}{z - 4 + 2i}\right) = \frac{1}{2}\pi$.

In questions 12 to 24 find the Cartesian equation of the locus of the point P representing the complex number z. Sketch the locus of P in each case.

12. $2|z + 1| = |z - 2|$

13. $|z + 4i| = 3|z - 4|$

14. $\left|\frac{z}{z - 4}\right| = 5$

15. $\left|\frac{z + i}{z - 5 - 2i}\right| = 1$

16. $\left|\frac{z}{z + 6}\right| = 5$

17. $\left|\frac{z - 1}{z + 1 - i}\right| = \frac{2}{3}$

18. $z - 5 = \lambda i(z + 5)$, where λ is a real parameter.

19. $\frac{z + 2i}{z - 2} = \lambda i$, where λ is a real parameter.

20. $z = 3i + \lambda(2 + 5i)$, where λ is a real parameter.

21. $\mathrm{Im}(z^2) = 2$

22. $\mathrm{Re}(z^2) = 1$

23. $\mathrm{Re}\left(z - \frac{1}{z}\right) = 0$

24. $\mathrm{Im}\left(z + \frac{9}{z}\right) = 0$.

In questions 25 to 34 shade in separate Argand diagrams the regions represented by the given inequalities. Indicate in each case whether the boundaries are to be included in the region.

25. $|z| > 5$

26. $1 \leqslant |z| \leqslant 2$

27. $|z - i| \leqslant 3$

28. $|z - 4 + 3i| < 4$

29. $0 \leqslant \arg z \leqslant \frac{1}{3}\pi$

30. $\frac{1}{4}\pi < \arg z < \frac{3}{4}\pi$

31. $-\frac{1}{6}\pi < \arg(z - 1) < \frac{1}{6}\pi$

32. $-\frac{1}{2}\pi \leqslant \arg(z + i) \leqslant \frac{2}{3}\pi$

33. $|z| > |z + 2|$

34. $|z + i| \leqslant |z - 3i|$

35. Represent each of the following loci in an Argand diagram.
(a) $\arg(z - 1) = \arg(z + 1)$, (b) $\arg z = \arg(z - 1 - i)$,
(c) $\arg(z - 2) = \pi + \arg z$, (d) $\arg(z - 1) = \pi + \arg(z - i)$.

36. Find the least value of $|z + 4|$ for which
(a) $\mathrm{Re}(z) = 5$, (b) $\mathrm{Im}(z) = 3$, (c) $|z| = 1$, (d) $\arg z = \frac{1}{4}\pi$.

37. Given that the complex number z varies such that $|z - 7| = 3$, find the greatest and least values of $|z - i|$.

38. Given that the complex numbers w and z vary subject to the conditions $|w - 12| = 7$ and $|z - 5i| = 4$, find the greatest and least values of $|w - z|$.

39. In an Argand diagram, the point P represents the complex number z, where $z = x + iy$. Given that $z + 2 = \lambda i(z + 8)$, where λ is a real parameter, find the Cartesian equation of the locus of P as λ varies. If also $z = \mu(4 + 3i)$, where μ is real, prove that there is only one possible position for P. (JMB)

40. (i) Represent on the same Argand diagram the loci given by the equations $|z - 3| = 3$ and $|z| = |z - 2|$. Obtain the complex numbers corresponding to the points of intersection of these loci. (ii) Find a complex number z whose argument is $\pi/4$ and which satisfies the equation $|z + 2 + i| = |z - 4 + i|$. (L)

14.6 De Moivre's theorem for an integral index

In §14.4 it was shown that

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2).$$

Writing $\theta_1 = \theta_2 = \theta$, we find that

$$(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$$
$$\therefore \quad (\cos\theta + i\sin\theta)^3 = (\cos 2\theta + i\sin 2\theta)(\cos\theta + i\sin\theta)$$
$$\text{Thus} \quad (\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$$

These results suggest a general formula,

$$\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta}$$

This statement, known as *De Moivre's*† *theorem*, may be proved for positive integral values of n by the method of induction.

Assuming that the theorem holds for $n = k$,

$$\therefore \quad \begin{aligned} (\cos\theta + i\sin\theta)^k &= \cos k\theta + i\sin k\theta \\ (\cos\theta + i\sin\theta)^{k+1} &= (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta) \\ &= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta) \\ &= \cos(k\theta + \theta) + i\sin(k\theta + \theta) \\ &= \cos(k+1)\theta + i\sin(k+1)\theta \end{aligned}$$

$\therefore$ if the theorem holds for $n = k$, it also holds for $n = k + 1$.

When $n = 1$, $\quad (\cos\theta + i\sin\theta)^n = \cos\theta + i\sin\theta$

and $\quad \cos n\theta + i\sin n\theta = \cos\theta + i\sin\theta$

$\therefore$ the theorem holds for $n = 1$.

Hence, by induction, De Moivre's theorem holds when n is a positive integer.

Example 1 Find the value of $(\cos\frac{1}{4}\pi + i\sin\frac{1}{4}\pi)^{12}$.

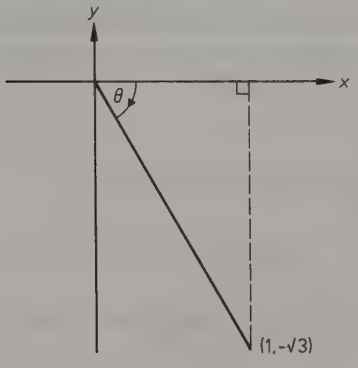
By De Moivre's theorem,

$$\begin{aligned} (\cos\tfrac{1}{4}\pi + i\sin\tfrac{1}{4}\pi)^{12} &= \cos(12 \times \tfrac{1}{4}\pi) + i\sin(12 \times \tfrac{1}{4}\pi) \\ &= \cos 3\pi + i\sin 3\pi = -1. \end{aligned}$$

Example 2 Express $(1 - i\sqrt{3})^4$ in the form $a + ib$.

$$|1 - i\sqrt{3}| = \sqrt{\{1^2 + (-\sqrt{3})^2\}} = \sqrt{\{1 + 3\}} = 2$$

If $\arg(1 - i\sqrt{3}) = \theta$, then $\tan\theta = \dfrac{-\sqrt{3}}{1} = -\sqrt{3}$



Since the point representing $1 - i\sqrt{3}$ lies in the fourth quadrant, $\arg(1 - i\sqrt{3}) = -\frac{1}{3}\pi$.

$$\therefore \quad 1 - i\sqrt{3} = 2\{\cos(-\tfrac{1}{3}\pi) + i\sin(-\tfrac{1}{3}\pi)\}$$

By De Moivre's theorem,

$$\begin{aligned} (1 - i\sqrt{3})^4 &= 2^4\{\cos(-\tfrac{1}{3}\pi) + i\sin(-\tfrac{1}{3}\pi)\}^4 \\ &= 16\{\cos(-\tfrac{4}{3}\pi) + i\sin(-\tfrac{4}{3}\pi)\} = 16\left\{-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right\} \end{aligned}$$

† *De Moivre, Abraham* (1667–1754) French mathematician of Huguenot extraction who settled in England, and was a friend of Newton. He published his theorem in *Miscellanea Analytica* in 1730.

Hence $(1 - i\sqrt{3})^4 = -8 + 8\sqrt{3}i$.

To prove De Moivre's theorem when n is a negative integer, let $n = -m$ where m is a positive integer, then

$$\begin{aligned}(\cos\theta + i\sin\theta)^n &= (\cos\theta + i\sin\theta)^{-m} \\ &= \frac{1}{(\cos\theta + i\sin\theta)^m}\end{aligned}$$

Applying De Moivre's theorem for positive integral index,

$$\begin{aligned}(\cos\theta + i\sin\theta)^n &= \frac{1}{\cos m\theta + i\sin m\theta} \\ &= \cos m\theta - i\sin m\theta \\ &= \cos(-m\theta) + i\sin(-m\theta) \\ &= \cos n\theta + i\sin n\theta\end{aligned}$$

Hence De Moivre's theorem holds for all integral values of n.

Example 3 Evaluate $\dfrac{1}{(1 - i\sqrt{3})^3}$.

As shown in Example 2, $1 - i\sqrt{3} = 2\{\cos(-\frac{1}{3}\pi) + i\sin(-\frac{1}{3}\pi)\}$
By De Moivre's theorem,

$$\begin{aligned}(1 - i\sqrt{3})^{-3} &= 2^{-3}\{\cos(-\tfrac{1}{3}\pi) + i\sin(-\tfrac{1}{3}\pi)\}^{-3} \\ &= \tfrac{1}{8}\{\cos[(-3)(-\tfrac{1}{3}\pi)] + i\sin[(-3)(-\tfrac{1}{3}\pi)]\} \\ &= \tfrac{1}{8}\{\cos\pi + i\sin\pi\}\end{aligned}$$

Hence $\dfrac{1}{(1 - i\sqrt{3})^3} = -\frac{1}{8}$

De Moivre's theorem may be used to obtain certain types of trigonometric identity.

Example 4 Express $\cos 4\theta$ in terms of $\cos\theta$ and $\tan 4\theta$ in terms of $\tan\theta$.

By De Moivre's theorem, $\cos 4\theta + i\sin 4\theta = (\cos\theta + i\sin\theta)^4$

Using the binomial expansion,

$$\begin{aligned}\cos 4\theta + i\sin 4\theta &= \cos^4\theta + 4\cos^3\theta(i\sin\theta) \\ &\quad + 6\cos^2\theta(i\sin\theta)^2 + 4\cos\theta(i\sin\theta)^3 + (i\sin\theta)^4 \\ &= (\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta) \\ &\quad + i(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)\end{aligned}$$

Equating real and imaginary parts,

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta \quad (1)$$
$$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta \quad (2)$$

From (1): $\cos 4\theta = \cos^4\theta - 6\cos^2\theta(1 - \cos^2\theta) + (1 - \cos^2\theta)^2$
$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$

$\therefore \quad \cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$

Using (1) and (2): $\tan 4\theta = \dfrac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}$

Dividing numerator and denominator by $\cos^4\theta$,

$$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$

Expressions for powers of $\sin\theta$ and $\cos\theta$ in terms of sines and cosines of multiples of θ can be derived using the following results.

If $z = \cos\theta + i\sin\theta$, then $\dfrac{1}{z} = \cos\theta - i\sin\theta$

$\therefore \quad z + \dfrac{1}{z} = 2\cos\theta \quad$ and $\quad z - \dfrac{1}{z} = 2i\sin\theta$

By De Moivre's theorem,

$$z^n = \cos n\theta + i\sin n\theta, \quad \text{so that} \quad \frac{1}{z^n} = \cos n\theta - i\sin n\theta$$

$$\boxed{z^n + \frac{1}{z^n} = 2\cos n\theta \quad \text{and} \quad z^n - \frac{1}{z^n} = 2i\sin n\theta}$$

Example 5 Express $\cos^4\theta$ in terms of cosines of multiples of θ.

$$\left(z + \frac{1}{z}\right)^4 = z^4 + 4z^3.\frac{1}{z} + 6z^2.\frac{1}{z^2} + 4z.\frac{1}{z^3} + \frac{1}{z^4}$$

$$= z^4 + 4z^2 + 6 + \frac{4}{z^2} + \frac{1}{z^4}$$

$$= \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6$$

If $z = \cos\theta + i\sin\theta$, $z^n + \dfrac{1}{z^n} = 2\cos n\theta$

$\therefore \quad (2\cos\theta)^4 = 2\cos 4\theta + 4 \times 2\cos 2\theta + 6$

i.e. $\quad 16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$

Hence $\quad \cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$

Exercise 14.6

1. Use De Moivre's theorem to simplify the following:

(a) $\left(\cos\dfrac{\pi}{5} + i\sin\dfrac{\pi}{5}\right)^{10}$ (b) $\left(\cos\dfrac{\pi}{8} + i\sin\dfrac{\pi}{8}\right)^{12}$

(c) $\left(\cos\frac{\pi}{9}+i\sin\frac{\pi}{9}\right)^{-3}$ (d) $\left\{\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right\}^{-4}$

2. Express $\sqrt{3}+i$ in modulus-argument form. Hence find $(\sqrt{3}+i)^{10}$ and $1/(\sqrt{3}+i)^7$ in the form $a+ib$.

3. Express $-1+i$ in modulus-argument form. Hence show that $(-1+i)^{16}$ is real and that $1/(-1+i)^6$ is purely imaginary, giving the value of each.

4. Simplify the following expressions:

(a) $\dfrac{\left(\cos\frac{2\pi}{7}-i\sin\frac{2\pi}{7}\right)^3}{\left(\cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}\right)^4}$ (b) $\dfrac{\left(\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}\right)^8}{\left(\cos\frac{3\pi}{5}-i\sin\frac{3\pi}{5}\right)^3}$

5. Find expressions for $\cos 3\theta$ in terms of $\cos\theta$, $\sin 3\theta$ in terms of $\sin\theta$ and $\tan 3\theta$ in terms of $\tan\theta$.

6. Express $\sin 5\theta$ and $\cos 5\theta/\cos\theta$ in terms of $\sin\theta$.

7. Prove that $\tan 5\theta = \dfrac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$. By considering the equation $\tan 5\theta = 0$, show that $\tan^2(\pi/5) = 5 - 2\sqrt{5}$.

8. Find expressions for $\cos 6\theta$ and $\sin 6\theta/\sin\theta$ in terms of $\cos\theta$ and for $\tan 6\theta$ in terms of $\tan\theta$.

9. Express in terms of cosines of multiples of θ:
(a) $\cos^5\theta$ (b) $\cos^7\theta$ (c) $\sin^2\theta\cos^3\theta$

10. Express in terms of sines of multiples of θ:
(a) $\sin^3\theta$ (b) $\sin^7\theta$ (c) $\cos^4\theta\sin^3\theta$

11. Prove that $\cos^6\theta + \sin^6\theta = \frac{1}{8}(3\cos 4\theta + 5)$.

12. Evaluate (a) $\displaystyle\int_0^{\pi}\sin^4\theta\,d\theta$, (b) $\displaystyle\int_0^{\pi/2}\cos^4\theta\sin^2\theta\,d\theta$.

Exercise 14.7 (miscellaneous)

1. Let $z = x + iy$ be any non-zero complex number. Express $\dfrac{1}{z}$ in the form $u + iv$.

Given that $z + \dfrac{1}{z} = k$ with k *real*, prove that either $y = 0$ or $x^2 + y^2 = 1$. Show (i) that if $y = 0$ then $|k| \geqslant 2$, (ii) that if $x^2 + y^2 = 1$ then $|k| \leqslant 2$. (JMB)

2. In the quadratic equation $x^2 + (p + iq)x + 3i = 0$, p and q are real. Given that the sum of the squares of the roots is 8, find all possible pairs of values of p and q. **(JMB)**

3. Express the complex number $z_1 = \dfrac{11 + 2i}{3 - 4i}$ in the form $x + iy$ where x and y are real. Given that $z_2 = 2 - 5i$, find the distance between the points in the Argand diagram which represent z_1 and z_2. Determine the real numbers α, β such that $\alpha z_1 + \beta z_2 = -4 + i$. **(JMB)**

4. (i) Find two complex numbers z satisfying the equation $z^2 = -8 - 6i$.
   (ii) Solve the equation $z^2 - (3 - i)z + 4 = 0$ and represent the solutions on an Argand diagram by vectors $\vec{OA}$ and $\vec{OB}$, where O is the origin. Show that triangle OAB is right-angled.

5. If z and w are complex numbers, show that

$$|z - w|^2 + |z + w|^2 = 2\{|z|^2 + |w|^2\}.$$

Interpret your results geometrically. **(AEB 1977)**

6. A regular octagon is inscribed in the circle $|z| = 1$ in the complex plane and one of its vertices represents the number $\dfrac{1}{\sqrt{2}}(1 + i)$. Find the numbers represented by the other vertices. **(AEB 1977)**

7. (i) Two complex numbers z_1 and z_2 each have arguments between 0 and π. If $z_1 z_2 = i - \sqrt{3}$ and $z_1/z_2 = 2i$ find the values of z_1 and z_2 giving the modulus and argument of each.
   (ii) Obtain in the form $a + ib$ the solutions of the equation $z^2 - 2z + 5 = 0$, and represent the solutions on an Argand diagram by the points A and B.

The equation $z^2 - 2pz + q = 0$ is such that p and q are real, and its solutions in the Argand diagram are represented by the points C and D. Find in the simplest form the algebraic relation satisfied by p and q in each of the following cases:
(a) $p^2 < q$, $p \neq 1$ and A, B, C, D are the vertices of a rectangle;
(b) $p^2 > q$ and $\angle CAD = \frac{1}{2}\pi$. **(O&C)**

8. (a) If $-\pi < \arg z_1 + \arg z_2 \leqslant \pi$ show that $\arg(z_1 z_2) = \arg z_1 + \arg z_2$. The complex numbers $a = 4\sqrt{3} + 2i$ and $b = \sqrt{3} + 7i$ are represented in the Argand diagram by points A and B respectively. O is the origin. Show that triangle OAB is equilateral and find the complex number c which the point C represents where $OABC$ is a rhombus. Calculate $|c|$ and $\arg c$. (You may leave answers in surd form.)

   (b) z is a complex number such that $z = \dfrac{p}{2 - i} + \dfrac{q}{1 + 3i}$ where p and q are real. If $\arg z = \pi/2$ and $|z| = 7$ find the values of p and q. **(SU)**

9. (i) Given that $z_1 = 3 + 4i$ and $z_2 = -1 + 2i$, represent z_1, z_2, $(z_1 + z_2)$ and $(z_2 - z_1)$ by vectors in the Argand diagram. Express $(z_1 + z_2)/(z_2 - z_1)$ in the form $a + ib$, where a and b are real. Find the magnitude of the angle between the vectors representing $(z_1 + z_2)$ and $(z_2 - z_1)$.
(ii) One root of the equation $z^3 - 6z^2 + 13z + k = 0$, where k is real, it is $z = 2 + i$. Find the other roots and the value of k. (L)

10. (a) Show that $(1 + 3i)^3 = -(26 + 18i)$.
(b) Find the three roots z_1, z_2, z_3 of the equation $z^3 = -1$.
(c) Find, in the form $a + ib$, the three roots z'_1, z'_2, z'_3 of the equation $z^3 = 26 + 18i$.
(d) Indicate in the same Argand diagram the points represented by z_r and z'_r for $r = 1, 2, 3$, and prove that the roots of the equations may be paired so that $|z_1 - z'_1| = |z_2 - z'_2| = |z_3 - z'_3| = 3$. (O&C)

11. Write down, or obtain, the non-real cube roots of unity, ω_1 and ω_2, in the form $a + ib$, where a and b are real. A regular hexagon is drawn in an Argand diagram such that two adjacent vertices represent ω_1 and ω_2, respectively, and the centre of the circumscribing circle of the hexagon is the point $(1, 0)$. Determine, in the form $a + ib$, the complex numbers represented by the other four vertices of the hexagon and find the product of these four complex numbers. (JMB)

12. A complex number ω is such that $\omega^3 = 1$ and $\omega \neq 1$. Show that

(i) $\omega^2 + \omega + 1 = 0$, (ii) $(x + a + b)(x + \omega a + \omega^2 b)(x + \omega^2 a + \omega b)$

is real for real x, a and b, and simplify this product. Hence, or otherwise, find the three roots of the equation $x^3 - 6x + 6 = 0$, giving your answers in terms of ω, ω^2 and cube roots of integers. (JMB)

13. (i) Find, without the use of tables, the two square roots of $5 - 12i$ in the form $x + iy$, where x and y are real.
(ii) Represent on an Argand diagram the loci $|z - 2| = 2$ and $|z - 4| = 2$. Calculate the complex numbers corresponding to the points of intersection of these loci. (L)

14. (i) Given that $(1 + 5i)p - 2q = 3 + 7i$, find p and q when (a) p and q are real, (b) p and q are conjugate complex numbers.
(ii) Shade on the Argand diagram the region for which $3\pi/4 < \arg z < \pi$ and $0 < |z| < 1$. Choose a point in the region and label it A. If A represents the complex number z, label clearly the points B, C, D and E which represent $-z$, iz, $z + 1$ and z^2 respectively. (L)

15. (i) Show that $z = 1 + i$ is a root of the equation $z^4 + 3z^2 - 6z + 10 = 0$. Find the other roots of this equation.
(ii) Sketch the curve in the Argand diagram defined by $|z - 1| = 1$, $\operatorname{Im} z \geqslant 0$. Find the value of z at the point P in which this curve is cut by the line $|z - 1| = |z - 2|$. Find also the value of $\arg z$ and $\arg(z - 2)$ at P. (L)

16. (i) If $z = 1 + i\sqrt{3}$, find $|z|$ and $|z^5|$, and also the values of $\arg z$ and $\arg(z^5)$ lying between $-\pi$ and π. Show that $\mathrm{Re}(z^5) = 16$ and find the value of $\mathrm{Im}(z^5)$.

(ii) Draw the line $|z| = |z - 4|$ and the half-line $\arg(z - i) = \pi/4$ in the Argand diagram. Hence find the complex number that satisfies both equations. (L)

17. (i) Without using tables, simplify $\dfrac{\left(\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}\right)^4}{\left(\cos\frac{\pi}{9} - i\sin\frac{\pi}{9}\right)^5}$.

(ii) Express $z_1 = \dfrac{7 + 4i}{3 - 2i}$ in the form $p + qi$, where p and q are real.

Sketch in an Argand diagram the locus of points representing complex numbers z such that $|z - z_1| = \sqrt{5}$. Find the greatest value of $|z|$ subject to this condition. (L)

18. (i) Given that $z = 1 - i$, find the values of $r(>0)$ and θ, $-\pi < \theta \leqslant \pi$, such that $z = r(\cos\theta + i\sin\theta)$. Hence, or otherwise, find $1/z$ and z^6, expressing your answers in the form $p + iq$, where $p, q \in \mathbb{R}$.

(ii) Sketch on an Argand diagram the set of points corresponding to the set A, where $A = \{z : z \in \mathbb{C},\ \arg(z - i) = \pi/4\}$. Show that the set of points corresponding to the set B, where $B = \{z : z \in \mathbb{C},\ |z + 7i| = 2|z - 1|\}$, forms a circle in the Argand diagram. If the centre of this circle represents the numbers z_1, show that $z_1 \in A$. (L)

19. Use De Moivre's theorem to show that

$$\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta.$$

20. (i) If $(1 + 3i)z_1 = 5(1 + i)$, express z_1 and z_1^2 in the form $x + iy$, where x and y are real. Sketch in an Argand diagram the circle $|z - z_1| = |z_1|$ giving the coordinates of its centre.

(ii) If $z = \cos\theta + i\sin\theta$. show that

$$z - \frac{1}{z} = 2i\sin\theta. \qquad z^n - \frac{1}{z^n} = 2i\sin n\theta.$$

Hence, or otherwise, show that

$$16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta. \qquad \text{(L)}$$

21. (i) Given that $(1 + i)^n = x + iy$. where x and y are real and n is an integer, prove that $x^2 + y^2 = 2^n$.

(ii) Given that $\left|\dfrac{z - 1}{z + 1}\right| = 2$, find the cartesian equation of the locus of z and represent the locus by a sketch in the Argand diagram. Shade the region for

which the inequalities $\left|\frac{z-1}{z+1}\right| > 2$ and $0 < \arg z < 3\pi/4$ are both satisfied. (L)

22. (i) Given that x and y are real, find the values of x and y which satisfy the equation $\frac{2y+4i}{2x+y} - \frac{y}{x-i} = 0$.

(ii) Given that $z = x + iy$, where x and y are real, (a) show that, when $\mathrm{Im}\left(\frac{z+i}{z+2}\right) = 0$, the point (x, y) lies on a straight line, (b) Show that, when $\mathrm{Re}\left(\frac{z+i}{z+2}\right) = 0$, the point (x, y) lies on a circle with centre $(-1, -\frac{1}{2})$ and radius $\frac{1}{2}\sqrt{5}$. (L)

23. (i) Find $|z|$ and $\arg z$ for each of the complex numbers z given by (a) $12 - 5i$, (b) $(1 + 2i)/(2 - i)$, giving the argument in degrees (to the nearest degree) such that $-180° < \arg z \leqslant 180°$.

(ii) By expressing $\sqrt{3} - i$ in modulus-argument form, or otherwise, find the least positive integer n such that $(\sqrt{3} - i)^n$ is real and positive.

(iii) The point P in the Argand diagram lies outside or on the circle of radius 4 with centre at $(-1, -1)$. Write down in modulus form the condition satisfied by the complex number z represented by the point P. (L)

24. Sketch the circle C with Cartesian equation $x^2 + (y - 1)^2 = 1$. The point P, representing the non-zero complex number z, lies on C. Express $|z|$ in terms of θ, the argument of z. Given that $z' = 1/z$, find the modulus and argument of z' in terms of θ. Show that, whatever the position of P on the circle C, the point P' representing z' lies on a certain line, the equation of which is to be determined. (JMB)

25. (a) The sum of the infinite series $1 + z + z^2 + z^3 + \ldots$ for values of z such that $|z| < 1$ is $1/(1 - z)$. By substituting $z = \frac{1}{2}(\cos\theta + i\sin\theta)$ in this result and using De Moivre's theorem, or otherwise, prove that

$$\frac{1}{2}\sin\theta + \frac{1}{2^2}\sin 2\theta + \ldots + \frac{1}{2^n}\sin n\theta + \ldots = \frac{2\sin\theta}{5 - 4\cos\theta}.$$

(b) Two variable complex numbers z and w are such that $|z - i| = |z - 1|$ and $|w - 4| = 2$. Show on the same Argand diagram the loci of the points representing z and w. Mark on the diagram the points A and B representing z and w respectively for which $|z - w|$ has its least value and find this value. (O&C)

26. The point representing the complex number z in an Argand diagram describes the circle $|z - 1| = 1$. Show that $z = 1 + \cos\theta + i\sin\theta$, where $-\pi < \theta \leqslant \pi$, and deduce that the point representing the complex number $1/z$ describes a straight line. Find the modulus and argument of z and hence, or otherwise, express

$(1 + \cos\theta + i\sin\theta)^n$ in the form $x + iy$, where n is a positive integer, and x and y are real. By writing $\cos\theta + i\sin\theta = \omega$, and using the binomial expansion of $(1 + \omega)^n$, prove that

$$1 + \binom{n}{1}\cos\theta + \binom{n}{2}\cos 2\theta + \ldots + \cos n\theta \equiv \left[2\cos\left(\frac{\theta}{2}\right)\right]^n \cos\left(\frac{n\theta}{2}\right). \qquad \text{(L)}$$

15 Probability distributions

15.1 Discrete probability distributions

The theory of probability is used to set up mathematical models of random experiments or trials. In its simplest form a *probability model* of an experiment consists of a description of the sample space (i.e. the set of possible outcomes) together with estimates of the probabilities associated with the outcomes. Let us consider, for instance, a trial in which a single unbiased die is thrown and the score noted. This can be described as a trial with a sample space containing 6 outcomes each with probability 1/6. We can use this model to make predictions about the behaviour of dice in experiments involving large numbers of such trials. The theory can be tested by comparing these predictions with the observed results of the experiments.

Since the score obtained in any particular throw of a single die cannot be predicted in advance, the trial is said to be *random* and the score obtained is called a *random variable.* In general, the values of a random variable are real numbers assigned to the elements of a sample space. This leads to the following formal definition.

> A random variable X is a function which assigns a real number x to each element s of a sample space S.

A random variable such as the height of students in a class, which may take any real value within a given interval, is said to be *continuous.* A random variable such as the score on a die, which takes only certain specified values, is described as *discrete.*

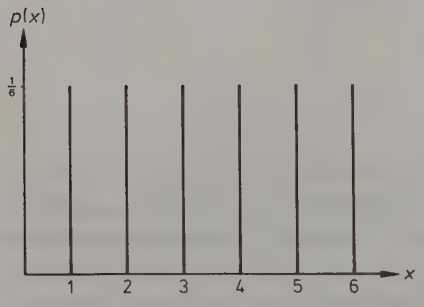
As we saw in Book 1, any subset A of a sample space S is called an event and the probability that event A occurs is denoted by $P(A)$. Thus the probability that the random variable X takes the value x may be written $P(X = x)$. We can now define a probability function on the set of values taken by X.

The probability function p of the discrete random variable X is defined by $p(x) = P(X = x)$, where $0 \leqslant p(x) \leqslant 1$ for all values of x.

The set of values $x_1, x_2, x_3, \ldots$ of a discrete random variable X, together with the corresponding values $p_1, p_2, p_3, \ldots$ of the probability function p, form a *discrete probability distribution.*

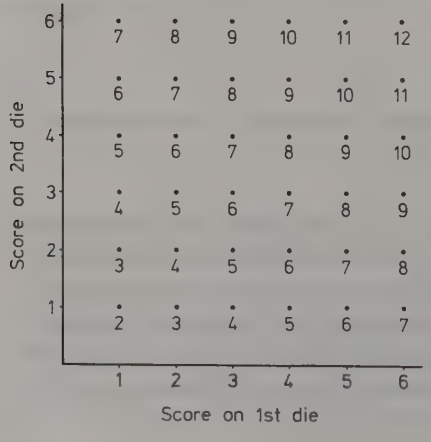
Since the sum of the probabilities of all possible outcomes of any experiment must be 1, all discrete probability distributions have the following property.

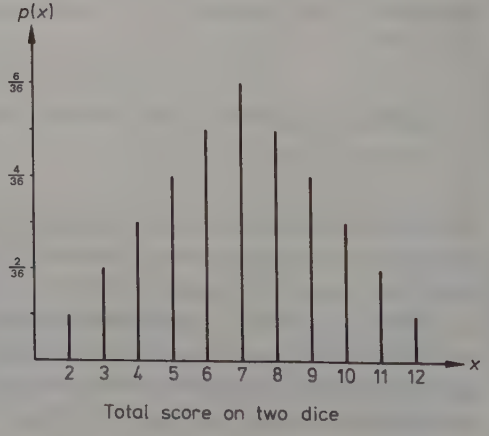
If a discrete random variable X takes the n values $x_1, x_2, \ldots x_n$ with probabilities $p_1, p_2, \ldots, p_n$ respectively, then $\sum_{r=1}^{n} p_r = 1$.



If X represents the score obtained when a single die is thrown, then $p(x) = 1/6$ for $x = 1, 2, 3, 4, 5$ and 6. A distribution such as this, in which all values of the random variable are equally likely, is said to be *rectangular* or *uniform.*

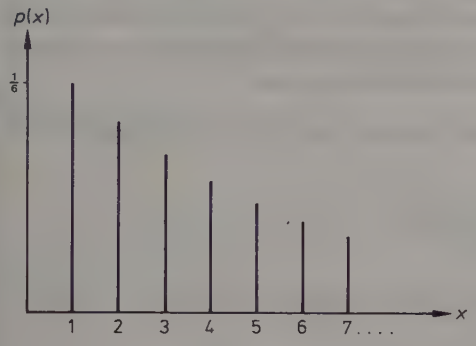
For a trial in which two dice are thrown and the total score noted the sample space S can be regarded as a set of 36 ordered pairs such as (1, 3), (4, 6) and (6, 4). The total score obtained in any throw is a discrete random variable defined on S and takes the values 2, 3, 4, ..., 12. The following diagrams show the sample space and corresponding probability distribution for this two dice trial.





A further type of discrete probability distribution is obtained when a die is thrown repeatedly and the random variable X is the number of throws needed to obtain a six. We then find that

$$p(1) = \frac{1}{6}, p(2) = \frac{5}{6} \times \frac{1}{6}, p(3) = \left(\frac{5}{6}\right)^2 \times \frac{1}{6}, p(4) = \left(\frac{5}{6}\right)^3 \times \frac{1}{6}, \ldots$$



Clearly X may take any one of the infinite set of values $1, 2, 3, 4, \ldots$ Since the corresponding probabilities form a geometric progression, this distribution is described as *geometric*.

In order to discuss the properties of probability distributions in more detail we need to define quantities similar to the measures of location and spread used to describe frequency distributions.

For a frequency distribution in which a set of n observations $x_1, x_2, \ldots, x_n$ occur with frequencies $f_1, f_2, \ldots, f_n$,

$$\text{the mean } \bar{x} = \frac{1}{n}\sum f_r x_r, \text{ where } n = \sum f_r,$$

$$\text{the variance} = \frac{1}{n}\sum f_r(x_r - \bar{x})^2 = \frac{1}{n}\sum f_r x_r^2 - \bar{x}^2$$

If n is fairly large the relative frequency f_r/n is approximately equal to the probability $p(x_r)$. Thus we obtain suitable definitions of the mean and variance of a probability distribution by replacing f_r/n by $p(x_r)$ in the formulae used for frequency distributions.

The mean of a probability distribution is called the *expectation* or *expected value* of the random variable X and is denoted by $E(X)$ or μ. The *variance* is denoted by $\text{Var}(X)$ or σ^2. These quantities are defined as follows:

$$\boxed{\begin{aligned} E(X) &= \mu = \sum x_r p(x_r) \\ \text{Var}(X) &= \sigma^2 = \sum (x_r - \mu)^2 p(x_r) \end{aligned}}$$

The expression for $\text{Var}(X)$ can be rearranged to give:

$$\boxed{\text{Var}(X) = \sigma^2 = \sum x_r^2 p(x_r) - \mu^2}$$

For the two dice trial discussed earlier:

$$E(X) = 2.\tfrac{1}{36} + 3.\tfrac{2}{36} + \ldots + 12.\tfrac{1}{36} = 7$$
$$\mathrm{Var}(X) = 2^2.\tfrac{1}{36} + 3^2.\tfrac{2}{36} + \ldots + 12^2.\tfrac{1}{36} - 7^2 = \tfrac{35}{6} \approx 5{\cdot}83.$$

Example 1 A box contains 10 coloured discs of which 2 are red. A man pays 10 pence to play a game in which discs are drawn one at a time from the box, without replacement, until a red disc is drawn. He will receive 25 pence if the first disc drawn is red, 20 pence if the second disc is red, 5 pence if the third disc is red, but nothing otherwise. Find the man's expected profit or loss in any game.

If the random variable X is the amount in pence received in any game, then X takes the values 25, 20, 5 and 0.

$p(25) = \tfrac{2}{10} = \tfrac{1}{5}$, $p(20) = \tfrac{8}{10} \times \tfrac{2}{9} = \tfrac{8}{45}$ and $p(5) = \tfrac{8}{10} \times \tfrac{7}{9} \times \tfrac{2}{8} = \tfrac{7}{45}$.

$\therefore$ the expected value of X

$$= 25 \times p(25) + 20 \times p(20) + 5 \times p(5) + 0 \times p(0)$$
$$= 25 \times \tfrac{1}{5} + 20 \times \tfrac{8}{45} + 5 \times \tfrac{7}{45} + 0 = 9\tfrac{1}{3}$$

Since the man pays 10 pence per game, his expected loss in pence is $\tfrac{2}{3}$.

In the next example we make use of the following results obtained in §9.1.

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1), \quad \sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1), \quad \sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2.$$

Example 2 The probability distribution of a random variable X is given by

$$P(X = r) = k(n - r) \quad \text{for} \quad r = 1, 2, 3, \ldots, n.$$

Find k in terms of n. Find also the mean and variance of X.

Since X is a discrete random variable, $\sum_1^n p(r) = 1$.

But $\sum_1^n p(r) = \sum_1^n k(n - r) = k\sum_1^n n - k\sum_1^n r$

$$= kn^2 - k.\tfrac{1}{2}n(n+1) = \tfrac{1}{2}kn(n-1)$$

Hence $k = 2/n(n-1)$.

$$E(X) = \sum_1^n rp(r) = \sum_1^n kr(n - r)$$

$$= kn\sum_1^n r - k\sum_1^n r^2$$

$$= kn.\frac{1}{2}n(n+1) - k.\frac{1}{6}n(n+1)(2n+1)$$

$$= \frac{2}{n(n-1)}.\frac{n(n+1)}{6}\{3n - (2n+1)\} = \frac{1}{3}(n+1)$$

$$\text{Var}(X) = \sum_1^n r^2 p(r) - \mu^2 = \sum_1^n kr^2(n-r) - \frac{1}{9}(n+1)^2$$

$$= kn\sum_1^n r^2 - k\sum_1^n r^3 - \frac{1}{9}(n+1)^2$$

$$= kn.\frac{1}{6}n(n+1)(2n+1) - k.\frac{1}{4}n^2(n+1)^2 - \frac{1}{9}(n+1)^2$$

$$= \frac{2}{n(n-1)}.\frac{n^2(n+1)}{12}\{2(2n+1) - 3(n+1)\} - \frac{1}{9}(n+1)^2$$

$$= \frac{1}{6}n(n+1) - \frac{1}{9}(n+1)^2$$

$$= \frac{1}{18}(n+1)\{3n - 2(n+1)\} = \frac{1}{18}(n+1)(n-2)$$

Thus the mean and variance of X are $\frac{1}{3}(n+1)$ and $\frac{1}{18}(n+1)(n-2)$ respectively.

Exercise 15.1

1. A student has a week-end job in a warehouse. The probability that he will be given 6 hours work is 1/6, the probability of 9 hours work is 1/3 and the probability of 12 hours work is 1/2. Find the expected number of hours the student will work.

2. A man pays x pence to play a game in which two unbiased dice are rolled. If the total score on the dice is an even number n, then the man receives n pence. Otherwise he receives nothing. Calculate the value of x which makes the game fair.

3. A bag contains one 50p coin, three 20p coins, seven 10p coins and several 5p coins. Given that when one coin is selected at random the expectation is 10p, find the number of 5p coins. Find also the expectation when two coins are selected at random without replacement.

4. Three discs are drawn at random, without replacement from a bag containing three black discs and five red discs. Find (a) the most probable number of black discs drawn, (b) the expected number of black discs drawn.

5. A bag X contains 3 red balls and 6 white balls and a second bag Y contains 5 red balls and 4 white balls. A ball is chosen at random from bag X and placed in bag Y. A ball is then chosen at random from bag Y and placed in bag X. If R is the number of red balls in bag X after these operations, find the probability distribution of R. Hence find the expected value of R.

6. A simple 'fruit machine' has three wheels on which there are pictures of fruit. When 10p is paid into the machine, the wheels spin and then come to rest independently, each showing one fruit. The probabilities that the three wheels show

a lemon are 0·4, 0·2 and 0·1 respectively. The machine pays out 10p for 1 lemon, 30p for 2 lemons and £1·60 for 3 lemons. Find the expected profit or loss per play.

7. In a certain gambling game a player nominates an integer x from 1 to 6 inclusive and he then throws three fair cubical dice. Calculate the probabilities that the number of x's thrown will be 0, 1, 2 and 3. The player pays 5 pence per play of the game and he receives 48 pence if the number of x's thrown is three, 15 pence if the number of x's thrown is two, 5 pence if only one x is thrown and nothing otherwise. Calculate the player's expected gain or loss per play of the game. (JMB)

8. In a certain snooker match the first man to win 3 frames wins the match. The winner of each frame receives £20 and the winner of the match receives an additional £40. Find the expected cost of the prizes in a match between players A and B if the probability that A wins a frame against B is (a) 0·5, (b) 0·2.

9. A player throws a die whose faces are numbered 1 to 6 inclusive. If the player obtains a six he throws the die a second time, and in this case his score is the sum of 6 and the second number; otherwise his score is the number obtained. The player has no more than two throws. Let X be the random variable denoting the player's score. Write down the probability distribution of X, and determine the mean of X. Show that the probability that the sum of two successive scores is 8 or more is 17/36. Determine the probability that the first of two successive scores is 7 or more, given that their sum is 8 or more. (C)

10. At a horse show there is a jumping competition in which each horse attempts to clear 4 successive fences, each fence being more difficult than the previous one. A particular horse has probability $(5 - i)/5$ of clearing the ith fence $(i = 1, 2, 3, 4)$, independent of its performance over the other fences. (i) Find the probability that the horse clears all four fences. (ii) Find the probability that the horse does not clear any fence. (iii) Determine the expected value of the number of fences that the horse clears. (C)

11. A disc is drawn from a bag containing 10 discs numbered $0, 1, 2, \ldots 9$. The random variable X is defined as the square of the number drawn. Find $E(X)$ and $\mathrm{Var}(X)$.

12. A box contains 6 cards all of different colours. A card is drawn at random from the box and its colour noted. If three cards are drawn, with replacement, find the mean and variance of the number of different colours obtained.

13. Let X denote the number of heads obtained when a fair coin is tossed three times. Find the mean and the variance of X. A pack of six cards consists of three cards labelled H and the other three labelled T. Three cards are drawn at random without replacement from the pack. Let Y denote the number of H's obtained. Calculate the probabilities that Y takes the values 0, 1, 2 and 3 respectively. Show that the mean of Y is equal to the mean of X and express the variance of Y as a percentage of the variance of X. (JMB)

14. In a certain field, each puffball which is growing in one year gives rise to a number, X, of new puffballs in the following year. None of the original puffballs is present in the following year. The probability distribution of the random variable X is as follows:

$$P(X = 0) = P(X = 2) = 0{\cdot}3, P(X = 1) = 0{\cdot}4.$$

Find the probability distribution of Y, the number of puffballs resulting from there being two puffballs in the previous year, and show that the variance of Y is 1·2. Hence, or otherwise, determine the probability distribution of the number, Z, of puffballs present in year 3, given that there was a single puffball present in year 1. Find also the mean and variance of Z. (C)

15. A cubical die is biased in such a way that the probability of scoring n($n = 1$ to 6) is proportional to n. Determine the mean value and variance of the score obtained in a single throw. What would be the mean and variance if the score showing were doubled? (O&C)

16. A random variable R takes values $1, 2, \ldots, n$ with equal probabilities. Determine the expectation μ of R and show that the variance σ^2 is given by $12\sigma^2 = n^2 - 1$. Find $P(|R - \mu| > \sigma)$ in the case $n = 100$. (C)

17. The probability distribution of a random variable R is defined by $P(R = r) = kr$ for $r = 1, 2, \ldots, n$. Find the value of k. Hence find the mean and variance of the distribution.

18. Ten identically shaped discs are in a bag; two of them are black, the rest white. Discs are drawn at random from the bag in turn and NOT replaced. Let X be the number of discs drawn up to AND INCLUDING the first black one. List the values of X and the associated theoretical probabilities. Calculate the mean value of X and its standard deviation. What is the most likely value of X? If instead each disc is replaced before the next is drawn, construct a similar list of values and probabilities and point out the chief differences between the two lists. (SU)

19. (a) Expand $(1 - x)^{-2}$ in a series of ascending powers of x, giving the first three terms and the term in x^{r-1}.
    (b) A boy is throwing stones at a tin can. The probability of hitting the can with any particular stone is p. Find the probability that the boy first hits the can with the rth stone. If the random variable R is the number of throws needed to hit the can, find the expected value of R in terms of p.

20. A faulty gas-lighter works successfully, on average, once in three attempts. When using it one goes on trying until it does work and the gas lights. What is the probability that the gas lights (i) first time, (ii) second time, (iii) at the fifth attempt? What is the probability that it will not have worked after five attempts? The sellers of the lighter promise that if one is returned, they will pay 25p for each time it fails to work before the gas is lit. What is one's expectation in taking up this offer? What is the probability of getting one's money back (or more) if the original price was £1? If

four people take up this offer, what is the probability that at least two get their money back (or more)? (AEB 1978)

21. An ordinary unbiased cubical die is to be thrown repeatedly. Show that the probability that a score of 6 will occur for the first time on the nth throw with the first $(n-1)$ throws resulting in one odd score and $(n-2)$ even scores (other than 6) is equal to $(n-1)(\frac{1}{4})(\frac{1}{3})^{n-1}$ for $n \geqslant 2$. If the die is to be thrown as many times as it is necessary for a score of 6 to occur, find, using the above result, or otherwise, the probability that only one of the throws will result in an odd score.

[Hint: For $|x| < 1$, $\sum_{r=0}^{\infty} (r+1)x^r = (1-x)^{-2}$] (JMB)

15.2 The binomial distribution

A *Bernouilli*† *trial* is one in which there are two possible outcomes, usually referred to as success and failure. If the probability of success is p and the probability of failure is q, then $p + q = 1$.

If a random variable X represents the number of successes in a sequence of n independent repetitions of a Bernouilli trial, then the resulting probability distribution is called a *binomial distribution.*

Consider, for example, a trial in which an unbiased die is rolled once and success is defined as obtaining a six. In one trial the probability of success is 1/6 and the probability of failure is 5/6. Suppose now that the random variable X is the number of sixes obtained in a sequence of 5 rolls of the die.

Clearly $$P(X = 0) = (\tfrac{5}{6})^5$$

The number of arrangements of a sequence of 4 failures and 1 success = 5.
The probability of each such arrangement = $(\frac{5}{6})^4(\frac{1}{6})$

$$\therefore \quad P(X = 1) = 5(\tfrac{5}{6})^4(\tfrac{1}{6})$$

The number of arrangements of a sequence of 3 failures and 2 successes = $\frac{5!}{3!2!} = 10$.
The probability of each such arrangement = $(\frac{5}{6})^3(\frac{1}{6})^2$

$$\therefore \quad P(X = 2) = 10(\tfrac{5}{6})^3(\tfrac{1}{6})^2$$

Using similar arguments it follows that

$$P(X = 3) = 10(\tfrac{5}{6})^2(\tfrac{1}{6})^3, P(X = 4) = 5(\tfrac{5}{6})(\tfrac{1}{6})^4, P(X = 5) = (\tfrac{1}{6})^5.$$

We see that the probabilities of 0, 1, 2, 3, 4, 5 sixes in a sequence of 5 rolls of a die are given by the terms in the binomial expansion of $(\frac{5}{6} + \frac{1}{6})^5$.

To obtain a more general result let us assume that the probability of success in a certain trial is p and that the probability of failure is q, where $q = 1 - p$.

†*Bernouilli, Jacques* (1654–1705) Swiss mathematician: one of the members of a large family who were all gifted mathematically. His *Ars Conjectandi* of 1713 contains a considerable quantity of work on probability.

The number of arrangements of a sequence of $(n - r)$ failures and r successes is $\dfrac{n!}{(n-r)!r!}$, written $\binom{n}{r}$ or ${}_nC_r$.

The probability of each such arrangement $= q^{n-r}p^r$.

Hence the probability of r successes in n independent trials is $\binom{n}{r}q^{n-r}p^r$.

Thus the probabilities that $r = 0, 1, 2, \ldots, n$ are the terms in the binomial expansion of $(q + p)^n$. This distribution is described as the binomial distribution with *parameters* n and p.

A random variable X is said to have a binomial distribution $B(n, p)$, if

$$P(X = r) = \binom{n}{r}q^{n-r}p^r \quad \text{for} \quad r = 0, 1, 2, \ldots, n,$$

where $0 < p < 1$ and $q = 1 - p$.

This binomial distribution arises when X represents the number of successes in n *independent* trials, each trial having the *constant* probability p of success.

Example 1 A doctor estimates that his treatment of a particular illness is successful in $66\frac{2}{3}\%$ of cases. Find the probability that he will treat successfully exactly 4 out of the 6 patients currently in his care. Find also the probability of success in at least 2 out of the 6 cases.

If X is the number of successes, then X has a binomial distribution with parameters $n = 6$ and $p = \frac{2}{3}$.

$$P(X = 4) = \binom{6}{4}\left(\tfrac{1}{3}\right)^2\left(\tfrac{2}{3}\right)^4 = \frac{6!}{2!4!} \times \frac{2^4}{3^6} = \tfrac{80}{243} \approx 0{\cdot}329$$

$\therefore$ the probability of success in exactly 4 out of 6 cases is 0·329.

$P(X = 0) = \left(\tfrac{1}{3}\right)^6 = \tfrac{1}{729}$, $P(X = 1) = \binom{6}{1}\left(\tfrac{1}{3}\right)^5\left(\tfrac{2}{3}\right) = \tfrac{4}{243}$,

$\therefore$ the probability of at least 2 successes

$= 1 - \tfrac{1}{729} - \tfrac{4}{243} = \tfrac{716}{729} \approx 0{\cdot}982.$

If a random variable X has the binomial distribution $B(n, p)$, then

$$\sum_{r=0}^{n} P(X = r) = \sum_{r=0}^{n}\binom{n}{r}q^{n-r}p^r = (q + p)^n = 1.$$

Thus we have verified that for the distribution $B(n, p)$,

$$\sum_{r=0}^{n} p_r = 1, \quad \text{where} \quad p_r = P(X = r).$$

We now obtain the mean and variance of X.

$$E(X) = \sum_{r=0}^{n} r\binom{n}{r} q^{n-r}p^r$$

$$= \sum_{r=0}^{n} \frac{r(n!)}{(n-r)!r!} q^{n-r}p^r$$

$$= \sum_{r=1}^{n} \frac{n!}{(n-r)!(r-1)!} q^{n-r}p^r$$

$$= np \sum_{r=1}^{n} \frac{(n-1)!}{(n-r)!(r-1)!} q^{n-r}p^{r-1} = np(q+p)^{n-1} = np.$$

$$\text{Var}(X) = \sum_{r=0}^{n} r^2\binom{n}{r} q^{n-r}p^r - \{E(X)\}^2$$

Writing $r^2 = r(r-1) + r$, we obtain:

$$\text{Var}(X) = \sum_{r=0}^{n} \frac{r(r-1)n!}{(n-r)!r!} q^{n-r}p^r + \sum_{r=0}^{n} \frac{r(n!)}{(n-r)!r!} q^{n-r}p^r - n^2p^2$$

$$= \sum_{r=2}^{n} \frac{n!}{(n-r)!(r-2)!} q^{n-r}p^r + np - n^2p^2$$

$$= n(n-1)p^2 \sum_{r=2}^{n} \frac{(n-2)!}{(n-r)!(r-2)!} q^{n-r}p^{r-2} + np - n^2p^2$$

$$= n(n-1)p^2(q+p)^{n-2} + np - n^2p^2$$
$$= n(n-1)p^2 + np - n^2p^2$$
$$= np - np^2 = np(1-p) = npq$$

$\therefore$ the mean and variance of the binomial distribution $B(n, p)$ are given by

$$\mu = np, \quad \sigma^2 = npq = np(1-p).$$

Example 2 The components produced by a particular machine are tested by taking samples containing 5 components and noting the number of rejects in each. The given table shows the results for 100 samples

Number of rejects (x)	0	1	2	3	4	5
Frequency (f)	10	27	31	20	9	3

(a) Calculate the mean and variance of the number of rejects.
(b) Estimate the probability p that a component selected at random is a reject.

(c) By assuming that the number of rejects in a sample of 5 components has a binomial distribution with parameters $n = 5$ and p, write down the theoretical mean and variance.

(d) Obtain the expected frequency distribution for 100 samples.

(a) $\sum f = 10 + 27 + 31 + 20 + 9 + 3 = 100$

$\sum fx = 0 \times 10 + 1 \times 27 + 2 \times 31 + 3 \times 20 + 4 \times 9 + 5 \times 3 = 200$

$\sum fx^2 = 0^2 \times 10 + 1^2 \times 27 + 2^2 \times 31 + 3^2 \times 20 + 4^2 \times 9 + 5^2 \times 3 = 550$

$$\therefore \quad \text{the mean } \bar{x} = \frac{\sum fx}{\sum f} = \frac{200}{100} = 2,$$

$$\text{the variance} = \frac{\sum fx^2}{\sum f} - \bar{x}^2 = \frac{550}{100} - 2^2 = 1{\cdot}5.$$

(b) Total number of rejects $= \sum fx = 200$

Total number of components $= 5 \times \sum f = 500$

$\therefore$ the probability p that any component is a reject $= \frac{200}{500} = \frac{2}{5}$.

(c) The mean and variance of the binomial distribution with parameters $n = 5$ and $p = 2/5$ are respectively:

$$\mu = np = 2, \quad \sigma^2 = np(1-p) = 1{\cdot}2.$$

(d) $$P(\text{no rejects}) = \left(\frac{3}{5}\right)^2 \approx 0{\cdot}08,$$

$$P(1 \text{ reject}) = \binom{5}{1}\left(\frac{3}{5}\right)^4\left(\frac{2}{5}\right) \approx 0{\cdot}26,$$

$$P(2 \text{ rejects}) = \binom{5}{2}\left(\frac{3}{5}\right)^3\left(\frac{2}{5}\right)^2 \approx 0{\cdot}35,$$

$$P(3 \text{ rejects}) = \binom{5}{3}\left(\frac{3}{5}\right)^2\left(\frac{2}{5}\right)^3 \approx 0{\cdot}23,$$

$$P(4 \text{ rejects}) = \binom{5}{4}\left(\frac{3}{5}\right)\left(\frac{2}{5}\right)^4 \approx 0{\cdot}08,$$

$$P(5 \text{ rejects}) = \left(\frac{2}{5}\right)^5 \approx 0{\cdot}01.$$

Multiplying these probabilities by 100 gives the expected frequencies in 100 samples (correct to the nearest integer).

Number of rejects	0	1	2	3	4	5
Expected frequency	8	26	35	23	8	1

[Note that because of rounding errors the sum of these expected frequencies is 101. This does not invalidate the results.]

We now consider an alternative way of finding the mean and variance of certain discrete probability distributions including the binomial distribution.

The *probability generating function* (p.g.f.) for the random variable X, which takes the values $0, 1, 2, \ldots, r, \ldots$ with probabilities $p_0, p_1, p_2, \ldots, p_r, \ldots$ is given by

$$G(t) = p_0 + p_1 t + p_2 t^2 + \ldots + p_r t^r + \ldots$$

Thus by using a "dummy" variable t, we are able to write down a single expression $G(t)$ representing a complete probability distribution. We note that since $\sum p_r = 1$, it follows that $G(1) = 1$.

For the binomial distribution with parameters n and p,

$$G(t) = q^n + \binom{n}{1} q^{n-1} pt + \binom{n}{2} q^{n-2} p^2 t^2 + \ldots + \binom{n}{r} q^{n-r} p^r t^r + \ldots$$

$\therefore$ the p.g.f. of the binomial distribution $B(n, p)$ is

$$G(t) = (q + pt)^n. \quad \text{where} \quad q = 1 - p.$$

If $G(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3 + \ldots + p_r t^r + \ldots$ then, differentiating with respect to t,

$$G'(t) = p_1 + 2p_2 t + 3p_3 t^2 + \ldots + rp_r t^{r-1} + \ldots$$
$$G''(t) = 2p_2 + 3.2p_3 t + \ldots + r(r-1)p_r t^{r-2} + \ldots$$
$$\therefore \quad G'(1) = p_1 + 2p_2 + 3p_3 + \ldots + rp_r + \ldots = \sum rp_r$$
$$G''(1) = 2p_2 + 3.2p_3 + \ldots + r(r-1)p_r + \ldots$$
$$= \sum r(r-1)p_r = \sum r^2 p_r - \sum rp_r$$

Hence $E(X) = \sum rp_r = G'(1)$

and $\text{Var}(X) = \sum r^2 p_r - [E(X)]^2 = G''(1) + G'(1) - [G'(1)]^2$

i.e. for a distribution with p.g.f. $G(t)$

$$\mu = G'(1), \quad \sigma^2 = G''(1) + G'(1) - [G'(1)]^2.$$

For the binomial distribution $B(n, p)$:

$$G(t) = (q + pt)^n$$
$$G'(t) = np(q + pt)^{n-1} \qquad \therefore \quad G'(1) = np$$
$$G''(t) = n(n-1)p^2(q + pt)^{n-2} \qquad \therefore \quad G''(1) = n(n-1)p^2$$

Hence $\mu = np$, $\sigma^2 = n(n-1)p^2 + np - n^2p^2 = np(1-p)$.

These results are the same as those obtained earlier by elementary methods.

Exercise 15.2

1. An unbiased die is thrown 9 times. Find the probability of (a) 2 sixes, (b) at least 2 sixes. Find also the mean and variance of the number of sixes thrown.

2. In a large batch of plastic mouldings 25% have faults. If a random sample of 10 mouldings is inspected, find the probability that the sample contains

(a) 3 faulty mouldings, (b) no more than 2 faulty mouldings. Find also the mean and variance of the number of faulty mouldings.

3. The probability that a certain darts player hits the "treble 20" with one dart is 0·3. Find the probability that the player scores at least one treble 20 with 3 darts. If the probability of scoring at least one treble 20 with n darts is greater than 0·9, find the least possible value of n.

4. One in eight of the torches assembled by a machine are imperfect. Eight torches are chosen at random. Find the probability that (a) one is imperfect, (b) at least 6 are imperfect. The probability that a box of n torches contains none that are imperfect is less than 0·1. Find the range of possible values of n.

5. Given that the binomial distribution $B(n, p)$ has mean 9·6 and standard deviation 2·4, find n and p. Hence obtain the probability generating function of the distribution.

6. Find the probability generating function of a binomial distribution whose mean is 1·8 and whose variance is 0·99. If the random variable X has this binomial distribution, (i) state the greatest possible value that X may take (with non-zero probability), (ii) write down a numerical expression for the probability that a randomly observed value of X will be equal to 3. (JMB)

7. A lake contains 1000 fish of species A, 2000 fish of species B and no other fish. Ten fish are taken from the lake at random. Explain why the probabilities of obtaining various numbers of fish of species A in the ten can be very well approximated by a binomial probability distribution, and hence estimate to three significant figures the probability of obtaining 4 fish of species A and six of species B. Estimate also to three significant figures the probability that in a catch of 5 fish altogether there are more fish of species A than there are of species B. (C)

8. A manufacturing company sends out invoices to its customers allowing 5% discount to those who settle their accounts within 15 days. In the past, 40% of the customers took advantage of the discount terms. On a particular day the company sent out 10 invoices. Examine critically the appropriateness of the binomial distribution model for this situation, discussing (a) the requirement of constant probability from trial to trial, (b) the independence of trials. Assuming the model to be appropriate, calculate the probability that, of the 10 invoices, less than 2 are settled within 15 days. For days when 10 invoices are sent out, calculate the mean and variance of the distribution of the number of invoices settled within 15 days. (L)

9. The numbers of males in a sample of 810 litters of 4 cats are recorded in the table below:

No. of males	0	1	2	3	4
No. of litters	156	326	242	74	12

Calculate the mean and variance of the number of males in this sample of litters. Obtain a corresponding table of the above form for a binomial distribution with $p = \frac{1}{3}$.

10. Six dice were thrown together 1000 times. At each throw, the number of ones obtained was noted and the frequency is shown in the table below:

No. of 1's in a single throw	0	1	2	3	4	5	6
No. of throws	370	380	150	90	9	1	0

Calculate the mean and variance of the number of ones obtained in the experiment. Produce a corresponding table of the expected frequencies, to the nearest integer in each case, based on the assumption that the six dice are unbiased and that the 1000 throws an independent. (L)

11. (a) Berg and Korner have a long rivalry in tennis: in the last 25 games, Berg has won 15 times. They start a new series of 8 games. Assuming that the binomial model may be applied, what is the probability that Berg will win at least 6 of these?

(b) Prove that the mean of the binomial distribution is np. A set of 100 pods, each containing 4 peas, was examined to see how many of the peas were good. The following were the results:

No. of good peas in pod	0	1	2	3	4
No. of pods	7	20	35	30	8

(i) What was the probability of getting a good pea? (ii) Calculate the theoretical frequencies of 0, 1, 2, 3, 4 good peas, using the associated theoretical binomial distribution. (SU)

12.

X	frequency
0	36
1	37
2	20
3	5
4	2

The number of wrong numbers (X) obtained in making a set of ten telephone calls is recorded in the given table which records the results obtained in making 100 sets of 10 telephone calls. Assuming that wrong numbers occur randomly, suggest a suitable statistical model to correspond to the making of 10 telephone calls and recording the number of wrong numbers obtained. Using this model estimate the percentage of sets of ten telephone calls in which 2 or more wrong numbers are obtained. What will this percentage be if the chance of getting a wrong number when making a call is doubled? (L)

13. (i) Find the probability of obtaining exactly two sixes in six throws of a single date. [You may assume that $(5/6)^5 = 0{\cdot}402$.]

(ii) If two dice are to be thrown together and the sum of their scores is recorded, find the probability that (a) the sum in one throw will be eleven, (b) a score of

eleven will occur exactly twice in six throws [assume that $17^4/18^5 = 0{\cdot}044$], (c) in one throw the score will be less than eight, given that exactly one die shows a four. (L)

14. A crossword puzzle is published in *The Times* each day of the week, except Sunday. A man is able to complete, on average, 8 out of 10 of the crossword puzzles. (i) Find the expected value and the standard deviation of the number of completed crosswords in a given week. (ii) Show that the probability that he will complete at least 5 in a given week is 0·655 (to 3 significant figures). (iii) Given that he completes the puzzle on Monday, find, to three significant figures, the probability that he will complete at least 4 in the rest of the week. (iv) Find, to three significant figures, the probability that, in a period of four weeks, he completes 4 or less in only one of the four weeks. (C)

15. 4 girls and 3 boys plan to meet together on the following Saturday. The probability that each boy will be present is $\frac{2}{3}$ independently of the other boys. Find the probabilities that (a) 0, (b) 1, (c) 2, (d) 3 boys will be present. The probability that each girl will be present is $\frac{1}{2}$ independently of the other girls and of the boys. (e) Find the probability that the number of girls present will equal the number of boys. (f) Find the probability that both sexes will be present. (g) Afterwards it was reported that the gathering had included at least one boy and at least one girl. What is the probability that there were equal numbers of boys and girls in the light of this additional information? (O&C)

16. Each trial of a binomial experiment has a constant probability p of yielding a success. (i) Given that 7 successes occurred in 15 independent trials show that the conditional probability that the first and last trials were successes is 1/5. (ii) Find the value of p given that in 20 independent trials the probability of exactly 4 successes is twice that of exactly 6 successes. (JMB)

17. The random variable X has the binomial distribution defined by

$$P(X = r) = \binom{n}{r}(1-p)^{n-r}p^r, \quad r = 0, 1, 2, \ldots, n.$$

Find an expression for $P(X = r)/P(X = r-1)$. Hence deduce the most likely values of X when (a) $n = 20$, $p = 0{\cdot}4$, (b) $n = 14$, $p = 0{\cdot}2$.

18. Cards are drawn at random and with replacement from an ordinary pack of playing cards until three spades have been drawn. Find the probability that the number of draws required is (a) exactly six, (b) at least six. Find an expression for P_n, the probability that exactly n cards have to be drawn to get three spades ($n \geqslant 3$). Hence, or otherwise, find the most likely value (or values) of n. (L)

19. A random variable X has p.g.f. $G(t) = 1/(2-t)^2$. Find the expressions for (a) $P(X = r)$, (b) $E(X)$, (c) $\mathrm{Var}(X)$.

20. A random variable X is defined as the number of throws of an unbiased die needed to obtain a six. Find the p.g.f. for X and hence determine $E(X)$ and $\mathrm{Var}(X)$.

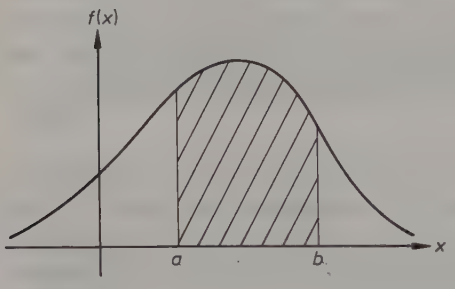
15.3 Continuous probability distributions

If X is a continuous random variable which may take any real value within some given interval, $0 \leqslant x \leqslant 10$ say, then it is difficult to give a meaning to probabilities such as $P(X = 2)$, $P(X = 7{\cdot}32054)$, $P(X = 17/39)$. If $P(X = x)$ were non-zero for all values of x, the sum of all such probabilities would be infinite. This means that we must regard the value of $P(X = x)$ as zero. Thus, in order to set up a suitable probability model we clearly need to define a new type of function.

The *probability density function* (p.d.f.) of the continuous random variable X is a function $f(x)$ such that

$$P(a \leqslant X \leqslant b) = \int_a^b f(x)\,dx.$$

[Since $P(X = a) = P(X = b) = 0$, the values of $P(a < X < b)$, $P(a \leqslant X < b)$ and $P(a < X \leqslant b)$ are also given by this integral.]



It follows that we can now think of probabilities as areas under the graph of $f(x)$. As probabilities are never negative, $f(x) \geqslant 0$ for all values of x. Since the sum of the probabilities of all possible outcomes of an experiment is 1, the total area under the graph must be 1.

i.e. $$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

For a discrete random variable X, $E(X) = \mu = \sum xp(x)$ and $\mathrm{Var}(X) = \sigma^2 = \sum(x - \mu)^2 p(x) = \sum x^2 p(x) - \mu^2$. To obtain similar formulae for the mean and variance of a continuous random variable X, let us consider the probability that the value of X lies between x and $x + \delta x$. Provided that δx is small, this probability is given approximately by $f(x)\delta x$. Thus we let

$$E(X) = \lim_{\delta x \to 0} \sum xf(x) = \int_{-\infty}^{\infty} xf(x)\,dx.$$

Applying similar arguments to the variance, we have:

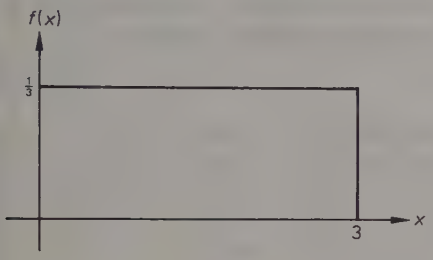
For a continuous random variable X,

$$E(X) = \mu = \int_{-\infty}^{\infty} xf(x)\,dx$$

$$\mathrm{Var}(X) = \sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)\,dx = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2.$$

One of the simplest types of continuous probability distribution is the *rectangular* or *uniform* distribution. A continuous random variable X is said to be *uniformly distributed* over the interval a to b if $f(x)$ has a constant non-zero value for $a \leqslant x \leqslant b$ and is zero otherwise.

Example 1 Find the mean and variance of the continuous random variable X which is uniformly distributed over the interval from 0 to 3.



Since the total area under the graph of the probability density function $f(x)$ must be 1,

$f(x) = \frac{1}{3}$ for $0 \leqslant x \leqslant 3$

$f(x) = 0$ otherwise.

$$\therefore \quad E(X) = \int_0^3 x \times \frac{1}{3}\,dx = \left[\frac{1}{6}x^2\right]_0^3 = \frac{3}{2}$$

$$\text{Var}(X) = \int_0^3 x^2 \times \frac{1}{3}\,dx - \left(\frac{3}{2}\right)^2 = \left[\frac{1}{9}x^3\right]_0^3 - \frac{9}{4} = 3 - \frac{9}{4} = \frac{3}{4}$$

Hence the mean of the distribution is $1\frac{1}{2}$ and the variance is $\frac{3}{4}$.

A *mode* or *modal value* of a continuous random variable X with probability density function $f(x)$ is a value of x for which $f(x)$ takes a maximum value. Thus the modes of a distribution are the x-coordinates of the maximum points on the graph of $f(x)$.

The *median* of a continuous random variable X with probability density function $f(x)$ is the value m such that $\int_{-\infty}^{m} f(x)\,dx = 0{\cdot}5$.

Example 2 A continuous random variable X has probability density function $f(x) = kx(9 - x^2)$ for $0 \leqslant x \leqslant 3$ and $f(x) = 0$ otherwise. Find (a) the value of k, (b) the mean, (c) the mode, (d) the median.

(a) $$\int_0^3 f(x)\,dx = k\int_0^3 (9x - x^3)\,dx = k\left[\frac{9x^2}{2} - \frac{x^4}{4}\right]_0^3 = \frac{81k}{4}$$

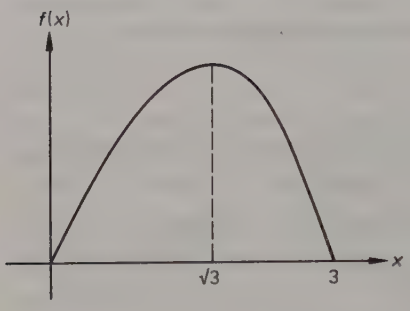
Since the total area under the graph of $f(x)$ must be 1, $k = 4/81$.

(b) $$E(X) = \int_0^3 xf(x)\,dx = \frac{4}{81}\int_0^3 (9x^2 - x^4)\,dx$$

$$= \frac{4}{81}\left[3x^3 - \frac{x^5}{5}\right]_0^3 = 4\left(1 - \frac{3}{5}\right) = \frac{8}{5}$$

Hence the mean of the distribution is 1·6.

(c)



$f(x) = \frac{4}{81}(9x - x^3)$,

$f'(x) = \frac{4}{81}(9 - 3x^2) = \frac{4}{27}(3 - x^2)$

$\therefore$ in the interval $0 \leqslant x \leqslant 3$, $f'(x) = 0$ when $x = \sqrt{3}$.
The sketch shows that this gives a maximum value of $f(x)$.

Hence the mode of the distribution is 1·73 (to 3 s.f.)

(d) If m is the median, $\int_0^m f(x)\,dx = \frac{1}{2}$

$$\therefore \quad \frac{4}{81}\int_0^m (9x - x^3)\,dx = \frac{4}{81}\left[\frac{9x^2}{2} - \frac{x^4}{4}\right]_0^m = \frac{1}{81}(18m^2 - m^4) = \frac{1}{2}$$

$$36m^2 - 2m^4 = 81$$
$$2m^4 - 36m^2 + 81 = 0$$

$$\therefore \quad m^2 = \frac{36 \pm \sqrt{(36^2 - 8.81)}}{4} = \frac{36 \pm 18\sqrt{(4 - 2)}}{4} = 9 \pm \frac{9}{2}\sqrt{2}$$

Since $0 < m < 3$, we must have $m^2 = 9 - \frac{9}{2}\sqrt{2}$

$$\therefore \quad m = \sqrt{\left\{9 - \frac{9}{2}\sqrt{2}\right\}} \approx 1{\cdot}62$$

Hence the median of the distribution is 1·62 (to 3 s.f.).

Another function used when modelling continuous probability distributions is defined as follows:

> The *cumulative distribution function* (c.d.f.) of the continuous random variable X is the function $F(x)$ such that
>
> $$F(x) = P(X \leqslant x).$$

If X has probability density function $f(x)$, then

$$F(t) = P(X \leqslant t) = \int_{-\infty}^{t} f(x)\,dx.$$

It follows that: $F'(x) = f(x)$.

Example 3 A random variable X has cumulative distribution function

$$F(x) = \begin{cases} 0 & \text{for } x \leqslant 0 \\ \frac{1}{6}x^2 & \text{for } 0 \leqslant x \leqslant 2 \\ -2 + 2x - \frac{1}{3}x^2 & \text{for } 2 \leqslant x \leqslant 3 \\ 1 & \text{for } x \geqslant 3. \end{cases}$$

Find (a) $P(X \leqslant 1{\cdot}5)$, (b) the median, (c) the probability density function $f(x)$.

(a) $P(X \leqslant 1{\cdot}5) = F(1{\cdot}5) = \frac{1}{6}(1{\cdot}5)^2 = 0.375$.

(b) The median is the value m such that $F(m) = \frac{1}{2}$.

Since $F(0) = 0$ and $F(2) = \frac{2}{3}$, m lies in the interval from 0 to 2 and thus $F(m) = \frac{1}{6}m^2$.

$\therefore$ $\frac{1}{6}m^2 = \frac{1}{2}$ which gives $m = \sqrt{3}$.

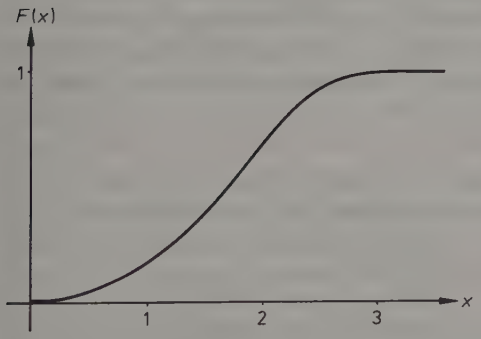
Hence the median of the distribution is 1·73 (to 3 s.f.).

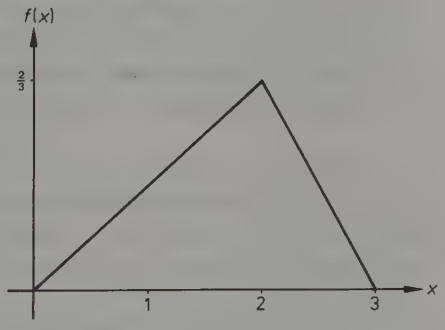
(c) The probability density function $f(x) = F'(x)$.

$\therefore$ for $0 \leqslant x \leqslant 2$, $f(x) = \dfrac{d}{dx}(\frac{1}{6}x^2) = \frac{1}{3}x$;

for $2 \leqslant x \leqslant 3$, $f(x) = \dfrac{d}{dx}(-2 + 2x - \frac{1}{3}x^2) = 2 - \frac{2}{3}x$;

otherwise, $f(x) = 0$.





Example 4 A random variable X has probability density function $f(x) = \frac{1}{2}x$ for $0 \leqslant x \leqslant 2$ and $f(x) = 0$ otherwise. Find the cumulative distribution function $F(x)$ of the variable X. Hence find the cumulative distribution function $G(y)$ of the variable $Y = X^2$. Deduce the probability density function $g(y)$.

For $0 \leqslant t \leqslant 2$, $F(t) = \displaystyle\int_0^t \tfrac{1}{2}x\,dx = [\tfrac{1}{4}x^2]_0^t = \tfrac{1}{4}t^2$.

$\therefore$ the cumulative distribution function of X is given by

$F(x) = 0\,(x \leqslant 0)$, $F(x) = \frac{1}{4}x^2\,(0 \leqslant x \leqslant 2)$, $F(x) = 1\,(x \geqslant 2)$.

$G(y) = P(Y \leqslant y) = P(X^2 \leqslant y) = P(X \leqslant \sqrt{y}) = F(\sqrt{y})$

$\therefore$ the cumulative distribution function of Y is given by

$G'(y) = 0 (y \leqslant 0)$, $G(y) = \frac{1}{4}y (0 \leqslant y \leqslant 4)$, $G(y) = 1 (y \geqslant 4)$.

Since $g(y) = G'(y)$, we find that $g(y) = \frac{1}{4} (0 \leqslant y \leqslant 4)$ and $g(y) = 0$ otherwise.

[We see that Y is uniformly distributed over the interval from 0 to 4.]

Exercise 15.3

1. Find the mean and variance of the continuous random variable X which is uniformly distributed over the interval
(a) from 0 to 1, (b) from 1 to 5,
(c) from 0 to k, (d) from a to b, where $b > a$.

2. A continuous random variable X has probability density function

$$f(x) = k(3x^2 - x^3) \quad \text{for} \quad 0 \leqslant x \leqslant 3 \quad \text{and} \quad f(x) = 0 \text{ otherwise.}$$

Find the value of k. Find also the mean, mode and standard deviation of X.

3. (a) A continuous variable x is distributed at random between the values 2 and 3 and has a probability density function of $6/x^2$. Find the median value of x.
(b) A continuous random variable X takes values between 0 and 1, with a probability density function of $Ax(1 - x)^3$. Find the value of A, and the mean and standard deviation of X. (SU)

4. A continuous variable X is distributed at random between two values, $x = 0$ and $x = 2$, and has a probability density function of $ax^2 + bx$. The mean is 1·25. (i) Show that $b = \frac{3}{4}$, and find the value of a. (ii) Find the variance of X. (iii) Verify that the median value of X is approximately 1·3. (iv) Find the mode. (SU)

5. The random variable X has a probability density function given by

$$P(x) = \begin{cases} kx(1 - x^2) & (0 \leqslant x \leqslant 1), \\ 0 & \text{elsewhere,} \end{cases}$$

k being a constant. Find the value of k and find also the mean and variance of this distribution. Find the median of the distribution. (O&C)

6. The continuous random variable X has probability density function given by

$$f(x) = \begin{cases} k(1 + x^2) & \text{for} \quad -1 \leqslant x \leqslant 1, \\ 0 & \text{otherwise,} \end{cases}$$

where k is a constant. Find the value of k, and determine $E(X)$ and $\text{Var}(X)$. A is the event $X > \frac{1}{2}$; B is the event $X > \frac{3}{4}$. Find (i) $P(B)$, (ii) $P(B|A)$. (C)

7. Petrol is delivered to a garage every Monday morning. At this garage the weekly demand for petrol, in thousands of units, is a continuous random variable X distributed with a probability density function of the form

$$f(x) = \begin{cases} ax^2(b - x) & 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise.} \end{cases}$$

(i) Given that the mean weekly demand is 600 units, determine the values of a and b. (ii) If the storage tanks at this garage are filled to their total capacity of 900 units every Monday morning, what is the probability that in any given week the garage will be unable to meet the demand for petrol? (AEB 1978)

8. A continuous random variable X has the probability density function f defined by

$$\begin{aligned} f(x) &= \tfrac{1}{3}cx, & 0 \leqslant x < 3, \\ f(x) &= c, & 3 \leqslant x \leqslant 4, \\ f(x) &= 0, & \text{otherwise,} \end{aligned}$$

where c is a positive constant. Find (i) the value of c, (ii) the mean of X, (iii) the value, a, for there to be a probability of 0·85 that a randomly observed value of X will exceed a. (JMB)

9. A continuous random variable X takes values between 0 and 1 only and its probability density function f is given by $f(x) = kx^2(1 - x)$. Prove that $k = 12$, and find $E(X)$. The cumulative distribution function F is such that $F(x) = P(X \leqslant x)$. Express $F(x)$ in terms of x, and sketch the graph of $y = F(x)$. Verify by calculation that the median of X lies between 0·60 and 0·62. (C)

10. The random variable X has the probability density function

$$f(x) = ae^{-ax} (0 < x < \infty).$$

Show that the cumulative distribution function of X is $F(x) = 1 - e^{-ax}$. The above distribution, with parameter $a = 0{\cdot}8$, is proposed as a model for the length of life, in years, of a species of bird. Find the expected frequencies, of a total of 50 birds, that would fall in the class intervals (years) 0−, 1−, 2−, 3−, 4−, 5 and over. (O)

11. A random variable X takes values in the interval $0 < x < 3$ and has probability density function $f(x)$, where

$$f(x) = \begin{cases} ax & (0 < x \leqslant 1), \\ \frac{1}{2}a(3 - x) & (1 < x < 3). \end{cases}$$

(i) Find the value of a. (ii) Find the expected value and the standard deviation of X. (iii) Find the (cumulative) distribution function $F(x)$. (iv) Find $P(|X - 1| < \frac{1}{2})$. (C)

12. The continuous random variable X has probability density function $f(x)$ defined by

$$f(x) = \begin{cases} c/x^4 & (x < -1) \\ c(2 - x^2) & (-1 \leqslant x \leqslant 1) \\ c/x^4 & (x > 1). \end{cases}$$

(i) Show that $c = \frac{1}{4}$. (ii) Sketch the graph of $f(x)$. (iii) Determine the cumulative distribution function $F(x)$. (iv) Determine the expected value of X and the variance of X. (C)

13. The continuous random variable X has (cumulative) distribution function given by

$$F(x) = \begin{cases} (1 + x)/8 & (-1 \leqslant x \leqslant 0), \\ (1 + 3x)/8 & (0 \leqslant x \leqslant 2), \\ (5 + x)/8 & (2 \leqslant x \leqslant 3), \end{cases}$$

with $F(x) = 0$ for $x < -1$, and $F(x) = 1$ for $x > 3$.

(i) Sketch the graph of the probability density function $f(x)$.
(ii) Determine the expectation of X and the variance of X.
(iii) Determine $P(3 \leqslant 2X \leqslant 5)$. (C)

14. The length X of an offcut of wooden planking is a random variable which can take any value up to 0·5 m. It is known that the probability of the length being not more than x metres $(0 \leqslant x \leqslant 0{\cdot}5)$ is equal to kx. Determine (i) the value of k, (ii) the probability density function of X, (iii) the expected value of X, (iv) the standard deviation of X (correct to 3 s.f.). (C)

15. The (cumulative) distribution function for a continuous random variable X is given by

$$F(x) = \begin{cases} 0 & \text{for} \quad x < -2 \\ k(4x - \frac{1}{3}x^3 + \frac{16}{3}) & \text{for} \quad -2 \leqslant x \leqslant 2, \\ 1 & \text{for} \quad x > 2. \end{cases}$$

Find (i) the value of k, (ii) $P(-1 < X < 1)$, (iii) the probability density function for X, (iv) the mean and variance of X. (C)

16. A random variable X has probability density function $f(x) = \frac{2}{3}x$ for $1 \leqslant x \leqslant 2$ and $f(x) = 0$ otherwise. Find the cumulative distribution function $F(x)$ of the variable X. Hence find the cumulative distribution function $G(y)$ of the variable $Y = X^{1/2}$. Deduce the probability density function $g(y)$.

17. X is a continuous random variable which is uniformly distributed in the interval $-1 \leqslant x \leqslant 1$. A second random variable Y is given by $Y = X^2$. Show that, if $0 < y < 1$, then $P(Y \leqslant y) = \sqrt{y}$, and deduce the probability density function of Y. Two independent observations of Y are made, and Z denotes the larger of the two observations. By considering the probability that both values of Y are

less than or equal to z, where $0 \leqslant z \leqslant 1$, or otherwise, prove that z is uniformly distributed. Determine the probability density function of W, the smaller of the two values of Y. (C)

18. Write down the probability density function and the cumulative distribution function of a random variable which is uniformly distributed over the interval $[a, b]$.

A person drives to work, a distance of 12 miles. The time he takes, in minutes, to cover the journey is uniformly distributed over the interval $[20, 30]$. Let V denote the car's average speed in miles per hour on such a journey. Find an expression for $P(V \leqslant v)$, where v is an arbitrary value between 24 and 36. Hence, or otherwise, find (i) the probability density function of V, (ii) the probability that on such a journey the car's average speed will be in excess of 30 m.p.h., (iii) the median average speed for the journey. (W)

19. A continuous random variable X has probability density function defined by

$$f(x) = 2x/a^2 \qquad 0 \leqslant x \leqslant a$$
$$f(x) = 0, \qquad \text{otherwise.}$$

(a) Find the variance of X. (b) Find the cumulative distribution function of X. (c) If $Y = 1 - (X/a)$, find (i) $P(Y > X)$, (ii) the probability density function of Y. (W)

20. A random variable X has probability density function $f(x) = |x|$ for $-1 \leqslant x \leqslant 1$ and $f(x) = 0$ otherwise. Find the probability density function $g(y)$ of the random variable $Y = X^2$.

15.4 The normal distribution

The *normal distribution* is the most important continuous probability distribution in theoretical statistics. It is found to provide a good probability model for the distributions of many variables such as the heights of people and other measurements of biological characteristics. One of the early uses of the normal distribution was as a model for the distribution of errors in experimental observations.

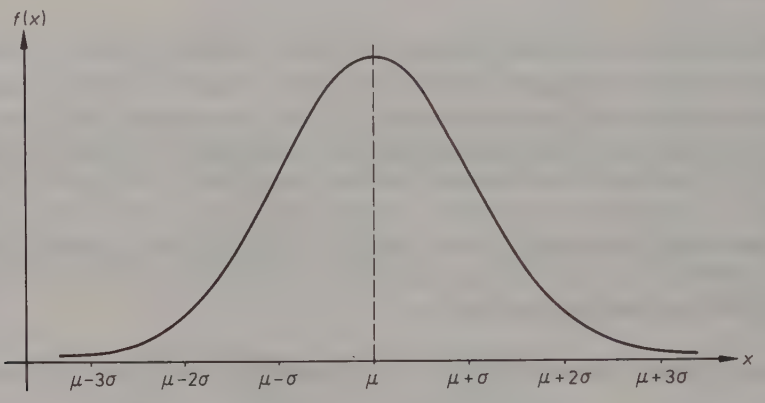
A continuous random variable X is said to be *normally distributed* with mean μ and variance σ^2 if it has probability density function.

$$f(x) = \frac{1}{\sigma\sqrt{(2\pi)}} e^{-(x-\mu)^2/2\sigma^2} \quad (\sigma > 0)$$

This distribution is denoted by $N(\mu, \sigma^2)$.

We cannot prove by elementary methods that this is a valid definition of a probability distribution. Clearly $f(x) \geqslant 0$ for all values of x, but for now we must

assume that the total area under the graph of $f(x)$ is 1 and that the distribution has mean μ and variance σ^2.



The graph of $f(x)$ is a bell-shaped curve called the *normal curve*. We see that
(1) the curve is symmetrical about the line $x = \mu$;
(2) the x-axis is an asymptote to the curve;
(3) when $x = \mu$, $f(x)$ takes its maximum value $1/\sigma\sqrt{(2\pi)}$ (i.e. $0{\cdot}4/\sigma$ approximately).
As described in the previous section, the probabilities associated with a continuous random variable X are represented by areas under the graph of the p.d.f. $f(x)$. Thus for the distribution $N(\mu, \sigma^2)$,

$$P(X \leqslant a) = \int_{-\infty}^{a} f(x)\,dx = \int_{-\infty}^{a} \frac{1}{\sigma\sqrt{(2\pi)}} e^{-(x-\mu)^2/2\sigma^2}\,dx.$$

This integral can be simplified by substituting

$$z = \frac{x-\mu}{\sigma} \quad \text{and} \quad dz = \frac{1}{\sigma}dx.$$

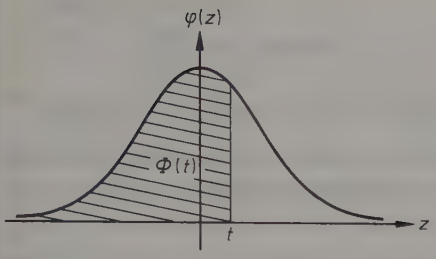
Writing $c = \dfrac{a-\mu}{\sigma}$, we have

$$P(X \leqslant a) = P(Z \leqslant c) = \int_{-\infty}^{c} \frac{1}{\sqrt{(2\pi)}} e^{-z^2/2}\,dz.$$

Hence the new random variable $Z = (X - \mu)/\sigma$ has probability density function $\phi(z) = \dfrac{1}{\sqrt{(2\pi)}} e^{-z^2/2}$. We deduce that Z is normally distributed with mean 0 and variance 1. We therefore call Z a *standardised* or *standard* normal variable. To summarise:

> If a normal variable X has the distribution $N(\mu, \sigma^2)$ then the random variable $Z = (X - \mu)/\sigma$ has the *standard normal distribution* $N(0, 1)$.

Using this change of variable any problem involving a normal distribution can be restated in terms of the standard distribution $N(0, 1)$.



The (cumulative) *distribution function* for the standard normal variable Z is denoted by $\Phi(z)$,
where $\Phi(z) = P(Z \leqslant z)$,

$$\therefore \quad \Phi(t) = \int_{-\infty}^{t} \phi(z)\,dz,$$

where $\phi(z) = \dfrac{1}{\sqrt{(2\pi)}} e^{-z^2/2}$.

Since these integrals are difficult to evaluate, tables of values have been compiled. Table 1, which appears towards the end of this book, gives values of $\Phi(z)$ for $0 \leqslant z \leqslant 4$. For example, $P(Z \leqslant 2{\cdot}18) = \Phi(2{\cdot}18) = 0{\cdot}98537$. If intermediate values of $\Phi(z)$ are required linear interpolation can be used.

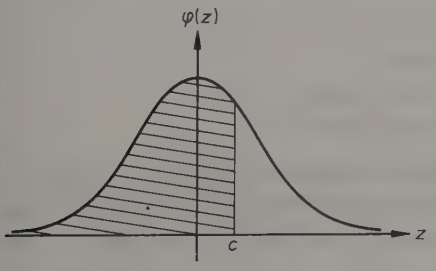
Example 1 Use Table 1 to estimate the value of $\Phi(0{\cdot}6354)$.

$\Phi(0{\cdot}64) - \Phi(0{\cdot}63) = 0{\cdot}7389 - 0{\cdot}7357 = 0{\cdot}0032$

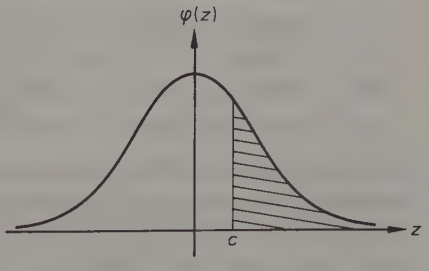
$\therefore$ by linear interpolation,

$\Phi(0{\cdot}6354) = 0{\cdot}7357 + \frac{54}{100} \times 0{\cdot}0032 \approx 0{\cdot}7374.$

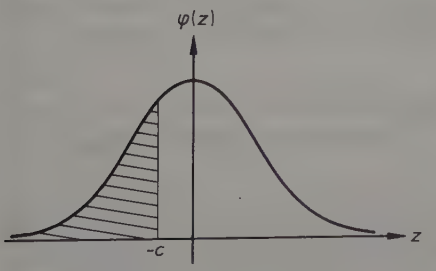
The following diagrams show how the symmetry of the normal distribution can be used to obtain $\Phi(z)$ for negative values of z.



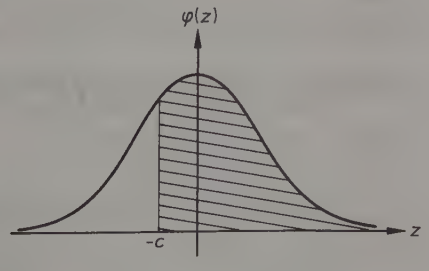
$P(Z \leqslant c) = \Phi(c)$



$P(Z \geqslant c) = 1 - \Phi(c)$



$P(Z \leqslant -c) = 1 - \Phi(c)$



$P(Z \geqslant -c) = \Phi(c)$

We may apply these results as follows:

$$P(Z \leqslant -0.72) = 1 - \Phi(0{\cdot}72) = 1 - 0{\cdot}7642 = 0.2358$$
$$P(-1{\cdot}5 \leqslant Z \leqslant -0{\cdot}5) = P(Z \geqslant -1{\cdot}5) - P(Z \geqslant -0.5)$$
$$= \Phi(1{\cdot}5) - \Phi(0{\cdot}5) = 0{\cdot}9332 - 0{\cdot}6915 = 0{\cdot}2417$$

In the next example we use our knowledge of the standard normal distribution to find probabilities associated with a different normal distribution.

Example 2 Given that a continuous random variable X is normally distributed with mean 20 and variance 16, find $P(15 \leqslant X \leqslant 24)$.

X has mean $\mu = 20$ and standard deviation $\sigma = 4$,
$\therefore$ the corresponding standard normal variable is

$$Z = (X - \mu)/\sigma = (X - 20)/4$$

$$P(X \leqslant 24) = P\left(Z \leqslant \frac{24 - 20}{4}\right) = P(Z \leqslant 1) = \Phi(1) = 0{\cdot}8413$$

$$P(X \leqslant 15) = P\left(Z \leqslant \frac{15 - 20}{4}\right) = P(Z \leqslant -1{\cdot}25) = 1 - \Phi(1{\cdot}25) = 0{\cdot}1056$$

$\therefore$ $P(15 \leqslant X \leqslant 24) = 0{\cdot}8413 - 0{\cdot}1056 = 0{\cdot}7357.$

We now demonstrate a different way of using values of the normal distribution function.

Example 3 Given that $\Phi(t) = 0{\cdot}81$, use Table 1 to estimate the value of t.

$\Phi(0{\cdot}87) = 0{\cdot}8078$, $\Phi(0{\cdot}88) = 0{\cdot}8106$,
$\therefore$ if $\Phi(t) = 0{\cdot}81$, then $0{\cdot}87 < t < 0{\cdot}88$.
$\Phi(t) - \Phi(0{\cdot}87) = 0{\cdot}0022$, $\Phi(0{\cdot}88) - \Phi(0{\cdot}87) = 0{\cdot}0028$,
$\therefore$ by linear interpolation, $t = 0{\cdot}87 + \frac{22}{28} \times 0.01 \approx 0.878$.

For certain values of $\Phi(z)$, the values of z can be obtained directly from Table 2, which gives *percentage points* of the normal distribution $N(0, 1)$. For instance, given that $\Phi(t) = 0{\cdot}8$, we know that 20% of the distribution lies "to the right" of the value $z = t$. Thus, using Table 2, $t = 0{\cdot}842$.

Example 4 A random variable X is normally distributed with mean μ and variance σ^2. Given that $P(X < 8) = 0{\cdot}05$ and $P(X > 15) = 0{\cdot}1$, find μ and σ^2.

Letting $Z = (X - \mu)/\sigma$ and using tables,
$0{\cdot}05 = P(Z > 1{\cdot}645) = P(Z < -1{\cdot}645)$
and $0{\cdot}1 = P(Z > 1{\cdot}282)$.

$$\therefore \quad -1{\cdot}645 = \frac{8 - \mu}{\sigma} \quad \text{i.e.} \quad 1{\cdot}645\sigma = \mu - 8 \tag{1}$$

$$1{\cdot}282 = \frac{15 - \mu}{\sigma} \quad \text{i.e.} \quad 1{\cdot}282\sigma = 15 - \mu \tag{2}$$

Adding (1) and (2): $2{\cdot}927\sigma = 7$

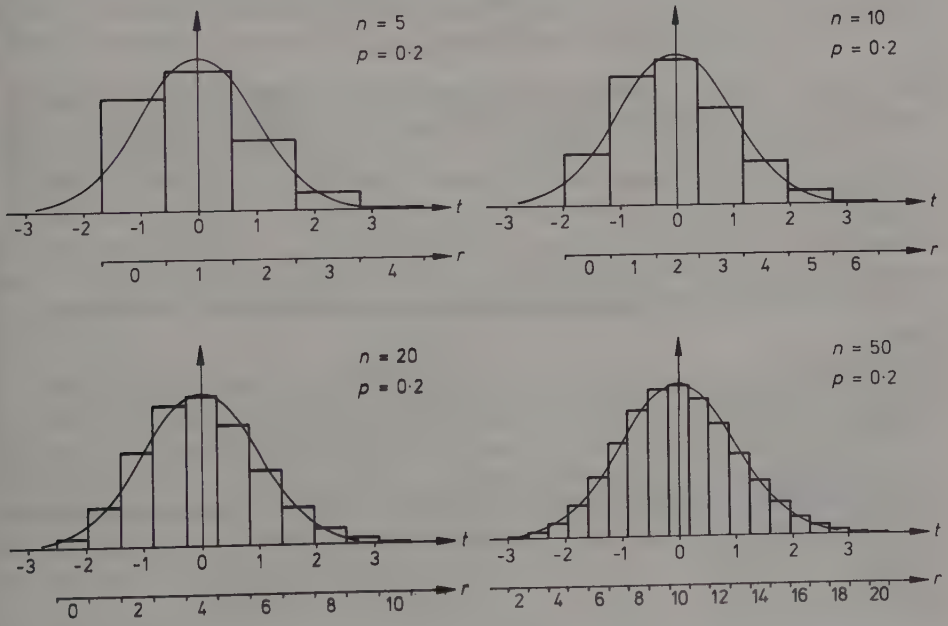
$\therefore \quad \sigma = \dfrac{7}{2{\cdot}927}$ and $\sigma^2 \approx 5{\cdot}72$

Substituting in (1): $\mu = 8 + \dfrac{1{\cdot}645 \times 7}{2{\cdot}927} \approx 11{\cdot}9$

The normal distribution arises theoretically as a limit of the binomial distribution, as described in the following theorem:

> If R is a random variable with distribution $B(n, p)$, then the distribution of the variable $T = (R - \mu)/\sigma$, where $\mu = np$ and $\sigma = \sqrt{(npq)}$, tends to $N(0, 1)$ as n tends to infinity.

The proof of this statement is rather difficult, but we can demonstrate the limiting process involved by means of an example. The given set of histograms shows the distribution of the standardised binomial variable T as the value of n increases, but p remains fixed.



Transforming back to variables with mean $\mu = np$ and variance $\sigma^2 = npq$ we draw the following important conclusion.

> If R is a random variable with distribution $B(n, p)$, then for large n, the distribution of R can be approximated by the normal distribution $N(np, npq)$, where $q = 1 - p$.

This approximation will be close enough for most practical purposes if both np and nq are greater than 5. However, more accurate results will be obtained if both np and nq are greater than 20.

Before considering a numerical example we note that when using the distribution of a continuous variable X as an approximation to the distribution of a discrete variable R, it is necessary to make a *continuity correction*. For instance, if R takes only the integral values $0, 1, 2, \ldots$, then these are assumed to correspond to the intervals $X \leqslant 0{\cdot}5$, $0{\cdot}5 \leqslant X \leqslant 1{\cdot}5$, $1{\cdot}5 \leqslant X \leqslant 2{\cdot}5, \ldots$.

Let us now suppose that R has a binomial distribution with $n = 50$, $p = 0{\cdot}2$, then R will have mean $\mu = np = 10$ and variance $\sigma^2 = npq = 8$. Thus the corresponding normal variable X also has mean 10 and variance 8. Using $N(10, 8)$ as an approximation to $B(50, 0{\cdot}2)$:

$$\begin{aligned} P(12 \leqslant R \leqslant 15) &= P(11{\cdot}5 \leqslant X \leqslant 15{\cdot}5) \\ &= P\left(\frac{11{\cdot}5 - 10}{\sqrt{8}} \leqslant \frac{X - 10}{\sqrt{8}} \leqslant \frac{15{\cdot}5 - 10}{\sqrt{8}}\right) \\ &= P(0{\cdot}5303 \leqslant Z \leqslant 1{\cdot}9445) \\ &= \Phi(1{\cdot}9445) - \Phi(0{\cdot}5303) \\ &= 0{\cdot}9741 - 0{\cdot}7020 = 0{\cdot}2721 \end{aligned}$$

$$\therefore \quad P(12 \leqslant R \leqslant 15) \approx 0{\cdot}27.$$

This turns out to be a fairly good approximation to the true value, which is 0·26 (to 2 d.p.). Even better results are obtained when the value of p is closer to 0·5.

Exercise 15.4

1. Given that X is normally distributed with mean 0 and variance 1, use Table 1 to find

(a) $P(X \leqslant 0{\cdot}3)$, (b) $P(X \geqslant 1{\cdot}2)$,
(c) $P(X > -2)$, (d) $P(X \leqslant -2{\cdot}23)$,
(e) $P(0{\cdot}8 \leqslant X \leqslant 1{\cdot}8)$, (f) $P(-1{\cdot}1 < X < 0{\cdot}5)$.

2. Given that Z is a standardised normal variable, use linear interpolation to find

(a) $P(Z \leqslant 1{\cdot}537)$, (b) $P(Z < -0.862)$,
(c) $P(Z > -1{\cdot}9035)$, (d) $P(Z \geqslant 2{\cdot}0067)$,
(e) $P(|Z| \geqslant 1{\cdot}263)$, (f) $P(|Z + 1| < 0{\cdot}616)$.

3. Given that X is normally distributed with mean 5 and standard deviation 1, find

(a) $P(X < 6{\cdot}7)$, (b) $P(X \leqslant 4{\cdot}2)$, (c) $P(3{\cdot}7 < X < 5{\cdot}7)$.

4. Given that X is normally distributed with mean 0·2 and standard deviation 0·01, find

(a) $P(X \geqslant 0{\cdot}205)$, (b) $P(X > 0{\cdot}198)$, (c) $P(0{\cdot}199 \leqslant X \leqslant 0{\cdot}201)$.

5. Given that X is normally distributed with mean 12 and variance 1·5, find
(a) $P(X \leqslant 14)$, (b) $P(X > 9{\cdot}6)$, (c) $P(9 \leqslant X \leqslant 10)$.

6. Use Table 1 to estimate the value of t given that
(a) $\Phi(t) = 0{\cdot}67$, (b) $\Phi(t) = 0{\cdot}72$, (c) $\Phi(t) = 0{\cdot}19$.

7. Use Table 2 to find the value of t such that
(a) $\Phi(t) = 0{\cdot}7$, (b) $\Phi(t) = 0{\cdot}99$, (c) $\Phi(t) = 0{\cdot}9995$.

8. The weights of a large number of cats are normally distributed with mean 1·55 kg and standard deviation 0·2 kg. Find the percentage of cats with weights (a) less than or equal to 1 kg, (b) between 1·2 and 1·3 kg, (c) between 1·5 and 1·75 kg, (d) greater than or equal to 2 kg. (L)

9. The weights of oranges are distributed normally with a mean of 96 g and standard deviation 25 g. For selling purposes, oranges are divided into three grades, I, II and III. Grade I oranges weigh more than 126 g, grade III oranges weigh less than 60 g. All remaining oranges are classed grade II. Calculate the percentage of oranges in each grade. Grade I, II and III oranges sell respectively for 12p, 8p and 4p each. Calculate the expected receipts from the sale of 80 000 oranges. (L)

10. Tests made on 2 types of electric light bulb show the following: Type A, lifetime distributed normally with an average life of 1150 hours and a standard deviation of 30 hours. Type B, long-life bulb, average lifetime of 1900 hours, with standard deviation of 50 hours. (i) What percentage of bulbs of type A could be expected to have a life of more than 1200 hours? (ii) What percentage of type B would you expect to last longer than 1800 hours? (iii) What lifetime limit would you estimate would contain the central 80% of the production of type A? (SU)

11. Batteries for a transistor radio have a mean life under normal usage of 160 hours, with a standard deviation of 30 hours. Assuming the battery life follows a normal distribution, (a) calculate the percentage of batteries which have a life between 150 hours and 180 hours; (b) calculate the range, symmetrical about the mean, within which 75% of the battery lives lie; (c) if a radio takes four of these batteries and requires all of them to be working, calculate the probability that the radio will run for at least 135 hours. (O&C)

12. Adult men in a certain country have heights which are normally distributed with a mean of 1·81 m and a standard deviation of 0.05 m.
(a) Estimate the number of men in a random sample of 100 men whose heights, when measured to the nearest 0·01 m, exceed 1·90 m.
(b) Find the probability that a man selected at random will have a height, when measured to the nearest 0·01 m, less than 1·72 m.
(c) Find the height which is exceeded by 90% of the men. (L)

13. Find the mean and variance of the normally distributed variable X in each of the following cases:
(a) $P(X > 3) = 0{\cdot}5$, $P(3 < X < 5) = 0{\cdot}1$.
(b) $P(X > 20) = P(X < 10) = 0{\cdot}05$.
(c) $P(X < 60) = 0{\cdot}01$, $P(X > 90) = 0{\cdot}3$.
(d) $P(X > 10) = 0{\cdot}1$, $P(9 < X < 10) = 0{\cdot}2$.

14. A machine is producing curtain rails whose lengths are normally distributed with mean 2·4 m. If 5% of the rails are longer than 2·42 m, find the standard deviation of the distribution. If the machine is adjusted so that 95% of the rails have lengths between 2·38 m and 2·42 m, find the new standard deviation of the distribution.

15. In an examination 30% of the candidates fail and 10% achieve distinction. Last year the pass mark (out of 200) was 84 and the minimum mark required for a distinction was 154. Assuming that the marks of the candidates were normally distributed, estimate the mean mark and the standard deviation. (O&C)

16. A random variable R has the binomial distribution $B(n, p)$. In each of the following cases find both the probability of r successes and the normal approximation to it.
(a) $n = 10, p = \frac{1}{2}, r = 6$,
(b) $n = 20, p = \frac{1}{2}, r = 12$,
(c) $n = 50, p = \frac{1}{2}, r = 30$,
(d) $n = 20, p = \frac{1}{4}, r = 4$,
(e) $n = 50, p = \frac{1}{4}, r = 10$,
(f) $n = 100, p = \frac{1}{4}, r = 20$.

17. Estimate the probability of obtaining more than 60 heads in 100 tosses of an unbiased coin.

18. In a large consignment of peaches 20% of the fruit are bruised. Estimate the probability of fewer than 10 bruised peaches in a box of 60.

19. Tutorial groups, each of 9 students, are picked at random from a large population of which two-thirds are men. (i) What is the probability that a group contains men only? (ii) Calculate the probability that the number of women in a group is either 1, 2, 3, 4 or 5. (iii) Compare this with the probability that a normal variable with the same mean and variance as the distribution appropriate to (ii) should lie between 0·5 and 5·5. (AEB 1979)

20. In a particular community the probability of a man having brown eyes is 2/5. If a random sample of 6 men is to be taken, find the probability that (a) exactly 3 men, (b) at least 3 men will have brown eyes. A second random sample of 100 men is to be taken. Use the normal distribution as an approximation to the binomial distribution, to estimate to 3 decimal places the probabilities that (a) at least half the men have brown eyes and (b) the number of men with brown eyes is between 30 and 45 inclusive. (L)

21. It is known that 30% of an apple crop has been attacked by insects. A random sample of 150 is selected from the crop. Assuming that the distribution of the

number of damaged apples in random samples of 150 may be approximated to by a normal distribution, estimate the probability that (a) more than half the sample is damaged, (b) less than 10% is damaged, (c) the number of damaged apples in the sample lies between 35 and 50, inclusive. (L)

22. In Urbania selection for the Royal Flying Corps (RFC) is by means of an aptitude test based on a week's intensive military training. It is known that the scores of potential recruits on this test follow a normal distribution with mean 45 and standard deviation 10. (i) What is the probability that a randomly chosen recruit will score between 40 and 60? (ii) What percentage of the recruits is expected to score more than 30? (iii) In a particular year 100 recruits take the test. Assuming that the pass mark is 50, calculate the probability that less than 35 recruits qualify for the RFC. (AEB 1978)

15.5 The Poisson distribution

The *Poisson*† *distribution* was originally derived as a limit of the binomial distribution. It is found to provide a useful model for distributions in which the variable is the number of occurrences of a particular event in a given interval of time or space. Some examples of variables which may have a Poisson distribution are: the number of customers entering a shop in an hour, the number of claims received by an insurance company in a day, the number of flaws in a given length of copper wire, the number of earthworms in a given area of land.

To demonstrate the way in which the Poisson distribution arises as a limit of the binomial distribution, we consider the number of telephone calls received in an hour by a small business. Assuming that the average number of calls received per hour is 4, we can construct a simple binomial model by dividing each hour into 12 five-minute "trials" and taking the probability of a call in any five-minute interval to be 1/3. However, since the firm may receive more than one call in a five-minute period, this model is not entirely satisfactory. It would be better to regard an hour as 60 one-minute trials, taking the probability of a call in any one-minute interval to be 1/15. Remembering the speed with which a modern switchboard can process calls, we should perhaps reduce the time interval to 10 seconds or even to 2 seconds. The given table shows that we are producing a sequence of binomial distributions with increasing numbers of trials but constant mean.

Interval	n	p	Mean	Variance
5 minutes	12	$\frac{1}{3}$	4	$4 \times \frac{2}{3} \approx 2{\cdot}67$
1 minute	60	$\frac{1}{15}$	4	$4 \times \frac{14}{15} \approx 3{\cdot}73$
10 seconds	360	$\frac{1}{90}$	4	$4 \times \frac{89}{90} \approx 3{\cdot}96$
2 seconds	1800	$\frac{1}{450}$	4	$4 \times \frac{449}{450} \approx 3{\cdot}99$

The limit of this sequence is the Poisson distribution with mean and variance 4.

†*Poisson, Siméon Denis* (1781–1840) French mathematician. His main contributions were to the theory of electricity and magnetism, but in 1837 he published *Recherches sur la probabilité des jugements*, dealing with probability distributions.

We obtain the general form of the Poisson distribution by considering the limit of the binomial distribution $B(n, p)$ as n tends to infinity in such a way that the mean $\mu = np$ remains constant.

$$P(X = r) = \binom{n}{r} p^r (1-p)^{n-r}$$

$$= \frac{n(n-1)\dots(n-r+1)}{r!} \left(\frac{\mu}{n}\right)^r \left(1 - \frac{\mu}{n}\right)^{n-r}$$

$$= \frac{n}{n}\left(\frac{n-1}{n}\right)\dots\left(\frac{n-r+1}{n}\right)\frac{\mu^r}{r!}\left(1-\frac{\mu}{n}\right)^n \left(1-\frac{\mu}{n}\right)^{-r}$$

$$= \left(1-\frac{1}{n}\right)\dots\left(1-\frac{r-1}{n}\right)\frac{\mu^r}{r!}\left(1-\frac{\mu}{n}\right)^n \Big/ \left(1-\frac{\mu}{n}\right)^r$$

$$\therefore \quad \lim_{n\to\infty} P(X = r) = \frac{\mu^r}{r!} \lim_{n\to\infty} \left(1 - \frac{\mu}{n}\right)^n$$

By the method of question 13, Exercise 9.4, it can be shown that

$$\lim_{n\to\infty} \left(1 - \frac{\mu}{n}\right)^n = e^{-\mu}$$

Thus, as $n \to \infty$, $P(X = r) \to \dfrac{\mu^r}{r!} e^{-\mu}$.

This result leads to the following definition:

A discrete random variable X is said to have a Poisson distribution with parameter μ, where $\mu > 0$, if

$$P(X = r) = \frac{\mu^r}{r!} e^{-\mu} \quad \text{for} \quad r = 0, 1, 2, \dots$$

Considering again the limit of the distribution $B(n, p)$ as $n \to \infty$, we see that if the mean $\mu = np$ remains constant, then $p \to 0$ and the variance $\sigma^2 = np(1-p)$ tends to μ. This suggests that the Poisson distribution has mean μ and variance μ. These results can proved either directly or by using the appropriate probability generating function. For instance:

$$E(X) = \sum_{r=0}^{\infty} r\left(\frac{\mu^r}{r!} e^{-\mu}\right) = \mu e^{-\mu} \sum_{r=1}^{\infty} \frac{\mu^{r-1}}{(r-1)!}$$

$$= \mu e^{-\mu}\left(1 + \mu + \frac{\mu^2}{2!} + \frac{\mu^3}{3!} + \dots\right) = \mu e^{-\mu} \times e^{\mu} = \mu.$$

[For other proofs see Exercise 15.5 questions 12 and 13.]

It follows from this discussion that the Poisson distribution is a suitable model for the distribution of the number of events occurring in a specified interval of continuous time or space, provided that

(a) the events occur at random and independently, but may not occur simultaneously;
(b) the expected number of events in any interval is proportional to the size of the interval.

Example 1 A small business receives an average of 4 telephone calls per hour. Use the appropriate Poisson distribution to find the probability that (a) 7 calls are received in a two hour period, (b) up to 3 calls are received in an interval of 30 minutes.

(a) The expected number of calls in 2 hours is 8.
Thus if X is the number of calls received in 2 hours, X has a Poisson distribution with parameter 8

$$\therefore \quad P(X = 7) = \frac{8^7}{7!}e^{-8} \approx 0{\cdot}140.$$

Hence the probability that 7 calls are received in a two hour period is 0.140.

(b) The expected number of calls in 30 minutes is 2.
Thus if Y is the number of calls received in 30 minutes, Y has a Poisson distribution with parameter 2

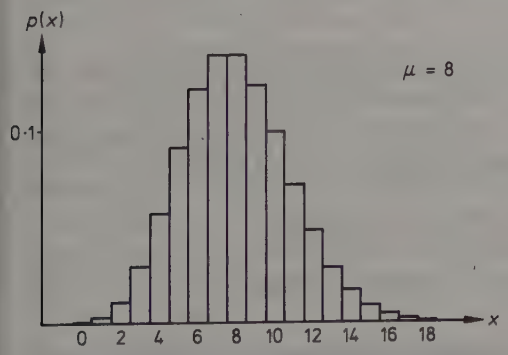
$$\therefore \quad P(Y \leqslant 3) = P(Y = 0) + P(Y = 1) + P(Y = 2) + P(Y = 3)$$

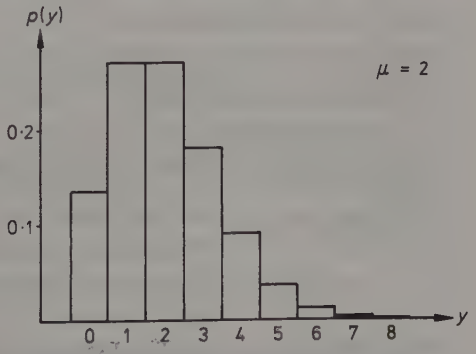
$$= e^{-2} + 2e^{-2} + \frac{2^2}{2!}e^{-2} + \frac{2^3}{3!}e^{-2}$$

$$= (1 + 2 + 2 + \tfrac{4}{3})e^{-2} \approx 0{\cdot}857.$$

Hence the probability that up to 3 calls are received in 30 minutes is 0.857.

The following diagrams show the Poisson distributions used in Example 1.





We have seen that the Poisson distribution with parameter $\mu = np$ is a limit of the binomial distribution $B(n, p)$. This means that under certain conditions a Poisson distribution can be used as an approximation to a binomial distribution with the same mean. In practice a Poisson approximation gives fairly accurate results when $n > 50$, $p < 0{\cdot}1$ and $np \leqslant 5$. It may also produce acceptable approximations to binomial distributions which do not satisfy these conditions. However, for a

binomial distribution such that both np and $n(1-p)$ are greater than 5, a normal approximation is usually more appropriate.

It can be shown that as μ tends to infinity, the distribution of the standardised Poisson variable $(X-\mu)/\sqrt{\mu}$ approaches the standard normal distribution $N(0, 1)$. It follows that for large μ, the normal distribution $N(\mu, \mu)$ can be used as an approximation to the Poisson distribution with parameter μ. Fairly good approximations are obtained for $\mu > 5$, but for accurate results it is necessary to have $\mu > 20$.

Exercise 15.5

1. In a school laboratory an average of 8 test-tubes are broken in 20 working days. Find the probability that (a) 5 test-tubes will be broken in the next 10 days; (b) fewer than 2 test-tubes will be broken in the next 3 days.

2. The number of heavy vehicles driving past a certain historic building in an hour is assumed to have a Poisson distribution with mean 3. Find the probability that (a) in the next hour more than 3 heavy vehicles will drive by; (b) in the next two hours fewer than 5 heavy vehicles will drive by.

3. A textile manufacturer estimates that there is an average of 2 faults in every 100 metres of fabric he produces. Find the probability that there is at least 1 fault in a 10-metre length of fabric. Write down an expression for the probability that there is no fault in an n-metre length of fabric. What is the greatest integer n such that this probability is greater than 0·9?

4. The number of plants of a certain species found in one square metre of woodland is assumed to have a Poisson distribution with mean 0·003. Find the probability of observing at least 2 specimens in an area of 400 m^2. A botanist is more than 99% certain that he will find at least one specimen in an area of n m^2. Find the least possible value of the integer n.

In questions 5 to 8 draw up a table showing the values, correct to 3 d.p., of the probabilities $p(0), p(1), p(2), \ldots$ for (a) the binomial distribution $B(n, p)$; (b) the Poisson distribution with parameter $\mu = np$; (c) the normal approximation to (a); (d) the normal approximation to (b). Comment briefly on your results.

5. $n = 20, p = 0{\cdot}5$.

6. $n = 20, p = 0{\cdot}1$.

7. $n = 50, p = 0{\cdot}1$.

8. $n = 100, p = 0{\cdot}08$.

9. It is found that on average 0·5% of the watches produced by a certain factory are faulty. Estimate the probability that (a) in a batch of 200 watches there are no faulty watches; (b) in a batch of 500 watches more than 2 are faulty.

10. On average 1 out of 50 patients treated with a certain drug suffer side effects. Estimate the probability that (a) if 80 patients take the drug then exactly 2 will have side effects, (b) if 200 patients take the drug then at least 4 will have side effects.

11. In a certain experiment the probability of success in any trial is 0·03. Use appropriate approximations to find the probabilities of (a) at least 2 successes in 50 trials, (b) at least 4 successes in 100 trials, (c) at least 20 successes in 500 trials.

12. Use the identity $r^2 \equiv r(r-1) + r$ to show that

$$\sum_{r=0}^{\infty} \frac{r^2 x^r}{r!} = x^2 e^x + x e^x.$$

Hence show that the Poisson distribution with parameter μ has variance μ.

13. Show that the probability generating function for the Poisson distribution with parameter μ is given by $G(t) = e^{\mu(t-1)}$. Hence show that the distribution has mean μ and variance μ.

14. The number of road accidents at a certain traffic roundabout has been found to follow a Poisson distribution with a mean of 0·8 accidents per week. Calculate, correct to 2 significant figures, the probabilities that (i) there will be at least 2 accidents in a particular week, (ii) there will be exactly 3 accidents in a particular three-week period. After road works have been carried out, the traffic is observed for a period of 8 weeks and in that time only 1 accident occurs. Calculate the probability of at most 1 accident in 8 weeks if the mean is still 0·8 accidents per week. Comment briefly on this result. (C)

15. (a) I recorded the number of phone calls I received over a period of 150 days:

No. of calls	0	1	2	3	4
No. of days	51	54	36	6	3

(i) Find the average number of calls per day. (ii) Calculate the frequencies of the comparable Poisson distribution.

(b) A firm selling electrical components packs them in boxes of 60. On average 2% of the components are faulty. What is the chance of getting more than 2 defective components in a box? (Use the Poisson distribution.) (SU)

16. An aircraft has 116 seats. The airline has found, from long experience, that on average 2·5% of people with tickets for a particular flight do not arrive for that flight. If the airline sells 120 seats for a particular flight, determine, using a suitable approximation, the probability that more than 116 people arrive for that flight. Determine also the probability that there are empty seats on the flight. (C)

17. A number of different types of fungi are distributed at random in a field. Eighty per cent of these fungi are mushrooms, and the remainder are toadstools. Five per cent of the toadstools are poisonous. A man who cannot distinguish between mushrooms and toadstools, wanders across the field and picks a total of 100 fungi. Determine, correct to 2 significant figures, using approximations, the probability that the man has picked (i) at least 20 toadstools, (ii) exactly two poisonous toadstools. (C)

18. A new telephone directory is to be published, and before publication takes place the entries have to be checked for misprints and any necessary corrections made. Previous experience suggests that, on average, 0·1% of the entries will require correction, and that entries requiring correction will be randomly distributed. The directory will contain 240 000 entries altogether, printed on 800 pages with 300 entries per page. There are two methods of making corrections: for method A the costs are £C per page containing one entry requiring correction and £3C per page containing two or more entries requiring correction; for method B the costs are £2C per page containing one or more entries requiring correction. Calculate the expected costs of correction for each method. (C)

19. In a certain office the number of reams of typing paper used per month has a Poisson distribution with mean 25. Estimate the probability that more than 25 reams of paper will be used in the next month. Given that it is more than 95% certain that there is enough paper in the stationery cupboard to last a month, find the minimum number of reams of paper in the cupboard.

20. Show that the variance of a Poisson distribution is equal to the mean. The number of bacteria in 1 ml of inoculum has a Poisson distribution with mean 2·0. If at least 3 bacteria are needed for a dose to be infective, find the probability that a dose of 1 ml will cause infection. Find approximate limits, symmetrical about the mean, between which lies 95% of the distribution of the number of bacteria in 100 ml of inoculum. (O&C)

15.6 The algebra of expectation and variance

The expectation $E(X)$ and the variance $\mathrm{Var}(X)$ of a random variable X were defined earlier in the chapter.

$$E(X) = \mu = \begin{cases} \sum x_r p(x_r), & \text{for discrete } X \\ \displaystyle\int_{-\infty}^{\infty} x f(x)\,dx, & \text{for continuous } X \end{cases}$$

$$\mathrm{Var}(X) = \sigma^2 = \begin{cases} \sum (x_r - \mu)^2 p(x_r), & \text{for discrete } X \\ \displaystyle\int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx, & \text{for continuous } X \end{cases}$$

If a new random variable $g(X)$ is defined in terms of X by a formula such as $g(X) = 2X + 3$ or $g(X) = X^2$, then the expectation of $g(X)$ is given by:

$$E[g(X)] = \begin{cases} \sum g(x_r) p(x_r), & \text{for discrete } X \\ \displaystyle\int_{-\infty}^{\infty} g(x) f(x)\,dx & \text{for continuous } X \end{cases}$$

If X is a discrete variable which takes the values $x_1, x_2, \ldots$ then we can show that this formula for $E[g(X)]$ is consistent with the definition of $E(X)$ by letting $Y = g(X)$.
(i) If Y takes distinct values $y_1 = g(x_1)$, $y_2 = g(x_2), \ldots$, then

$$P(Y = y_r) = P(g(X) = g(x_r)) = P(X = x_r) = p(x_r),$$

$$\therefore \quad E[g(X)] = E(Y) = \sum_{\text{all } r} y_r P(Y = y_r) = \sum_{\text{all } r} g(x_r)p(x_r).$$

(ii) If the values $g(x_1), g(x_2), \ldots$ are not all distinct then we have relationships such as $y_k = g(x_i) = g(x_j) = \ldots$. In these cases

$$P(Y = y_k) = P(X = x_i) + P(X = x_j) + \ldots = p(x_i) + p(x_j) + \ldots$$

$$\therefore \quad y_k P(Y = y_k) = y_k p(x_i) + y_k p(x_j) + \ldots$$
$$= g(x_i)p(x_i) + g(x_j)p(x_j) + \ldots$$

$$\therefore \quad E[g(X)] = E(Y) = \sum_{\text{all } k} y_k P(Y = y_k) = \sum_{\text{all } r} g(x_r)p(x_r).$$

Thus the formula for $E[g(X)]$ is valid for discrete X. We will assume that it is also valid for continuous X.

Example 1 A discrete random variable X takes the values 0, 1, 2, 3 and $p(0) = 0{\cdot}1$, $p(1) = p(3) = 0{\cdot}2$, $p(2) = 0{\cdot}5$. Find $E(X^2 - X)$.

$$E(X^2 - X) = \sum (x^2 - x)p(x)$$
$$= (0^2 - 0)0{\cdot}1 + (1^2 - 1)0{\cdot}2 + (2^2 - 2)0{\cdot}5 + (3^2 - 3)0{\cdot}2$$
$$= 0 + 0 + 2 \times 0{\cdot}5 + 6 \times 0{\cdot}2 = 2{\cdot}2.$$

Example 2 Find $E(X^3)$ for the continuous random variable X which has probability density function $f(x) = 2(1 - x)$ for $0 \leqslant x \leqslant 1$ and $f(x) = 0$ otherwise.

$$E(X^3) = \int_0^1 x^3 f(x)\,dx = \int_0^1 x^3 . 2(1 - x)\,dx$$

$$= 2\int_0^1 (x^3 - x^4)\,dx = 2\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1 = 2\left(\frac{1}{4} - \frac{1}{5}\right) = 0{\cdot}1.$$

Two useful properties of expectation follow from the definition of $E[g(X)]$.

> If X is a random variable, discrete or continuous,
> (i) $E(aX + b) = aE(X) + b$, for constants a and b.
> (ii) $E[g(X) + h(X)] = E[g(X)] + E[h(X)]$.

We give here the proof of (i) for discrete X. The remaining proofs are left as an exercise.

$$E(aX + b) = \sum (ax_r + b)p(x_r)$$
$$= \sum \{ax_r p(x_r) + bp(x_r)\} = a\sum x_r p(x_r) + b\sum p(x_r)$$

But $\quad \sum x_r p(x_r) = E(X) \quad$ and $\quad \sum p(x_r) = 1$

Hence $\quad E(aX + b) = aE(X) + b.$

It is now possible to express the variance of a random variable in terms of expected values:

$$\text{Var}(X) = E[(X - \mu)^2], \quad \text{where} \quad \mu = E(X)$$

Thus, using results (i) and (ii):

$$\text{Var}(X) = E(X^2 - 2\mu X + \mu^2) = E(X^2) - 2\mu E(X) + \mu^2 = E(X^2) - \mu^2$$

i.e.

$$\boxed{\text{Var}(X) = E(X^2) - [E(X)]^2.}$$

Hence if a and b are constants,

$$\begin{aligned}\text{Var}(aX + b) &= E[(aX + b)^2] - [E(aX + b)]^2 \\ &= E(a^2X^2 + 2abX + b^2) - [aE(X) + b]^2 \\ &= a^2E(X^2) + 2abE(X) + b^2 - a^2[E(X)]^2 - 2abE(X) - b^2 \\ &= a^2E(X^2) - a^2[E(X)]^2 = a^2\{E(X^2) - [E(X)]^2\}\end{aligned}$$

i.e.

$$\boxed{\text{Var}(aX + b) = a^2\,\text{Var}(X)}$$

To develop the theory of expectation further we must now consider the distribution of two discrete random variables X and Y with probability functions $p_X(x)$ and $p_Y(y)$ respectively. Let X take the values $x_1, x_2, \ldots, x_i, \ldots$ and let Y take the values $y_1, y_2, \ldots, y_j, \ldots$

The *joint probability distribution* of X and Y is the set of all probabilities $p(x_i, y_j)$, where $p(x_i, y_j) = P(X = x_i, Y = y_j)$.

For example if the possible values of X are 0, 1, 2 and the values of Y are 0, 1, 2, 3, then the joint distribution will be a set of 12 probabilities, which can be displayed in a two-way table as follows:

X \ Y	0	1	2	3	Total
0	$p(0,0)$	$p(0,1)$	$p(0,2)$	$p(0,3)$	$p_X(0)$
1	$p(1,0)$	$p(1,1)$	$p(1,2)$	$p(1,3)$	$p_X(1)$
2	$p(2,0)$	$p(2,1)$	$p(2,2)$	$p(2,3)$	$p_X(2)$
Total	$p_Y(0)$	$p_Y(1)$	$p_Y(2)$	$p_Y(3)$	1

As indicated in the table, the row totals give the values of $p_X(x)$. For instance

$$p(0,0) + p(0,1) + p(0,2) + p(0,3) = p_X(0).$$

Similarly the column totals are the values of $p_Y(y)$. Since these probabilities appear

in the margins, they are called the *marginal distributions* of X and Y. Using this table we could obtain the probability distribution of a random variable such as $X + Y$. We see that $P(X + Y = 0) = p(0, 0)$, $P(X + Y = 1) = p(1, 0) + p(0, 1)$, and so on. However, it is possible to obtain general formulae for the expected values of $X + Y$ and other functions of X and Y.

Taking into account all ordered pairs (x_i, y_j):

$$E(X + Y) = \sum_i \sum_j (x_i + y_j)p(x_i, y_j)$$

$$= \sum_i \left[\sum_j x_i p(x_i, y_j) \right] + \sum_j \left[\sum_i y_j p(x_i, y_j) \right]$$

$$= \sum_i \left[x_i \sum_j p(x_i, y_j) \right] + \sum_j \left[y_j \sum_i p(x_i, y_j) \right]$$

$$= \sum_i x_i p_X(x_i) + \sum_j y_j p_Y(y_j) = E(X) + E(Y).$$

Similar arguments can be used to extend this result to any *linear combination* of random variables.

If X and Y are random variables and a and b are constants, then

$$E(aX + bY) = aE(X) + bE(Y).$$

If $X_1, X_2, \ldots, X_n$ are random variables, then

$$E(X_1 + X_2 + \ldots + X_n) = E(X_1) + E(X_2) + \ldots + E(X_n).$$

[These results are valid for both discrete and continuous variables, but in the continuous case the proofs require integration methods not covered in this course.]

If random variables X and Y are *independent*, then

$$P(X = x_i, Y = y_j) = P(X = x_i) \times P(Y = y_j)$$

i.e.
$$p(x_i, y_j) = p_X(x_i)p_Y(y_j)$$

$$E(XY) = \sum_i \sum_j x_i y_j p(x_i, y_j)$$

$$= \sum_i \left[\sum_j x_i y_j p_X(x_i) p_Y(y_j) \right]$$

$$= \sum_i \left[x_i p_X(x_i) \sum_j y_j p_Y(y_j) \right]$$

$$= \left[\sum_i x_i p_X(x_i) \right] \left[\sum_j y_j p_Y(y_j) \right] = E(X)E(Y).$$

$\therefore$ If X and Y are independent random variables, then

$$E(XY) = E(X)E(Y).$$

We are now ready to consider the variance of a linear combination of random variables. In particular:

$$\begin{aligned}\mathrm{Var}(X+Y) &= E[(X+Y)^2] - [E(X+Y)]^2 \\ &= E(X^2 + 2XY + Y^2) - [E(X) + E(Y)]^2 \\ &= E(X^2) + 2E(XY) + E(Y^2) - [E(X)]^2 - 2E(X)E(Y) - [E(Y)]^2 \\ &= \{E(X^2) - [E(X)]^2\} + 2\{E(XY) - E(X)E(Y)\} \\ &\quad + \{E(Y^2) - [E(Y)]^2\}\end{aligned}$$

The quantity $E(XY) - E(X)E(Y)$ is called the *covariance* of X and Y, and is denoted by $\mathrm{Cov}(X, Y)$. [We will see in Chapter 17 that it is a measure of the relationship between the distributions of X and Y.]
Thus, in general:

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + 2\,\mathrm{Cov}(X, Y) + \mathrm{Var}(Y).$$

However, if the variables X and Y are independent,

$$\mathrm{Cov}(X, Y) = E(XY) - E(X)E(Y) = 0$$

$$\therefore \quad \mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y).$$

As before, this basic result can be extended:

If X and Y are independent random variables and a and b are constants, then

$$\mathrm{Var}(aX + bY) = a^2\,\mathrm{Var}(X) + b^2\,\mathrm{Var}(Y).$$

Two important special cases are given by:

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X-Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$$

If $X_1, X_2, \ldots, X_n$ are independent random variables,

$$\mathrm{Var}(X_1 + X_2 + \ldots + X_n) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \ldots + \mathrm{Var}(X_n).$$

Example 3 In the ice-cream dessert served in a motorway cafeteria, the masses of the portions of ice-cream are distributed about a mean of 80 g with standard deviation 6 g. The fruit topping has mean mass 20 g and standard deviation 4 g. If the cost of 100 g of ice-cream is 5 p and the cost of 100 g of topping is 10 p, find the mean and standard deviation of the cost of a dessert.

Let the random variables X and Y represent the masses in grammes of portions of ice-cream and topping respectively. If Z is the total cost in pence of the dessert, then

$$Z = \tfrac{5}{100}X + \tfrac{10}{100}Y = \tfrac{1}{20}X + \tfrac{1}{10}Y$$

$$\therefore \quad E(Z) = \tfrac{1}{20}E(X) + \tfrac{1}{10}E(Y) = \tfrac{80}{20} + \tfrac{20}{10} = 6,$$

$$\text{Var}(Z) = (\tfrac{1}{20})^2 \text{Var}(X) + (\tfrac{1}{10})^2 \text{Var}(Y)$$
$$= \frac{6^2}{20^2} + \frac{4^2}{10^2} = \frac{9+16}{100} = \frac{1}{4}$$

Hence the cost of a dessert has mean 6 p and standard deviation $\frac{1}{2}$ p.

We now state without proof two important theorems concerning linear combinations of normal random variables.

If X and Y are independent normal random variables with distributions $N(\mu_x, \sigma_x^2)$ and $N(\mu_y, \sigma_y^2)$ respectively, then $Z = aX + bY$, where a and b are constants, is also a normal random variable and has mean $a\mu_x + b\mu_y$, variance $a^2\sigma_x^2 + b^2\sigma_y^2$.

Sums and differences of independent normal variables are of particular interest:

$Z = X + Y$ has the distribution $N(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)$,
$Z = X - Y$ has the distribution $N(\mu_x - \mu_y, \sigma_x^2 + \sigma_y^2)$.

If $X_1, X_2, \ldots, X_n$ are independent observations of a random variable with distribution $N(\mu, \sigma^2)$, then $Z = X_1 + X_2 + \ldots + X_n$ is normally distributed with mean $n\mu$, variance $n\sigma^2$.

Example 4 Brand X beefburgers have masses normally distributed about a mean of 65 g with variance 15 g^2. The masses of Brand Y beefburgers are normally distributed with mean 100 g, variance 25 g^2. Find the probability that the mass of 3 Brand X beefburgers is greater than the mass of 2 Brand Y beefburgers.

[It is tempting to think that what we need to find here is $P(3X - 2Y > 0)$. However, in the expression $3X - 2Y$, $3X$ represents 3 times the mass of one beefburger, not the mass of three beefburgers.]

Let $D = (X_1 + X_2 + X_3) - (Y_1 + Y_2)$, where X_1, X_2, X_3 represent the masses (in grammes) of 3 Brand X beefburgers and Y_1, Y_2 represent the masses of 2 Brand Y beefburgers, then

$$E(D) = E(X_1) + E(X_2) + E(X_3) - E(Y_1) - E(Y_2)$$
$$= 3 \times 65 - 2 \times 100 = -5$$
$$\text{Var}(D) = \text{Var}(X_1) + \text{Var}(X_2) + \text{Var}(X_3) + \text{Var}(Y_1) + \text{Var}(Y_2)$$
$$= 3 \times 15 + 2 \times 25 = 95$$

Letting Z be the standardised normal variable $(D + 5)/\sqrt{95}$,

$$P(D > 0) = P(Z > 5/\sqrt{95})$$
$$= P(Z > 0{\cdot}5130)$$
$$= 1 - \Phi(0{\cdot}5130) = 1 - 0{\cdot}6961 = 0{\cdot}3039$$

$\therefore$ the required probability is approximately 0·304.

Two further results are sometimes useful.

If independent random variables X and Y have distributions $B(m, p)$ and $B(n, p)$, then $X + Y$ has distribution $B(m + n, p)$.

If independent random variables X and Y have Poisson distributions with means λ and μ, then $X + Y$ has a Poisson distribution with mean $\lambda + \mu$.

Methods of proof are indicated in Exercise 15.6 questions 23 and 25.

Exercise 15.6

1. A discrete random variable X takes the values 1, 2, 3, 4, 5 with probabilities $p(1) = p(2) = 0{\cdot}1, p(3) = 0{\cdot}15, p(4) = 0{\cdot}4, p(5) = 0{\cdot}15$. Find $E(X)$, $E(X^2)$ and $E(1/X)$. Hence find $E(2X^2 - 1)$ and $E[(X + 1)^2/X]$.

2. A discrete random variable X takes the values $-1, 0, 1, 2$ with probabilities $p(-1) = 0{\cdot}2, p(0) = 0{\cdot}3, p(1) = 0{\cdot}4, p(2) = 0{\cdot}1$. Find $E(X)$, $E(X^2)$ and $E(X^4)$. Hence find $\mathrm{Var}(X)$ and $\mathrm{Var}(X^2)$.

3. A continuous random variable X has probability density function

$$f(x) = \tfrac{1}{2}(3 - x) \text{ for } 1 \leqslant x \leqslant 3 \text{ and } f(x) = 0 \text{ otherwise.}$$

Find $E(X)$, $E(X^2)$ and $E(X^3)$. Hence find $E(3X^3 + 2X)$ and $\mathrm{Var}(X)$.

4. A continuous random variable X has probability density function

$$f(x) = 6x(1 - x) \text{ for } 0 \leqslant x \leqslant 1 \text{ and } f(x) = 0 \text{ otherwise.}$$

Find $E(X)$, $E(X^2)$, and $E(\sqrt{X})$. Hence find $E[X(X - 1)]$ and $\mathrm{Var}(\sqrt{X})$.

5. A random variable X takes the values 5, 8, 4, 7 with equal probability and an independent variable Y takes the values 4, 7, 1 with equal probability. Find the mean and variance of the following:

(a) X (b) Y, (c) $4X + 3$, (d) $5 - 2Y$,
(e) $X + Y$, (f) $X - Y$, (g) $3X - Y$, (h) $2X + 3Y$.

6. Independent random variables X and Y are such that $E(X) = 10$, $\mathrm{Var}(X) = 9$ and $E(Y) = 7$, $\mathrm{Var}(Y) = 1$. Find (a) the mean and variance of $5X - 3Y + 2$, (b) the values of $E(X^2)$, $E(Y^2)$ and $E(X^2 + Y^2)$.

7. A factory produces nuts with masses distributed about a mean of 5 g with a standard deviation of 0.2 g. These are screwed on to bolts whose masses have mean

20 g and standard deviation 0·1 g. If each bolt is fitted with two nuts, find the mean and standard deviation of the resulting total mass.

8. In a can containing baked beans and 5 small sausages, the mass of each sausage is distributed about a mean of 20 g with standard deviation 2 g. The mean mass of the baked beans is 100 g with standard deviation 5 g. Find the mean mass of the total contents of a can and the standard deviation.

9.

X \ Y	0	1	2
1	0·06	0·12	0·02
2	0·15	0·3	0·05
3	0·09	0·18	0·03

The given table shows the joint probability distribution of two discrete random variables X and Y.

(a) Use the marginal distributions of X and Y to find $E(X)$, $\text{Var}(X)$ and $E(Y)$, $\text{Var}(Y)$.

(b) Are there any values of x and y for which

$$P(X = x, Y = y) \neq P(X = x) \times P(Y = y)?$$

Use your answer to determine whether or not X and Y are independent.

(c) Find the distribution of XY i.e. $P(XY = 0)$, $P(XY = 1)$, $P(XY = 2)$,... Hence determine whether or not $E(XY) = E(X)E(Y)$.

(d) Find the distribution of $X + Y$. Hence determine whether or not

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y).$$

10. Repeat question 9 for the following joint distributions.

(i)

X \ Y	0	1	2	3
0	0·03	0·1	0.05	0·02
1	0·05	0	0·25	0·1
2	0·02	0·2	0·1	0·08

(ii)

X \ Y	1	2	3
0	0·05	0·2	0·05
1	0·2	0	0·2
2	0·05	0·2	0·05

11. A random variable X takes the values 0 and 2 each with probability 0·5. Find $E(X)$, $E(X^2)$, $E(X^3)$, $E(X^4)$. Use the formula $\text{Var}(Y) = E(Y^2) - [E(Y)]^2$ to verify that (a) if $Y = 3X - 5$, $\text{Var}(Y) = 9\,\text{Var}(X)$,

(b) if $Y = X^2 + X$, $\text{Var}(Y) \neq \text{Var}(X^2) + \text{Var}(X)$.

12. The daily demand, x kilogrammes, for fresh fish at a certain fishmonger's shop is a continuous random variable with probability density function f, where f is given by $f(x) = (30 - x)/450$, $0 \leqslant x \leqslant 30$, $f(x) = 0$, otherwise. The fishmonger receives a delivery of 20 kg of fresh fish each morning. He makes a profit of 40 pence on each kilogramme of fish sold on the day the fish is delivered; fish left unsold on the day it is

delivered is sold to a fertiliser manufacturer at a loss of 5 pence per kg. Show that on a day when x is less than 20, the fishmonger's profit is $(45x - 100)$ pence and that on a day when x is 20 or more, the fishmonger's profit is £8. Hence determine the fishmonger's expected daily profit from the sale of fresh fish. (JMB)

13 A sample of four mice is to be chosen at random from a litter of ten mice, of which six are male and four are female. Let X denote the number of male mice in the sample. Calculate the values of $P(X = r)$ for $r = 0, 1, 2, 3, 4$. Hence find the mean and the standard deviation of X. Deduce the mean and the standard deviation of the number of female mice in the sample. Independently of the above, a sample of four mice is to be chosen at random from another litter of ten mice, of which four are male and six are female. Calculate the mean and the standard deviation of the total number of male mice in the combined sample of eight mice. (W)

14. The joint probability distribution of two discrete random variables X and Y is displayed in the following table, the entries in the body of the table being $P(X = x, Y = y)$.

		x		
		0	1	2
y	0	0·1	0·1	0
	1	0·1	0·2	0·1
	2	0·1	0·2	0·1

(i) Find $E(X)$ and $E(Y)$. (ii) Determine the distribution of $Z = 2X - Y$ and verify that $E(Z) = 2E(X) - E(Y)$. (iii)Let (X_1, Y_1) and (X_2, Y_2) denote two independent observations of (X, Y). By using the distribution of Z, or otherwise, find the probability that

$$2(X_1 - X_2) = Y_1 - Y_2.$$ (W)

15. On a piece-work operation a company pays a bonus to any employee who processes in excess of 300 kg of raw material in a day. The daily amounts processed by two employees A and B are independent and normally distributed, the mean and standard deviation of the amounts processed by A being 291 kg and 9 kg, respectively, and of the amounts processed by B being 282 kg and 12 kg, respectively. Calculate, to two significant figures in each case, (i) the proportion of days on which both A and B will earn the bonus, (ii) the probability that the combined daily amount processed by A and B will exceed 600 kg, (iii) the proportion of days on which the amount processed by B will exceed that processed by A.

(W)

16. The diameters of axles supplied by a factory have a mean value of 19·92 mm and a standard deviation of 0·05 mm. The inside diameters of bearings supplied by another factory have a mean of 20·04 mm and a standard deviation of 0·03 mm. What is the mean and standard deviation of the random variable defined to be the diameter of a bearing less the diameter of an axle? Assuming that both dimensions are normally distributed, what percentage of axles and bearings taken at random will not fit? (O&C)

17. The time of arrival of a bus at a bus stop varies in a normal distribution with a mean of 09.00 a.m. and a standard deviation of 2 minutes. Independently a second bus departs from this stop at a time which varies in a normal distribution with a mean of 09.01 a.m. (i.e. one minute past 09.00 a.m.) and a standard deviation of 1 minute. Find the probability that (a) the first bus arrives before the second bus leaves; (b) this happens on 5 given consecutive days. (O&C)

18. The weight of a large loaf of bread is a normal variable with mean 420 g and standard deviation 30 g. The weight of a small loaf of bread is a normal variable with mean 220 g and standard deviation 10 g. (i) Find the probability that 5 large loaves weigh more than 10 small loaves. (ii) Find the probability that the total weight of 5 large loaves and 10 small loaves lies between 4·25 kg and 4·4 kg. (C)

19. The mass of a cheese biscuit has a normal distribution with mean 6 g and standard deviation 0·2 g. Determine the probability that (i) a collection of twenty-five cheese biscuits has a mass of more than 149 g, (ii) a collection of thirty cheese biscuits has a mass of less than 180 g, (iii) twenty-five times the mass of a cheese biscuit is less than 149 g. The mass of a ginger biscuit has a normal distribution with mean 10 g and standard deviation 0·3 g. Determine the probability that a collection of seven cheese biscuits has a mass greater than a collection of four ginger biscuits. (C)

20. Four sprinters A, B, C and D each run 100 m. The times (in seconds) that they take may be regarded as independent observations from a normal distribution with mean 14 and standard deviation 0·2. The athlete E runs 400 m. The time (in seconds) that this athlete takes may be regarded as an observation from a normal distribution with mean 58 and standard deviation 1·0 and is independent of the times of the sprinters. Determine the probability that (i) the time taken by E is less than 3 seconds greater than the sum of the times taken by the four sprinters, (ii) the time taken by E is less than 4 times as great as the time taken by A. For the four sprinters A, B, C and D, find the probability that A is the fastest sprinter of the four. Find also the probability that C and D (in either order) are the two slowest sprinters. (C)

21. P is a fixed point on the circumference of a circle, centre O, of radius r and Q is a point on the circumference such that $\angle POQ = \theta$, where θ is a random variable with a rectangular distribution in $[0, 2\pi]$. Find the mean and median values for the length of the shorter arc PQ. The length of the chord PQ is X. Find $E(X)$ and show that $\mathrm{Var}(X) = 2r^2(1 - 8/\pi^2)$. Find also $P(X > r\sqrt{3})$. (C)

22. X and Y are independent random variables with means and variances μ_x, σ_x^2 and μ_y, σ_y^2 respectively, and you may assume that for any such independent variables $E(XY) = E(X)E(Y)$. Find expressions for (i) the expectation of the area of a square of side X, (ii) the expectation and variance of the area of a rectangle of sides X and Y, (iii) the expectation of the volume of a cuboid with a square base of side X and with height Y. (C)

23. Two independent random variables X and Y have Poisson distributions with means λ and μ respectively. Obtain an expression for $P(X = r, Y = n - r)$. Hence express $P(X + Y = n)$ as the sum of a series. Deduce that $X + Y$ has a Poisson distribution with mean $\lambda + \mu$.

24. Telephone calls reach a secretary independently and at random, internal ones at a mean rate of 2 in any 5 minute period, and external ones at a mean rate of 1 in any 5 minute period. Calculate the probability that there will be more than 2 calls in any period of 2 minutes. (O&C)

25. (a) Verify that if a random variable X has probability generating function $G_X(t)$, then $G_X(t) = E(t^X)$.
    (b) Two independent random variables X and Y have distributions $B(m, p)$ and $B(n, p)$ respectively. Write down expressions for $E(t^X)$ and $E(t^Y)$. Assuming that t^X and t^Y can be regarded as independent random variables, write down $E(t^X \times t^Y)$ and hence find the p.g.f. of the variable $Z = X + Y$. Deduce that Z has the distribution $B(m + n, p)$.

Exercise 15.7 (*miscellaneous*)

1. A random variable R takes the integer values $1, 2, \ldots, n$ each with probability $1/n$. Find the mean and variance of R. A pack of 15 cards bearing the numbers 1 to 15 is shuffled. Find the probability that the number on the top card is larger than that on the bottom card, giving reasons for your answer. If the sum of these two numbers is S, find (a) the probability that $S \leqslant 4$, (b) the expected value of S. (O&C)

2. The number X of radios sold per week by a certain shop is a binomially distributed random variable with $n = 10$ and $p = 0{\cdot}3$. The number Y of television sets sold per week by the same shop may be 0, 1 or 2 with $P(Y = 0) = 0{\cdot}6$, $P(Y = 1) = 0.3$, $P(Y = 2) = 0{\cdot}1$. The shopkeeper makes a profit of £2 on each radio sold and a profit of £20 on each television set sold. Find the shopkeeper's expected weekly profit from the sale of radios and television sets. (JMB)

3. (a) In a certain manufacturing process, it is known that approximately 10% of the items produced are defective. A quality control scheme is set up, by selecting 20 items out of a large batch, and rejecting the whole batch if 3 or more are defective. Find the probability that the batch is rejected.
    (b) Two boys, John and David, play a game with a die. The die will be thrown 4 times. David will give John £x if there is an odd number of sixes, otherwise John will give David £1. If the game is to be a fair one to both John and David, find the value of x. (SU)

4. The probability of there being X unusable matches in a full box of Surelite matches is given by $P(X = 0) = 8k$, $P(X = 1) = 5k$, $P(X = 2) = P(X = 3) = k$,

$P(X \geqslant 4) = 0$. Determine the constant k and the expectation and variance of X. Two full boxes of Surelite matches are chosen at random and the total number Y of unusable matches is determined. Calculate $P(Y > 4)$, and state the values of the expectation and variance of Y. (C)

5. Fergus Lightfingers, an educated thief, has broken into a house and has come upon a large safe. Lying upon the table is a bunch of k similar keys, only one of which will open the safe. He considers two possible strategies: (i) randomly select keys, one at a time *without replacement*, until successful; (ii) randomly select keys, one at a time *with replacement*, until successful. Let N, a random variable, be the number of keys including the successful one tried by Fergus in order to open the safe. Derive the probability distribution of N for both strategies. Calculate the expected value of N in each case. Use your statistical judgment in order to select the strategy that Fergus should use. (You may assume that $\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}$ for $|x| < 1$.) (AEB 1978)

6. A batsman's performance is described by the following probability distribution. For each ball that he receives he has probability $\frac{1}{2}$ of scoring no runs, probability 1/5 of scoring one run and otherwise he is equally likely to score two runs, four runs, or to be bowled out. In the last case he receives no further balls.

(i) Determine the probability that the batsman's total score is 0.
(ii) Given that the batsman is not bowled by a particular ball, calculate the expectation of his score from that ball.
(iii) Calculate the expectation of the number of balls that the batsman receives.
(iv) Deduce the expectation of the batsman's total score. (C)

7. In a series of N independent trials the probability of success in any trial is constant and equal to p. Show that the probability of r successes is given by

$$P(r) = \binom{N}{r} p^r (1-p)^{N-r}.$$

Find $P(3)$ if $N = 5$ and $p = 0{\cdot}2$. Write a flow diagram for a program to print values of $P(r)$ if $N = 25$ and $p = 0{\cdot}4$ for $r = 0(1)25$. (L)

8. In a certain factory 70% of the microprocessors produced are found to be imperfect and have to be discarded. Find the probability that, in a randomly chosen sample of 5 microprocessors produced, the number of perfect specimens will be (i) 0, (ii) 1, (iii) 2, (iv) 3, (v) 4, (vi) 5. Verify that the probability of more than 3 perfect specimens is less than 1/30. If a sample of 200 is taken from the same set of microprocessors, find the probability that there are at least 50 perfect specimens. (AEB 1980)

9. Derive the mean and variance of the binomial distribution. Mass production of miniature hearing aids is a particularly difficult process and so the quality of these products is monitored carefully. Samples of size six are selected regularly and tested for correct operation. The number of defectives in each sample is

recorded. During one particular week 140 samples are taken and the distribution of the number of defectives per sample is given in the following table.

Number of defectives per samples (x)	0	1	2	3	4	5	6
Number of samples with x defectives (f)	27	36	39	22	10	4	2

Find the frequencies of the number of defectives per sample given by a binomial distribution having the same mean and total as the observed distribution.

(AEB 1978)

10. Show that the probability generating function for the binomial distribution arising from n independent trials, in each of which the probability of a success is p, is $(1 - p + pt)^n$. Show that the probability generating function for the number of trials needed to obtain a success is $pt/(1 - t + pt)$, and find the mean and variance of this distribution.

When a gun first fires at a target the probability of a hit is 1/5. Firing continues until the first hit, after which the gun is resighted and the probability of a hit becomes 2/5. Find the probability generating function for the number of shots required to obtain 2 hits and hence find the probability that fewer than 5 shots will be needed.

(O&C)

11. A random variable X takes values in the intervals 0 to 1 and 2 to 3, and has probability density function given by

$$f(x) = \begin{cases} 1 - x & (0 \leqslant x \leqslant 1), \\ x - a & (2 \leqslant x \leqslant 3), \\ 0 & \text{elsewhere.} \end{cases}$$

Determine (i) the value of a, (ii) the expectation, μ, of X, (iii) the variance of X, (iv) the (cumulative) distribution function $F(x)$, (v) the probability that $|X - \mu|$ exceeds 1.

(C)

12. The random variable X is the distance, in metres, that an inexperienced tight-rope walker has moved along a given tight-rope before falling off. It is given that

$$P(X > x) = 1 - x^3/64, \quad 0 \leqslant x \leqslant 4.$$

(i) Show that $E(X) = 3$. (ii) Find the standard deviation, σ, of X. (iii) Show that $P(|X - 3| < \sigma) = \frac{69}{80}\sqrt{\frac{3}{5}}$. (C)

13. Find the mean and the variance of the distribution whose probability density function f is given by

$$\begin{aligned} f(x) &= x, & 0 \leqslant x < 1, \\ f(x) &= 2 - x, & 1 \leqslant x \leqslant 2, \\ f(x) &= 0, & \text{otherwise.} \end{aligned}$$

A rectangle is constructed with adjacent sides of lengths x cm and $(2 - x)$ cm where x is a random value from the above distribution. Find the probability that the

area of the rectangle will exceed 0·75 cm^2. Also find the expected value of the area of the rectangle. (JMB)

14. The operational lifetime in hundreds of hours of a battery-operated minicalculator may be regarded as a continuous random variable having probability density function

$$f(x) = cx(10 - x), \quad 5 \leqslant x \leqslant 10,$$
$$f(x) = 0, \quad \text{otherwise.}$$

(a) Find the value of c and of the expected operational lifetime of such a minicalculator.

(b) The purchase price of such a minicalculator is £20 and its running cost (for batteries) amounts to 20 pence per hundred hours operation. Thus, the overall average cost in pence per hundred hours operation of a minicalculator whose operational lifetime is X hundred hours is given by $Y = 20 + (2000/X)$. (i) Evaluate $E(Y)$, the expected overall average cost per hundred hours. (ii) Find the probability that the overall average cost per hundred hours will exceed £2.70. (W)

15. A random variable X has (cumulative) distribution function

$$F(x) = \begin{cases} 0 & (x \leqslant 0) \\ \frac{1}{4}x^2 & (0 \leqslant x \leqslant 1) \\ \alpha x + \beta & (1 \leqslant x \leqslant 2) \\ \frac{1}{4}(5 - x)(x - 1) & (2 \leqslant x \leqslant 3) \\ 1 & (3 \leqslant x) \end{cases}$$

Find (i) the values of the constants α and β, (ii) the probability that $1{\cdot}5 \leqslant X \leqslant 2{\cdot}5$, (iii) the probability density function $f(x)$. Sketch the graph of $f(x)$, and hence, or otherwise, deduce the mean of X. Determine also the variance of X. (C)

16. A failure time, X, has density function $\lambda e^{-\lambda x}$ for $x \geqslant 0$, $\lambda > 0$. Find (a) the distribution function; (b) the mean and variance; (c) the median; (d) the time below which it is 95% certain that there will be no failure. (O)

17. Henri de Lade regularly travels from his home in the suburbs to his office in Paris. He always tries to catch the same train, the 08.05 from his local station. He walks to the station from his home in such a way that his arrival times form a normal distribution with mean 08.00 hours and standard deviation 6 minutes. (i) Assuming that his train always leaves on time, what is the probability that on any given day Henri misses his train? (ii) If Henri visits his office in this way 5 days each week and if his arrival times at the station each day are independent, what is the probability that he misses his train once and only once in a given week? (AEB 1980)

18. The score, S, gained by an expert rifleman with a single shot, is a random variable with the following probability distribution:

$$P(S = 8) = 0{\cdot}01, \quad P(S = 9) = 0{\cdot}29, \quad P(S = 10) = 0{\cdot}70.$$

(i) Use a normal approximation to determine the probability that the rifleman obtains six or more scores of 8 in a series of 900 independent shots. (ii) Find the expectation and variance of S. (iii) Use a normal approximation to determine the probability that the rifleman scores less than 96 with ten independent shots. (C)

19. The prices of houses on an estate agent's books are approximately normally distributed with a mean price of £18 000. Given that 90% of the houses are priced at less than £30 000, find the standard deviation of the house prices and the percentage of the houses with a price of less than £10 000. Find the mean price and the standard deviation of the prices when the price of each house is increased by (a) £2000, (b) 10%. (L)

20. The weights of grade A oranges are normally distributed with mean 200 g and standard deviation 12 g. Determine, correct to 2 significant figures, the probability that (i) a grade A orange weighs more than 190 g but less than 210 g, (ii) a sample of 4 grade A oranges weighs more than 820 g. The weights of grade B oranges are normally distributed with mean 175 g and standard deviation 9 g. Determine, correct to 2 significant figures, the probability that (iii) a grade B orange weighs less than a grade A orange, (iv) a sample of 8 grade B oranges weighs more than a sample of 7 grade A oranges. (C)

21. Mass-produced right circular cylindrical pipes have internal diameters that are normally distributed with a mean of 10 cm and a standard deviation of 0·4 cm. (i) Find the probability that a randomly chosen pipe will have an internal diameter greater than 10·3 cm. (ii) Find the expected number of pipes in a random sample of 100 pipes that have internal diameters in the range from 9·7 cm to 10·3 cm. (iii) Find the expected value of the internal cross-sectional area of a randomly chosen pipe; give your answer correct to three significant figures. (iv) Mass-produced pistons have diameters that are normally distributed with a mean of 9·9 cm and a standard deviation of 0·3 cm. Find the probability that a randomly selected piston will have a diameter less than the internal diameter of a randomly selected pipe. (W)

22. Two gauges are used to test the thickness of manufactured metal sheets. Over a long period it is found that 1·25% of the sheets will pass through the smaller gauge of 1·4 mm while 95·4% will pass through the larger gauge of 1·6 mm. Assuming that the distribution of sheet thickness is normal, find its mean and standard deviation. At the next stage of manufacture 2 sheets are clamped together to make a thick plate. Find the mean and the standard deviation of the thickness of these plates. If a plate will pass through a gauge of 3·2 mm but not through one of 2·8 mm, find the probability that both the sheets of which it is composed will pass through the 1·6 mm gauge but not through the 1·4 mm gauge. Find also the size of a gauge which will allow 99·5% of these plates to pass through. (O&C)

23. Next May, an ornithologist intends to trap one male cuckoo and one female cuckoo. The mass M of the male cuckoo may be regarded as being a normal random variable with mean 116 g and standard deviation 16 g. The mass F of the female cuckoo may be regarded as being independent of M and as being a normal random

variable with mean 106 g and standard deviation 12 g. Determine (i) the probability that the mass of the two birds together will be more than 230 g. (ii) the probability that the mass of the male will be more than the mass of the female. By considering $X = 9M - 16F$, or otherwise, determine the probability that the mass of the female will be less than nine-sixteenths of that of the male. Suppose that one of the two trapped birds escapes. Assuming that the remaining bird will be equally likely to be the male or the female, determine the probability that its mass will be more than 118 g. (C)

24. (a) The number of accidents notified in a factory per day over a period of 200 days gave rise to the following table:

No. of accidents	0	1	2	3	4	5
No. of days	127	54	14	3	1	1

(i) Calculate the mean number of accidents per day. (ii) Assuming that this situation can be represented by a suitable Poisson distribution, calculate the corresponding frequencies.

(b) Of items produced by a machine, approximately 3% are defective, and these occur at random. What is the probability that, in a sample of 144 items, there will be at least two which are defective? (SU)

25. Define a Poisson distribution giving the general probability formula; state the relationship between its mean and variance. The distribution of the number of vehicles observed passing under a motorway bridge in 100 successive intervals of 12 seconds, at a time when the traffic flow is 720 vehicles per hour, may be considered to be Poissonian. During how many of the intervals would one expect (i) no cars to have passed? (ii) 2 cars to have passed, (iii) more than 2 cars to have passed? Find the smallest value of N such that the probability of at most N cars in an interval is greater than 0·999. (AEB 1978)

26. Each play of a one-armed bandit machine has probability 0·04 of yielding a win. It may be assumed that this probability applies independently for each play of the machine. (a) Find the probability that in ten plays of the machine there will be (i) at least two wins, (ii) exactly two wins. (b) For a sequence of n plays of the machine write down an expression for the probability that there will be no win. Hence find the smallest number of plays of the machine for there to be a probability of at least 0·95 of winning at least once. (c) Use an appropriate method to find an approximate value for the probability that there will be six or fewer wins in 80 plays of the machine. (W)

27. Gnat larvae are distributed at random in pond water so that the number of larvae contained in a random sample of 10 cm^3 of pond water may be regarded as a random variable having a Poisson distribution with mean 0·2. Ten independent random samples, each of 10 cm^3, of pond water, are taken by a zoologist. Determine (correct to three significant figures) (i) the probability that none of the samples contain larvae, (ii) the probability that one sample contains a single larva and the remainder contain no larvae, (iii) the probability that one sample contains two or

more larvae and the remainder contain no larvae, (iv) the expectation of the total number of larvae contained in the ten samples, (v) the expectation of the number of samples containing no larvae. (C)

28. A prospector for gold examines exactly 800 pans of material every month. The contents of each pan may be assumed to be independent of the others with the probability of a pan containing gold being 0·005 for each pan. The prospector has a good month when 4 or more of the pans that he examines contain gold. Show that, to three significant figures, the probability that a randomly chosen month is a good month is 0·567. Determine the probability that (i) a randomly chosen period of four months contains more than two good months, (ii) a randomly chosen period of 24 months contains more than 12 good months. (C)

29. The number of organic particles suspended in a volume V cm^3 of a certain liquid follows a Poisson distribution with mean $0{\cdot}1V$. (a) Find the probabilities that a sample of 1 cm^3 of the liquid will contain (i) at least one organic particle, (ii) exactly one organic particle. (b) Use an appropriate approximate procedure to find the probability that a sample of 1000 cm^3 of the liquid will contain at least 90 organic particles. (c) The liquid is sold in vials, each vial containing 10 cm^3 of the liquid. The vials are dispatched for sale in boxes, each box containing 100 vials. Find the probability that a vial will contain at least one organic particle. Hence find the mean and the standard deviation of the number of vials per box of 100 vials that contain at least one organic particle. (W)

30. When a number is rounded off to its nearest integer value, the rounding off error X may be regarded as a random variable which is uniformly distributed over the interval $(-\frac{1}{2}, \frac{1}{2})$. Find the variance of X. If n numbers are rounded off to their nearest integer values and then summed, the total error in the sum is given by $Y = X_1 + X_2 + \ldots + X_n$, where $X_1, X_2, \ldots, X_n$ are the individual errors in the n numbers, which may be assumed to be independent and uniformly distributed over $(-\frac{1}{2}, \frac{1}{2})$. Write down the mean and the variance of Y in terms of n. Suppose n is large enough to justify approximating the distribution of Y by a normal distribution. Show that there is a probability in excess of 0·9 that Y will be numerically less than $\frac{1}{2}\sqrt{n}$. Find the largest integer value of n for which there is a probability of at least 0·6 that Y will be numerically less than 1·0. (W)

31. The joint probability function of two discrete random variables X and Y is given by $P(X = r,\ Y = s) = c|r + s|$ for $r = -1, 0, 1$ and $s = -2, -1, 0, 1, 2$,
$P(X = r, Y = s) = 0$, otherwise.
(a) Show that (i) $c = 0{\cdot}05$, (ii) $\mathrm{Var}(Y) = 2 + \mathrm{Var}(X)$.
(b) Evaluate $\mathrm{Var}(Y - X)$. (W)

32. A continuous random variable X is distributed with probability density function $f(x) = 3x^{-4}$, $x \geqslant 1$, $f(x) = 0$, otherwise. Find (i) the cumulative distribution function of X, (ii) the mean and the variance of X. The random variable Y is defined by $Y = X^{-1}$. By first finding the cumulative distribution function of Y, or otherwise, determine the probability density function of Y. Verify that $E(X)E(Y)$

and $E(XY)$ are not equal and explain why they have different values in this particular example. (W)

33. A continuous random variable X has probability density function

$$f(x) = 4x(1 - x^2), \qquad 0 \leqslant x \leqslant 1,$$
$$f(x) = 0, \qquad \text{otherwise.}$$

(i) Verify that $f(x)$ satisfies the requirements to be a probability density function. (ii) Find the mean value of X. (iii) Find the cumulative distribution function of X. (iv) If X denotes the area of a square of side Y find the probability density function of Y and use it to verify that $E(Y^2) = E(X)$. (W)

34. The probability density function of a continuous random variable X is given by

$$f(x) = \tfrac{2}{3}x, \qquad 1 \leqslant x \leqslant 2,$$
$$f(x) = 0, \qquad \text{otherwise.}$$

(a) Find (i) $E(X^2)$, (ii) $E(X^4)$, (iii) the cumulative distribution function of X.
(b) Suppose X is the length of the side of a square. (i) Find the mean and the variance of the area of the square. (ii) Show that the area of the square has a uniform distribution over a certain interval which should be specified. (W)

16 Samples and significance

16.1 Sampling distributions

In this chapter we will be considering the interpretation of statistical data. This usually involves analysing the data, then setting up an appropriate probability model. The known properties of the theoretical distribution can then be used to draw conclusions or make predictions about the observed distribution. Later we will look at tests which help us to measure the significance of the available data. However, first we define some of the terms to be used.

A *population* is the set of all items under discussion in a particular survey or experiment, e.g. the ages of all the inhabitants of Bournemouth, the lengths of all components produced by a certain machine. A *sample* is a set of observations taken from a population, i.e. a subset of the population. For instance, a set of 10 scores obtained by tossing a single die is a sample of size 10 drawn from the theoretically infinite population of all possible tosses. A *random sample* is one in which each element of the population has an equal chance of being selected.

A *statistic* is a quantity calculated using the observations in a sample. The mean, the median and the variance of a sample are all examples of statistics. By considering all the samples of a given size which can be drawn from a particular population, it is possible to obtain the probability distribution of a statistic such as the sample mean. These theoretical distributions are called *sampling distributions.*

In order to investigate the properties of sampling distributions we will regard a *random sample of size n*, from a population with mean μ and variance σ^2, as a set of random variables $X_1, X_2, \ldots, X_n$ each with the same distribution as the parent population. We will assume that these variables are *independent* if there is little or no change in the population as elements of a sample are selected,

i.e. if (a) the population is infinite,
or (b) the population is finite, but very large,
or (c) the population is finite, but the sampling is *with replacement* (each item is replaced before the next is selected).

The variables $X_1, X_2, \ldots, X_n$ cannot be treated as independent if samples are taken *without replacement* from a relatively small population.

If $\bar{X}$ represents the sample mean $\frac{1}{n}(X_1 + X_2 + \ldots + X_n)$, then, using results obtained in §15.6:

$$E(\bar{X}) = E\left[\frac{1}{n}(X_1 + X_2 + \ldots + X_n)\right]$$

$$= \frac{1}{n}[E(X_1) + E(X_2) + \ldots + E(X_n)] = \frac{1}{n}(n\mu) = \mu.$$

Provided that the variables $X_1, X_2, \ldots, X_n$ are independent,

$$\mathrm{Var}(\bar{X}) = \mathrm{Var}\left[\frac{1}{n}(X_1 + X_2 + \ldots + X_n)\right]$$

$$= \frac{1}{n^2}[\mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \ldots + \mathrm{Var}(X_n)] = \frac{1}{n^2}(n\sigma^2) = \frac{\sigma^2}{n}$$

$\therefore$ If $\bar{X}$ is the mean of a random sample of size n from a population with mean μ, variance σ^2

$$E(\bar{X}) = \mu.$$

For a large population or sampling with replacement

$$\mathrm{Var}(\bar{X}) = \sigma^2/n.$$

[It can be shown that for a random sample of size n drawn without replacement from a population of size v, the appropriate formula is

$$\mathrm{Var}(\bar{X}) = \left(\frac{v-n}{v-1}\right)\frac{\sigma^2}{n}.$$

We see that as $v \to \infty$, $\mathrm{Var}(\bar{X}) \to \sigma^2/n$.]

If the mean, $\bar{x}$, of a particular sample is regarded as an approximate value of the mean μ of the whole population, then the deviation $\bar{x} - \mu$ is the error in this approximation. For this reason the standard deviation of the sampling distribution of a statistic is often called the *standard error* of the statistic.

Thus, in general, the standard error of the mean of a random sample of size n is $\sigma/\sqrt{n}$.

Since linear combinations of normal random variables are also normally distributed we can extend these results as follows.

If $\bar{X}$ is the mean of a random sample of size n from a population which is normally distributed with mean μ, variance σ^2, then $\bar{X}$ is normally distributed with mean μ, variance σ^2/n.

It is more difficult to make general statements about the sampling distribution of the mean for populations which are not normally distributed. However, there is an

important theorem which tells us a great deal about the behaviour of $\bar{X}$ when n is large.

The central limit theorem. Let X be a random variable with mean μ, variance σ^2. If $\bar{X}$ is the mean of a random sample of size n drawn from the distribution of X then the distribution of the standardised variable $(\bar{X} - \mu)/(\sigma/\sqrt{n})$ tends to the standard normal distribution $N(0, 1)$ as n tends to infinity.

In practice we use a less formal statement of the result.

If $\bar{X}$ is the mean of a random sample of size n from *any* population with mean μ, variance σ^2, then for large n, $\bar{X}$ has an approximately normal distribution with mean μ, variance σ^2/n.

If the population distribution is reasonably symmetrical, good approximations may be obtained for $n \geqslant 25$. However, when sampling from a rather skew (i.e. unsymmetrical) distribution, we may need $n > 100$ for good results.

Example 1 The metal discs produced by a certain machine have mean diameter 30·01 mm with standard deviation 0·08 mm. A sample of 100 discs is taken. Find the probability that the mean diameter of the discs in this sample is less than 30 mm.

Using the central limit theorem, the distribution of the means of samples of size 100 has an approximately normal distribution with mean 30·01 mm and standard deviation $0{\cdot}08/\sqrt{100}$ mm i.e. 0·008 mm.
Let $\bar{X}$ be the mean of a sample of size 100 and let Z be a standard normal variable, then

$$P(\bar{X} < 30) = P\left(Z < \frac{30 - 30{\cdot}01}{0{\cdot}008}\right)$$

$$= P(Z < -1{\cdot}25) = 1 - \Phi(1{\cdot}25) = 0{\cdot}1056$$

Hence the probability that the sample mean is less than 30 mm is 0·11 approximately.

We now consider briefly the sampling distribution of a *difference between two means.* Let us suppose that $\bar{X}$ and $\bar{Y}$ are the means of random samples of sizes n_x, n_y, taken from two populations with means μ_x, μ_y and variances σ_x^2, σ_y^2 respectively. The distributions of $\bar{X}$ and $\bar{Y}$ will then have means μ_x, μ_y and variances $\sigma_x^2/n_x, \sigma_y^2/n_y$.

$\therefore$ the sampling distribution of the difference $\bar{X} - \bar{Y}$ has mean $\mu_x - \mu_y$ and variance $\dfrac{\sigma_x^2}{n_x} + \dfrac{\sigma_y^2}{n_y}$.

If both parent populations are normally distributed then $\bar{X} - \bar{Y}$ will also have a normal distribution. If the populations are not normally distributed, then it follows from the central limit theorem that, for large values of n_x and n_y, $\bar{X} - \bar{Y}$ will have an approximately normal distribution.

One of the most useful statistics that can be obtained from a sample is a *proportion*. If r members of a sample of size n have a certain attribute, then the corresponding proportion p is given by r/n. For instance, if 4 students in a class of 25 are left-handed, the proportion of left-handed students is 0·16. If the proportion of the total population having the stated attribute is π, then the value π gives the probability that any one member of the population has the attribute. This means that if R is the number of "successes" in a random sample of size n, then R has the binomial distribution $B(n, \pi)$. Thus if P is the proportion of successes in the sample,

$$P(P = p) = P(R = r) = \binom{n}{r}\pi^r(1 - \pi)^{n-r}.$$

These probabilities form the *sampling distribution of a proportion* P and using our knowledge of the binomial distribution we have:

$$E(P) = E(R/n) = \frac{1}{n}E(R) = \frac{1}{n}(n\pi) = \pi$$

$$\text{Var}(P) = \text{Var}(R/n) = \frac{1}{n^2}\text{Var}(R) = \frac{1}{n^2}[n\pi(1 - \pi)] = \frac{\pi(1 - \pi)}{n}$$

For large n, R will have an approximately normal distribution,

$\therefore$ If a proportion π of the members of a population have a certain attribute, then the proportion P in a random sample of size n, where n is large, has an approximately normal distribution with mean π and variance $\pi(1 - \pi)/n$.

Elementary problems involving sample proportions can be solved using either the appropriate sampling distribution or the corresponding binomial distribution.

Example 2 It is estimated that in a certain large city the proportion of families that own a freezer is 0·35. Find the probability that in a random sample of n families over 40% own freezers, where (a) $n = 6$, (b) $n = 60$, (c) $n = 600$.

Let R be the number of families owning a freezer in a sample of size n, then R has the distribution $B(n, \pi)$, where $\pi = 0{\cdot}35$, and the proportion P of families with freezers is given by $P = R/n$.

(a) $P(P > 0{\cdot}4) = P(R > 2{\cdot}4)$

$$= 1 - \{P(R = 0) + P(R = 1) + P(R = 2)\}$$

$$= 1 - \left\{(0{\cdot}65)^6 + \binom{6}{1}(0{\cdot}65)^5(0{\cdot}35) + \binom{6}{2}(0{\cdot}65)^4(0{\cdot}35)^2\right\}$$

$$\approx 0{\cdot}353$$

Hence, when $n = 6$ the required probability is 0·353.

(b) Using a normal approximation to $B(60, 0{\cdot}35)$ and a suitable continuity correction,

$$P(P > 0{\cdot}4) = P(R > 24)$$

$$\approx P\left(Z > \frac{24{\cdot}5 - 21}{\sqrt{(60 \times 0{\cdot}35 \times 0{\cdot}65)}}\right)$$

$$= P(Z > 0{\cdot}9473) = 1 - \Phi(0{\cdot}9473) \approx 0{\cdot}172$$

Hence, when $n = 60$ the required probability is 0.172.

(c) When $n = 600$, the sampling distribution of the proportion P is approximately normal with mean $\mu_p = 0{\cdot}35$ and variance

$$\sigma_p^2 = (0{\cdot}35 \times 0{\cdot}65)/600.$$

$$\therefore \quad P(P > 0{\cdot}4) \approx P\left(Z > \frac{0{\cdot}4 - \mu_p}{\sigma_p}\right)$$

$$= P(Z > 2{\cdot}5678) = 1 - \Phi(2{\cdot}5678) \approx 0{\cdot}005$$

Hence, when $n = 600$ the required probability is 0·005.

[Note that when n has a very large value such as 600, applying a continuity correction would not produce a significant improvement in accuracy.]

Another important statistic is the *sample variance*, but the mathematics involved in studying its sampling distribution is rather difficult. All we can do for the moment is find the expected value of the sample variance.

Using the notation introduced earlier in the section, the variance V of a random sample of size n is given by

$$V = \frac{1}{n}\sum(X_r - \bar{X})^2 = \frac{1}{n}\sum X_r^2 - \bar{X}^2$$

$$\therefore \quad E(V) = E\left[\frac{1}{n}\sum X_r^2 - \bar{X}^2\right] = \frac{1}{n}\sum E(X_r^2) - E(\bar{X}^2)$$

But $\mathrm{Var}(X_r) = E(X_r^2) - [E(X_r)]^2$, so $E(X_r^2) = \sigma^2 + \mu^2$

and $\mathrm{Var}(\bar{X}) = E(\bar{X}^2) - [E(\bar{X})]^2$, so $E(\bar{X}^2) = \dfrac{\sigma^2}{n} + \mu^2$

$$\therefore \quad E(V) = \sigma^2 + \mu^2 - \left(\frac{\sigma^2}{n} + \mu^2\right) = \left(\frac{n-1}{n}\right)\sigma^2$$

i.e. If V is the variance of a random sample of size n from a population with mean μ, variance σ^2,

$$E(V) = \left(\frac{n-1}{n}\right)\sigma^2.$$

[Note that this result is not valid for small populations sampled without replacement, since under those conditions, $\mathrm{Var}(\bar{X}) \neq \sigma^2/n$.]

Exercise 16.1

1. A random variable X takes the values $0, 1, 2, 3$ with probabilities $p(0) = 0{\cdot}3$, $p(1) = 0{\cdot}5$, $p(2) = p(3) = 0{\cdot}1$. Find the mean μ and the variance σ^2 of this distribution. Random samples of size 2 are drawn from the distribution. List all 10 possible samples and their probabilities. In each case find the sample mean $\bar{X}$ and the sample variance V. Hence obtain the sampling distributions of the mean and variance. Verify that $E(\bar{X}) = \mu$, $\text{Var}(\bar{X}) = \frac{1}{2}\sigma^2$ and $E(V) = \frac{1}{2}\sigma^2$.

2. Using the distribution given in question 1, obtain the sampling distributions of the mean $\bar{X}$ and variance V of random samples of size 3. Verify that in this case $E(\bar{X}) = \mu$, $\text{Var}(\bar{X}) = \frac{1}{3}\sigma^2$ and $E(V) = \frac{2}{3}\sigma^2$.

3.

Family	*A*	*B*	*C*	*D*	*E*
No. of children	2	0	0	6	2

The table shows the number of children in 5 families A, B, C, D, E. Find the mean μ and the variance σ^2 of the distribution. Random samples of size 2 are selected, *without replacement*, from the population of 5 families. List all 10 possible samples and find in each case the sample mean $\bar{X}$ and the sample variance V. Remembering that each sample occurs with probability $0{\cdot}1$, obtain the sampling distributions of $\bar{X}$ and V. Hence find $E(\bar{X})$, $\text{Var}(\bar{X})$ and $E(V)$. Write down the values of $E(\bar{X})$, $\text{Var}(\bar{X})$ and $E(V)$ for samples of size 2 selected *with replacement*.

4. Using the distribution given in question 3, obtain the sampling distributions of the mean $\bar{X}$ and variance V of random samples of size 3 selected without replacement. Hence find $E(\bar{X})$, $\text{Var}(\bar{X})$ and $E(V)$. Write down the values of $E(\bar{X})$, $\text{Var}(\bar{X})$ and $E(V)$ for samples of size 3 selected with replacement.

5. The masses of a large consignment of melons have an approximately normal distribution with mean $1{\cdot}5$ kg and standard deviation $0{\cdot}3$ kg. Find the probability that (a) the mass of one melon differs from $1{\cdot}5$ kg by more than $0{\cdot}5$ kg, (b) the mean mass of 25 melons differs from $1{\cdot}5$ kg by more than $0{\cdot}1$ kg.

6. A random variable X is normally distributed with mean 20 and variance 32. Find the probability that X lies between 15 and 25. If a random sample of size 50 is drawn from this distribution find the probability that the sample mean lies between 19 and 21.

7. The diameters x of 110 steel rods were measured in centimetres and the results were summarised as follows:

$$\sum x = 36{\cdot}5, \quad \sum x^2 = 12{\cdot}49.$$

Find the mean and standard deviation of these measurements. Assuming these measurements are a sample from a normal distribution with this mean and this

variance, find the probability that the mean diameter of a sample of size 110 is greater than 0·345 cm (O&C)

8. A class of 18-year-old pupils contains 18 boys and 15 girls. The measured heights in metres of the boys are $x_i (i = 1, \ldots, 18)$, and those of the girls are $y_j (j = 1, \ldots, 15)$. It is found that $\sum x_i = 31{\cdot}50$, $\sum y_j = 24{\cdot}75$, $\sum x_i^2 = 55{\cdot}305$, $\sum y_j^2 = 41{\cdot}175$.

(a) Find the means and the sample variances of the heights of (i) the boys, (ii) the girls.

(b) Assuming that the heights of boys and of girls of age 18 are independent and both are normally distributed with the same means and variances as in these samples, find the difference in mean heights of boys and girls. Find also the standard deviation of this difference in a large population of 18-year-olds.

(O&C)

9. Two random variables X and Y have independent normal distributions $N(3, 7)$ and $N(5, 11)$ respectively. A random sample of 12 observations is selected from each distribution. Find the sampling distribution of $\bar{X} - \bar{Y}$, the difference between the sample means. Hence find the probability that (a) $\bar{X} < \bar{Y}$, (b) $|\bar{X} - \bar{Y}| < 1$.

10. A random variable X has the distribution $N(5, 5)$ and $\bar{X}$ is the mean of a random sample of size n drawn from the distribution of X. Find the least value of n such that (a) $P(\bar{X} > 6) < 0{\cdot}01$, (b) $P(\bar{X} > 4{\cdot}9) > 0{\cdot}9$.

11. The random variables $X_1, X_2, \ldots, X_n$ are independent and each has a normal distribution with mean μ and variance 1. The random variable $\bar{X}$ is defined to be $(X_1 + X_2 + \ldots + X_n)/n$. Determine, in terms of n, the value v which is such that, when $\mu = 0$, the probability of $\bar{X}$ exceeding v is 0·05. For this value of v it is desired that the probability of $\bar{X}$ being less than v when $\mu = 0{\cdot}2$ should be at most 0·10. Calculate the smallest possible value of n which satisfies this requirement. (C)

12. In a presidential election the proportion of voters who voted for the socialist candidate was 0·48. Find the probability that in a random sample of n voters over 50% voted for the socialist, where (a) $n = 5$, (b) $n = 50$, (c) $n = 500$, (d) $n = 5000$.

13. It is known that in a large consignment of hyacinth bulbs 24% are expected to produce pink blooms. For each of the following samples find the probability that at least 20% of the bulbs will produce pink blooms. (a) 8 bulbs in a window box, (b) 80 bulbs in a rockery, (c) 800 bulbs in a municipal park.

16.2 Unbiased estimators

Having studied the properties of various sampling distributions we are now ready to consider the use of sample statistics to estimate population parameters, such as the mean μ, the variance σ^2 or a proportion π. When producing such an estimate we can calculate either (i) a single value or *point estimate* for the parameter, or (ii) two

values which define a *confidence interval* within which the parameter is expected to lie. Thus there are two basic types of estimation, *point estimation* and *interval estimation*. In this section we concentrate on the former.

A function of the observations in a sample, which is used to estimate a population parameter θ is called an *estimator* of θ. The value taken by this estimator in a particular sample is an *estimate* of θ. For instance, the sample mean $\bar{X}$ is an estimator of the population mean μ and the mean $\bar{x}$ of a particular sample is an estimate of μ. To determine whether a statistic T is likely to be a "good" estimator of a parameter θ, we need to know something about its sampling distribution. Clearly the distribution of T should be centred on the true value of θ. An estimator with this property is said to be *unbiased*.

A statistic T obtained from a random sample and used to estimate a population parameter θ is an *unbiased estimator* of θ if $E(T) = \theta$.

It is also desirable that the spread of values produced by an estimator should be as small as possible.

If T_1 and T_2 are two unbiased estimators of a population parameter θ, both obtained from samples of size n, then if $\text{Var}(T_1) < \text{Var}(T_2)$, T_1 is said to be a *better* or *more efficient* estimator of θ than T_2.

It was shown in §16.1 that if $\bar{X}$ is the mean of a random sample of size n taken from a population with mean μ, then $E(\bar{X}) = \mu$.

$\therefore$ $\bar{X}$ is an unbiased estimator of μ.

Similarly, if a proportion P of the elements of a random sample of size n have a certain attribute and π is the proportion of the population having that attribute, then $E(P) = \pi$.

$\therefore$ P is an unbiased estimator of π.

However, if V is the variance of a random sample of size n from a population with variance σ^2, $E(V) \neq \sigma^2$. Thus V is a biased estimator of σ^2. Under most conditions,

$$E(V) = \left(\frac{n-1}{n}\right)\sigma^2$$

$\therefore$
$$E\left[\left(\frac{n}{n-1}\right)V\right] = \left(\frac{n}{n-1}\right)E(V) = \sigma^2$$

$\therefore$ $\left(\frac{n}{n-1}\right)V$ is an unbiased estimator of σ^2.

The standard notation for a typical value of this estimator is s^2.

Thus $s^2 = \left(\frac{n}{n-1}\right)\frac{1}{n}\sum(x_r - \bar{x})^2$ or $\left(\frac{n}{n-1}\right)\left(\frac{1}{n}\sum x_r^2 - \bar{x}^2\right)$

$\therefore$ an unbiased estimate of population variance from a random sample of size n with mean $\bar{x}$ is given by

$$s^2 = \frac{1}{n-1}\sum(x_r - \bar{x})^2 \quad \text{or} \quad \frac{1}{n-1}\left(\sum x_r^2 - n\bar{x}^2\right).$$

The corresponding formulae for data given in the form of a frequency distribution are

$$s^2 = \frac{1}{n-1}\sum f_r(x_r - \bar{x})^2 \quad \text{or} \quad \frac{1}{n-1}(\sum f_r x_r^2 - n\bar{x}^2).$$

Example 1 A random sample drawn from a large population contains 10 observations $x_1, x_2, \ldots, x_{10}$ such that $\sum x = 24{\cdot}5$ and $\sum x^2 = 61{\cdot}93$. Obtain unbiased estimates for the population mean and variance.

$$\bar{x} = \frac{1}{n}\sum x = \tfrac{1}{10} \times 24{\cdot}5 = 2{\cdot}45$$

$$s^2 = \frac{1}{n-1}\left[\sum x^2 - n\bar{x}^2\right] = \tfrac{1}{9}[61{\cdot}93 - 10(2{\cdot}45)^2] \approx 0{\cdot}212$$

Hence unbiased estimates of the mean and variance of the population are 2·45 and 0·212 respectively.

Example 2 A continuous random variable X has probability density function $f(x) = 2x/\theta^2$ for $0 \leqslant x \leqslant \theta$ and $f(x) = 0$ otherwise. Find the mean μ of the distribution of X. Given that $\bar{X}$ denotes the mean of a random sample of size n drawn from this distribution, show that $3\bar{X}/2$ is an unbiased estimator of θ.

$$\mu = E(X) = \int_0^\theta xf(x)\,dx = \int_0^\theta \frac{2x^2}{\theta^2}\,dx = \left[\frac{2x^3}{3\theta^2}\right]_0^\theta = \tfrac{2}{3}\theta$$

$$\therefore \quad E\left(\frac{3\bar{X}}{2}\right) = \tfrac{3}{2}E(\bar{X}) = \tfrac{3}{2}\mu = \tfrac{3}{2}(\tfrac{2}{3}\theta) = \theta$$

Hence $3\bar{X}/2$ is an unbiased estimator of θ.

Finally we show that data from two samples can be *pooled* to produce unbiased *two-sample estimates* of population parameters. We will suppose that two random samples of sizes m and n are drawn from the same population. These samples can be combined to form a single sample of size $m + n$. If the means of the original samples are $\bar{X}$ and $\bar{Y}$ respectively, then the mean $\bar{Z}$ of the combined sample is given by

$$\bar{Z} = \frac{m\bar{X} + n\bar{Y}}{m+n}$$

If the mean of the parent population is μ, then

$$E(\bar{Z}) = \frac{mE(\bar{X}) + nE(\bar{Y})}{m+n} = \frac{m\mu + n\mu}{m+n} = \mu$$

$\therefore$ If $\bar{X}$ and $\bar{Y}$ are the means of random samples of sizes m and n respectively, drawn from a population with mean μ, then an unbiased estimator of μ is given by:

$$\frac{m\bar{X} + n\bar{Y}}{m+n}$$

The next result is obtained using similar arguments:

Random samples of sizes m and n are drawn from the same population and proportions P_x and P_y of these samples have a certain attribute. If π is the proportion of the population having that attribute, then an unbiased estimator of π is given by:

$$\frac{mP_x + nP_y}{m+n}$$

If V_x and V_y are the variances of the samples under consideration and σ^2 is the population variance, then

$$\begin{aligned} E(mV_x + nV_y) &= mE(V_x) + nE(V_y) \\ &= (m-1)\sigma^2 + (n-1)\sigma^2 = (m+n-2)\sigma^2 \end{aligned}$$

$\therefore$ If V_x and V_y are the variances of random samples of sizes m and n respectively, drawn from a population with variance σ^2, then an unbiased estimator of σ^2 is given by:

$$\frac{mV_x + nV_y}{m+n-2}$$

In terms of the unbiased variance estimates, s_x^2 and s_y^2, obtained from random samples of sizes m and n, the two-sample unbiased estimate of σ^2 is:

$$\frac{(m-1)s_x^2 + (n-1)s_y^2}{m+n-2}$$

[The results obtained in §§16.1 and 16.2 are summarised in the Formulae for Reference section towards the end of the book.]

Exercise 16.2

1. The following random sample of size 12 was drawn from a large population: 2·2, 3·1, 4·7, 6·3, 5·0, 4·4, 1·9, 6·8, 3·2, 5·7, 2·5, 3·4. Obtain unbiased estimates of

(a) the population mean, (b) the population variance, (c) the proportion of the population having values between 2 and 6.

2. The following random sample of size 8 was drawn from a large population: 42·0, 35·7, 52·3, 35·9, 45·2, 54·1, 48·6, 31·8. Obtain unbiased estimates of (a) the population mean, (b) the population variance, (c) the proportion of the population having values less than 40.

3. A random sample drawn from a large population contains 20 observations $x_1, x_2, \ldots, x_{20}$ such that $\sum x = 48{\cdot}2$ and $\sum x^2 = 160{\cdot}66$. Obtain unbiased estimates for the population mean and variance. Find also the expected mean and variance of samples of size 100.

4. A random sample drawn from a large population contains 25 observations $x_1, x_2, \ldots, x_{25}$ such that $\sum x = 1492$ and $\sum x^2 = 95\,030$. Obtain unbiased estimates for the population mean and variance. Find also the expected mean and variance of samples of size 10.

5. The given table shows the distribution of days lost through illness in a three-month period for a random sample of 60 workers in a large factory.

No. of days lost	0	1	2	3	4	5	6
No. of workers	8	12	19	14	4	2	1

Find unbiased estimates for the mean and variance of the numbers of days lost per worker for the whole work force. Estimate also the proportion of workers losing more than 2 days work.

6. A continuous random variable X is uniformly distributed over the interval from 0 to θ. Find the mean μ and the variance σ^2 of the distribution of X. If a random sample of size 9 drawn from this distribution has mean $\bar{X}$ and variance V, show that (a) $2\bar{X}$ is an unbiased estimator of θ, (b) $13{\cdot}5V$ is an unbiased estimator of θ^2. Given that one random sample of size 9 contains the observations 5·7, 0·2, 1·7, 3·8, 2·5, 4·2, 5·1, 1·4, 3·3, calculate unbiased estimates of θ and θ^2.

7. A continuous random variable X has probability density function $f(x) = 3x^2/\theta^3$ for $0 \leqslant x \leqslant \theta$ and $f(x) = 0$ otherwise. Find the mean μ and the variance σ^2 of the distribution of X. If a random sample of size 6 drawn from this distribution has mean $\bar{X}$ and variance V, show that (a) $4\bar{X}/3$ is an unbiased estimator of θ, (b) $32V$ is an unbiased estimator of θ^2. Given that one random sample of size 6 contains the observations 3·7, 5·9, 7·1, 8·1, 9·0, 9·4, calculate unbiased estimates of θ and θ^2.

8. Two random sample of size 8 drawn from the same large population contain the following observations:

Sample I:	13,	37,	66,	72,	27,	33,	47,	61.
Sample II:	29,	46,	40,	68,	19,	60,	30,	44.

(a) Find the sample means and variances. Hence obtain unbiased two-sample estimates of the population mean and variance.

(b) Treating the observations in the two samples as a single sample of size 16, find further unbiased estimates of the population mean and variance.

9. Two random samples drawn from the same large population are of sizes 12 and 6.

Sample I:	3,	8,	10,	12,	6,	5,	9,	13,	11,	15,	6,	13.
Sample II:	14,	4,	11,	6,	9,	10.						

(a) Find the sample means and variances. Hence obtain unbiased two-sample estimates of the population mean and variance.
(b) Treating the observations in the two samples as a single sample of size 18, find further unbiased estimates of the population mean and variance.

10. A discrete random variable X can take the values 0, 1 and 2 only, with respective probabilities $\frac{1}{2}\theta$, $1 - \theta$, and $\frac{1}{2}\theta$, where θ is an unknown number between 0 and 1. Let X_1 and X_2 denote two randomly observed values of X. List the possible values of $\{X_1, X_2\}$ that may arise and calculate the probability of each possibility; verify that your probabilities sum to unity. By calculating the value of $(X_1 - X_2)^2$ for each possible $\{X_1, X_2\}$ determine the sampling distribution of $(X_1 - X_2)^2$. Hence show that $Y = \frac{1}{2}(X_1 - X_2)^2$ is an unbiased estimator of θ and express its sampling variance in terms of θ. Since θ is the probability that X will not take the value 1, another possible estimator of θ is the proportion of sample values not equal to 1; for a sample of two observations this estimator is given by $Z = \frac{1}{2}N$, where N is the number of observations (0, 1 or 2) not equal to 1. State, giving your reasons, which of Y and Z you would prefer as the estimator of θ. (W)

11. Each trial of a random experiment has probability p, $(0 < p < 1)$, of yielding a success. In n_1 independent trials of the experiment the number of successes obtained was r_1. Write down an unbiased estimate, $\hat{p}_1$, of p and find its standard error in terms of n_1 and p. In further n_2 independent trials of the same experiment the number of successes obtained was r_2. Let $\hat{p}_2$ denote the unbiased estimate of p from these n_2 trials. Verify that $\frac{1}{2}(\hat{p}_1 + \hat{p}_2)$ is an unbiased estimate of p, and find its standard error in terms of n_1, n_2 and p. Determine the range of values of the ratio n_1/n_2 for which the estimate $\frac{1}{2}(\hat{p}_1 + \hat{p}_2)$ is to be preferred to each of $\hat{p}_1$ and $\hat{p}_2$. (W)

16.3 Confidence intervals

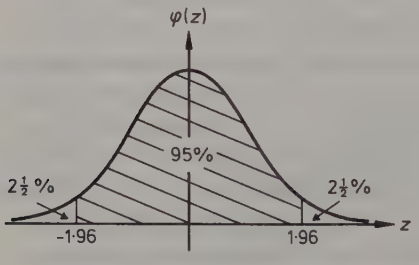
In the previous section we saw that a sample statistic, such as a sample mean, can be used to obtain estimates of a population parameter. However, since these point estimates may vary widely from sample to sample, it is important to know how close a particular estimate is likely to be to the true value of the parameter. This information is often given in the form of an interval estimate. Using the sampling distribution of a statistic we can determine a *confidence interval* for the corresponding population parameter. As the following definition shows, the probability that the chosen interval contains this parameter is expressed as a percentage.

If a and b are two quantities obtained using the observations in a random sample and θ is an unknown population parameter, such that

$$P(a \leqslant \theta \leqslant b) = 0{\cdot}95,$$

then the interval $[a, b]$ is said to be a *95% confidence interval* for θ. The numbers a and b are called *95% confidence limits* for θ.

Let us suppose that $\bar{X}$ is the mean of a random sample of size n from a population which is normally distributed with mean μ and variance σ^2. As shown in §16.1, the sampling distribution of $\bar{X}$ is $N(\mu, \sigma^2/n)$. Hence we can produce a variable Z with the standard normal distribution $N(0, 1)$, by letting $Z = \dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}}$



Using the 2·5% value given in Table 2 (percentage points of the normal distribution), we find that

$$P(-1{\cdot}96 \leqslant Z \leqslant 1{\cdot}96) = 0{\cdot}95$$

$$\therefore \quad P\left(-1{\cdot}96\frac{\sigma}{\sqrt{n}} \leqslant \bar{X} - \mu \leqslant 1{\cdot}96\frac{\sigma}{\sqrt{n}}\right) = 0{\cdot}95$$

$$\therefore \quad P\left(-1{\cdot}96\frac{\sigma}{\sqrt{n}} \leqslant \mu - \bar{X} \leqslant 1{\cdot}96\frac{\sigma}{\sqrt{n}}\right) = 0{\cdot}95$$

$$\therefore \quad P\left(\bar{X} - 1{\cdot}96\frac{\sigma}{\sqrt{n}} \leqslant \mu \leqslant \bar{X} + 1{\cdot}96\frac{\sigma}{\sqrt{n}}\right) = 0{\cdot}95$$

Thus the (symmetric) 95% confidence interval for μ is $\left[\bar{x} - 1{\cdot}96\dfrac{\sigma}{\sqrt{n}}, \bar{x} + 1{\cdot}96\dfrac{\sigma}{\sqrt{n}}\right]$, that is:

If $\bar{x}$ is the mean of a random sample of size n from a normal population with *unknown* mean μ and *known* variance σ^2, then the 95% confidence limits for μ are $\bar{x} \pm 1{\cdot}96\dfrac{\sigma}{\sqrt{n}}$.

Using the 0·5% value given in Table 2, the 99% confidence limits for μ are $\bar{x} \pm 2{\cdot}576\dfrac{\sigma}{\sqrt{n}}$.

It follows from the central limit theorem that these results can also be used to give approximate confidence limits for the mean of a population which is not normal, provided that n is fairly large.

Example 1 The diameters of a large consignment of ball-bearings are normally distributed with standard deviation 0·3 mm. The mean diameter of a random sample of size 16 is found to be 5·4 mm. Find (a) 95% confidence limits for the mean diameter of the whole consignment, (b) the smallest sample size needed to give a 99% confidence interval of width less than 0·2 mm.

(a) The 95% confidence limits for the mean diameter are (in mm):

$$\bar{x} \pm 1{\cdot}96\frac{\sigma}{\sqrt{n}} = 5{\cdot}4 \pm 1{\cdot}96 \times \frac{0{\cdot}3}{4} \approx 5{\cdot}4 \pm 0.15$$

Hence the required limits are 5·25 mm and 5·55 mm.

(b) The 99% confidence interval for the mean diameter has limits:

$$\bar{x} \pm 2{\cdot}576\frac{\sigma}{\sqrt{n}} \quad \text{(in mm)}$$

Hence this interval will have width less than 0·2 mm if

$$2{\cdot}576 \times \frac{0.3}{\sqrt{n}} < 0{\cdot}1$$

i.e. $$\frac{2.576 \times 0{\cdot}3}{0{\cdot}1} < \sqrt{n}$$

i.e. $$n > (2{\cdot}576 \times 3)^2 \approx 59{\cdot}7$$

Thus the smallest sample size needed is 60.

In practice it is often necessary to find confidence intervals for the mean of an approximately normal population whose variance is *unknown*. If in our earlier working we replace σ^2 by its estimated value s^2, then the standard normal variable $Z = (\bar{X} - \mu)/(\sigma/\sqrt{n})$ is replaced by a new variable which is not simply a linear function of $\bar{X}$, but a function of both $\bar{X}$ and the sample variance. In general, this new variable has what is called a *t-distribution*, which is not covered in this course. However, as n increases this t-distribution approaches the standard normal distribution. Thus, for large n we can again use percentage points of $N(0, 1)$ to obtain confidence intervals.

If $\bar{x}$ is the mean of a random sample of size n from a normal population with mean μ and variance σ^2 *both unknown*, then for large n the approximate 95% confidence for μ are

$$\bar{x} \pm 1{\cdot}96\frac{s}{\sqrt{n}},$$

where s^2 is the unbiased estimate of σ^2.

[This result usually provides good approximations for $n \geqslant 30$. For $n \geqslant 200$ the sample variance $\frac{1}{n}\sum(x - \bar{x})^2$ can be used as an estimate of σ^2 with little loss of accuracy.]

More general results can be obtained by considering the $C\%$ confidence interval for a population parameter λ. Let us assume that a sample statistic L is an unbiased estimator of λ with standard error σ_l, and that L has a normal (or nearly normal) distribution. For a particular unbiased estimate l the corresponding $C\%$ confidence limits for λ are given by $l \pm z\sigma_l$, where z is the value of the standard normal variable Z such that $P(-z \leqslant Z \leqslant z) = C/100$. We have already seen that $C = 95$, $z = 1{\cdot}96$ and for $C = 99$, $z = 2{\cdot}576$.

For large samples drawn from a normal population it is usually possible to replace σ_l by an estimate s_l to give the approximate $C\%$ confidence limits $l \pm zs_l$.

Thus if p is the estimate of a proportion obtained using a large sample of size n, then approximate 95% confidence limits for the population proportion π are given by:

$$p \pm 1{\cdot}96\sqrt{\left\{\frac{p(1-p)}{n}\right\}}$$

If $\bar{x}_1, \bar{x}_2$ are the means of samples of sizes n_1, n_2 drawn from normal populations with means μ_1, μ_2 respectively, but the same variance σ^2, the 95% confidence limits for the difference between means, $\mu_1 - \mu_2$, are given by:

$$(\bar{x}_1 - \bar{x}_2) \pm 1{\cdot}96\sqrt{\left(\frac{\sigma^2}{n_1} + \frac{\sigma^2}{n_2}\right)} \quad \text{i.e.} \quad \bar{x}_1 - \bar{x}_2 \pm 1{\cdot}96\sigma\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$

If s_1^2, s_2^2 are unbiased estimates of σ^2 obtained from large samples of sizes n_1, n_2, then approximate 95% confidence limits are given by:

$$\bar{x}_1 - \bar{x}_2 \pm 1{\cdot}96s\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}, \quad \text{where} \quad s^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}.$$

[This two-sample estimate of variance was introduced in §16.2.]

Exercise 16.3

1. Over a long period it is found that the mass of soap powder in certain packets is normally distributed with standard deviation 15 g. Find 95% confidence limits for the mean of the distribution given that (a) the mean of a random sample of size 25 is 842 g, (b) the mean of a random sample of size 100 is 843 g.

2. A set of 50 observations $x_1, x_2, \ldots, x_{50}$ is a random sample from a normal distribution with mean μ. Given that $\sum x = 163$ and $\sum x^2 = 548$ find (a) approximate 95% confidence limits for μ, (b) approximate 99% confidence limits for μ.

3. The masses of a certain type of sausage are normally distributed with standard deviation 4 g. The masses of packs containing 8 sausages have a distribution with mean μ. Find 99% confidence limits for μ, given that (a) the mass of one pack selected at random is 468 g, (b) the total mass of a random sample of 10 packs is 4605 g.

4. In a certain survey 26 shoppers out of a random sample of 200 stated that they preferred brown bread to white. If π is the proportion of all shoppers having a

preference for brown bread find (a) approximate 95% confidence limits for π, (b) approximate 98% confidence limits for π.

5. The heights of young trees in a large plantation have an approximately normal distribution with mean μ and the proportion of trees damaged by deer is π. A random sample of size 300 is selected and the height, x m, of each tree recorded. It is found that $\sum x = 1185$ and $\sum x^2 = 5165$. The number of damaged trees in the sample is 28. Find (a) 95% confidence limits for μ, (b) 98% confidence limits for μ, (c) 96% confidence limits for π, (d) 99% confidence limits for π.

6. It is known that 78 of the 90 eleven year old children entering a certain school have already had chickenpox. Treating these children as a random sample of all eleven year old children in a certain city, obtain 99% confidence limits for the proportion who have had chickenpox. Find also the minimum sample size that would be needed to give a 95% confidence interval of width less than 0·1.

7. A random sample of size n is to be drawn from a normal population with unknown mean μ and known standard deviation 1·8. Find the smallest value of n which will give (a) a 90% confidence interval for μ for width less than 1, (b) a 95% confidence interval for μ of width less than 1.

8. Machines X and Y fill cartons with orange drink. The amounts of orange drink in 10 randomly selected cartons filled by X were measured in ml and recorded as follows: 223, 247, 229, 252, 243, 254, 238, 241, 232, 251. The amounts in 8 randomly selected cartons filled by Y were as follows: 226, 248, 236, 257, 242, 238, 253, 244. Given that both machines dispense quantities of orange drink normally distributed with a standard deviation of 10 ml, find (a) 90% confidence limits for the mean amount in a carton filled by X, (b) 95% confidence limits for the mean amount in a carton filled by Y, (c) 99% confidence limits for the difference, $\mu_Y - \mu_X$, between the mean amounts dispensed by the two machines.

9. A fruit farmer is comparing two methods of cultivation. Using method I, 60 fruit bushes yield crops with mean mass 0·81 kg and variance 0·042 kg^2. Using method II, 80 fruit bushes yield crops with mean mass 0·89 kg and variance 0·036 kg^2. Find 99% confidence limits for the mean mass of fruit per bush (a) for method I, (b) for method II. Treating both crops as random samples from normal populations with common variance σ^2, find (c) a two-sample unbiased estimate of σ^2, (d) 95% confidence limits for the difference, $\mu_2 - \mu_1$, between the mean crops for methods II and I respectively.

10. In a sample of 400 shops taken in 1972, it was discovered that 136 of them sold carpets at below the list prices which had been recommended by manufacturers (i) Estimate the percentage of all carpet selling shops selling below list price. (ii) Calculate the 95% confidence limits for this estimate, and explain briefly what these mean. (iii) What size sample would have to be taken in order to estimate the percentage to within $\pm 2\%$? (SU)

11. The table below gives the distribution of the age in years at marriage of 200 men.

Age (centre of interval)	17·5	22·5	27·5	32·5	37·5	42·5	47·5	52·5	57·5	62·5
Frequency	31	73	48	21	12	6	3	2	1	3

Calculate estimates of the mean and standard deviation of these ages. If these ages may be assumed to be a random sample from a large population, obtain 95% confidence limits for the population mean, and explain carefully what your limits signify. (C)

12. Two random samples, each consisting of 50 Bramley apples, are selected from a very large collection of these apples. It is known that the standard deviation of the mass of a Bramley apple is 15 g. Give a symmetrical 95% probability interval for X, where X is the mass of the first sample minus that of the second sample, stating your assumptions. (C)

16.4 Introduction to hypothesis testing

In this section we consider an elementary way of measuring the significance of statistical evidence. The principle of the method is that if we have only a small number of observations, then it is easier to reject a theory than to support one. For example, one black Manx cat is sufficient evidence to reject the theory that all cats have tails, but cannot be considered strong evidence that all cats are black. Thus, to test the significance of data, a theory is formulated and the evidence used to decide whether or not the theory can be rejected. This theory is called the *null hypothesis*, often denoted by H_0. We now look at the test procedure in more detail.

Illustration A A new drug is to be used to treat 10 patients suffering from a serious illness. It is known that with established forms of treatment the probability of recovery is 0·3.

Let the null hypothesis, H_0, be that using the new drug the probability of recovery is still 0·3. The research team hopes to reject this theory in favour of the *alternative hypothesis*, H_1, that using the new treatment the probability of a cure is greater than 0·3.

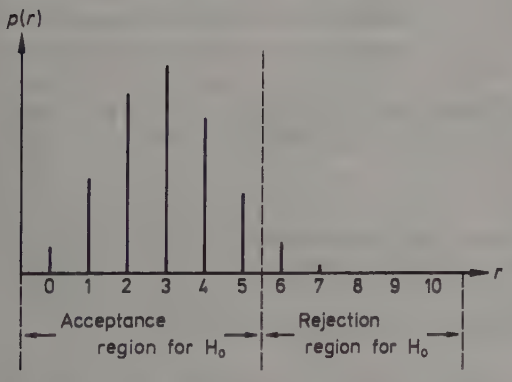
Assuming that H_0 is true, the random variable R, representing the number of patients who recover, will have the binomial distribution $B(10, 0{\cdot}3)$. The values of $p(r)$ for $r = 0, 1, \ldots, 10$ are given correct to 3 d.p. in the table below.

r	0	1	2	3	4	5	6	7	8	9	10
$p(r)$	0·028	0·121	0·233	0·267	0·200	0·103	0·037	0·009	0·001	0·000	0·000

The problem now is to determine which values of the *test statistic* R can be regarded as significant evidence that H_0 should be rejected. Even the recovery of all 10 patients, although extremely unlikely, could still be consistent with the null

hypothesis. To decide where to draw the line between acceptance and rejection of H_0 we must consider the probabilities $P(R > r)$. In practice events with a probability of 0·05 or less are regarded as *unlikely*. Thus when testing a hypothesis such events are said to provide *significant* evidence against the null hypothesis H_0, or more formally, to provide sufficient evidence *at the 5% level* to reject H_0. The "jargon" of significance tests can be summarised as follows:

Probability	Event	Evidence	Significance
0·05 or less	unlikely	significant	5% level
0·001 or less	very unlikely	strong or highly significant	1% level
0·0001 or less	extremely unlikely	very strong or very highly significant	0·1% level



In our example the research team decide to look for evidence significant at the 5% level.

Since $P(R > 5) \approx 0{\cdot}047$
and $P(R > 4) \approx 0{\cdot}150$,

the appropriate *decision rule* is: "reject H_0 if $R > 5$".

Thus, given that H_0 is true, the probability of rejecting it in favour of H_1 is 0·047 or 4·7%. This quantity, the probability of wrongly rejecting the null hypothesis, is called the *significance level* of the test.

Hence we have constructed a test with a significance level of 4·7%, in which 6 or more recoveries will be regarded as evidence that the new treatment is more effective than the old.

Illustration B A production line is designed to produce bicycle wheels with mean diameter 42 cm. It is known that the diameters are normally distributed with standard deviation 1·5 cm. In order to test the hypothesis that the mean diameter is indeed 42 cm, a random sample of 25 wheels is selected. The sample mean is found to be 41·5 cm.

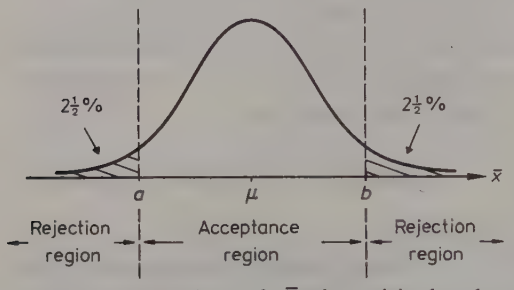
Using standard notation, we suggest a suitable test:

$$H_0: \mu = 42. \quad H_1: \mu \neq 42. \quad \text{Significance level: } 5\%.$$

Test statistic: $\bar{X}$, the mean of a sample of size 25.

It was shown in §16.1 that if a random sample of size n is drawn from the distribution $N(\mu, \sigma^2)$, then the sample mean has distribution $N(\mu, \sigma^2/n)$. Hence, on the assumption that H_0 is true, $\bar{X}$ is normally distributed with mean 42 and standard deviation 0·3.

Since we can reject H_0 if there is significant evidence that $\mu > 42$ or that $\mu < 42$, we require a test with a rejection region that covers *both* ends of the distribution of $\bar{X}$. This type of test is called a *two-tailed test.* (In our earlier example we used a *one-tailed test* which involved only *one* end of the distribution of the test statistic R.) For a test with a 5% significance level there is a 5% probability that the observed value of the test statistic will lie in the rejection region. In a two-tailed test the probability of rejection is divided equally between the two "tails".



In our present example we now need to find critical values a, b, which lie at the boundaries of the rejection region for H_0.

Using the 2½% value given in Table 2, we find that the appropriate critical values for the distribution $N(0, 1)$ would be $\pm 1{\cdot}96$. Hence for the distribution of $\bar{X}$ the critical values are $42 \pm 1{\cdot}96 \times 0{\cdot}3$ i.e. 41·412 and 42·588. It follows that for this test a suitable decision rule is:

"reject H_0 if $\bar{X} < 41{\cdot}4$ or if $\bar{X} > 42{\cdot}6$".

Since the observed value of $\bar{X}$ is 41·5 we do not have sufficient evidence at the 5% level to reject the hypothesis that the mean diameter of a bicycle wheel is 42 cm.

[Note that because we have rounded the critical values the true significance level of this test is a little less than 5%.]

We now summarise the steps that must be taken when conducting a statistical test of this kind.

(1) Formulate a null hypothesis H_0.
(2) Formulate an alternative hypothesis H_1 and hence decide whether the test is to be one-tailed or two-tailed.
(3) Choose the significance level of the test.
(4) Choose a suitable test statistic and describe its sampling distribution on the assumption that H_0 is true.
(5) Formulate a decision rule either by defining a suitable rejection region or by obtaining the critical value(s) of the test statistic.
(6) Carry out the test.

Example 1 A die is rolled 150 times and 35 sixes are obtained. Test, at the the 1% level, the hypothesis that the die is unbiased.

Let p be the probability of obtaining a six in one roll of a die.
$H_0: p = 1/6$. $H_1: p \neq 1/6$. Two-tailed test.

Significance level: 1%.
Test statistic: R, the number of sixes obtained in 150 rolls of the die.
Assuming that H_0 is true, R has a binomial distribution with parameters $n = 150$, $p = 1/6$. Since n is large, this can be approximated by the normal distribution with mean $np = 25$ and variance $np(1 - p) = 125/6$.
In a two-tailed test with 1% significance level the critical values for the distribution $N(0, 1)$ are $\pm 2{\cdot}576$.
Hence the critical values of R are $25 \pm 2{\cdot}576 \times \sqrt{(125/6)}$ i.e. 13 and 37 (to the nearest integer).
Decision rule: reject H_0 if the number of sixes is either less than 13 or greater than 37.
Since 35 sixes were obtained, we do not have significant evidence (at the 1% level) that the die is biased.

Example 2 Two schools entered candidates for the same public examination. The marks obtained by 80 candidates in school A had mean 59·7 and standard deviation 12. The marks obtained by 60 candidates in school B had mean 56·5 and standard deviation 15. Regarding the sets of marks as large random samples from normal populations with the same variance σ^2, determine whether the performance of school A was significantly better than that of school B.

Let μ_1, μ_2 be the population means for marks obtained by students from schools A and B respectively.
$H_0: \mu_1 = \mu_2$. $H_1: \mu_1 > \mu_2$. . One-tailed test.
Significance level: 5%.
Test statistic: $D = \bar{X}_1 - \bar{X}_2$, where $\bar{X}_1$ is the mean mark of 80 candidates from school A and $\bar{X}_2$ is the mean mark of 60 candidates from school B.
The sampling distributions of $\bar{X}_1$ and $\bar{X}_2$ are $N(\mu_1, \sigma^2/80)$ and $N(\mu_2, \sigma^2/60)$.
However, since the value of σ^2 is unknown we will have to use estimated variances based on the two-sample unbiased estimate $s^2 = \dfrac{80 \times 12^2 + 60 \times 15^2}{80 + 60 - 2}$.
Thus, on the assumption that H_0 is true, D is normally distributed with mean 0 and estimated variance $(s^2/80 + s^2/60)$
i.e. $7s^2/240$.

[Note that for large samples the statistic $D/\sqrt{(7s^2/240)}$ has an approximately normal distribution. This means that we can use values obtained from normal distribution tables to calculate the critical value of D.]

Using Table 2, the 5% value in the distribution $N(0, 1)$ is 1·645.
Hence the critical value of D is given approximately by

$$1{\cdot}645 \times \sqrt{\left(\frac{7s^2}{240}\right)} = 1{\cdot}645 \times \sqrt{\left(\frac{7(80 \times 12^2 + 60 \times 15^2)}{240 \times 138}\right)} \approx 3{\cdot}78.$$

Decision rule: reject H_0 if the difference $\bar{X}_1 - \bar{X}_2$ exceeds 3·78.
The observed value of $D = 59{\cdot}7 - 56{\cdot}5 = 3{\cdot}2$.
Thus we do not have sufficient evidence at the 5% level to suggest that the performance of school A was significantly better than that of school B.

Finally, we consider briefly the two types of error which may arise when testing a hypothesis.

A *Type I error* is the rejection of the null hypothesis when it is true. Since the probability of a Type I error is the probability of wrongly rejecting the null hypothesis, it must always be equal to the significance level of the test. For instance in Illustration A the probability of a Type I error was 0·047 or 4·7%.

A *Type II error* is the acceptance of a null hypothesis which is false. The probability of a Type II error is the probability of accepting the null hypothesis given that the alternative hypothesis is true. The value of this probability depends on the form of the alternative hypothesis and it is difficult to estimate for the tests discussed in this section.

Exercise 16.4

1. Anne thinks that the probability of growing a healthy plant from an avocado pear stone is $\frac{1}{3}$. Barbara has succeeded with 4 stones out of 5. Assuming that Anne is right, find the probability of succeeding with 4 stones or more. Hence decide whether Barbara's success provides sufficient evidence at the 5% level to suggest that the probability of growing a healthy plant is (a) not equal to $\frac{1}{3}$, (b) greater than $\frac{1}{3}$.

2. A driving school claims that 90% of its pupils pass the driving test at the first or second attempt. This hypothesis is to be tested against the alternative hypothesis that the success rate is less than 90%. Given that a random sample of 8 pupils will be taken, devise tests with significance levels of approximately (a) 5%, (b) 1%, (c) 0·1%. In each case state the decision rule and calculate the true significance level of the test correct to 2 d.p.

3. A cricketer estimates that the probability p that he will score more than 30 runs in an innings is 5/12. He intends to test this theory against the alternative hypothesis that $p \neq 5/12$ by noting the number of runs he scores in his next 7 innings. Obtain a suitable decision rule on the assumption that the significance level of the test is approximately 5%. Calculate the probability of a Type I error giving your answer correct to 2 s.f.

4. Just before an election the Conservative party claims that 60% of the voters support their candidate. In an opinion poll 281 people out of a random sample of 500 state that they will vote for the Conservative candidate. Decide whether there is sufficient evidence at the 5% level to suggest that less than 60% of the electorate will vote Conservative.

5. A random sample of size 80 is drawn from a normal distribution with unknown mean and variance 16. The sample mean is found to be 63·1. Does this result provide (a) significant evidence at the 1% level that the population mean is not equal to 62, (b) significant evidence at the 3% level that the population mean is less than 64.

6. The mean height of a large population of women is 1·65 m with a standard deviation of 0·07 m. Random samples of 100 are taken from this population.

Describe the distribution of the means of these samples, giving the mean and the standard deviation of this distribution. A particular group of 100 women has a mean height of 1·68 m. Should this be considered significantly different from the expected mean of a random sample? Give reasons for your answer. (AEB 1980)

7. A machine is designed to produce articles of mass 2·4 kg. A random sample of size 120 is found to have mean 2·52 kg and variance 0·8 kg^2. Treating this as a large sample from a normal population, decide whether the data suggests that the population mean is greater than 2·4 kg.

8. A keen gardener wishes to compare two types of seed potato. He finds that 50 plants of variety *C* (Champion Chipper) yield crops of potatoes with a mean mass of 1·59 kg and 60 plants of variety *F* (Fancy Fryer) have yields with a mean mass of 1·71 kg. Given that the population variances for varieties *C* and *F* are 0·060 kg^2 and 0·084 kg^2 respectively, test the hypothesis that there is no difference between their mean yields (a) at the 5% level, (b) at the 1% level.

9. In an experiment to investigate extra-sensory perception one card is drawn at random from an ordinary pack of 52 by the "sender"; he then concentrates on the suit of the card drawn, trying to communicate the name of the suit to a second person, the "receiver", who is in another room. The card is returned to the pack and the pack is shuffled after each trial. In an extended series of trials the receiver correctly identifies the suit of the card in 291 trials out of 1000. Explain carefully, with supporting calculations, whether you regard this result as contradicting the supposition that results are obtained entirely by guesswork. (C)

10. (a) After a survey a market research company asserted that 75% of T.V. viewers watched a certain programme. Another company interviewed 75 viewers and found that 51 watched the programme and 24 did not. Does this provide evidence at the 5% level of significance that the first company's figure of 75% was incorrect?

(b)

Tree	*A*	*B*
Sample size	60	80
Mean	11·4	10·7
S.D.	2·6	3·1

Samples of leaves were collected from two oak trees *A* and *B*. The number of galls was counted on each leaf and the mean and standard deviation of the number of galls per leaf was calculated with the given results. Assuming normal distributions, do the data provide evidence at the 5% significance level of different population means for the two trees? (SU)

11.

	number	$\sum x$	$\sum x^2$
males	120	198	327
females	160	248	385

From a large population of students, 120 males and 160 females are chosen at random.

Their heights x in metres are summarised in the given table. The males and females may be treated as random samples from two independent populations. (a) Find the sample means and variances. (b) Assuming that in both populations the heights are normally distributed with these means and variances, find the probability that a randomly-chosen female will be taller than a randomly-chosen male. (c) Assuming only that height is distributed normally in both populations, test the hypothesis that the mean height of the population of male students exceeds the mean height of the population of female students by less than 0·08 metres. (O&C)

12. A large number of tomato plants are grown under controlled conditions. Half of the plants, chosen at random, are treated with a new fertilizer, and the other half of the plants are treated with a standard fertilizer. Random samples of 100 plants are selected from each half, and records are kept of the total crop mass of each plant. For those treated with the new fertilizer, the crop masses (in suitable units) are summarised by the figures $\sum x = 1030{\cdot}0$, $\sum x^2 = 11\,045{\cdot}59$. Obtain an unbiased estimate of the population variance, and, treating the sample as a large sample from a normal distribution, obtain a symmetric 96% confidence interval for the mean crop mass.

The corresponding figures for those plants treated with the standard fertilizer are $\sum y = 990{\cdot}0$, $\sum y^2 = 10079{\cdot}19$. Treating the sample as a large sample from a normal distribution, and assuming that the population variances of both distributions are equal, obtain a two-sample pooled estimate of the common population variance. Assuming that it is impossible for the new fertilizer to be less efficacious than the old fertilizer and assuming that both distributions are normal, test whether the results provide significant evidence (at the 3% level) that the new fertilizer is associated with a greater mean crop mass, stating clearly your null and alternative hypothesis. (C)

13. A large group of sunflowers is growing in the shady side of a garden. A random sample of 36 of these sunflowers is measured. The sample mean height is found to be 2·86 m, and the sample standard deviation is found to be 0·60 m. Treating the sample as a large sample and assuming the heights to be normally distributed, give a symmetric 99% confidence interval for the mean height of the sunflowers in the shady side of the garden.

A second group of sunflowers is growing in the sunny side of the garden. A random sample of 26 of these sunflowers is measured. The sample mean height is found to be 3·29 m and the sample standard deviation is found to be 0·90 m. Treating the samples as large samples from normal distributions having the same variance but possibly different means, obtain a pooled estimate of the variance, and test whether the results provide significant evidence (at the 5% level) that the sunny-side sunflowers grow taller, on average, than the shady-side sunflowers. (C)

14. According to some recent accident statistics, a random sample of 800 car drivers injured in road accidents comprised 250 who were wearing seat belts and 550 who were not wearing seat belts. Determine a symmetric 99% confidence interval for the proportion of injured drivers wearing seat belts.

The injuries of the car drivers were described as either slight or serious. Of those wearing seat belts, 50 were seriously injured; of those not wearing seat belts, 150

were seriously injured. Determine an unbiased estimate of the overall proportion of serious injuries amongst the injured drivers. Test, at the 5% significance level, the hypothesis that the proportion of serious injuries is greater amongst those injured drivers not wearing seat belts than amongst those injured drivers wearing seat belts. (C)

15. Countries A and B contain large numbers of elm trees, many of which are affected by Dutch elm disease. It is found that in country A, out of a random sample of 100 elm trees, 67 are affected by the disease and that in country B, out of a random sample of 150 elm trees, 93 are affected by the disease. Tree experts have a theory that the proportions of affected trees in the two countries are the same, although there is a possibility that, since the disease affected the trees in country A before those in country B, the proportion of trees affected in country A may be greater than the proportion affected in country B. Using the sample results, test, at the 10% significance level, the theory of the experts. For the test that you perform, state clearly the hypotheses under comparison. Assuming that the proportions of affected trees in the two countries are the same, give an approximate symmetric 98% confidence interval for this common proportion. (C)

16. A person claims that he has an almost foolproof system for identifying the suit of a playing card placed face downwards. To test whether the person is merely guessing he is asked to identify the suits of 4 playing cards placed face downwards. It is agreed to accept his claim only if he correctly identifies at least 3 of the 4 cards. Viewing this as a hypothesis testing problem, write down an appropriate null hypothesis and an appropriate alternative hypothesis. Calculate the significance level of the test and the probability that the person will be regarded as merely guessing when in fact his system has a probability of 0·8 of correctly identifying the suit of a card. (JMB)

17. Two types of seed, X and Y, appear identical but have different probabilities of germination, these being 50% for type X seeds and 60% for type Y seeds. The germination, or otherwise, of each seed may be regarded as being independent of the germination, or otherwise, of all other seeds. A packet contains 100 type X seeds, which are sown. Show, using a normal approximation, that the smallest number n, which is such that $P(n \text{ or more germinate}) < 0{\cdot}05$, is 59. A second packet also contains 100 seeds, which are sown. It is not known whether these seeds (which are all of the same type) are type X or type Y. There are two hypotheses, H_0: the seeds are of type X, H_1: the seeds are of type Y. Construct a test based on n, which is such that $P(\text{Accept } H_1 \mid H_0 \text{ true}) < 0{\cdot}05$. State this test clearly and determine $P(\text{Accept } H_0 \mid H_1 \text{ true})$. (C)

Exercise 16.5 (*miscellaneous*)

1. What is meant by the 'sampling distribution of a statistic'? If random samples of size n are drawn from an unspecified distribution with mean μ and variance σ^2, what is the sampling distribution of the mean as n tends to infinity?

The distribution of the energy provided by certain fuel cells has mean 20 units and standard deviation 2 units. A power pack made up of 64 cells is assumed to have an

output which is equal to the total energy supplied by all the cells in it. Find the probability that such a pack will provide 1300 units of energy. The manufacturers wish to quote an energy output for such packs that will be satisfied by 99% of the packs. Determine what energy level should be quoted.

2. (a) Explain briefly, in your own words, the idea of the Central Limit Theorem.
Samples of 100 pebbles are taken from a large number of pebbles on a certain beach. The mean mass of the pebbles overall is known to be 283 g with a standard deviation of 23 g. Describe the distribution of the means of such samples, giving its mean and standard deviation. What effect, if any, is the type of distribution of the pebbles on the beach likely to have on the distribution of means?

(b) How does the variance of the means of small samples of n elements compare with the variance of the population from which it is taken. A random sample of 10 lengths of straw was picked from a field. The lengths, in metres, were 0·8, 0·8, 0·8, 0·9, 1, 1, 1·1, 1·1, 1·2, 1·3. Give an estimate of the mean length of straw in the field and the standard error of this estimate. (AEB 1979)

3. In a certain examination with a very large entry, the percentage marks obtained by the male candidates were found to follow a normal distribution with a mean of 54 and a standard deviation of 16. Let $\bar{X}$ denote the mean of the percentage marks scored by a random sample of 4 male candidates. What is the sampling distribution of $\bar{X}$? Calculate the probability that $\bar{X}$ will exceed 70 and the value c such that there is a probability of 0·95 that $\bar{X}$ will be within c marks of the mean mark of 54. In the same examination the percentage marks obtained by the female candidates were found to follow a normal distribution with a mean of 59 and a standard deviation of 20. Let $\bar{Y}$ denote the mean of the percentage marks scored by a random sample of 5 female candidates. What is the sampling distribution of $\bar{Y} - \bar{X}$? Calculate the probability that the value of $\bar{Y}$ will be greater than the value of $\bar{X}$. (W)

4. A carton contains 10 boxes of matches. It was found that 3 of the boxes contained 46 matches, 6 contained 47 matches, and the remaining box contained 48 matches. Calculate the mean number of matches per box in the carton. Suppose a random sample of 3 boxes are chosen without replacement from the carton. Let m denote the mean number of matches per box in the sample. By considering the contents of all possible samples of 3 boxes and their corresponding probabilities, obtain the sampling probability distribution of m. Verify that m is an unbiased estimate of the mean number of matches per box in the carton and calculate its standard error. [The arithmetic can be reduced substantially by operating on a suitably chosen linear function of m and using properties of expectation and variance.] (W)

5. Show that, if $X_1, \ldots, X_n$ are n independent random variables with mean μ and variance σ^2, then $\sum X_i/n$ has mean μ and variance σ^2/n. The diameters of 25 steel rods are measured and found to have a mean of 0·980 cm and a standard deviation of 0·015 cm. Assuming this is a sample from a normal distribution with the same variance, find 99% confidence limits for the population mean. (O&C)

6.

Time	Frequency
$0 < t \leqslant 50$	31
$50 < t \leqslant 100$	24
$100 < t \leqslant 150$	20
$150 < t \leqslant 200$	13
$200 < t \leqslant 300$	11
$300 < t \leqslant 500$	1
	100

On 1 January 1979, 100 new "Eternity" light-bulbs were installed in a certain building, together with a device which records how long each light-bulb is in use. By 1 March all 100 bulbs had failed, and the recorded life-times, t, (in hours of use since 1 January) are summarised in the given table. Obtain estimates of the sample mean and the sample standard deviation for this set of data. Assuming that the bulbs constituted a random sample of "Eternity" light-bulbs, obtain a symmetric 99% confidence interval for the mean life-time of "Eternity" light-bulbs. (C)

7. If $\bar{X}$ is the mean of a random sample of size n_1 from a population distribution having mean μ_1 and variance ${\sigma_1}^2$, and n_1 is very large, what can you say about the sampling distribution of $\bar{X}$? If $\bar{Y}$ is the mean of a random sample of size n_2 from another population distribution having mean μ_2 and variance ${\sigma_2}^2$, and n_2 is also very large, what can you say about the sampling distribution of $\bar{Y} - \bar{X}$?

The heights, measured in metres, of 100 women drawn at random from a certain tribe had a sum of 160 and the sum of their squares was 265. Use this information to construct a 90% confidence interval for the mean height of women of this tribe. The heights, measured in metres, of 200 men drawn at random from the same tribe, had a sum of 360 and the sum of their squares was 712. Use the information contained in the two samples to construct a 95% confidence interval for the amount by which the average height of the men in the tribe exceeds that of the women. (W)

8. Let p denote the probability of obtaining a head when a certain coin is tossed. (i) If $p = 0{\cdot}4$, find the probability of obtaining at least 3 heads in 10 independent tosses of the coin. (ii) If $p = 0{\cdot}6$, find the probability of obtaining exactly 12 heads in 20 independent tosses of the coin. (iii) Write down an appropriate null hypothesis and an appropriate alternative hypothesis for testing whether the coin is unbiased. To carry out this test 20 independent tosses of the coin are made and the number of heads that occurs is observed. Given that 15 heads occurred, carry out the test, assuming a 5% significance level. Write down a statement of the conclusion you draw about the value of p for this coin. (JMB)

9. The masses of loaves from a certain bakery are normally distributed with mean 500 g and standard deviation 20 g. (a) Determine what percentage of the output would fall below 475 g and what percentage would be above 530 g. (b) The bakery produces 1000 loaves daily at a cost of 8 p per loaf and can sell all those above 475 g for 20 p each but is not allowed to sell the rest. Calculate the expected daily profit. (c) A sample of 25 loaves yielded a mean mass of 490 g. Does this provide evidence of of a reduced population mean? Use the 5% level of significance and state whether the test is one-tailed or two. (SU)

10. A man, Mr Knowall, claims to be able to recognize a well known brand, A, of whisky blindfolded. On 8 occasions he is given two glasses in random order, one containing brand A and the other containing brand X. On 7 occasions he names the brands correctly. Would you consider his claim justified? The same experiment is repeated the following day, and this time he names the brand correctly in 6 out of 8 trials. How does this affect your opinion of his claim? (AEB 1977)

11. Experimental data concerning a variable X, which measures the reliability of a certain electric component, is as follows: $\sum x = 1164{\cdot}2$, $\sum x^2 = 13911{\cdot}6$, $n = 100$. Calculate the sample mean and standard deviation from these figures. Explain whether, on the evidence of this sample, you would reject the hypothesis that the mean value of X is 12. Figures collected over a long period have established that the mean and standard deviation of X are 12 and 2 respectively. After a change in the manufacturing process it is expected that the mean will have been *increased*, but it may be assumed that the standard deviation remains equal to 2. A sample of n values of X is taken, with sample mean m; if m is greater than some critical value it will be accepted that the mean has in fact increased, but if m is less than this critical value the increase is not established. State carefully appropriate null and alternative hypotheses for this situation, and find, in terms of n, the critical value for a 1% significance level. (C)

12. The current medical treatment for a particular ailment has a cure rate of 80% (i.e. a patient with this ailment has a probability of 0·8 of being cured by the treatment). (i) Write down the mean and the variance of the number of patients that will be cured per 100 patients treated. (ii) Calculate the probability that, of 4 patients treated, at least 3 will be cured. (iii) A pharmaceutical company claims that it has developed a new treatment for the ailment which has a higher cure rate than the current treatment. To test this claim the new treatment is to be given to 20 patients. It is decided to accept the new treatment as being better than the current treatment only if at least 19 of the 20 patients are cured. Write down an appropriate null hypothesis and an appropriate alternative hypothesis for this test. Calculate the significance level of the test. Also calculate the probability that the new treatment will not be accepted as being better than the current treatment when the new treatment has a cure rate of 90%. (JMB)

13. The lengths (in millimetres) of nine screws selected at random from a large consignment are found to be 7·99, 8·01, 8·00, 8·02, 8·03, 7·99, 8·00, 8·01, 8·01. Calculate unbiased estimates of the population mean and variance. Assuming a normal distribution with variance 0·0001, test, at the 5% level, the hypothesis that the population mean is 8·00 against the alternative hypothesis that the population mean is not 8·00. From a second large consignment, sixteen screws are selected at random, and their mean length (in millimetres) is found to be 7·992. Assuming a normal distribution with variance 0·0001, test, at the 5% level, the hypothesis that this population has the same mean as the first population, against the alternative hypothesis that this population has a smaller mean than the first population. (C)

14. (a) A sample of 200 similar packets of breakfast cereal was examined, and the mass of the contents in each packet was noted. The following results were

obtained: sample mean = 341·2 g. sample standard deviation = 0·92 g. Calculate a 95% confidence interval for the mean mass of contents in all packets of this type, stating any assumptions that you make.

(b) Samples of packets of cereal filled by two different machines were examined, and the following results about the mass (x grams) of the contents were obtained:

	Sample size	$\bar{x}$	$\sum(x-\bar{x})^2$
Machine 1	20	340·9	16·82
Machine 2	10	341·4	8·63

Test whether the mean masses of contents in packets filled by the two machines are significantly different at the 5% level. Assume that the masses of contents in packets filled by each machine are normally distributed, and that the two normal distributions have equal variances. (C)

15. (a) In one county in England, a random sample of 225 12-year-old boys and 250 12-year-old girls was given an arithmetic test. The average mark for the boys was 57 with a standard deviation of 12, whilst the average for the girls was 60 with a standard deviation of 15. Assuming that the distributions are normal, does this provide evidence at the 2% level that 12-year-old girls are superior to 12-year-old boys at arithmetic?

(b) An IQ test which had been standardised giving a mean of 100 and a standard deviation of 12 was given to a random sample of 50 children in one area. The average mark obtained was 105. Does this provide evidence, at the 5% level, that children from this area are generally more intelligent than other children? (SU)

16. An organisation interviews a randomly chosen sample of 1000 adults from the population of the United Kingdom, and 517 of those interviewed claim to support the Conservative party. A second organisation independently interviews a random sample of 2000 adults, of whom 983 claim to support the Conservative party. (i) Verify that the results of the two organisations do not differ significantly at the 5% level. (ii) Obtain a symmetric 99% confidence interval (based on the combined results) for the proportion of the population who claim to support the Conservative party. (C)

17. Determine the probability that a six-digit number (leading zeros being relevant) chosen at random from the range 000000 to 999999 inclusive has no two successive digits equal to one another. An electronic device is supposed to generate six-digit numbers (leading zeros being relevant) in the range 000000 to 999999 inclusive, so that each of the one million possibilities is equally probable. However, it is believed that the machine may be biased so that it under-represents numbers having at least one pair of successive digits equal to one another. A random selection of twenty numbers generated by the device is examined and it is found that precisely r of these numbers contain at least one pair of successive digits equal to one another. (i) Given that $r = 6$, does this provide significant evidence, at the 10% level, that the device is biased in the manner described? (ii) How small would r have to be in order that

there was significant evidence, at the 1% level, that the device was biased in the manner described? (C)

18. A machine produces nails whose lengths are normally distributed with mean μ cm and standard deviation 0·1 cm. When the machine is working correctly $\mu = 3{\cdot}0$, but occasionally the machine goes wrong, in which case $\mu = 3{\cdot}05$, the standard deviation remaining 0·1 cm. In order to decide whether the machine is working correctly, the lengths of the nails in a sample of n nails are measured, and the sample mean $\bar{x}$ is found. If the value of $\bar{x}$ exceeds a predetermined value v then it is concluded that the machine has gone wrong, otherwise the machine is presumed to be working correctly. It is required that there should be a probability of no more than 5% of presuming that the machine has gone wrong when in fact it is working correctly, and that there should also be a probability of no more than 10% of presuming the machine to be working correctly when it has gone wrong. Determine appropriate values of n and v, if n is to be as small as possible. (C)

19. The mass, in grams, of marmalade in a randomly chosen jar from a large batch filled by a machine is normally distributed. The mean, μ, for a batch can be adjusted but the standard deviation is always 4. Each batch produced is tested by determining the mean mass, $\bar{x}$, of the marmalade in a random sample of n jars. Under the null hypothesis $\mu = 454$, a batch is rejected if $\bar{x}$ is such that there is significant evidence, at the 1% level, that the value of μ is less than 454. The sample size, n, is chosen so that, in the case $\mu = 452$, the probability of rejecting the batch is at least 0·95. Find the smallest value of n to achieve this, and the corresponding critical value of $\bar{x}$ for rejection. (C)

20. Two hypotheses concerning the probability density function of a random variable are

$$H_0: f(x) = \begin{cases} 1 & 1 < x < 2, \\ 0 & \text{otherwise;} \end{cases} \qquad H_1: f(x) = \begin{cases} 3x^2/7 & 1 < x < 2, \\ 0 & \text{otherwise.} \end{cases}$$

Give a sketch of the probability density function for each case.
The following test procedure is decided upon:

A single observation of X is made and if X is less than a particular value a, where $1 < a < 2$, then H_0 is accepted; otherwise H_1 is accepted.

Find a such that, when H_0 is true, the test procedure leads, with probability 0·1, to the acceptance of H_1. With this value of a, find the probability that, when H_1 is true, the test procedure leads to the acceptance of H_0. (C)

17 Correlation and regression

17.1 Bivariate distributions

In most of our earlier work on probability and statistics we have considered distributions involving values of one variable. However, it is often possible to measure two or more characteristics of the members of a sample. For instance, we may record both the height and the weight of each child admitted to a certain school. The pairs of observations obtained form a *bivariate distribution.* This chapter is devoted to the study of such distributions.

The main purpose of collecting bivariate data is to investigate any relationship between the variables. A possible first step in this process is to represent the data graphically. This is done by plotting each pair of observations as a single point on a graph called a *scatter diagram.*

Illustration A Ten students seeking careers advice were given two aptitude tests. Their test scores and the corresponding scatter diagram were as follows:

Student	Verbal test	Spatial test
1	69	52
2	80	71
3	57	55
4	52	48
5	85	89
6	66	64
7	49	37
8	76	60
9	63	56
10	73	78

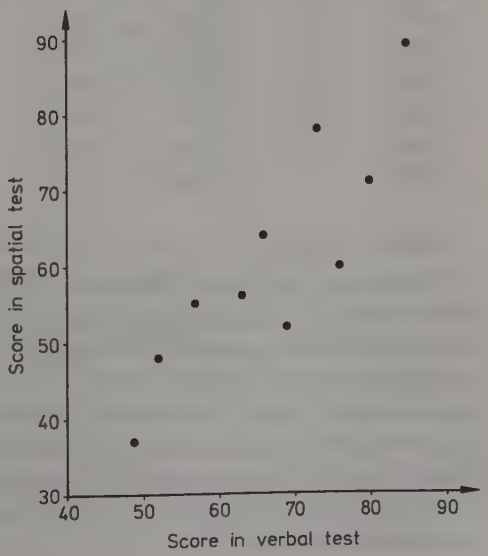


Illustration B The heights and batting averages of the members of a college cricket team were recorded. The resulting data and the scatter diagram are given below.

Player	Height in metres	Batting average
1	1·72	3
2	1·89	33
3	1·76	10
4	1·92	8
5	1·66	28
6	1·94	16
7	1·84	22
8	1·68	9
9	1·77	47
10	1·86	6
11	1·76	16

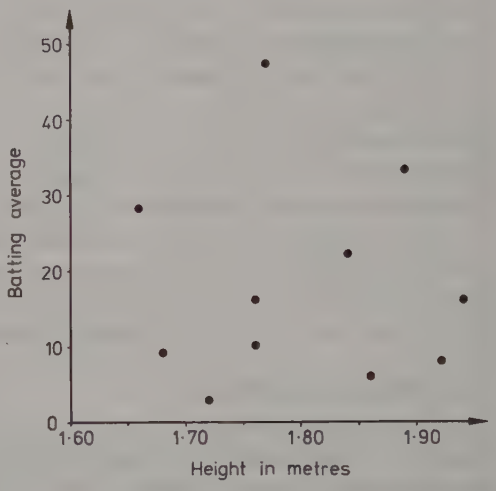
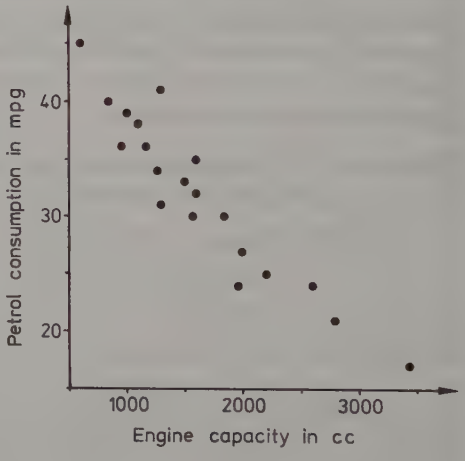


Illustration C Twenty types of car were tested by a motoring magazine. The engine capacity and fuel consumption of each car were recorded as follows:

Capacity in cc	Petrol in mpg	Capacity in cc	Petrol in mpg
1000	39	3440	17
1300	31	950	36
600	45	840	40
2000	27	1100	38
1170	36	1290	41
1600	32	2600	24
1500	33	1980	24
1580	30	1260	34
2800	21	1850	30
1600	35	2200	25

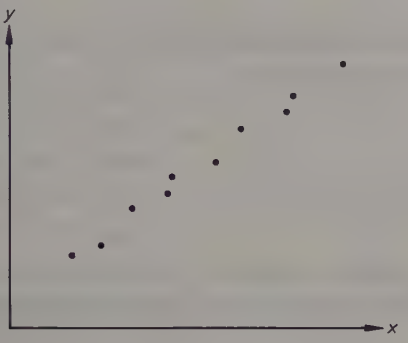


Considering the scatter diagrams obtained in these examples, we see that in Illustration A the points plotted tend to lie in a fairly narrow belt along a straight line with positive gradient. This suggests that high marks in the spatial test are usually associated with high marks in the verbal test and that there is an approximately linear relationship between the variables. In Illustration B the diagram shows little evidence of any association between the variables, whereas in Illustration C the points plotted lie close to a straight line with negative gradient, indicating a linear relationship in which low mileage per gallon is associated with high engine capacity.

Any relationship between the variables in a bivariate distribution is called a *correlation.* The *linear* correlation observed in Illustration A is described as a *positive* or *direct* correlation. Similarly the relationship in Illustration C is a *negative* or *inverse* correlation. In Illustration B the variables appear to be *uncorrelated.*

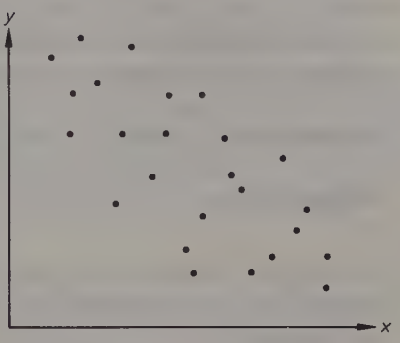
The following diagrams illustrate some further types of relationship which may rise.

(1)



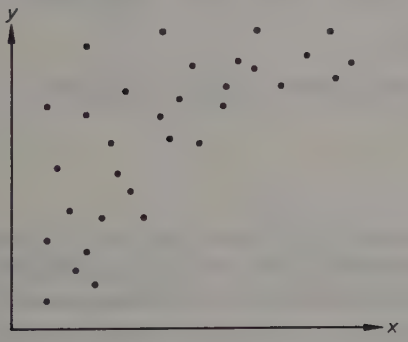
High positive correlation

(2)



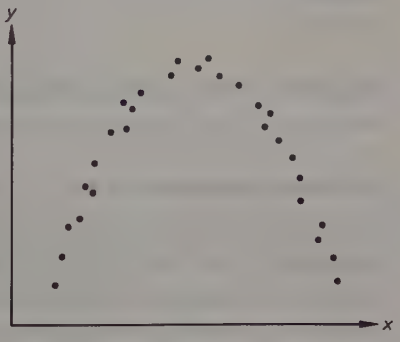
Low negative correlation

(3)



Weak correlation
(possibly non-linear)

(4)



Strong correlation
(possibly quadratic)

Exercise 17.1

In each of the following questions use the given data to plot a scatter diagram. Describe any apparent relationship between the variables.

1.

x	46	61	53	55	32	66	42	31	49	73
y	17	23	19	21	11	26	14	8	18	29

2.

x	14	11	5	12	9	15	8	15	10	6
y	11	9	16	11	12	8	14	4	10	12

3. The given table shows the oil production and oil consumption of various parts of the world in millions of metric tons per annum.

Oil production	68	538	277	23	299	690	390	0	67
Oil consumption	72	697	135	628	43	51	346	200	116

4. The given table shows the face values of the British coins minted in 1953 together with the selling prices quoted by a coin dealer.

Face value (in d.)	$\frac{1}{4}$	$\frac{1}{2}$	1	3	6	12	12	24	30	60
Selling price (in £)	0·40	1·00	1·80	0·80	0·50	0·60	0·90	1·30	1·60	2·00

5. The given table shows the average distance from the sun (in millions of kilometres) of each planet in the solar system together with its equatorial diameter (in kilometres).

Distance from sun	60	110	150	230	780	1430	2870	4500	5900
Diameter	4880	12 100	12 760	6790	142 800	120 000	51 800	49 500	6000

17.2 Linear correlation

In §17.1 we saw that a scatter diagram indicates the general nature of a bivariate distribution. We now consider some numerical measures of the relationship between two variables. When dealing with the distribution of a single variable X we use the deviations of the observed values $x_1, x_2, \ldots$ from the sample mean $\bar{x}$ to obtain useful measures such as the variance and standard deviation. Similarly for a bivariate distribution of variables X and Y the values of the deviations $x - \bar{x}$, $y - \bar{y}$ can be used to obtain measures of correlation.

Let us consider again the data given in Illustration A.

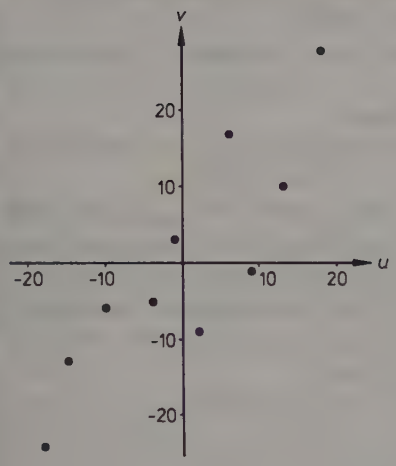
Verbal test score, x	69	80	57	52	85	66	49	76	63	73
Spatial test score, y	52	71	55	48	89	64	37	60	56	78

$$\bar{x} = \tfrac{1}{10}\sum x = 67, \quad \bar{y} = \tfrac{1}{10}\sum y = 61$$

We now calculate the values of $u = x - \bar{x}$, $v = y - \bar{y}$ and plot the corresponding scatter diagram.

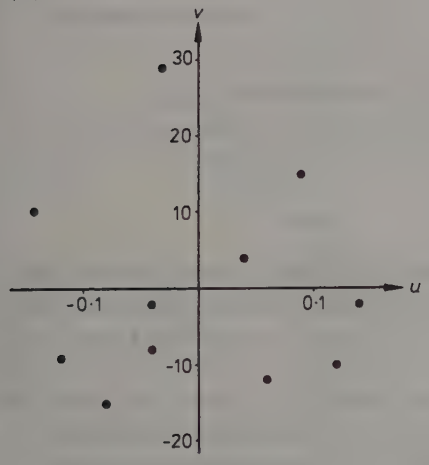
u	2	13	−10	−15	18	−1	−18	9	−4	6
v	−9	10	−6	−13	28	3	−24	−1	−5	17

(A)



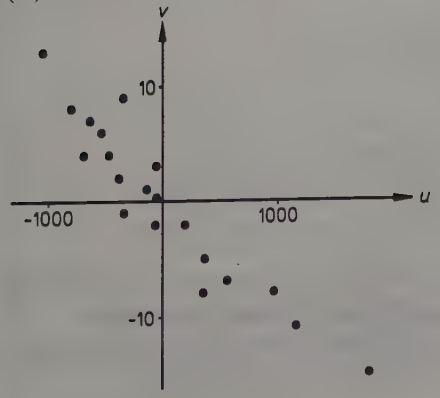
Because of the positive correlation between x and y, most of the points in this new scatter diagram lie in the 1st and 3rd quadrants. For these points the product uv takes positive values, whereas for the remaining points uv takes negative values. Thus it is not surprising to find that $\sum uv$ has the fairly large positive value 1413.

(B)



Performing similar calculations for the data in Illustration B we find that there is no definite pattern in the u, v scatter diagram and that the product uv takes a set of positive and negative values which tend to cancel each other out. In this case $\sum uv = -0{\cdot}28$.

(C)



In illustration C there is a high negative correlation between the variables and nearly all the points in the u, v scatter diagram lie in the 2nd and 4th quadrants. It follows that most of the values of the product uv will be negative and thus we find that $\sum uv$ takes the negative value $-92\,304$.

Considering the results obtained in these examples we see that the value of $\sum uv$, i.e. $\sum(x-\bar{x})(y-\bar{y})$, is influenced by three main factors:

(i) the strength of the linear relationship or linear correlation between the variables,

(ii) the number of pairs of observations in the sample,

(iii) the spread or variation in the values of each variable (indicated here by the scales used on the u- and v-axes.)

We conclude that the sum $\sum(x-\bar{x})(y-\bar{y})$, often denoted by S_{xy}, can be used as a measure of the linear correlation between x and y only when comparing samples of equal size from similar bivariate distributions. A more useful measure is the *covariance* of x and y, which is defined as the mean value of the product $(x-\bar{x})(y-\bar{y})$. More formally, if (x_1, y_1), $(x_2, y_2), \ldots, (x_n, y_n)$ are the observed values of variables X and Y in a sample of size n, then:

$$\text{sample covariance} = \frac{1}{n}S_{xy} = \frac{1}{n}\sum(x-\bar{x})(y-\bar{y})$$

We now have a measure of correlation that can be used to compare samples of different sizes.

	S_{xy}	Covariance (to 3 s.f.)
Illustration A	1413	141
Illustration B	−0·28	−0·0255
Illustration C	−92 304	−4620

However, further consideration of Illustrations A, B and C shows that we still need to allow for differences in "spread". For instance, we must remember that in B the values of $x-\bar{x}$ range from $-0{\cdot}14$ to $+0{\cdot}14$, whereas in C we have $-1033 \leqslant x-\bar{x} \leqslant +1807$.

We can solve this problem by dividing the values of $x-\bar{x}$ by the standard deviation of x and dividing the values of $y-\bar{y}$ by the standard deviation of y. The covariance of these standardised variables is a measure of linear correlation independent of both sample size and spread. This new measure, the *correlation coefficient* r, is defined as follows:

$$r = \frac{1}{n}\sum\left(\frac{x-\bar{x}}{\text{s.d. of } x}\right)\left(\frac{y-\bar{y}}{\text{s.d. of } y}\right)$$

i.e.
$$\text{correlation coefficient } r = \frac{\text{covariance of } x \text{ and } y}{\sqrt{(\text{variance of } x)}\sqrt{(\text{variance of } y)}}$$

(To avoid confusion with other measures of correlation r may also be called the *product moment correlation coefficient* or the *coefficient of linear correlation.*)

We obtain a simpler formula for r by extending the notation introduced to deal with covariance.

$$\text{Covariance} = \frac{1}{n}S_{xy} = \frac{1}{n}\sum(x-\bar{x})(y-\bar{y})$$

$$\text{Variance of } x = \frac{1}{n}S_{xx} = \frac{1}{n}\sum(x-\bar{x})^2$$

$$\text{Variance of } y = \frac{1}{n}S_{yy} = \frac{1}{n}\sum(y-\bar{y})^2$$

$$\therefore \quad r = \frac{\frac{1}{n}S_{xy}}{\sqrt{\left(\frac{1}{n}S_{xx}\right)}\sqrt{\left(\frac{1}{n}S_{yy}\right)}} = \frac{S_{xy}}{\sqrt{(S_{xx}S_{yy})}}$$

We are already familiar with the results:

$$S_{xx} = \sum(x-\bar{x})^2 = \sum x^2 - n\bar{x}^2, \quad S_{yy} = \sum(y-\bar{y})^2 = \sum y^2 - n\bar{y}^2.$$

It can similarly be shown that:

$$S_{xy} = \sum(x-\bar{x})(y-\bar{y}) = \sum xy - n\bar{x}\bar{y}$$

$$\therefore \quad r = \frac{S_{xy}}{\sqrt{(S_{xx}S_{yy})}} = \frac{\sum xy - n\bar{x}\bar{y}}{\sqrt{(\sum x^2 - n\bar{x}^2)}\sqrt{(\sum y^2 - n\bar{y}^2)}}$$

For Illustration A the calculation of r using this formula may be set out as follows:

x	y	xy	x^2	y^2
69	52	3588	4761	2704
80	71	5680	6400	5041
57	55	3135	3249	3025
52	48	2496	2704	2304
85	89	7565	7225	7921
66	64	4224	4356	4096
49	37	1813	2401	1369
76	60	4560	5776	3600
63	56	3528	3969	3136
73	78	5694	5329	6084
670	610	42 283	46 170	39 280

$$n = 10, \quad \bar{x} = \frac{1}{n}\sum x = \tfrac{670}{10} = 67, \quad \bar{y} = \frac{1}{n}\sum y = \tfrac{610}{10} = 61$$

$$\therefore \quad S_{xy} = \sum xy - n\bar{x}\bar{y} = 42\,283 - 10 \times 67 \times 61 = 1413$$
$$S_{xx} = \sum x^2 - n\bar{x}^2 = 46\,170 - 10 \times 67^2 = 1280$$
$$S_{yy} = \sum y^2 - n\bar{y}^2 = 39\,280 - 10 \times 61^2 = 2070$$

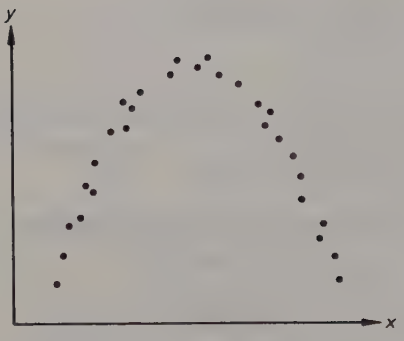
Hence $r = \dfrac{S_{xy}}{\sqrt{(S_{xx}S_{yy})}} = \dfrac{1413}{\sqrt{(1280 \times 2070)}} \approx 0{\cdot}868.$

For Illustrations B and C the approximate values of r turn out to be $-0{\cdot}022$ and $-0{\cdot}939$ respectively. These results tend to confirm that the correlation coefficient is a satisfactory measure of the linear relationship between two variables.

It can be shown that r always lies between -1 and $+1$. (Methods of proof are suggested in Exercise 17.2, questions 12 and 13.)

$r = +1$ means perfect positive correlation.
$r = \ \ 0$ means that there is no linear correlation.
$r = -1$ means perfect negative correlation.

It is difficult to lay down more general rules for the interpretation of r as the significance of any particular value depends on the size and nature of the sample taken.



However, it must be emphasised that r is a measure of *linear* correlation. The given scatter diagram shows that it is possible to find a distribution in which there is a strong relationship between the variables, but a value of r close to zero.

We must also point out that a value of r close to -1 or to $+1$ does not necessarily imply a *causal* relationship between the variables. For instance, suppose that over a period of several years we had selected random samples of 10 year old children and recorded their consumption of frozen food and their levels of tooth decay. We would probably have found a negative correlation between these variables but it would clearly be wrong to assume that eating frozen food prevents tooth decay. Even if the variables measured had been use of "fluoride" toothpaste and levels of tooth decay, it would have been unwise to assume a connection without further evidence.

When working with *bivariate frequency distributions* a modified version of the formula for the correlation coefficient can be used. If a typical pair of observations (x, y) occurs with frequency f then:

$$S_{xx} = \sum fx^2 - n\bar{x}^2, \quad S_{yy} = \sum fy^2 - n\bar{y}^2 \quad \text{and}$$
$$S_{xy} = \sum fxy - n\bar{x}\bar{y}, \quad \text{where} \quad \sum f = n,$$

$$\therefore \quad r = \frac{S_{xy}}{\sqrt{(S_{xx}S_{yy})}} = \frac{\sum fxy - n\bar{x}\bar{y}}{\sqrt{(\sum fx^2 - n\bar{x}^2)}\sqrt{(\sum fy^2 - n\bar{y}^2)}}$$

Finally, we extend the work of Chapter 15 by considering briefly linear correlation between random variables. To set up a probability model of a bivariate experiment it is necessary to construct a joint probability distribution for two suitable random variables X and Y. As shown in §15.6:

$$\text{Var}(X) = E[\{X - E(X)\}^2] = E(X^2) - [E(X)]^2$$
$$\text{Var}(Y) = E[\{Y - E(Y)\}^2] = E(Y^2) - [E(Y)]^2$$

It therefore seems reasonable to define the covariance of X and Y as follows:

$$\text{Cov}(X, Y) = E[\{X - E(X)\}\{Y - E(Y)\}]$$

It can then be shown that:

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y)$$

We now define the correlation coefficient ρ of random variables X and Y by writing:

$$\rho = \frac{\text{Cov}(X, Y)}{\sqrt{\{\text{Var}(X)\text{Var}(Y)\}}}$$

It was further shown in §15.6 that if X and Y are independent random variables then $E(XY) = E(X)E(Y)$. Thus for independent variables $\rho = 0$. However, it is also possible for the value of ρ to be zero when X and Y are *not* independent.

Exercise 17.2

Questions 1 to 5. Find the correlation coefficients for the data given in Exercise 17.1 questions 1 to 5.

6. From 12 pairs of values of x and y the following were calculated:

$$\sum x = 112, \quad \sum y = 94, \quad \sum xy = 784, \quad \sum x^2 = 1116, \quad \sum y^2 = 1020.$$

Find (a) the covariance of x and y, (b) the correlation coefficient.

7.

		x				
		12	13	14	15	16
y	4	0	0	0	1	1
	5	0	1	3	6	1
	6	0	3	8	4	0
	7	0	5	2	0	0
	8	3	1	0	1	0

For the bivariate frequency distribution in the given table calculate (a) the mean and variance of each of x and y, (b) the covariance of x and y, (c) the correlation coefficient.

8.

		Claire's "guess"				
		1	2	3	4	5
Assistant's choice	1	11	4	0	0	0
	2	1	16	8	1	0
	3	0	2	18	6	1
	4	0	0	2	13	5
	5	0	0	0	4	8

Claire Voyant claims that she possesses "extra-sensory perception". A small room is divided by a metal screen, with Claire seated on one side and an experimental assistant on the other. Each is given a set of cards numbered 1 to 5. At each trial of the experiment the assistant selects one of these cards and thinks of that number; Claire than records her "guess" by showing the appropriate card.
100 trials are completed and the results are given in the form of a bivariate table. Calculate the product-moment correlation coefficient for these data. (AEB 1978)

9. A set of pairs of values of variables x and y satisfy the linear relationship $y = mx + c$. Show that (a) if $m > 0$ then $r = +1$, (b) if $m < 0$ then $r = -1$.

10. A sample is drawn from a certain population and three characteristics with values represented by x, y and z are measured. The three coefficients of linear correlation between these variables are $r(x, y)$, $r(y, z)$ and $r(x, z)$. What values are possible for $r(x, z)$ given that (a) $r(x, y) = r(y, z) = +1$, (b) $r(x, y) = r(y, z) = -1$, (c) $r(x, y) = r(y, z) = 0$?

11. On a certain island there are large numbers of each of two clans, the Fatties and the Thinnies. Two random samples of 50 adult males are taken, one sample from each clan. Each of the clansmen is weighed and measured. For each clan, the value of the correlation coefficient between the heights and weights of the clansmen is found to be near $+1$. However, for the combined sample of 100 adult males, the value of the correlation coefficient is found to be near -1. Show, by a sketch of the scatter diagram for the combined sample, how this could arise. (C)

12. Using the definitions of u, v, S_{xx} and S_{yy} given in the text, let $U = u/\sqrt{S_{xx}}$ and $V = v/\sqrt{S_{yy}}$. Show that $\sum(U + V)^2 = 2n(1 + r)$ and hence prove that $r \geqslant -1$. By considering both $\sum(U + V)^2$ and $\sum(U - V)^2$, prove that $-1 \leqslant r \leqslant 1$.

13. Express $\sum(\lambda u + v)^2$ as a quadratic function of λ. Using the fact that this expression is never negative, prove that $(S_{xy})^2 \leqslant S_{xx}S_{yy}$. Deduce that $-1 \leqslant r \leqslant 1$.

Questions 14 to 16. Find the values of $\mathrm{Cov}(X, Y)$ and the correlation coefficient ρ for the joint probability distributions given in Exercise 15.6 questions 9, 10(i) and 10(ii).

17.3 Rank correlation

Let us suppose that two gardening experts, Mrs. Stone and Mr. Turvy are judging 8 entries in a prize cabbage competition at a village fair. They arrange the entries in

order of merit and the positions or *ranks* allotted to the cabbages by each judge are given in the following table.

Cabbage	A	B	C	D	E	F	G	H
Mrs. Stone	3	7	4	5	8	1	2	6
Mr. Turvy	4	7	2	6	5	3	1	8

We can see that the rankings produced by the judges are broadly similar. However, to make a more precise statement about the level of agreement we need a numerical measure of *rank correlation.*

One such measure, *Spearman's*† *rank correlation coefficient* r_S, is simply the correlation coefficient introduced in the previous section calculated using the ranks assigned to sample members as data.

Mrs. Stone's rank, x	Mr. Turvy's rank, y	xy	x^2	y^2
3	4	12	9	16
7	7	49	49	49
4	2	8	16	4
5	6	30	25	36
8	5	40	64	25
1	3	3	1	9
2	1	2	4	1
6	8	48	36	64
36	36	192	204	204

Thus we have $n = 8$ and $\bar{x} = \bar{y} = 4{\cdot}5$

$$\therefore \qquad r_S = \frac{\sum xy - n\bar{x}\bar{y}}{\sqrt{(\sum x^2 - n\bar{x}^2)}\sqrt{(\sum y^2 - n\bar{y}^2)}} \approx 0{\cdot}714$$

Considering the values at the foot of the table we realise that for any ranking of 8 objects with *no tied ranks*, i.e. no equal placings, we will have $\sum x = 36$, $\sum x^2 = 204$. Similarly, for a ranking of n objects we can express $\sum x$ and $\sum x^2$ in terms of n and hence obtain a simpler formula for r_S.

Let us assume that ranks x and y take the integral values $1, 2, \ldots, n$ in some order. Using results obtained in §9.1 we have:

$$\sum x = \sum y = \tfrac{1}{2}n(n+1), \quad \sum x^2 = \sum y^2 = \tfrac{1}{6}n(n+1)(2n+1)$$

$$\therefore \qquad \bar{x} = \bar{y} = \tfrac{1}{2}(n+1)$$

If $d = x - y$, then $\sum d^2 = \sum (x-y)^2 = \sum x^2 + \sum y^2 - 2\sum xy$

$$\therefore \qquad \sum xy = \tfrac{1}{6}n(n+1)(2n+1) - \tfrac{1}{2}\sum d^2$$

†*Spearman, Charles Edward* (1863–1945) English psychologist. He was one of the first to apply correlation techniques to the analysis of data obtained from testing human abilities. He proposed his rank correlation method in 1906.

Hence
$$r_S = \frac{\frac{1}{6}n(n+1)(2n+1) - \frac{1}{2}\sum d^2 - \frac{1}{4}n(n+1)^2}{\frac{1}{6}n(n+1)(2n+1) - \frac{1}{4}n(n+1)^2}$$
$$= 1 - \frac{6\sum d^2}{n(n+1)[2(2n+1) - 3(n+1)]}$$
$$= 1 - \frac{6\sum d^2}{n(n+1)(n-1)}$$

For two rankings of n objects, with no tied ranks, Spearman's rank correlation coefficient is given by
$$r_S = 1 - \frac{6\sum d^2}{n(n^2-1)}$$
where d is the difference $x - y$ between ranks.

Applying this result to the cabbage rankings:

Cabbage	A	B	C	D	E	F	G	H
d	−1	0	2	−1	3	−2	1	−2

Thus $\sum d^2 = 1 + 4 + 1 + 9 + 4 + 1 + 4 = 24$

$\therefore$
$$r_S = 1 - \frac{6 \times 24}{8(64-1)} = 1 - \tfrac{2}{7} \approx 0{\cdot}714$$

We now calculate r_S for Illustration A of §17.1, denoting the ranks of the x and y values by R_x and R_y respectively.

x	y	R_x	R_y	$d = R_x - R_y$	d^2
69	52	5	8	−3	9
80	71	2	3	−1	1
57	55	8	7	1	1
52	48	9	9	0	0
85	89	1	1	0	0
66	64	6	4	2	4
49	37	10	10	0	0
76	60	3	5	−2	4
63	56	7	6	1	1
73	78	4	2	2	4

$n = 10$ and $\sum d^2 = 24$,

$\therefore$
$$r_S = 1 - \frac{6\sum d^2}{n(n^2-1)} = 1 - \frac{6 \times 24}{10 \times 99} \approx 0{\cdot}855$$

Strictly speaking, when there are *tied ranks*, the simplified formula for r_S is not valid and the original formula should be used. However, if the number of ties is small

the "difference" formula can be used with little loss of accuracy provided that we adopt the usual ranking convention. For instance, if two objects are placed equal 3rd, both receive the rank 3·5 i.e. the mean of the ranks 3 and 4. We demonstrate the method using Illustration B.

x	y	R_x	R_y	$d = R_x - R_y$	d^2
1·72	3	9	11	−2	4
1·89	33	3	2	1	1
1·76	10	7·5	7	0·5	0·25
1·96	8	2	9	−7	49
1·66	28	11	3	8	64
1·94	16	1	5·5	−4·5	20·25
1·84	22	5	4	1	1
1·68	9	10	8	2	4
1·77	47	6	1	5	25
1·86	6	4	10	−6	36
1·76	16	7·5	5·5	2	4

$$n = 11 \text{ and } \sum d^2 = 208{\cdot}5,$$

$$\therefore \qquad r_S \approx 1 - \frac{6\sum d^2}{n(n^2-1)} = 1 - \frac{6 \times 208{\cdot}5}{11 \times 120} \approx 0{\cdot}052$$

[Note that the true correlation coefficient of R_x and R_y is 0·048 (to 3 d.p.).]

Kendall's† *rank correlation coefficient* r_K is a different measure of rank correlation. It is also designed to take values ranging from $+1$ for perfect agreement to -1 for total disagreement. To calculate r_K for two rankings of a sample, every pair of objects that can be selected from the sample is considered. For instance, in the case of the cabbage competition we must take into account the ranks given to AB, AC, AD, ..., BC, BD, BE, ... i.e. 28 pairs in all. Each pair is given a score of $+1$ if the judges have ranked the two cabbages in the same order, but -1 if the judges disagree. Kendall's coefficient is obtained by dividing the total score by the total number of pairs. Referring to the table of ranks we see that AB scores $+1$, AC scores -1 and so on. Clearly, determining all 28 scores one at a time would be rather tedious. A better method involves rearranging the table so that for the 1st judge the cabbages are in rank order.

Cabbage	Mrs. Stone	Mr. Turvy	Agreements	Disagreements
F	1	3	5	2
G	2	1	6	0
A	3	4	4	1
C	4	2	4	0
D	5	6	2	1
H	6	8	0	2
B	7	7	0	1
E	8	5		
			21	7

†*Kendall, Sir Maurice George* (b. 1907) English statistician. He proposed his rank correlation coefficient in 1938. *Rank Correlation Methods* was published in 1948.

The first entry in Mr. Turvy's column is 3. Below this 3 there are 5 ranks greater than 3, which correspond to 5 pairs FA, FD, FH, FB, FE, ranked in the same order by both judges. Similarly, below the 3 there are 2 ranks less than 3, which correspond to 2 disagreements. The second entry in Mr. Turvy's column is 1. Below the 1 there are 6 ranks greater than 1, but no ranks less than 1. Thus we have 6 further agreements, but no more disagreements. Working down the table in this way we eventually obtain the total numbers of agreements and disagreements.

Hence $$r_K = \frac{\text{no. of agreements} - \text{no. of disagreements}}{\text{total no. of pairs}} = \frac{21 - 7}{28} = 0{\cdot}5$$

For a set of n objects the total number of pairs is $\frac{1}{2}n(n-1)$. Thus the general formula for Kendall's rank correlation coefficient is

$$r_K = \frac{P - Q}{\frac{1}{2}n(n-1)}$$

where P is the number of agreements and Q is the number of disagreements.

This formula can also be used when there are tied ranks, by giving a pair tied in one or both rankings a score of zero. However, results obtained in this way must be treated with caution if the number of ties is large.

We can now calculate r_K for Illustration B using the ranks obtained earlier.

R_x	R_y	Agreements	Disagreements	Ties
1	5·5	5	4	1 (R_y col.)
2	9	2	7	0
3	2	7	1	0
4	10	1	6	0
5	4	4	2	0
6	1	5	0	0
7·5	7	2	1	1 (R_x col.)
7·5	5·5	2	1	0
9	11	0	2	0
10	8	0	1	0
11	3			
		28	25	

$$\therefore \quad r_K = \frac{P - Q}{\frac{1}{2}n(n-1)} = \frac{28 - 25}{\frac{1}{2} \times 11 \times 10} = \tfrac{3}{55} \approx 0{\cdot}055$$

Exercise 17.3

1. In a cookery competition three judges X, Y and Z place 6 sponge cakes in order of merit as follows.

	A	B	C	D	E	F
X	2	6	1	3	5	4
Y	1	5	4	2	3	6
Z	4	3	5	1	2	6

Calculate Spearman's rank correlation coefficient for the placings awarded by (a) judges X and Y, (b) judges Y and Z, (c) judges X and Z.

2. Calculate Kendall's rank correlation coefficient for the pairs of rankings given in question 1.

3. Three persons, P, Q and R, were asked to place in order of importance nine features of a house (A, B, C, ..., I). Calculate Spearman's rank correlation coefficients between the pairs of preferences as shown in the following table:

	A	B	C	D	E	F	G	H	I
P	1	2	4	8	9	7	6	3	5
Q	1	4	5	8	7	9	2	3	6
R	1	9	6	8	7	4	2	3	5

How far does this help to decide which pair from the three would be most likely to be able to compromise on a suitable house? (AEB 1978)

4. Calculate Kendall's rank correlation coefficient for the pairs of preferences given in question 3. Comment briefly on your results.

Questions 5 to 9. Find r_S and r_K for the data given in Exercise 17.1 questions 1 to 5.

10.

x	24	15	8	12	5	16	9
y	26	24	31	39	14	15	36

Use the given data to plot a scatter diagram. Find the product moment correlation coefficient r. Find also the rank correlation coefficients r_S and r_K. Comment briefly on your results.

11. Repeat question 10 for the following data.

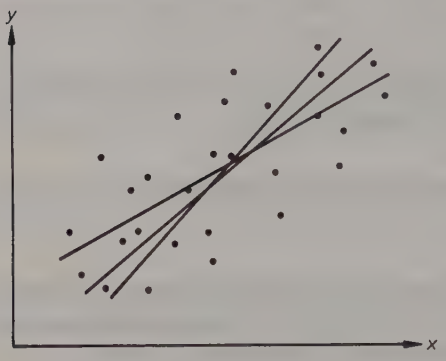
x	38	7	14	21	19	4	11	24
y	4	27	11	6	7	43	13	5

12. Find r_S and r_K for the data given in Illustration C §17.1.

17.4 Regression lines

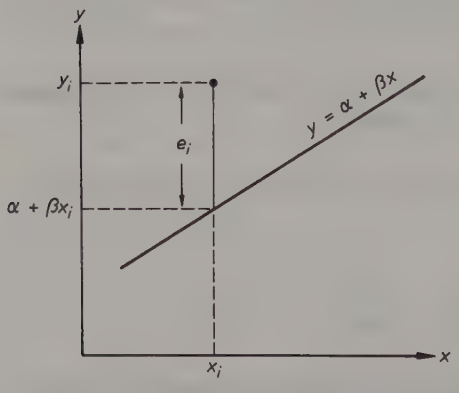
In §17.1 we found that the points in a scatter diagram representing a bivariate distribution sometimes tend to lie in a fairly narrow band along a straight line.

Such lines are called *regression*† *lines*. The equation of a regression line forms a linear approximation to any relationship which may exist between the variables.



For the scatter diagram shown here it is difficult to decide which line best fits the data. All three lines illustrated might be suitable. We clearly need a numerical procedure for finding the equations of regression lines.

Let us consider a set of n pairs of observations, $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$. We will assume that we need to find the regression line best suited to estimating unknown values of y for given values of x. This line is called the *regression line of* y *on* x and its equation will be denoted by $y = \alpha + \beta x$.



Using this line the estimated value of y corresponding to a typical observation x_i would be $\alpha + \beta x_i$. The deviation of the observed value y_i from this predicted value is called the *error of estimation* (or prediction) and is denoted by e_i. Thus we have:

$$y_i = \alpha + \beta x_i + e_i.$$

Our aim now is to obtain estimates for α and β such that the errors $e_1, e_2, \ldots, e_n$ (also called *residuals*) are as small as possible. The most satisfactory way of doing this is to choose the values which minimise the sum of the squares of the errors. This procedure is known as the *method of least squares.*

Let S be the sum of the squares of the (vertical) deviations of the points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ from a variable line $y = a + bx$,
i.e. let $S = \sum\{y - (a + bx)\}^2$.

[Since S is a function of two variables, namely a and b, we cannot use elementary calculus to find its minimum value but must use an algebraic method. Readers with a knowledge of advanced calculus may prefer to use partial differentiation.]

†*Regression* Sir Francis Galton (1822–1911), English anthropologist, published *Natural Inheritance* in 1889. In this he called attention to the tendency of naturally inherited characteristics to revert to the mean. This tendency was later called regression.

We can simplify the algebra by writing:

$$y - (a + bx) = (y - \bar{y}) - b(x - \bar{x}) + (\bar{y} - a - b\bar{x}),$$

then letting $u = x - \bar{x}$, $v = y - \bar{y}$ and $c = \bar{y} - a - b\bar{x}$ to give:

$$\begin{aligned} S &= \sum(v - bu + c)^2 \\ &= \sum(v^2 + b^2u^2 + c^2 - 2buv - 2bcu + 2cv) \\ &= \sum v^2 + b^2\sum u^2 + nc^2 - 2b\sum uv - 2bc\sum u + 2c\sum v. \end{aligned}$$

But $\sum u = \sum v = 0$ and in the notation of §17.2

$$\sum u^2 = S_{xx}, \quad \sum v^2 = S_{yy}, \quad \sum uv = S_{xy},$$

$$\therefore \qquad S = b^2S_{xx} - 2bS_{xy} + S_{yy} + nc^2$$

"Completing the square" for the terms involving b:

$$S = S_{xx}\left(b - \frac{S_{xy}}{S_{xx}}\right)^2 + S_{yy} - \frac{(S_{xy})^2}{S_{xx}} + nc^2$$

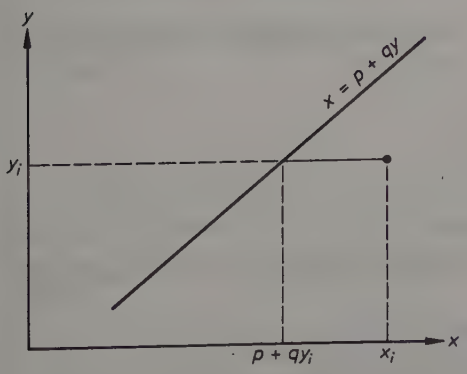
Hence S takes its minimum value $S_{yy} - \dfrac{(S_{xy})^2}{S_{xx}}$ when $b - \dfrac{S_{xy}}{S_{xx}} = 0$ and $c = 0$

i.e. $\bar{y} - a - b\bar{x} = 0$.

$\therefore$ the *least squares* regression line of y on x has equation $y = a + bx$ where

$$b = \frac{S_{xy}}{S_{xx}} = \frac{\sum xy - n\bar{x}\bar{y}}{\sum x^2 - n\bar{x}^2} \quad \text{and} \quad a = \bar{y} - b\bar{x}.$$

We deduce that this regression line has gradient S_{xy}/S_{xx} and passes through the point $(\bar{x}, \bar{y})$. Its equation can also be expressed in the form $y - \bar{y} = b(x - \bar{x})$. The quantity b is referred to as the coefficient of regression of y on x or simply the *regression coefficient.*



Provided that we are considering a bivariate experiment in which both x and y are allowed to vary freely, we can obtain a second least squares regression line called the *regression line of x on y.* This line is used to estimate values of x for given values of y. If we let its equation be $x = p + qy$, then the estimation errors take the form $x_i - (p + qy_i)$ and are represented graphically by horizontal deviations from the line. The values of p and q which minimise the sum of the squares of these errors are given by:

$$q = \frac{S_{xy}}{S_{yy}} \quad \text{and} \quad p = \bar{x} - q\bar{y}$$

Thus this regression line also passes through the point $(\bar{x}, \bar{y})$. In this case the gradient of the line is $1/q$ i.e. S_{yy}/S_{xy}.

Note that where less precision is required long calculations can be avoided by sketching regression lines on a scatter diagram "by eye". When doing this remember that

(i) *both* lines pass through the point $(\bar{x}, \bar{y})$,
(ii) the "y on x" line minimises *vertical* deviations,
(iii) the "x on y" line minimises *horizontal* deviations.

We now obtain the equations of both regression lines for the data given in Illustration A of §17.1. When in §17.2 we calculated the correlation coefficient r for this distribution we obtained the following results:

$$\bar{x} = 67, \quad \bar{y} = 61, \quad S_{xy} = 1413, \quad S_{xx} = 1280, \quad S_{yy} = 2070,$$

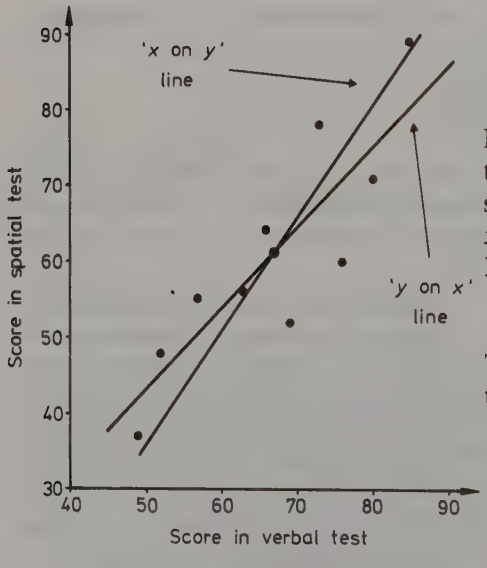
$$\therefore \qquad b = \tfrac{1413}{1280} \approx 1{\cdot}10, \quad a = 61 - (\tfrac{1413}{1280})67 \approx -12{\cdot}96.$$

Hence the regression line of y on x has equation

$$y = 1{\cdot}10x - 12{\cdot}96 \tag{1}$$

Similarly, $q = \frac{1413}{2070} \approx 0{\cdot}68$, $p = 67 - (\frac{1413}{2070})61 \approx 25{\cdot}36$.
Hence the regression line of x on y has equation

$$x = 0{\cdot}68y + 25{\cdot}36 \tag{2}$$



If a further student was given the verbal test and scored 61, we could predict his score in the spatial test using the "y on x" line.
From (1), if $x = 61$,

$$y = 1{\cdot}10 \times 61 - 12{\cdot}96 = 54{\cdot}14$$

Thus his predicted score in the spatial test is 54.

To estimate the verbal test score of a student who scored 74 in the spatial test we must use the "x on y" line.
From (2), if $y = 74$, $x = 0{\cdot}68 \times 74 + 25{\cdot}36 = 75{\cdot}68$.
Hence the predicted verbal test score is 76.

Finally we consider briefly the sampling distribution of the regression coefficient. For this purpose we will regard each value x_i as a given constant and the cor-

responding value y_i as an observed value of a random variable Y_i, such that $E(Y_i) = \alpha + \beta x_i$. It then follows that $\bar{x}$ is also a constant and that $E(\bar{Y}) = \alpha + \beta\bar{x}$. The random variable B representing the regression coefficient is now defined by writing:

$$B = \frac{\sum(x_i - \bar{x})(Y_i - \bar{Y})}{\sum(x_i - \bar{x})^2}$$

Using the rules of expectation algebra, (see §15.6),

$$\begin{aligned} E(B) &= \frac{\sum(x_i - \bar{x})E(Y_i - \bar{Y})}{\sum(x_i - \bar{x})^2} \\ &= \frac{\sum(x_i - \bar{x})\{(\alpha + \beta x_i) - (\alpha + \beta\bar{x})\}}{\sum(x_i - \bar{x})^2} \\ &= \frac{\sum\beta(x_i - \bar{x})^2}{\sum(x_i - \bar{x})^2} = \beta \end{aligned}$$

Thus the regression coefficient b is an unbiased estimate of the population parameter β.

If we also assume that each error $e_i = y_i - (\alpha + \beta x_i)$ is drawn at random from a population with variance σ^2, then $\text{Var}(Y_i) = \sigma^2$ for all values of i. It can then be shown (see Exercise 17.4 question 10) that

$$\text{Var}(B) = \sigma^2/S_{xx}, \quad \text{where} \quad S_{xx} = \sum(x_i - \bar{x})^2.$$

The quantity σ^2 is called the *residual variance* and when its value is known it is possible to obtain confidence limits for β by the methods of §16.3. We do not intend to discuss the estimation of σ^2, but give for reference the formula used.

$$\text{Estimated residual variance} = \frac{1}{n-2}\left\{S_{yy} - \frac{(S_{xy})^2}{S_{xx}}\right\} = \frac{S_{yy}(1 - r^2)}{n-2}.$$

Exercise 17.4

1. Calculate the equation of the regression line of y on x for the following distribution:

x	25	30	35	40	45	50
y	78	70	65	58	48	42

Is it possible to calculate from the equation you have just found (a) an estimate for the value of x when $y = 54$? (b) an estimate for the value of y when $x = 37$? In each case, if the answer is "Yes", calculate the estimate. If the answer is "No", say why not. (SU)

2. Fifteen boys took two exam papers in the same subject and the marks as percentages were as follows, where each boy's marks are in the same column.

Paper I (x)	65	73	42	52	84	60	70	79	60	83	57	77	54	66	89
Paper II (y)	78	88	60	73	92	77	84	89	70	99	73	88	70	85	89

Given that $\sum xy = 83\,734$, calculate the equation of the line of regression of y on x. Two boys were each absent for one paper. One scored 63 on Paper I, the other scored 81 on Paper II. In which case can you use your regression line to estimate the mark that the boy should be allocated for the paper he did not take, and what is that mark? (SU)

3. The 1962 and 1982 values, in pounds, of some "first editions" were as follows:

1962 value (x)	10	15	30	20	35
1982 value (y)	200	240	470	370	620

(i) Plot these values on a scatter diagram. (ii) Find the coordinates of a point through which the regression line of y on x must pass. (iii) Fit this regression line by eye. (iv) Find the equation of your line in the form $y = a + bx$, giving the constants a and b to two significant figures. Use the equation to estimate the 1982 value of a book valued at £25 in 1962.

Questions 4 to 8. Find the equations of the regression line of y on x and the regression line of x on y for the data given in Exercise 17.1 questions 1 to 5.

9. Using the usual notation, express in terms of S_{xx}, S_{yy} and r (a) the gradient of the regression line of y on x, (b) the gradient of the regression line of x on y. Use your results to prove that when $r = \pm 1$ the regression lines coincide. What form do the regression lines take when $r = 0$?

10. Using the definition of the random variable B given in the text, prove that

(a) $B = \dfrac{\sum (x_i - \bar{x}) Y_i}{S_{xx}}$, (b) $\text{Var}(B) = \dfrac{\text{residual variance}}{S_{xx}}$.

Exercise 17.5 (*miscellaneous*)

1. (a) Find the product moment correlation coefficient for the following distribution:

x	18	33	26	25	28	22	32	24
y	53	40	45	44	40	51	37	50

(b) Two judges in a baby competition rank the 8 babies as follow:

Baby	A	B	C	D	E	F	G	H
Judge 1	2	6	1	5	7	4	3	8
Judge 2	3	1	5	2	7	8	4	6

Calculate a rank correlation coefficient between the two judges. (SU)

2. (a) The marks of eight candidates in English and Mathematics are:

Candidate	1	2	3	4	5	6	7	8
English (x)	50	58	35	86	76	43	40	60
Mathematics (y)	65	72	54	82	32	74	40	53

Rank the results and hence find a rank correlation coefficient between the two sets of marks.

(b) Using the data in part (a), obtain the product-moment correlation coefficient. (SU)

3. (a) The following marks were awarded by two judges at a music competition:

	Child 1	Child 2	Child 3	Child 4	Child 5
Judge 1	10	5	8	7	9
Judge 2	9	6	10	5	8

Calculate a coefficient of rank correlation.

(b) Determine, by calculation, the equation of the regression line of x on y based on the following information about 8 children:

Child	1	2	3	4	5	6	7	8
Arithmetic mark (x)	45	33	27	23	18	14	8	0
English mark (y)	31	33	18	20	19	9	13	1

(SU)

4. 12 students were given a prognostic test at the beginning of a course and their scores X_i in the test were compared with their scores Y_i obtained in an examination at the end of the course ($i = 1, 2, \ldots, 12$). The results were as follows:

X_i	1	2	2	4	5	5	6	7	8	8	9	9
Y_i	3	4	5	5	4	8	6	6	6	7	8	10

Find the equation of the regression line of Y on X and determine the correlation coefficient between X and Y. (SU)

5. The heights h, in cm, and weights W, in kg, of 10 people are measured. It is found that $\sum h = 1710$, $\sum W = 760$, $\sum h^2 = 293\,162$, $\sum hW = 130\,628$ and $\sum W^2 = 59\,390$. Calculate the correlation coefficient between the values of h and W. What is the equation of the regression line of W on h? (O&C)

6. Explain briefly how the principle of least squares is used to find a regression line based on a sample of size n. Illustrate on a rough sketch the distances whose squares are minimised, taking care to distinguish the dependent and independent variates.

The table below shows the yields (y) obtained in an agricultural experiment in tonnes per hectare after using x tonnes of fertiliser per hectare.

Fertiliser (x)	0	0·2	0·4	0·6
Yield (y)	1·26	1·47	1·87	2·00

(a) Find the means of x and y. (b) Plot the measurements and their means on a graph of yield (y) against fertiliser (x). (c) Calculate the regression line of y on x in the form $y = a + bx$. (d) Draw the regression line you have calculated on the graph. (e) Estimate the total yield in tonnes from a 2·2 hectare field if 0·5 tonne of fertiliser is used altogether. (O&C)

7. Each of the variables x and y takes the values $1, 2, \ldots, n$ but not necessarily in the same order as each other. Prove that the covariance of x and y is

$$\text{cov}(x, y) = \frac{n^2 - 1}{12} - \frac{1}{2n} \sum_{i=1}^{n} (x_i - y_i)^2.$$

Hence show that Spearman's coefficient of rank correlation between x and y may be written as

$$1 - \frac{6}{n(n^2 - 1)} \sum_{i=1}^{n} (x_i - y_i)^2.$$

Seven army recruits (A, B, ..., G) were given two separate aptitude tests. Their orders of merit in each test were

Order of Merit	1st	2nd	3rd	4th	5th	6th	7th
1st test	G	F	A	D	B	C	E
2nd test	D	F	E	B	G	C	A

Find Spearman's coefficient of rank correlation between the two orders and comment briefly on the correlation obtained. (O&C)

8. The numbers of pupils, in a class of 20, gaining various scores in two tests are shown in the table below.

		Scores x in test A				Row totals
		1	2	3	4	
Score y in test B	1	1	—	—	—	1
	2	2	5	3	—	10
	3	—	2	4	2	8
	4	—	—	—	1	1
Column totals		3	7	7	3	20

(Thus, for example, 3 pupils gained a score of 3 in test A and a score of 2 in test B.) Find (a) the average score in each test, (b) the variances of x and y, (c) the covariance of x and y, (d) the equation of the regression line of y on x in the form $y = a + bx$. (e) If a new pupil scores 3·5 in test A, predict his score in test B. (O&C)

9. Explain the meaning of Spearman's rank correlation coefficient, giving particular attention to the cases when this coefficient takes the values (i) $+1$, (ii) -1.

The scores of ten batsmen in two successive innings were

Batsman	A	B	C	D	E	F	G	H	I	J
First innings	1	0	52	73	21	5	18	10	2	3
Second innings	36	0	5	21	46	32	11	2	6	1

Calculate Spearman's rank correlation coefficient. (AEB)

10. The body and heart masses of fourteen 10-month-old male mice are tabulated below:

Body mass (x) (grams)	27	30	37	38	32	36	32	32	38	42	36	44	33	38
Heart mass (y) (milligrams)	118	136	156	150	140	155	157	114	144	159	149	170	131	160

(i) Draw a scatter diagram of these data. (ii) Calculate the equation of the regression line of y on x and draw this line on the scatter diagram. (iii) Calculate the product-moment coefficient of correlation. (AEB 1977)

11. Explain clearly what is meant by the statistical term "correlation".

Vegboost Industries, a small chemical firm specializing in garden fertilizers, set up an experiment to study the relationship between a new fertilizer compound and the yield from tomato plants. Eight similar plants were selected and treated regularly throughout their life with x grams of fertilizer diluted in a standard volume of water. The yield y, in kilograms, of good tomatoes was measured for each plant. The following table summarises the results.

Plant	A	B	C	D	E	F	G	H
Amount of fertilizer x (g)	1·2	1·8	3·1	4·9	5·7	7·1	8·6	9·8
Yield y (kg)	4·5	5·9	7·0	7·8	7·2	6·8	4·5	2·7

(i) Calculate the product-moment correlation coefficient for these data. (ii) Calculate Spearman's rank correlation coefficient for these data. (iii) Is there any evidence of a relationship between these variables? Justify your answer. (No formal test is required.) (AEB 1978)

12. A scientist, working in an agricultural research station, believes there is a relationship between the hardness of the shells of eggs laid by chickens and the amount of a certain food supplement put into the diet of the chickens. He selects ten chickens of the same breed and collects the following data:

Chicken	A	B	C	D	E	F	G	H	I	J
Amount of food supplement x (g)	7·0	9·8	11·6	17·5	7·6	8·2	12·4	17·5	9·5	19·5
Hardness of shells y	1·2	2·1	3·4	6·1	1·3	1·7	3·4	6·2	2·1	7·1

(Hardness is measured on a 0 – 10 scale, 10 being the hardest. There are no units attached.)

(i) Calculate the equation of the regression line of y on x.
(ii) Calculate the product-moment correlation coefficient.
(iii) Do you believe that this linear model will continue to be appropriate no matter how large or small x becomes? Justify your reply. (AEB 1978)

13. The state of Tempora demands that every household in the country shall have a reliable clock; inspectors are being introduced throughout the country to

implement the policy. The Chief Inspector has the following data on the population size of towns, where Inspection Units have been set up, and the number of man-hours spent on inspection.

Population (thousands)	3	4	5	9	13	15	18	20	21	22
Man-hours (thousands)	8	11	13	18	24	26	31	32	34	33

(i) Calculate the regression line for predicting the number of man-hours from the population size (note that the mean value of each variate is a whole number).
(ii) Predict the manpower required (in man-hours) for a new Inspection Unit to be installed in a town with a population of 17 000. (O)

14. In each of the following sections, which purport to be extracts from reports, the second sentence is an inference from the statement made in the first sentence. State whether the inferences are valid or invalid, and give the reasons for your decisions.

(a) The amount of fertilizer applied was varied from plot to plot and the correlation coefficient between the yield of corn from a plot and the amount of fertilizer applied was found to be 0·02. This shows that there was no relation between yield and amount of fertilizer applied

(b) Inspection of the police reports of car accidents in the town during 1975 revealed that, when the number of accidents involving drivers of a particular age was correleated with that age, the correlation coefficient was $-0{\cdot}72$. We conclude that older drivers are less likely to have an accident than younger drivers.

(c) The correlation coefficient between the sugar content s of the peas and the length of time t they have been in the greengrocer's shop was negative. It follows that the regression lines of s on t and of t on s have negative slopes.

(d) The correlation coefficient of percentage of children over sixteen at school and the Gross National Product, over the years for which we have records, is 0·91, which is a high correlation. It is obvious that if many more children can be persuaded to stay on at school after sixteen then the Gross National Product will increase substantially. (O)

15. A response variable y is known to be linearly related to the value of a variable x. To estimate the precise form of this relationship the response y is measured for each of six specified values of x, the results being as shown in the following table:

$x =$	10	20	30	40	50	60
$y =$	19	27	26	32	35	41

Whereas the specified values of x are known to be accurate, the observed values of y are subject to independent normally distributed errors with mean zero and standard deviation 1·5.

(i) Calculate the least squares estimate of the linear relationship connecting x and y.

(ii) Calculate 95 per cent confidence limits for the slope of the relationship. Hence, or otherwise, find 95 per cent confidence limits for the increase in the value of y when x is increased by 20 units. (W)

18 Matrices and determinants

18.1 Linear transformations in three dimensions

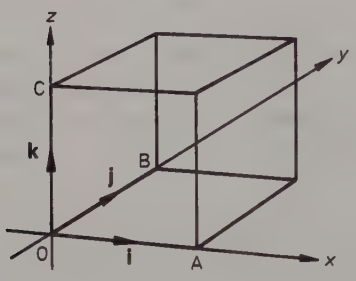
In this section we consider the effect of linear transformations on three-dimensional space. The effects of such transformations on two-dimensional space were examined in Chapter 19 of Book 1. We recall that a linear transformation is a mapping under which linear relationships between position vectors remain unchanged. Let us consider a linear transformation such that the images of the points $A(1,0,0)$, $B(0,1,0)$, $C(0,0,1)$ with position vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the points A', B', C' with position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ respectively. Since the transformation preserves linear relationships, the image of a point $P(x, y, z)$ with position vector $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ will be the point $P'(x', y', z')$ with position vector $x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$. Thus, writing $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$, we have:

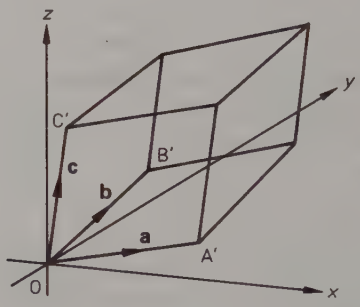
$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = x\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + y\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + z\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} a_1x + b_1y + c_1z \\ a_2x + b_2y + c_2z \\ a_3x + b_3y + c_3z \end{pmatrix}$$

$$= \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$\therefore$ The matrix $\mathbf{M} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ represents the the linear transformation in which $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$.

To interpret this result geometrically we regard the origin O and the points A, B, C as vertices of a unit cube. In general, the transformation with matrix $\mathbf{M}$ maps this unit cube onto a parallelepiped with O, A', B', C' among its vertices.

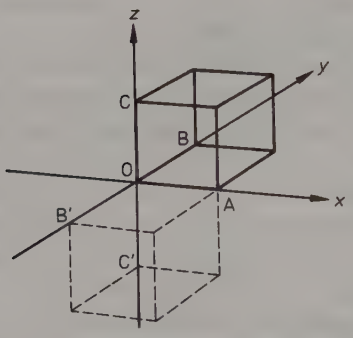




Example 1 Describe the transformations represented by the matrices

(a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

(a)



$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

$\therefore$ the images of the points $A(1,0,0)$, $B(0,1,0)$ and $C(0,0,1)$ are $A(1,0,0)$, $B'(0,-1,0)$ and $C'(0,0,-1)$ respectively.

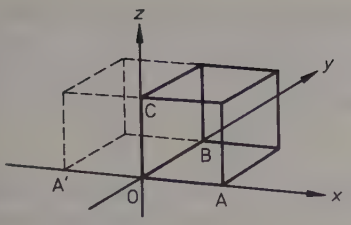
The diagram shows that the matrix represents a rotation through 180° about the x-axis.

(b) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

This means that the transformation maps A to B, B to C and C to A. Hence the matrix represents a rotation about an axis through the origin and the point $(1,1,1)$. Since three such rotations will return A, B and C to their original positions, the angle of rotation must be 120°.

We now consider some other basic transformations and their matrices.

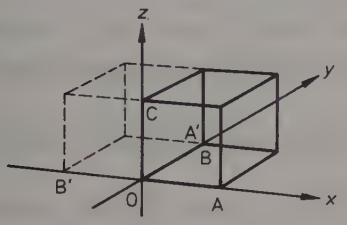
(1) *Reflection in the y, z plane* (i.e. the plane $x = 0$).



$$\begin{pmatrix}1\\0\\0\end{pmatrix} \to \begin{pmatrix}-1\\0\\0\end{pmatrix}, \quad \begin{pmatrix}0\\1\\0\end{pmatrix} \to \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \begin{pmatrix}0\\0\\1\end{pmatrix} \to \begin{pmatrix}0\\0\\1\end{pmatrix}$$

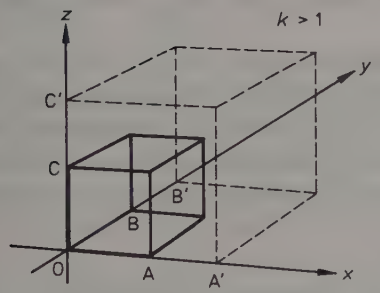
$\therefore$ the matrix of the transformation is $\begin{pmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$

(2) *Rotation* through 90° about the z-axis, so that the image of A is B.



$$\begin{pmatrix}1\\0\\0\end{pmatrix} \to \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \begin{pmatrix}0\\1\\0\end{pmatrix} \to \begin{pmatrix}-1\\0\\0\end{pmatrix}, \quad \begin{pmatrix}0\\0\\1\end{pmatrix} \to \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$\therefore$ the matrix of the transformation is $\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$



(3) *Enlargement* with scale factor k.
The matrix of the transformation is

$$\begin{pmatrix}k & 0 & 0\\ 0 & k & 0\\ 0 & 0 & k\end{pmatrix}$$

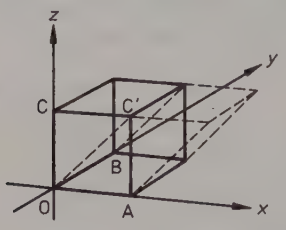
When $k = 1$ this matrix represents the identity transformation, which maps every point onto itself. If $k < 0$ the transformation reverses the direction of every position vector. This effect is sometimes called *central inversion.*

(4) *Orthogonal projection* on the plane $z = 0$.
Under this transformation the image of the point $P(x, y, z)$ is the point $P'(x, y, 0)$, which lies in the plane $z = 0$. This is called an orthogonal projection, because PP' is perpendicular (orthogonal) to the given plane.

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \to \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \begin{pmatrix}0\\1\\0\end{pmatrix} \to \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \begin{pmatrix}0\\0\\1\end{pmatrix} \to \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$\therefore$ the matrix of the transformation is $\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}$

(5) *Shear* with plane $z = 0$ invariant, $(0, 0, 1) \rightarrow (1, 0, 1)$.



$$\begin{pmatrix}1\\0\\0\end{pmatrix} \rightarrow \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \begin{pmatrix}0\\1\\0\end{pmatrix} \rightarrow \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \begin{pmatrix}0\\0\\1\end{pmatrix} \rightarrow \begin{pmatrix}1\\0\\1\end{pmatrix}$$

$\therefore$ the matrix of the transformation is $\begin{pmatrix}1 & 0 & 1\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$

The matrix $\mathbf{I} = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$ is the 3×3 *identity matrix* and represents the identity transformation under which every point of three-dimensional space is mapped onto itself.

The *inverse* of a 3×3 matrix $\mathbf{M}$ is the matrix $\mathbf{M}^{-1}$ which has the property that

$$\mathbf{M}\mathbf{M}^{-1} = \mathbf{M}^{-1}\mathbf{M} = \mathbf{I}.$$

If it is impossible to find such a matrix, $\mathbf{M}$ is said to be *singular*. A matrix $\mathbf{M}$ which has an inverse is *non-singular*.

If the linear transformation under which the point $P(x, y, z)$ has image $P'(x', y', z')$ is represented by the matrix $\mathbf{M}$, then the inverse transformation, if it exists, maps P' to P and has matrix $\mathbf{M}^{-1}$. General methods for finding the inverses of 3×3 matrices will be discussed later in the chapter. However, in some cases the inverse of a matrix $\mathbf{M}$ can be found by considering the inverse of the transformation represented by $\mathbf{M}$.

We illustrate the method using the matrices

$$\mathbf{A} = \begin{pmatrix}-1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix}0 & -1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 2\end{pmatrix}.$$

As shown in (1), $\mathbf{A}$ represents a reflection in the y, z plane. However, any reflection is its own inverse

$$\therefore \quad \mathbf{A}^{-1} = \mathbf{A} = \begin{pmatrix}-1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}.$$

As shown in (2), $\mathbf{B}$ represents a rotation through 90° about the z-axis. The inverse of this transformation is an equal rotation about the z-axis in the opposite sense. Hence $\mathbf{B}^{-1}$ represents the transformation under which the images of the

points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ are $(0,-1,0)$, $(1,0,0)$, $(0,0,1)$ respectively.

$$\therefore \qquad \mathbf{B}^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Matrix **C** represents an enlargement, scale factor 2. Thus $\mathbf{C}^{-1}$ must represent an enlargement, scale factor $\frac{1}{2}$.

i.e. $$\mathbf{C}^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

As in the case of transformations of the x, y plane, if two linear transformations with matrices **P** and **Q** are performed in succession using first **P** then **Q**, the matrix of the composite transformation is **QP**.
Considering the matrices **A** and **B** defined earlier we have:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{\mathbf{B}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \xrightarrow{\mathbf{A}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \xrightarrow{\mathbf{B}} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{\mathbf{A}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \xrightarrow{\mathbf{B}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \xrightarrow{\mathbf{A}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and $$\mathbf{AB} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus the matrix **AB** has the same effect as **B** followed by **A** and represents a reflection in the plane $y = x$.

Certain linear transformations map every point in three-dimensional space onto a single straight line through the origin. There are other linear transformations, such as orthogonal projections, which map every point onto a single plane through the origin.

For instance under the transformation with matrix $\begin{pmatrix} 4 & -2 & 6 \\ -6 & 3 & -9 \\ 2 & -1 & 3 \end{pmatrix}$ the position vectors of the images of the points $A(1,0,0)$, $B(0,1,0)$, $C(0,0,1)$ are all multiples of the vector $\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$. Hence this transformation maps every point onto the straight line which passes through the origin and the point $(2,-3,1)$. The vector equation of this line is $\mathbf{r} = \lambda(2\mathbf{i} - 3\mathbf{j} + \mathbf{k})$.

Under the transformation with matrix $\begin{pmatrix} 2 & -1 & 3 \\ -1 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix}$ the position vectors of the images A', B', C' of the points A, B, C are

$$\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$$

respectively. Since $\mathbf{c} = 2\mathbf{a} + \mathbf{b}$, C' must lie in the plane through the origin which contains A' and B'. Hence this transformation maps every point onto the plane through O and the points $A'(2, -1, 1)$ and $B'(-1, 3, 2)$. The vector equation of the plane is $\mathbf{r} = \lambda(2\mathbf{i} - \mathbf{j} + \mathbf{k}) + \mu(-\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})$. The corresponding Cartesian equation is found to be $x + y - z = 0$.

Exercise 18.1

1. Find the images of the points $(1, 0, -2)$ and $(1, 1, 1)$ under the linear transformations with matrices

(a) $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 3 & -2 \\ 0 & 0 & 5 \\ 0 & -4 & 0 \end{pmatrix}$

2. Write down the 3×3 matrices representing the following transformations:
(a) Reflection in the x, y plane,
(b) Enlargement, scale factor 5,
(c) Rotation through 180° about the y-axis,
(d) One way stretch parallel to the z-axis, $\times 2$,
(e) Rotations through 90° about the x-axis.

3. Describe geometrically the transformations represented by the following matrices:

(a) $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (e) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ (f) $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

4. Describe geometrically the transformations represented by

(a) $\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ (e) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ (f) $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

5. Write down the 3×3 matrices representing the following transformations:
(a) Reflection in the plane $x + y = 0$,
(b) Orthogonal projection on the plane $x = y$,

(c) Rotation through θ about the y-axis,
(d) Shear with x, y plane invariant, $(0, 0, 1) \rightarrow (1, 1, 1)$
(e) Rotation through 180° about the line $x + y = z = 0$ (i.e. the line through the origin and the point $(1, -1, 0)$.)

6. If $\mathbf{M} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, find $\mathbf{M}^2$ and $\mathbf{M}^4$. Describe geometrically the transformations represented by $\mathbf{M}$, $\mathbf{M}^2$ and $\mathbf{M}^4$.

7. If $\mathbf{P} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\mathbf{Q} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, find $\mathbf{QP}$ and $\mathbf{PQ}$. Describe geometrically the transformations represented by $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{QP}$ and $\mathbf{PQ}$. Verify that $\mathbf{QP}$ has the same effect as $\mathbf{P}$ followed by $\mathbf{Q}$ and that $\mathbf{PQ}$ is equivalent to $\mathbf{Q}$ then $\mathbf{P}$.

8. Find, where possible, the inverses of the matrices given in question 3. Verify for each matrix $\mathbf{A}$ which has an inverse $\mathbf{A}^{-1}$ that $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$, where $\mathbf{I}$ is the 3×3 identity matrix.

9. Find, where possible, the inverses of the matrices given in question 4.

10. Show that the linear transformation with matrix $\begin{pmatrix} 2 & -5 & 3 \\ -2 & 5 & -3 \\ 4 & -10 & 6 \end{pmatrix}$ maps every point of three-dimensional space onto a single line and write down a vector equation of this line. Describe the set of points mapped onto the origin under this transformation.

11. Show that the vectors $\begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix}$, $\begin{pmatrix} 14 \\ 0 \\ 14 \end{pmatrix}$ are coplanar. Deduce that the transformation with matrix $\begin{pmatrix} 1 & 3 & 14 \\ -3 & 5 & 0 \\ 4 & -2 & 14 \end{pmatrix}$ maps all the points of three-dimensional space onto a plane. Write down vector and Cartesian equations of this plane.

12. If $\mathbf{M} = \begin{pmatrix} 0 & 1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & 0 \end{pmatrix}$, find $\mathbf{M}^2$ and $\mathbf{M}^3$. Describe geometrically the transformations represented by $\mathbf{M}$, $\mathbf{M}^2$ and $\mathbf{M}^3$.

18.2 3 × 3 matrices and determinants

The basic properties of matrix addition and multiplication were established in Book 1. We now consider the application of this theory to 3 × 3 matrices.

Example 1 If $\mathbf{A} = \begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -1 \\ 0 & 2 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 0 & 1 \\ 4 & -2 & 1 \end{pmatrix}$,

find $3\mathbf{A} - 2\mathbf{B}$ and $\mathbf{A}^2\mathbf{B}$.

$$3\mathbf{A} - 2\mathbf{B} = 3\begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -1 \\ 0 & 2 & 4 \end{pmatrix} - 2\begin{pmatrix} 2 & -1 & 3 \\ 0 & 0 & 1 \\ 4 & -2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 & -12 \\ 9 & 3 & -5 \\ -8 & 10 & 10 \end{pmatrix}$$

$$\mathbf{A}^2\mathbf{B} = \begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -1 \\ 0 & 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -1 \\ 0 & 2 & 4 \end{pmatrix}\begin{pmatrix} 2 & -1 & 3 \\ 0 & 0 & 1 \\ 4 & -2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -4 & -10 \\ 6 & -1 & 11 \\ 6 & 10 & 14 \end{pmatrix}\begin{pmatrix} 2 & -1 & 3 \\ 0 & 0 & 1 \\ 4 & -2 & 1 \end{pmatrix} = \begin{pmatrix} -38 & 19 & -11 \\ -32 & 16 & 6 \\ 68 & -34 & 42 \end{pmatrix}$$

Example 2 If $\mathbf{P} = \begin{pmatrix} 4 & 4 & -2 \\ -1 & 0 & 1 \\ 3 & 6 & -1 \end{pmatrix}$, show that $\mathbf{P}^2 = \mathbf{P} + k\mathbf{I}$ for some constant k, where $\mathbf{I}$ is the 3 × 3 unit matrix. Hence find the inverse matrix $\mathbf{P}^{-1}$.

$$\mathbf{P}^2 = \begin{pmatrix} 4 & 4 & -2 \\ -1 & 0 & 1 \\ 3 & 6 & -1 \end{pmatrix}\begin{pmatrix} 4 & 4 & -2 \\ -1 & 0 & 1 \\ 3 & 6 & -1 \end{pmatrix} = \begin{pmatrix} 6 & 4 & -2 \\ -1 & 2 & 1 \\ 3 & 6 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 4 & -2 \\ -1 & 0 & 1 \\ 3 & 6 & -1 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \mathbf{P} + 2\mathbf{I}$$

Rearranging: $\mathbf{P}^2 - \mathbf{P} = 2\mathbf{I}$

$\therefore \quad (\mathbf{P} - \mathbf{I})\mathbf{P} = 2\mathbf{I}$

$\therefore \quad \frac{1}{2}(\mathbf{P} - \mathbf{I}) \times \mathbf{P} = \mathbf{I}$

Hence $\mathbf{P}^{-1} = \frac{1}{2}(\mathbf{P} - \mathbf{I}) = \frac{1}{2}\left\{\begin{pmatrix} 4 & 4 & -2 \\ -1 & 0 & 1 \\ 3 & 6 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}$

$$= \frac{1}{2}\begin{pmatrix} 3 & 4 & -2 \\ -1 & -1 & 1 \\ 3 & 6 & -2 \end{pmatrix} \text{ or } \begin{pmatrix} \frac{3}{2} & 2 & -1 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & 3 & -1 \end{pmatrix}$$

Consider now two 3 × 3 matrices **A** and **B** with inverses $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$ respectively.

$$(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = [\mathbf{A}(\mathbf{B}\mathbf{B}^{-1})]\mathbf{A}^{-1} = (\mathbf{AI})\mathbf{A}^{-1} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$

i.e. $(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \mathbf{I}$ and similarly $(\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB}) = \mathbf{I}$

Hence $$\boxed{(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}}$$

It can also be shown that if the transposes of **A** and **B** are $\mathbf{A}^T$ and $\mathbf{B}^T$ respectively, then

$$\boxed{(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T}$$

The *determinant* of a square matrix **A** is denoted by det **A**, $|\mathbf{A}|$ or simply Δ.

If $$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det \mathbf{A} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

In Book 1 we saw that if **A** represents a linear transformation of the x, y plane, then under this transformation the image of a unit square is a parallelogram of area $|\det \mathbf{A}|$.

The determinant of a 3 × 3 matrix $\mathbf{A} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ is defined as

$$\boxed{\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}$$

It can be shown that under the linear transformation of three-dimensional space with matrix **A** the image of a unit cube is a parallelepiped of volume $|\det \mathbf{A}|$.

By considering the volume changes produced by the composite transformation with matrix **AB** we obtain the result:

$$\boxed{\det(\mathbf{AB}) = (\det \mathbf{A}).(\det \mathbf{B})}$$

Example 3 Evaluate $\begin{vmatrix} 1 & 2 & -1 \\ -1 & 3 & 0 \\ 2 & 1 & 2 \end{vmatrix}$

$$\begin{vmatrix} 1 & 2 & -1 \\ -1 & 3 & 0 \\ 2 & 1 & 2 \end{vmatrix} = 1\begin{vmatrix} 3 & 0 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} -1 & 0 \\ 2 & 2 \end{vmatrix} + (-1)\begin{vmatrix} -1 & 3 \\ 2 & 1 \end{vmatrix}$$

$$= 1 \times 6 - 2(-2) + (-1)(-7) = 17.$$

Let us consider now the effect produced when each element of a row or column of a determinant is multiplied by a constant.

$$\begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = kad - kbc = k(ad - bc) = k\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$\begin{vmatrix} a_1 & kb_1 & c_1 \\ a_2 & kb_2 & c_2 \\ a_3 & kb_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} kb_2 & c_2 \\ kb_3 & c_3 \end{vmatrix} - kb_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & kb_2 \\ a_3 & kb_3 \end{vmatrix}$$

$$= ka_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - kb_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + kc_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

In general, if each element of a row or column of a determinant is multiplied by a constant k then the value of the determinant is multiplied by k.

Example 4 Evaluate $\begin{vmatrix} 12 & 6 & 10 \\ 2 & 3 & 15 \\ 6 & 0 & 5 \end{vmatrix}$

$$\begin{vmatrix} 12 & 6 & 10 \\ 2 & 3 & 15 \\ 6 & 0 & 5 \end{vmatrix} = 2\begin{vmatrix} 6 & 3 & 5 \\ 2 & 3 & 15 \\ 6 & 0 & 5 \end{vmatrix} = 2.2.3.5\begin{vmatrix} 3 & 1 & 1 \\ 1 & 1 & 3 \\ 3 & 0 & 1 \end{vmatrix}$$

$$= 60\left\{3\begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix}\right\}$$

$$= 60(3 + 8 - 3) = 480.$$

In the expression $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$ the determinants $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, $\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$ and $\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$ are called the *minors* of the elements a_1, b_1 and c_1 respectively.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

In general the minor of an element of a determinant is obtained by deleting the row and column in which the element lies. For example the minor of c_2 is obtained by crossing out the second row and third column to give $\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$. If the

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

minors are allotted signs alternately + and −, as shown in the diagram, the resulting quantities are the *cofactors* of the corresponding elements of the determinant. Thus the cofactor of c_2 is $-\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$

Denoting the cofactors of $a_1, b_1, \ldots, b_3, c_3$ by $A_1, B_1, \ldots, B_3, C_3$ respectively, the definition of Δ becomes

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1A_1 + b_1B_1 + c_1C_1.$$

But
$$\begin{aligned} \Delta &= a_1(b_2c_3 - c_2b_3) - b_1(a_2c_3 - c_2a_3) + c_1(a_2b_3 - b_2a_3) \\ &= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 \\ &= -a_2b_1c_3 + a_3b_1c_2 + a_1b_2c_3 - a_3b_2c_1 - a_1b_3c_2 + a_2b_3c_1 \end{aligned}$$

$$= -b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + b_2\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - b_3\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

$$= b_1B_1 + b_2B_2 + b_3B_3$$

Using this and other rearrangements of the original expression for Δ, it is found that Δ can be evaluated using the elements of any row or column together with the corresponding cofactors:

$$\begin{aligned} \Delta &= a_1A_1 + b_1B_1 + c_1C_1, & \Delta &= a_1A_1 + a_2A_2 + a_3A_3, \\ \Delta &= a_2A_2 + b_2B_2 + c_2C_2, & \Delta &= b_1B_1 + b_2B_2 + b_3B_3, \\ \Delta &= a_3A_3 + b_3B_3 + c_3C_3, & \Delta &= c_1C_1 + c_2C_2 + c_3C_3. \end{aligned}$$

These alternative expressions for Δ are particularly useful when one or more elements of the determinant are zero.

Example 5 Evaluate (a) $\begin{vmatrix} 2 & 4 & 3 \\ 0 & 5 & 4 \\ 0 & 2 & 3 \end{vmatrix}$ (b) $\begin{vmatrix} 2 & 4 & 5 \\ 0 & 1 & 0 \\ 3 & 9 & 7 \end{vmatrix}$

(a) Expanding down the first column

$$\begin{vmatrix} 2 & 4 & 3 \\ 0 & 5 & 4 \\ 0 & 2 & 3 \end{vmatrix} = 2\begin{vmatrix} 5 & 4 \\ 2 & 3 \end{vmatrix} - 0 + 0 = 2(15 - 8) = 14$$

(b) Expanding along the second row

$$\begin{vmatrix} 2 & 4 & 5 \\ 0 & 1 & 0 \\ 3 & 9 & 7 \end{vmatrix} = -0 + 1\begin{vmatrix} 2 & 5 \\ 3 & 7 \end{vmatrix} - 0 = 14 - 15 = -1$$

If two rows or two columns of a determinant are equal then the cofactors of the remaining row or column must all be zero. For instance, using the second column,

$$\begin{vmatrix} p & x & p \\ q & y & q \\ r & z & r \end{vmatrix} = -x\begin{vmatrix} q & q \\ r & r \end{vmatrix} + y\begin{vmatrix} p & p \\ r & r \end{vmatrix} - z\begin{vmatrix} p & p \\ q & q \end{vmatrix} = 0$$

Thus if two rows or columns of a determinant are equal then the value of the determinant is zero.

Example 6 Evaluate $\begin{vmatrix} -1 & 3 & 4 \\ 2 & -6 & -8 \\ 3 & 4 & 1 \end{vmatrix}$

$$\begin{vmatrix} -1 & 3 & 4 \\ 2 & -6 & -8 \\ 3 & 4 & 1 \end{vmatrix} = -2\begin{vmatrix} -1 & 3 & 4 \\ -1 & 3 & 4 \\ 3 & 4 & 1 \end{vmatrix} = 0$$

Exercise 18.2

1. If $\mathbf{P} = \begin{pmatrix} 2 & 1 & -4 \\ 0 & -3 & 1 \\ 5 & 0 & -1 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} -1 & 0 & -1 \\ 2 & 0 & 3 \\ 0 & 1 & -2 \end{pmatrix}$, find (a) $\mathbf{P} - \mathbf{Q}$, (b) $2\mathbf{P} + 3\mathbf{Q}$, (c) $\mathbf{PQ}$, (d) $\mathbf{QP}$, (e) $\mathbf{Q}^2$.

2. If $\mathbf{A} = \begin{pmatrix} 1 & 3 & -9 \\ 5 & -1 & 3 \\ 1 & 3 & 7 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 3 & 0 \\ 2 & -1 & 3 \\ -1 & 0 & 1 \end{pmatrix}$, find $\mathbf{AB}$ and hence write down $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$.

3. If $\mathbf{M} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$, find $\mathbf{M}^2$ and $\mathbf{M}^3$. Hence find $\mathbf{M}^{-1}$.

4. If $\mathbf{A} = \begin{pmatrix} 0 & 2 & -1 \\ -1 & -3 & 1 \\ 1 & 2 & -2 \end{pmatrix}$, calculate $\mathbf{A}^2 + 4\mathbf{A}$ and hence find $\mathbf{A}^{-1}$.

5. If $\mathbf{A} = \begin{pmatrix} 1 & -2 & 0 \\ -3 & 0 & 2 \\ 2 & 1 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & -1 & 2 \\ 0 & 4 & -1 \\ -2 & 1 & 0 \end{pmatrix}$, write down $\mathbf{A}^T$ and $\mathbf{B}^T$. Verify that $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$ and that $(\mathbf{BA})^T = \mathbf{A}^T\mathbf{B}^T$.

6. If $\mathbf{P} = \begin{pmatrix} 1 & 2 & -2 \\ 2{\cdot}8 & -1 & 0{\cdot}4 \\ 0{\cdot}4 & 2 & 2{\cdot}2 \end{pmatrix}$, find $\mathbf{PP}^T$ and hence write down $\mathbf{P}^{-1}$ in terms of $\mathbf{P}^T$.

7. Evaluate the following determinants:

(a) $\begin{vmatrix} 2 & -1 & 1 \\ 0 & 3 & -2 \\ 1 & -1 & 0 \end{vmatrix}$ (b) $\begin{vmatrix} 0 & 2 & -1 \\ 1 & 0 & -3 \\ -2 & 1 & 0 \end{vmatrix}$ (c) $\begin{vmatrix} 1 & 2 & 0 \\ 4 & 0 & 1 \\ 3 & 1 & 2 \end{vmatrix}$

(d) $\begin{vmatrix} 4 & 10 & 7 \\ 7 & 9 & 8 \\ 5 & 9 & 7 \end{vmatrix}$ (e) $\begin{vmatrix} 8 & -2 & 7 \\ -2 & 2 & 7 \\ 5 & -3 & 6 \end{vmatrix}$ (f) $\begin{vmatrix} 6 & -8 & 11 \\ 2 & 1 & -2 \\ 3 & -6 & 7 \end{vmatrix}$

8. Evaluate the following determinants:

(a) $\begin{vmatrix} 2 & 3 & -1 \\ 5 & -4 & 3 \\ 2 & 3 & -1 \end{vmatrix}$ (b) $\begin{vmatrix} 12 & -8 & 10 \\ -9 & 0 & 2 \\ 15 & 1 & 0 \end{vmatrix}$ (c) $\begin{vmatrix} 2 & 1 & -4 \\ -1 & 0 & 2 \\ 1 & 3 & -2 \end{vmatrix}$

(d) $\begin{vmatrix} 6 & -2 & 0 \\ -3 & 2 & -7 \\ 30 & 5 & 35 \end{vmatrix}$ (e) $\begin{vmatrix} -21 & 0 & 14 \\ 24 & 40 & 0 \\ 0 & -15 & 18 \end{vmatrix}$ (f) $\begin{vmatrix} 9 & -3 & 6 \\ 15 & 7 & 20 \\ -6 & 2 & -4 \end{vmatrix}$

9. Evaluate the following determinants using the elements a, b, c and their corresponding cofactors.

(a) $\begin{vmatrix} a & 2 & 1 \\ b & 4 & 3 \\ c & 3 & 2 \end{vmatrix}$ (b) $\begin{vmatrix} -1 & 2 & -3 \\ a & b & c \\ 2 & -5 & 1 \end{vmatrix}$ (c) $\begin{vmatrix} 1 & 6 & a \\ 2 & 7 & b \\ 1 & 4 & c \end{vmatrix}$

10. Evaluate the following determinants

(a) $\begin{vmatrix} 0 & 4 & 1 \\ -1 & 2 & -3 \\ 0 & 3 & 2 \end{vmatrix}$ (b) $\begin{vmatrix} 2 & -4 & 6 \\ 5 & -3 & -7 \\ 0 & 0 & 2 \end{vmatrix}$ (c) $\begin{vmatrix} 4 & 0 & -1 \\ -5 & 0 & 2 \\ 1 & 3 & -5 \end{vmatrix}$

11. $\mathbf{A} = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 3 & -2 \\ 6 & -3 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 0 & 4 & -2 \\ 0 & -1 & 1 \\ 3 & 2 & -1 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} -1 & 0 & 3 \\ 0 & 2 & 0 \\ 2 & 4 & -5 \end{pmatrix}$

Find $\det \mathbf{A}$, $\det \mathbf{B}$ and $\det \mathbf{C}$. Verify that $\det(\mathbf{AB}) = (\det \mathbf{A})(\det \mathbf{B}) = \det(\mathbf{BA})$ and that $\det(\mathbf{ABC}) = (\det \mathbf{A})(\det \mathbf{B})(\det \mathbf{C})$.

18.3 Inverse matrices: adjoint method

We consider again the determinant $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$. As we saw in the previous section, when the elements in any row or column are combined with the corresponding cofactors then the value of Δ is obtained e.g. $a_3A_3 + b_3B_3 + c_3C_3 = \Delta$, $c_1C_1 + c_2C_2 + c_3C_3 = \Delta$. However, if the elements of any row (or column) are combined with *alien cofactors*, i.e. with the cofactors of a different row (or column), the result is zero. For instance,

$$a_2A_1 + b_2B_1 + c_2C_1 = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

but since this determinant has two equal rows,

$$a_2A_1 + b_2B_1 + c_2C_1 = 0.$$

Using these results we find that

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix} = \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix} \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = \begin{pmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{pmatrix}$$

If $\mathbf{A} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ then the matrix $\begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}$, obtained by replacing each element of $\mathbf{A}$ by its cofactor and then *transposing*, is called the *adjoint* (or adjugate) of $\mathbf{A}$, written adj $\mathbf{A}$. Thus we have shown that

$$\boxed{\mathbf{A}(\text{adj}\,\mathbf{A}) = (\text{adj}\,\mathbf{A})\mathbf{A} = \Delta\mathbf{I}, \quad \text{where} \quad \Delta = \det \mathbf{A}}$$

Example 1 If $\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ -2 & 0 & 2 \end{pmatrix}$, find adj $\mathbf{A}$ and verify that

$$\mathbf{A}(\text{adj}\,\mathbf{A}) = (\text{adj}\,\mathbf{A})\mathbf{A} = (\det \mathbf{A})\mathbf{I}.$$

Using the minors with their appropriate signs, the cofactors of the elements of $\mathbf{A}$ are found to be:

$$\text{1st row:} \quad + \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2, \quad - \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = -5, \quad + \begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix} = 1,$$

$$\text{2nd row:} \quad - \begin{vmatrix} 2 & -1 \\ 0 & 2 \end{vmatrix} = -4, \quad + \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} = 1, \quad - \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} = -2,$$

$$\text{3rd row:} \quad + \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 3, \quad - \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = -3, \quad + \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3$$

Hence $\operatorname{adj}\mathbf{A} = \begin{pmatrix} 2 & -4 & 3 \\ -5 & 1 & -3 \\ 1 & -2 & -3 \end{pmatrix}$

$$\mathbf{A}(\operatorname{adj}\mathbf{A}) = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ -1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & -4 & 3 \\ -5 & 1 & -3 \\ 1 & -2 & -3 \end{pmatrix} = \begin{pmatrix} -9 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & -9 \end{pmatrix} = -9\mathbf{I}$$

$$(\operatorname{adj}\mathbf{A})\mathbf{A} = \begin{pmatrix} 2 & -4 & 3 \\ -5 & 1 & -3 \\ 1 & -2 & -3 \end{pmatrix}\begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ -1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} -9 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & -9 \end{pmatrix} = -9\mathbf{I}$$

Hence $\det\mathbf{A} = -9$ and $\mathbf{A}(\operatorname{adj}\mathbf{A}) = (\operatorname{adj}\mathbf{A})\mathbf{A} = (\det\mathbf{A})\mathbf{I}$.

Suppose now that $\mathbf{A}$ is a matrix with zero determinant, then for any matrix $\mathbf{B}$,

$$\det(\mathbf{AB}) = (\det\mathbf{A})(\det\mathbf{B}) = 0.$$

This means that it is impossible to find a matrix $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{I}$.

$\therefore$ if $\det\mathbf{A} = 0$, then $\mathbf{A}$ is a singular matrix with no inverse and $\mathbf{A}(\operatorname{adj}\mathbf{A}) = (\operatorname{adj}\mathbf{A})\mathbf{A} = 0$.

However, if $\mathbf{A}$ is a non-singular matrix, then

$$\mathbf{A}(\operatorname{adj}\mathbf{A}) = (\operatorname{adj}\mathbf{A})\mathbf{A} = \Delta\mathbf{I}, \quad \text{where} \quad \Delta \neq 0$$

$$\therefore \quad \mathbf{A}\left(\frac{1}{\Delta}\operatorname{adj}\mathbf{A}\right) = \left(\frac{1}{\Delta}\operatorname{adj}\mathbf{A}\right)\mathbf{A} = \mathbf{I}$$

Hence, provided that $\Delta \neq 0$, the inverse of $\mathbf{A}$ is the matrix $\frac{1}{\Delta}\operatorname{adj}\mathbf{A}$.

i.e. if $\det\mathbf{A} \neq 0$, then $\mathbf{A}^{-1} = \dfrac{\operatorname{adj}\mathbf{A}}{\det\mathbf{A}}$.

Example 2 Find the inverse of $\mathbf{A} = \begin{pmatrix} -1 & 3 & 2 \\ 2 & -6 & 0 \\ 1 & -2 & -4 \end{pmatrix}$

$$\operatorname{adj}\mathbf{A} = \begin{pmatrix} +24 & -(-8) & +2 \\ -(-8) & +2 & -(-1) \\ +12 & -(-4) & +0 \end{pmatrix}^T = \begin{pmatrix} 24 & 8 & 12 \\ 8 & 2 & 4 \\ 2 & 1 & 0 \end{pmatrix}$$

$\det\mathbf{A} = (-1)24 + 3 \times 8 + 2 \times 2 = 4$

Hence $\mathbf{A}^{-1} = \dfrac{\operatorname{adj}\mathbf{A}}{\det\mathbf{A}} = \dfrac{1}{4}\begin{pmatrix} 24 & 8 & 12 \\ 8 & 2 & 4 \\ 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 6 & 2 & 3 \\ 2 & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{4} & 0 \end{pmatrix}$.

Exercise 18.3

In questions 1 to 12 find the adjoint and, where possible, the inverse of the given matrix.

1. $\begin{pmatrix} 2 & -1 & -2 \\ 0 & 2 & 3 \\ -3 & 0 & 1 \end{pmatrix}$ 2. $\begin{pmatrix} 0 & -1 & 2 \\ 3 & 1 & 0 \\ -4 & 1 & -5 \end{pmatrix}$ 3. $\begin{pmatrix} 3 & 0 & -2 \\ 1 & -2 & 4 \\ 5 & -4 & 6 \end{pmatrix}$

4. $\begin{pmatrix} 1 & 2 & 1 \\ 3 & 0 & 4 \\ 2 & 1 & 1 \end{pmatrix}$ 5. $\begin{pmatrix} -2 & -3 & 1 \\ 4 & 1 & -2 \\ 1 & -3 & -2 \end{pmatrix}$ 6. $\begin{pmatrix} 2 & -1 & 3 \\ 0 & -3 & 2 \\ -1 & 0 & -2 \end{pmatrix}$

7. $\begin{pmatrix} -6 & 9 & 3 \\ 10 & -15 & -5 \\ -8 & 12 & 4 \end{pmatrix}$ 8. $\begin{pmatrix} 2 & 1 & 2 \\ 3 & 1 & 2 \\ 1 & 6 & 3 \end{pmatrix}$ 9. $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & 1 \\ 2 & 2 & -3 \end{pmatrix}$

10. $\begin{pmatrix} 1 & -2 & 3 \\ 1 & 0 & 1 \\ 1 & 2 & 3 \end{pmatrix}$ 11. $\begin{pmatrix} 5 & -2 & 3 \\ 1 & 1 & -5 \\ -10 & 4 & -6 \end{pmatrix}$ 12. $\begin{pmatrix} 1 & -6 & 2 \\ -2 & 5 & 0 \\ 4 & -1 & -5 \end{pmatrix}$

13. If $\mathbf{A} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 & 3 \\ 7 & 5 & 0 \\ 3 & 2 & 1 \end{pmatrix}$, find $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$.

Verify that (a) $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$ and $(\mathbf{B}^T)^{-1} = (\mathbf{B}^{-1})^T$, (b) $\det(\mathbf{AB}) = (\det \mathbf{A})(\det \mathbf{B})$, (c) $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ and $(\mathbf{BA})^{-1} = \mathbf{A}^{-1}\mathbf{B}^{-1}$.

[For further practice in finding inverses see Exercise 18.5.]

18.4 Sets of linear equations

In this section we consider sets of simultaneous linear equations of the form

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

Writing $\mathbf{A} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$, $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$, we obtain a single matrix equation:

$$\mathbf{Ar} = \mathbf{d}$$

If $\det \mathbf{A} \neq 0$, the matrix $\mathbf{A}$ has an inverse $\mathbf{A}^{-1}$,

$$\therefore \quad \mathbf{A}^{-1}(\mathbf{Ar}) = \mathbf{A}^{-1}\mathbf{d}$$
$$(\mathbf{A}^{-1}\mathbf{A})\mathbf{r} = \mathbf{A}^{-1}\mathbf{d}$$
$$\mathbf{Ir} = \mathbf{A}^{-1}\mathbf{d}$$
Thus $$\mathbf{r} = \mathbf{A}^{-1}\mathbf{d}$$

This result suggests one way of solving simultaneous linear equations.

Example 1 Solve the equations

$$\begin{aligned} x + 3y - 2z &= 1 \\ 2x \qquad + z &= 5 \\ - y + 2z &= 4 \end{aligned}$$

In matrix form these equations become:

$$\mathbf{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}, \quad \text{where } \mathbf{A} = \begin{pmatrix} 1 & 3 & -2 \\ 2 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$\operatorname{adj}\mathbf{A} = \begin{pmatrix} +1 & -4 & +(-2) \\ -4 & +2 & -(-1) \\ +3 & -5 & +(-6) \end{pmatrix}^T = \begin{pmatrix} 1 & -4 & 3 \\ -4 & 2 & -5 \\ -2 & 1 & -6 \end{pmatrix}$$

$$\det \mathbf{A} = 1 \times 1 + 3(-4) + (-2)(-2) = -7$$

$$\therefore \quad \mathbf{A}^{-1} = \frac{\operatorname{adj}\mathbf{A}}{\det \mathbf{A}} = \frac{1}{7}\begin{pmatrix} -1 & 4 & -3 \\ 4 & -2 & 5 \\ 2 & -1 & 6 \end{pmatrix}$$

$$\therefore \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} -1 & 4 & -3 \\ 4 & -2 & 5 \\ 2 & -1 & 6 \end{pmatrix}\begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 7 \\ 14 \\ 21 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Hence the solution of the given set of equations is $x = 1$, $y = 2$, $z = 3$.

Since the method of Example 1 involves a great deal of numerical calculation it is not regarded as an efficient way of solving simultaneous equations. Elementary methods involving elimination of variables are often better.

$$\begin{aligned} x + 3y - 2z &= 1 && (1) \\ 2x \qquad + z &= 5 && (2) \\ - y + 2z &= 4 && (3) \\ x \qquad + 4z &= 13 && (4) \\ 7z &= 21 \\ z &= 3 \end{aligned}$$

(1) + 3 × (3):
2 × (4) − (2):
$\therefore$

Substituting $z = 3$ in (2) and (3) we find that, as before, $x = 1$, $y = 2$, $z = 3$. This approach can be developed into a systematic elimination process which can be applied to any set of linear equations in three or more variables.

Suppose that
$$2x + y - 3z = 3 \quad (1)$$
$$x + 3y - 5z = 1 \quad (2)$$
$$6x - 2y + z = 9 \quad (3)$$

Using (1) to eliminate x from (2) and (3):

(1): $2x + y - 3z = 3$ (4)
$2 \times (2) - (1)$: $5y - 7z = -1$ (5)
$(3) - 3 \times (1)$: $-5y + 10z = 0$ (6)

Using (5) to eliminate y from (6) we produce the following set of equations:

(4): $2x + y - 3z = 3$
(5): $5y - 7z = -1$
(5) + (6): $3z = -1$

This new set of three equations is said to be in *echelon form*. In this context echelon means stepwise arrangement or ladder. To solve the equations we now work up the "ladder" obtaining $z = -\frac{1}{3}$, then $y = -\frac{2}{3}$ and $x = \frac{4}{3}$.

In reducing these equations to echelon form we have changed the matrix of coefficients from

$$\begin{pmatrix} 2 & 1 & -3 \\ 1 & 3 & -5 \\ 6 & -2 & 1 \end{pmatrix} \text{ to } \begin{pmatrix} 2 & 1 & -3 \\ 0 & 5 & -7 \\ 0 & 0 & 3 \end{pmatrix}$$

The new matrix has non-zero elements on or above the *leading diagonal* and zeros below it. Any matrix which either has zeros below the leading diagonal or has zeros above the leading diagonal is called a *triangular matrix*. In general, a set of linear equations is said to be in echelon form if the corresponding matrix is triangular.

Any systematic reduction method can be set out in table form. Each stage in the process is described by expressing the new rows r'_1, r'_2, r'_3 in terms of the previous rows r_1, r_2, r_3.

	Coefficients of x	y	z	Constant term
	2	1	−3	3
	1	3	−5	1
	6	−2	1	9
	2	1	−3	3
$r'_2 = 2r_2 - r_1$	0	5	−7	−1
$r'_3 = r_3 - 3r_1$	0	−5	10	0
	2	1	−3	3
	0	5	−7	−1
$r'_3 = r_2 + r_3$	0	0	3	−1

$\therefore$ as before, $z = -\frac{1}{3}$, $y = -\frac{2}{3}$, $x = \frac{4}{3}$.

We now consider a geometrical interpretation of the equations

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

These can be regarded as the equations of three planes in three-dimensional space.

If $\det \mathbf{A} \neq 0$, the equations have a unique solution and thus the planes must intersect in a single point. As we have seen, the coordinates of this point can be found either by using the inverse matrix $\mathbf{A}^{-1}$ or by reducing the equations to echelon form.

If $\det \mathbf{A} = 0$, the nature of the solution set of the equations depends on the way in which the three planes intersect.

Case (i): The planes have no common point. This situation arises when at least two of the planes are parallel (but not coincident), or when one plane is parallel to the line of intersection of the other two. The equations are then said to be inconsistent and have no solution.

Case (ii): The planes have a common line. This means that the equations will have an infinite solution set corresponding to the set of points on the line of intersection of the planes.

Case (iii): The planes are coincident. Again the equations have an infinite solution set, but this time the solutions are the coordinates of the points in the plane represented by all three equations.

Example 2 Discuss the solutions of the given equations when (a) $k = 5$, (b) $k = 6$.

$$\begin{aligned} 2x - 3y + z &= 1 \quad (1) \\ x + 2y - z &= 3 \quad (2) \\ 2x + 4y - 2z &= k \quad (3) \end{aligned}$$

(a) When $k = 5$, equations (2) and (3) represent parallel planes. Hence the equations are inconsistent i.e. have no solution.

(b) When $k = 6$, equations (2) and (3) represent the same plane. Hence any solution of equations (1) and (2) also satisfies (3). Substituting $x = \lambda$ in (1) and (2) we have:

$$\begin{aligned} 2\lambda - 3y + z &= 1 \quad (4) \\ \lambda + 2y - z &= 3 \quad (5) \end{aligned}$$

(4) + (5): $3\lambda - y = 4 \quad \therefore \quad y = 3\lambda - 4$

From (5): $\lambda + 6\lambda - 8 - z = 3 \quad \therefore \quad z = 7\lambda - 11$

Hence the solutions are $x = \lambda$, $y = 3\lambda - 4$, $z = 7\lambda - 11$, where λ may take any real value.

Example 3 Discuss the solutions of the given equations when (a) $k = 0$, (b) $k = 3$.

$$\begin{aligned} 2x - y + z &= 1 \quad (1) \\ 2x - 3y - z &= 3 \quad (2) \\ 4x - 3y + z &= k \quad (3) \end{aligned}$$

	x	y	z	constant
	2	−1	1	1
	2	−3	−1	3
	4	−3	1	k
	2	−1	1	1
$r_2' = r_1 - r_2$	0	2	2	−2
$r_3' = 2r_1 - r_3$	0	1	1	$2 - k$
	2	−1	1	1
$r_2' = \frac{1}{2}r_2$	0	1	1	−1
$r_3' = r_3 - \frac{1}{2}r_2$	0	0	0	$3 - k$

(a) If $k = 0$, there is a contradiction in the new third row. Hence the equations are inconsistent.
(b) If $k = 3$, we have a line of zeros in the new third row together with the equations

$$\begin{aligned} 2x - y + z &= 1 \\ y + z &= -1 \end{aligned}$$

Hence in parametric form the solutions are $x = \lambda$, $y = \lambda - 1$, $z = -\lambda$.
This set of solutions represents the line of intersection of the planes with equations (1), (2) and (3).

Example 4 Discuss the solutions of the equations

$$\begin{aligned} 4x - 2y + 6z &= 2 \\ -2x + y - 3z &= -1 \\ 6x - 3y + 9z &= 3 \end{aligned}$$

Since all three equations represent the same plane their solution set represents the points in this plane. The solutions may be expressed in parametric form by writing:

$$x = \lambda, \quad y = \mu, \quad z = \tfrac{1}{3}(1 - 2\lambda + \mu).$$

Exercise 18.4

1. Find the inverse of the matrix $\begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 2 & 0 & -1 \end{pmatrix}$ and hence solve the equations:

(a) $\begin{aligned} x - y - 2z &= 1 \\ y + z &= 1 \\ 2x - z &= 1 \end{aligned}$ (b) $\begin{aligned} p - q - 2r &= 2 \\ q + r &= -1 \\ 2p - r &= 3 \end{aligned}$ (c) $\begin{aligned} a - b - 2c &= 0 \\ b + c &= 0 \\ 2a - c &= 0 \end{aligned}$

2. Find the inverse of the matrix $\begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 3 & 1 & 2 \end{pmatrix}$ and hence solve the equations:

(a) $\begin{aligned} 2x - y + z &= 1 \\ x + 2y - z &= 1 \\ 3x + y + 2z &= -2 \end{aligned}$

(b) $\begin{aligned} 2p - q + r &= 0 \\ p + 2q - r &= 3 \\ 3p + q + 2r &= -1 \end{aligned}$

(c) $\begin{aligned} 2a - b + c &= -1 \\ a + 2b - c &= 3 \\ 3a + b - 2c &= 2 \end{aligned}$

3. Use any method to solve for x, y and z the equations

(a) $\begin{aligned} 2x + 6y + z &= 0 \\ -x + 2y - z &= 10 \\ 4x + 3y + z &= 1 \end{aligned}$

(b) $\begin{aligned} 3x - 2y + 4z &= -7 \\ x + y - 6z &= -10 \\ 2x + 3y + 2z &= 3 \end{aligned}$

(c) $\begin{aligned} 4x + y + z &= 3 \\ x + 4y + z &= -1 \\ x + y + 4z &= -2 \end{aligned}$

(d) $\begin{aligned} x + 2y + z &= 0 \\ 3x - 2y - 5z &= 8 \\ 2x + 5y - z &= 6 \end{aligned}$

(e) $\begin{aligned} 3x - 2y + z &= -1 \\ 4x + y + 3z &= 2 \\ -x + 5y + 2z &= 5 \end{aligned}$

(f) $\begin{aligned} 9x - 10y + 2z &= -6 \\ x + 3y + 6z &= 13 \\ 4x - y + 8z &= 13 \end{aligned}$

4. Discuss the solutions of the given equations when (a) $k = 4$, (b) $k = 8$.

$$\begin{aligned} 3x + 3y + z &= 10 \\ x - y - z &= -4 \\ -2x + 2y + 2z &= k \end{aligned}$$

5. Discuss the solutions of the given equations when (a) $k = 9$, (b) $k = -9$.

$$\begin{aligned} x + y - z &= k \\ 3x - 2y + z &= 16 \\ 9x - y - z &= 5 \end{aligned}$$

6. Discuss the solutions of the given equations when (a) $k = -15$, (b) $k = -10$

$$\begin{aligned} 12x - 9y + 3z &= k \\ -4x + 3y - z &= 5 \\ 8x - 6y + 2z &= -10 \end{aligned}$$

7. Discuss the solutions of the given equations when (a) $k = -1$, (b) $k = +1$.

$$\begin{aligned} -4x + 12y + 8z &= -16 \\ kx - 3y - (k+1)z &= 4 \\ 2x - 6y - 4z &= 8 \end{aligned}$$

8. Show that the equations
$$\begin{aligned} 2x - y + z &= 3 \\ x + 2y - 3z &= 1 \\ 3x + y - 2z &= k \end{aligned}$$
are inconsistent when $k = 2$, but have an infinite set of solutions when $k = 4$. Obtain these solutions in parametric form.

9. Find the values of k for which the following sets of equations have solutions and find these solutions

(a)
$$\begin{aligned} 2x + y - z &= 5 \\ x - y + z &= 1 \\ x - 2y + 2z &= k \end{aligned}$$

(b)
$$\begin{aligned} 3x + y - z &= 1 \\ x + y + 3z &= 4 \\ x - 2z &= k \end{aligned}$$

10. Evaluate, in terms of k, the determinant $\begin{vmatrix} 1 & 1 & -3 \\ 3 & -1 & -1 \\ 5 & -3 & k \end{vmatrix}$.

Hence find the value of k such that the equations
$$\begin{aligned} x + y - 3z &= 1 \\ 3x - y - z &= 7 \\ 5x - 3y + kz &= 13 \end{aligned}$$
do not have a unique solution and obtain the general solution in this case.

11. Show that the equations
$$\begin{aligned} 2x + y + 2z &= 1 \\ 3x - z &= 3 \\ 4x - y + az &= b \end{aligned}$$
(i) have a unique solution unless $a = -4$; (ii) are inconsistent when $a = -4$ and $b = 2$; (iii) have an infinite number of solutions when $a = -4$ and $b = 5$. Interpret each case geometrically.

12. Solve the simultaneous equations
$$\begin{aligned} x + 2y - z &= 6, \\ 2x - y - z &= 1, \\ y + z &= 1. \end{aligned}$$

Find a if the set of equations
$$\begin{aligned} x + 2y - z &= 6, \\ 2x - y - z &= 1, \\ ay + z &= 1, \end{aligned}$$

has no solutions. Find a and b if the set of equations

$$\begin{aligned} x + 2y - z &= 6, \\ 2x - y - z &= 1, \\ ay + z &= b, \end{aligned}$$

has more than one solution. Find the solution for which $y = \lambda$. (AEB 1980)

18.5 Inverse matrices: row reduction method

The systematic reduction of a set of linear equations by means of row operations on the matrix of coefficients suggests another way of obtaining the inverse of a matrix. We illustrate the method by finding the inverse of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 2 & -1 & 2 \end{pmatrix}$$

Let $$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

then

$$\begin{aligned} x - y &= p && (1) \\ -x + 2y + z &= q && (2) \\ 2x - y + 2z &= r && (3) \end{aligned}$$

Using (1) to eliminate x from (2) and (3):

$$\begin{aligned} &\text{(1):} && x - y = p && (4) \\ &\text{(2) + (1):} && y + z = p + q && (5) \\ &\text{(3) − (2) × (1):} && y + 2z = -2p + r && (6) \end{aligned}$$

Using (5) to eliminate y from (4) and (6):

$$\begin{aligned} &\text{(4) + (5):} && x + z = 2p + q && (7) \\ &\text{(5):} && y + z = p + q && (8) \\ &\text{(6) − (5):} && z = -3p - q + r && (9) \end{aligned}$$

Using (9) to eliminate z from (7) and (8):

$$\begin{aligned} &\text{(7) − (9):} && x = 5p + 2q - r \\ &\text{(8) − (9):} && y = 4p + 2q - r \\ &\text{(9):} && z = -3p - q + r \end{aligned}$$

Thus $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 & 2 & -1 \\ 4 & 2 & -1 \\ -3 & -1 & 1 \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

Hence $$\mathbf{A}^{-1} = \begin{pmatrix} 5 & 2 & -1 \\ 4 & 2 & -1 \\ -3 & -1 & 1 \end{pmatrix}$$

Using the notation of the previous section, it is quicker to set out the working as follows:

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 2 & -1 & 2 \end{pmatrix} \quad \vdots \quad \mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$r'_2 = r_2 + r_1$, $r'_3 = r_3 - 2r_1$

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \quad \vdots \quad \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

$r'_2 = r_1 + r_2$, $r'_3 = r_3 - r_2$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \vdots \quad \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ -3 & -1 & 1 \end{pmatrix}$$

$r'_1 = r_1 - r_3$, $r'_2 = r_2 - r_3$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I} \quad \vdots \quad \begin{pmatrix} 5 & 2 & -1 \\ 4 & 2 & -1 \\ -3 & -1 & 1 \end{pmatrix} = \mathbf{A}^{-1}$$

In the following more typical example a further step in the process is needed.

Example 1 Find the inverse of the matrix $\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ -1 & 0 & 2 \end{pmatrix}$

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ -1 & 0 & 2 \end{pmatrix} \quad \vdots \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$r'_2 = r_2 - 2r_1$, $r'_3 = r_3 + r_1$

$$\begin{pmatrix} 1 & 2 & -1 \\ 0 & -3 & 3 \\ 0 & 2 & 1 \end{pmatrix} \quad \vdots \quad \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$r'_1 = 3r_1 + 2r_2$, $r'_3 = 3r_3 + 2r_2$

$$\begin{pmatrix} 3 & 0 & 3 \\ 0 & -3 & 3 \\ 0 & 0 & 9 \end{pmatrix} \quad \vdots \quad \begin{pmatrix} -1 & 2 & 0 \\ -2 & 1 & 0 \\ -1 & 2 & 3 \end{pmatrix}$$

$r'_1 = 3r_1 - r_3$, $r'_2 = 3r_2 - r_3$

$$\begin{pmatrix} 9 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & 9 \end{pmatrix} \quad \vdots \quad \begin{pmatrix} -2 & 4 & -3 \\ -5 & 1 & -3 \\ -1 & 2 & 3 \end{pmatrix}$$

$r'_1 = r_1/9$, $r'_2 = -r_2/9$, $r'_3 = r_3/9$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \vdots \quad \begin{pmatrix} -\frac{2}{9} & \frac{4}{9} & -\frac{3}{9} \\ \frac{5}{9} & -\frac{1}{9} & \frac{3}{9} \\ -\frac{1}{9} & \frac{2}{9} & \frac{3}{9} \end{pmatrix}$$

Hence $\mathbf{A}^{-1} = \begin{pmatrix} -\frac{2}{9} & \frac{4}{9} & -\frac{1}{3} \\ \frac{5}{9} & -\frac{1}{9} & \frac{1}{3} \\ -\frac{1}{9} & \frac{2}{9} & \frac{1}{3} \end{pmatrix}$ or $\frac{1}{9}\begin{pmatrix} -2 & 4 & -3 \\ 5 & -1 & 3 \\ -1 & 2 & 3 \end{pmatrix}$

In some examples it is necessary to interchange rows to avoid zeros on the main diagonal.

Example 2 Find the inverse of the matrix $\mathbf{A} = \begin{pmatrix} -1 & 3 & 2 \\ 2 & -6 & 0 \\ 1 & -2 & -4 \end{pmatrix}$

$$\begin{pmatrix} -1 & 3 & 2 \\ 2 & -6 & 0 \\ 1 & -2 & -4 \end{pmatrix} \quad \vdots \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$r_2' = r_2 + 2r_1$, $r_3' = r_3 + r_1$

$$\begin{pmatrix} -1 & 3 & 2 \\ 0 & 0 & 4 \\ 0 & 1 & -2 \end{pmatrix} \quad \vdots \quad \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$r_2' = r_3$, $r_3' = r_2$

$$\begin{pmatrix} -1 & 3 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 4 \end{pmatrix} \quad \vdots \quad \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

$r_1' = r_1 - 3r_2$

$$\begin{pmatrix} -1 & 0 & 8 \\ 0 & 1 & -2 \\ 0 & 0 & 4 \end{pmatrix} \quad \vdots \quad \begin{pmatrix} -2 & 0 & -3 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

$r_1' = r_1 - 2r_3$, $r_2' = 2r_2 + r_3$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \quad \vdots \quad \begin{pmatrix} -6 & -2 & -3 \\ 4 & 1 & 2 \\ 2 & 1 & 0 \end{pmatrix}$$

$r_1' = -r_1$, $r_2' = \frac{1}{2}r_2$, $r_3' = \frac{1}{4}r_3$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \vdots \quad \begin{pmatrix} 6 & 2 & 3 \\ 2 & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{4} & 0 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} 6 & 2 & 3 \\ 2 & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{4} & 0 \end{pmatrix} \quad \text{or} \quad \frac{1}{4}\begin{pmatrix} 24 & 8 & 12 \\ 8 & 2 & 4 \\ 2 & 1 & 0 \end{pmatrix}$$

The next example shows how the method breaks down for a matrix which has no inverse.

Example 3 Find, if possible, the inverse of the matrix $\mathbf{A} = \begin{pmatrix} 2 & -1 & 3 \\ -1 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix}$.

$$\left(\begin{array}{ccc} 2 & -1 & 3 \\ -1 & 3 & 1 \\ 1 & 2 & 4 \end{array}\right) \quad \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

$r_2' = 2r_2 + r_1$, $r_3' = 2r_3 - r_1$

$$\left(\begin{array}{ccc} 2 & -1 & 3 \\ 0 & 5 & 5 \\ 0 & 5 & 5 \end{array}\right) \quad \left(\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -1 & 0 & 2 \end{array}\right)$$

$r_1' = 5r_1 + r_2$

$$\left(\begin{array}{ccc} 10 & 0 & 20 \\ 0 & 5 & 5 \\ 0 & 0 & 0 \end{array}\right) \quad \left(\begin{array}{ccc} 6 & 2 & 0 \\ 1 & 2 & 0 \\ -2 & -2 & 2 \end{array}\right)$$

$r_3' = r_3 - r_2$

Since a complete row of zeros has been obtained, no inverse of the matrix **A** can be found.

[Note that the method of finding inverses using elementary row operations can be extended to square matrices of any order. For large matrices this process is usually more efficient than the adjoint method introduced in §18.3.]

Exercise 18.5

In questions 1 to 6 find, if possible, the inverses of the given matrices.

1. (a) $\left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{array}\right)$ (b) $\left(\begin{array}{ccc} 1 & -1 & 1 \\ -1 & 2 & -2 \\ 0 & 0 & 1 \end{array}\right)$

   (c) $\left(\begin{array}{ccc} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{array}\right)$ (d) $\left(\begin{array}{ccc} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 5 & -2 & 1 \end{array}\right)$

2. (a) $\left(\begin{array}{ccc} 1 & -1 & 1 \\ -3 & 4 & -2 \\ 2 & 0 & 5 \end{array}\right)$ (b) $\left(\begin{array}{ccc} 1 & -3 & 1 \\ -1 & 4 & -2 \\ 2 & -1 & -2 \end{array}\right)$

   (c) $\left(\begin{array}{ccc} 1 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{array}\right)$ (d) $\left(\begin{array}{ccc} 1 & -2 & 0 \\ -2 & 3 & -1 \\ 1 & 1 & 2 \end{array}\right)$

3. (a) $\left(\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & -1 & -3 \end{array}\right)$ (b) $\left(\begin{array}{ccc} 1 & -2 & 0 \\ 3 & 1 & 1 \\ 0 & 1 & 0 \end{array}\right)$

   (c) $\left(\begin{array}{ccc} 1 & -5 & -3 \\ -3 & 2 & -4 \\ 4 & 1 & 9 \end{array}\right)$ (d) $\left(\begin{array}{ccc} -3 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & 0 & 0 \end{array}\right)$

4. (a) $\begin{pmatrix} -1 & 0 & 1 \\ -2 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & -2 & 1 \\ -1 & 0 & -3 \\ 4 & -3 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 3 & 2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{pmatrix}$ (d) $\begin{pmatrix} 4 & 2 & 3 \\ 0 & -1 & -2 \\ 2 & 4 & 7 \end{pmatrix}$

5. (a) $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 0 & -3 \\ 0 & 0 & -1 \\ 2 & 2 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 1 & 0 \\ 5 & 2 & 1 \\ 3 & 4 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & 2 & -10 \\ 4 & 2 & 2 \\ 7 & 3 & 6 \end{pmatrix}$

6. (a) $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$

7. Given that $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -3 & -2 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 3 & 5 \\ 0 & -1 & -1 \end{pmatrix}$, find $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$.

Use your answers to verify that
(a) $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$, (b) $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$, (c) $(\mathbf{B}^2)^{-1} = (\mathbf{B}^{-1})^2$.

8. Find the inverses of the matrices $\mathbf{A} = \begin{pmatrix} 1 & -1 & 3 \\ 1 & 0 & 4 \\ 2 & -4 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

Hence find $\mathbf{P}^{-1}$ and $\mathbf{Q}^{-1}$, where $\mathbf{P} = \mathbf{AB}$ and $\mathbf{Q} = \mathbf{B}^2\mathbf{A}$.

[For further practice in finding inverses see Exercise 18.3.]

Exercise 18.6 (*miscellaneous*)

1. Describe geometrically the transformations represented by the following matrices.

(a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} \frac{1}{2}\sqrt{3} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2}\sqrt{3} \end{pmatrix}$

2. Write down the 3×3 matrices representing the following transformations of three-dimensional space. (a) Reflection in the plane $y = x \tan\theta$, (b) $180°$ rotation about the line $y = x \tan\theta$ in the x, y plane.

3. T_1, T_2, T_3 are linear transformations of three-dimensional space. T_1 is reflection in the xy-plane; T_2 is rotation about the z-axis through $90°$ so that the positive x-axis is turned into the positive y-axis; T_3 is rotation through $180°$ about the line which lies in the xy-plane and has equation $y = x$. Give the coordinates of the images of the point (x, y, z) under each of these transformations, and hence find the matrix of each transformation. Find the matrices of the combined transformations $T_3 T_2$ and $T_2 T_3$ and describe each of these transformations geometrically. Describe also the transformations $T_3 T_2 T_1$ and $T_1 T_2 T_3$. (L)

4. Show that the linear transformation with matrix $\begin{pmatrix} 1 & -3 & 0 \\ 2 & -4 & 3 \\ 1 & 1 & 6 \end{pmatrix}$ maps every point of three-dimensional space onto a single plane and give the Cartesian equation of this plane. Describe the set of points mapped onto the origin under this transformation.

5. The transformations $\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$ and $\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$ represent reflections of the plane; find the axes of these two reflections. Denote the matrices of reflections by R and S respectively; what is the transformation of the plane determined by the matrix SR? (W)

6. If $\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & -1 \\ -1 & 2 & -1 \end{pmatrix}$, find (a) $\mathbf{AB}$, (b) a matrix $\mathbf{X}$ such that $\mathbf{AX} + \mathbf{B} = \mathbf{A}$. (O&C)

7. $\mathbf{P}$ is a 3×3 matrix such that $\mathbf{P}^2 = \mathbf{P} - \mathbf{I}$. (a) Find $\mathbf{P}^{-1}$ in terms of $\mathbf{P}$ and $\mathbf{I}$. (b) Show that $\mathbf{P}^3 + \mathbf{I} = \mathbf{0}$. (c) If $\mathbf{PQ} = 2\mathbf{I} - \mathbf{P}$, find $\mathbf{Q}$ in the form $\lambda\mathbf{I} + \mu\mathbf{P}$, where λ and μ are constants to be determined.

8. If $\mathbf{A}$ is a 3×3 matrix with transpose $\mathbf{A}^T$, then $\mathbf{A}$ is said to be *symmetric* if $\mathbf{A}^T = \mathbf{A}$ and *skew-symmetric* if $\mathbf{A}^T = -\mathbf{A}$. Prove that
(a) if $\mathbf{A}$ is both symmetric and skew-symmetric, then $\mathbf{A}$ is the zero matrix;
(b) if $\mathbf{A}$ is skew-symmetric then $\mathbf{A}^2$ is symmetric;
(c) if $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{AB}$ are symmetric, then $\mathbf{AB} = \mathbf{BA}$. [You may assume that $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.]

9. (a) Find the inverse of the matrix $A = \begin{pmatrix} 2 & 3 & -5 \\ 1 & 2 & -1 \\ -1 & -1 & 5 \end{pmatrix}$.

(b) Calculate the matrix product BC, where

$$B = \begin{pmatrix} 1 & 1 & -2 \\ 2 & -1 & 1 \\ 2 & -2 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & -1 & 1 \\ 4 & -7 & 5 \\ 2 & -4 & 3 \end{pmatrix},$$

and hence or otherwise find the matrix X such that $CX = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$. (W)

10. Obtain the multiplicative inverse, where it exists, of each of the following matrices. If there is no inverse, explain briefly why this is so.

(i) $\begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & -2 \\ -3 & 6 \end{pmatrix}$ (iii) $\begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$

(iv) $\begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 4 \end{pmatrix}$ (v) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ (AEB 1979)

11. Find the inverse of the matrix $\begin{pmatrix} 1 & -3 & 0 \\ 2 & 0 & 1 \\ 4 & 1 & 3 \end{pmatrix}$. Hence solve the equations $x - 3y - a = 0, 2x + z - b = 0$ and $4x + y + 3z - c = 0$ for x, y, z in terms of a, b, c. (O&C)

12. Find the inverse of the matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 & -1 \\ 3 & 4 & -2 \\ -1 & 1 & 4 \end{pmatrix}$ by row reduction, or otherwise. Hence solve the simultaneous equations

$$\begin{aligned} x + y - z &= 1 \\ 3x + 4y - 2z &= 3 \\ -x + y + 4z &= 2, \end{aligned}$$

and find the matrix $\mathbf{X}$ such that $\mathbf{XA} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}$. (W)

13. Find the inverse of the matrix $A = \begin{pmatrix} 2 & -1 & 1 \\ 2 & -2 & 3 \\ 1 & 1 & -2 \end{pmatrix}$ by row reduction, or

otherwise. Hence solve the simultaneous equations

$$\begin{aligned} 2x - y + z &= 1 \\ 2x - 2y + 3z &= -1 \\ x + y - 2z &= 3. \end{aligned}$$

Calculate the matrix $B = A^{-1}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} A$. *Without calculating* B^2 *directly*, show that $B^2 = I_3$ (the 3×3 identity matrix). (W)

14. Find the values of α for which the equations

$$\begin{aligned} x + 2y + 3z &= 1 \\ 5x + y + 3z &= \alpha \\ 3x + 9y + 13z &= \alpha^2 \end{aligned}$$

have solutions. Find all the solutions for each value of α. (AEB 1977)

15. Matrices **A**, **B** and vectors **X**, **K** are given by

$$\mathbf{A} = \begin{pmatrix} 1 & -3 & 1 \\ 2 & -4 & 3 \\ 3 & -1 & 4 \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & -3 & -4 \\ -1 & 3 & 5 \\ 2 & -3 & -5 \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix}.$$

(i) Use a *matrix* method to solve the system of equations $\mathbf{AX} = \mathbf{K}$.

(ii) Find values of the constants λ, μ such that the rows of matrix **B** are connected by the relation $R_3 = \lambda R_1 + \mu R_2$. Hence explain why the system of equations $\mathbf{BX} = \mathbf{K}$ would be consistent (i.e. they do not possess a solution set.) (W)

16. Find the matrix of the linear transformation which maps $U(2, 1, 1)$ to $U'(4, 2, 2)$, $V(3, 2, 1)$ to $V'(-3, -2, 1)$ and $W(0, 0, 1)$ to $W'(0, 0, -1)$.

17. For the matrix equation $\mathbf{M}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 4 + k \end{pmatrix}$, where $\mathbf{M} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & k \end{pmatrix}$, find the value of k for which the equation does not have a unique solution. For this value of k, solve the equation and interpret the solution geometrically in relation to the linear transformation with matrix **M**.

18. The linear transformation $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \mathbf{M}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, where **M** is a 3×3 matrix, maps the points with position vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ to the points with position

vectors $\begin{pmatrix}3\\2\\1\end{pmatrix}$, $\begin{pmatrix}2\\1\\0\end{pmatrix}$, $\begin{pmatrix}1\\0\\0\end{pmatrix}$ respectively. Write down the matrix $\mathbf{M}$ and find the inverse matrix $\mathbf{M}^{-1}$. Show that the transformation with matrix $\mathbf{M}$ maps points of the plane $x + y + z = 0$ to points of the plane $x = y$, and verify that the inverse transformation with matrix $\mathbf{M}^{-1}$ maps points of the plane $x = y$ to points of the plane $x + y + z = 0$. (L)

19. $\mathbf{M}$ is the matrix $\begin{pmatrix}3 & 1 & -3\\1 & 2a & 1\\0 & 2 & a\end{pmatrix}$.

(i) Find the two values of a for which $\mathbf{M}$ is singular.

(ii) Solve the equation $\mathbf{M}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-3\frac{1}{2}\\5\frac{1}{2}\\5\end{pmatrix}$ in the case $a = 2$, and also determine whether or not solutions exist for each of the two values of a found in (i). (C)

19 Binary operations and groups

19.1 Definition of a group

A *binary operation* $*$ is a rule for combining two objects, a and b, to give a third object $c = a * b$.

We have already met many examples of binary operations e.g. addition of real numbers, intersection of sets, multiplication of matrices, composition of functions.

A set S is said to be *closed* under a binary operation $*$, if $a * b \in S$ for all elements $a, b \in S$.

For instance, if a and b are integers, then $a - b$ is also an integer
$\therefore$ the set of integers $\mathbb{Z}$ is closed under subtraction.
However, although 2 and 7 are positive integers, $2 - 7$ is not a positive integer,
$\therefore$ the set of positive integers $\mathbb{Z}^+$ is not closed under subtraction.

An element e of a set S is said to be an *identity element* for the binary operation $*$ if $e * a = a * e = a$ for all $a \in S$.

For example, since $\varnothing \cup A = A \cup \varnothing = A$ for any set A, the empty set $\varnothing$ is an identity element for the set union operation.
However, we can find no real number e such that $e \div a = a \div e = a$ for all $a \in \mathbb{R}$. Thus there is no identity element for division of real numbers.

[It will be shown in §19.4 that a set contains at most one identity element for any given binary operation.]

An *inverse* of an element a with respect to a binary operation $*$ is an element a^{-1} such that $a * a^{-1} = a^{-1} * a = e$.

We already know that the inverse of a non-zero real number a with respect to multiplication is $1/a$ and that the inverse of a non-singluar matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with respect to matrix multiplication is $\frac{1}{\Delta}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, where $\Delta = ad - bc$.
However, for a non-empty set A there is no set B such that $A \cup B = B \cup A = \varnothing$. Therefore the set A has no inverse with respect to the operation $\cup$.

A binary operation $*$ on a set S is *associative* if $a*(b*c) = (a*b)*c$ for all $a, b, c \in S$. The operation is *commutative* if

$$a*b = b*a \quad \text{for all} \quad a, b \in S.$$

For instance, union of sets is both associative and commutative; multiplication of matrices is associative but not commutative; subtraction of integers is neither associative nor commutative.

When investigating the properties of binary operation, it is often helpful to draw up an *operation table*. The following tables show two operations on the set $S = \{0, 1, 2, 3\}$, subtraction $-$ and the operation $*$, where $a*b = |a - b|$.

(1)

		Second element			
	$-$	0	1	2	3
	0	0	−1	−2	−3
First	1	1	0	−1	−2
element	2	2	1	0	−1
	3	3	2	1	0

(2)

		Second element			
	$*$	0	1	2	3
	0	0	1	2	3
First	1	1	0	1	2
element	2	2	1	0	1
	3	3	2	1	0

Since table (1) contains elements such as -1 which do not belong to S, the set S is not closed under $-$. All the entries in table (2) are elements of S, so S is closed under $*$.

There is no identity element for $-$ and hence the elements of S do not have inverses with respect to $-$. However, 0 is an identity element for $*$ and table (2) shows that every element of S is self-inverse.

For a commutative operation the operation table is symmetrical about the main diagonal. Thus we see that table (2) has this symmetry, but table (1) does not.

We cannot use the tables directly to decide whether or not the operations $-$ and $*$ are associative. In each case we must either prove the associative law or produce a counter-example to show that it does not hold.

Since $1 - (2 - 3) = 2$, whereas $(1 - 2) - 3 = -4$
and $1*(2*3) = 0$, whereas $(1*2)*3 = 2$,

we must conclude that neither operation is associative.

When studying the properties of a set and its associated binary operations it is useful to be able to identify patterns which can also be seen in other algebraic systems. One structure which occurs in many branches of mathematics is called a group.

A *group* $(G, *)$ is a set G with binary operation $*$ such that,
(1) G is closed under $*$ i.e. $a * b \in G$ for all $a, b \in G$.
(2) $*$ is associative i.e. $a * (b * c) = (a * b) * c$ for all $a, b, c \in G$.
(3) There is an identity element e in G such that $e * a = a * e = a$ for all $a \in G$.
(4) Every element a of G has an inverse a^{-1} in G such that $a * a^{-1} = a^{-1} * a = e$.

One familiar example of a group is $(\mathbb{Z}, +)$, the set of integers under addition.
$\mathbb{Z}$ is closed under addition.
Addition of integers is associative.
0, the identity element for addition, belongs to $\mathbb{Z}$.
Every integer a has an additive inverse $(-a)$, which is also an integer.
Since $\mathbb{Z}$ is an infinite set, $(\mathbb{Z}, +)$ is called an *infinite group*.

In the worked examples which follow we consider some finite groups. The *order* of a *finite group* is the number of elements it contains.

Example 1 Show that the set $A = \{1, i, -1, -i\}$, where $i^2 = -1$, forms a group under multiplication.

$\times$	1	i	-1	$-i$
1	1	i	-1	$-i$
i	i	-1	$-i$	1
-1	-1	$-i$	1	i
$-i$	$-i$	1	i	-1

(1) The operation table shows that A is closed under multiplication.
(2) Multiplication of complex numbers is associative.
(3) A contains 1, the identity element for multiplication.
(4) Since 1 appears in every row and column of the table, every element of A has an inverse in A. [1 and -1 are self-inverse, the inverse of i is $-i$ and the inverse of $-i$ is i.]

Hence A forms a group under multiplication.

Example 2 Given that $f(x) = x$, $g(x) = 1 - \frac{1}{x}$, $h(x) = \frac{1}{1-x}$, for all real x except $x = 0$, $x = 1$, show that the set $S = \{f, g, h\}$ forms a group under composition of functions.

$$f^2(x) = f(x), \quad fg(x) = gf(x) = g(x), \quad fh(x) = hf(x) = h(x),$$

$$g^2(x) = g\left(1 - \frac{1}{x}\right) = 1 - 1\Big/\left(1 - \frac{1}{x}\right) = 1 - \frac{x}{x-1} = \frac{1}{1-x} = h(x)$$

$$gh(x) = g\left(\frac{1}{1-x}\right) = 1 - 1\Big/\left(\frac{1}{1-x}\right) = 1 - (1-x) = x = f(x)$$

$$hg(x) = h\left(1 - \frac{1}{x}\right) = 1\Big/\left\{1 - \left(1 - \frac{1}{x}\right)\right\} = 1\Big/\left\{\frac{1}{x}\right\} = x = f(x)$$

$$h^2(x) = h\left(\frac{1}{1-x}\right) = 1\Big/\left\{1 - \left(\frac{1}{1-x}\right)\right\} = \frac{1-x}{-x} = 1 - \frac{1}{x} = g(x)$$

	f	g	h
f	f	g	h
g	g	h	f
h	h	f	g

(1) The table shows that the set S is closed under function composition.
(2) For any functions F, G, H,

$$F[GH](x) = F(GH(x)) = F(G(H(x))) = FG(H(x)) = [FG]H(x)$$
$$\therefore \quad F[GH] = [FG]H$$

Hence composition of functions is associative.
(3) The identity function f belongs to S.
(4) The table shows that $f^{-1} = f$, $g^{-1} = h$ and $h^{-1} = g$
$\therefore$ every element of S has an inverse in S.

A group $(G, *)$ is said to be *Abelian*† if the operation $*$ is commutative in G i.e. if $a * b = b * a$ for all $a, b \in G$.

Since addition of integers is commutative, the group $(\mathbb{Z}, +)$ is Abelian. Similarly, the group A in Example 1 has a commutative binary operation and is therefore Abelian. In Example 2, although composition of functions is generally non-commutative, the symmetry of the operation table shows that within the set S the operation is commutative. Thus, this group is also Abelian.

For an example of a non-Abelian group, we consider a set of 2×2 matrices under multiplication.

Example 3 Show that the matrices $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\mathbf{E} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\mathbf{F} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathbf{G} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ form a group S under matrix multiplication.

First element \ Second element	**I**	**A**	**B**	**C**	**D**	**E**	**F**	**G**
I	**I**	**A**	**B**	**C**	**D**	**E**	**F**	**G**
A	**A**	**B**	**C**	**I**	**E**	**F**	**G**	**D**
B	**B**	**C**	**I**	**A**	**F**	**G**	**D**	**E**
C	**C**	**I**	**A**	**B**	**G**	**D**	**E**	**F**
D	**D**	**G**	**F**	**E**	**I**	**C**	**B**	**A**
E	**E**	**D**	**G**	**F**	**A**	**I**	**C**	**B**
F	**F**	**E**	**D**	**G**	**B**	**A**	**I**	**C**
G	**G**	**F**	**E**	**D**	**C**	**B**	**A**	**I**

(1) The operation table shows that S is closed under matrix multiplication.
(2) Matrix multiplication is associative.
(3) S contains the identity matrix **I**.
(4) Since **I** appears in every row and column of the table, every element of S has an inverse in S. [**I**, **B**, **D**, **E**, **F** and **G** are self-inverse, $\mathbf{A}^{-1} = \mathbf{C}$ and $\mathbf{C}^{-1} = \mathbf{A}$.]

Hence S forms a group under matrix multiplication.

[Since the operation table is not symmetrical about the main diagonal, the group is non-Abelian.]

† *Abel, Niels Henrik* (1802–1829) Norwegian mathematician. His work on the theory of equations (he proved that an algebraic solution of a quintic was impossible) helped to lay the foundations of modern group theory.

To prove that a set does *not* form a group under a given binary operation we must find a counter-example which demonstrates that one of the group axioms does not hold. It may not be necessary to consider all four group axioms.

Example 4 Show that the set $A = \{p, q, r, s\}$ does not form a group under either of the operations defined as follows:

(a)

*	p	q	r	s
p	q	p	s	r
q	p	q	r	s
r	s	r	s	p
s	r	s	p	q

(b)

$\circ$	p	q	r	s
p	p	q	r	s
q	q	p	q	r
r	r	q	p	q
s	s	r	q	p

(a) A is closed under $*$.
The identity element for $*$ is q.
However, since q does not appear in the third column of the table, r has no inverse.
$\therefore$ A does not form a group under $*$.

(b) A is closed under $\circ$.
The identity element for $\circ$ is p.
Each element is its own inverse under $\circ$.
Considering the associative law:

$$(q \circ r) \circ s = q \circ s = r, \quad q \circ (r \circ s) = q \circ q = p$$

$\therefore$ the associative law does not hold for $\circ$.
Hence A does not form a group under $\circ$.

Exercise 19.1

In questions 1 to 16 decide whether the given set forms a group under the stated binary operation, giving reasons for your answer. If the set does not form a group state which of the group axioms do not hold.

1. The set of even integers under addition.

2. The set of odd integers under multiplication.

3. $\mathbb{Q}$, the set of rational numbers, under multiplication.

4. The set $A = \{\frac{1}{4}, \frac{1}{2}, 1, 2, 4\}$ under multiplication.

5. The set of 2×3 matrices with real elements under matrix addition.

6. The set of numbers of the form p/q, where p and q are odd integers, under multiplication.

7. The set of all subsets of the set $S = \{a, b, c, d\}$ under the operation intersection $\cap$.

8. The set containing $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} -i & 0 \\ 0 & 0 \end{pmatrix}$ under matrix multiplication.

9. The set of non-negative integers under $*$, where $a * b = |a - b|$.

10. The set $\mathbb{R}^+$, the positive real numbers, under $\circ$, where $a \circ b = ab/(a + b)$.

11. The set of integers $\mathbb{Z}$ under $*$, where $a * b = a$.

12. The set of real numbers $\mathbb{R}$ under $\oplus$, where $a \oplus b = a + b - 1$.

13. The set of non-negative rationals under $*$, where $a * b = \sqrt{(a^2 + b^2)}$.

14. The set $\mathbb{R}$ under $\circ$, where $a \circ b = 2(a + b)$.

15. The set of functions $e: x \to x$, $f: x \to -x$, $g: x \to \frac{1}{x}$, $h: x \to -\frac{1}{x}$ under composition of functions.

16. The set of matrices of the form $\begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}$, where $p \in \mathbb{Z}$, under matrix multiplication.

17. S is the set of all real numbers except 1 and the binary operation $*$ on S is defined by $a * b = ab - a - b + 2$. Show that $(S, *)$ is an Abelian group and explain why 1 must be excluded from the set S.

18. S is the set of ordered pairs (a, b), where $a, b \in \mathbb{R}$ and $a \neq 0$. The operation $*$ on S is defined by $(a, b) * (c, d) = (ac, bc + d)$. Show that $(S, *)$ is a non-Abelian group.

19. Given that, for all real x except $x = 0$ and $x = 1$, $e(x) = x$, $p(x) = \dfrac{x - 1}{x}$, $q(x) = \dfrac{1}{1 - x}$, $f(x) = 1 - x$, $g(x) = \dfrac{1}{x}$ and $h(x) = \dfrac{x}{x - 1}$, show that the set $A = \{e, p, q, f, g, h\}$ forms a group under composition of functions.

20. A set of six complex numbers forms a group under multiplication. Given that one element of the set is $\frac{1}{2}(1 + i\sqrt{3})$, find the other elements.

21. The following tables show a set $A = \{p, q, r, s\}$ under a binary operation $*$. In each case decide whether $(A, *)$ is a group. If $(A, *)$ does not form a group, state which of the group axioms do not hold.

(a)

$*$	p	q	r	s
p	p	q	r	s
q	q	t	p	r
r	r	p	s	q
s	s	r	q	t

(b)

$*$	p	q	r	s
p	q	r	s	p
q	p	q	r	s
r	s	p	q	r
s	r	s	p	q

(c)

$*$	p	q	r	s
p	r	s	p	q
q	s	s	q	p
r	p	q	r	s
s	q	p	s	r

(d)

$*$	p	q	r	s
p	r	s	q	p
q	s	r	p	q
r	q	p	s	r
s	p	q	r	s

22. For each of the systems $(A, *)$ defined in question 21, find the solution sets of the equations $p * x = q$ and $(x * r) * q = s$.

23. Prove that for any elements a, b, c of a group $(G, *)$,
(i) $(a * c = b * c) \Rightarrow (a = b)$,
(ii) $(a * c = c * b) \Rightarrow (a = c * b * c^{-1})$.

24.

$*$	I	A	B	C
I	I	A	B	C
A	A			I
B	B		I	
C	C	I		

Copy and complete the given table so that the set $\{I, A, B, C\}$ forms a group under $*$. Use the table to simplify
(a) $B * (C * A)$, (b) $(A * B) * (B * C)$.

25. (i) Prove that the set of non-zero real numbers forms a group under the operation $\circ$ defined by $x \circ y = 2xy$. State the identity element for this system, and give the inverse of x.
(ii) A set of four 2×2 matrices forms a group under matrix multiplication. Two members of the set are $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Find the other members and write out the group table. (L)

26. Given that S is the set of all matrices of the form $\begin{pmatrix} a & b \\ 2b & a \end{pmatrix}$, where a and b are real, prove that S is closed under addition and multiplication. If a and b can take all real values, show that (i) S forms a group under addition, (ii) S does not form a group under multiplication. (The associative properties for addition and for multiplication may be assumed.) If a and b are rational and a is not zero, show that (iii) S does not form a group under addition, (iv) no member of S has zero determinant, (v) S forms a group under multiplication. (JMB)

19.2 The integers modulo n

The set of *integers modulo n*, often denoted by $\mathbb{Z}_n$, is the set of remainders or *residues* obtained when members of the set of integers are divided by n, where n is a positive integer. For instance, the set of integers modulo 5, $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$. Arithmetic is performed in $\mathbb{Z}_n$ by replacing any sum or product by the appropriate remainder, e.g. $3 + 4 = 2 \pmod 5$ and $2 \times 4 = 3 \pmod 5$. The system consisting of the sets $\mathbb{Z}_n$ with their associated operations of addition and multiplication is given a variety of names, such as modulo arithmetic, modular arithmetic, finite arithmetic, and, in more abstract treatments, the algebra of residues.

[The elements of $\mathbb{Z}_n = \{0, 1, 2, \ldots, (n-1)\}$ are sometimes considered to represent infinite sets of integers. For instance, in $\mathbb{Z}_5$ the element 2 represents the set $\{\ldots, -8, -3, 2, 7, 12, 17, \ldots\}$ whose elements all leave remainder 2 when divided by 5.]

In this section we will be mainly concerned with group structures which arise in modulo arithmetic. Consider the addition and multiplication tables for the integers modulo 5.

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$\times_5$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

$\times_5$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

$\mathbb{Z}_5$ is closed under both $+$ and $\times$.
Since addition and multiplication of integers is associative, these operations must also be associative in $\mathbb{Z}_5$.
$\mathbb{Z}_5$ contains the additive identity element 0 and the multiplicative identity element 1.
For addition, the inverses of 1, 2, 3 and 4 are 4, 3, 2 and 1 respectively.
For multiplication, the inverse of 2 is 3 and vice versa, 4 is self-inverse, but 0 has no inverse.
This means that $\mathbb{Z}_5$ forms a group under addition modulo 5 but not under multiplication. However, excluding 0, the set $\{1, 2, 3, 4\}$ does form a group under multiplication modulo 5.

Equations in $\mathbb{Z}_5$ can be solved using the additive and multiplicative inverses already obtained.

Example 1 Solve in $\mathbb{Z}_5$ the equation $2x + 3 = 1 \pmod 5$.

Performing all operations modulo 5:

$$
\begin{aligned}
2x + 3 = 1 &\Rightarrow (2x + 3) + 2 = 1 + 2 && \text{Adding inverse of 3 for } + \\
&\Rightarrow 2x + (3 + 2) = 1 + 2 && \text{Associativity of } + \\
&\Rightarrow 2x + 0 = 3 \\
&\Rightarrow 2x = 3 \\
&\Rightarrow 3 \times 2x = 3 \times 3 && \text{Using inverse of 2 for } \times \\
&\Rightarrow (3 \times 2)x = 3 \times 3 && \text{Associativity of } \times \\
&\Rightarrow 1 \times x = 4 \\
&\Rightarrow x = 4
\end{aligned}
$$

In general, the addition table for any set $\mathbb{Z}_n$ takes a similar form to the table for $\mathbb{Z}_5$. However, multiplication tables vary greatly in structure.

(1)

$\times_8$	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	2	4	6	0	2	4	6
3	3	6	1	4	7	2	5
4	4	0	4	0	4	0	4
5	5	2	7	4	1	6	3
6	6	4	2	0	6	4	2
7	7	6	5	4	3	2	1

(2)

$\times_8$	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

Table (1) shows that the set $\{1, 2, 3, 4, 5, 6, 7\}$ is not closed under multiplication modulo 8. However, from table (2) we see that the set $\{1, 3, 5, 7\}$ forms a group under this operation.

$\times_{10}$	2	4	6	8
2	4	8	2	6
4	8	6	4	2
6	2	4	6	8
8	6	2	8	4

It is more surprising to find that the set $\{2, 4, 6, 8\}$ forms a group under multiplication modulo 10. The identity element for the group is 6.

Exercise 19.2

1. Show that the set of integers $\{0, 1, 2, 3, 4, 5\}$ forms a group under addition modulo 6 but not under multiplication modulo 6. Decide whether the set $\{1, 2, 3, 4, 5\}$ forms a group under multiplication modulo 6.

2. Show that the set $\{0, 1, 2, 3, 4, 5, 6\}$ forms a group under addition modulo 7 and that the set $\{1, 2, 3, 4, 5, 6\}$ forms a group under multiplication modulo 7.

3. Solve the following equations in $\mathbb{Z}_5$ using the inverses of the elements with respect to addition and multiplication modulo 5.

(a) $x + 2 = 1$ (b) $4x = 3$ (c) $3x + 4 = 2$
(d) $4x + 3 = 0$ (e) $3(x + 1) = 0$ (f) $4x = 2x + 1$

4. Solve the following equations in $\mathbb{Z}_7$ by performing additions and multiplications modulo 7.

(a) $x + 4 = 3$ (b) $2x = 5$ (c) $3x + 2 = 3$
(d) $5x + 6 = 0$ (e) $6x + 3 = 5$ (f) $4x + 1 = 2x$

5. By investigating the subsets of the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ which are closed under multiplication modulo 10, determine which of them form groups under this operation.

6. Find all possible groups of order 2 in $\mathbb{Z}_{12}$ under multiplication modulo 12.

7. Find all possible groups of order 4 in $\mathbb{Z}_{15}$ under multiplication modulo 15.

8. Show that the set $\mathbb{Z}_n$, where n is a positive integer, forms a group under addition modulo n.

9. Show that if $n = pq$, where both p and q are integers greater than 1, then the set $\mathbb{Z}_n$ does not form a group under multiplication modulo n.

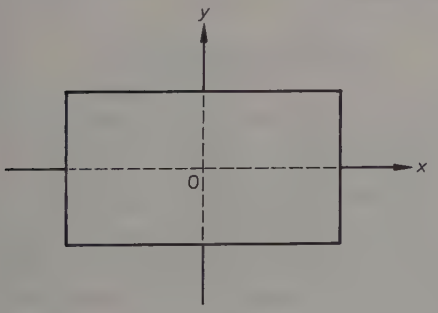
19.3 Symmetries and permutations

The *symmetries* of a geometrical figure in two or three dimensional space are transformations of the figure in which distances between points and the region of space occupied by the figure remain unchanged. The two main types of symmetry are rotation about an axis of symmetry and reflection in a line or plane of symmetry.

The product AB of two symmetries A and B is defined as B followed by A. As composition of mappings is an associative operation, $A(BC) = (AB)C$ for all symmetries A, B, C. However, in general this operation is not commutative, $AB \neq BA$.

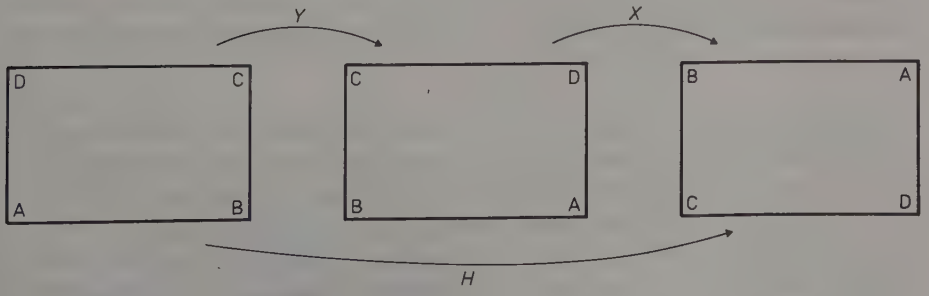
We demonstrate that the set of all symmetries of a given figure forms a group under composition by considering various examples.

(1) *The symmetries of a rectangle*



I: identity transformation
X: reflection in x-axis
Y: reflection in y-axis
H: half turn about O (i.e. about an axis through O perpendicular to the plane of the rectangle.)

Products of these symmetries can be obtained by labelling the corners of the rectangle then performing one transformation followed by another.



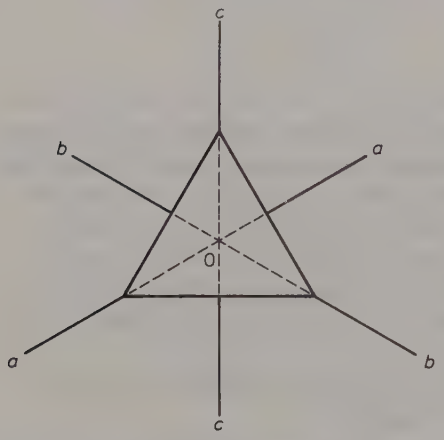
These diagrams show that Y followed by X has the same effect as H i.e. that $XY = H$.

	I	X	Y	H
I	I	X	Y	H
X	X	I	H	Y
Y	Y	H	I	X
H	H	Y	X	I

The operation table shows that the set $\{I, X, Y, H\}$ is closed under composition of symmetries.
The operation is associative.
The set contains the identity transformation. Each element is self-inverse.

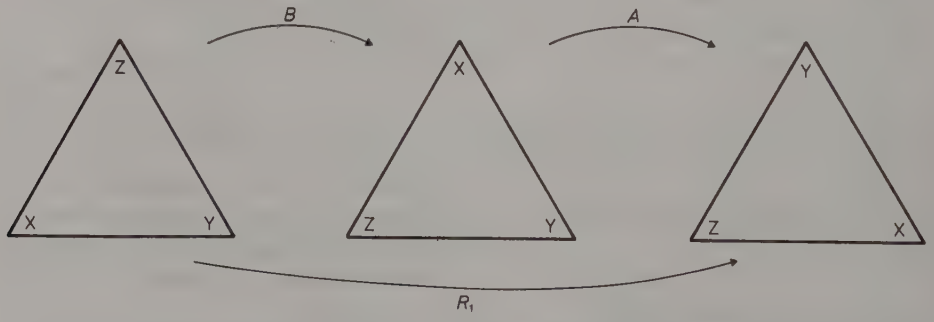
Thus the symmetries of a rectangle form a group of order 4.

(2) *The symmetries of an equilateral triangle*



I: identity transformation
R_1: rotation through 120° anti-clockwise about O
R_2: rotation through 120° clockwise about O
A: reflection in a
B: reflection in b
C: reflection in c
[Note that the lines a, b, c are fixed in space and do not move with the triangle.]

In this more difficult example it may be helpful to make a cardboard copy of the triangle and use this to complete the operation table.

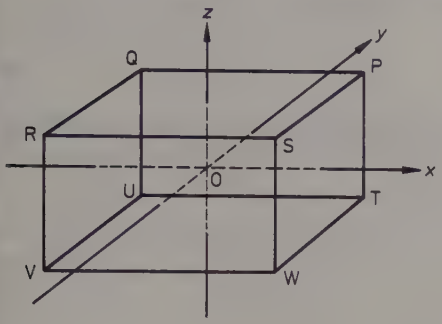


As shown in these diagrams, $AB = R_1$. However, it is also found that $BA = R_2$. Thus $AB \neq BA$.

	I	R_1	R_2	A	B	C
I	I	R_1	R_2	A	B	C
R_1	R_1	R_2	I	C	A	B
R_2	R_2	I	R_1	B	C	A
A	A	B	C	I	R_1	R_2
B	B	C	A	R_2	I	R_1
C	C	A	B	R_1	R_2	I

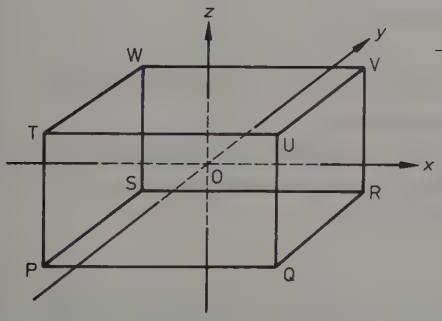
The table confirms that the set $\{I, R_1, R_2, A, B, C\}$ forms a group.
I, A, B, C are self-inverse. R_1 is the inverse of R_2 and vice-versa.
As the table is not symmetrical about the main diagonal, this group is not commutative.

(3) *The symmetries of the cuboid*



Initial position

I: identity transformation
X: reflection in y, z plane
Y: reflection in x, z plane
Z: reflection in x, y plane
H_1: half turn about x-axis
H_2: half turn about y-axis
H_3: half turn about z-axis
C: central inversion, (as shown in the second diagram).



After central inversion

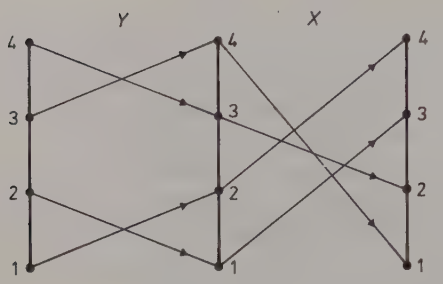
	I	H_1	H_2	H_3	X	Y	Z	C
I	I	H_1	H_2	H_3	X	Y	Z	C
H_1	H_1	I	H_3	H_2	C	Z	Y	X
H_2	H_2	H_3	I	H_1	Z	C	X	Y
H_3	H_3	H_2	H_1	I	Y	X	C	Z
X	X	C	Z	Y	I	H_3	H_2	H_1
Y	Y	Z	C	X	H_3	I	H_1	H_2
Z	Z	Y	X	C	H_2	H_1	I	H_3
C	C	X	Y	Z	H_1	H_2	H_3	I

The operation table shows that the symmetries of the cuboid form an Abelian group in which every element is self-inverse.

The symmetries of geometrical figures can be regarded as elements of a larger set of transformations called *isometries*. An isometry is a transformation of the points of two or three dimensional space in which distances between points remain unchanged. Thus under an isometry a geometrical figure retains its shape and size but may change its position in space. There are four types of *plane isometry*: rotation, reflection, translation and glide-reflection (i.e. reflection together with a translation in the direction of the mirror line). It can be shown that the set of all plane isometries form a group.

Any symmetry transformation of a geometrical figure results in a rearrangement of the vertices of the figure. These rearrangements form a subset of the set of all rearrangements or permutations of the vertices. We next consider a more abstract approach to transformations which rearrange the objects in a set.

A *permutation* is defined as a mapping of a finite set onto itself. If X is the permutation of the set $\{1, 2, 3, 4\}$ which maps $1 \to 3, 2 \to 4, 3 \to 2, 4 \to 1$, then X may be represented by the array $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$.



If $X = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$ and

$$Y = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix},$$

then the product XY, which means "first Y then X", maps $1 \to 2 \to 4$, $2 \to 1 \to 3$, $3 \to 4 \to 1$, $4 \to 3 \to 2$.

Hence $XY = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}$.

Writing $Z = XY$, we can produce an operation table for the set of permutations $A = \{I, X, Y, Z\}$, where I is the identity permutation $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$.

	I	X	Y	Z
I	I	X	Y	Z
X	X	Y	Z	I
Y	Y	Z	I	X
Z	Z	I	X	Y

The set A is closed under composition of permutations.
Since composition of mappings of any type is associative, this operation is associative.

A contains the identity I and each element has an inverse in A. Hence the set A forms a group.

It can also be shown that the set S_4 of *all* permutations of the set $\{1, 2, 3, 4\}$ forms a group. As before, we can assume the associative law for composition of mappings in S_4. Since S_4 contains all possible permutations, it contains the identity permutation and must also contain the product of any given pair of permutations. To demonstrate the existence of inverses we consider a typical element of S_4. If the permutation T maps $1 \to 2$, $2 \to 4$, $3 \to 3$, $4 \to 1$, then the inverse permutation T^{-1} maps $2 \to 1$, $4 \to 2$, $3 \to 3$, $1 \to 4$,

i.e. if $T = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$, then $T^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}$.

Clearly the inverse of any element can be found in this way. Since $XT = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$ and $TX = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$, composition of permutations is not usually commutative.

Thus the set S_4 forms a non-Abelian group. Since the number of different arrangements of 4 objects is 4!, the order of the permutation group S_4 is 24.

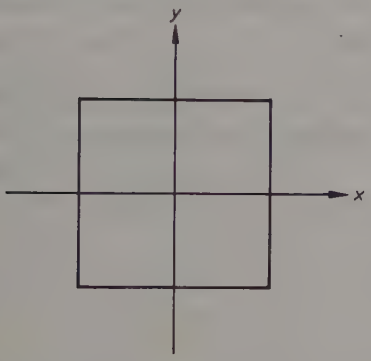
In general, the set of all permutations of a set with n elements forms a permutation group of order $n!$ called the *symmetric group* S_n.

Exercise 19.3

1. Use the group table for the symmetries of an equilateral triangle to simplify
(a) ABC (b) AR_1BR_2
(c) $(R_1)^2R_2$ (d) $R_1^{-1}(R_2^{-1})^2$

2. Use the group table for the symmetries of a cuboid to verify that
(a) $X(YZ) = (XY)Z$ (b) $H_1(CX) = (H_1C)X$

3. Make a group table for the group of symmetries of a square using the following notation:



I: identity
R_1: rotation through 90° anti-clockwise
R_2: rotation through 180° anti-clockwise
R_3: rotation through 270° anti-clockwise
X: reflection in the x-axis
Y: reflection in the y-axis
A: reflection in the line $y = x$
B: reflection in the line $y = -x$

4. Discuss the symmetry groups of the following plane figures, stating the order of the group in each case.
(a) a parallelogram, (b) a rhombus,
(c) a regular pentagon, (d) a circle.

5. Discuss the symmetry groups of the following solids, stating the order of the group in each case.
(a) a pyramid with a square base,
(b) a prism whose cross-section is an isosceles triangle,
(c) a cylinder with an elliptical cross-section,
(d) a prism whose cross-section is an equilateral triangle.

6. Are the following statements about plane isometries true in all cases? If false, justify your answer.
(a) A reflection in one line followed by a reflection in another line results in a rotation.
(b) A rotation followed by a translation is equivalent to a rotation about a different centre.
(c) A reflection followed by a translation is equivalent to a glide-reflection.
(d) A rotation about A followed by a rotation about B results in a rotation about a third point C.
(e) A rotation followed by a reflection results in a reflection in a different mirror line.

7. Find the following products of permutations:

(a) $\begin{pmatrix}1&2&3&4\\1&3&2&4\end{pmatrix}\begin{pmatrix}1&2&3&4\\3&2&4&1\end{pmatrix}$ (b) $\begin{pmatrix}1&2&3&4\\2&3&1&4\end{pmatrix}\begin{pmatrix}1&2&3&4\\4&2&1&3\end{pmatrix}$

(c) $\begin{pmatrix}1&2&3&4\\3&4&2&1\end{pmatrix}\begin{pmatrix}1&2&3&4\\3&2&4&1\end{pmatrix}\begin{pmatrix}1&2&3&4\\3&1&4&2\end{pmatrix}$

(d) $\begin{pmatrix}1&2&3&4\\1&4&2&3\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}\begin{pmatrix}1&2&3&4\\4&3&2&1\end{pmatrix}$

8. Write down the six arrays representing the elements of the set of permutations of the integers $\{1,2,3\}$. Prove that this set forms a group under composition of mappings.

9. Given below are arrays representing certain elements of the symmetric group S_5. In each case determine the order of the smallest group containing the given element(s).

(a) $\begin{pmatrix}1&2&3&4&5\\1&4&3&5&2\end{pmatrix}$ (b) $\begin{pmatrix}1&2&3&4&5\\2&1&5&4&3\end{pmatrix}$ (c) $\begin{pmatrix}1&2&3&4&5\\4&5&1&2&3\end{pmatrix}$

(d) $\begin{pmatrix}1&2&3&4&5\\4&5&1&3&2\end{pmatrix}$ (e) $\begin{pmatrix}1&2&3&4&5\\3&1&5&4&2\end{pmatrix}, \begin{pmatrix}1&2&3&4&5\\3&5&1&4&2\end{pmatrix}$

10. The symmetries of the square $ABCD$ are related to the permutations of its vertices. For instance, the permutation $\begin{pmatrix}A&B&C&D\\B&C&D&A\end{pmatrix}$ is equivalent to a rotation of the square through 90°. Describe the transformations of the square $ABCD$ equivalent to the following:

(a) $\begin{pmatrix}A&B&C&D\\C&D&A&B\end{pmatrix}$ (b) $\begin{pmatrix}A&B&C&D\\C&B&A&D\end{pmatrix}$ (c) $\begin{pmatrix}A&B&C&D\\D&C&B&A\end{pmatrix}$

Do all permutations of the vertices A, B, C, D correspond to symmetries of the square $ABCD$?

19.4 Properties of groups

First we use the group axioms to obtain two basic properties of groups. To simplify the working we will now write the product of elements a and b as ab instead of $a * b$.

Example 1 Prove that in any group G, (a) the identity element is unique, (b) the inverse of any element is unique.

(a) Let e_1 and e_2 be two identity elements in G,
then since e_1 is an identity element, $e_1e_2 = e_2$
and since e_2 is an identity element, $e_1e_2 = e_1$
Hence $e_1 = e_2$.

(b) Let an element a of G have two inverses b and c
then $ab = ba = e$ and $ac = ca = e$,
where e is the identity element in G.

$$\begin{aligned} ab = e &\Rightarrow c(ab) = ce \\ &\Rightarrow (ca)b = c \\ &\Rightarrow eb = c \\ &\Rightarrow b = c \end{aligned}$$

Thus the element a has a unique inverse.

If a subset H of a group $(G, *)$ also forms a group under $*$, then H is said to be a *subgroup* of G.

Since the associative law for $*$ holds in G, it must hold in any subset of G. Thus to prove that a subset H is a subgroup, it is necessary to show that
(i) H is closed under $*$,
(ii) H contains the identity element,
(iii) if $x \in H$, then $x^{-1} \in H$.

The given operation table defines a group $(G, *)$. Suppose that we wish to list all the subgroups of G. To satisfy condition (ii) we need consider only subsets of G which contain the identity element e, namely: $\{e\}$, $\{e, a\}$, $\{e, b\}$, $\{e, c\}$, $\{e, a, b\}$, $\{e, a, c\}$, $\{e, b, c\}$ and G itself. From the table we see that the sets $\{e\}$, $\{e, a\}$, $\{e, b\}$, $\{e, c\}$ and G are closed under $*$, but that the remaining sets are not. Since every element of G is self-inverse, property (iii) is also satisfied in these five sets. Hence G has five subgroups, $\{e\}$, $\{e, a\}$, $\{e, b\}$, $\{e, c\}$ and G itself.

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Every group G has two subgroups, $\{e\}$ and G. Any other subgroup is described as a *proper subgroup*. In this example, G has three proper subgroups: $\{e, a\}$, $\{e, b\}$, $\{e, c\}$.

Let us consider now the group formed by the set $\{1, 2, 3, 4, 5, 6\}$ under multiplication modulo 7. Since 6 is self-inverse the set $\{1, 6\}$ is a subgroup of order 2. The table also shows that the set $\{1, 2, 4\}$ forms a subgroup of order 3. Thus this group has two proper subgroups. This example illustrates a general result which is often useful when studying the subgroups of a finite group.

$\times_7$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

Lagrange's† *theorem*: The order of any subgroup of a finite group is a factor of

[A method of proof is indicated in Exercise 19.4, question 27.]

† *Lagrange, Joseph Louis* (1736–1813) French mathematician. His main work was *Mécanique Analytique* (1788). He also made advances in the theory of equations which helped in the formulation of group theory.

Example 2 A set S is defined as the set of all elements of a group G which commute with every element of G i.e. $a \in S$ if and only if $ax = xa$ for every element $x \in G$. Prove that S is a subgroup of G.

If $a, b \in S$, then $ax = xa$, $bx = xb$ for all $x \in G$
$\therefore \quad (ab)x = a(bx) = a(xb) = (ax)b = (xa)b = x(ab)$
i.e. $(ab)x = x(ab)$ for all $x \in G$
$\therefore$ if a and b belong to S, ab also belongs to S.
Hence S is closed under the group operation.
If e is the identity element in G,

$$ex = xe = x \quad \text{for all} \quad x \in G$$

Hence the identity element e belongs to S.

$$\begin{aligned} ax = xa &\Rightarrow a^{-1}(ax)a^{-1} = a^{-1}(xa)a^{-1} \\ &\Rightarrow (a^{-1}a)(xa^{-1}) = (a^{-1}x)(aa^{-1}) \\ &\Rightarrow e(xa^{-1}) = (a^{-1}x)e \\ &\Rightarrow xa^{-1} = a^{-1}x \end{aligned}$$

Hence if a belongs to S, a^{-1} also belongs to S.
Therefore the set S is a subgroup of the group G.

In a group G with identity element e, the *order* or *period* of any element x is the least positive integer n such that $x^n = e$.

The only element of order 1 in any group is the identity itself. If an element x has order 2, then $x^2 = e$. It follows that any element of order 2 is self-inverse.

A group whose elements can be expressed in the form $e, x, x^2, x^3, \ldots, x^{n-1}$ is called a *cyclic group* of order n and denoted by C_n. The element x is said to *generate* the group and is described as a *generator* of the group.

We now give two examples of cyclic groups.

(1) The rotational symmetries of an equilateral triangle.

	I	R_1	R_2
I	I	R_1	R_2
R_1	R_1	R_2	I
R_2	R_2	I	R_1

Using the notation of §19.3, the elements of the group can be expressed either as $\{I, R_1, R_1^2\}$ or as $\{I, R_2, R_2^2\}$.
Hence this group is a cyclic group of order 3, which may be generated by R_1 or R_2.

(2) The set $\{1, 2, 3, 4\}$ under multiplication modulo 5.

$\times_5$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

The elements of the group can be expressed either as $\{1, 2, 2^2, 2^3\}$ or as $\{1, 3, 3^2, 3^3\}$.
Hence this group is a cyclic group of order 4, which may be generated by 2 or 3. Both 2 and 3 are elements of order 4, whereas the element 4 is of order 2 and generates the subgroup $\{1, 4\}$.

Example 3 Prove that every cyclic group is Abelian.

If a and b are elements of a cyclic group with generator x, then there are positive integers p and q such that $a = x^p$ and $b = x^q$.
Thus $ab = (x^p)(x^q) = x^{p+q} = x^{q+p} = (x^q)(x^p) = ba$.
Hence every cyclic group is Abelian.

Example 4 Prove that the cyclic group C generated by an element x of order n has n distinct elements.

The elements of C are all of the form x^p, where p is a positive integer. Since x is of order n, $x^n = e$, where e is the identity.
Considering the division of p by n, we see that there are integers q and r such that

$$p = nq + r, \quad \text{where} \quad q \geqslant 0, \quad 0 \leqslant r \leqslant n - 1$$

If $r = 0$, $x^p = x^{nq} = (x^n)^q = e^q = e$.
If $r \neq 0$, $x^p = x^{nq+r} = (x^n)^q(x^r) = e(x^r) = x^r$.
Thus, for all values of p, x^p is an element of the set $\{e, x, x^2, x^3, \ldots, x^{n-1}\}$.
If $x^s = x^t$, where $0 < t < s \leqslant n$
then $x^{s-t} = e$
$\therefore$ $s - t$ is a multiple of n.
Since this is impossible if s and t are both less than or equal to n, the n elements $e, x, x^2, x^3, \ldots, x^{n-1}$ must be distinct. Hence the group C generated by x is of order n.

It follows from Example 4 that:

> A group of order n is cyclic if and only if it contains an element of order n.

	I	X	Y	H
I	I	X	Y	H
X	X	I	H	Y
Y	Y	H	I	X
H	H	Y	X	I

Considering again the symmetries of a rectangle as defined in §19.3, we find that each of the elements X, Y, H is of order 2 and generates a cyclic subgroup of order 2. There is no element of order 4 which generates the whole group. Hence this group is not cyclic.

The elements of a group which is not cyclic can be expressed in terms of two or more generators. Although the group $\{I, X, Y, H\}$ has no single generator, the whole group is generated by the two elements X and Y. Referring back to Example 3, §19.1 we find that the elements of the group $\{\mathbf{I}, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}, \mathbf{F}, \mathbf{G}\}$ can be expressed as $\mathbf{I}, \mathbf{A}, \mathbf{A}^2, \mathbf{A}^3, \mathbf{D}, \mathbf{AD}, \mathbf{A}^2\mathbf{D}, \mathbf{A}^3\mathbf{D}$ respectively. In this case two elements **A** and **D** generate a group of order 8.

Exercise 19.4

1. Given that a and b are elements of a group G, show that
(i) $(ab)^{-1} = b^{-1}a^{-1}$, (ii) $(a^{-1})^{-1} = a$, (iii) $(a^2)^{-1} = (a^{-1})^2$.

2. Given that a, b, x and y belong to a group G, show that
(i) $ax = ay \Rightarrow x = y$, (ii) $ax = b \Rightarrow x = a^{-1}b$.

In questions 3 to 10 list the proper subgroups of the given groups.

3. The set containing the sets $\varnothing$, $X = \{1\}$, $Y = \{2\}$, $Z = \{1, 2\}$ under the operation symmetric difference Δ, where $A \Delta B = (A \cap B') \cup (A' \cap B)$.

4. The set of functions $\{e, f, g, h\}$ under function composition, where $e(x) = x$, $f(x) = 1 - \frac{1}{2x}$, $g(x) = \frac{1-x}{1-2x}$, $h(x) = \frac{1}{2-2x}$.

5. The set containing the matrices, $\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$,
$\mathbf{B} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $\mathbf{E} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, under matrix multiplication.

6. The symmetry group of the equilateral triangle as defined in §19.3.

7. The set $\{2, 4, 8, 10, 14, 16\}$ under multiplication modulo 18.

8. The set $\left\{\pm 1, \pm i, \pm\frac{1}{\sqrt{2}}(1+i), \pm\frac{1}{\sqrt{2}}(1-i)\right\}$ under multiplication.

9. The group $\mathrm{S} = \{\mathbf{I}, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}, \mathbf{F}, \mathbf{G}\}$ defined in Example 3 §19.1.

10. The symmetry group of the cuboid as defined in §19.3.

11. Decide whether the following sets are subgroups of the group $(\mathbb{Z}, +)$.
(a) the set of even integers,
(b) the set of odd integers,
(c) the set $\{0, 10, 20, 30, 40, \ldots\}$
(d) the set $\{\ldots, -6, -3, 0, 3, 6, 9, \ldots\}$.

12. Decide whether the following sets are subgroups of the group formed by the set of non-zero real numbers under multiplication.
(a) the set of non-zero integers,
(b) the set of positive rational numbers,
(c) the set $\{x \in \mathbb{R}: 0 < x \leqslant 1\}$,
(d) the set of real numbers of the form $a\sqrt{3} + b$, where a and b are non-zero rational numbers.

In questions 13 to 20 determine whether the given group is cyclic and if so list the elements which are generators.

13. The set $\{1, 3, 5, 7\}$ under multiplication modulo 8.

14. The symmetries of a square.

15. The set $\{0, 1, 2, 3, 4\}$ under addition modulo 5.

16. The set $\{1, i, -1, -i\}$ under multiplication.

17. The functions which map x onto x, $1/x$, $1 - x$, $1 - 1/x$, $1/(1 - x)$, $x/(x - 1)$ under composition of functions.

18. The rotations of a regular hexagon.

19. The set $\{1, 2, 3, 4, 5, 6\}$ under multiplication modulo 7.

20. The permutations of the set $\{1, 2, 3, 4\}$ in which 2 is mapped onto itself.

21. A cyclic group has elements $e, x, x^2, x^3, x^4, x^5, x^6, x^7$. Which of these elements generate the whole group? Find the orders of the remaining elements.

22. Repeat question 21 for the cyclic group with elements $e, a, a^2, a^3, a^4, a^5, a^6$.

23.

	I	A	B	C	D	E	F	G
I	I	A	B	C	D	E	F	G
A	A	B	C	I	E	F	G	D
B	B	C	I	A	F	G	D	E
C	C	I	A	B	G	D	E	F
D	D	G	F	E	B	A	I	C
E	E	D	G	F	C	B	A	I
F	F	E	D	G	I	C	B	A
G	G	F	E	D	A	I	C	B

Find the orders of the elements of the group defined in the given table. Deduce that the group is not cyclic. Show that the group has four proper subgroups, which are all cyclic.

24. The matrices $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $C = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$, $D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $E = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$, $F = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, $G = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, where $i^2 = -1$, form a group of order 8 under matrix multiplication. (Do not prove this result.)

(i) Determine the order of each element.
(ii) Write down a subgroup of order 2.
(iii) Write down a cyclic subgroup of order 4 and explain why it is cyclic.
(iv) Write down a non-cyclic subgroup of order 4 and explain why it is non-cyclic.
(v) Prove that the given group is non-commutative.
(vi) Write down two elements which do commute with every element of the group.

(JMB)

25. Two elements a and b of a group G are such that $a^2 = (ab)^2 = b^4$. Deduce that $aba = b^3$ and $bab = a$. Given that a is of order 4, find the orders of the elements a^3b and b^3a.

26. Prove that if S and T are distinct subgroups of a group G then $S \cap T$ is also a subgroup of G. Illustrate this result by taking G to be the set $\{1, 2, 3, \ldots, 12\}$ under multiplication modulo 13 and letting S and T be the subgroups of order 4 and 6 respectively. Demonstrate that in this case $S \cup T$ is not a subgroup of G.

27. A finite group G of order n has a proper subgroup H of order p. The set xH, where $x \in G$, is defined as the set containing all products of the form xh, where $h \in H$. Prove that (a) xH has p distinct elements, (b) if xH and yH have a common element z, then $xH = yH$, (c) G can be expressed as the union of disjoint sets of the form xH, (d) p is a factor of n.

19.5 Isomorphisms

Two groups G and H which have precisely the same structure are said to be *isomorphic* and we write $G \cong H$.

One elementary way of showing that two groups are isomorphic is to demonstrate that their operation tables are equivalent. For instance, if $G = \{1, 2, 3, 4\}$ under multiplication modulo 5 and $H = \{2, 4, 6, 8\}$ under multiplication modulo 10, we obtain the tables:

$\times_5$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

$\times_{10}$	2	4	6	8
2	4	8	2	6
4	8	6	4	2
6	2	4	6	8
8	6	2	8	4

We now rearrange the tables:

$\times_5$	1	2	4	3
1	1	2	4	3
2	2	4	3	1
4	4	3	1	2
3	3	1	2	4

$\times_{10}$	6	2	4	8
6	6	2	4	8
2	2	4	8	6
4	4	8	6	2
8	8	6	2	4

Since the tables now have exactly the same structure the groups G and H must be isomorphic.

More formally, two groups $(G, *)$ and $(G', \circ)$ are said to be isomorphic if there is a one-one correspondence between the elements $a, b, \ldots \in G$ and $a', b', \ldots \in G'$, such

that if $a \leftrightarrow a'$ and $b \leftrightarrow b'$, then $a * b \leftrightarrow a' \circ b'$. This one-one correspondence between G and G' is called an *isomorphism.*

In the above example, the rearrangement of the tables was equivalent to setting up the isomorphism $1 \leftrightarrow 6, 2 \leftrightarrow 2, 4 \leftrightarrow 4, 3 \leftrightarrow 8$.

$\times_5$	1	3	4	2
1	1	3	4	2
3	3	4	2	1
4	4	2	1	3
2	2	1	3	4

A further rearrangement of the first table shows that there is a different isomorphism,

$$1 \leftrightarrow 6, 3 \leftrightarrow 2, 4 \leftrightarrow 4, 2 \leftrightarrow 8.$$

Thus it may be possible to set up an isomorphism between two groups in more than one way.

Consider now two further groups of order 4: the set of symmetries of a rectangle, $S = \{I, X, Y, H\}$, and the set of integers $\{1, 3, 5, 7\}$ under multiplication modulo 8.

	I	X	Y	H
I	I	X	Y	H
X	X	I	H	Y
Y	Y	H	I	X
H	H	Y	X	I

$\times_8$	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

The tables show that the mapping $I \leftrightarrow 1, X \leftrightarrow 3, Y \leftrightarrow 5, H \leftrightarrow 7$ is an isomorphism,
$\therefore$ the groups S and T are isomorphic, $S \cong T$.

In any rearrangement of these group tables, the identity elements always appear along the main diagonal. It is therefore impossible to set up an isomorphism between S and T and the earlier groups of order 4, G and H. These pairs of isomorphic groups can be regarded as examples of two distinct abstract groups.

	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

The first is the cyclic group of order 4, C_4, and consists of an identity element, one element of order 2 and two elements of order 4. The second is called the Klein† four-group and has an identity and three elements of order 2. These are the only possible groups of order 4.

This discussion of groups of order 4 suggests that we can test a pair of groups for isomorphism by comparing the orders of the elements in each group. It follows from the definition of an isomorphism that if $a \leftrightarrow a'$ then $a^n \leftrightarrow (a')^n$. Hence in any isomorphism between groups G and G', any element of G must correspond to an element of the same order in G'.

† *Klein, Felix* (1849–1925) German mathematician. In his *Erlangen Programm* he attempted a unification of mathematics in terms of group theory.

Example 1 Decide whether any of the groups, G_1, G_2, G_3 defined below are isomorphic.

G_1 is the set $\{0, 1, 2, 3, 4, 5\}$ under addition modulo 6.
G_2 is the set $\{1, 2, 3, 4, 5, 6\}$ under multiplication modulo 7.
G_3 is the set $\{I, R_1, R_2, A, B, C\}$ of symmetries of an equilateral triangle, as defined in §19.3.

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

$\times_7$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

	I	R_1	R_2	A	B	C
I	I	R_1	R_2	A	B	C
R_1	R_1	R_2	I	C	A	B
R_2	R_2	I	R_1	B	C	A
A	A	B	C	I	R_1	R_2
B	B	C	A	R_2	I	R_1
C	C	A	B	R_1	R_2	I

From the group tables we see that G_1 and G_2 are both Abelian with one element of order 2, whereas G_3 is non-Abelian with three elements of order 2. Hence G_3 is isomorphic to neither G_1 nor G_2, but G_1 may be isomorphic to G_2.

In G_1 the element 1 is of order 6. To set up an isomorphism between G_1 and G_2 we must find an element of order 6 in G_2.

In G_2 1 is the identity and, since $2^3 = 1 \pmod 7$, the element 2 is of order 3. However, working modulo 7:

$$3^1 = 3, \quad 3^2 = 2, \quad 3^3 = 2 \times 3 = 6,$$
$$3^4 = 6 \times 3 = 4, \quad 3^5 = 4 \times 3 = 5, \quad 3^6 = 5 \times 3 = 1.$$

Thus in G_2 the element 3 is of order 6. Hence the required isomorphism between G_1 and G_2 is given by:

$$1 \leftrightarrow 3^1 \quad \text{i.e.} \quad 1 \leftrightarrow 3$$
$$1+1 \leftrightarrow 3^2 \quad 2 \leftrightarrow 2$$
$$1+1+1 \leftrightarrow 3^3 \quad 3 \leftrightarrow 6$$
$$1+1+1+1 \leftrightarrow 3^4 \quad 4 \leftrightarrow 4$$
$$1+1+1+1+1 \leftrightarrow 3^5 \quad 5 \leftrightarrow 5$$
$$1+1+1+1+1+1 \leftrightarrow 3^6 \quad 0 \leftrightarrow 1$$

$\times_7$	1	3	2	6	4	5
1	1	3	2	6	4	5
3	3	2	6	4	5	1
2	2	6	4	5	1	3
6	6	4	5	1	3	2
4	4	5	1	3	2	6
5	5	1	3	2	6	4

We can now rearrange the operation table for G_2 so that it has the same structure as the table for G_1.

Example 2 Show that the mapping $f: x \to 2^x$ from the set of integers $\mathbb{Z}$ to the set $S = \{\ldots, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, \ldots\}$ sets up an isomorphism between the groups $(\mathbb{Z}, +)$ and $(S, \times)$.

$f(x) = f(y) \Rightarrow 2^x = 2^y \Rightarrow x = y$

Hence f is a one-one correspondence between $\mathbb{Z}$ and S.

$f(x + y) = 2^{x+y} = (2^x) \times (2^y) = f(x) \times f(y)$

Hence f maps the sum of two elements in $(\mathbb{Z}, +)$ to the product of the corresponding elements in $(S, \times)$.
Thus f sets up an isomorphism between $(\mathbb{Z}, +)$ and $(S, \times)$.

Exercise 19.5

1. Determine which of the following groups of order 4 are isomorphic.

(a) The set of integers $\{0, 1, 2, 3\}$ under addition modulo 4.

(b) The set of matrices $\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $\mathbf{Q} = \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}$, $\mathbf{R} = \begin{pmatrix} i & i \\ 0 & 0 \end{pmatrix}$, $\mathbf{S} = \begin{pmatrix} -i & -i \\ 0 & 0 \end{pmatrix}$ under matrix multiplication.

(c) The functions $e(x) = x, f(x) = -\frac{1}{x}, g(x) = \frac{1+x}{1-x}, h(x) = \frac{x-1}{x+1}$ under function composition.

(d) The set of symmetries of a cube containing the identity transformation and the 180° rotations about the three axes of symmetry which pass through the mid-points of opposite faces.

(e) The set of integers $\{1, 3, 7, 9\}$ under multiplication modulo 10.

(f) The functions $a: x \to 2 - x$, $b: x \to x$, $c: x \to \frac{x-2}{x-1}$, $d: x \to \frac{x}{x-1}$ under function composition.

2. Determine which of the following groups of order 6 are isomorphic.

(a) The symmetry group of the equilateral triangle.

(b) The set $\{1, 2, 4, 5, 7, 8\}$ under multiplication modulo 9.

(c) The matrices $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}$ and their products under matrix multiplication.

(d) The permutation group with elements $I = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$,

$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 3 & 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 4 & 1 & 5 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 1 & 3 & 5 \end{pmatrix}$

$D = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 1 & 4 & 2 \end{pmatrix}$, $E = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 4 & 3 & 2 \end{pmatrix}$.

(e) The set $\{2, 4, 6, 8, 10, 12\}$ under multiplication modulo 14.

(f) The matrices $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{pmatrix}$ and their products under matrix multiplication.

3. Show that the set $\{3, 6, 9, 12\}$ forms a group under multiplication modulo 15, which may be assumed to be associative. Show that this group is isomorphic to that formed by the set $\{1, -1, i, -i\}$ under multiplication of complex numbers. Name a geometrical figure which has a symmetry group of order four which is not isomorphic to these groups. By considering the orders of the elements, give a reason for this non-isomorphism. (JMB)

4. Given that $S = \{1, 2, 3, 4\}$, show that S forms a group under multiplication modulo 5 and that three elements of S form a group under multiplication modulo 7. Deduce that there are elements $x, y, z, \in S$ such that the set $T = \{(x, x), (x, y), (x, z), (y, x), (y, y), (y, z)\}$ forms a group under the operation $*$, where $(a, b) * (c, d) = (ac \text{ [mod 5]}, bd \text{ [mod 7]})$.

Show that the set of permutations on S which map 1 onto itself forms a group G. Explain why the set of permutations on S which map 2 onto 3 does not form a group.

Show that the group T is not isomorphic to the group G. Show further that the symmetry group of the regular hexagon has both a subgroup isomorphic to T and a subgroup isomorphic to G.

5. Given that f is an isomorphism from a group $(G, *)$ to a group $(H, \circ)$, prove that (a) if e is the identity in G, then $f(e)$ is the identity in H, (b) if x^{-1} is the inverse of x in G, then $f(x^{-1})$ is the inverse of $f(x)$ in H.

6. Prove that the mapping $f : x \to \ln x$ from the set of positive real numbers $\mathbb{R}^+$ to the set of real numbers $\mathbb{R}$ sets up an isomorphism between the groups $(\mathbb{R}^+, \times)$ and $(\mathbb{R}, +)$.

7. Prove that the mapping $f : x \to 2x$ from the set of integers $\mathbb{Z}$ to the set of even integers E sets up an isomorphism between the groups $(\mathbb{Z}, +)$ and $(E, +)$.

8. Decide which of the following mappings sets up an isomorphism of the group of non-zero real numbers under multiplication with itself.

$$a : x \to \frac{1}{x}, \quad b : x \to -x, \quad c : x \to 3x, \quad d : x \to x^3.$$

9. Prove that the mapping $f: a \to a^{-1}$ is an isomorphism of a group G onto itself if and only if the group is Abelian.

Exercise 19.6 (miscellaneous)

1. Prove that the set of the form $\dfrac{1+2m}{1+2n}$, where m and n are integers $(0, \pm 1, \pm 2, \ldots)$, forms a group under the operation of multiplication. (AEB 1976)

2. S is the subset of the real numbers given by $S = \{x : 0 \leqslant x < 1\}$. A binary operation $*$ is defined on S as follows: $x * y = \dfrac{x+y}{1+xy}$. Prove that $*$ is closed and associative. [Hint for closure: simplify $1 - (x*y)$.]

Find the identity element for $*$, and show that no element of S (other than the identity element) has an inverse in S. Determine the truth or falsity of the following proposition: given $x, y \in S$ then $((x * x) = (y * y)) \Rightarrow (x = y)$. (C)

3. The multiplication table for the set $\{e, a, b, c, d\}$ is given below.

	e	a	b	c	d
e	e	a	b	c	d
a	a	e	c	d	b
b	b	d	e	a	c
c	c	b	d	e	a
d	d	c	a	b	e

Using this table, determine $(ab)c$, $a(bc)$, $(bc)d$, $b(cd)$. Ascertain which group axioms are satisfied by the given set under the given multiplication. Find two subsets from the above set which form a group under the given multiplication. (L)

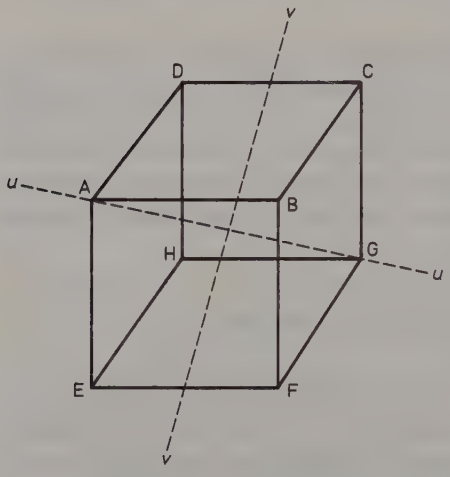
4. S is the set of functions having the form $f: x \to \dfrac{x}{nx+1}$ where n is an integer, and the domain of f is all real x except numbers of the form $x = 1/m$ where m is an integer. Show that S forms a group under the operation of composition of functions. (The Associative Law for composition of functions should not be assumed.) (JMB)

5. Show that H, the set of all non-singular 2×2 real matrices, forms a group under matrix multiplication. (It may be assumed that matrix multiplication is associative and that $\det \mathbf{A} \times \det \mathbf{B} = \det \mathbf{AB}$.) Given that S is the set of all non-singular 2×2 matrices which commute with the matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, show that each member of S must be of the form $\begin{pmatrix} a & 0 \\ b & a \end{pmatrix}$, where $a \neq 0$. Show that under the operation of matrix multiplication, S is a commutative group. (L)

6. Prove that the set S of matrices of the form $\begin{pmatrix} 1 & p & q \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix}$, where p, q, and r are integers, form a non-Abelian group under matrix multiplication. Show that the subset of S containing matrices in which $p = 0$ is an Abelian subgroup of S, but that the subset in which $q = 0$ is not a subgroup.

7. Prove that the set of integers $\{1, 3, 4, 5, 9\}$ forms a cyclic group under multiplication modulo 11. List the elements which generate the whole group.

8.



U and V are rotational symmetries of a cube. U is a rotation through 120° about the line u. V is a rotation through 180° about the line v, which passes through the mid-points of two opposite edges. Show that U and V generate a group of order 6. Find the order of the group containing all rotations of the cube.

9. Let S be the set of permutations of the numbers 1, 2 and 3. The elements of S may be listed as follows:

$$p_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad p_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \quad p_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$p_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad p_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \quad p_6 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix},$$

and $p_1 * p_2$ is defined to mean "p_2 followed by p_1". *It is given that* the elements of S under $*$ form a group. State the order of each element of S. Construct the group table for $(S, *)$. Show that the group $(S, *)$ is not commutative. Write down a cyclic sub-group of $(S, *)$ of order three. Name a generator of this sub-group. List all other non-trivial sub-groups of $(S, *)$. (JMB)

10. (i) Show that the set $\{2, 4, 6, 8, 10, 12\}$ forms a group under the operation of multiplication modulo 14. Name the identity element and the inverse of each of the other elements. Show that the set $\{1, 3, 5, 7, 9, 11, 13\}$ does NOT form a group under multiplication modulo 14. (Only one reason need be given, but the fact that one of the group postulates is not satisfied must be clearly demonstrated.)

(ii) The tables of two groups G_1, G_2 of 4 are given.

G_1	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

G_2	p	q	r	s
p	p	q	r	s
q	q	r	s	p
r	r	s	p	q
s	s	p	q	r

Name a plane figure whose group of symmetries is isomorphic to G_1 and another plane figure whose group of symmetries contains a subgroup isomorphic to G_2. (L)

11. (a) (i) Show that the set of all matrices of the form

$$\begin{pmatrix} \exp(a) & b \\ 0 & \exp(-a) \end{pmatrix}, \quad a, b \in \mathbb{R}$$

forms a group under matrix multiplication. State the values of a and b which give the identity element and the inverse of a general element. (ii) Find a subgroup of the group in (i).

(b) Prove that the symmetry group of the equilateral triangle is not isomorphic to the cyclic group of order 6. (AEB 1975)

12. Construct operation tables for two *distinct* groups of order 4, explaining why they are not isomorphic.

Write down a table for the symmetry group of the square and show that this group contains subgroups of order 4 of each type. Show further that two such subgroups of different types contain two elements in common and that both of these elements commute with all other elements of the whole group. (AEB 1978)

13. **A** and **B** are square matrices of the same order, such that $\mathbf{A}^4 = \mathbf{I}$, $\mathbf{A}^2 = \mathbf{B}^2$, $\mathbf{AB} = \mathbf{BA}^3$, where **I** is the identity matrix. Prove that $\mathbf{A}^3\mathbf{B} = \mathbf{BA}$ and $\mathbf{A}^2\mathbf{B} = \mathbf{BA}^2$. Complete the combination table (where $\otimes$ denotes matrix multiplication, which may be assumed associative).

$\otimes$	**I**	**A**	$\mathbf{A}^2$	$\mathbf{A}^3$	**B**	**BA**	$\mathbf{BA}^3$	$\mathbf{BA}^3$
I	**I**	**A**	$\mathbf{A}^2$	$\mathbf{A}^3$	**B**	**BA**	$\mathbf{BA}^2$	$\mathbf{BA}^3$
A	**A**	$\mathbf{A}^2$	$\mathbf{A}^3$	**I**	$\mathbf{BA}^3$			
$\mathbf{A}^2$	$\mathbf{A}^2$	$\mathbf{A}^3$	**I**	**A**	$\mathbf{BA}^2$			
$\mathbf{A}^3$	$\mathbf{A}^3$	**I**	**A**	$\mathbf{A}^2$	**BA**			
B	**B**	**BA**	$\mathbf{BA}^2$	$\mathbf{BA}^3$	$\mathbf{A}^2$	$\mathbf{A}^3$	**I**	**A**
BA	**BA**				**A**	$\mathbf{A}^2$	$\mathbf{A}^3$	**I**
$\mathbf{BA}^2$	$\mathbf{BA}^2$				**I**	**A**	$\mathbf{A}^2$	$\mathbf{A}^3$
$\mathbf{BA}^3$	$\mathbf{BA}^3$				$\mathbf{A}^3$	**I**	**A**	$\mathbf{A}^2$

Hence prove that the set $\{\mathbf{I}, \mathbf{A}, \mathbf{A}^2, \mathbf{A}^3, \mathbf{B}, \mathbf{BA}, \mathbf{BA}^2, \mathbf{BA}^3\}$ forms a group under matrix multiplication. (L)

14. If ω is one of the complex cube roots of 1, prove that the set of numbers of the form $a + b\omega$ (where a and b are real numbers, not both zero), with the operation of multiplication, forms a group G. [You need not verify the associative law, because this is valid for complex numbers in general.] State, with reasons, whether the subsets of G, defined as follows, are subgroups:
(i) a and b are rational numbers (ii) a and b are integers. (O)

15. Show that the set of permutations of three symbols (a, b, c) forms a group under the operation of successive performance of permutations. By writing the symbols as a column matrix, $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, the permutation $(a, b, c) \to (b, a, c)$, for example, can be represented by a matrix A, where $A \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b \\ a \\ c \end{pmatrix}$. Write down this matrix A and find the set of matrices which, under matrix multiplication, forms a group isomorphic to the permutation group. Find two subgroups of the permutation group which each contain two elements, and one subgroup which contains three elements. (AEB 1977)

16. Given the matrices

$$\mathbf{M}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{M}_2 = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad \mathbf{M}_3 = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix},$$

$$\mathbf{M}_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{M}_5 = \begin{pmatrix} 0 & \omega^2 \\ \omega & 0 \end{pmatrix}, \quad \mathbf{M}_6 = \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix},$$

where $\omega^3 = 1$, $\omega \neq 1$, complete the multiplication table, where $\otimes$ denotes matrix multiplication.

$\otimes$	$\mathbf{M}_1$	$\mathbf{M}_2$	$\mathbf{M}_3$	$\mathbf{M}_4$	$\mathbf{M}_5$	$\mathbf{M}_6$
$\mathbf{M}_1$	$\mathbf{M}_1$	$\mathbf{M}_2$	$\mathbf{M}_3$	$\mathbf{M}_4$	$\mathbf{M}_5$	$\mathbf{M}_6$
$\mathbf{M}_2$	$\mathbf{M}_2$				$\mathbf{M}_4$	$\mathbf{M}_5$
$\mathbf{M}_3$	$\mathbf{M}_3$				$\mathbf{M}_6$	$\mathbf{M}_4$
$\mathbf{M}_4$	$\mathbf{M}_4$				$\mathbf{M}_2$	$\mathbf{M}_3$
$\mathbf{M}_5$	$\mathbf{M}_5$	$\mathbf{M}_6$	$\mathbf{M}_4$	$\mathbf{M}_3$	$\mathbf{M}_1$	$\mathbf{M}_2$
$\mathbf{M}_6$	$\mathbf{M}_6$	$\mathbf{M}_4$	$\mathbf{M}_5$	$\mathbf{M}_2$	$\mathbf{M}_3$	$\mathbf{M}_1$

Deduce that the above set of matrices forms a group under matrix multiplication, which may be assumed to be associative. Find a subset of three matrices which forms a group and is isomorphic to the group of mappings F, G, H of $S \to S$, where $S = \{x : x \in \mathbb{R}, x \neq 0, x \neq 1\}$, under composition of mappings where $F : x \to x$, $G : x \to 1 - 1/x$, $H : x \to 1/(1 - x)$. Indicate the corresponding elements of the elements of the isomorphism. (L)

17. Show that the set of matrices of the form $\begin{pmatrix} 1 & \ln a \\ 0 & 1 \end{pmatrix}$, where a is real and positive, forms an Abelian group under matrix multiplication. Find two groups, whose elements are real numbers, which are isomorphic to the given group.

18. Given the set S of six distinct elements, where $S = \{x_0, x_1, \ldots, x_5\}$, the law of composition is defined $\circ$ is defined by $x_i \circ x_j = x_k$, where $i + j \equiv k \pmod 6$. Prove that $(S, \circ)$ forms a group. Prove that the group has an element of order 6.

The group $(T, *)$ is given by the set T, where $T = \{0, 2, 3, 4, 5, 6\}$, and the law of composition $*$, where $a * b \equiv a + b - ab \pmod 7$. Determine whether or not $(S, \circ)$ and $(T, *)$ are isomorphic. (JMB)

19. $(G, .)$ is a group. The element $x \,.\, y$, where $x \in G$, $y \in G$, is denoted by xy and the inverse of x is denoted by x^{-1}.

(a) S is a non-empty subset of G such that

$$(a \in S \quad \text{and} \quad b \in S) \Rightarrow (ab^{-1} \in S).$$

(This includes the case where a and b are the same element.) Prove that S is a subgroup of G.

(b) If H is a subgroup of G and p is a *given* element of G, prove that the set $\{p^{-1}hp : h \in H\}$ is a subgroup of G isomorphic with H. (L)

20. G is a multiplicative group with identity element e. Every element $g \in G$ has the property that $g^2 = e$. Use this property applied to the element ab to prove that $ab = ba$ for any elements $a, b \in G$. H is the subset of G containing the three elements e, a and b. Prove that H cannot be a subgroup of G. Determine whether it is possible for a subset of G containing four distinct elements to be a subgroup of G. (C)

21. Let G be a group generated by elements a and b satisfying the relations $a^2 = b^3 = e$ and $ab = ba$, where e is the unit element of G. Show that G is isomorphic to the cyclic group of order 6. (C)

Formulae for reference

Logarithms

$$x = a^p \quad \Leftrightarrow \quad \log_a x = p; \quad \log_a x = \log_b x / \log_b a.$$

Series

Sum of A.P. $= \frac{1}{2}n\{2a + (n-1)d\}$, sum of G.P. $= \dfrac{a(1-r^n)}{1-r}$

For $|r| < 1$, sum to infinity of G.P. $= a/(1-r)$

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + b^n,$$

where n is a positive integer.

$$\sum_1^n r = \tfrac{1}{2}n(n+1); \quad \sum_1^n r^2 = \tfrac{1}{6}n(n+1)(2n+1); \quad \sum_1^n r^3 = \tfrac{1}{4}n^2(n+1)^2$$

Trigonometry

$$\cos^2\theta + \sin^2\theta = 1; \quad 1 + \tan^2\theta = \sec^2\theta; \quad \cot^2\theta + 1 = \operatorname{cosec}^2\theta.$$

$$\sin(A+B) = \sin A\cos B + \cos A\sin B \qquad \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\cos(A-B) = \cos A\cos B + \sin A\sin B \qquad \tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

$\sin 2A = 2\sin A\cos A$; $\tan 2A = 2\tan A/(1-\tan^2 A)$
$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$
$\cos^2 A = \frac{1}{2}(1+\cos 2A)$, $\sin^2 A = \frac{1}{2}(1-\cos 2A)$

$$\sin\theta = \frac{2t}{1+t^2}, \quad \cos\theta = \frac{1-t^2}{1+t^2}, \quad \tan\theta = \frac{2t}{1-t^2}, \quad \text{where} \quad t = \tan\tfrac{1}{2}\theta.$$

$$2\sin A\cos B = \sin(A+B) + \sin(A-B)$$
$$2\cos A\cos B = \cos(A+B) + \cos(A-B)$$
$$2\sin A\sin B = -\{\cos(A+B) - \cos(A-B)\}$$
$$\sin X + \sin Y = 2\sin\tfrac{1}{2}(X+Y)\cos\tfrac{1}{2}(X-Y)$$
$$\sin X - \sin Y = 2\cos\tfrac{1}{2}(X+Y)\sin\tfrac{1}{2}(X-Y)$$
$$\cos X + \cos Y = 2\cos\tfrac{1}{2}(X+Y)\cos\tfrac{1}{2}(X-Y)$$
$$\cos X - \cos Y = -2\sin\tfrac{1}{2}(X+Y)\sin\tfrac{1}{2}(X-Y)$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R, \quad a^2 = b^2 + c^2 - 2bc\cos A$$

Area of $\triangle ABC = \frac{1}{2}bc\sin A = \sqrt{\{s(s-a)(s-b)(s-c)\}}$,
where $s = \frac{1}{2}(a+b+c)$.
Length of circular arc $= r\theta$, area of sector $= \frac{1}{2}r^2\theta$ (θ in radians)

Calculus

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}, \qquad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

y	$\sin x$	$\cos x$	$\tan x$	$\cot x$	$\sec x$	$\operatorname{cosec} x$
dy/dx	$\cos x$	$-\sin x$	$\sec^2 x$	$-\operatorname{cosec}^2 x$	$\sec x\tan x$	$-\operatorname{cosec} x\cot x$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int e^{kx}dx = \frac{1}{k}e^{kx} + c, \qquad \int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + c,$$

$$\int \frac{dx}{\sqrt{(a^2-x^2)}} = \sin^{-1}\frac{x}{a} + c, \quad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c.$$

If $\dfrac{d^2y}{dx^2} + n^2y = k$, then $y = A\cos nx + B\sin nx + \dfrac{k}{n^2}$.

If $\dfrac{d^2y}{dx^2} - n^2y = k$, then $y = Ae^{nx} + Be^{-nx} - \dfrac{k}{n^2}$.

Expansions

$$(1+x)^n = 1 + nx + \frac{1}{2!}n(n-1)x^2 + \ldots + \binom{n}{r}x^r + \ldots \quad (|x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^r}{r!} + \ldots \quad (\text{all } x)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + \frac{(-1)^{r+1}x^r}{r} + \ldots \quad (-1 < x \leqslant 1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots, \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

Maclaurin's series: $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$

Hyperbolic functions

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}), \quad \sinh x = \tfrac{1}{2}(e^x - e^{-x})$$

$$\cosh^2 x - \sinh^2 x = 1.$$

Numerical methods

Newton-Raphson process: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

Trapezium rule:

$$\int_a^b f(x)\,dx \approx \tfrac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\}$$

Simpson's rule:

$$\int_a^b f(x)\,dx \approx \tfrac{1}{3}h\{(y_0 + y_n) + 4(y_1 + y_3 + \ldots) + 2(y_2 + y_4 + \ldots)\}$$

Coordinate geometry

Point dividing line joining (x_1, y_2) to (x_2, y_2) in ratio $\mu:\lambda$ is

$$\left(\frac{\lambda x_1 + \mu x_2}{\lambda + \mu}, \quad \frac{\lambda y_1 + \mu y_2}{\lambda + \mu}\right)$$

Distance of (x_1, y_1) from $ax + by + c = 0$ is $\dfrac{|ax_1 + by_1 + c|}{\sqrt{(a^2 + b^2)}}$

Area of triangle $= \frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$

Circle: $(x - h)^2 + (y - k)^2 = a^2$, centre (h, k), radius a;
$x^2 + y^2 + 2gx + 2fy + c = 0$, centre $(-g, -f)$, radius $\sqrt{(f^2 + g^2 - c)}$.

Rectangular hyperbola: $xy = c^2$, $(ct, c/t)$

Semicubical parabola: $ay^2 = x^3$, (at^2, at^3)

Parabola: $y^2 = 4ax$, $(at^2, 2at)$ focus $(a, 0)$, directrix $x = -a$.

Ellipse: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $(a\cos\theta, b\sin\theta)$, where $b^2 = a^2(1 - e^2)$, $e < 1$, foci $(\pm ae, 0)$ or $(\pm\sqrt{\{a^2 - b^2\}}, 0)$, directrices $x = \pm a/e$ or $\pm a^2/\sqrt{(a^2 - b^2)}$

Hyperbola: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, $(a\sec\theta, b\tan\theta)$, where $b^2 = a^2(e^2 - 1)$, $e > 1$, foci $(+ae, 0)$, directrices $x = \pm a/e$, asymptotes $y = \pm bx/a$

Mechanics

Straight line motion with constant acceleration

$$v = u + at; \quad s = ut + \tfrac{1}{2}at^2; \quad s = \tfrac{1}{2}(u + v)t; \quad v^2 = u^2 + 2as.$$

Hooke's law: $T = \dfrac{\lambda}{l}x$; energy stored in string $= \dfrac{\lambda x^2}{2l}$.

Uniform body	Length, area or volume	Centre of mass
Triangular lamina, height h	$\frac{1}{2}h \times$ base	$\frac{1}{3}h$ from base
Circular arc, radius r, angle at centre 2α	$2r\alpha$	$\dfrac{r\sin\alpha}{\alpha}$ from centre
Sector of circle, radius r, angle at centre 2α	$r^2\alpha$	$\dfrac{2r\sin\alpha}{3\alpha}$ from centre
Solid cone or pyramid, height h	$\frac{1}{3}h \times$ base area	$\frac{1}{4}h$ from base
Solid hemisphere, radius r	$\frac{2}{3}\pi r^3$	$\frac{3}{8}r$ from centre
Hemispherical shell, radius r	$2\pi r^2$	$\frac{1}{2}r$ from centre

Probability

$$P(A \cup B) = P(A) + P(B) - P(A \cap B), \quad P(A \cap B) = P(A) \times P(B \mid A)$$

$$E(X) = \mu = \begin{cases} \sum x_r p(x_r) & X \text{ discrete} \\ \int_{-\infty}^{\infty} x f(x)\,dx & X \text{ continuous} \end{cases}$$

$$\mathrm{Var}(X) = \sigma^2 = \begin{cases} \sum (x_r - \mu)^2 p(x_r) = \sum x_r^2 p(x_r) - \mu^2, & X \text{ discrete} \\ \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2, & X \text{ continuous} \end{cases}$$

Binomial distribution B(n, p):

$$P(X = r) = \binom{n}{r} p^r (1-p)^{n-r} \quad (r = 0, 1, 2, \ldots, n)$$

$$E(X) = np, \quad \mathrm{Var}(X) = np(1-p)$$

Poisson distribution, parameter μ:

$$P(X = r) = \frac{\mu^r}{r!} e^{-\mu} \quad (r = 0, 1, 2, \ldots)$$

$$E(X) = \mu, \quad \mathrm{Var}(X) = \mu$$

Properties of expectation and variance:

$$E[g(X)] = \begin{cases} \sum g(x_r) p(x_r) & X \text{ discrete} \\ \int_{-\infty}^{\infty} g(x) f(x)\,dx & X \text{ continuous} \end{cases}$$

$$E(aX + b) = aE(X) + b, \qquad E(aX + bY) = aE(X) + bE(Y)$$

$$\mathrm{Var}(X) = E(X^2) - [E(X)]^2, \quad \mathrm{Var}(aX + b) = a^2 \mathrm{Var}(X)$$

For independent random variables $E(XY) = E(X)E(Y)$ and $\mathrm{Var}(aX + bY) = a^2 \mathrm{Var}(X) + b^2 \mathrm{Var}(Y)$

$$\mathrm{Cov}(X, Y) = E[\{X - E(X)\}\{Y - E(Y)\}] = E(XY) - E(X)E(Y)$$

Correlation coefficient $\rho = \dfrac{\mathrm{Cov}(X, Y)}{\sqrt{\{\mathrm{Var}(X)\mathrm{Var}(Y)\}}}$

Sampling distributions (large populations):

$$E(\bar{X}) = \mu, \qquad \mathrm{Var}(\bar{X}) = \sigma^2/n$$

$$E(\bar{X} - \bar{Y}) = \mu_x - \mu_y, \quad \mathrm{Var}(\bar{X} - \bar{Y}) = \frac{\sigma_x^2}{n_x} + \frac{\sigma_y^2}{n_y}$$

For sample variance V, $E(V) = \left(\dfrac{n-1}{n}\right)\sigma^2$

Statistics

Sample mean, $\bar{x} = \frac{1}{n}\sum x$

Sample variance, $\text{var}(x) = \frac{1}{n}\sum (x - \bar{x})^2 = \frac{1}{n}\sum x^2 - \bar{x}^2$

Sample covariance, $\text{cov}(x, y) = \frac{1}{n}\sum (x - \bar{x})(y - \bar{y}) = \frac{1}{n}\sum xy - \bar{x}\bar{y}$

Unbiased estimate of $\mu = \bar{x}$

Unbiased estimate of $\sigma^2 = s^2 = \frac{1}{n-1}\sum (x - \bar{x})^2$

Two sample estimate of $\mu = \frac{m\bar{x} + n\bar{y}}{m+n}$

Two sample estimate of $\sigma^2 = \frac{(m-1)s_x^2 + (n-1)s_y^2}{m+n-2}$

$$S_{xy} = \sum (x - \bar{x})(y - \bar{y}) = \sum xy - n\bar{x}\bar{y}$$
$$S_{xx} = \sum (x - \bar{x})^2 = \sum x^2 - n\bar{x}^2$$
$$S_{yy} = \sum (y - \bar{y})^2 = \sum y^2 - n\bar{y}^2$$

Correlation coefficient $r = \frac{\text{cov}(x, y)}{\sqrt{\{\text{var}(x)\text{var}(y)\}}} = \frac{S_{xy}}{\sqrt{(S_{xx}S_{yy})}}$

Spearman's rank correlation coefficient $r_S = 1 - \frac{6\sum d^2}{n(n^2-1)}$

Kendall's rank correlation coefficient $r_K = \frac{P - Q}{\frac{1}{2}n(n-1)}$

Least squares regression line of y on x:

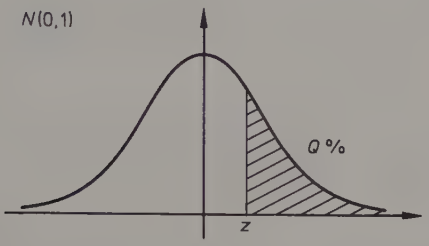
$$y = a + bx, \quad \text{where} \quad b = \frac{S_{xy}}{S_{xx}}, \quad a = \bar{y} - b\bar{x}.$$

Table 1 The normal distribution function $\Phi(z)$

z	·00	·01	·02	·03	·04	·05	·06	·07	·08	·09
0·0	·5000	·5040	·5080	·5120	·5160	·5199	·5239	·5279	·5319	·5359
01	·5398	·5438	·5478	·5517	·5557	·5596	·5636	·5675	·5714	·5753
0·2	·5793	·5832	·5871	·5910	·5948	·5987	·6026	·6064	·6103	·6141
0·3	·6179	·6217	·6255	·6293	·6331	·6368	·6406	·6443	·6480	·6517
0·4	·6554	·6591	·6628	·6664	·6700	·6736	·6772	·6808	·6844	·6879
0·5	·6915	·6950	·6985	·7019	·7054	·7088	·7123	·7157	·7190	·7224
0·6	·7257	·7291	·7324	·7357	·7389	·7422	·7454	·7486	·7517	·7549
0·7	·7580	·7611	·7642	·7673	·7704	·7734	·7764	·7794	·7823	·7852
0·8	·7881	·7910	·7939	·7967	·7995	·8023	·8051	·8078	·8106	·8133
0·9	·8159	·8186	·8212	·8238	·8264	·8289	·8315	·8340	·8365	·8389
1·0	·8413	·8438	·8461	·8485	·8508	·8531	·8554	·8577	·8599	·8621
1·1	·8643	·8665	·8686	·8708	·8729	·8749	·8770	·8790	·8810	·8830
1·2	·8849	·8869	·8888	·8907	·8925	·8944	·8962	·8980	·8997	·9015
1·3	·9032	·9049	·9066	·9082	·9099	·9115	·9131	·9147	·9162	·9177
1·4	·9192	·9207	·9222	·9236	·9251	·9265	·9279	·9292	·9306	·9319
1·5	·9332	·9345	·9357	·9370	·9382	·9394	·9406	·9418	·9429	·9441
1·6	·9452	·9463	·9474	·9484	·9495	·9505	·9515	·9525	·9535	·9545
1·7	·9554	·9564	·9573	·9582	·9591	·9599	·9608	·9616	·9625	·9633
1·8	·9641	·9649	·9656	·9664	·9671	·9678	·9686	·9693	·9699	·9706
1·9	·9713	·9719	·9726	·9732	·9738	·9744	·9750	·9756	·9761	·9767
2·0	·97725	·97778	·97831	·97882	·97932	·97982	·98030	·98077	·98124	·98169
2·1	·98214	·98257	·98300	·98341	·98382	·98422	·98461	·98500	·98537	·98574
2·2	·98610	·98645	·98679	·98713	·98745	·98778	·98809	·98840	·98870	·98899
2·3	·98928	·98956	·98983	·99010	·99036	·99061	·99086	·99111	·99134	·99158
2·4	·99180	·99202	·99224	·99245	·99266	·99286	·99305	·99324	·99343	·99361
2·5	·99379	·99396	·99413	·99430	·99446	·99461	·99477	·99492	·99506	·99520
2·6	·99534	·99547	·99560	·99573	·99585	·99598	·99609	·99621	·99632	·99643
2·7	·99653	·99664	·99674	·99683	·99693	·99702	·99711	·99720	·99728	·99736
2·8	·99744	·99752	·99760	·99767	·99774	·99781	·99788	·99795	·99801	·99807
2·9	·99813	·99819	·99825	·99831	·99836	·99841	·99846	·99851	·99856	·99861
3·0	·99865	·99869	·99874	·99878	·99882	·99886	·99889	·99893	·99896	·99900

z	3·1	3·2	3·3	3·4	3·5	3·6	3·7	3·8	3·9	4·0
Φ	·99903	·99931	·99952	·99966	·99977	·99984	·99989	·99993	·99995	·99997

Table 2 Percentage points of the normal distribution



The shaded area represents $Q\%$ of the total distribution.

$$Q(z) = 100P(Z \geqslant z) = 100\{1 - \Phi(z)\}$$

Q	z	Q	z	Q	z
50	0·000	1·5	2·170	0·2	2·878
40	0·253	1·0	2·326	0·15	2·968
30	0·524	0·9	2·366	0·1	3·090
20	0·842	0·8	2·409	0·05	3·291
10	1·282	0·7	2·457	0·01	3·719
5	1·645	0·6	2·512	0·005	3·891
4	1·751	0·5	2·576	0·001	4·265
3	1·881	0·4	2·652	0·0005	4·417
2·5	1·960	0·3	2·748	0·0001	4·753
2	2·054	0·25	2·807	0·00005	4·892

Answers

Exercise 1.1

1. (a) Most people disapprove of the Knave of Hearts.
   (b) N is not prime.
   (c) Boycott took over as England cricket captain.
2. (a) Not valid. There could be swallow-tail butterflies in Norfolk, but there may not be.
   (b) Not valid. We are not told that absent-minded people do wear odd socks.
   (c) Not valid. $ABCD$ could be an isosceles trapezium.
3. Since $x - y = 0$, we cannot divide by $(x - y)$ to obtain $x + y = y$.
4. (1) Either, (2) Either, (3) Either, (4) True, (5) Either, (6) Either, (7) True, (8) Either, (9) Either, (10) Either, (11) Either, (12) True, (13) Either, (14) False, (15) False (16) True (17) False (18) Either.

Exercise 1.2

1. (a) $p \Rightarrow q, p \Rightarrow r, q \Rightarrow r$ (b) $p \Rightarrow r$ (c) $p \Rightarrow q, p \Rightarrow r$ (d) $p \Leftrightarrow q, p \Leftrightarrow r, q \Leftrightarrow r$
2. (a) $q \not\Rightarrow p, r \not\Rightarrow p, r \not\Rightarrow q$ (b) $p \not\Rightarrow q, q \not\Rightarrow p, q \not\Rightarrow r, r \not\Rightarrow p, r \not\Rightarrow q$ (c) $q \not\Rightarrow p, q \not\Rightarrow r, r \not\Rightarrow p, r \not\Rightarrow q$ (d) none
3. (a) $\sim q \Rightarrow \sim p, \sim r \Rightarrow \sim p, \sim r \Rightarrow \sim q$ (b) $\sim r \Rightarrow \sim p$ (c) $\sim q \Rightarrow \sim p, \sim r \Rightarrow \sim p$ (d) $\sim p \Leftrightarrow \sim q, \sim p \Leftrightarrow \sim r, \sim q \Leftrightarrow \sim r$
4. (a) $q \Rightarrow p, \sim p \Rightarrow \sim q$ (b) $p \Rightarrow \sim q, q \Rightarrow \sim p$ (c) $p \Leftrightarrow \sim q, \sim p \Leftrightarrow q$ (d) $p \Leftrightarrow \sim q, \sim p \Leftrightarrow q$
5. (a) $q \Rightarrow p$ (b) $\sim p \Leftrightarrow q$ (c) $\sim q \not\Rightarrow p$ (d) $\sim p \not\Leftrightarrow \sim q$ (e) $q \Rightarrow \sim p$ (f) $p \Leftrightarrow \sim q$
6. (a) $q \Rightarrow p$ (b) $q \Rightarrow p$ (c) $q \Rightarrow p$ (d) $q \Rightarrow p$ (e) $p \Rightarrow q$ (f) $p \Rightarrow q$
7. $AB = AC$ if $\triangle ABC$ is equilateral.
   $\triangle ABC$ is equilateral only if $AB = AC$.
   $\triangle ABC$ being equilateral is a sufficient condition for $AB = AC$.
   $AB = AC$ is a necessary condition for $\triangle ABC$ to be equilateral.

Exercise 1.3

1. If $\angle ABC = \angle CAT$, then AT must be a tangent to the circle.
3. (a) If a diameter of a circle is perpendicular to a chord then it bisects that chord; (i) True, (ii) True.
   (b) If $2x^2 - x - 3 = 0$, then $2x - 3 = 0$; (i) True, (ii) False.
   (c) If a pentagon is equilateral, it must be regular; (i) True, (ii) False.
   (d) If x^2 is greater than y^2, then x is greater than y; (i) False, (ii) False.
   (e) If the sum of the digits of a positive integer N is divisible by 9, then N is divisible by 9; (i) True, (ii) True.
   (f) If there are five pairs of socks in a drawer, then any set of six socks taken from the drawer will contain a pair; (i) False, (ii) True.
4. (a) True, (b) False. 5. (a) True, (b) False.
6. (a) False, (b) True.

Exercise 1.4

9. (a) 2870 (b) 19 270 (c) 11 480 (d) 10 660 10. $\frac{1}{4}$
11. (a) $(n, 2n - 1)$ (b) $(n + 1)y = (2n + 1)x$
12. $\frac{x}{2x + 1}, \frac{x}{3x + 1}, \frac{x}{nx + 1}$

Exercise 1.5

1. If $(x - 3)^2 = (x + 1)^2$, then $x - 3 = \pm(x + 1)$.
2. Since $\lg(0{\cdot}2) < 0$, $2\lg(0{\cdot}2) < \lg(0{\cdot}2)$.
3. No value of x satisfies the initial statement.
4. $a + b > 0$ implies $a^2 + ab > 0$ only if $a > 0$.
5. The diagram is incorrect. P lies outside $\triangle ABC$.
6. There is no value of n for which a_n is divisible by 6.
7. It is necessary to prove that $r \Rightarrow q$ and that $s \Rightarrow q$.
8. It is necessary to prove that $P(1)$ and $P(2)$ are true.
14. $\begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 7 \\ 0 & 8 \end{pmatrix}, \begin{pmatrix} 1 & 2^n - 1 \\ 0 & 2^n \end{pmatrix}$ 15. 1.
18. (i) False, (ii) True, (iii) False. 19. One example is $f(x) \equiv x$.
20. One possibility is $\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \mathbf{X} = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}$; $\begin{pmatrix} -2 + 2^n.3 & -2 + 2^{n+1} \\ 3 - 2^n.3 & 3 - 2^{n+1} \end{pmatrix}$.

Exercise 2.1

1. (a) Even, (b) neither, (c) odd. 2. (a) Even, (b) even, (c) odd.
3. (a) Even, (b) odd, (c) neither. 4. (a) Neither, (b) neither, (c) even.

5. (a) Yes, $2\pi/3$; (b) yes, $\pi/2$; (c) yes, 8π.
6. (a) No; (b) yes, 2π; (c) yes, π.
7. (a) Yes, 2π; (b) no; (c) yes, 2π.
8. (a) Yes, π; (b) no; (c) yes, π.
9. f: even, $f(x) \geqslant -1$; g: neither, $\mathbb{R}$; fg: neither, $fg(x) \geqslant -1$; gf: even, $gf(x) \geqslant -1$.
10. f: neither, $\mathbb{R}$; g: odd, $-1 \leqslant g(x) \leqslant 1$, 2π; fg: neither, $\frac{1}{2}(\pi - 1) \leqslant fg(x) \leqslant \frac{1}{2}(\pi + 1)$, 2π; gf: even, $-1 \leqslant gf(x) \leqslant 1$, 4π.
11. f: even, $-1 \leqslant f(x) \leqslant 1$, 2π; g: even, $g(x) \geqslant 0$; fg: even, $-1 \leqslant fg(x) \leqslant 1$; gf: even, $0 \leqslant gf(x) \leqslant 1$, π.
12. f: even, $f(x) \geqslant 0$; g: odd, $\mathbb{R}$; fg: even, $fg(x) \geqslant 0$; gf: even, $gf(x) \geqslant -2$.
17. (a) -1, 1, (b) -1, 1. 18. (a) none, (b) -1, 1.
19. (a) $x \in \mathbb{Z}$, (b) $x \in \mathbb{Z}$. 20. (a) 1, (b) -1, 1. 22. 3.

Exercise 2.2

7. $(-1, -1)$ min; $(-1, -1)$ max, $x = -2$, $x = 0$, $y = 0$.
8. $(-1, 0)$ min; $x = -1$, $y = 0$. 9. $(-1, 1)$ min; $(-1, 1)$ max, $y = 0$.
10. $(-3, 36)$ max, $(5/3, -400/27)$ min; $(-3, 1/36)$ min, $(5/3, -27/400)$ max, $x = -5$, $x = 0$, $x = 3$, $y = 0$.
11. $x = -2$, $y = 2$; $x = \frac{1}{2}$, $y = \frac{1}{2}$. 12. None, $(3/2, 0)$ min.
13. (0.4) max; $(-2, 0)$ min, $(0, 16)$ max, $(2, 0)$ min.
14. $(2n\pi + \frac{1}{2}\pi, 1)$ max, $(2n\pi - \frac{1}{2}\pi, -1)$ min; $(n\pi, 0)$ min, $(n\pi + \frac{1}{2}\pi, 1)$ max.
15. $x = 1$, $y = -2$; $(0, 0)$ min, $x = 1$, $y = 4$.
16. $(0, 0)$ min, $(2, 4)$ max; $(2, -2)$ min, $(2, 2)$ max.

Exercise 2.3

1. Yes; $\frac{1}{2}(x - 1)$; $\mathbb{R}$. 2. Yes; $5 - x$; $\mathbb{R}$. 3. No; e.g. -2, 0.
4. Yes; $\sqrt{(x + 2)}$; $\{x \in \mathbb{R} : x \geqslant -2\}$.
5. Yes; $1/(x - 1)$; $\mathbb{R} - \{1\}$.
6. Yes, $9 - 1/x$; $\mathbb{R} - \{0\}$. 7. Yes; $3 - x^2$; $\{x \in \mathbb{R} : x \geqslant 0\}$.
8. No; e.g. -1, 1. 9. Yes; $\log_3 x$; $\mathbb{R}^+$. 10. Yes, $10^x - 2$; $\mathbb{R}$.
11. No; e.g. -1, 1. 12. Yes; $x/(x - 2)$; $\mathbb{R} - \{2\}$.
13. No; e.g. 0, 2π. 14. Yes; $\sin x$; $\{x \in \mathbb{R} : -\frac{1}{2}\pi \leqslant x \leqslant \frac{1}{2}\pi\}$
15. $1 + 4x^2$; $\frac{1}{2}\sqrt{(x - 1)}$. 16. $gf : x \to 3 - 9/x$.
17. (a) $f^{-1} : x \to 2 + \sqrt{x}$, $x \geqslant 0$, (b) $f^{-1} : x \to 1 + 10^{x-1}$,

(c) $f^{-1} : x \to \dfrac{1 - x}{1 + x}$, $x \neq -1$ (d) $f^{-1} : x \to \sqrt{\left(1 + \dfrac{1}{x^2}\right)}$, $x > 0$.

18. $k = 2$, $\{x \in \mathbb{R} : x \geqslant 4\}$. 19. $k = 3$, $\{x \in \mathbb{R} : x \leqslant 9\}$.
20. $k = 4$, $\{x \in \mathbb{R} : x \geqslant 0\}$. 21. $k = 1$, $\{x \in \mathbb{R} : x \geqslant -1\}$.
22. $k = 1$, $\{x \in \mathbb{R} : x \geqslant 2\}$. 23. $k = \frac{1}{4}$, $\{x \in \mathbb{R} : x \geqslant -\frac{1}{4}\}$.
24. (a) No, (b) Yes, (c) Yes, (d) No, (e) No, (f) Yes.

Exercise 2.4

1. (a) 5, (b) 4, (c) -2, (d) $x-7$, (e) 0, (f) $-4x+5$.
2. 5, -12. 3. $-3x+4$. 4. 2, -8. 5. 4, -2, -5.
6. $3(a+b)(b+c)(c+a)$. 7. $3(x-y)(y-z)(z-x)$.
8. $k=-7$, $(x+1)^2(2x-7)$; $k=20$, $(x-2)^2(2x+5)$.
9. $(x+2)^2(3x^2-4x+4)$. 10. -2, 13; $(x-1)^2(x-6)$.
11. -3, -3, 12. 12. $k=-5$: $-1/2$, $-1/2$, $5/2$; $k=27$: $3/2$, $3/2$, $-3/2$.
13. -11; -3, $-\frac{1}{2}$, 2. 14. -14; 3, -2, $4/3$. 15. 1, 4, 2.
16. 1, 3, 1, 6. 17. 1, -3, 1, 0. 19. -3, $\frac{1}{4}$. 20. 0, 2.
21. $(x-a)^2g'(x)+2(x-a)g(x)+A$; $f'(a)$, $f(a)-af'(a)$; $k=-1$: -3, 0, 0; $k=1$; -1, 2, 2.

Exercise 2.5

7. $y\leqslant 1/5$ or $y\geqslant 5$; $x=-4$, $x=1$, $y=0$; $(-1,5)$ min, $(11,1/5)$ max.
8. $-\frac{1}{2}\leqslant y\leqslant 1$; $y=0$; $(-2,-\frac{1}{2})$ min, $(1,1)$ max.
9. All real values; $x=-3$, $x=0$, $y=0$; no turning points.
10. $y\leqslant 1/9$ or $y\geqslant 1$; $x=-3$, $x=0$, $y=0$; $(-1,1)$ min, $(3,1/9)$ max.
11. $y\leqslant 1$ or $y\geqslant 9$; $x=1$, $x=2$, $y=0$; $(0,1)$ max, $(4/3,9)$ min.
12. $-5/4\leqslant y\leqslant 5$; $y=0$; $(4,-5/4)$ min, $(-1,5)$ max.
13. $-\frac{1}{3}\leqslant y\leqslant 1$; $y=0$; $(0,1)$ max, $(2,-\frac{1}{3})$ min.
14. $y\leqslant -1$ or $y\geqslant -1/9$; $x=-5$, $x=-2$, $y=0$; $(-8,-1/9)$ min, $(-4,-1)$ max.
15. $y\leqslant 0$ or $y\geqslant 4/3$; $x=1$, $x=4$, $y=0$; $(5/2,4/3)$ min.
16. All real values; $x=1$, $x=4$, $y=0$; no turning points.
20. $x=3$, $y=x$; $(1,-1)$ max, $(5,7)$ min.

Exercise 2.6

1. $\dfrac{1}{x+1}-\dfrac{1}{x+2}$. 2. $\dfrac{1}{x-1}-\dfrac{2}{x+2}$. 3. $\dfrac{2}{x-2}+\dfrac{1}{x-4}$.

4. $\dfrac{3}{x}+\dfrac{1}{x-3}$. 5. $\dfrac{3}{x-2}-\dfrac{2}{x+1}$. 6. $\dfrac{2}{x+1}+\dfrac{1}{2x+1}$.

7. $\dfrac{1}{2(x-1)}+\dfrac{1}{2(x+1)}$. 8. $\dfrac{1}{3+x}+\dfrac{1}{2-x}$. 9. $\dfrac{1}{4x-1}+\dfrac{2}{2x+1}$.

10. $\dfrac{1}{2(2x-1)}-\dfrac{3}{2(2x+5)}$. 11. $\dfrac{3}{x-2}-\dfrac{4}{x+3}+\dfrac{1}{x+1}$.

12. $\dfrac{6}{x-1}+\dfrac{2}{x+1}-\dfrac{3}{x-2}$. 13. $\dfrac{x+2}{x^2+1}-\dfrac{1}{x+1}$. 14. $\dfrac{2}{2x-1}-\dfrac{x}{x^2+1}$.

15. $\dfrac{1}{x}+\dfrac{3-2x}{2x^2+1}$.

16. $\dfrac{2x+1}{x^2+3}-\dfrac{3}{2x-1}$.

17. $\dfrac{2}{x-1}-\dfrac{4}{(x-1)^2}-\dfrac{1}{x+2}$.

18. $\dfrac{1}{x}+\dfrac{2}{x+1}+\dfrac{3}{(x+1)^2}$.

19. $\dfrac{5}{x}+\dfrac{2}{x^2}-\dfrac{3}{x-1}$.

20. $\dfrac{1}{x-2}-\dfrac{1}{x+1}-\dfrac{2}{(x+1)^2}-\dfrac{8}{(x+1)^3}$.

21. $\dfrac{x}{2x^2+5}-\dfrac{1}{4x-5}$.

22. $\dfrac{1}{x-2}-\dfrac{x}{x^2-x+2}$.

23. $1+\dfrac{1}{x-1}-\dfrac{1}{x+1}$.

24. $1+\dfrac{4}{3(x-2)}-\dfrac{1}{3(x+1)}$.

25. $\dfrac{1}{3}+\dfrac{5}{6(3x-1)}-\dfrac{1}{2(x-1)}$.

26. $x+\dfrac{2}{x-2}+\dfrac{2}{x+2}$.

27. $\dfrac{x+3}{x^2+3}-\dfrac{x+2}{x^2+2}$.

28. $\dfrac{2x-1}{x^2+2}+\dfrac{1}{x+1}-\dfrac{2}{(x+1)^2}$.

29. $\dfrac{x}{2}+1-\dfrac{1}{4(2x-1)}-\dfrac{1}{4(2x-1)^2}+\dfrac{2}{x-2}$.

30. $x^2-x-3+\dfrac{3}{x^2+1}+\dfrac{3x}{x^2+2}$.

31. $\dfrac{2}{2x+3}+\dfrac{1}{x+1}; -\dfrac{4}{(2x+3)^2}-\dfrac{1}{(x+1)^2}; \dfrac{16}{(2x+3)^3}+\dfrac{2}{(x+1)^3}$.

32. (a) $(0,0)$, (b) $(-7,-3/7)$.

Exercise 2.7

1. f: even, periodic, $\frac{1}{3}\pi$; g: neither, not periodic; h: even, not periodic; k: odd, periodic, 4π.
2. f: none; 2; g: x an even integer; x an even integer; h: $-1, 2; -1, 2$; k: none; 1.
3. $\pi, \pi; \pi; 0 \leqslant f(x).g(x) < 2$.
4. f: neither; g: even; h: even; $12\pi, 6\pi$.
5. $4 \leqslant f(x) \leqslant 20$, f^{-1} exists; $-4 \leqslant g(x) \leqslant 0$, no inverse; $-5 \leqslant h(x) \leqslant 5$, no inverse; $k(x) \leqslant 1$, k^{-1} exists.
6. $a = 1$; $g^{-1}(x) = (2-\sqrt{\{4-x^2\}})/x$, $-2 \leqslant x \leqslant 2$, $-1 \leqslant g^{-1}(x) \leqslant 1$.
7. 1, 1/5.
8. $-5 < f(x) \leqslant 4$; $-2 < x \leqslant 1$: $f^{-1}(y) = \frac{1}{2}(y+1)$, $1 < x \leqslant 2$: $f^{-1}(y) = \sqrt{y}$, $2 \leqslant x < 3$: $f^{-1}(y) = \frac{1}{3}(10-y)$.

9. $-\frac{1}{4}\pi \leqslant x \leqslant \frac{1}{4}\pi, f^{-1}(y) = \frac{1}{2}\sin^{-1} y$; $\frac{1}{4}\pi \leqslant x \leqslant \frac{3}{4}\pi, f^{-1}(y) = \frac{1}{4}\pi(2 - y)$.
10. (i) F: odd, periodic, 6π; G: even, periodic, 2π; f_3: odd; f_4: even, period $\frac{1}{3}\pi$, (ii) 0.
11. $\{y \in \mathbb{R} : y \geqslant 0, y \neq 2r + 1, r \in \mathbb{Z}\}$; not one-one; $g^{-1} : x \to x - \frac{1}{2}[x]$; $\{x \in \mathbb{R}^+ : 2r < x < 2r + 1, r \in \mathbb{Z}\}$.
12. $-1 < x < 2, 3 < x < 6$. 14. $x < -1$ or $x > 0$.
16. $-12\frac{1}{24}$; $-3/2, 4/3$; $-1{\cdot}56, -1{\cdot}44, 1{\cdot}27, 1{\cdot}39$.
17. $a = b = 1$; $x^3 - 2x^2 + 3$. 18. (a) $-6x + 2$, (b) $-2x$.
19. $2, \pm\sqrt{5}$.
20. (a) $(x + y + z)(x - y)(y - z)(z - x)$, (b) $3(a - b - c)(b - c - a)(c - a - b)$.
21. $a = -2, b = -1, c = 16$. 23. $-2, -2, 4 \pm 2\sqrt{2}$. 24. $0, -3, 125$.
26. $4, -5$.
27. (i) 2, (ii) HCF of $A(x)$ and $B(x)$ is $8x + 8$; $\alpha : g(x)$, $\beta : A(x)$ and $B(x)$ have no common factor.
28. $-\frac{1}{3} \leqslant y \leqslant 1, x = -1$; $-\frac{1}{3} \leqslant f(x) \leqslant 1$; $(-2, -\frac{1}{3}), (0, 1)$.
29. (a) $\mathbb{R} - \{\frac{1}{2}\}$; $x = -\frac{1}{2}$, $y = \frac{1}{2}$; none, (b) $\{y \in \mathbb{R} : y < 0$ or $y \geqslant 1\}$; $x = -3$, $x = 1$, $y = 0$; $(-1, 1)$, (c) $\{y \in \mathbb{R} : y \leqslant -3$ or $y \geqslant -\frac{1}{3}\}$; $x = 0$, $x = 3$, $y = 0$; $(-3, -\frac{1}{3}), (1, -3)$, (d) $\mathbb{R}$; $x = -2$, $x = 2$, $y = 0$; none.
30. (a) none, (b) (0, 0) max, (2, 4) min, (c) (0, 0) min, (d) (3, 1/8) max.
31. $\{x \in \mathbb{R} : x \neq -1, 3\}$; $\{y \in \mathbb{R} : y \leqslant -1$ or $y \geqslant \frac{1}{2}\}$; $x = -1$, $x = 3$, $y = 1$.
32. (a) max $(1, \frac{1}{2})$, min $(-1, -\frac{1}{2})$; inflexions $(-\sqrt{3}, -\frac{1}{4}\sqrt{3}), (0, 0), (\sqrt{3}, \frac{1}{4}\sqrt{3})$.

33. $\dfrac{3}{3x - 2} - \dfrac{1}{2x + 3}$. 34. $\dfrac{1}{x - 1} + \dfrac{1}{(x - 1)^2} - \dfrac{2}{2x - 1}$.

35. $\dfrac{1}{2(x - 1)} + \dfrac{1 - x}{2(x^2 + 1)}$. 36. $\dfrac{1}{3(x + 1)} - \dfrac{1}{x - 1} + \dfrac{2}{3(x - 2)}$.

37. $-\dfrac{1}{2} + \dfrac{3}{2(2x + 1)} + \dfrac{2}{3 - x}$. 38. $\dfrac{1}{1 - x} - \dfrac{1 - x}{1 + x + x^2}$.

39. $x - 2 - \dfrac{2}{3(x - 1)} + \dfrac{8}{3(x + 2)}$.

40. $\dfrac{1}{x - 1} + \dfrac{2}{(x - 1)^2} - \dfrac{1}{x + 1} + \dfrac{3}{(x + 1)^2}$.

Exercise 3.1

1. (a) $2e^{2x}$, (b) $-5e^{-x}$, (c) $3e^{3x+5}$.
2. (a) $6x^2e^{2x^3}$, (b) $\dfrac{1}{2\sqrt{x}}e^{\sqrt{x}}$, (c) $\dfrac{1}{x^2}e^{-1/x}$.
3. (a) $\cos x e^{\sin x}$, (b) $-2\sin 2x e^{\cos 2x}$, (c) $4\sec^2 x e^{4\tan x}$.
4. (a) $(2x^2 + 1)e^{x^2}$, (b) $(5x + 2)xe^{5x}$, (c) $(\cos x - x\sin x)e^{x\cos x}$.
5. (a) $e^{2x}(2\cos 3x - 3\sin 3x)$, (b) $e^{-x^2}(\cos x - 2x\sin x)$, (c) $\frac{1}{2}(x + 1)^2(x + 7)e^{x/2}$.
6. (a) $-2e^{4x}(1 - e^{4x})^{-\frac{1}{2}}$, (b) $e^{(x+e^x)}$, (c) $e^x\cos(e^x)$.
7. (a) $e^{2x} - e^{-2x}$, (b) $(2 + e^x)/(1 + e^{-x})^2$, (c) 0.

8. (a) $\frac{1}{3}e^{3x} + c$, (b) $-e^{-x} + c$, (c) $e^{x+4} + c$.
9. (a) $3(e^2 - 1)$, (b) $2(e - 1)$, (c) $e^6 - 1$.
10. (a) $\frac{1}{x}$, (b) $-\frac{1}{x}$, (c) $\frac{2}{x} - 1$. 12. $y = e(3x - 5)$. 13. $e - 1$.
15. $(1, e^{-1})$ max. 16. $(-2, -e^{-2})$ min.
17. $(-1, -e^{-\frac{1}{2}})$ min, $(1, e^{-\frac{1}{2}})$ max. 18. $(0, 1)$ max, $(1, 0)$ min.
19. $(1, e)$ min. 20. $(2k\pi, 1)$ min, $(\{2k + 1\}\pi, e^2)$ max, $k \in \mathbb{Z}$.
21. 4 m, -2 m s^{-2}. 22. $x = -1$, max; $x = 2$, min.
23. $x = \pi/4$, max; $x = 5\pi/4$, min.
24. $\frac{1}{2}(e^2 - 3 + 2e^{-1}) \approx 2{\cdot}56$; $\frac{1}{4}\pi(e^4 - 3 + 2e^{-2}) \approx 40{\cdot}7$.

Exercise 3.2

1. (a) $\frac{1}{x}$, (b) $\frac{1}{x + 3}$, (c) $\frac{2}{2x - 1}$.
2. (a) $\frac{3x^2}{x^3 + 4}$, (b) $2\cot 2x$, (c) $\tan x$.
3. (a) $x(2\ln x + 1)$, (b) $(1 - 2\ln x)/x^3$, (c) $1/x\ln x$.
4. (a) $\frac{3}{x}$, (b) $\frac{2}{4x + 5}$, (c) $-1 - \ln x$.
5. (a) $\sec x$, (b) $\frac{4x}{x^2 + 4}$, (c) $(\log_{10} e)/x$.
6. (a) $\frac{3x - 4}{x(3x - 2)}$, (b) $\frac{1}{2x(x + 1)}$, (c) $\frac{x}{1 - x^2} - \tan x$.
7. (a) $2x$, (b) $\sec^2 x$, (c) $\frac{2}{x} - 1$. 8. $y = 3x - 6$. 9. $t/(1 + t)$.
11. (a) $2p + q + 1$, (b) p/q, (c) $1/(p + q)$, (d) e^{p+q}
12. (a) $t = 0$ or $\ln 2$, (b) $x = -\ln 2$.
13. (a) none, (b) none, (c) $(1, 0)$ min. 14. $(1, 1)$ min.
17. (a) $3^x \ln 3$, (b) $5^{2x} 2\ln 5$, (c) $\ln 10 \cos x(10^{\sin x})$.
18. (a) $x^x(1 + \ln x)$, (b) $(2\ln x)x^{(\ln x - 1)}$, (c) $(\ln x)^x\left\{\ln(\ln x) + \frac{1}{\ln x}\right\}$.
19. (a) $\frac{2(x - 2)}{(2x + 1)^3}$, (b) $\frac{(6 - 5x)x^2}{2(1 - x)^{3/2}}$, (c) $\frac{4x^2(x - 1)e^{4x}}{(x + 1)^3}$.
20. (a) $\frac{1}{2}(13 - 10x)\sqrt{\left\{\frac{x + 2}{(1 - 5x)^3}\right\}}$, (b) $\frac{4x}{(x - 2)^{2/3}(x + 4)^{4/3}}$.
(c) $\frac{4\sin^2 x + 1}{\cos^3 x\sqrt{(\sin 2x)}}$.

Exercise 3.4

1. (a) $\frac{1}{2}\ln(2x+7)+c$, (b) $\frac{1}{5}\ln(5x-2)+c$.
2. (a) $-\ln(1-x)+c$, (b) $\frac{1}{2}\ln(1-2x)+c$.
3. (a) $\ln(2-3x)+c$, (b) $-2\ln(3-4x)+c$.
4. (a) $\ln|x+3|+c$, (b) $-3\ln|3-x|+c$, (c) $\frac{1}{2}\ln|4x+5|+c$.
5. (a) $x+\ln|x-1|+c$, (b) $x+2\ln|x-4|+c$, (c) $\frac{1}{2}x+\frac{1}{4}\ln|2x+1|+c$.
6. (a) $\frac{1}{2}x^2-\ln|x|+c$, (b) $x-\ln(1+x^2)+c$, (c) $-\frac{1}{2}x^2-\ln|1-x|+c$.
7. (a) $-\frac{1}{2}\ln|1-x^2|+c$, (b) $\frac{1}{3}\ln|x^3+8|+c$, (c) $\frac{1}{4}\ln|2x^2+4x-1|+c$.
8. (a) $\ln(1+\sin x)+c$, (b) $\ln|\sin x|+c$, (c) $x+\frac{1}{2}\ln|\sec 2x|+c$.
9. (a) $\ln(1+e^x)+c$, (b) $\ln|\ln x|+c$, (c) $2\ln(1+\sqrt{x})+c$.
10. (a) $5\ln 2$, (b) $\ln 4$, (c) $\ln 3$.
11. $\frac{1}{3}\ln\frac{5}{2}$, (b) $-\ln 4$, (c) $2+\ln\frac{2}{3}$.
12. (a) $\ln 2$, (b) $\ln 5$, (c) $\ln\frac{4}{3}$.
13. (a) 0·805, (b) 1·10.
14. $\dfrac{5(1-x)}{(2x-1)(3-x)}$; $-0{\cdot}167$.
15. $\frac{3}{4}-\ln 2$.
16. $\pi(8+\ln 3)$.

Exercise 3.5

1. (a) $2\cosh 2x$, (b) $3\sinh 3x$, (c) $2\sinh x\cosh x$, (d) $\dfrac{\sinh x}{2\sqrt{(\cosh x)}}$.
2. (a) $\cosh x+c$, (b) $\frac{1}{2}\sinh 2x+c$, (c) $\ln(\cosh x)+c$.
3. (a) $\operatorname{sech}^2 x$, (b) $-\operatorname{cosech}^2 x$, (c) $-\operatorname{sech} x\tanh x$, (d) $-\operatorname{cosech} x\coth x$.
5. (a) 0 or $\ln 5$, (b) $-\ln 2$.

Exercise 3.6

1. (a) $e^{3x}(3x+1)$, (b) $e^{2x}\left(\dfrac{1}{x}+2\ln 4x\right)$, (c) $a^x\ln a$.
2. (a) $\dfrac{1}{x}\cos(\ln x)$, (b) $\dfrac{1}{x}-x-2x\ln x$, (c) $\operatorname{cosec} x$.
3. (a) $\dfrac{8}{(x-3)(3x-1)}$, (b) $2\sec x$, (c) $\dfrac{3-4x^2}{1-x^2}$.
4. (a) $(\ln 2, 4)$ min, (b) $(0,0)$ inflexion, $(3, 27e^{-3})$ max.
5. $x=e$, max.
6. 24/7
9. $(-1, -2\sqrt{e})$ min, $(4, 18e^{-2})$ max.
10. (a) $\ln 3$, (b) $\frac{1}{2}\ln 6$, (c) $\frac{3}{2}\ln 3$.
11. (a) $\frac{1}{3}x-\frac{1}{9}\ln|3x+1|+c$, (b) $\ln|x|+\dfrac{2}{x^2}+c$,

    (c) $2x-\ln(x^2+x+1)+c$.
12. $\frac{1}{2}\ln(1+x)-\frac{1}{2}\ln(1-x)-x$; 0.
14. $2xe^{2x}$; 59·5.

16. $\frac{5}{6} + 2\ln 2$. 17. $2 - \frac{2}{e}$; $2\pi\left(1 - \frac{1}{e^2}\right)$.

18. $(2, 3), (3, 0)$; $y + 3x = \pm 6\sqrt{2}$.
19. $(-1, -\frac{1}{2})$ min, $(1, \frac{1}{2})$ max; $\frac{1}{2}\ln(1 + a^2)$.
20. Inflexion at $(-\{\ln b\}/c, a/2b)$.
21. Asymptote $x = 6$; $(4, 1)$ max, $(8, 9)$ min; $\frac{15}{2} - 8\ln 2$.
22. $\frac{1}{2}\pi(e^{2h} + 8e^h + 8h - 9)$; $50/\{\pi(e^{1\cdot5} + 2)^2\} \approx 0{\cdot}379$.

Exercise 4.1

1. $\frac{3}{2x^2} - \frac{1}{x} + c$. 2. $\frac{1}{7}x^7 - \frac{1}{2}x^4 + x + c$.
3. $\frac{2}{5}x^{5/2} + \frac{2}{3}x^{3/2} + c$. 4. $\frac{1}{3}\sin 3x + c$. 5. $\frac{1}{4}e^{(4x-3)} + c$.
6. $\frac{1}{2}\cos(1 - 2x) + c$. 7. $-\frac{1}{5}(1 - x)^5 + c$. 8. $-\frac{1}{x + 4} + c$.
9. $-\frac{1}{4(2x - 5)^2} + c$. 10. $\frac{1}{2}\ln|2x - 5| + c$. 11. $\frac{1}{5}\tan 5x + c$.
12. $2\ln|\sin\frac{1}{2}x| + c$. 13. $\frac{1}{2}x - \frac{1}{4}\sin 2x + c$.
14. $\frac{1}{4}\cos 2x - \frac{1}{8}\cos 4x + c$. 15. $-x - \cot x + c$.
16. $-\frac{1}{3}\cos^3 x + c$. 17. $\sec x + c$. 18. $\frac{1}{3}\cos^3 x - \cos x + c$.
19. $x - \ln|x + 2| + c$. 20. $x + \frac{1}{2}x^2 + \ln|x - 1| + c$.
21. $\frac{1}{2}x^2 - 3\ln(x^2 + 5) + c$. 22. $\frac{2}{5}\sqrt{(5x + 3)} + c$.
23. $\frac{1}{4}\ln(1 + x^4) + c$. 24. $\frac{1}{16}(1 + x^4)^4 + c$. 25. $e - 1$.
26. $40\frac{1}{2}$. 27. $3/16$. 28. $-2/21$. 29. $(\pi + 2)/24$. 30. $2/15$.
31. $3/2$. 32. $1/4$. 33. $2 - \frac{1}{2}\pi$. 34. $\ln 2$. 35. 2.
36. $\frac{1}{2}\ln(4/3)$. 38. $\sec x$; $\frac{1}{2}\ln 3$. 39. $\tan\frac{1}{2}x + c$. 40. $3\pi/8$.

Exercise 4.2

1. $\ln\left|\frac{x - 1}{x + 2}\right| + c$. 2. $\frac{1}{2}\ln\left|\frac{1 + x}{1 - x}\right| + c$. 3. $\frac{1}{3}\ln\left|\frac{x - 3}{x}\right| + c$.
4. $\frac{1}{2}\ln|x^2 - 4| + c$. 5. $\ln|(x - 3)^3(x + 1)| + c$.
6. $\ln|(x - 2)(x - 3)| + c$. 7. $\ln|(x + 3)^2(x + 1)| + c$.
8. $\frac{1}{2}\ln\left|\frac{x^2}{2x - 1}\right| + c$. 9. $\ln\left|\frac{(x - 1)^3}{x - 4}\right| + c$. 10. $\frac{1}{2}\ln\left|\frac{1 + x}{1 - x}\right| - x + c$.
11. $\frac{1}{2}\ln|x^2 - 4x - 8| + c$. 12. $\frac{1}{2}x^2 + 2\ln|x^2 - 4| + c$.

13. $\ln|(x+1)(x-2)| + \dfrac{1}{x+1} + c$. 14. $3\ln\left|\dfrac{x-1}{x}\right| + \dfrac{2}{x} + c$.

15. $\ln|x+3| + \dfrac{3}{x+3} + c$. 16. $\ln\left|\dfrac{x^2+1}{x+1}\right| + c$.

17. $\frac{1}{2}\ln\dfrac{(x-1)^2}{x^2+1} + c$. 18. $\frac{1}{4}\ln\dfrac{x^2+3}{x^2+5} + c$.

19. $\frac{1}{2}\ln(2x+1)^2(x^2+2) + c$. 20. $x + \ln\dfrac{x^2+x+1}{(x-1)^2} + c$.

21. $\ln\frac{32}{27}$. 22. $\ln\frac{12}{7}$. 23. $\ln 6$. 24. $\frac{3}{2}\ln\frac{9}{8}$. 25. $\frac{1}{3}\ln\frac{27}{2}$.
26. $\ln\frac{4}{3}$. 27. $\ln\frac{9}{10}$. 28. $2 + \ln\frac{5}{9}$. 29. $\frac{1}{2} + \ln\frac{3}{4}$.
30. $\frac{3}{2} + \ln\frac{8}{3}$. 31. $\frac{13}{2} + \ln\frac{512}{3}$. 32. $11 + 2\ln\frac{6}{5}$.
33. $\frac{1}{2}\ln\frac{40}{17}$. 34. $1 + \ln\frac{5}{2}$. 35. $\ln 2$. 36. $\ln\frac{12}{7}$.

37. (a) $\ln\left(\dfrac{x-1}{x+3}\right) + c$, (b) $\ln\left(\dfrac{1-x}{x+3}\right) + c$, (c) $\ln\left(\dfrac{x-1}{x+3}\right) + c$.

38. (a) $\frac{1}{2}\ln\dfrac{(x+1)^2}{2x-3} + c$, (b) $\frac{1}{2}\ln\dfrac{(x+1)^2}{3-2x} + c$.

Exercise 4.3

1. $\frac{1}{12}(3x-2)^4 + c$. 2. $-\frac{1}{10}(1-2x)^5 + c$.
3. $-\frac{1}{42}(1-x)^6(1+6x) + c$. 4. $\frac{1}{6}(x^3-1)^4 + c$. 5. $\frac{1}{3}\tan^3 x + c$.
6. $-\frac{1}{6}\cos^6 x + c$. 7. $\frac{1}{3}(x^2+4)^{3/2} + c$.
8. $-\frac{2}{15}(1-x)^{3/2}(2+3x) + c$. 9. $\frac{1}{3}(x-1)\sqrt{(2x+1)} + c$.
10. $\ln(1+e^x) + c$. 11. $\sin\theta - \frac{1}{3}\sin^3\theta + c$.
12. $\frac{1}{5}\cos^5\theta - \frac{1}{3}\cos^3\theta + c$. 13. $\frac{1}{2}\sec\theta + c$. 14. $2\tan(\sqrt{x}) + c$.

15. 12/5. 16. 16/105. 17. $\ln 2$. 18. $\dfrac{e-1}{2(e+1)}$. 19. $\pi/2$.

20. $\pi/18$. 21. 4/5. 22. $\frac{1}{24}(2\pi + 3\sqrt{3})$. 23. $\frac{1}{3}\ln 2$.
24. $\frac{1}{4}\ln 3$. 25. $\frac{1}{120}(2x+3)^5(10x-3) + c$.
26. $\frac{1}{40}(x^2-1)^4(4x^2+1) + c$. 27. $\frac{2}{9}(1+3x)^{3/2} + c$.

28. $\frac{2}{3}(x-6)\sqrt{(x+3)} + c$. 29. $\dfrac{1}{3(1-x^3)} + c$. 30. $-\dfrac{x}{(x-1)^2} + c$.

31. $\frac{1}{2}e^{x^2} + c$. 32. $2\sqrt{(\ln x)} + c$. 33. $\frac{1}{4}\ln\left|\dfrac{2+e^x}{2-e^x}\right| + c$.

34. 6. 35. 9. 36. $9\pi/4$. 37. $\sqrt{3}$. 38. $\frac{1}{6}(4\pi + 3\sqrt{3})$.
39. $\ln(4/3)$. 40. $23\frac{1}{5}$. 41. $8; 5\pi$. 42. 3π. 44. $\frac{1}{2}\pi$.

Exercise 4.4

1. $\sin x - x\cos x + c$. 2. $2x\sin\frac{1}{2}x + 4\cos\frac{1}{2}x + c$.
3. $-e^{-x}(x+1) + c$. 4. $\frac{1}{4}x^2(2\ln 2x - 1) + c$.
5. $\frac{1}{4}e^{2x}(2x-1) + c$. 6. $x\tan x + \ln(\cos x) + c$.
7. $\frac{1}{8}\sin 2x - \frac{1}{4}x\cos 2x + c$. 8. $x\tan x + \ln(\cos x) - \frac{1}{2}x^2 + c$.
9. $\frac{1}{4}x^2 + \frac{1}{4}x\sin 2x + \frac{1}{8}\cos 2x + c$. 10. $(x^2-2)\sin x + 2x\cos x + c$.
11. $e^x(x^3 - 3x^2 + 6x - 6) + c$. 12. $\frac{1}{16}x^4(4\ln x - 1) + c$.
13. $\frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2\ln x + \frac{1}{4}x^2 + c$. 14. $(x-1)\ln(x-1) - x + c$.
15. $-\frac{1}{2}e^{-x}(\sin x + \cos x) + c$. 16. -2. 17. $2/e$. 18. $\pi/9$.
19. 1. 20. $\frac{1}{4}\pi^2$. 21. $\frac{8}{3}\ln 2 - \frac{7}{9}$. 22. $10e^{-2} - 17e^{-3}$.
23. $\frac{1}{4} - \frac{3}{4}e^{-2}$. 24. $\pi - 2$. 25. $\frac{1}{2}(e^{\pi} + 1)$. 26. $\frac{5}{4}e^4 - \frac{1}{4}e^2$.
27. $\frac{1}{2}e^2$. 28. $\frac{1}{30}(x+1)^5(5x-1) + c$.

29. (a) $\frac{2}{15}(x-1)^{3/2}(3x+2) + c$, (b) $-\dfrac{(x+1)}{(x+2)^2} + c$.

Exercise 4.5

1. $\sin^{-1}\dfrac{x}{4} + c$. 2. $\frac{1}{5}\tan^{-1}\dfrac{x}{5} + c$. 3. $\frac{1}{2}\sin^{-1}2x + c$.

4. $\frac{1}{2}\ln(x^2+1) + \tan^{-1}x + c$. 5. $\frac{1}{5}\sin^{-1}\dfrac{5x}{2} + c$.

6. $\frac{1}{6}\tan^{-1}\dfrac{3x}{2} + c$. 7 $\ln|x+1| - \tan^{-1}x + c$.

8. $\ln\left(\dfrac{x^2+1}{(1-x)^2}\right) + 2\tan^{-1}x + c$. 9. $\ln\left(\dfrac{x^2+9}{x^2}\right) + 2\tan^{-1}\dfrac{x}{3} + c$.

10. $3\ln(x^2+4) - \ln|x+2| - 4\tan^{-1}\dfrac{x}{2} + c$. 11. $\frac{1}{2}\tan^{-1}\dfrac{(x+1)}{2} + c$.

12. $\sin^{-1}\dfrac{(x-2)}{3} + c$. 13. $\pi/18$. 14. $\pi/2$. 15. $\pi/8$.

16. $\pi/2$. 17. $\ln 2 + \pi/2$. 18. $\pi/4$. 19. $\pi/4$.
20. $\frac{1}{2}\ln(1+e^{2x}) + \tan^{-1}(e^x) + c$. 21. $\tan^{-1}(e^x) + c$.
22. $\frac{1}{2}\sin^{-1}x + \frac{1}{2}x\sqrt{(1-x^2)} + c$. 23. $\sin^{-1}x + \sqrt{(1-x^2)} + c$.

24. $\ln(x^2 - 2x + 10) + \frac{2}{3}\tan^{-1}\dfrac{(x-1)}{3} + c$. 25. $\frac{1}{2}\tan^{-1}(x^2+1) + c$.

26. $-\sin^{-1}\left(\dfrac{1}{x}\right) + c$. 27. (a) $-0{\cdot}123$, (b) $0{\cdot}545$.

Exercise 4.6

1. (a) 2/5, (b) 0, (c) π, (d) $\sqrt{3}$, (e) 0, (f) 0. 2. 0.
3. (a) -2, (b) 4, (c) 4. 4. (a) $-1\frac{1}{2}$, (b) $1\frac{1}{2}$, (c) $2\frac{1}{2}$.
5. (a) 0, (b) 8/3. 7. $e-1$. 8. $\frac{2}{3}(2-\sqrt{2})$.
9. $I=\frac{1}{2}\pi$, $J=2(e-1)$. 10. (a) 2, (b) 1, (c) π.

Exercise 4.7

1. $2-\frac{1}{2}\ln 3$. 2. $\ln 2-1$. 3. $\frac{1}{4}\ln\frac{9}{5}$. 4. 2.

5. $1+\ln\frac{8}{3}$. 6. $\dfrac{64\sqrt{2}}{21}$. 7. 8/3. 8. π. 9. 7/6.

10. $\frac{1}{2}(1-\ln 2)$. 11. 1/24. 12. $\ln 2$. 13. $\ln(4/3)$. 14. $\pi/16$.

15. $1/(\ln 2)$. 16. $\frac{1}{5}\ln\left(\dfrac{x-3}{x+2}\right)-\dfrac{1}{x-3}+c$. 18. $\pi/12$.

19. $(\pi-2)/4$. 20. $(\pi+2)/8$. 21. $\frac{1}{2}\ln 3$.
23. $(\frac{1}{4},\frac{8}{3})$ min; $\frac{3}{2}-\ln 4$. 24. $(-4,\frac{4}{3})$ max; $2\ln 3$.

25. $\dfrac{2}{x-1}-\dfrac{x}{x^2+1}$; 1·04, $-0{\cdot}23$.

27. (i) a^2+b^2, (ii) $2\log_{10}2-\log_{10}e$, (iii) $(2\pi+3\sqrt{3})/48$.
28. $y=x-a$, $x+y=5a$; $8a^2/3$.
29. $-3\sin t\cos t$; $y+3x\sin t_1\cos t_1=3\sin t_1-2\sin^3 t_1$.
31. $\frac{1}{2}x^2\sin 2x+\frac{1}{2}x\cos 2x-\frac{1}{4}\sin 2x+c$; π; $\pi^2(2\pi^2-3)/12$; $\pi^4/6$.

32. $\dfrac{1}{x+1}+\dfrac{x+2}{x^2+3}$; $\ln(\sqrt{2}+\sqrt{6})+\dfrac{\pi}{2\sqrt{3}}$; $\dfrac{1}{\sqrt{3}}\ln(\sqrt{2}+\sqrt{6})+\dfrac{\pi}{6}$.

33. $p=2$, $q=1$; $\dfrac{1}{4}\pi$; $\ln 2$; $\pi+\ln 2$.

34. (a) $\dfrac{1-t^2}{1+t^2}$ (b) $\dfrac{1}{1-t}+\dfrac{1}{1+t}+\dfrac{8}{1+9t^2}$.

35. (a) $x(\log_e x-1)+c$; $x\{(\log_e x)^2-2\log_e x+2\}+c$, (b) $\pi(e-2)$.

36. (i) $\frac{1}{2}\pi-1$, (ii) $\frac{1}{2}(1-e^{-2})$. 37. π.

38. $\dfrac{a^2}{12}(4\pi-3\sqrt{3})$; $\dfrac{5\pi a^3}{24}$.

39. (i) $\pi(2\ln 2-1)$, (ii) $\frac{1}{2}\pi-1$.

40. $\frac{22}{15}a^2$, $\frac{7}{12}\pi a^3$. 41. $\frac{16}{3}$, $\dfrac{64\pi}{15}$.

42. $x=\dfrac{\pi}{4},\dfrac{9\pi}{4}$: max, $x=\dfrac{5\pi}{4},\dfrac{13\pi}{4}$: min; $\frac{1}{2}(1+e^{-\pi})$; $\frac{1}{2}e^{-2\pi}(1+e^{-\pi})$.

Exercise 5.1

1. -42 N m, 14 N m, -21 N m, 0 N m.
2. 42 N m, -30 N m, -42 N m, 30 N m, 0 N m.
3. $(4+\sqrt{3})$ N m, $3\sqrt{3}$ N m, $(2+5\sqrt{3})$ N m.
4. (a) 6 N, 2 m, (b) 8 N, 3 m.
5. (a) 1 N, 10 m, (b) 1 N, 15 m.
6. 50 N m. 7. $F = G = 4$.
8. (a) $5\sqrt{2}$ N, 4 m, (b) 15 N, 3 m.
9. (a) $4\sqrt{2}$ N, 5 m, (b) 6·5 N, 12 m. 10. $6\sqrt{2}$ N along BD.
11. $20\sqrt{3}$ N m, $10\sqrt{3}$ N m, 0 N m, 0 N m, $10\sqrt{3}$ N m, $20\sqrt{3}$ N m; 10 N along CD.

Exercise 5.2

1. 4 N at 90° to AB, 4 m. 2. 2 N at 90° to AB, 5 m.
3. $6\sqrt{3}$ N at 60° to AB, 1 m. 4. 6 N at 60° to AB, 2 m.
5. (a) $5P$ at 53° 8′ to AB, (b) $6a$. 6. $3x - 7y + 7 = 0$.
7. $2x - 5y + 10 = 0$. 8. a, b. 9. c. 10. a, c. 11. b, c.
12. c. 13. c. 14. a, c. 15. b, c. 16. c. 17. a.
18. b. 19. a, c. 20. 6 N m clockwise; $y = x + 1$.
21. (a) 5, 13, (b) 2·5, 19·5. 22. $7\sqrt{5}$ N, 1 m; 28 N m in sense CBA.
23. (a) 10; 28 N along AD, (b) 18 N, $14\sqrt{3}$ N m in sense ABC.
24. (a) $25P$, $24y = 7x$, (b) (i) $S = -18P$, $T = 15P$, (ii) $72Pa$, (iii) clockwise.

Exercise 5.3

1. 18 cm from A. 2. 1·05 m from P. 3. 5 cm. 4. 3 cm, 4 cm.
5. (2, 1). 6. $(3, -\frac{1}{2})$. 7. $3\mathbf{i} + \mathbf{j}$. 8. $2\mathbf{i} + 2{\cdot}5\mathbf{j}$. 9. (3, 0).
10. (0·4, 1·6). 11. 10 cm, 3 cm. 12. 10 cm. 13. $1\frac{2}{3}$ cm from AB.
14. $6\frac{1}{3}$ cm from AD, $4\frac{4}{5}$ cm from AB. 15. 11 cm from both AB and AF.
16. $3\frac{1}{2}$ cm from AE, 4 cm from AB. 17. 4 cm from AD, 3·5 cm from AB.
18. $7a/5$ above base, $16a/11$ above base.

Exercise 5.4

3. $\dfrac{2a(14+3\pi)}{3(8+\pi)}$. 4. $\dfrac{2a(26-3\pi)}{3(16-\pi)}$. 5. $\dfrac{28r}{9\pi}$.

6. $\dfrac{27\sqrt{3}a}{3\sqrt{3}+4\pi}$. 7. $\dfrac{11h}{28}$.

8. (a) $\dfrac{7\sqrt{3}r}{8\pi+3\sqrt{3}}$, (b) $\dfrac{7\sqrt{3}r}{4\pi-3\sqrt{3}}$. 9. $28\pi a^3/3$, $12a/7$. 11. $3r$.

Exercise 5.5

1. 6, 103/15; (9/7, 103/70). 2. 153/4, 414/5; (17/8, 23/5).
3. $\ln 2$, 7/48; ($2\ln 2$, 7/24). 4. 1/4, 2/5; (3/8, 3/5). 5. (20/7, 0).
6. (3/5, 0). 7. $(0, \pi/8)$. 8. (0, 7/3). 9. $\left(\frac{1}{e-1}, \frac{e+1}{4}\right)$.
10. $(1, \pi/8)$. 11. (2, 0). 12. (2/3, 0). 13. $(2\ln 2, 0)$.
14. $\left(\frac{e^2-3}{2(e^2-1)}, 0\right)$. 15. (9/7, 0). 16. (0, 14/5).
17. $2\ln 2 - 1$, $2\ln 2 - \frac{3}{4}$, $2(\ln 2)^2 - 4\ln 2 + 2$; (1·65, 0·24).
18. 1/6; (1/2, 2/5).

Exercise 5.6

1. (a) 13*P*, 12/5, (b) 3*a*; 12*Pa*, sense *ABC*. 2. $\lambda = 5/2$; 5*a*/2; 5*a*/7.
3. (c) (i) $3\sqrt{2}$ along *AC*, $\sqrt{2}$ along *BD*; 7*a*;
   (ii) 7 along *AD*, $4\sqrt{2}$ along *CA*, $6\sqrt{2}$ along *BD*.
4. (a) $4\sqrt{3}P$, $\frac{\sqrt{3}}{3}P$; $\frac{7\sqrt{3}}{3}P$; (b) $\frac{5\sqrt{3}}{3}P$, $\frac{\sqrt{3}}{3}P$; $\frac{7\sqrt{3}}{3}Pa$ in sense *ABC*.
5. $p = 11$, $q = 14$; (a) $2\sqrt{19}$ N, (b) $12\sqrt{3}$ N m.
6. (i) $m = 2$, $n = 6$, (iii) $x = 3a\sqrt{3}$; $\sqrt{31}$, $5Pa\sqrt{3}$.
7. (7*a*/18, 4*a*/9).
8. 4·8 cm from *OA*, 8·8 cm from *OC*.
9. $\frac{11}{24}\pi a^3$, $\frac{5}{24}\pi a^3$; $\frac{21}{88}a$, $\frac{27}{40}a$ from centre of hemisphere.
11. $\frac{3a}{8}$. 12. $\frac{32}{15}\sqrt{2}$; $\frac{8\pi}{5}$. 13. 16/3; 8π; (0, 20/3).
14. (0·864, 0·333). 15. 4/3, $104\pi/15$; (15/13, 0). 16. $3\pi a^2$, $(\pi a, 5a/6)$.

Exercise 6.1

1. 360 N, 2·5 m. 2. 41 N at *A*, 33 N at *B*. 3. 1·6 m.
4. 0·6 m, 20 N. 5. (a) 1200 N (b) 3 m. 6. 0·4 m.
7. 5*W*/8 at *A*, 3*W*/8 at *B*. 8. 5. 9. 5*a*/3, 5*a*/2; $\tan^{-1} 10/21 \approx 25° 28'$.
10. (a, a); 1/2. 11. $2/(\pi + 2)$. 12. $1/\pi$.
13. $\tan^{-1} 3/4 \approx 36° 52'$; 3*aW*. 14. $\tan^{-1} 11/13 \approx 40° 14'$; 26*aW*/9.
15. $\sqrt{2}$. 16. 2*W*/3.
17. $h + \left(\frac{3a^2 - h^2}{8a + 4h}\right)$ from vertex of cone; (a) 17°, (b) $h = a\sqrt{3}$.
18. 30°; $1/\sqrt{3}$. 19. 36° 52′. 20. (a) 53° 8′, (b) 26° 34′.

Exercise 6.2

1. (a) 10, (b) $10\sqrt{5}$ N at 26° 34′ to vertical.
2. $\frac{\sqrt{3}}{2}mg$ in AC, $\frac{1}{2}mg$ in BC; $\frac{1}{2}a$. 3. 2 N at A, $2\sqrt{17}$ N at C.
4. 30 N at 60° to vertical. 5. 45 cm, 16 N.
6. (a) $20\sqrt{3}$ N, 90°, (b) $15\sqrt{5}$ N, 63° 26′. 7. 5/6; 5W/12, 13W/12.
8. (a) $W\sqrt{3}$ at both A and C, (b) $2a$.
9. (a) 15 N, (b) 9 N at 53° 8′ to vertical. 10. $5W$, 75 cm. 11. $\frac{1}{2}$; 30 cm.
12. 5/24. 13. (a) $\frac{1}{2}mg\cot\alpha$, (b) $\frac{1}{2}\cot\alpha$.
14. 45°, $\frac{1}{2}W$; half way up, all the way. 15. (a) 30°, (b) $1/\sqrt{3}$.
17. 1/3.

Exercise 6.3

1. (a) 24 N, (b) $12\sqrt{3}$ N, 13 N.
2. (a) 60 N, (b) 48·2 N at 41° 38′ to horizontal.
3. (a) 15, (b) 47·4 N at 71° 34′ to horizontal.
4. $4\sqrt{3}$ N perpendicular to rod; $2\sqrt{3}$ N, 10 N.
5. (a) 14 N, 30 N (b) 1·92 m. 6. (a) 4/7, (b) $\tan^{-1}\frac{2}{3} \approx 33° 41'$.
7. 5 N, 1/2. 8. 121 N, 17·5 N; 7/48. 9. $\frac{1}{2}\tan\alpha$, $2W\tan\alpha$.
10. $3W/10$, $W/10$; 0·8. 11. 1/5. 12. $8W/15$; $24W/75$, $43W/75$.
13. $(5\sqrt{3})/18$; $(W\sqrt{3})/2$. 14. $5a/3$. 15. $\tan^{-1}(4/3)$; 40, 900 N.
16. 10, 6/17. 17. $W/4$, 1/3. 18. $11W/4$. 19. 5 m.

Exercise 6.4

1. (a) $5\sqrt{10}$ N, $5\sqrt{5}$ N (b) 21°48′, 45°. 2. $\frac{1}{2}W\sqrt{5}$, 63° 26′.
3. $\frac{1}{3}W\sqrt{3}$; $\frac{1}{2}W$.
4. (ii) $W\sqrt{(1+\mu^2)}$ at $\tan^{-1}\mu$ to vertical, (iii) $\mu W = \frac{1}{2}W\cot\theta$.
5. (a) 7/10, (b) $5W/2$, $3W/4$. 6. C; 67° 23′. 7. C will slip.
8. (a) $9W/5$, $12W/5$, (b) 3° 11′. 9. $\sqrt{3}/2$. 10. $2-\sqrt{3}$.
11. $\frac{3}{4}W$, $\frac{1}{4}W\tan\alpha$; $1/\sqrt{3}$.
13. (i) $\frac{9w}{25}$, (ii) $W+\frac{27w}{125}$, (iii) $\frac{36w}{125}$.

Exercise 6.5

1. AB: $\sqrt{3}$ kN tension; BC: $4\sqrt{3}$ kN tension; CD: $2\sqrt{3}$ kN thrust; AD: $2\sqrt{3}$ kN thrust; BD: $2\sqrt{3}$ kN tension.
2. AB and BC: 5 kN thrust; BD: no stress; CD: 4 kN tension; AD: 3 kN tension.
3. AB: 2 kN thrust; BC: $2\sqrt{2}$ kN tension; CD: 4 kN tension; AC: $2\sqrt{2}$ kN thrust.

4. AB: 8 kN thrust; BC: $4\sqrt{3}$ kN thrust; CD: $2\sqrt{3}$ kN tension; BD: 4 kN tension.
5. AB: $10\sqrt{2}$ kN thrust; BC: $5\sqrt{2}$ kN thrust; CD: 5 kN tension; DE: $5\sqrt{2}$ kN tension; AE: 10 kN tension; BD: 5 kN thrust; BE: 5 kN tension.
6. AB: 4 kN tension; BC: 3 kN thrust; CD: 3 kN thrust; AD: 4 kN tension; AC: 15 kN tension.
7. AB: $4\sqrt{3}$ kN tension; BC: $2\sqrt{3}$ kN tension; CD: $4\sqrt{3}$ kN thrust; DE: $3\sqrt{3}$ kN thrust; AE: $8\sqrt{3}$ kN thrust; BD: $2\sqrt{3}$ kN tension; BE: $2\sqrt{3}$ kN thrust.
8. AB and BC: 6 kN tension; CD and AE: $6\sqrt{3}$ kN thrust; BD and BE: $2\sqrt{3}$ kN tension; DE: $4\sqrt{3}$ kN thrust.
9. $P = \frac{1}{3}W\sqrt{2}$; AB: $\frac{2}{3}W$ thrust; BC: $\frac{1}{3}W\sqrt{2}$ tension; CD: $\frac{1}{3}W$ thrust; AC: $\frac{2}{3}W\sqrt{2}$ tension; AD: $\frac{2}{3}W$ thrust.
10. (ii) AB: 390 N; BC: 650 N; CD: 150 N; AD: 250 N; AC: $120\sqrt{5}$ N
    (iii) AC and CD.

Exercise 6.6

2. 4/11; 7:2. 4. $5a/2$, $5a/2$; $\tan^{-1}(1/6) \approx 9° 28'$; $W/6$.
7. (a) $12\frac{1}{2}$ N; $7\frac{1}{2}$ N in the direction CA, (b) $6\frac{1}{4}$ N; $6\frac{1}{4}$ N at 53° 8′ to CA produced; 8 N acting 0·625 m from A.
8. $\frac{1}{2}W\sqrt{5}$; $7l/12$.
9. (a) 15 N, (b) 8/3 m, (c) 42·7 N at 73° 41′ to the horizontal.
10. (ii) $\frac{1}{4}\sqrt{3}$, (iii) $24c/7$. 11. 3/46. 12. 3/8, $5l/3$.
14. $\frac{3}{2}mg\cot\alpha$; $mg\cot\alpha$. 15. $\cos^{-1}(2m/M)$.
16. $W(1 - \sin 2\theta)$; 18° 26′.
17. $AX = \frac{2}{3}a$, $13W$ at 67° 23′ to XA; $CY = 11a/12$.
18. (i) $25W/37$, $20W/37$, (ii) 13/40.
20. $(8a/15, 3a/16)$; $128/45 \approx 2{\cdot}84$.
21. $h + \left(\dfrac{6l^2 - h^2}{4h + 12l}\right)$ from vertex of cone. 22. $\sin^{-1}\left(\dfrac{k+3}{3k-24}\right)$.
23. (i) $35\sqrt{2}$ N, (ii) $5\sqrt{2}$ N.
24. Force at the ground. 25. $\frac{1}{4}Wl\sqrt{3}$.
26. $\frac{1}{2}W/5$; $\frac{1}{2}W$, 0. 27. Slipping at A.
28. (i) $\frac{1}{2}W$, (ii) $\frac{1}{2}W$, W, (iv) $\frac{1}{2}$.
29. $2W\sqrt{3}$ along BA; $\frac{1}{3}W\sqrt{3}$, $3W/2$ at A. $\frac{1}{3}W\sqrt{3}$, $\frac{1}{2}W$ at C.
30. $\frac{2}{3}W\sqrt{3}$ between cylinders, $\frac{1}{3}W\sqrt{3}$ at both walls. 31. $1\sqrt{3}$.
32. (i) $35W/4$, (ii) 0, $5W$ at A; $7W$, $5W/4$ at B, (iii) $21aW/2$.
33. (i) 2400 N, (ii) 2600 N at 67° 23′ to vertical,
    (iii) AB: 2400 N; BC: 700 N; CD: $700\sqrt{2}$ N; DE: 1400 N; BD: $700\sqrt{2}$ N; BE: $1000\sqrt{2}$ N; rods AB, BC, BD in compression.
34. (i) 14, (ii) AB: $4/\sqrt{3}$ kN tension; BC: $8/\sqrt{3}$ kN thrust; CD: $8/\sqrt{3}$ kN thrust; DE: $16/\sqrt{3}$ kN thrust; AE: $2/\sqrt{3}$ kN thrust; BD: $4/\sqrt{3}$ kN tension; BE: $4\sqrt{3}$ kN thrust, (iii) 2 kN vertically downwards.

Exercise 7.1

1. $\mathbf{a} - \mathbf{b} + \mathbf{c}$. 2. $\mathbf{a} - 2\mathbf{b} + 2\mathbf{c}$. 3. $60°$. 4. $7, \sqrt{19}$.
5. (a) $s = -2, t = 6$, (b) $s = 6/5, t = 3/5$; $s = -2, t = -1$.
6. (a) 9, (b) -1. 7. (a) $\frac{3}{8}\mathbf{a} + \frac{5}{8}\mathbf{b}$, (b) $-\frac{1}{2}\mathbf{a} + \frac{3}{2}\mathbf{b}$, (c) $2\mathbf{a} - \mathbf{b}$.
8. (a) $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, (b) $-7\mathbf{i} - 2\mathbf{j} - 5\mathbf{k}$.
9. $2\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$, $7\mathbf{i} - 2\mathbf{k}$; $(4, -3, 1)$, $(-1, -8, 6)$.
10. P is a vertex of parallelogram $ABCP$; Q divides AC internally in the ratio 4:3; R divides BC externally in the ratio 2:3.
11. When produced BA and CD meet a point Q such that $BQ:AQ = 7:4$ and $CQ:DQ = 5:2$.
12. $\mathbf{r} = 3\mathbf{q} - 2\mathbf{p}$; 3:2.

Exercise 7.2

1. (a) $\mathbf{r} = (1 + 3\lambda)\mathbf{i} + 2\mathbf{j} - (1 + \lambda)\mathbf{k}$, (b) $\mathbf{r} = (\lambda + 4)\mathbf{i} - 3\lambda\mathbf{j} + 3(\lambda - 1)\mathbf{k}$.
2. (a) $\mathbf{r} = 2(1 - 2\lambda)\mathbf{i} + (4\lambda - 3)\mathbf{j} + \mathbf{k}$, (b) $\mathbf{r} = (1 + 2\lambda)\mathbf{i} + (5 - 6\lambda)\mathbf{j} + 2\lambda\mathbf{k}$,
   (c) $\mathbf{r} = 5\lambda\mathbf{i} + (6 - 13\lambda)\mathbf{j} + 2(5\lambda - 4)\mathbf{k}$, (d) $\mathbf{r} = 5\lambda\mathbf{i} - 2\lambda\mathbf{j} + 3\lambda\mathbf{k}$.
3. $\mathbf{r} = (1 - \lambda)\mathbf{a} + \frac{1}{3}\lambda\mathbf{b} + \frac{2}{3}\lambda\mathbf{c}$, $\mathbf{r} = -\mu\mathbf{a} + (1 - \mu)\mathbf{b} + 2\mu\mathbf{c}$; $-\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b} + \mathbf{c}$.
4. $\frac{1}{2}\mathbf{a}, \frac{1}{2}\mathbf{a} + \mathbf{b}, \frac{1}{3}\mathbf{a} + \frac{2}{3}\mathbf{b}$. 5. $\frac{3}{2}\mathbf{a}$. 6. 1:3.
7. (a) Intersect at $2\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$, (b) skew, (c) parallel,
   (d) intersect at $6\mathbf{i} + 13\mathbf{j} - 4\mathbf{k}$.
8. (a) A, B, (b) B, (c) C, (d) A, B, C.
9. (a) $\mathbf{r} = (\lambda + 1)\mathbf{i} + (3\lambda - \mu - 2)\mathbf{j} + 2\mu\mathbf{k}$,
   (b) $\mathbf{r} = (2\lambda - \mu)\mathbf{i} + (2\mu - \lambda)\mathbf{j} - 7\mu\mathbf{k}$,
   (c) $\mathbf{r} = (\mu + 3)\mathbf{i} + (\lambda + \mu + 1)\mathbf{j} + (\mu - 1)\mathbf{k}$.
10. $\mathbf{r} = (\lambda + 5\mu)\mathbf{i} + (2\mu - 3\lambda)\mathbf{j} + (2 - \lambda - 2\mu)\mathbf{k}$.
11. $-\mathbf{i} + 3\mathbf{k}$, $\mathbf{j}$; $\mathbf{r} = (2 - \lambda)\mathbf{i} + (5 + \mu)\mathbf{j} + 3\lambda\mathbf{k}$.
12. $\mathbf{r} = (2t + 1)\mathbf{i} + (2s - t)\mathbf{j} - (2 + s - 3t)\mathbf{k}$; $(5, 0, 3)$.

Exercise 7.3

1. $6\mathbf{i} - 8\mathbf{j}$, $4\mathbf{i} + 3\mathbf{j}$; $10\mathbf{i} - 5\mathbf{j}$.
2. $3\sqrt{2}\,\mathrm{N}, 5\sqrt{2}\,\mathrm{N}$. 3. 27 N.
4. (a) $\mathbf{r} = (1 + \lambda)\mathbf{i} + (1 + 2\lambda)\mathbf{j}$, (b) $\mathbf{r} = -2\lambda\mathbf{i} + (5\lambda + 2)\mathbf{j} + (3\lambda - 1)\mathbf{k}$,
   (c) $\mathbf{r} = 3\mathbf{i} - 2\lambda\mathbf{j} + 3\lambda\mathbf{k}$.
5. $\mathbf{r} = (2 + 3\lambda)\mathbf{i} + (\lambda - 1)\mathbf{j} - 2\lambda\mathbf{k}$, $\mathbf{r} = (9 + \mu)\mathbf{i} - \mu\mathbf{j} + (3\mu - 1)\mathbf{k}$; $8\mathbf{i} + \mathbf{j} - 4\mathbf{k}$;
   $\mathbf{r} = (8 + 4t)\mathbf{i} + \mathbf{j} + (t - 4)\mathbf{k}$.
6. 11 N, $\mathbf{r} = (3 + 7\lambda)\mathbf{i} + (5 + 6\lambda)\mathbf{j} + 6\lambda\mathbf{k}$.
7. $3x + y = 26$.
8. $3\sqrt{13}\,\mathrm{N}, \frac{2}{3}$; $2x - 3y + 2 = 0$.
9. (a) 15**YX**, where $BX : XC = 3:2$, $AY:YB = 2:1$,
   (b) 24**XY**, where $AX:XC = 1:3$, $AY:YB = 1:5$,
   (c) 6**YX**, where $AX:XC = 2:1$ and $AY = YB$,
   (d) 21**XY**, where $AX:XC = 1:2$, $AY:YB = 3:4$.

Exercise 7.4

1. $\mathbf{r}_A = (3 + 5t)\mathbf{i} + (4 + 2t)\mathbf{j}$, $\mathbf{r}_B = (5 + 2t)\mathbf{i} + (8 + t)\mathbf{j}$; $\mathbf{AB} = (2 - 3t)\mathbf{i} + (4 - t)\mathbf{j}$; $\sqrt{10}$ m.
2. $4\mathbf{i} - 6\mathbf{j}$; $(4t - 2)\mathbf{i} + (3 - 6t)\mathbf{j}$; $t = \frac{1}{2}$.
3. $(2 - 5t)\mathbf{i} + (4t - 18)\mathbf{j}$; $2\sqrt{41}$ km; $16\mathbf{i} + 19\mathbf{j}$, $8\mathbf{i} + 9\mathbf{j}$.
4. $8\cos\omega t\mathbf{i} + 8\sin\omega t\mathbf{j}$. 5. $\mathbf{i} - 2\mathbf{j}$, 3 m; 6 m s^{-1}.
6. (a) $\pi/4$, (b) $5\pi/4$. 7. (a) 1/3, (b) 3.
8. (a) $8t\mathbf{i} + (9t^2 - 2)\mathbf{j}$, $m(8\mathbf{i} + 18t\mathbf{j})$, (b) $-6t^2\mathbf{j} + 2t\mathbf{k}$, $m(-12t\mathbf{j} + 2\mathbf{k})$, (c) $(2 - 2t)\mathbf{i} + (6\cos 2t)\mathbf{j}$, $m\{-2\mathbf{i} - (12\sin 2t)\mathbf{j}\}$, (d) $e^{-t}(1 - t)\mathbf{i} - 2\mathbf{j}$, $me^{-t}(t - 2)\mathbf{i}$.
9. (a) $\sqrt{19}$ m s^{-1}, (b) 20i. 10. $12\mathbf{i} + 30\mathbf{j}$, $18\mathbf{i} + 45\mathbf{j}$.
11. 40 m s^{-1}, $18\mathbf{i} - 35\mathbf{j}$. 12. $5\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$.
13. $2\mathbf{i} + 6\mathbf{j}$; $9\sqrt{10}$ m.
14. $(16t - 5t^2)\mathbf{j}$, $12t\mathbf{i} + (25 - 5t^2)\mathbf{j}$; 15 m.

Exercise 7.5

1. (a) $p^2 + 2\mathbf{p}.\mathbf{q} + q^2$, (b) $p^2 - q^2$, (c) $2\mathbf{p}.\mathbf{r}$, (d) $p^2 - 2\mathbf{p}.\mathbf{q} + q^2$.
2. (a) x^2, (b) $2x^2 - 3y^2$. 3. 81, 49; 9, 7. 4. 13, 9. 8. $-3/4$.
9. $3\mathbf{a} - 2\mathbf{b}$. 14. (a) 68° 16′, (b) 60°, (c) 90°, (d) 35° 16′.
15. (a) 1/3, (b) 5/11. 16. (a) 3, (b) 2, (c) 1, (d) -1. 17. 65/7, 7/9.
18. (a) $4\mathbf{i} - 3\mathbf{j}$, (b) $2\mathbf{i} + 5\mathbf{j}$, (c) $3\mathbf{i} - 2\mathbf{j}$.
19. $\angle A = 90°$, $\angle B = 36° 52′$, $\angle O = 53° 8′$; 30 sq. units.
20. 11/21, $8\sqrt{5}$ sq. units. 21. $5\mathbf{i} + 2\mathbf{k}$.
22. (a) $(2, -1, 1)$, (b) 70° 32′.
23. $\mathbf{i} - \mathbf{j}$, $\mathbf{j} + \mathbf{k}$; 60°. 24. -11, -7; 7; 7/57.

Exercise 7.6

1. $-2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$; 3 units. 2. $\mathbf{i} - \mathbf{k}$; 9 sq. units.
3. $3\mathbf{i} - \mathbf{j} + 4\mathbf{k}$; $6\mathbf{i} - 2\mathbf{j} + 8\mathbf{k}$. 4. $5\mathbf{i} - \mathbf{j} + 6\mathbf{k}$; 13 units.
5. (a) $\mathbf{r}.(4\mathbf{i} - \mathbf{k}) = 3$, (b) $\mathbf{r}.(\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = 1$.
6. (a) $\mathbf{r}.\mathbf{k} = 1$, (b) $\mathbf{r}.(3\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) = 0$.
7. (a) 5, (b) 3, (c) 4, (d) 0.
8. $2\mathbf{i} - 5\mathbf{j} - \mathbf{k}$, $\mathbf{r}.(2\mathbf{i} - 5\mathbf{j} - \mathbf{k}) = 7$. 9. $\mathbf{r}.(\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) = 1$.
10. $\mathbf{r}.(\mathbf{j} - 2\mathbf{k}) = -8$. 11. (a) 60°, (b) 48° 11′, (c) 70° 32′.
12. (a) 11/18, (b) 5/6, (c) 6/7. 13. $-\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$, $2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$.
14. $\mathbf{r} = (3\lambda + 5)\mathbf{i} - (4\lambda + 3)\mathbf{j} + (\lambda + 4)\mathbf{k}$; $2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$.
15. (a) $\dfrac{x-1}{2} = \dfrac{y+2}{3} = \dfrac{z-1}{4}$, (b) $\dfrac{x}{1} = \dfrac{y-2}{-1} = \dfrac{z-3}{2}$,
(c) $\dfrac{x-2}{5} = \dfrac{y+3}{-6} = \dfrac{z-1}{3}$, (d) $\dfrac{x-5}{1} = \dfrac{y}{3} = \dfrac{z+4}{0}$
16. (a) $2x - 3y + z - 5 = 0$, (b) $3x + y - 5z - 9 = 0$.
17. $5x + y + z - 6 = 0$. 18. $2x - y + 5z - 3 = 0$.

19. $\mathbf{i}-3\mathbf{j}+2\mathbf{k}$; (a) $\mathbf{r}.(\mathbf{i}-3\mathbf{j}+2\mathbf{k})=-5$, (b) $x-3y+2z+5=0$.
20. $3x+y-4z-1=0$. 21. (a) 70° 32′, (b) 65° 54′, (c) 28° 8′.
22. (a) Plane through O perpendicular to OA,
(b) Plane through A perpendicular to OA,
(c) Sphere with centre A, radius 1 unit,
(d) Plane through O perpendicular to AB,
(e) Sphere with diameter OA,
(f) Sphere with diameter AB.

Exercise 7.7

1. 31° 35′, $5\,\mathrm{m\,s^{-1}}$. 2. $\sqrt{41}\,\mathrm{N}\approx 6{\cdot}40\,\mathrm{N}$, 50° 39′.
3. (a) 2 N, (b) −3 N, (c) $\sqrt{2}$ N, (d) $-5/3$ N.
4. (a) $8\mathbf{i}-6\mathbf{k}$, $3\mathbf{i}+4\mathbf{j}+4\mathbf{k}$, (b) $-10\mathbf{j}+24\mathbf{k}$, $5\mathbf{i}-12\mathbf{j}-5\mathbf{k}$,
(c) $-6\mathbf{i}+18\mathbf{j}-27\mathbf{k}$, $24\mathbf{i}+17\mathbf{j}+6\mathbf{k}$, (d) $\frac{2}{3}(4\mathbf{i}-8\mathbf{j}+\mathbf{k})$, $\frac{1}{3}(\mathbf{i}+4\mathbf{j}+28\mathbf{k})$.
5. (a) 2 J, (b) 0 J, (c) 21 J, (d) −11 J.
6. (a) $8m$ J, (b) $24m$ J, (c) $360m$ J, (d) 0 J.
7. (a) $4m\mathbf{i}$, $4t\mathbf{i}-5\mathbf{k}$; $16mt$, (b) $6m(t\mathbf{j}-\mathbf{k})$, $3t^2\mathbf{j}-6t\mathbf{k}$; $18mt(t^2+2)$,
(c) $-me^{-t}\mathbf{j}$, $\mathbf{i}+e^{-t}\mathbf{j}$; $-me^{-2t}$.
(d) $-m(\sin t\mathbf{j}+3\cos t\mathbf{k})$, $\cos t\mathbf{j}-3\sin t\mathbf{k}$; $4m\sin 2t$.
8. 14 J; $8\,\mathrm{m\,s^{-1}}$. 9. −114 J; 6; −96 W, −18 W.
10. $t^2\mathbf{i}+(1-3t)\mathbf{j}+\mathbf{k}$; 20 J; 31 W.
11. $(4t+6)\mathbf{i}-2\sin t\mathbf{j}+\frac{1}{2}t^2\mathbf{k}$, $t^3+32t+48+4\sin 2t$. 12. 24 J.
13. $10t^4+6t^2+2t$; 544 J. 14. $\int_1^2 \mathbf{F}.(\mathbf{i}-2\lambda\mathbf{j})d\lambda$; 12.

Exercise 7.8

1. $\mathbf{b}-\mathbf{a}$, $\frac{1}{2}(\mathbf{a}+\mathbf{c})$, $\mathbf{a}-\mathbf{b}+\mathbf{c}$; $2\mathbf{k}$, $2\mathbf{j}$, $2\mathbf{i}+2\mathbf{j}+2\mathbf{k}$, $\mathbf{i}+\mathbf{j}+2\mathbf{k}$, $2\mathbf{j}+2\mathbf{k}$; $(2\sqrt{2})/3$.
2. $\frac{1}{2}(\mathbf{a}+\mathbf{b}),\frac{1}{2}(\mathbf{b}+\mathbf{c}),\frac{1}{2}(\mathbf{c}+\mathbf{d}),\frac{1}{2}(\mathbf{d}+\mathbf{a}),\frac{1}{2}(\mathbf{a}+\mathbf{c}),\frac{1}{2}(\mathbf{b}+\mathbf{d})$; $(\mathbf{i}+\mathbf{j})/\sqrt{2}$.
3. Mid-point of XC, where X divides AB in ratio 1:2; $\frac{1}{3}\mathbf{a}+\frac{2}{3}(\frac{1}{4}\mathbf{b}+\frac{3}{4}\mathbf{c})$;
R divides AY in ratio 2:1, where Y divides BC in ratio 3:1.
4. $\{(1+\lambda)\mathbf{a}+\mathbf{b}+\lambda\mathbf{c}\}/2(1+\lambda)$, $\{\lambda\mathbf{a}+(1+\lambda)\mathbf{b}+\mathbf{c}\}/2(1+\lambda)$,
$\{\mathbf{a}+\lambda\mathbf{b}+(1+\lambda)\mathbf{c}\}/2(1+\lambda)$; $\frac{1}{2}(1+\sqrt{5})$.
6. $[r(1-p)\mathbf{c}+p(r-1)\mathbf{a}]/(r-p)$, $[p(1-q)\mathbf{a}+q(p-1)\mathbf{b}]/(p-q)$.
7. $3\mathbf{i}+2\mathbf{j}+4\mathbf{k}$; $\mathbf{r}=3\mathbf{i}+s(2\mathbf{i}+5\mathbf{j}+4\mathbf{k})+t(\mathbf{j}+2\mathbf{k})$.
8. $\mathbf{r}=(1-\lambda)\mathbf{i}+(6\lambda+3)\mathbf{j}+(4\lambda+2)\mathbf{k}$. 10. 70° 32′.
12. $\frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})$. 13. $\lambda:\mu=1:6$; $\mathbf{r}=(1+t)\mathbf{a}+6t\mathbf{b}$; $-12\mathbf{i}-6\mathbf{j}+6\mathbf{k}$.
14. (i) $\mathbf{r}=(1+\lambda+\mu)\mathbf{i}-2\lambda\mathbf{j}-3\mu\mathbf{k}$ or $\mathbf{r}.(6\mathbf{i}+3\mathbf{j}+2\mathbf{k})=6$,
(ii) $\mathbf{r}=\lambda(6\mathbf{i}+3\mathbf{j}+2\mathbf{k})$, (iii) $6(6\mathbf{i}+3\mathbf{j}+2\mathbf{k})/49$.
15. (i) $(1,-2,0),(4,4,-3)$, (ii) 90°, 90°,
(iii) $x-y-z-3=0, x+z=1$, (iv) 90°.
16. $(p-\mathbf{a}.\mathbf{n})/(\mathbf{b}.\mathbf{n})$; $(2\frac{1}{3},2\frac{1}{3},2\frac{2}{3})$.
17. -1, 1; $\mathbf{i}+5\mathbf{j}-\mathbf{k}$, $-3\mathbf{i}+3\mathbf{j}+\mathbf{k}$; $2\sqrt{6}$.
18. (i) $-\mathbf{i}+\mathbf{j}+3\mathbf{k}$, (ii) $3\mathbf{j}-\mathbf{k}$, (iii) 55°, (iv) 35°.

19. $61° 52'$, $5\sqrt{3}$ N; $\mathbf{r} = \lambda\mathbf{i} + (7\lambda + 1)\mathbf{j} + (5\lambda - 1)\mathbf{k}$, $\frac{x}{1} = \frac{y-1}{7} = \frac{z+1}{5}$.

20. (a) AD, (b) the straight line perpendicular to AB which divides AB in the ratio 1:5.
21. (i) 26·7 N in the direction of the vector $3\mathbf{i} + 13\mathbf{j}$; $-36/65$, (ii) $6\mathbf{i} + 26\mathbf{j}$.
22. $(2T^4 - 2)\mathbf{i} + (3T^2 + 3)\mathbf{j}$; $94\mathbf{i} + 30\mathbf{j}$.
23. $6\mathbf{i} + 8\mathbf{j}$, 6; $6\mathbf{i} + 8\mathbf{j} + 6(\mathbf{i}\cos(\pi t/12) + \mathbf{j}\sin(\pi t/12))$; $-\pi^2/24$, $-\pi^2\sqrt{3}/24$; $(9 - \sqrt{3}\pi)\mathbf{i} + (8 + 3\sqrt{3} + \pi)\mathbf{j}$. 24. $1/(1 + \cos\theta)$; towards O.
25. $2/\sqrt{13}$. 26. (i) $(10 - 2t)\mathbf{i} + (5 + 4t)\mathbf{j}$, (ii) 14.00 hrs, (iii) $4\sqrt{5}$ km.
27. $(260 - 35t)\mathbf{i} - 5\sqrt{3}t\mathbf{j}$; 62 km, 19.00 hrs.
28. (a) $\mathbf{v} - \mathbf{u}$, (b) $\mathbf{p} + t(\mathbf{v} - \mathbf{u})$; $\left\{p^2 - \frac{[\mathbf{p}.(\mathbf{u} - \mathbf{v})]^2}{|\mathbf{u} - \mathbf{v}|^2}\right\}^{1/2}$.
29. (i) $\mathbf{F}.\mathbf{v}$, (ii) $\int_0^T \mathbf{F}.\mathbf{v}\,dt$; $-4a\sin 2t\mathbf{i} + 2a\cos 2t\mathbf{j}$, $-8ma\cos 2t\mathbf{i} - 4ma\sin 2t\mathbf{j}$; $12ma^2\sin 4t$; $6ma^2$, $\frac{1}{4}\pi$.
30. $\frac{1}{2}ma^2(2 - 2\cos\theta)\dot{\theta}^2 + mga(1 + \cos\theta) = 2mga$.

Exercise 8.1

1. (a) $8\mathbf{i} - 28\mathbf{j}$, (b) $2\mathbf{i} - \frac{2}{3}\mathbf{j} + \mathbf{k}$, (c) 65, (d) 4.
2. (a) $10\mathbf{i} + 6\mathbf{j}$, (b) $2\mathbf{i} - 52\mathbf{j} + 8\mathbf{k}$, (c) $3\mathbf{j}$.
3. (a) 480 N s, (b) 30 000 N s, (c) 12 N s.
4. (a) $12\mathbf{j}$, (b) $-2\mathbf{i} + 6\mathbf{k}$, (c) $-60\mathbf{k}$. 5. 54 N s. 6. 4 N s.
7. 600 N. 8. (a) 16·8 m s^{-1}, (b) 7·2 m s^{-1}. 9. 7·5 s.
10. $12\mathbf{i} + 15\mathbf{j}$; $2\mathbf{i} + 2{\cdot}5\mathbf{j}$. 11. $8\mathbf{i} + 4\mathbf{j}$; 10 J
12. (a) 14 m s^{-1}, 240 J gain, (b) 2 m s^{-1}, 48 J loss, (c) 10 m s^{-1}, 96 J gain.
13. 2·5 N s. 14. $20\sqrt{3}$ N s. 15. 600 N. 16. 12 m s^{-1}.
17. 2 N at $53° 8'$ to horizontal
18. (a) 2 m s^{-1}, $6\sqrt{3}$ N s, (b) $(4\sqrt{3})/3$ m s^{-1}, $4\sqrt{6}$ N s.
19. 1·5 m s^{-1} along AB, 5·22 m s^{-1} at $73° 18'$ to AB; 4·5 N s.
20. (a) $2\sqrt{2}$ m s^{-1}, $2\sqrt{2}$ m s^{-1}, 4 m s^{-1}, (b) 2 m s^{-1}, $2\sqrt{3}$ m s^{-1}, 4 m s^{-1}.

Exercise 8.2

1. 5 m s^{-1}. 2. 720 m s^{-1}. 3. 7·5 m s^{-1} towards A; 78·75 J.
4. 2·5 m s^{-1}, 2512·5 J. 5. 5 m s^{-1}. 6. 7 m s^{-1}. 7. $2\mathbf{i} - 5\mathbf{j}$.
8. $5\mathbf{i} + \frac{5}{2}\mathbf{j}$. 9. 28 J. 10. $3u$, $2u$. 11. 6 m s^{-1}, 26 kJ.
12. 3·5 m s^{-1}, 2000 N. 13. 0·6 m s^{-1}, 720 N s. 14. 30.
15. (a) 14 m s^{-1}, (b) 23·8 m s^{-1}. 16. $u/2g\sin\alpha$.

Exercise 8.3

1. 4800 N, 5 s. 2. 9000 N, 1/600 s. 3. 2 m s^{-1}, 25 m. 4. 1.
5. (a) 70 N s, (b) 77 N s. 6. 10/7 s, $14\sqrt{2}$ m s^{-1}.

7. $4{\cdot}2\,\mathrm{m\,s^{-1}}$, 78 400 N. 8. 90 160 N, 1/7 s. 9. 1·25 m.
10. $469\,\mathrm{m\,s^{-1}}$. 11. h, $\frac{1}{4}h$. 12. 0·1 m. 13. $2{\cdot}1\,\mathrm{m\,s^{-1}}$, 6·3 N s.
14. $7{\cdot}35\,\mathrm{m\,s^{-1}}$, $2\frac{1}{4}$ s. 15. 3 s, $4{\cdot}2\,\mathrm{m\,s^{-1}}$. 16. (a) $\frac{7}{3}h$, (b) $\frac{2}{3}m\sqrt{(\frac{2}{3}gh)}$.
17. $2{\cdot}8\,\mathrm{m\,s^{-1}}$, 20·6 J. 18. (i) $\frac{3}{4}mgh$, (ii) $9m^2gh/4M$.
19. (ii) $\frac{12}{5}m\sqrt{(gc)}$, (iii) $\frac{33}{25}c$, (iv) $\frac{12}{5}mgc$.
20. $40\,\mathrm{m\,s^{-1}}$, 80 J; 1200 J; $(d + 14)$ cm; 1, 8000 N.

Exercise 8.4

1. (a) 2·5 m, (b) 14·7 J, (c) 4·2 N s. 2. $\frac{3}{4}$, 4·05 m. 3. $\frac{2}{3}$.
4. $8\sqrt{5}\,\mathrm{m\,s^{-1}}$, 60 N s. 5. $1/\sqrt{3}$; 45°. 6. 60°.
7. $4\frac{1}{6}\,\mathrm{m\,s^{-1}}$ at 53° 8′ to horizontal. 8. $2u$, $4u$; $18mu^2$. 9. $\frac{1}{2}$.
10. (a) $1\,\mathrm{m\,s^{-1}}$, $4\,\mathrm{m\,s^{-1}}$, same direction.
(b) $\frac{1}{2}\,\mathrm{m\,s^{-1}}$, $5\frac{1}{2}\,\mathrm{m\,s^{-1}}$, opposite directions.
11. $\frac{1}{4}$, 2:3. 12. $5\,\mathrm{m\,s^{-1}}$, $-4\,\mathrm{m\,s^{-1}}$, $16\,\mathrm{m\,s^{-1}}$; yes.
13. $u/4$, $3u/16$, $9u/16$; yes. 14. 2/9. 16. $u(1 - e)$, u.
17. $\frac{1}{2}\sqrt{(2ag)}$, $\frac{1}{4}\sqrt{(2ag)}$ in same direction.
18. (a) $\frac{1}{3}V(1 - 2e)$, $\frac{1}{3}V(1 + e)$, 0 (b) $\frac{1}{3}V(1 - 2e)$, $\frac{1}{9}V(1 + e)(2 - e)$, $\frac{2}{9}V(1 + e)^2$.

Exercise 8.5

1. (i) $9mu$, (ii) $81mu^2/4$. 2. (a) $12\mathbf{i} + 4\mathbf{j}$, (b) $5\mathbf{i} - 2\mathbf{j}$, (c) 32 J.

3. $-\dfrac{mv}{2}\mathbf{i} + \dfrac{mv}{2\sqrt{3}}\mathbf{j}$; $\dfrac{v}{3\sqrt{3}}$ at 30° to the x-axis in a clockwise sense.

4. (i) 283 kg, (ii) 11 500 J, (iii) 39 kW, (iv) 56 kW; 2500 N.
5. (ii) $u/6$, $(u\sqrt{3})/4$; $mu/6$. 6. $|m_1u_1 - m_2u_2|/(m_1 + m_2)$.
8. $\frac{1}{2}mgh$, $m(1 + \sqrt{2})\sqrt{(gh)}$. 9. $1{\cdot}6\,\mathrm{m\,s^{-1}}$, $0{\cdot}4\,\mathrm{m\,s^{-1}}$; 131 mm, 8 mm.
10. (i) $5{\cdot}6\,\mathrm{m\,s^{-1}}$, (ii) $3{\cdot}2\,\mathrm{m\,s^{-1}}$, (iii) 13 440 J, (iv) 123·9 kN.
11. (i) $\frac{6}{5}m\sqrt{(2ga)}$, (ii) $\frac{6}{5}mga$, (iii) $\frac{2}{5}\sqrt{(17ga)}$.
12. $1{\cdot}96\,\mathrm{m\,s^{-2}}$; $1{\cdot}12\,\mathrm{m\,s^{-1}}$, 3·36 N s.
14. (i) $g/5$, $12mg/5$, (iii) $9mv/8$, (iv) $15mv^2/16$, (v) $1/\sqrt{3}$.
15. $mMu^2/2a(m + M)$. 16. (b) 16° 16′, (c) 5:6.

17. $\dfrac{u\sin\alpha}{\sin\beta}$; $\frac{1}{2}u\left|\dfrac{\sin(\alpha - \beta)}{\sin\beta}\right|$; $\dfrac{u^2}{g\sqrt{3}}$.

19. (a) $\cot\alpha\tan\beta$, (b) $u\cos\alpha\sec\beta$, (c) $mu\sin(\alpha + \beta)\sec\beta$.
20. (a) $\frac{1}{2}mu^2(1 - e^2)\cos^2\alpha$, (b) $u^2e\sin 2\alpha/g(1 + e)$.
21. $\frac{1}{2}$; 2:7; $2\,\mathrm{m\,s^{-1}}$, $7\,\mathrm{m\,s^{-1}}$ in opposite directions.
22. (i) u along $\overrightarrow{BA}$, (ii) 4/5, (iii) $3mu^2$,
(iv) u, $4u/3$ along $\overrightarrow{BC}$, (v) 1/9.
23. (i) $\frac{1}{2}$, (ii) $\frac{1}{4}mu^2$, (iii) $\frac{2}{3}$, (iv) u. 24. $4u/3$, $u/6$; $7mu^2/12$.
26. $\frac{1}{3}(7 - 8e)u$, $\frac{1}{3}(7 + 16e)u$ in direction $\overrightarrow{AB}$; 1/8;
$3u/(1 + k)$, $k > \frac{1}{2}$.

27. $e = 1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots, \dfrac{1}{1 + 2n}, \ldots$.

Exercise 9.1

1. (a) $n(n+2)$, (b) $\frac{1}{6}n(n-1)(2n+5)$, (c) $2^{n+1}-2$,
(d) $\frac{1}{12}n(n+1)(n+2)(3n+1)$, (e) $\frac{3}{2}(1-3^{-n})$,
(f) $\frac{1}{4}(n+4)^2(n+5)^2-100$.

2. (a) $\dfrac{n}{2(n+2)}$, (b) $\frac{3}{4}-\dfrac{1}{2(n+1)}-\dfrac{1}{2(n+2)}=\dfrac{n(3n+5)}{4(n+1)(n+2)}$,

(c) $\frac{1}{12}-\dfrac{1}{2(n+2)}+\dfrac{1}{2(n+3)}$.

3. (a) (i) 0, (ii) 1, (b) (i) $-n$, (ii) $n+1$. 4. $\frac{1}{3}n(n+1)(n+2)$
7. $\frac{1}{6}n(n+1)(2n+1)(3n^2+3n-1)$. 8. $(n+1)!-1$.
9. (a) $\frac{5}{2}n(n+1)$, (b) $\frac{1}{6}n(n+1)(38n+1)$.
10. (a) n^2, (b) $\frac{1}{3}n(2n-1)(2n+1)$.
11. (a) 352 800, (b) 56 953, (c) $\frac{1}{4}n^2(3n+1)(5n+3)$, (d) $-n^2(4n+3)$.
12. $r(r+3)$; $\frac{1}{3}n(n+1)(n+5)$.
13. $(3r-1)^2$; $\frac{1}{2}n(6n^2+3n-1)$.
14. $2r(2r-1)$; $\frac{1}{3}n(n+1)(4n-1)$.
15. $r(r+1)(r+4)$; $\frac{1}{12}n(n+1)(n+2)(3n+17)$.

16. $\dfrac{1}{r(r+3)}$; $\dfrac{11}{18}-\dfrac{1}{3}\left(\dfrac{1}{n+1}+\dfrac{1}{n+2}+\dfrac{1}{n+3}\right)$.

17. $\dfrac{1}{(2r-1)(2r+1)}$; $\dfrac{n}{2n+1}$.

18. $\dfrac{1}{(2r-1)(2r+1)(2r+3)}$; $\dfrac{1}{12}-\dfrac{1}{8(2n+1)}+\dfrac{1}{8(2n+3)}$.

19. $\dfrac{r}{(r+1)(r+2)(r+3)}$; $\dfrac{1}{4}+\dfrac{1}{2(n+2)}-\dfrac{3}{2(n+3)}$.

20. $\dfrac{x^{n+1}-1}{x-1}$; $\dfrac{nx^{n+1}-(n+1)x^n+1}{(x-1)^2}$.

21. $\dfrac{nx^{n+2}-(n+1)x^{n+1}+x}{(1-x)^2}$.

22. $\{1-a-(2n+3)(-a)^{n+1}+(2n+1)(-a)^{n+2}\}/(1+a)^2$.

Exercise 9.2

1. $1-\frac{1}{2}x^2+\frac{1}{24}x^4$, 2. $x+\frac{1}{3}x^3$. 3. $1+\frac{1}{2}x^2+\frac{5}{24}x^4$.
4. $-\frac{1}{2}x^2-\frac{1}{12}x^4$. 5. $x^2-\frac{1}{3}x^4$. 6. $x-\frac{1}{2}x^2+\frac{1}{6}x^3-\frac{1}{12}x^4$.
7. $x-\frac{1}{3}x^3$. 8. $1+x+\frac{1}{2}x^2-\frac{1}{8}x^4$. 9. $x+\frac{1}{6}x^3$.
10. $x+x^2+\frac{1}{3}x^3-\frac{1}{30}x^5-\frac{1}{90}x^6$.

Exercise 9.3

1. $1 - 3x + 6x^2 - 10x^3$. 2. $1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3$.
3. $\frac{1}{8} - \frac{3}{64}x + \frac{15}{1024}x^2 - \frac{35}{8192}x^3$. 4. $1 + 4x + 8x^2 + 16x^3$.
5. $\frac{1}{4} - \frac{1}{2}x + \frac{7}{16}x^2 - \frac{5}{16}x^3$. 6. $1 + 2x^2 + 8x^3$. 7. $1 - x + x^3$.
8. $1 - \frac{1}{2}x + \frac{11}{8}x^2 - \frac{29}{16}x^3$. 9. $1 + \frac{1}{3}x + \frac{8}{9}x^2 - \frac{49}{81}x^3$,
10. $1 - x + \frac{1}{2}x^2, |x| < 1$. 11. $2 + \frac{25}{12}x + \frac{1175}{288}x^2, |x| < \frac{1}{3}$.
12. $1 + \frac{1}{2}x^2, |x| < \frac{1}{5}$. 13. $\frac{1}{3} - \frac{2}{9}x + \frac{7}{27}x^2, |x| < 1$
14. $1 - 3x + 9x^2, |x| < \frac{1}{2}$. 15. $1 + \frac{5}{2}x + \frac{43}{8}x^2, |x| < \frac{1}{2}$.
16. $3 + x + 11x^2 + 25x^3 + 83x^4, |x| < \frac{1}{3}$.
17. $\frac{4}{3}x + \frac{8}{9}x^2 + \frac{28}{27}x^3 + \frac{80}{81}x^4 + \frac{244}{243}x^5, |x| < 1$.
18. $1 + 2x + x^2 + x^4 + 2x^5, |x| < 1$.
19. $4 - x + 12x^2 - 11x^3 + 38x^4, |x| < \frac{1}{2}$.
20. $2 - 2x + 2x^4 - 2x^5 + 2x^8, |x| < 1$.
21. $-1 + x + \frac{7}{4}x^2 + \frac{1}{2}x^3 - \frac{3}{16}x^4, |x| < \sqrt{2}$.
22. $3 + 3x + 9x^2; \{(-1)^r + 2^{r+1}\}x^r$.
23. $4x - 8x^2 + 28x^3; \{1 - (-3)^r\}x^r$.
24. $2 - \frac{3}{2}x + \frac{17}{4}x^2; \{(-2)^r + 2^{-r}\}x^r$.
25. $1 + \frac{1}{2}x - \frac{3}{8}x^2 + \frac{7}{16}x^3; |x| < \frac{1}{2}$; (a) 1·0885, (b) 1·019 428.
26. $(a + b) + 2(1 - a + b)x + (-4 - 2a + 3b)x^2 + 4(-1 - a + b)x^3$; 13, −8.
27. $n = 6, a = \frac{1}{2}$. 28. $a = 3, b = 1$; 70/13.

Exercise 9.4

1. (a) $1 - x + \frac{1}{2}x^2; \dfrac{(-1)^r x^r}{r!}$, (b) $1 + 3x + \frac{9}{2}x^2; \dfrac{3^r x^r}{r!}$,

   (c) $1 + x^2 + \frac{1}{2}x^4; \dfrac{x^{2r}}{r!}$, (d) $e + ex + \frac{1}{2}ex^2; \dfrac{ex^r}{r!}$.

2. (a) $1 + 4x + 6x^2 + \frac{16}{3}x^3$, (b) $1 - 3x + \frac{7}{2}x^2 - \frac{13}{6}x^3$,
   (c) $1 - x + \frac{3}{2}x^2 - \frac{7}{6}x^3$.
3. 2·7183; 0·3679.
4. (a) $3x - \frac{9}{2}x^2 + 9x^3; -\frac{1}{3} < x \leqslant \frac{1}{3}$. (b) $-\frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{24}x^3; -2 \leqslant x < 2$.
   (c) $\ln 2 + \frac{1}{3}x - \frac{1}{18}x^2; -3 < x \leqslant 3$. (d) $1 - \dfrac{x}{e} - \dfrac{x^2}{2e^2}; -e \leqslant x < e$.

5. (a) $x - \frac{3}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4$, (b) $x - \frac{5}{2}x^2 + \frac{7}{3}x^3 - \frac{17}{4}x^4$,
   (c) $-4x + 3x^2 - \frac{34}{3}x^3 + \frac{63}{2}x^4$. 6. 1·0986.
7. (a) $(1 - r)/r!$, (b) $(4 + 5r - 3r^2)2^{r-2}/r!$, (c) $(4r^2 + 1)/2^r r!$.
8. (a) $x + \frac{1}{2}x^2$, (b) $4 - 4x + 3x^2$, (c) $1 + \frac{15}{2}x + \frac{193}{8}x^2$, (d) $2x$,
   (e) $1 + 2x + \frac{5}{2}x^2$, (f) $2 + x + \frac{11}{6}x^2$.
9. 0·6931. 10. $a = 2, n = -2$. 11. $a = \frac{1}{2}, b = 2$; 31/170.

Exercise 9.5

1. (a) 0·2955, (b) 0·9211, (c) −0·4161, (d) 0·8415.
2. (a) $3x - \frac{9}{2}x^3 + \frac{81}{40}x^5$, (b) $1 - 2x^2 + \frac{2}{3}x^4$, (c) $x - \frac{1}{2}x^2 + \frac{1}{6}x^3$,
   (d) $e - \frac{e}{2}x^2 + \frac{e}{6}x^4$, (e) $1 + \frac{1}{6}x^2 + \frac{7}{360}x^4$, (f) $x + \frac{1}{3}x^3 + \frac{2}{15}x^5$.
3. (a) $x + x^2 - \frac{2}{3}x^3 - \frac{1}{6}x^4$, (b) $2x - 2x^2 + \frac{4}{3}x^3$, (c) $\frac{4}{3}x^4$,
   (d) $1 + 2x + x^2 - \frac{2}{3}x^3 - x^4$.

Exercise 9.6

1. (a) $\frac{1}{3}n(n^2 - 1)$, (b) $\frac{1}{3}(n + 1)(4n^2 + 14n + 9)$, (c) $\dfrac{n + 1}{n(2n + 1)}$,
   (d) $\frac{3}{4} - \dfrac{1}{2n} - \dfrac{1}{2(n + 1)}$.
2. (a) $\frac{1}{12}n(n + 1)(n + 2)(3n + 5)$, (b) $n(n + 1)(2n^2 + 2n - 1)$.
3. $\frac{1}{6}n^2(n + 1)^2(n - 1)$. 4. $\dfrac{1 - (n + 1)x^{2n} + nx^{2n+2}}{(1 - x^2)^2}$
5. $(x + 1)(x + 2)(x + 3)$; $\dfrac{1}{x + 1} + \dfrac{2}{x + 2} - \dfrac{3}{x + 3}$.
6. (a) $2^{n+1} - 2 + n^2$, (b) $a = -2, b = -1$; 73/74.
7. $1 + \dfrac{1}{2x^2} - \dfrac{1}{8x^4}, 1 - \dfrac{1}{2x^2} - \dfrac{1}{8x^4}$.
8. $2 - \frac{1}{4}x - \frac{1}{64}x^2; 2 + \frac{7}{4}x + \frac{175}{64}x^2, |x| < \frac{1}{2}$. 9. $2 + x - \frac{3}{2}x^3$.
10. (a) $3x - 3x^2 + 9x^3, \{1 - (-2)^r\}x^r, |x| < \frac{1}{2}$,
    (b) $\frac{1}{6}x + \frac{5}{36}x^2 + \frac{19}{216}x^3, \left(\dfrac{1}{2^r} - \dfrac{1}{3^r}\right)x^r, |x| < 2$,
    (c) $1 + 5x + 18x^2, \frac{1}{2}(5 . 3^r - 2^{r+1} - 1)x^r, |x| < \frac{1}{3}$.
11. (a) $1 - 2x - x^2 + 2x^3 + \frac{3}{2}x^4; |x| < 1/\sqrt{2}$,
    (b) $1 + 4x + 14x^2 + 40x^3 + 104x^4; |x| < \frac{1}{2}$,
    (c) $3 + 6x + 6x^2 + 12x^3 + 36x^4; |x| < \frac{1}{2}$.
12. (a) $x + \frac{5}{24}x^3 - \frac{1}{8}x^4$; for $|x| < 1; a = -\frac{1}{4}$,
    (b) $1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3, \dfrac{1 . 3 . 5 \ldots . (2r - 1)}{2^r r!}(-1)^r x^r$; 0·099 504.
13. (a) (i) (0, 1), (ii) (0, 1), (iii) (1, 0), (iv) (0, 0);
    (b) $7x + \frac{1}{2}x^2 + \frac{43}{3}x^3 + \frac{49}{4}x^4, -\frac{1}{3} \leqslant x < \frac{1}{3}; \{(-2)^{n+1} + 3^n\}x^n/n$.
14. $1 + x + x^2 + \frac{1}{3}x^3$; 0·287.
15. (i) $x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots + (-1)^{n+1}\dfrac{x^n}{n} + \ldots$; (ii) 0·6931; 1/3, 23/60.

16. $p^2 - 2$, $p^3 - 3p$; $-px - \frac{1}{2}(p^2 - 2)x^2 - \frac{1}{3}(p^3 - 3p)x^3$.
17. $1, \sqrt{3}, 2$.

Exercise 10.1

1. 0·697. 2. 5·731. 3. 45·992. 4. 4·154. 5. 0·144.
6. 0·825. 7. Greater. 8. Greater. 9. Greater.
10. Less. 11. Greater. 12. Less. 13. 0·8358. 14. 0·4783.
15. 0·6807. 16. 0·8945. 17. 0·2763. 18. 3·2413.
19. $1 + 12x^2 + 66x^4 + 220x^6 + 495x^8$; 1·7655; 1·9253.
20. 0·0711; 0·0715.
21. (a) 10·075, (b) 9·99. 22. (a) $7{\cdot}175\,\mathrm{m\,s^{-1}}$, (b) $11{\cdot}85\,\mathrm{m\,s^{-1}}$; 131 m.
23. (a) 422·5 J. (b) 423 J. 24. 0·7444. 25. 1·4241.
26. 1·68. 27. 0·524.

Exercise 10.2

1. 2·7, −0·7. 2. 0·7. 3. −2·1. 4. −0·6, 1, 1·6.
5. −1·8, −0·4, 1·2. 6. 0·8. 7. 0, ±1·9. 8. −3·6, −2·1, 1·3.
9. 1·1. 10. 1, 4·2. 11. One. 12. Two. 13. Three.
14. Five. 15. One; −3, −2. 16. One; 1, 2.
17. Three; −4, −3; 1, 2; 2, 3. 18. Three; −3, −2; −1, 0; 1, 2.
19. One; 1, 2. 20. One; 1, 2. 21. Two; −3, −2; 1, 2.
22. One; 2, 3. 23. 3·2. 24. −0·8, 1·6. 25. −3, 1; 1·1.
26. 2·2. 27. −5·2, −0·6, 2·9. 28. −0·2, 0·6, 2·0. 29. −4·1.
30. 1·8, 4·5. 31. −1·3, 1·9, 4·0.
32. (a) −1·75, 0·43, 1·32, (b) −1·6, 0·7, 1·1.
33. (a) −0·1, (b) 0·25, (c) 0·4, (d) 0·2.

Exercise 10.3

1. 1·20. 2. −1·11. 3. −1·52. 4. 2·39. 5. 1·56.
6. 0·79. 7. −1·90. 8. 1·17. 9. 2·803. 10. 1·650.
11. −0·830. 12. −1·834. 13. 1·278. 14. 2·219. 15. 0·453.
16. 3·6; 3·6913. 17. Two; 3·98, −1·06.
18. 2, $a := 2$, $h := 0{\cdot}01$, $g(x) := e^x - 2$; 1·68.

Exercise 10.4

1. (a) 2·24, (b) 1·26, (c) 0·79, 3·87. 2. (a) 1·414, (b) 3·162, (c) 3·606.
3. 1·71. 4. 0·769. 5. 1·821. 6. −3·794.
7. 0·1926; for $x_0 = -6$ the sequence diverges, for $x_0 = -5$ the sequence converges to the positive root.
8. 1; 1·260, 1·312. 9. 1·732.

10. 1·087 using $x_{n+1} = (4 - \frac{5}{2}x_n)^{1/3}$, $x_{n+1} = \dfrac{8}{2x_n^2 + 5}$ or Newton's method.

11. 2·618; decision box should ensure maximum error of 0·1. 12. 0·7.

Exercise 10.5

1. 3·69, 6·71. 2. 4, 3; (a) 4·826, (b) 4·8224. 3. 0·4778; 0·4783.
4. 0·410. 5. (i) 2·09, (ii) $-1·53$, 0·26, 1·27; 0·5.
6. (i) two, (ii) $m < -\frac{1}{2}$. 7. $(0, 0), (2, 4e^{-2})$. 8. $q < 0$.
9. 2·4; underestimate. 10. Three; 1·43, larger. 11. (ii) output "2, 3".
12. (a) 2·208, (b) 2·208. 13. 1, 3; -4, 0; $a < -4$ or $a > 0$; 4·1.
14. 3·43; 3·75; 15. 15. $(x - 2)(x - 3)(x^2 + x + 2)$; 3·065.
16. (a) 0·818, (b) 0·810. 17. 0·74. 18. 0·5, 0·73.
19. 1·2. 20. 0·77. 21. $x = (3x - 1)^{1/3}$; 1·53.

Exercise 11.1

3. (a) and (d). 4. 2, -1. 5. 1, 0, 2. 6. $\dfrac{dy}{dx} = 3x^2$.

7. $x\dfrac{dy}{dx} = y + x^2$. 8. $x\dfrac{dy}{dx} + y = 0$. 9. $\dfrac{dy}{dx} + y = 0$.

10. $x + y\dfrac{dy}{dx} = 0$. 11. $2x\dfrac{dy}{dx} = y$. 12. $x\dfrac{dy}{dx} = 2x - y$.

13. $x\dfrac{dy}{dx} = 2y + x^2$. 14. $2\dfrac{dy}{dx} + y\tan x = 0$.

15. $x\ln x\dfrac{dy}{dx} = y\ln y$. 16. $\dfrac{d^2y}{dx^2} + y = 0$.

17. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} - 3y = 0$. 18. $3\dfrac{dy}{dx}\left(\dfrac{d^2y}{dx^2}\right) + y\dfrac{d^3y}{dx^3} = 0$.

19. $\dfrac{d^3y}{dx^3} - 2\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 0$.

Exercise 11.2

1. $y = x^3 + x + c$. 2. $y = 2\sin\frac{1}{2}x + c$. 3. $y = \frac{1}{2}x^3 + \ln|x| + c$.
4. $y = x + \ln(x - 1)^2 + c$. 5. $y = \cos x + \frac{1}{3}x^3 + Ax + B$.

6. $y = e^{-x} - e^x + Ax + B$. 7. $y = \dfrac{1}{2x + k}$. 8. $y = \tan^{-1}(x + k)$.

9. $y = \ln|x + k|$. 10. $y = (c - x)^{-1/3}$. 11. $y = \frac{1}{3}(x + 1)^3 - 9$.
12. $y = 1 + \sqrt{(2x + 3)}$. 13. $y = x \sin x + \cos x + 1$.
14. $y = \ln \frac{1}{2}(1 + x^2)$. 15. $y = \frac{1}{4}(x - \sin 2x)$.
16. $y = 2 \ln x + x^2 - 4x + 3$. 17. $y = 3e^{-2x}$. 18. $y = 1 - e^x$.
19. $y = \tan(x + \frac{1}{4}\pi)$. 20. $y = (3 + e^x)/(3 - e^x)$. 21. $y = x/(1 + kx)$.
22. $y = kx + x \sin x$. 23. $y = kx^2 - x$. 24. $y = x - e^{1-x} + 2$.

25. $xe^y \dfrac{dy}{dx} + e^y$; $y = \ln\left(x - \dfrac{2}{x}\right)$.

Exercise 11.3

1. $y = (x^2 - x + c)^{1/3}$. 2. $y = 1/(c - 3x^2)$. 3. $y = -\ln(\cos x + c)$.
4. $y = \ln(e^x + c)$. 5. $y = \sin^{-1}(\frac{1}{2}x^2 + c)$. 6. $y = \tan^{-1}(3x + c)$.
7. $y = Ax$. 8. $y = A/(1 - x)$. 9. $y = A(x^2 + 1)^2$.
10. $y = A(x - 1)/(x + 1)$. 11. $y = x/(1 - 2x)$. 12. $y = 5e^{-x^2}$.
13. $y = 2 \sec x$. 14. $y = \frac{1}{2}\ln(x^2 + 1)$. 15. $y = 2e^{x^2} - 1$.
16. $y = xe^{x-1}$. 17. $y = \sin^{-1}(1 + x^3)$. 18. $y = \sin^{-1}(x - 1)$.
19. $y = 1/(1 - 2x^2)$. 20. $y = (x - 1)/(x + 1)$. 21. $y = Ae^{-x}/(1 - x)$.
22. $y = Ae^x(x - 1)/(x + 1)$. 23. $y = \tan(e^x + c)$. 24. $y = \ln(1 + Ae^{x^2})$.
25. $y = e^{-x}\sqrt{(2x + 1)}$. 26. $y = \sqrt{\{(\ln x)^2 + 9\}}$.
27. $x^2 - y^2 - 2xy + 2x + 2y = A$.
28. $x^2(x^2 - y^2) = A$.

Exercise 11.4

1. $x^2 + y^2 = 5$. 2. $y = 1/x^2$.

3. $\dfrac{dC}{dt} = kC$; $C = C_0e^{kt}$; $t = \dfrac{1}{k}\ln\frac{3}{2}$. 4. $\dfrac{d\theta}{dt} = -k\theta$.

5. $(1 + \sqrt{2})t$. 6. $\dfrac{dx}{dt} = -kx$; 93 days; 320 mg. 7. 40 min.

8. $x = 100(1 + e^{-t/10})$; 137 litres; 6 min 56 sec. 9. $30 \ln 2$ min., $100e^{-1}$ kg.

10. 2·99. 11. $t = \dfrac{\ln 3}{V\beta}$; $m = \dfrac{V}{\alpha(1 + 3e^{-V\beta t})}$.

Exercise 11.5

1. $y = A \cos 4x + B \sin 4x$. 2. $y = A \cos \frac{\sqrt{7}}{2}x + B \sin \frac{\sqrt{7}}{2}x$.
3. $y = 2 + A \cos 3x + B \sin 3x$. 4. $y = 1 + A \cos \sqrt{10}x + B \sin \sqrt{10}x$.
5. $y = Ae^{6x} + Be^{-6x}$. 6. $y = Ae^{2\sqrt{2}x} + Be^{-2\sqrt{2}x}$.
7. $y = -\frac{1}{27} + Ae^{\sqrt{3}x} + Be^{-\sqrt{3}x}$. 8. $y = \frac{1}{4} + Ae^{2x} + Be^{-2x}$.
9. $y = 4 \cos 2x - 3 \sin 2x$. 10. $y = 3e^{4x} + 7e^{-4x}$. 11. $y = 6 \cos \frac{1}{6}x$.

12. $y = 4\sin\frac{4}{3}x - 5\cos\frac{4}{3}x$. 13. $y = -2e^{5x/2} - 7e^{-5x/2}$.
14. $y = 4e^{-2\sqrt{3}x}$. 15. $y = -\frac{4}{7} + 3e^{7x/3} + 6e^{-7x/3}$.
16. $y = \frac{8}{5} + 4\cos\frac{5}{2}x + 9\sin\frac{5}{2}x$. 17. $y = 2 + \sin\sqrt{3}x$.
18. $y = -3 + e^{9x/4} + e^{-9x/4}$. 19. $y = -3 + \sqrt{2}e^{3\sqrt{2}x}$.
20. $y = \frac{1}{2} - 4\sqrt{3}\cos 2\sqrt{3}x - 3\sqrt{3}\sin 2\sqrt{3}x$.

Exercise 11.6

1. $\left(\frac{dy}{dx}\right)^2 - 4x\frac{dy}{dx} + 4y = 0.$

2. $(2x+1)\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - (2x+3)y = 0.$

3. $y = x^2$; $y = x^2 + 2x$.
4. (a) $\ln\frac{1}{3}(3 - 7u) = 7(t - 5)$, (b) $3(y + 18) = (x^2 + 9)^{3/2}$.
5. (a) $2x^3 + 11x^2 + 20x = 37 - 2t$, (b) $y = \{\frac{1}{2}(5e^{x+1} - 3)\}^{1/2}$.

6. $y = 36e^{-t/6} + 27$; 11 minutes; 27 degrees. 7. $y = \sin x + \dfrac{\cos x - 1}{1 + x}$.

8. (a) $x^2 - 4y^2 = 4$; $x - y\sqrt{3} = 1$.
9. (a) $\tan y = 1 - \cos x$, (b) $y = \ln\frac{1}{2}(1 + e^{2x})$.
10. (i) $y = e^{-x}(1 + x)^2$, (ii) $y^2 = 2x$.

12. $y = \dfrac{2x}{x^2 + cx + 1}$; (a) $y = \dfrac{2x}{x^2 + 1}$, (b) $y = \dfrac{2x}{x^2 + x + 1}$,

(c) $y = \dfrac{2x}{(x+1)^2}$.

13. $\dfrac{dy}{dx} = \dfrac{y}{x}$, $y = 2x$; $\dfrac{dx}{dt} = -2x^3$, $x = \dfrac{1}{\sqrt{(4t+1)}}$.

14. (a) $y\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 = 0.$

15. (i) (a) $y = A + \ln[\ln(1/x)]$, (b) $y = A + \ln(\ln x)$, (iii) $y = x$, $y = 2 - x$.

16. $20°, 53\frac{3}{4}°$. 17. (a) $x = 2 + \dfrac{1}{12}\ln\left(\dfrac{5(3y-2)}{3y+2}\right)$, (b) $\dfrac{dp}{dt} = kp$.

18. (a) $y = \frac{1}{3}(5e^{3x-6} - 2)$, (b) $\dfrac{dm}{dt} = -km$; 75·98%.

19. $\dfrac{dP}{dt} = kP(N - P)$, $P = N/(1 + Ae^{-Nkt})$; $N = 400\,000$; N is the limiting size of P.

20. $\dfrac{dh}{dt} = -\lambda$, $\pi h\dfrac{dh}{dt} = -(Ck + \pi\lambda h)$.

21. (i) $\dfrac{dQ}{dt} = -kQ$; $Q = Q_0e^{-kt}$; $Q = 2{\cdot}5 \times 10^{-4}$, (ii) $y = 3e^{5x} + e^{-5x}$.
22. $y = 2 - 4\sin 3x$; $-2, \frac{2}{3}n\pi + \frac{1}{18}\pi, \frac{2}{3}n\pi + \frac{5}{18}\pi$.
23. (i) $(y + 1)e^{-y} = (1 - x)e^{x}$, (ii) $y = 4 - 2\sin 2x$; $\frac{1}{2}r\pi$, r an integer.
24. (i) $y = Ae^{x}/(1 + x)$, (ii) $y = \frac{1}{2}(1 - \cos 2x - \sin 2x)$.
25. $y = \frac{1}{2} + A\cos(2x^2) + B\sin(2x^2)$; $y = -\frac{1}{2} + Ae^{2x^2} + Be^{-2x^2}$.

Exercise 12.1

1. $\ln 3\,\mathrm{m\,s^{-1}}$. 2. $\ln 5$ seconds. 3. $6\,\mathrm{m\,s^{-1}}$. 4. $0{\cdot}585\,\mathrm{m\,s^{-1}}$.
5. $2a$. 6. $\sqrt{(gR/3)}$. 7. (a) $2/x$, (b) $\sqrt{(4t + h^2)}$. 8. $-\frac{1}{2}e^{x}$.

Exercise 12.2

1. $\ln 3$. 2. $\frac{1}{2}\ln(1 + u^2)$. 3. $-\dfrac{1}{k}\{v + \ln(1 - v)\}$.

4. $\dfrac{a}{b^2}(2 - \ln 3)$. 5. $\dfrac{2u}{2 + su}$; $\dfrac{s^2}{4} + \dfrac{s}{u}$. 6. (i) 482 m, (ii) 24·1 s.

8. $\left[\dfrac{g}{k}(1 - e^{-2ks}) + u^2e^{-2ks}\right]^{1/2}$. 9. (iii) 11·1 s.

10. $\dfrac{dv}{dt} = \dfrac{k(U^2 - v^2)}{mv}$; $v\dfrac{dv}{dx} = \dfrac{k(U^2 - v^2)}{mv}$.

11. $\dfrac{m}{2k}\tan^{-1}(\frac{1}{2}u)$; $\dfrac{m}{2k}ln(1 + \frac{1}{4}u^2)$. 12. $\dfrac{P^2}{k}\left(1 - \dfrac{1}{e}\right)$.

15. $\dfrac{m}{R^2}\left\{H\ln\left(\dfrac{H - Ru}{H - 2Ru}\right) - Ru\right\}$;

$\dfrac{m}{R^3}\left\{H^2\ln\left(\dfrac{H - Ru}{H - 2Ru}\right) - HRu - \frac{3}{2}R^2u^2\right\}$.

16. $\dfrac{mU^2}{2S}\ln\left(\dfrac{U^2}{U^2 - V^2}\right)$; $\dfrac{mU^2}{2S}\left\{U\ln\left(\dfrac{U + V}{U - V}\right) - 2V\right\}$.

Exercise 12.3

1. $5\,\mathrm{m\,s^{-1}}$, $10\,\mathrm{m\,s^{-2}}$. 2. (a) 2 m, (b) $\frac{2}{3}\pi$ s, (c) $3\sqrt{3}\,\mathrm{m\,s^{-1}}$.
3. 6π s; 2·5 N. 4. (i) $270/\pi$, (ii) $\frac{2}{3}\sqrt{13}$ m, (iii) $540\sqrt{13}$ N.
5. $6\,\mathrm{m\,s^{-1}}$, 2 m; $5{\cdot}9\,\mathrm{m\,s^{-1}}$. 6. (i) π s, (ii) 2π s, (iii) $3\pi/2$ s.

7. (i) $\dfrac{5}{3\pi}$ m, (ii) $\frac{5}{2}\sqrt{3}\,\mathrm{m\,s^{-1}}$, (iii) $\dfrac{1}{2\pi}$ m, (iv) $\frac{1}{18}$ s.

8. $2\sqrt{3}\,\mathrm{m\,s^{-1}}$, 1/6 s; 8 J.
9. (a) $x = 3\cos 2t + 3\sin 2t$, (b) -3, 6, (c) $3\pi/8$.
10. (a) $x = 3\cos 4t + \sqrt{3}\sin 4t$, (b) $\pi/24$, (c) $2\sqrt{3}$.

Exercise 12.4

1. $5a/3$. 2. (i) $2l$, (ii) $\frac{1}{2}\pi\sqrt{(3l/g)}$, (iii) $15l/8$.
3. $\frac{1}{2}mg$, $\frac{1}{2}mg$; (i) $3l/2$, (ii) $\pi(2l/g)^{1/2}$. 4. 4.
5. $2\pi\sqrt{(2a/g)}$, a; $\frac{1}{2}a$ above 0. 6. $\sqrt{(2a\lambda/m)}$.
7. $5l/2$, $\pi(lm/\lambda)$, $\sqrt{(\lambda l/m)}$. 8. $2l$; $\pi\sqrt{\left(\frac{2l}{g}\right)}, \left(\frac{2u^2l + gl^2}{4g}\right)^{1/2}$.
9. $u\sqrt{\left(\frac{lm}{\lambda}\right)}; \frac{\pi}{2}\sqrt{\left(\frac{lm}{\lambda}\right)}; \pi\sqrt{\left(\frac{lm}{\lambda}\right)} + \frac{2l}{u}$. 10. $\frac{16}{9}a$; $\sqrt{(5ag)}$; $\frac{\pi}{6}\sqrt{\left(\frac{a}{g}\right)}$.
11. (a) 994 mm, (b) 991 mm, (c) 996 mm, (d) 164 mm.
12. (a) 143 cm, (b) 56 cm, (c) 36 cm, (d) 14 cm.

Exercise 12.5

1. $49\cos\theta - 24$, $58{\cdot}8\cos\theta - 19{\cdot}2$; 3·68, 25·8; 61°.
2. $6{\cdot}18\,\mathrm{m\,s^{-1}}$, 19·35 N. 3. $\frac{1}{3}$, $\frac{1}{3}ag$.
5. $mg|3\cos\theta - 2|$; $\frac{2}{3} < \cos\theta < 1$. 6. $\sqrt{(2ag/3)}$.
7. $\frac{mu^2}{a} - 2mg + 3mg\cos\theta$; mg. 8. (ii) $\frac{7}{2}ag$, $\sqrt{(\frac{1}{2}ag)}$.
9. $\cos\alpha\sqrt{(2gl\sin\alpha)}$; no. 10. $\{2ag(1 + \sin\theta)\}^{1/2}$; (b) $\pi/6$, $(3ag)^{1/2}$; $(a/g)^{1/2}$.
11. $\{u^2 - 6ag(1 - \cos\theta)\}^{1/2}$; $\frac{mu^2}{2a} - 3mg(1 + \sin\phi)$.
12. $\frac{2}{3}mg(1 + \cos\theta)$, $\frac{1}{3}mg(5\sin\theta - 4\theta)$.

Exercise 12.6

1. Particle escapes from the earth.
2. $mv\frac{dv}{dx} = Ke^{-\alpha x} - \lambda m$; $\left\{\frac{2K}{\alpha m}\left(1 - e^{-\alpha x}\right) - 2\lambda x\right\}^{1/2}$.
3. $\frac{1}{2}(\mu/a)^{1/2} \leqslant V \leqslant \frac{3}{4}(\mu/a)^{1/2}$.
4. $5\cos 3t + 12\sin 3t + 2$; 15, 39; $t \approx 0{\cdot}967$. 5. $\frac{2k^2}{g}\left(1 - e^{-gt/k}\right) - kt$.
6. $\frac{1}{800}$ s; $\frac{dv}{dt} = -16 \times 10^3(2v)^{1/2}$, $\frac{1}{400}$ s; $v = Ae^{-kt}$, A and k positive constants.

7. (i) $2uT$, (ii) $T\ln 3$, (iii) $4mu^2$. 8. $v = gTe^{-x/gT^2}$

9. $v^2 = u^2\tan\left(\frac{\pi}{4} - \frac{2gz}{u^2}\right)$. 10. (ii) $u\sqrt{(1 - e^{-2kx})}$.

13. (b) (i) $3\pi/8$ s, (ii) $26\frac{2}{3}$ m s^{-1}, (iii) $142\frac{2}{9}$ m s^{-2}, (iv) $\pi/16$ s.

14. $\frac{\pi}{4}$ m/s, $\frac{\pi^2}{8}$ m/s^2; $\frac{\pi\sqrt{3}}{8}$ m/s, $\frac{2}{3}$ s.

15. (i) 12 s, $24/\pi$ m, (ii) $t = 5$; $12\sqrt{2}$ s, $\sqrt{2}$ m s^{-1}.
16. $\ddot{x} = -\omega^2 x$, $2\pi/\omega$; $t = 12$ s, $t = 24$ s. 17. $4u/\pi$.

18. (i) 5·09 m, (ii) 3·60 m, (iii) $\frac{4}{\pi}\sin^{-1}\left(\frac{\pi}{16}\right)$; 2π N, $\mu \geqslant \pi/g$.

19. $\frac{400}{3}$ N; $\frac{2}{\sqrt{3}}$ m/s. 20. $2mg + \frac{2\lambda}{3l}x$. 21. $\sqrt{(\frac{1}{3}gl)}$.

23. $\frac{5\sqrt{2}}{24}l$. 25. $\{ag(2\cos\theta - 1)\}^{1/2}$, $2mg(3\cos\theta - 1)$; $8a/25$.

28. $\sqrt{(3ga/2)}$. 29. $\ddot{\mathbf{r}} = \mathbf{g} - k\dot{\mathbf{r}}$.

Exercise 13.1

1. (a) 1/7, (b) 1/8, (c) 21/13, (d) 7/4, (e) 15/8, (f) 3/4.
2. (a) (5, 0), (b) (−2, −2), (c) (2, −2), (d) $(-\frac{1}{2}, 1\frac{1}{4})$.
3. (a) (12, −5), (b) (6, −19), (c) (−4, −1), (d) $(7, -8\frac{1}{2})$.
4. (a) 1, (b) 2, (c) $\sqrt{2}$, (d) $\sqrt{13}$, (e) 9, (f) $2\sqrt{10}$.
5. (a) 6, (b) 24, (c) 18, (d) $11\frac{1}{2}$, (e) 0, (f) $36\frac{1}{2}$.
6. (a) $2x + y - 1 = 0$, $x - 2y - 2 = 0$. (b) $x + y - 1 = 0$, $x - y - 1 = 0$.
   (c) $x = -\frac{1}{3}$, $y = \frac{1}{3}$. (d) $2x - y + 4 = 0$, $x + 2y - 13 = 0$.
7. 45°, 45°, 90°. 8. 45°, 26° 34′, 108° 26′. 9. (−3/7, 12/7).
10. $(5\frac{5}{18}, 3\frac{1}{2})$. 11. 3. 12. 30° 58′. 13. (2, 3).
14. $(-\frac{1}{2}, -\frac{1}{2})$. 15. (i) (6, 7), (ii) 16/15, (iii) $8\sqrt{2}$, (iv) 80.

Exercise 13.2

4. $x^2 + y^2 - 6x - 2y - 10 = 0$, (5, 5); $x^2 + y^2 - 6x - 2y - 70 = 0$, (−1, −7).
5. (11/5, 13/5).
6. $x^2 + y^2 - 2x - 2y - 23 = 0$, $x^2 + y^2 - 18x - 14y + 105 = 0$, $4x + 3y = 32$, $3x - 4y + 26 = 0$, $3x - 4y = 24$.
7. $x^2 + y^2 - 6x + 10y - 16 = 0$, $x^2 + y^2 + 12x - 14y - 40 = 0$; $3x - 4y = 4$, 10
8. $4x - 3y = 3$.
9. $x^2 + y^2 - 6x + 4y - 12 = 0$; $x^2 + y^2 + 14x - 6y - 42 = 0$.
12. (a) $2x - 2y - 1 = 0$, (b) $x^2 + y^2 - 2y - 1 = 0$.

13. (a) $x^2 + y^2 + x - 4y + 1 = 0$, (b) $x^2 + y^2 - 3y = 0$,
(c) $x^2 + y^2 + 4x - 7y + 4 = 0$.
14. (a) $x^2 + y^2 + x - 4y - 9 = 0$, (b) $5x^2 + 5y^2 + 23x - 29y = 0$,
(c) $2x^2 + 2y^2 - 7y - 23 = 0$.
15. $x + 1 = 0, x + 2\sqrt{6}y = 11 - 2\sqrt{6}, x - 2\sqrt{6}y = 11 + 2\sqrt{6}$.

Exercise 13.3

1. (a) $(3t^2, 6t)$, (b) $(3t, 3/t)$, (c) (t^2, t^3), (d) $(5t/2, 5/2t)$, (e) $(4t^2, 4t^3)$,
(f) $(5t^2/2, 5t)$.
2. (a) $(3\cos\theta, 2\sin\theta)$, (b) $(3\cos\theta, 3\sin\theta)$, (c) $(\sqrt{5}\cos\theta, \sin\theta)$,
(d) $(\cos\theta, \frac{2}{3}\sin\theta)$, (e) $(\cos\theta, 3\sin\theta)$, (f) $(5\cos\theta, 1 + 5\sin\theta)$.
3. $x + t^2y = 8t$; $ty = t^3x + 4 - 4t^4$. 4. $x + 4y = 4, 2y = 8x - 15$.
5. $x + 2\sqrt{3}y = 4, 4\sqrt{3}x - 2y = 3\sqrt{3}$.
7. $4x + y = 4c$ at $(\frac{1}{2}c, 2c)$, $4x + y + 4c = 0$ at $(-\frac{1}{2}c, -2c)$.

9. $x = t, y = \dfrac{3}{2t} - \dfrac{3t^3}{2}$.

10. $x + pqy = a(p + q), x + p^2y = 2ap; \left(\dfrac{2apq}{p+q}, \dfrac{2a}{p+q}\right)$. 14. $xy = 2c^2$.

15. $2y = 3tx - at^3; \dfrac{at^2}{4}, -\dfrac{at^3}{8}; t = 0$.

16. $(3\{p^2 + pq + q^2\}, \frac{9}{2}pq\{p + q\})$; $9y^2 = 12x - 16$.

18. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 2$. 19. $x\sin\theta + y = 2\tan\theta$; $x^2 - y^2 = 4$.

20. (a) $d^2\sin^2\theta$, (b) $-1, 3/2$.

Exercise 13.4

1. (a) $4/5$; $(\pm 4, 0)$; $x = \pm 25/4$, (b) $3/5$; $(\pm 3/2, 0)$; $x = \pm 25/6$,
(c) $\frac{1}{2}\sqrt{3}$; $(\pm\sqrt{3}, 0)$; $x = \pm(4\sqrt{3})/3$, (d) $2/3$; $(\pm 2, 0)$; $x = \pm 9/2$,
(e) $\frac{1}{3}\sqrt{5}$; $(0, \pm\sqrt{5})$; $y = \pm(9\sqrt{5})/5$, (f) $\frac{1}{2}\sqrt{3}$; $(0, \pm\frac{1}{2}\sqrt{3})$; $y = \pm\frac{2}{3}\sqrt{3}$.
2. (a) $5/4$; $(\pm 5, 0)$; $x = \pm 16/5$; $y = \pm\frac{3}{4}x$, (b) $\sqrt{2}$; $(\pm 2\sqrt{2}, 0)$; $x = \pm\sqrt{2}$; $y = \pm x$,
(c) $\frac{1}{2}\sqrt{5}$; $(\pm\sqrt{5}, 0)$; $x = \pm 4/\sqrt{5}$; $y = \pm\frac{1}{2}x$.
3. (a) $\frac{1}{2}\sqrt{3}$, (b) $4/5$, (c) $\frac{1}{3}\sqrt{5}$, (d) $4/5$.

4. (a) $\dfrac{x^2}{16} + \dfrac{y^2}{12} = 1$, (b) $\dfrac{x^2}{225} + \dfrac{y^2}{144} = 1$.

5. $5x^2 + 9y^2 - 40x - 18y + 44 = 0$.
6. (a) $x = 3, y = 0$, (b) $2y = x + 4, 2x + y + 3 = 0$,
(c) $9x + 4y = 20, 8x - 18y = 7$, (d) $2x + y + 12 = 0, 2y = x + 1$.

7. (a) $4xx_1 + 9yy_1 = 36$, (b) $\dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} = 1$, (c) $\dfrac{xx_1}{3} - \dfrac{yy_1}{2} = 1$.

8. $2xy + 2kx + 2ky = 7k^2$.

11. (i) $2ax - 2by = a^2 - b^2$; (ii) $(0, -b)$, $\left(\frac{3\sqrt{3}}{5}b, \frac{4}{5}b\right)$.

12. $a^2 = 2c^2$.

Exercise 13.5

1. 7.
2. $x - 2y + 9 = 0$, $x + 3y = 16y$; $x^2 + y^2 - 2x - 10y + 1 = 0$, $x^2 + y^2 - 7x - 144 = 0$.
3. $\frac{1}{5}|a + 4|$; $a = 1$, $3x + 4y = 7$.
4. $(5, 5)$, $\sqrt{10}$; $3m^2 - 10m + 3 = 0$; $y = \frac{1}{3}x$, $y = 3x$; (i) $53° 8'$, (ii) $x^2 + y^2 + 2x - 14y + 40 = 0$.
5. $2x + y + 4 = 0$, $(2, -8)$, $(-4, 4)$; 108.
6. 2, 5; 3/10, $\pm 2/5$.
7. $(0, 8)$, $4\sqrt{2}$; $(9, -1)$, $5\sqrt{2}$; $(4, 4)$; 167°.
8. $-1, 5$; 3/4; 0, $-8/9$.
9. (i) $3x - 4y = 9$, (ii) $(-1, -3)$, $(\frac{23}{25}, -\frac{39}{25})$, $2x^2 + 2y^2 + x + 8y + 5 = 0$.
10. $(8, 2)$; 16/3.
12. $\left(ct + \frac{c}{t}, ct + \frac{c}{t}\right)$.
13. $-4c^2/a^2$; 1/2; 2:3.
14. $\left(\frac{8}{9}, \frac{16\sqrt{2}}{27}\right)$, $\left(\frac{2}{9}, -\frac{2\sqrt{2}}{27}\right)$.
15. $8\frac{2}{3}$.
16. $x \cos t + y \sin t = 2 \sin t \cos t$; $3\pi/8$.
17. (a) $ay + bx = ab\sqrt{2}$, (b) $(ax - by)\sqrt{2} = a^2 - b^2$; $\frac{1}{4}\pi ab$, πab.
18. 0°, 60°; $(2, 0)$, $(1, 3/2)$.
19. $bx \sec\theta + ay \operatorname{cosec}\theta = ab$; $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \left(\frac{ab}{a^2 - b^2}\right)^2$.
20. $t_1t_2 + t_2t_3 + t_3t_1 = 0$; $-2t$.

Exercise 14.1

1. (a) $5i$ (b) $i\sqrt{5}$, (c) $3 - 4i$, (d) $2 - 3\sqrt{2}i$.
2. (a) ± 2, (b) $\pm 2i$, (c) $\pm i\sqrt{2}$, (d) $\pm\frac{3}{2}i$, (e) $0, \pm i\sqrt{3}$, (f) $\pm 1, \pm i$.
3. (a) $1 \pm i$, (b) $3 \pm i$, (c) $2 \pm 3i$, (d) $\frac{1}{2} \pm i$, (e) $\frac{1}{4}(-3 \pm i\sqrt{7})$, (f) $\frac{1}{2}(1 \pm i\sqrt{3})$.
4. (a) $3 - 5i$, (b) $4 + 7i$, (c) $-3i$, (d) -4, (e) $-1 - i$.
5. (a) $7 + 9i$, $3 + 7i$, (b) $3 - i$, $1 + 7i$, (c) $8 - 4i$, $-2 + 2i$, (d) $2 + 7i$, $-4 - 3i$.
6. (a) $3 + 11i$, (b) $7 - 9i$, (c) $4 - 7i$, (d) 25, (e) $-21 - 20i$, (f) $-2 - 2i$
7. (a) $-i$, (b) i, (c) 1 (d) -1, (e) $-i$, (f) i.
8. (a) $-1 + 2i$, (b) $\frac{1}{5} + \frac{3}{5}i$, (c) $\frac{5}{2} + \frac{1}{2}i$, (d) $\frac{1}{2} - \frac{\sqrt{3}}{2}i$, (e) $-\frac{1}{2}i$, (f) $\frac{4}{25} - \frac{3}{25}i$.
9. (a) $\frac{4}{13}$, (b) $\frac{7}{25} + \frac{1}{25}i$, (c) 6.

10. (a) $a = -1, b = 1$, (b) $a = 2, b = \frac{1}{2}$.
11. (a) $\pm(2 - i)$, (b) $\pm(1 + i)$, (c) $\pm(3 + i)$, (d) $\pm(5 - 2i)$.
12. (a) $-4, 5$, (b) $2, 10$, (c) $0, 16$, (d) $-6, 34$.
13. (a) $(x + i)(x - i)$, (b) $(x + 5i)(x - 5i)$,
    (c) $(3x + 2i)(3x - 2i)$, (d) $(x + 1 + 2i)(x + 1 - 2i)$,
    (e) $(x + 3 + 2i)(x + 3 - 2i)$, (f) $(2x - 3 + 4i)(2x - 3 - 4i)$.
14. $0, 0$; $1, 0$; $-\frac{1}{2}, \frac{1}{2}\sqrt{3}$; $-\frac{1}{2}, -\frac{1}{2}\sqrt{3}$. 15. $x = 4, y = -3$.
16. $2 - i$, $2 + 3i$. 17. $32 + 47i$, $32 - 47i$; $53, 61$.
18. $-5 - 4i$, $1 + 7i$; $-3, -1$.
19. (a) $0, \pm 1$, (b) $\pm(1 + \sqrt{2}), \pm(1 - \sqrt{2})$. 20. $\pm(3 + 2i), 2 + 4i, -1 + 2i$.

Exercise 14.2

5. -2. 6. $3 - i, -3$. 7. $1 - i, -2; 4$. 8. $2 + 3i, -1; -3, 9$.

9. $-i, \frac{1}{2}(-1 \pm \sqrt{5})$. 10. $-1 - i, 2 \pm i$. 11. $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$.

12. $\alpha, \alpha\omega, \alpha\omega^2$; (a) $3, -\frac{3}{2} \pm \frac{3\sqrt{3}}{2}i$, (b) $-2, 1 \pm i\sqrt{3}$,

    (c) $-i, \frac{\sqrt{3}}{2} + \frac{1}{2}, -\frac{\sqrt{3}}{2} + \frac{1}{2}i$.

13. (a) $2, -1 \pm i\sqrt{3}$, (b) $-1, \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$, (c) $0, -\frac{3}{2} \pm i\frac{\sqrt{3}}{2}$.

14. (a) 0, (b) -1, (c) -1. 15. (a) $-\omega$, (b) $1 - \omega$, (c) $\omega - 1$.
16. $a^3 + b^3 + c^3 - 3abc$. 17. $9z^3 - 12z^2 + 6z - 1$. 18. $-1, -1$.

Exercise 14.3

1. (a) $3 + 2i$, (b) $-2 + 4i$, (c) $5i$, (d) $1 - 2i$, (e) -2.
2. (a) 5, (b) $\sqrt{29}$, (c) 3, (d) $\sqrt{2}$, (e) 4.
3. $-2 - i, 2 - i, -2 + i, -1 + 2i, 1 - 2i$. 4. $2i, -2 + 2i, \frac{1}{2} - \frac{1}{2}i$.
5. $6 + 8i, i$. 6. (a) $\pm i$, (b) $\pm(1 - i)$, (c) $\pm(2 + i)$, (d) $\pm(4 + i)$.
8. (a) $k(1 + 2i), k \geqslant 0$, (b) $k(1 + 2i), -1 \leqslant k \leqslant 0$, (c) $k(1 + 2i), k \leqslant -1$.

10. (a) 5, (b) $\sqrt{85}$, (c) $\sqrt{13}$. 12. $\frac{4}{3} + \frac{2\sqrt{2}}{3}i, \frac{2}{3} + \frac{4\sqrt{2}}{3}i$.

Exercise 14.4

1. (a) $2, \pi/6$, (b) $\sqrt{2}, -\pi/4$, (c) $4, \pi/2$, (d) $3, \pi$.
2. (a) $5, -0{\cdot}927$, (b) $\sqrt{5}, 2{\cdot}678$, (c) $\sqrt{10}, -1{\cdot}893$, (d) $\sqrt{34}, -0{\cdot}540$.
3. (a) $5, \pi/6$, (b) $1/2, -\pi/3$, (c) $1/\sqrt{2}, \pi/4$, (d) $\sqrt{2}, 3\pi/4$.
4. (a) $5, \pi/3$, (b) $1, -\pi/2$, (c) $1, -2\pi/3$, (d) $2, -3\pi/4$.

5. (a) $2\sqrt{3}\{\cos(-\pi/3) + i\sin(-\pi/3)\}$, (b) $8\{\cos 0 + i\sin 0\}$,
(c) $2\sqrt{2}\{\cos(-3\pi/4) + i\sin(-3\pi/4)\}$, (d) $1\{\cos(-\pi/2) + i\sin(-\pi/2)\}$
6. (a) $r, \theta - \pi$, (b) $r, \theta + \frac{1}{2}\pi$, (c) $r^2, 2\theta$, (d) $1, 2\theta$.
7. (a) -1, (b) $-\dfrac{\sqrt{3}}{2} + \frac{1}{2}i$, (c) $-\dfrac{1}{\sqrt{2}} - \dfrac{1}{\sqrt{2}}i$, (d) $-\frac{1}{2} + \dfrac{\sqrt{3}}{2}i$,
(e) $-i$, (f) $\frac{1}{2} + \dfrac{\sqrt{3}}{2}i$.
8. (a) $10, \pi/2$; $2, \pi/3$; $20, 5\pi/6$; $5, \pi/6$,
(b) $4, 5\pi/6$; $\sqrt{2}, -\pi/4$; $4\sqrt{2}, 7\pi/12$; $2\sqrt{2}, -11\pi/12$.
9. $\sec\theta, \theta$.
10. $2\sin\frac{1}{2}\theta, \frac{1}{2}(\theta - \pi)$; $2\sin\frac{1}{2}\theta, \frac{1}{2}(\theta + \pi)$; $\frac{1}{2}\operatorname{cosec}\frac{1}{2}\theta, \frac{1}{2}(\pi - \theta)$; $\frac{1}{2}\operatorname{cosec}\frac{1}{2}\theta, \frac{1}{2}(\theta - \pi)$.
11. $2, \pi/3$; $2, \pi/2$. 12. $\pm\pi/2$.
13. (a) $\frac{1}{13} + \frac{5}{13}i$, 1·37; 2·75, $\frac{1}{13}$, (b) $|z|^2 : 1$.

Exercise 14.5

12. $x^2 + y^2 + 4x = 0$. 13. $x^2 + y^2 - 9x - y + 16 = 0$. 14. $x = 2$.
15. $5x + 3y = 14$. 16. $2x^2 + 2y^2 + 25x + 75 = 0$.
17. $5x^2 + 5y^2 - 26x + 8y + 1 = 0$. 18. $x^2 + y^2 = 25$, excluding $(-5, 0)$.
19. $x^2 + y^2 - 2x + 2y = 0$, excluding $(2, 0)$. 20. $5x - 2y + 6 = 0$.
21. $xy = 1$. 22. $x^2 - y^2 = 1$. 23. $x(x^2 + y^2 - 1) = 0$, excluding $(0, 0)$.
24. $y(x^2 + y^2 - 9) = 0$, excluding $(0, 0)$. 36. (a) 9, (b) 3, (c) 3, (d) 4.
37. $5\sqrt{2} + 3, 5\sqrt{2} - 3$. 38. 24, 2. 39. $x^2 + y^2 + 10x + 16 = 0$.
40. (i) $1 \pm i\sqrt{5}$, (ii) $1 + i$.

Exercise 14.6

1. (a) 1, (b) $-i$, (c) $\frac{1}{2} - \dfrac{\sqrt{3}}{2}i$, (d) $-\frac{1}{2} + \dfrac{\sqrt{3}}{2}i$.
2. $2\left(\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right)$; $512 - 512\sqrt{3}i$, $-\dfrac{\sqrt{3}}{256} + \frac{1}{256}i$.
3. $\sqrt{2}\left(\cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}\right)$; $256, -\frac{1}{8}i$. 4. (a) 1, (b) -1.
5. $4\cos^3\theta - 3\cos\theta$, $3\sin\theta - 4\sin^3\theta$, $\dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$.
6. $16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$, $1 - 12\sin^2\theta + 16\sin^4\theta$.
8. $32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$, $32\cos^5\theta - 32\cos^3\theta + 6\cos\theta$,
$\dfrac{6\tan\theta - 20\tan^3\theta + 6\tan^5\theta}{1 - 15\tan^2\theta + 15\tan^4\theta - \tan^6\theta}$.

9. (a) $\frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$,
(b) $\frac{1}{64}(\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta)$,
(c) $\frac{1}{16}(2\cos\theta - \cos 3\theta - \cos 5\theta)$.
10. (a) $\frac{1}{4}(3\sin\theta - \sin 3\theta)$, (b) $\frac{1}{64}(35\sin\theta - 21\sin 3\theta + 7\sin 5\theta - \sin 7\theta)$,
(c) $\frac{1}{64}(3\sin\theta + 3\sin 3\theta - \sin 5\theta - \sin 7\theta)$.
12. (a) $3\pi/8$, (b) $\pi/32$.

Exercise 14.7

1. $\dfrac{x}{x^2+y^2} - \dfrac{y}{x^2+y^2}i$. 2. $3, 1; -3, -1$. 3. $1 + 2i; 5\sqrt{2}; -2, -1$.

4. (i) $\pm(1 - 3i)$, (ii) $2 - 2i, 1 + i$.
5. Sum of squares of diagonals of a parallelogram = sum of squares of sides.

6. $\pm 1, \pm i, \pm\dfrac{1}{\sqrt{2}}(1 - i), -\dfrac{1}{\sqrt{2}}(1 + i)$.

7. (i) $-1 + i\sqrt{3}, 2, 2\pi/3; \dfrac{\sqrt{3}}{2} + \frac{1}{2}i, 1, \pi/6$,

(ii) $1 \pm 2i$; (a) $p^2 = q - 4$, (b) $2p = q + 5$.
8. (a) $-3\sqrt{3} + 5i; 2\sqrt{13}, 2{\cdot}38$ rad, (b) $5, -20$.
9. (i) $-1 - i, 3\pi/4$, (ii) $2 - i, 2; -10$.

10. (b) $-1, \frac{1}{2} \pm \dfrac{\sqrt{3}}{2}i$, (c) $-1 - 3i, \frac{1}{2}(1 - 3\sqrt{3}) + \frac{1}{2}(3 + \sqrt{3})i$,

$\frac{1}{2}(1 + 3\sqrt{3}) + \frac{1}{2}(3 - \sqrt{3})i$.

11. $-\frac{1}{2} \pm \dfrac{\sqrt{3}}{2}i; 1 \pm i\sqrt{3}, \frac{5}{2} \pm \dfrac{\sqrt{3}}{2}i$; 28.

12. (ii) $x^3 - 3abx + a^3 + b^3$; $-\sqrt[3]{2} - \sqrt[3]{4}, -\omega\sqrt[3]{2} - \omega^2\sqrt[3]{4}, -\omega^2\sqrt[3]{2} - \omega\sqrt[3]{4}$.
13. (i) $\pm(3 - 2i)$, (ii) $3 \pm i\sqrt{3}$. 14. (i) (a) $7/5, -4/5$; (b) $2 \pm i$.
15. (i) $1 - i, -1 \pm 2i$, (ii) $\frac{1}{2}(3 + i\sqrt{3}); \pi/6, 2\pi/3$.
16. (i) $2, 32; \pi/3, -\pi/3; -16\sqrt{3}$, (ii) $2 + 3i$. 17. (i) -1, (ii) $1 + 2i, 2\sqrt{5}$.
18. (i) $\sqrt{2}, -\pi/4; \frac{1}{2} + \frac{1}{2}i, 8i$. 20. (i) $2 - i, 3 - 4i; (2, -1)$.
21. (ii) $3x^2 + 3y^2 + 10x + 3 = 0$. 22. (i) $x = 1, y = 2$ or $x = -1, y = -2$.
23. (i) (a) 13, $-23°$, (b) 1, $90°$; (ii) 12; (iii) $|z + 1 + i| \geqslant 4$.
24. $2\sin\theta; \frac{1}{2}\operatorname{cosec}\theta, -\theta; y = -\frac{1}{2}$. 25. $2\sqrt{2} - 2$.

26. $2\cos\frac{1}{2}\theta, \frac{1}{2}\theta; \left[2\cos\left(\dfrac{\theta}{2}\right)\right]^n \cos\left(\dfrac{n\theta}{2}\right) + i\left[2\cos\left(\dfrac{\theta}{2}\right)\right]^n \sin\left(\dfrac{n\theta}{2}\right)$.

Exercise 15.1

1. 10. 2. $3\frac{1}{2}$. 3. 14; 20 p. 4. (a) 1, (b) 9/8.
5. $p(2) = 2/15, p(3) = 8/15, p(4) = 1/3$; 3·2. 6. 0·8 p loss.

7. $\frac{125}{216}, \frac{75}{216}, \frac{15}{216}, \frac{1}{216}$; 2 p loss. 8. (a) £122.50, (b) £112·672.
9. $p(x) = 1/6, x = 1, 2, 3, 4, 5; p(6) = 0; p(x) = 1/36, x = 7, 8, 9, 10, 11, 12; 4\frac{1}{12}$; 6/17.
10. (i) 24/625, (ii) 24/625, (iii) 2. 11. 28·5, 721·05. 12. 2·53, 0·305.
13. $1\frac{1}{2}, \frac{3}{4}; \frac{1}{20}, \frac{9}{20}, \frac{9}{20}, \frac{1}{20}$; 60%.
14.

r	0	1	2	3	4
$P(Y=r)$	0·09	0·24	0·34	0·24	0·09

r	0	1	2	3	4
$P(Z=r)$	0·447	0·232	0·222	0·072	0·027

$E(Z) = 1$, $\text{Var}(Z) = 1{\cdot}2$.

15. $4\frac{1}{3}, 2\frac{2}{9}; 8\frac{2}{3}, 8\frac{8}{9}$. 16. $\frac{1}{2}(n+1)$, 0·42.
17. $2/n(n+1); \frac{1}{3}(2n+1), \frac{1}{18}(n-1)(n+2)$.
18. $x = 1, 2, 3, \ldots, 9$; $p(x) = \frac{9}{45}, \frac{8}{45}, \frac{7}{45}, \ldots, \frac{1}{45}; \frac{11}{3}, \frac{2}{3}\sqrt{11}$; 1;

$x = 1, 2, 3, \ldots$; $p(x) = \dfrac{10 - x}{45}$.

19. (a) $1 + 2x + 3x^2 + \ldots + rx^{r-1} + \ldots$, (b) $p(1-p)^{r-1}$; $1/p$.
20. (i) $\frac{1}{3}$, (ii) $\frac{2}{9}$, (iii) $\frac{16}{243}; \frac{32}{243}$; 50 p; $\frac{16}{81}$; 0·177. 21. 3/16.

Exercise 15.2

1. (a) 0·279, (b) 0·457; 1·5, 1·25. 2. (a) 0·250, (b) 0·526; 2·5, 1·875.
3. 0·657; 7. 4. (a) 0·393, (*b*) 0·000 085 2; $n \geqslant 18$.
5. 24, 0·4; $(0{\cdot}6 + 0{\cdot}4t)^{24}$.
6. $(0{\cdot}55 + 0{\cdot}45t)^4$; (i) 4, (ii) $4(0{\cdot}55)(0{\cdot}45)^3$. 7. 0·228, 0·210.
8. 0·0464; 4, 2·4.
9. $\frac{4}{3}, \frac{356}{405}$,

No. of males	0	1	2	3	4
Expected no. of litters	160	320	240	80	10

10. 0·991, 0·977;

No. of ones	0	1	2	3	4	5	6
Expected no. of throws	335	402	201	54	8	1	0

11. (a) 0·315, (b) (i) 0·53, (ii) 5, 22, 37, 28, 8. 12. 26·4%, 62·4%.
13. (i) 0·201; (ii) (a) 0·0556, (b) 0·0367, (c) 0·6.
14. (i) 4·8, 0·98, (iii) 0·737, (iv) 0·388.
15. (a) $\frac{1}{27}$, (b) $\frac{2}{9}$, (c) $\frac{4}{9}$, (d) $\frac{8}{27}$, (e) 0·299, (f) 0·903, (g) 0·328.

16. (ii) 1/5. 17. $\dfrac{(n-r+1)p}{r(1-p)}$; (a) 8, (b) 2, 3.

18. (a) $\dfrac{135}{2048}$; (b) $\dfrac{459}{512}$; $\binom{n-1}{2}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^{n-3}$; 8, 9.

19. (a) $(r+1)/2^{r+2}$, (b) 2, (c) 4. 20. $t/(6-5t)$; 6, 30.

Exercise 15.3

1. (a) 1/2, 1/12, (b) 3, 4/3, (c) $k/2$, $k^2/12$, (d) $(a+b)/2$, $(b-a)^2/12$.
2. 4/27; 9/5, 2, 3/5. 3. (a) 2·4, (b) 20; 1/3, 0·178.
4. (i) $-3/16$, (ii) 19/80, (iv) 2. 5. 4, 8/15, 11/225; 0·541.
6. 3/8, 0, 2/5; (i) 85/512, (ii) 85/152. 7. (i) 12, 1, (ii) 0·0523.
8. (i) 2/5, (ii) 13/5, (iii) 3/2.
9. 0·6; $F(x) = 0$ for $x \leqslant 0$, $F(x) = 4x^3 - 3x^4$ for $0 \leqslant x \leqslant 1$, $F(x) = 1$ for $x \geqslant 1$.
10. 28, 12, 6, 2, 1, 1.
11. (i) 2/3, (ii) 4/3, $(\sqrt{14})/6$, (iii) $F(x) = 0$ for $x \leqslant 0$, $F(x) = \frac{1}{3}x^2$ for $0 \leqslant x \leqslant 1$, $F(x) = \frac{1}{6}(-3 + 6x - x^2)$ for $1 \leqslant x \leqslant 3$, $F(x) = 1$ for $x \geqslant 3$, (iv) 13/24.
12. (iii) $F(x) = -\frac{1}{12}x^3$ for $x \leqslant -1$, $F(x) = \frac{1}{2} + \frac{1}{2}x - \frac{1}{12}x^3$ for $-1 \leqslant x \leqslant 1$, $F(x) = 1 - \frac{1}{12}x^3$ for $x \geqslant 1$, (iv) $E(X) = 0$, $\mathrm{Var}(X) = 11/15$.
13. (ii) 1, 5/6, (iii) 1/4.
14. (i) $k = 2$, (ii) $f(x) = 2$ for $0 \leqslant x \leqslant 0{\cdot}5$, $f(x) = 0$ otherwise, (iii) 0·25, (iv) 0·144.
15. (i) 3/32, (ii) 11/16, (iii) $f(x) = 3(4 - x^2)/32$ for $-2 \leqslant x \leqslant 2$, $f(x) = 0$ otherwise, (iv) 0, 4/5.
16. $F(x) = 0$ for $x \leqslant 1$, $F(x) = \frac{1}{3}(x^2 - 1)$ for $1 \leqslant x \leqslant 2$, $F(x) = 1$ for $x \geqslant 2$; $G(y) = 0$ for $y \leqslant 1$, $G(y) = \frac{1}{3}(y^4 - 1)$ for $1 \leqslant y \leqslant \sqrt{2}$, $G(y) = 1$ for $y \geqslant \sqrt{2}$; $g(y) = 4y^3/3$ for $1 \leqslant y \leqslant \sqrt{2}$, $g(y) = 0$ otherwise.
17. $1/(2\sqrt{y})$; $(1/\sqrt{w}) - 1$.
18. $f(x) = 1/(b-a)$ for $a \leqslant x \leqslant b$, $\delta f(x) = 0$ otherwise; $F(x) = 0$ for $x \leqslant a$, $F(x) = (x-a)/(b-a)$ for $a \leqslant x \leqslant b$, $F(x) = 1$ for $x \geqslant b$; $P(V \leqslant v) = 3 - 72/v$; (i) $f(v) = 72/v^2$, (ii) 0·4, (iii) 28·8 m.p.h.
19. (a) $a^2/18$, (b) $F(x) = 0$ for $x \leqslant a$, $F(x) = x^2/a^2$ for $0 \leqslant x \leqslant a$, $F(x) = 1$ for $x \geqslant a$, (c) (i) $1/(a+1)^2$, (ii) $2(1-y)$ for $0 \leqslant y \leqslant 1$, 0 otherwise.
20. $g(y) = 1$ for $0 \leqslant y \leqslant 1$, $g(y) = 0$ otherwise.

Exercise 15.4

1. (a) 0·6179, (b) 0·1151, (c) 0·97725, (d) 0·01287, (e) 0·1760, (f) 0·5558.
2. (a) 0·9378, (b) 0·1944, (c) 0·9715, (d) 0·02239, (e) 0·2065, (f) 0·2976.
3. (a) 0·9554, (b) 0·2119, (c) 0·6612.
4. (a) 0·3085, (b) 0·5793, (c) 0·0796.
5. (a) 0·9487, (b) 0·9750, (c) 0·04412
6. (a) 0·44, (b) 0·583, (c) $-0{\cdot}878$.
7. (a) 0·524, (b) 2·326, (c) 3·291.

8. (a) 0·3%, (b) 6·6%, (c) 44%, (d) 1·2%.
9. 11·51%, 81%, 7·49%; £6528.64.
10. (i) 4·8%, (ii) 97·7%; 1111·5 to 1188·5.
11. (a) 37·82%, (b) 125·5 hrs to 194·5 hrs, (c) 0·405.
12. (a) 4 men, (b) 0·0446, (c) 1·746 m.
13. (a) 3, 62·3, (b) 15, 9·24, (c) 84·5, 111, (d) 8·31, 1·74.
14. 0·0122, 0·0102. 15. 104, 38·8.
16. (a) 0·205, 0·205, (b) 0·120, 0·119, (c) 0·0419, 0·0416, (d) 0·190, 0·179, (e) 0·0985, 0·0932, (f) 0·0493, 0·0474.
17. 0·0287. 18. 0·210. 19. (i) 0·026, (ii) 0·932, (iii) 0·923.
20. (a) 4320/15625; (b) 7120/15625; (a) 0·026, (b) 0·853.
21. (a) Negligible, (b) negligible, (c) 0·806.
22. (i) 0·648, (ii) 92·65%, (iii) 0·654.

Exercise 15.5

1. (a) 0·156, (b) 0·663. 2. (a) 0·353, (b) 0·285.
3. 0·181; $e^{-0.02n}$; 5. 4. 0·337; 1536.

5.

r	0	1	2	3	4	5	6	7	8	9	10
(a)				0·001	0·005	0·015	0·037	0·074	0·120	0·160	0·176
(b)			0·002	0·008	0·019	0·038	0·063	0·090	0·113	0·125	0·125
(c)				0·001	0·005	0·015	0·037	0·073	0·119	0·160	0·177
(d)	0·001	0·002	0·005	0·011	0·021	0·036	0·057	0·080	0·103	0·120	0·126

r	11	12	13	14	15	16	17	18	19	20	21
(a)	0·160	0·120	0·074	0·037	0·015	0·005	0·001				
(b)	0·114	0·095	0·073	0·052	0·035	0·022	0·013	0·007	0·004	0·002	0·001
(c)	0·160	0·119	0·073	0·037	0·015	0·005	0·001				
(d)	0·120	0·103	0·080	0·057	0·036	0·021	0·011	0·005	0·002	0·001	

6.

r	0	1	2	3	4	5	6	7	8	9
(a)	0·122	0·270	0·285	0·190	0·090	0·032	0·009	0·002		
(b)	0·135	0·271	0·271	0·180	0·090	0·036	0·012	0·003	0·001	
(c)	0·132	0·223	0·291	0·223	0·101	0·027	0·004			
(d)	0·144	0·217	0·276	0·217	0·106	0·032	0·006	0·001		

7.

r	0	1	2	3	4	5	6	7	8	9
(a)	0·005	0·029	0·078	0·139	0·181	0·185	0·154	0·108	0·064	0·033
(b)	0·007	0·034	0·084	0·140	0·175	0·175	0·146	0·104	0·065	0·036
(c)	0·017	0·033	0·070	0·121	0·167	0·186	0·167	0·121	0·070	0·033
(d)	0·022	0·037	0·073	0·119	0·160	0·177	0·160	0·119	0·073	0·037

r	10	11	12	13
(a)	0·015	0·006	0·002	0·001
(b)	0·018	0·008	0·003	0·001
(c)	0·012	0·004	0·001	
(d)	0·015	0·005	0·001	

8.

r	0	1	2	3	4	5	6	7	8	9
(a)		0·002	0·009	0·025	0·054	0·090	0·123	0·144	0·146	0·129
(b)		0·003	0·011	0·029	0·057	0·092	0·122	0·140	0·140	0·124
(c)	0·003	0·005	0·013	0·027	0·050	0·080	0·112	0·137	0·146	0·137
(d)	0·004	0·007	0·015	0·030	0·052	0·080	0·110	0·132	0·140	0·132

r	10	11	12	13	14	15	16	17	18
(a)	0·102	0·073	0·047	0·028	0·015	0·007	0·003	0·001	0·001
(b)	0·099	0·072	0·048	0·030	0·017	0·009	0·005	0·002	0·001
(c)	0·112	0·080	0·050	0·027	0·013	0·005	0·002	0·001	
(d)	0·110	0·080	0·052	0·030	0·015	0·007	0·003	0·001	

9. (a) 0·368, (b)0·456. 10. (a) 0·258, (b) 0·567.
11. (a) Poisson approximation: 0·442,
(b) Poisson approximation: 0·353,
(c) Normal approximation: 0·119.
14. (i) 0·19, (ii) 0·21, (iii) 0·012.
15. (a) (i) 1·04, (ii) 53, 55, 29, 10, 3; (b) 0·121.
16. 0·647, 0·185. 17. (i) 0·55, (ii) 0·18.
18. £266C, £415C. 19. 0·460; 33.
20. 0·323; 172 to 228.

Exercise 15.6

1. 3·1, 12, 0·33; 23, 5·43. 2. 0·4, 1, 2·2; 0·84, 1·2.
3. $1\frac{2}{3}$, 3, $5\frac{4}{5}$; $20\frac{11}{15}$, $\frac{2}{9}$.
4. $\frac{1}{2}$, $\frac{3}{10}$, $\frac{24}{35}$; $-\frac{1}{5}$, $\frac{73}{2450} \approx 0{\cdot}0298$.
5. (a) 6, 2·5, (b) 4, 6, (c) 27, 40, (d) -3, 24,
(e) 10, 8·5, (f) 2, 8·5, (g) 14, 28·5, (h) 24, 64.
6. (a) 31, 234, (b) 109, 50, 159. 7. 30 g, 0·3 g. 8. 200 g, 6·71 g.
9. (a) 2·1, 0·49; 0·8, 0·36, (b) no; independent,

(c)

xy	0	1	2	3	4	6
$p(xy)$	0·3	0·12	0·32	0·18	0·05	0·03

; yes,

(d)

$x + y$	1	2	3	4	5
$p(x + y)$	0·06	0·27	0·41	0·23	0·03

; yes.

10. (i) (a) 1·2, 0·56; 1·7, 0·81, (b) all values; not independent,

(c)

xy	0	1	2	3	4	6
$p(xy)$	0·27	0	0·45	0·1	0·1	0·08

; no,

(d)

$x+y$	0	1	2	3	4	5
$p(x+y)$	0·03	0·15	0·07	0·47	0·2	0·08

; no.

(ii) (a) 1, 0·6; 2, 0·6, (b) all values, not independent,

(c)

xy	0	1	2	3	4	6
$p(xy)$	0·3	0·2	0·05	0·2	0·2	0·05

; yes,

(d)

$x+y$	1	2	3	4	5
$p(x+y)$	0·05	0·4	0·1	0·4	0·05

; yes.

11. 1, 2, 4, 8. 12. £3.33.
13. $\frac{1}{210}, \frac{4}{35}, \frac{3}{7}, \frac{8}{21}, \frac{1}{14}$; 2·4, 0·8; 1·6, 0·8; 4, 1·13.
14. (i) 0·9, 1·2,

(ii)

z	−2	−1	0	1	2	3	4
$p(z)$	0·1	0·1	0·3	0·2	0·2	0·1	0

,

(iii) 0·2.

15. (i) 0·011, (ii) 0·036, (iii) 0·27. 16. 0·12 mm, 0·0583 mm; 1·98%
17. (a) 0·673, (b) 0·138. 18. (i) 0·089, (ii) 0·661.
19. (i) 0·841, (ii) 0·500, (iii) 0·421; 0·994.
20. (i) 0·823, (ii) 0·059; 1/4, 1/6. 21. $\frac{1}{2}r\pi, \frac{1}{2}r\pi$; $4r/\pi$; 1/3.
22. (i) $\mu_x^2 + \sigma_x^2$; (ii) $\mu_x\mu_y$; $\mu_x^2\sigma_y^2 + \mu_y^2\sigma_x^2 + \sigma_x^2\sigma_y^2$; (iii) $\mu_y(\mu_x^2 + \sigma_x^2)$.
23. $\dfrac{\lambda^r\mu^{n-r}}{r!(n-r)!}e^{-(\lambda+\mu)}$; $\displaystyle\sum_{r=0}^{n}\frac{\lambda^r\mu^{n-r}}{r!(n-r)!}e^{-(\lambda+\mu)}$. 24. 0·121.
25. (b) $(1-p+pt)^m$, $(1-p+pt)^n$; $(1-p+pt)^{m+n}$.

Exercise 15.7

1. $(n+1)/2$, $(n^2-1)/12$; 1/2; (a) 2/105, (b) 16.
2. £16. 3. (a) 0·323, (b) $£1\frac{32}{65} \approx £1.49$.
4. 1/15; 2/3, 34/45; 1/75; 4/3, 68/45.
5. (i) $P(N=r) = \dfrac{1}{k}(1 \leqslant r \leqslant k)$, $E(N) = \frac{1}{2}(k+1)$,

(ii) $P(N=r) = \dfrac{1}{k}\left(\dfrac{k-1}{k}\right)^{r-1}$ $(r \geqslant 1)$, $E(N) = k$; first strategy.

6. (i) 1/5, (ii) 8/9, (iii) 10, (iv) 8. 7. 0·0512.
8. (i) 0·168, (ii) 0·360, (iii) 0·309, (iv) 0·132, (v) 0·028, (vi) 0·002; 0·947.

9.

No. of defectives	0	1	2	3	4	5	6
Expected frequency	16	42	45	26	8	1	0

10. $\frac{1}{p}, \frac{1-p}{p^2}; \frac{2t^2}{(5-3t)(5-4t)}; \frac{194}{625} = 0{\cdot}3104.$

11. (i) 2, (ii) $1\frac{1}{2}$, (iii) $1\frac{5}{12}$,
(iv) $F(x) = 0$ for $x \leqslant 0$, $F(x) = x - \frac{1}{2}x^2$ for $0 \leqslant x \leqslant 1$, $F(x) = \frac{1}{2}$ for $1 \leqslant x \leqslant 2$, $F(x) = \frac{1}{2}x^2 - 2x + \frac{5}{2}$ for $2 \leqslant x \leqslant 3$, $F(x) = 1$ for $x \geqslant 3$, (v) $\frac{3}{4}$.
12. (ii) $\sqrt{(3/5)}$. 13. 1, 1/6; 3/4; 5/6.
14. (a) 3/250, $6\frac{7}{8}$, (b) (1) £3.20, (ii) 0·792.
15. (i) 1/2, −1/4, (ii) 7/16,
(iii) $f(x) = \frac{1}{2}x$ for $0 \leqslant x \leqslant 1$, $f(x) = \frac{1}{2}$ for $1 \leqslant x \leqslant 2$, $f(x) = \frac{3}{2} - \frac{1}{2}x$ for $2 \leqslant x \leqslant 3$, $f(x) = 0$ otherwise; 3/2; 5/12.
16. (a) $1 - e^{-\lambda x}$, (b) $\frac{1}{\lambda}, \frac{1}{\lambda^2}$, (c) $\frac{1}{\lambda}\ln 2$ (d) $\frac{1}{\lambda}\ln\left(\frac{20}{19}\right)$.
17. (i) 0·202, (ii) 0·410.
18. (i) 0·880, (ii) 9·69, 0·2339, (iii) 0·180.
19. £9360, 19·6%; (a) £20 000, £9360; (b) £19 800, £10 296.
20. (i) 0·60, (ii) 0·20, (iii) 0·95, (iv) 0·50.
21. (i) 0·227, (ii) 54·7, (iii) 78·7 cm^2, (iv) 0·579.
22. 1·514 mm, 0·051 mm; 3·028 mm, 0·072 mm; 0·895; 3·214 mm.
23. (i) 0·345, (ii) 0·692; 0·0033; 0·304.

24. (a) (i) 0·5, (ii)

No. of accidents	0	1	2	3	4	5
Expected no. of days	121	61	15	3	0	0

(b) Binomial model: 0·932, Poisson approximation: 0·929.
25. (i) 9, (ii) 26, (iii) 43; 8.
26. (a) (i) 0·0581, (ii) 0·0519; (b) $(0{\cdot}96)^n$, 74; (c) 0·955.
27. (i) 0·135, (ii) 0·271, (iii) 0·0290, (iv) 2, (v) 8·19.
28. (i) 0·419, (ii) 0·676.
29. (a) (i) 0·0952, (ii) 0·0905; (b) 0·853; (c) 0·632; 63·2, 4·82.
30. 1/12; 0, $n/12$; 16. 31. (b) 2·2.
32. (i) $F(x) = 0$ for $x \leqslant 1$, $F(x) = 1 - x^{-3}$ for $x \geqslant 1$;
(ii) 3/2, 3/4; p.d.f. of Y: $g(y) = 3y^2$ for $0 \leqslant y \leqslant 1$, $g(y) = 0$ otherwise.
33. (ii) 8/15,
(iii) $F(x) = 0$ for $x \leqslant 0$, $F(x) = x^2(2 - x^2)$ for $0 \leqslant x \leqslant 1$, $F(x) = 1$ for $x \geqslant 1$;
(iv) $g(y) = 8y^3(1 - y^4)$ for $0 \leqslant y \leqslant 1$, $g(y) = 0$ otherwise.
34. (a) (i) 5/2, (ii) 7,
(iii) $F(x) = 0$ for $x \leqslant 1$, $F(x) = \frac{1}{3}(x^2 - 1)$ for $1 \leqslant x \leqslant 2$, $F(x) = 1$ for $x \geqslant 2$;
(b) (i) 5/2, 3/4, (ii) interval from 1 to 4.

Exercise 16.1

1. 1, 0·8;

sample	0,0	0,1	0,2	0,3	1,1	1,2	1,3	2,2	2,3	3,3
probability	0·09	0·30	0·06	0·06	0·25	0·10	0·10	0·01	0·02	0·01
$\bar{X}$	0	0·5	1	1·5	1	1·5	2	2	2·5	3
V	0	0·25	1	2·25	0	0·25	1	0	0·25	0

$\bar{x}$	0	0·5	1	1·5	2	2·5	3
$p(\bar{x})$	0·09	0·30	0·31	0·16	0·11	0·02	0·01

v	0	0·25	1	2·25
$p(v)$	0·36	0·42	0·16	0·06

2.

$\bar{x}$	0	1/3	2/3	1	4/3	5/3	2	7/3	8/3	3
$p(\bar{x})$	0·027	0·135	0·252	0·242	0·174	0·108	0·040	0·018	0·003	0·001

v	0	2/9	2/3	8/9	14/9	2
$p(v)$	0·154	0·456	0·120	0·126	0·108	0·036

3. 2, 4·8;

sample	A,B	A,C	A,D	A,E	B,C	B,D	B,E	C,D	C,E	D,E
$\bar{X}$	1	1	4	2	0	3	1	3	1	4
V	1	1	4	0	0	9	1	9	1	4

$\bar{x}$	0	1	2	3	4
$p(\bar{x})$	0·1	0·4	0·1	0·2	0·2

v	0	1	4	9
$p(v)$	0·2	0·4	0·2	0·2

2, 1·8, 3; 2, 2·4, 2·4.

4.

$\bar{x}$	2/3	4/3	2	8/3	10/3
$p(\bar{x})$	0·2	0·2	0·1	0·4	0·1

v	8/9	32/9	56/9	8
$p(v)$	0·4	0·1	0·4	0·1

2, 0·8, 4; 2, 1·6, 3·2.

5. (a) 0·096, (b) 0·096. 6. 0·623, 0·789. 7. 0·332, 0·0587; 0·0092.
8. (a) (i) 1·75, 0·01, (ii) 1·65, 0·0225; (b) 0·1, 0·180.
9. $N(-2, 1{\cdot}5)$; (a) 0·949, (b) 0·200.
10. (a) 28, (b) 822. 11. $1{\cdot}645/\sqrt{n}$; 215.
12. (a) 0·463 using binomial distribution,
(b) 0·336 using normal approximation with continuity correction,
(c) 0·185 without continuity correction (0·174 with correction),
(d) 0·0023 without continuity correction.
13. (a) 0·608, (b) 0·834, (c) 0·996.

Exercise 16.2

1. (a) 4·1, (b) 2·66, (c) 0·75. 2. (a) 43·2, (b) 68·0, (c) 0·375.
3. 2·41, 2·34; 2·41, 2·32. 4. 59·7, 249; 59·7, 225.
5. 2·07, 1·83; 0·35. 6. $\theta/2$, $\theta^2/12$; 6·2, 39·2.
7. $3\theta/4$, $3\theta^2/80$; 9·6, 122.
8. (a) I: 44·5, 373; II: 42, 233·25; 43·25, 346; (b) 43·25, 325.
9. (a) I: 9·25, 12·7; II: 9, 10·7; 9·17, 13·5; (b) 9·17, 12·7.
10.

$\{X_1, X_2\}$	$\{0,0\}$	$\{0,1\}$	$\{0,2\}$	$\{1,1\}$	$\{1,2\}$	$\{2,2\}$
Probability	$\frac{1}{4}\theta^2$	$\theta - \theta^2$	$\frac{1}{2}\theta^2$	$(1-\theta)^2$	$\theta - \theta^2$	$\frac{1}{4}\theta^2$

$(X_1 - X_2)^2$	0	1	4
Probability	$1 - 2\theta + \frac{3}{2}\theta^2$	$2\theta - 2\theta^2$	$\frac{1}{2}\theta^2$

;
$\frac{1}{2}\theta + \frac{1}{2}\theta^2$; Z, $\mathrm{Var}(Z) = \frac{1}{2}\theta - \frac{1}{2}\theta^2 < \mathrm{Var}(Y)$.

11. $\frac{r_1}{n_1}$, $\sqrt{\left\{\frac{p(1-p)}{n_1}\right\}}$; $\sqrt{\left\{\frac{p(1-p)(n_1+n_2)}{4n_1n_2}\right\}}$; $\frac{1}{3} < \frac{n_1}{n_2} < 3$.

Exercise 16.3

1. (a) 836·1 g, 847·9 g, (b) 840·1 g, 845·9 g. 2. (a) 3·10, 3·42, (b) 3·05, 3·47.
3. (a) 439 g, 497 g, (b) 451·3 g, 469·7 g.
4. (a) 0·083, 0·177, (b) 0·075, 0·185.
5. (a) 3·81 m, 4·09 m, (b) 3·78 m, 4·12 m, (c) 0·059, 0·128, (d) 0·050, 0·137.
6. 0·774, 0·959; 178. 7. (a) 36, (b) 50.
8. (a) 235·8 ml, 246·2 ml, (b) 236·1 ml, 249·9 ml, (c) −10·2 ml, 14·2 ml.
9. (a) 0·741 kg, 0·879 kg, (b) 0·835 kg, 0·945 kg, (c) 0·0391 kg^2, (d) 0·014 kg, 0·146 kg.
10. (i) 34%, (ii) 29·4%, 38·6%, (iii) 2156. 11. 26·9, 8·65; 25·7, 28·1.
12. [−294 g, 294 g].

Exercise 16.4

1. 0·045; (a) no, (b) yes.
2. Reject claim if number of passes is less than (a) 6, (b) 5, (c) 4; true significance level (a) 3·81%, (b) 0·50%, (c) 0·04%.
3. Reject hypothesis if cricketer scores more than 30 runs in 0, 6 or 7 innings; 0·047.
4. Yes. 5. (a) No, (b) Yes.
6. Approximately normal with mean 1·65 m, s.d. 0·007 m; yes, significant at 5% level.
7. No, not significant at 5% level.

8. (a) Reject hypothesis, (b) accept hypothesis.
9. Strong evidence against supposition (probability of more than 290 successes by guesswork is less than 0·002).
10. (a) No, (b) No.
11. (a) Males: $\mu_m = 1{\cdot}65$, variance = 0·0025; females: $\mu_f = 1{\cdot}55$, variance = 0·00375;
    (b) 0·103;
    (c) H_0: $\mu_m - \mu_f = 0{\cdot}08$, H_1: $\mu_m - \mu_f > 0{\cdot}08$; may reject H_0 at 5% level and at 1% level.
12. 4·41; [9·87, 10·73]; 3·61; H_0: $\mu_x - \mu_y = 0$, H_1: $\mu_x > \mu_y$; evidence not significant at 3% level.
13. [2·60, 3·12]; 0·567; yes, results significant.
14. [0·270, 0·355]; 0·25; accept hypothesis that proportion of serious injuries is greater amongst drivers not wearing seat belts.
15. H_0: $\pi_A - \pi_B = 0$, H_1: $\pi_A - \pi_B > 0$; accept theory of experts; 0·569, 0·711.
16. H_0: $\pi = 0{\cdot}25$, H_1: $\pi > 0{\cdot}25$, where π is probability of correctly identifying the suit of a card; 5·08%; 0·181.
17. Accept H_0 if fewer than 59 germinate; 0·380.

Exercise 16.5

1. $N(\mu, \sigma^2/n)$; 0·106; 1240 units.
2. (a) Approximately normal with mean 283 g, s.d. 2·3 g; (b) 1, 0·0558.
3. $N(54, 64)$; 0·023, 15·7; $N(5, 144)$; 0·662.
4. 46·8;

m	46	$46\frac{1}{3}$	$46\frac{2}{3}$	47	$47\frac{1}{3}$
$p(m)$	1/120	3/20	2/5	19/60	1/8

   0·3055.
5. 0·9723, 0·9877.
6. 105, 78·1; [84·8, 125·2].
7. [1·550, 1·650] using unbiased estimate of variance (or [1·551, 1·649] using sample variance); [0·102, 0·298] in both cases.
8. (i) 0·833,
   (ii) 0·180 using binomial distribution (0·181 using normal approximation),
   (iii) H_0: $p = 0{\cdot}5$, H_1: $p \neq 0{\cdot}5$; conclusion: $p \neq 0{\cdot}5$.
9. (a) 10·56%, 6·68%, (b) £98.88, (c) yes, using one-tailed test.
10. Accept claim, since probability of 7 or more successes is less than 5%; stronger evidence to support claim, since probability of 13 or more successes is approximately 1%.
11. 11·6, 1·89; would not reject at 5% level; H_0: mean = 12, H_1: mean > 12; $12 + 4{\cdot}652/\sqrt{n}$.
12. (i) 80, 16, (ii) 0·8192, (iii) H_0: cure rate = 80%, H_1: cure rate > 80%; 6·92%; 0·608.
13. 8·00667, 0·000175; accept alternative hypothesis; accept alternative hypothesis.
14. (a) [341·07, 341·33]; (b) not significantly different.
15. (a) Yes, (b) Yes.
16. (ii) [0·476, 0·524].
17. 0·590; (i) no, (ii) $r \leqslant 2$.
18. 35, 3·028.
19. 64, 452·84.
20. 1·9, 0·837.

Exercise 17.1

1. High positive correlation. 2. Negative correlation.
3. Probably uncorrelated. 4. Fairly low positive correlation.
5. No linear correlation, but possibly some non-linear relationship.

Exercise 17.2

1. 0·994. 2. −0·816. 3. 0·050. 4. 0·661. 5. −0·032.
6. (a) −7·78, (b) −0·659.
7. (a) 14, 1·02; 6·05, 1·07, (b) −0·75, (c) −0·683. 8. 0·878.
10. (a) $r(x, z) = 1$, (b) $r(x, z) = 1$, (c) $-1 \leqslant r(x, z) \leqslant 1$.
13. $\left(\sum u^2\right)\lambda^2 + 2\left(\sum uv\right)\lambda + \sum v^2$ 14. 0, 0. 15. 0·04, 0·059. 16. 0, 0.

Exercise 17.3

1. (a) 0·429, (b) 0·543, (c) −0·314. 2. (a) 0·333, (b) 0·467, (c) −0·2.
3. P, Q: 0·75; Q, R: 0·567; P, R: 0·317.
4. P, Q: 0·556; Q, R: 0·444; P, R: 0·222.
5. 1, 1. 6. −0·858, −0·711. 7. −0·15, −0·111.
8. 0·530, 0·444. 9. 0·317, 0·167. 10. −0·049, 0, 0·048.
11. −0·781; −1, −1. 12. −0·922, −0·8.

Exercise 17.4

1. $y = 114{\cdot}38 - 1{\cdot}45x$; (a) no, (b) yes: 61.
2. $y = 32{\cdot}05 + 0{\cdot}73x$; first case: 78.
3. (ii) (22, 380), (iv) $y = 22 + 16x$; £422.
4. $y = -5{\cdot}35 + 0{\cdot}47x$; $x = 11{\cdot}81 + 2{\cdot}10y$.
5. $y = 18{\cdot}63 - 0{\cdot}76x$; $x = 19{\cdot}93 - 0{\cdot}88y$.
6. $y = 240{\cdot}82 + 0{\cdot}051x$; $x = 248{\cdot}84 + 0{\cdot}049y$.
7. $y = 0{\cdot}798 + 0{\cdot}020x$; $x = -9{\cdot}35 + 22{\cdot}22y$.
8. $y = 46576 - 0{\cdot}78x$; $x = 1841{\cdot}5 - 0{\cdot}0013y$.
9. (a) $r\sqrt{\left(\frac{S_{yy}}{S_{xx}}\right)}$, (b) $\frac{1}{r}\sqrt{\left(\frac{S_{yy}}{S_{xx}}\right)}$; "$y$ on x" line: $y = \bar{y}$; "x on y" line: $x = \bar{x}$.

Exercise 17.5

1. (a) −0·930, (b) $r_S = 0{\cdot}143$, $r_K = 0$.
2. (a) $r_S = 0{\cdot}143$, $r_K = 0{\cdot}071$, (b) 0·152.
3. (a) $r_S = 0{\cdot}6$, $r_K = 0{\cdot}4$, (b) $x = -1{\cdot}17 + 1{\cdot}23y$.
4. $y = 2{\cdot}90 + 0{\cdot}56x$; 0·792. 5. 0·603; $W = -75{\cdot}90 + 0{\cdot}89h$.

6. (a) 0·3, 1·65, (c) $y = 1{\cdot}257 + 1{\cdot}31x$, (e) 3·42 tonnes. 7. $-0{\cdot}036$.
8. (a) 2·5, 2·45, (b) 0·85, 0·4475, (c) 0·425, (d) $y = 1{\cdot}2 + 0{\cdot}5x$, (e) 2·95.
9. 0·309. 10. (ii) $y = 48{\cdot}35 + 2{\cdot}75x$, (iii) 0·787.
11. (i) $-0{\cdot}374$, (ii) $-0{\cdot}256$, (iii) scatter diagram suggests non-linear relationship.
12. (i) $y = -2{\cdot}40 + 0{\cdot}49x$, (ii) 0·997, (iii) no.
13. (i) $y = 5680 + 1{\cdot}33x$, (ii) 28300.
14. (a) Invalid, (b) invalid, (c) valid, (d) invalid.
15. (i) $y = 16 + 0{\cdot}4x$, (ii) 0·330 to 0·470; 6·60 to 9·40.

Exercise 18.1

1. (a) (3, 0, 2), (0, 2, 0), (b) (2, −1, −1), (0, 2, 0), (c) (5, −10, 0), (2, 5, −4).
2. (a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, (b) $\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$,

   (c) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, (d) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$,

   (e) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$.
3. (a) 90° rotation about z-axis, $(0, 1, 0) \to (1, 0, 0)$. (b) Central inversion.
   (c) Orthogonal projection onto x, z plane. (d) Reflection in x, z plane.
   (e) 120° rotation about line $x = y = z$, $(1, 0, 0) \to (0, 0, 1)$.
   (f) Two-way stretch, × 3, in both x- and y-directions.
4. (a) Rotation through θ about z-axis, from x-axis towards y-axis.
   (b) Reflection in plane $y = z$.
   (c) Shear with y, z plane invariant, $(1, 0, 0) \to (1, 0, 1)$.
   (d) 120° rotation about line $x = -y = -z$, $(0, 1, 0) \to (0, 0, 1)$.
   (e) 180° rotation about line $x = z$, $y = 0$.
   (f) Mapping onto plane $x = y$.
5. (a) $\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, (b) $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

   (c) $\begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$ or $\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$,

   (d) $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, (e) $\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

6. $\begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix}, \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$; **M**: 90° rotation about y-axis, $(1,0,0) \to (0,0,1)$, $\mathbf{M}^2$: 180° rotation about y-axis, $\mathbf{M}^4$: identity transformation.

7. $\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}, \begin{pmatrix}0&0&-1\\0&1&0\\-1&0&0\end{pmatrix}$; **P**: reflection in y, z plane, **Q**: 90° rotation about y-axis, $(1,0,0) \to (0,0,-1)$,

QP: reflection in plane $x = z$, **PQ**: reflection in plane $x + z = 0$.

8. (a) $\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix}$, (b) $\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}$, (c) no inverse,

(d) $\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}$, (e) $\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$, (f) $\begin{pmatrix}\frac{1}{3}&0&0\\0&\frac{1}{3}&0\\0&0&1\end{pmatrix}$.

9. (a) $\begin{pmatrix}\cos\theta&\sin\theta&0\\-\sin\theta&\cos\theta&0\\0&0&1\end{pmatrix}$, (b) $\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$, (c) $\begin{pmatrix}1&0&0\\0&1&0\\-1&0&1\end{pmatrix}$,

(d) $\begin{pmatrix}0&-1&0\\0&0&1\\-1&0&0\end{pmatrix}$, (e) $\begin{pmatrix}0&0&1\\0&-1&0\\1&0&0\end{pmatrix}$, (f) no inverse.

10. $\mathbf{r} = \lambda(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$; the plane $2x - 5y + 3z = 0$.
11. $\mathbf{r} = \lambda(\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) + \mu(3\mathbf{i} + 5\mathbf{j} - 2\mathbf{k})$; $x - y - z = 0$.

12. $\begin{pmatrix}-2&2&-2\\-4&4&-4\\-2&2&-2\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$; **M**: mapping onto plane $x - y + z = 0$, $\mathbf{M}^2$: mapping onto line $\mathbf{r} = \lambda(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$. $\mathbf{M}^3$: mapping onto origin.

Exercise 18.2

1. (a) $\begin{pmatrix}3&1&-3\\-2&-3&-2\\5&-1&1\end{pmatrix}$, (b) $\begin{pmatrix}1&2&-11\\6&-6&11\\10&3&-8\end{pmatrix}$, (c) $\begin{pmatrix}0&-4&9\\-6&1&-11\\-5&-1&-3\end{pmatrix}$,

(d) $\begin{pmatrix}-7&-1&5\\19&2&-11\\-10&-3&3\end{pmatrix}$, (e) $\begin{pmatrix}1&-1&3\\-2&3&-8\\2&-2&7\end{pmatrix}$.

2. $\begin{pmatrix}16&0&0\\0&16&0\\0&0&16\end{pmatrix}$; $\frac{1}{16}\begin{pmatrix}1&3&0\\2&-1&3\\-1&0&1\end{pmatrix}$, $\frac{1}{16}\begin{pmatrix}1&3&-9\\5&-1&3\\1&3&7\end{pmatrix}$.

3. $\begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; $\begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$.

4. $\begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$; $\frac{1}{3}\begin{pmatrix} -4 & -2 & 1 \\ 1 & -1 & -1 \\ -1 & -2 & -2 \end{pmatrix}$.

5. $\begin{pmatrix} 1 & -3 & 2 \\ -2 & 0 & 1 \\ 0 & 2 & 3 \end{pmatrix}$, $\begin{pmatrix} 3 & 0 & -2 \\ -1 & 4 & 1 \\ 2 & -1 & 0 \end{pmatrix}$

6. $\begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}$; $\frac{1}{9}\mathbf{P}^T$.

7. (a) -5, (b) 11, (c) -11, (d) 0, (e) 142, (f) -35.
8. (a) 0, (b) -354, (c) 0, (d) 840, (e) $-20\,160$, (f) 0.
9. (a) $-a - b + 2c$, (b) $13a + 5b - c$, (c) $a + 2b - 5c$.
10. (a) 5, (b) 28, (c) -9. 11. 6, 6, -2.

Exercise 18.3

1. $\begin{pmatrix} 2 & 1 & 1 \\ -9 & -4 & -6 \\ 6 & 3 & 4 \end{pmatrix}$, $\begin{pmatrix} 2 & 1 & 1 \\ -9 & -4 & -6 \\ 6 & 3 & 4 \end{pmatrix}$.

2. $\begin{pmatrix} -5 & -3 & -2 \\ 15 & 8 & 6 \\ 7 & 4 & 3 \end{pmatrix}$, $\begin{pmatrix} 5 & 3 & 2 \\ -15 & -8 & -6 \\ -7 & -4 & -3 \end{pmatrix}$.

3. $\begin{pmatrix} 4 & 8 & -4 \\ 14 & 28 & -14 \\ 6 & 12 & -6 \end{pmatrix}$, no inverse.

4. $\begin{pmatrix} -4 & -1 & 8 \\ 5 & -1 & -1 \\ 3 & 3 & -6 \end{pmatrix}$, $\frac{1}{9}\begin{pmatrix} -4 & -1 & 8 \\ 5 & -1 & -1 \\ 3 & 3 & -6 \end{pmatrix}$.

5. $\begin{pmatrix} -8 & -9 & 5 \\ 6 & 3 & 0 \\ -13 & -9 & 10 \end{pmatrix}$, $\frac{1}{15}\begin{pmatrix} 8 & 9 & -5 \\ -6 & -3 & 0 \\ 13 & 9 & -10 \end{pmatrix}$.

6. $\begin{pmatrix} 6 & -2 & 7 \\ -2 & -1 & -4 \\ -3 & 1 & -6 \end{pmatrix}$, $\frac{1}{5}\begin{pmatrix} 6 & -2 & 7 \\ -2 & -1 & -4 \\ -3 & 1 & -6 \end{pmatrix}$.

7. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, no inverse.

8. $\begin{pmatrix} -9 & 9 & 0 \\ -7 & 4 & 2 \\ 17 & -11 & -1 \end{pmatrix}, \frac{1}{9}\begin{pmatrix} -9 & 9 & 0 \\ -7 & 4 & 2 \\ 17 & -11 & -1 \end{pmatrix}$.

9. $\begin{pmatrix} -11 & -3 & -1 \\ -1 & -6 & -2 \\ -8 & -6 & 5 \end{pmatrix}, \frac{1}{21}\begin{pmatrix} 11 & 3 & 1 \\ 1 & 6 & 2 \\ 8 & 6 & -5 \end{pmatrix}$.

10. $\begin{pmatrix} -2 & 12 & -2 \\ -2 & 0 & 2 \\ 2 & -4 & 2 \end{pmatrix}, \frac{1}{4}\begin{pmatrix} -1 & 6 & -1 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{pmatrix}$.

11. $\begin{pmatrix} 14 & 0 & 7 \\ 56 & 0 & 28 \\ 14 & 0 & 7 \end{pmatrix}$, no inverse.

12. $\begin{pmatrix} -25 & -32 & -10 \\ -10 & -13 & -4 \\ -18 & -23 & -7 \end{pmatrix}, \begin{pmatrix} 25 & 32 & 10 \\ 10 & 13 & 4 \\ 18 & 23 & 7 \end{pmatrix}$.

13. $\begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 5 & 6 & -15 \\ -7 & -8 & 21 \\ -1 & -2 & 5 \end{pmatrix}$.

Exercise 18.4

1. $\begin{pmatrix} -1 & -1 & 1 \\ 2 & 3 & -1 \\ -2 & -2 & 1 \end{pmatrix}$; (a) $-1, 4, -3$, (b) $2, -2, 1$, (c) $0, 0, 0$.

2. $\frac{1}{10}\begin{pmatrix} 5 & 3 & -1 \\ -5 & 1 & 3 \\ -5 & -5 & 5 \end{pmatrix}$; (a) $1, -1, -2$, (b) $1, 0, -2$, (c) $\frac{1}{5}, \frac{7}{5}, 0$.

3. (a) $2, 1, -10$, (b) $-3, 2, 3/2$, (c) $1, -1/3, -2/3$,
   (d) $1/4, 3/4, -7/4$, (e) $-1, 0, 2$, (f) $-2/7, 5/7, 13/7$.
4. (a) No solution, (b) $\lambda, 3 - 2\lambda, 1 + 3\lambda$.
5. (a) No solution, (b) $\lambda, 4\lambda - 7, 2 + 5\lambda$.
6. (a) $\lambda, \mu, -5 - 4\lambda + 3\mu$, (b) no solution
7. (a) $\lambda, -\frac{1}{3}(4 + \lambda), \lambda$, (b) $\lambda, \mu, \frac{1}{2}(\lambda - 3\mu - 4)$.
8. $\lambda, 7\lambda - 10, 5\lambda - 7$.
9. (a) $k = 0$; $2, \lambda, \lambda - 1$, (b) $k = -3/2$; $\lambda, \frac{1}{4}(7 - 10\lambda), \frac{1}{4}(3 + 2\lambda)$.
10. $4 - 4k$; $k = 1$; $\lambda, 2\lambda - 5, \lambda - 2$.

11. (i) Three planes meeting in a single point;
(ii) each plane parallel to the line of intersection of the other two;
(iii) three planes with common line of intersection.
12. $1, 2, -1; -5; -5, -11; 3\lambda - 5, \lambda, 5\lambda - 11$.

Exercise 18.5

1. (a) $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & -1 \\ -1 & -1 & 1 \end{pmatrix}$, (b) $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$,

(c) $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$, (d) $\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$.

2. (a) $\begin{pmatrix} 20 & 5 & -2 \\ 11 & 3 & -1 \\ -8 & -2 & 1 \end{pmatrix}$, (b) $\begin{pmatrix} -10 & -7 & 2 \\ -6 & -4 & 1 \\ -7 & -5 & 1 \end{pmatrix}$,

(c) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, (d) $\begin{pmatrix} 7 & 4 & 2 \\ 3 & 2 & 1 \\ -5 & -3 & -1 \end{pmatrix}$.

3. (a) $\begin{pmatrix} -3 & 2 & 1 \\ 9 & -5 & -2 \\ -5 & 3 & 1 \end{pmatrix}$, (b) $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ -3 & 1 & -7 \end{pmatrix}$,

(c) no inverse, (d) $\begin{pmatrix} 0 & 0 & 1 \\ -3 & 1 & -11 \\ 1 & 0 & 3 \end{pmatrix}$.

4. (a) $\frac{1}{2}\begin{pmatrix} -2 & 0 & 1 \\ -4 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix}$, (b) no inverse,

(c) $\frac{1}{6}\begin{pmatrix} 2 & -4 & 7 \\ 0 & 6 & -12 \\ 0 & 0 & 3 \end{pmatrix}$, (d) $\frac{1}{2}\begin{pmatrix} 1 & -2 & -1 \\ -4 & 22 & 8 \\ 2 & -12 & -4 \end{pmatrix}$.

5. (a) $\begin{pmatrix} 0 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{pmatrix}$, (b) $\frac{1}{2}\begin{pmatrix} 1 & -3 & 0 \\ -1 & 3 & 1 \\ 0 & -2 & 0 \end{pmatrix}$,

(c) $\frac{1}{3}\begin{pmatrix} -4 & 0 & 1 \\ 3 & 0 & 0 \\ 14 & 3 & -5 \end{pmatrix}$, (d) no inverse

6. (a) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, (b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$, (c) $\begin{pmatrix} 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 1 & 0 & 0 \end{pmatrix}$, (d) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$.

7. $\mathbf{A}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -5 & 2 & 1 \end{pmatrix}$, $\mathbf{B}^{-1} = \frac{1}{4}\begin{pmatrix} 2 & 2 & 10 \\ 1 & -1 & -5 \\ -1 & 1 & 1 \end{pmatrix}$.

8. $\begin{pmatrix} -16 & 9 & 4 \\ -5 & 3 & 1 \\ 4 & -2 & -1 \end{pmatrix}$, $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$;

$\begin{pmatrix} -11 & 6 & 3 \\ -9 & 5 & 2 \\ 4 & -2 & -1 \end{pmatrix}$, $\begin{pmatrix} -16 & 41 & -30 \\ -5 & 13 & -10 \\ 4 & -10 & 7 \end{pmatrix}$.

Exercise 18.6

1. (a) Reflection in the plane $y + z = 0$.
   (b) 180° rotation about z-axis and enlargement × 2.
   (c) 30° rotation about y-axis, from x-axis to z-axis.

2. (a) $\begin{pmatrix} \cos 2\theta & \sin 2\theta & 0 \\ \sin 2\theta & -\cos 2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$, (b) $\begin{pmatrix} \cos 2\theta & \sin 2\theta & 0 \\ \sin 2\theta & -\cos 2\theta & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

3. $(x, y, -z), (-y, x, z), (y, x, -z)$;

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$;

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ 180° rotation about x-axis,

$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ 180° rotation about y-axis;

reflection in x, z plane, reflection in y, z plane.

4. $3x - 2y + z = 0$; the line $\mathbf{r} = \lambda(9\mathbf{i} + 3\mathbf{j} - 2\mathbf{k})$.
5. The line $x = 1$, the line $y = -x$; anti-clockwise rotation through 90° about $(1, -1)$.

6. (a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, (b) $\begin{pmatrix} 1 & -1 & -1 \\ -1 & 3 & -1 \\ 0 & 1 & 3 \end{pmatrix}$.

7. (a) $\mathbf{I} - \mathbf{P}$, (c) $\mathbf{I} - 2\mathbf{P}$.

9. (a) $\begin{pmatrix} 9 & -10 & 7 \\ -4 & 5 & -3 \\ 1 & -1 & 1 \end{pmatrix}$, (b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} -2 & 1 & -1 \\ 1 & 2 & 1 \\ 3 & 2 & 2 \end{pmatrix}$.

10. (i) $\begin{pmatrix} -2 & -1 \\ 1\frac{1}{2} & \frac{1}{2} \end{pmatrix}$, (ii) no inverse, $\Delta = 0$, (iii) no inverse, $\Delta = 0$,
(iv) no inverse (not square), (v) $\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$.

11. $\frac{1}{5}\begin{pmatrix} -1 & 9 & -3 \\ -2 & 3 & -1 \\ 2 & -13 & 6 \end{pmatrix}$; $\frac{1}{5}(-a + 9b - 3c)$, $\frac{1}{5}(-2a + 3b - c)$, $\frac{1}{5}(2a - 13b + 6c)$.

12. $\begin{pmatrix} 18 & -5 & 2 \\ -10 & 3 & -1 \\ 7 & -2 & 1 \end{pmatrix}$; 7, −3, 3; $\begin{pmatrix} 5 & -1 & 1 \\ 4 & -1 & 0 \\ -3 & 1 & 0 \end{pmatrix}$.

13. $\begin{pmatrix} -1 & 1 & 1 \\ -7 & 5 & 4 \\ -4 & 3 & 2 \end{pmatrix}$; 1, 0, −1; $\begin{pmatrix} 3 & -4 & 6 \\ 11 & -21 & 33 \\ 6 & -12 & 19 \end{pmatrix}$.

14. $2, -7/3$; $\alpha = 2$: $\lambda, 4\lambda - 1, 1 - 3\lambda$; $\alpha = -7/3$: $\lambda, 4\lambda + 10/3, -3\lambda - 17/9$.

15. (i) $\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}$, (ii) 5, 3; $\mathbf{K}$ does not satisfy $R_3 = 5R_1 + 3R_2$.

16. $\begin{pmatrix} 11 & -18 & 0 \\ 6 & -10 & 0 \\ 6 & -9 & -1 \end{pmatrix}$.

17. 5; $\lambda, 3 - 2\lambda, \lambda$; line mapped onto point $(3, 6, 9)$.

18. $\begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -2 & 1 \end{pmatrix}$.

19. (i) $-4/3$, $3/2$, (ii) 0, 1, 3/2; $a = -4/3$, no solutions; $a = 3/2$, solutions exist.

Exercise 19.1

1. Group. 2. No inverses. 3. 0 has no inverse.
4. Not closed. 5. Group. 6. Group. 7. No inverses.
8. Group 9. Not associative. 10. No identity or inverses.
11. No identity or inverses. 12. Group. 13. Not closed, no inverses.
14. Not associative, no identity or inverses. 15. Group. 16. Group.
17. 1 has no inverse. 20. $1, -1, \frac{1}{2}(-1 + i\sqrt{3}), \frac{1}{2}(-1 - i\sqrt{3}), \frac{1}{2}(1 - i\sqrt{3})$.

21. (a) Not closed, not associative, s has no inverse.
(b) Not associative, no identity or inverses.
(c) Not associative, q has no inverse. (d) Group.
22. (a) $\{q\}, \varnothing$, (b) $\{p\}, \{p\}$, (c) $\{s\}, \{p, q\}$, (d) $\{r\}, \{q\}$.

24.

$*$	I	A	B	C
I	I	A	B	C
A	A	B	C	I
B	B	C	I	A
C	C	I	A	B

(a) B, (b) I.

15. (i) $\frac{1}{2}$; $1/4x$; (ii) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

Exercise 19.2

1. No. 3. (a) 4, (b) 2, (c) 1, (d) 3, (e) 4, (f) 3.
4. (a) 6, (b) 6, (c) 5, (d) 3, (e) 5, (f) 3.
5. $\{1\}, \{5\}, \{6\}, \{1, 9\}, \{6, 4\}, \{6, 2, 4, 8\}, \{1, 3, 7, 9\}$.
6. $\{1, 5\}, \{1, 7\}, \{1, 11\}, \{4, 8\}, \{9, 3\}$.
7. $\{1, 2, 4, 8\}, \{1, 4, 11, 14\}, \{1, 4, 7, 13\}, \{6, 3, 9, 12\}$.

Exercise 19.3

1. (a) B, (b) R_2, (c) R_1, (d) R_1.

3.

	I	R_1	R_2	R_3	X	Y	A	B
I	I	R_1	R_2	R_3	X	Y	A	B
R_1	R_1	R_2	R_3	I	A	B	Y	X
R_2	R_2	R_3	I	R_1	Y	X	B	A
R_3	R_3	I	R_1	R_2	B	A	X	Y
X	X	B	Y	A	I	R_2	R_3	R_1
Y	Y	A	X	B	R_2	I	R_1	R_3
A	A	X	B	Y	R_1	R_3	I	R_2
B	B	Y	A	X	R_3	R_1	R_2	I

4. (a) 2, (b) 4, (c) 10, (d) infinite. 5. (a) 8, (b) 4, (c) 8, (d) 12.
6. (a) False; parallel mirror lines produce a translation.
(b) True.
(c) False: a translation perpendicular to mirror line produces a reflection.
(d) False: equal and opposite rotations produce a translation.
(e) False: only true if centre of rotation is on mirror line, otherwise result is a glide-reflection.

7. (a) $\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}$, (b) $\begin{pmatrix}1&2&3&4\\4&3&2&1\end{pmatrix}$,
   (c) $\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}$, (d) $\begin{pmatrix}1&2&3&4\\1&3&2&4\end{pmatrix}$.
8. $\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}, \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}, \begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}, \begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}, \begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}, \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$.
9. (a) 3, (b) 2, (c) 5, (d) 6, (e) 8.
10. (a) Rotation through 180°. (b) Reflection in BD.
    (c) Reflection in line through mid-points of AD and BC.
    No.

Exercise 19.4

3. $\{\varnothing, X\}, \{\varnothing, Y\}, \{\varnothing, Z\}$. 4. $\{e, g\}$.
5. $\{\mathbf{I}, \mathbf{A}, \mathbf{B}\}, \{\mathbf{I}, \mathbf{C}\}, \{\mathbf{I}, \mathbf{D}\}, \{\mathbf{I}, \mathbf{E}\}$. 6. $\{I, R_1, R_2\}, \{I, A\}, \{I, B\}, \{I, C\}$.
7. $\{10, 8\}, \{10, 4, 16\}$. 8. $\{1, -1\}, \{1, -1, i, -i\}$.
9. $\{I, B\}, \{I, D\}, \{I, E\}, \{I, F\}, \{I, G\}, \{I, A, B, C\}$.
10. $\{I, H_1\}$, $\{I, H_2\}$, $\{I, H_3\}$, $\{I, X\}$, $\{I, Y\}$, $\{I, Z\}$, $\{I, C\}$, $\{I, H_1, H_2, H_3\}$, $\{I, H_1, X, C\}$, $\{I, H_1, Y, Z\}$, $\{I, H_2, Y, C\}$, $\{I, H_2, X, Z\}$, $\{I, H_3, Z, C\}$, $\{I, H_3, X, Y\}$.
11. (a) Yes, (b) No, (c) No, (d) Yes.
12. (a) No, (b) Yes, (c) No, (d) Yes. 13. No. 14. No.
15. 1, 2, 3, 4. 16. $i, -i$. 17. No. 18. Rotations of 60° or 300°.
19. 3, 5. 20. No.
21. x, x^3, x^5, x^7; e: 1, x^2: 4, x^4: 2, x^6: 4.
22. All except e which has order 1.
23. I: 1, A: 4, B: 2, C: 4, D: 4, E: 4, F: 4, G: 4.
24. (i) I: 1, A: 4, B: 2, C: 4, D: 2, E: 2, F: 2, G: 2.
    (ii) $\{I, B\}, \{I, D\}, \{I, E\}, \{I, F\}, \{I, G\}$.
    (iii) $\{I, A, B, C\}$; $B = A^2$, $C = A^3$, $I = A^4$.
    (iv) $\{I, B, D, F\}$ or $\{I, B, E, G\}$, no element of order 4.
    (v) I, B.
25. 4, 4.

Exercise 19.5

1. (a), (b), (c) and (e); (d) and (f).
2. (a), (d) and (f); (b), (c) and (e).
3. Rectangle or rhombus; given groups each have element of order 4, suggested symmetry groups do not.
4. The set does not contain the identity and is not closed.
8. a, d.

Exercise 19.6

2. Identity is 0; true.
3. e, e, b, d; closed, identity, inverses; $\{e\}$, $\{e, a\}$, $\{e, b\}$, $\{e, c\}$, $\{e, d\}$.
7. 3, 4, 5, 9. 8. 24.
9. p_1: 1, p_2: 3, p_3: 3, p_4: 2, p_5: 2, p_6: 2; $\{p_1, p_2, p_3\}$; p_2 or p_3; $\{p_1, p_4\}$, $\{p_1, p_5\}$, $\{p_1, p_6\}$.
10. (i) Identity 8, inverse elements 2 and 4, 10 and 12, 6 is self-inverse; 7 has no inverse.
    (ii) Rectangle or rhombus; square.
11. (a) (i) $a = b = 0$, $\begin{pmatrix} \exp(-a) & -b \\ 0 & \exp(a) \end{pmatrix}$, (ii) Two possibilities are given by $a \in \mathbb{R}$, $b = 0$ and $a \in \mathbb{Z}$, $b \in \mathbb{R}$.
13. Top right: $\mathbf{B}$ $\mathbf{BA}$ $\mathbf{BA}^2$ / $\mathbf{BA}^3$ $\mathbf{B}$ $\mathbf{BA}$ / $\mathbf{BA}^2$ $\mathbf{BA}^3$ $\mathbf{B}$; Bottom left: $\mathbf{BA}^2$ $\mathbf{BA}^3$ $\mathbf{B}$ / $\mathbf{BA}^3$ $\mathbf{B}$ $\mathbf{BA}$ / $\mathbf{B}$ $\mathbf{BA}$ $\mathbf{BA}^2$
14. (i) Yes, (ii) no, set does not contain inverses.
15. $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$;

    I, A, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$;

    $\left\{\begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix}, \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix}\right\}$, $\left\{\begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix}, \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix}\right\}$,

    $\left\{\begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix}, \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix}\right\}$; $\left\{\begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix}, \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}, \begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix}\right\}$.
16. $\mathbf{M}_3$ $\mathbf{M}_1$ $\mathbf{M}_6$ / $\mathbf{M}_1$ $\mathbf{M}_2$ $\mathbf{M}_5$ / $\mathbf{M}_5$ $\mathbf{M}_6$ $\mathbf{M}_1$ $\quad$ $\{\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3\}$, $\mathrm{F} \leftrightarrow \mathbf{M}_1$, $\mathrm{G} \leftrightarrow \mathbf{M}_2$, $\mathrm{H} \leftrightarrow \mathbf{M}_3$ or $\mathrm{F} \leftrightarrow \mathbf{M}_1$, $\mathrm{G} \leftrightarrow \mathbf{M}_3$, $\mathrm{H} \leftrightarrow \mathbf{M}_2$.
17. Two possibilities are: $(\mathbb{R}, +)$, $\begin{pmatrix} 1 & \ln a \\ 0 & 1 \end{pmatrix} \leftrightarrow \ln a$; $(\mathbb{R}^+, \times)$, $\begin{pmatrix} 1 & \ln a \\ 0 & 1 \end{pmatrix} \leftrightarrow a$.
18. Yes. 20. Yes.

Index

Abel, Niels Henrik, 567
Abelian group, 567
Acceleration, as a function of displacement, 332
  as a function of velocity, 335
  vector form, 194
Acceptance region, 497
Adjoint, 546
Adjugate, 546
Algorithm, 292
Alien cofactors, 546
Alternative hypothesis, 496
Amplitude (of a complex number), 403
Amplitude (of S.H.M.), 342
Angle between two vectors, 180, 201
Angles between lines and planes, 203, 205
Angles in coordinate geometry, 367
Apollonius' circle, 412
Approximate methods, integration, 273, 596
  solution of equations, 281, 287
  using series, 259, 266, 278
Arc, length of circular, 595
Argand, Jean Robert, 399
Argand diagram, 399, 409
Argument (of a complex number), 403
Arithmetic progression, 251, 594
Associative binary operation, 565
Asymptotes, 34
  to a hyperbola, 384, 597

Base vectors, 179
Bernouilli, Jacques, 434
Bernouilli trial, 434
Binary operation, 564
Binomial distribution, 434, 483, 598
  normal approximation, 453, 484
  Poisson approximation, 459
  probability generating function, 438
  sum of binomial variables, 468
Binomial series, 257, 258, 596
Binomial theorem, 594
  proof of, 11
Bisection process, 292
Bivariate distribution, 509
Bivariate frequency distribution, 516

Cartesian equation of a plane, 207
Cartesian equations of a line, 207
Central inversion, 535, 575
Central limit theorem, 482
Centre of gravity, 113, 118
  composite bodies, 116
  derivation of standard results, 120
  set of particles, 114
  triangular lamina, 113
Centre of mass, 118
  standard results, 597
Centroid, 118
  plane area, 128, 130
  triangle, 181
  volume of revolution, 130
Change of variable, in differential equations, 310
  in integration, 77
Circles, 372, 376, 382, 597
  in the Argand diagram, 410, 412
Circular motion, 194, 351, 354
Closed binary operation, 564
Codomain, 17
Coefficients, comparing, 32
Cofactor, 543, 546
Common chord, 373
Commutative binary operation, 565
Complex numbers, 391
  geometrical representation, 399
Complex roots of equations, 391, 395
Component vectors, 179, 210
Components in given directions, 200, 210
Composite bodies, 116, 138
Composite function, 19
Composite transformation, 537, 541
Confidence interval, 487, 491
Confidence limits, 492, 527
Conic section, 382
Conjugate, complex, 392, 393, 400
Conservation of linear momentum, 228, 241
Conservation of mechanical energy, 233
Constant acceleration formulae, 597
Continuity correction, 454, 484
Continuous function, 20
Continuous probability distribution, 442
Continuous random variable, 427
Contradiction, proof by, 7
Converse, 8
Coplanar forces, 105
  see also Equilibrium conditions
Correlation, 511
  linear, 511, 512
  negative or inverse, 511
  non-linear, 511
  positive or direct, 511

quadratic, 511
rank, 518
Correlation coefficient, 514, 516, 599
of random variables, 517, 598
Correlation coefficients, rank, 519, 521, 599
cosech x, 65
cosh x, 65, 596
coth x, 65
Counter-example, 8, 565, 568
Couple, 101, 103
Covariance, of random variables, 466, 517, 598
using sample data, 514, 599
Critical values, 498
Cube roots of unity, 396
Cubic equations, sum and product of roots, 31
Cumulative distribution function, 444
standard normal variable, 451, 600
Curve sketching, *see* Sketch graphs
Cusp, 377
Cyclic group, 580

Decision rule, 497
Deduction, 2, 6
Definite integral, properties of, 90
De Moivre, Abraham, 418
De Moivre's theorem, 417
Density function, *see* Probability density function
Derivative, 20
Derived function, 21
Determinant, 541, 543, 546
Difference between means, 482, 494, 598
Differences, method of, 253
Differentiable function, 21
Differential equations, 303
$d^2y/dx^2 \pm n^2y =$ constant, 321, 595
formulation, 318
separable variables, 313
solution curves, 303, 305
use of a substitution, 310
Differentiation, standard formulae, 595
Direct impact, 239, 241
Direction ratios, 207
Direction vector, 184
Directrix, 383
Discontinuity, 20
Discrete probability distribution, 428
Discrete random variable, 427
Distance from a point to a line, 369, 597
Distribution function, (cumulative), 444
standard normal variable, 451, 600
Division of line in a given ratio, 181, 368, 596
Domain, 17
Dot product, 198

e, definition of, 50
Eccentricity, 383
Echelon form, 550
Efficient estimator, 487
Elastic impact, 238
Elastic strings (in S.H.M.), 346
Ellipse, 377, 382, 597
Energy equations, 212, 232, 233
Energy stored in an elastic string, 597
Enlargement matrix, 535
Equating coefficients, 32
Equating real and imaginary parts, 393
Equations, *see* Solution of equations
Equilibrium conditions, 136, 153, 157
three forces, 144, 146
two forces, 137
Equivalent statements, 4
Equivalent systems of forces, 107
Error, of estimation, 524
Errors, Types I and II, 500
Estimate, 487
of mean and variance, 599
using pooled sample data, 488, 599
Estimator, 487
see also Unbiased estimator
Even function, 17, 91
exp x, 50, 59
Expansions, *see* Series
Expectation, continuous variable, 442, 462, 598
discrete variable, 429, 462, 598
function of a random variable, 462, 598
linear combination of random variables, 465, 598
linear function of a random variable, 463, 598
product of independent random variables, 465, 598
sum of random variables, 465
using probability generating function, 438
Expectation algebra, 462, 598
Expected frequency, 437
Expected value, 429
Exponent, 49
Exponential functions, 49, 60
Exponential series, 257, 264, 596

Factor theorem, 30
Fermat, Pierre de, 1
Finite arithmetic, 571
Finite group, 566
Finite series, 251, 594
First moment of area, 128
Focus, 383
Frameworks, 167
Frequency, of S.H.M., 342
Frequency distribution, 429
bivariate, 516
Function, 17, 23

Galton, Sir Francis, 524
General solution of a differential equation, 303
Generator of a group, 580
Geometric distribution, 429
Geometric progresssion, 251, 594
Graphs, *see* Sketch graphs
Group, 566
Abelian, 567
cyclic, 580
non-Abelian, 567
Groups, isomorphic, 584
Growth and decay, laws of, 50, 318

Half-line in the Argand diagram, 410
Hooke's law, 597
Hyperbola, 382, 384, 597
rectangular, 34, 376, 385, 597
Hyperbolic function, 65, 596
Hypothesis, alternative, 496
null, 496

Identity element, 564, 578
Identity matrix, 536
Identity transformation, 536
Image, 17
Imaginary axis, 399
Imaginary part, 391, 393
Imaginary number, 391
Impact, 223, 228
direct, 239, 241
indirect, oblique, 239, 240

Implication, 4, 6
Impulse of a force, 221
Impulse-momentum equation, 222, 232
Impulsive tension, 224
Independent random variables, 465, 480, 517
Indirect impact, 239, 240
Induction, proof by, 9
Inductive reasoning, 1
Inelastic impact, 239
Infinite group, 566
Integers modulo *n*, 571
Integral curves, 303
Integration, by parts, 83
  by substitution, 77
  numerical, 273, 596
  standard forms, 69, 595
  using inverse trigonometric functions, 87
  using logarithms, 61
  using partial fractions, 73
Interval bisection, 292
Interval estimation, 487, 491
Inverse element, 564, 578
Inverse function, 26
Inverse matrix, 536, 540, 541, 546, 555
Inverse transformation, 536
Inverse trigonometric functions, 87
Isometries, 575
Isomorphism, 584
Iterative processes, 288, 291
  general formula, 292

Joint probability distribution, 464, 517

Kendall, Sir Maurice George, 521
Kendall's rank correlation coefficient, 521, 599
Klein, Felix, 585
Klein four-group, 585

Lagrange, Joseph Louis, 579
Lagrange's theorem, 579
Lami's theorem, 147
Least squares, method of, 524
  regression line, 525, 599
Level of significance, 497
Light frameworks, 167
Light rods, 167
Line, *see* Straight line
Line of action of resultant, 99, 105
  parallel forces, 100, 105
  vector equation, 190
Linear combinations, of normal variables, 467
  of random variables, 465, 466, 598
Linear correlation, 511, 512
  coefficient of, 514, 516, 598, 599
Linear equations, simultaneous, 548
Linear interpolation, 284, 451
Linear transformations, 533
ln x, 54, 59
Location of roots, 282
Loci, in the Argand diagram, 409
  using parametric equations, 376
Logarithmic differentiation, 56
Logarithmic functions, 54
Logarithmic series, 257, 265, 596
Logarithms, laws of, 54, 594
  Napierian, 54
  natural, 54

Maclaurin, Colin, 256
Maclaurin's series, 256, 596
Maclaurin's theorem, 256
Major axis, 383
Marginal distribution, 465
Matrix, adjoint, 546
  determinant of, 541, 543, 546
  inverse, 536, 540, 541, 546, 555
  non-singular, 536, 547
  of a linear transformation, 533
  singular, 536, 547
  transpose of, 541
  triangular, 550
Mean, binomial distribution, 436, 598
  continuous probability distribution, 442, 462
  discrete probability distribution, 429, 462
  frequency distribution, 429
  Poisson distribution, 458, 598
  using probability generating function, 438
  see also Expectation
Mean of a population, confidence limits, 492, 493
  unbiased estimator, 487
Median, 443
Method of differences, 253
Method of least squares, 524
Minor, 542
Minor axis, 383
Mode, 443
Modulo arithmetic, 571
Modulus, of a complex number, 401, 403
Modulus-argument form, 405
Moment, of a couple, 103
  of a force, 101
  of area, 128
Moments, principle of, 103
Momentum, 222
  conservation of, 228, 241
  continuous change in, 223

Napier, John, 54
Napierian logarithms, 54
Natural logarithms, 54
Necessary condition, 5
Negation, 5
Newton-Raphson method, 287, 596
Newton's experimental law, 239
Newton's law of cooling, 50, 318
Newton's law of restitution, 239, 241
Newton's second law, vector form, 195
Non-Abelian group, 567
Non-singular matrix, 536, 547
Normal approximation,
  to binomial distribution, 453, 484
  to Poisson distribution, 460
Normal curve, 450
Normal distribution, 449
  as a limit of the binomial distribution, 453
  linear combination of normal variables, 467
  sum and difference of normal variables, 467
  table of percentage points, 601
Normal distribution function, 451
  table of values, 600
Null hypothesis, 496
Numerical methods, 273, 596

Oblique impact, 239, 240
Odd function, 17, 91
One-one function, 26
One-tailed test, 498
Operation table, 565
Order of, differential equation, 303
  element, 580

group, 566
subgroup, 579
Orthogonal circles, 373
Orthogonal projection, 535, 537
Oscillations, simple harmonic, 341, 346

Parabola, 376, 382, 597
Parallel forces, 100, 105, 136
Parameter, of a population, 486
Parametric equations of curves, 376
Partial fractions, 39
Partial fractions, applications
to integration, 73
to series expansions, 262
to summation of series, 253
Particular integral, 303
Pendulum, 351
Percentage points of the normal distribution, 452, 601
Period, of element, 580
of function, 18
of S.H.M., 342
Periodic function, 18
Permutation groups, 575
Perpendicular vectors, 198, 201
Plane, Cartesian equation, 207
vector equations, 187, 204
Plane isometries, 575
Planes, angle between, 205
intersections between, 551
Point dividing line in given ratio, 181, 368, 596
Point estimate, 486
Poisson, Siméon Denis, 457
Poisson approximation to the binomial distribution, 459
Poisson distribution, 457, 598
as a limit of the binomial distribution, 457
normal approximation, 460
probability generating function, 461
sum of Poisson variables, 468
Polynomial equations, 31, 395
repeated roots, 31
Polynomials, 30
Pooled sample estimates, 488, 599
Population, 480
Population parameter, 486
Power, vector approach, 213
Principal argument, 403
Probability density function, 442
normal variable, 449
standard normal variable, 450
Probability distribution, continuous, 442
discrete, 428
joint, 464, 517
marginal, 465
Probability function, 428
Probability generating function, 438, 461
Probability model, 427, 480
Product moment correlation coefficient, 514
Projection of a vector, 200
Proof, 1, 7
by contradiction, 7
by induction, 9
Proper fraction, 39
Proper subgroup, 579
Proportion, 483, 487, 494

Quadratic approximation, 260, 267, 285

Random sample, 480
Random trial, 427
Random variables, 427
independent, 465, 480, 517
see also Expectation, Variance
Range, 17
of a rational function, 35
Rank correlation, 518
Kendall's coefficient, 521, 599
Spearman's coefficient, 519, 599
Ranks, 519
tied, 520, 522
Raphson, Joseph, 288
Ratio theorem, 180, 191
Rational function, 34
Real and imaginary parts, 391, 393
Real axis, 399
Rectangular distribution, 428, 443
Rectangular hyperbola, 34, 376, 385, 597
Reduction method (inverse matrices), 555
Reduction to echelon form, 550
Regression, 523
Regression coefficient, 525
sampling distribution, 526
Regression lines, 523
y on x, 524, 525, 599
x on y, 525
Rejection region, 497
Relation, 22
Relative frequency, 429
Relative motion, 193
Remainder theorem, 30
Repeated factor, 31
Residual variance, 527
Residuals, 524
Residues, algebra of, 571
Restitution, coefficient of, 239
Newton's law of, 239, 241
Resultant, 107
line of action, 99, 105, 190
using ratio theorem, 191
Roots of equations, location, 282
Newton-Raphson approximation, 287
real and complex, 395
sums and products, 31
Row operations, 550, 555
Row reduction method, 555

Sample, 480
covariance, 514, 599
mean, 481
proportion, 483
variance, 484, 599
Sample space, 427
Sampling distribution, 480
difference between means, 482, 598
mean, 481, 598
proportion, 483
regression coefficient, 526
variance, 484, 598
Scalar product, 198
Scatter diagram, 509
sech x, 65
Seconds pendulum, 352
Sector, area of, 595
Semi-cubical parabola, 377, 597
Series, binomial, 257, 596
exponential, 257, 264, 596
finite, 251, 594
logarithmic, 257, 264, 596
Maclaurin's, 256, 596
sin x and cos x, 258, 269, 596

Shear, matrix of, 536
Significance level, 497
Simple harmonic motion, 341
  forces producing S.H.M., 346
  standard formulae, 342
Simple pendulum, 351
Simpson, Thomas, 275
Simpson's rule, 275, 596
Simultaneous linear equations, 548
Singular matrix, 536, 547
sinh x, 65, 596
Sketch graphs, odd and even functions, 17
  periodic functions, 18
  rational functions, 34
  $y = |f(x)|$, 23
  $y = 1/f(x)$, 23
  $y = \{f(x)\}^2$, 24
  $y^2 = f(x)$, 25
Skew distribution, 482
Skew lines, 185
Skew-symmetric matrix, 560
Smooth surfaces, 136
Smoothly hinged rods, 162
Solution curves, 303, 305
Solution of equations, graphical methods, 281
  iterative processes, 288, 291
  linear interpolation, 284
  location of roots, 282
  Newton-Raphson method, 287
  quadratic approximation, 285
Solution of sets of linear equations, 548
Spearman, Charles Edward, 519
Spearman's rank correlation coefficient, 519, 599
Springs (in S.H.M.), 346
Standard error, 481
Standard normal distribution, 450
Statistic, 480, 486
Statistical tables, 600
Straight line, Cartesian equations, 207
  vector equation, 184, 203
Subgroup, 579
  proper, 579
Sufficient condition, 5
Sum and product of roots, 31
Summation of finite series, 251
Symmetric group, 576
Symmetric matrix, 560
Symmetries, groups of, 573
Symmetry, of graphs, 17
Systematic elimination (or reduction), 549, 555
Systems of, coplanar forces, 105
  forces in vector form, 189
  rigid bodies, 161

t-distribution, 493
Tables, statistical, 600
tanh x, 65
Tension in a light rod, 168
Test statistic, 496
Testing a hypothesis, 496, 498
Tests of significance, 496
  one-tailed, 498
  two-tailed, 498
Thrust in a light rod, 168
Tied ranks, 520, 522
Toppling and sliding, 139, 141
Torque, 101
Transformations, composite, 537, 541
  inverse, 536
  isometries, 575
  linear, 533
  symmetries, 573
Transpose of a matrix, 541
Trapezium rule, 274, 596
Triangle, area of, 200, 370, 595, 597
Triangle inequalities, 401
Triangular matrix, 550
Trigonometry, standard formulae, 594
Two-sample estimates, 488, 599
Two-tailed test, 498
Type I error, 500
Type II error, 500

Unbiased estimate, of mean and variance, 487, 599
Unbiased estimator, 487
  of mean, 487
  of proportion, 487
  of regression coefficient, 527
  of variance, 487
  using two samples, 488
Uncorrelated variables, 511
Uniform distribution, 428, 443
Unity, cube roots of, 396

Variance, binomial distribution, 436, 598
  continuous probability distribution, 442, 462, 598
  discrete probability distribution, 429, 462, 598
  frequency distribution, 429
  in terms of expected values, 464, 598
  linear combination of random variables, 466, 598
  linear function of a random variable, 464, 598
  Poisson distribution, 458, 598
  sums and differences of random variables, 466
Variance of a population, unbiased estimator, 487
Vector equation of a line, 184
Vector equation of a plane, parametric form, 187
  scalar product form, 204
Vectors, angle between, 180, 201
  basic properties, 179
  scalar product, 198
  use in kinematics, 193
  use in mechanics, 195, 210
Velocity, as a vector, 193
Vertical circle, motion in, 354

Weight, 112
Work done, vector approach, 210
Work-energy equation, 212, 232

This is the second of two books covering most single-subject Advanced Level Mathematics syllabuses. It provides the remaining work (about 40%) to be studied in the second year of the course.

The two volumes together will provide a thorough treatment of the Pure Mathematics and Mechanics required at this level, and also cover the options in Modern Algebra, Numerical Methods and Probability and Statistics.

Topics are presented in a logical sequence and are arranged in short chapters to allow flexibility. The text contains many worked examples and numerous exercises providing both routine practice and opportunities to develop understanding.

Bell & Hyman Limited

ISBN 0 7135 1322 5